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Jean Claude Dutailly

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SPINORS AND GRAVITATIONAL FIELD

Jean Claude Dutailly Paris (France)

4 July 2015

Abstract

Spinors have been used in Particle Physics since the Dirac's equation. However their physical meaning is still obscure. In this paper we show that Spinors, vectors of a 4 dimensional complex vector space, can be used to represent the kinematic characteristics of particles, encompassing both their transversal and their rotational parts. The framework used is the geometry of General Relativity, which is presented in a comprehensive and consistent way, by the use of fiber bundles. Spinors can be used at any scale, and the definition of a deformable solid body is introduced. The gravitational field is treated as a gauge field, through a connection on a fiber bundle. Using the spinor representation the action of the gravitation field on a material body, with the covariant derivative, takes then a simple form. The propagation of the field is studied by a two form valued in the Lie algebra, similar to the Riemann tensor. In this formulation, more general but fully compatible with the more traditional approach starting from the metric and the scalar curvature, the structure of the gravitational field can be explored, showing the existence of a rotational and a transversal component, and quantized, the spin being similar to the usual 3 dimensional gravity.

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Mechanics is the mother of Physics : it states the rules linking the motion of material bodies to the forces exercised by the fields. Material bodies oppose a resistance in the change of their motion, their inertia is represented by the momenta which combine the motion and characteristics of the material body similar to mass. So they make the link between Geometry, where the motion of material bodies are represented, and Particle Physics, in that particles are the ultimate components of material bodies and provide the intrinsic characteristics (independent from the motion) which are the other part of the momenta (mass and spin). Kinematics is the part of Mechanics which deals with inertia, and thus with momenta, and the purpose of this paper is to provide a comprehensive and consistent formalism to kinematics.

To achieve this we need first a strong formalism of Geometry, to represent the motion of material bodies. The first part of this paper is dedicated to Geometry. Relativity has deeply changed the way we see the physical universe, and is an unavoidable basis for a Physical Geometry. The issue is not to give a picture of the whole Universe, which is the topic of Cosmology, but to provide a theory which tells us how we can represent the location and the motion of material bodies, what are the key variables which can be used and measured, and how different observers can compare their measures. This last point is essential because Relativity underlines the fact that each observer has his own window to observe the world, and what they see through their window is different from what see their neighbours. We will do it in the framework of General Relativity (GR), because it is the most general, and also because, by compelling us to leave the usual formalism of cartesian frames it helps to focus on understanding the physical meaning of the concepts of location and velocity. General Relativity encompasses both a theory of the Geometry of the Universe and a theory of gravitation, which are linked but actually distinct. We will focus first only on the Geometry. General Relativity is often seen as difficult to understand and use, but this can be greatly alleviated by the use of the right mathematical tools, which are fiber bundles. In this first part there will be not new results : all the assumptions and the consequences which are drawn are in line with the commonly accepted GR, the tetrad formalism is well proven and the use of fiber bundles is standard in Gauge Theories. What is new is more the compact and consistent framework which is presented.

This framework is necessary to introduce spinors. Momenta can be easily understood and formalised for translational motion, or for rigid solids. But issues appear when one considers rotation. Actually, even in the formalism of Newtonian Mechanics there is a discrepancy between the way Physicists and Mathematicians see a rotation : the latter have a clear and consistent representation, through orthonormal frames, Lie groups and algebras, but what matters for the Physicists is the rotational momentum itself, which is a different quantity. This discrepancy is not really acknowledged and has led to a formalism and computational rules which are, perhaps efficient, but not really consistent : think about the generalised usage of cross-product or the definition of a torque. The situation is worse when one considers particles which have no internal structure, for which geometric rotation cannot be defined, or the relativist geometry

: rotation in a 4 dimensional universe is totally different from a rotation in a 3 dimensional one. The transposition of the usual computational practices has lead to the introduction of new concepts, the spin of particles to represent the impossible rotation of bodies without internal structure, and spinors to provide a variable which could represent the momentum in the relativist context. Meanwhile a great emphasize has been given to the Poincaré group, which is just the extension of the mathematicians point of view in the Minkovski space, but without real physical meaning. We propose here a new point of view : acknowledging fully the need to define a quantity which can represent the momentum of a particle, in the GR geometry, this quantity must be intrinsic : it represents the inertia of a particle, it combines motion and physical characteristics which are independant of the observer (motion is absolute in GR), but whose measure changes as a geometric quantity. The fiber bundle formalism leads to the definition of this relativist momentum and its representation by a spinor : a vector belonging to a 4 dimensional vector space E . This is fully similar to the definition used in Quantum Theory of Fields, but we will introduce the vector space E as a representation of a Clifford algebra. This mathematical structures is not new, it provides many efficient tools to represent physical quantities such as the motion of a material body, incorporating both the translational and the rotational motion in a single object and the spin. The screening of E leads naturally to the distinction between particles and antiparticles. In the formalism of fiber bundles, matter fields are naturally represented as sections of a vector bundle, and this leads to the definition of the spin, which can then be quantized.

The introduction of spinors will be done for particles, defined here as any material body which, at the scale of the study, can be considered as occupying a single point of the Universe, and without any internal structure (but for which we will be able to define a spin). They are the equivalent of the material points of classical Mechanics. However the issue of composite bodies stays : if particles are the bricks of matter we must find a way to represent easily (that is other than through tensor products and Fock spaces) solid bodies. The usual concept of solid cannot be transposed in Relativity, however we wil introduce a consistent definition, first of a material body (that is with a spatial extension larger than a point) and then of a deformable solid through spinors, which enables to do manageable computations in the GR framework, a solution which can be useful in Astro-Physics.

The third part is dedicated to the gravitational field. The formalism introduced previously leads naturally to treat the gravitational field as a gauge field, with the introduction of a connection which is more general than the Levy-Civita connection. Similarly the propagation of the field is studied through a 2 form valued in the Lie algebra, which is similar to the Riemann tensor, but more general and easier to use. In this framework it is possible to study the structure of the gravitational field, with its transversal and rotational component, to define its spin and proceed to its quantization.

This paper is actually part of a larger study. It uses many, well known and some new, mathematical results whcih can be found in my book "Mathematics

for Theoretical Physics” and will be referred to as (Maths.XXX). The quantization of spinors is done using theorems which are proven in my book ”Quantum Mechanics Revisited v.3” and will be referred to as (QMR.XXX). The usage of spinors in gauge fields theory, and notably with gravitation, is exposed in my book ”Mathematics in Physics”. All are freely available.

1 GEOMETRY OF GENERAL RELATIVITY

Almost all, if not all, measures rely on measures of lengths and times. These concepts are expressed in theories about the geometry of the universe, meaning of the container in which live the objects of physics. The issue here is not a model of the Universe, seen in its totality, which is the topic of Cosmology, but a model which tells us how to measure lengths and times, and how to compare measures done by different observers. Such a model is a prerequisite to any physical theory. Geometry, as a branch of Mathematics, is the product of this quest of a theory of the universe, and naturally a physical geometry is formalized with the tools of Mathematical Geometry. There are several Geometries used in Physics : Galilean Geometry, Special Relativity (SR) and General Relativity (GR).

In this first section we will see how such a geometry can be built, from simple observations. We will go directly to the General Relativity model. This is the one which is the most general and will be used in the rest of the paper. It is said to be difficult, but actually these difficulties can be overcome with the right formalism. Moreover it forces us to leave usual representations, which are often deceptive.

1.1 Manifold structure

1.1.1 The Universe has the structure of a manifold

The first question is how do we measure a location ?

For the spatial location one can use a chart, meaning any procedure which relates a point to a system of coordinates. We need three scalar coordinates. There are many maps which are in use : on Earth the geographic coordinates with longitude, latitude and elevation, in Astronomy the celestial coordinates system, and in an experiment in the laboratory the position with respect to a trajectory. Of course a system of Cartesian coordinates, which measures distances with respect to a point (which itself must be located) provides a system of coordinates, even if it is rarely used.

For the temporal location one uses the coincidence with any agreed upon event. For millennia men used the position of celestial bodies for this purpose. Say "See you at Stonehenge at the spring's equinox" and you will be understood. Of course one can use a clock, but the purpose of a clock is to measure elapsed time, so one needs a clock and a starting point, which are agreed upon, to locate an event in time.

When necessary, one can use several charts to cover an area. The key point is that the charts are compatible : there are mathematical functions, transitions maps, which relate the coordinates of the same point in different charts.

A collection of compatible charts, each defined in an open subset of a vector space and valued in a given set M, is an atlas. A collection of compatible atlas over a set M defines the structure of a manifold.

1.1.2 What is a manifold ?

Let M be a set, E a topological vector space, an atlas, denoted $A = (O_i, \varphi_i, E)_{i \in I}$ is a collection of :

subsets $(O_i)_{i \in I}$ of M such that $\cup_{i \in I} O_i = M$ (this is a cover of M)

maps $(\varphi_i)_{i \in I}$ called **charts**, such that :

i) $\varphi_i : O_i \rightarrow U_i :: \xi = \varphi_i(m)$ is bijective and ξ are the coordinates of M in the chart

ii) U_i is an open subset of E

iii) $\forall i, j \in I : O_i \cap O_j \neq \emptyset :$

$\varphi_i(O_i \cap O_j), \varphi_j(O_i \cap O_j)$ are open subsets of E, and there is a bijective, continuous map, called a transition map :

$\varphi_{ij} : \varphi_i(O_i \cap O_j) \rightarrow \varphi_j(O_i \cap O_j)$

Notice that no mathematical structure of any kind is required on M. A topological structure can be imported on M, by telling that all the charts are continuous, and conversely if there is a topological structure on M the charts must be compatible with it.

Two atlas $A = (O_i, \varphi_i, E)_{i \in I}, A' = (O'_j, \varphi'_j, E)_{j \in J}$ of M are said to be compatible if their union is still an atlas. Which implies that :

$\forall i \in I, j \in J : O_i \cap O'_j \neq \emptyset : \exists \varphi_{ij} : \varphi_i(O_i \cap O'_j) \rightarrow \varphi'_j(O_i \cap O'_j)$ which is a homeomorphism

The relation A, A' are compatible atlas of M , is a relation of equivalence. A class of equivalence is a **structure of manifold** on the set M .

The key points are :

- there can be different structures of manifold on the same set. On \mathbb{R}^4 there are unaccountably many non equivalent structures of smooth manifolds (this is special to \mathbb{R}^4 : on $\mathbb{R}^n, n \neq 4$ all the smooth structures are equivalent !).

- all the interesting properties on M come from E : the dimension of M is the dimension of E (possibly infinite); if E is a Fréchet space we have a Fréchet manifold, if E is a Banach space we have a Banach manifold and then we can have differentials, if E is a Hilbert space we have a Hilbert manifold, but these additional properties require that the transition maps φ_{ij} meet additional properties.

- for many sets several charts are required (a sphere requires at least two charts) but an atlas can have only one chart, then the manifold structure is understood as the same point M will be defined by a set of compatible charts.

The charts define over M a topology, deduced from the vector space. The manifold is differentiable (resp. smooth) if the transition maps are differentiable (resp. smooth) (Maths.15.1.1).

1.1.3 The manifold structure of the Universe

So, at least in a vast area which is accessible to our observations, that we will denote by Ω , where there is no singularity such as black hole, we can say that the physical universe can be represented as a manifold, that we will denote M .

Proposition 1 *The Universe can be represented as a four dimensional real manifold M*

In Galilean Geometry the manifold is the product of \mathbb{R} with a 3 dimensional affine space, and in SR this is a 4 dimensional affine space (affine spaces have a manifold structure).

We will limit ourselves to an area Ω of the universe, which can be large, so that one can assume that one chart suffices. We will represent such a chart as a map :

$$\varphi_M : \mathbb{R}^4 \rightarrow \Omega :: \varphi_M (\xi^0, \xi^1, \xi^2, \xi^3) = m$$

which is assumed to be bijective and smooth, where $\xi = (\xi^0, \xi^1, \xi^2, \xi^3)$ are the coordinates of m in the chart φ_M .

We will assume that Ω is a relatively compact open in M , so that the manifold structure on M is the same as on Ω , and Ω is bounded.

A change of chart is represented by a bijective smooth map (the transition map) :

$$\chi : \mathbb{R}^4 \rightarrow \mathbb{R}^4 :: \eta^\alpha = \chi^\alpha (\xi^0, \xi^1, \xi^2, \xi^3)$$

such that the new map $\tilde{\varphi}_M$ and the initial map φ_M locate the same point : $\tilde{\varphi}_M (\chi^\alpha (\xi^0, \xi^1, \xi^2, \xi^3), \alpha = 0, ..3) = \varphi_M (\xi^0, \xi^1, \xi^2, \xi^3)$

Notice that there is no algebraic structure on M : $am + bm'$ has no meaning. This is illuminating in GR, but still holds in SR or Galilean Geometry. There

is a clear distinction between coordinates, which are scalars depending on the choice of a chart, and the point they locate on the manifold (affine space or not).

The idea that the Universe could be 4 dimensional is not new. R.Penrose remarked in his book “The road to reality” that Galileo considered this possibility. The true revolution of Relativity has been to acknowledge that, if the physical universe is 4 dimensional, it becomes necessary to dissociate the abstract representation of the world, the picture given by a mathematical model, from the actual representation of the world as it can be seen through measures. And this dissociation goes through the introduction of a new object in Physics : the observer. Indeed, if the physical Universe is 4 dimensional, the location of a point is absolute : there is a unique material body, in space and time, which can occupy a location. Then, does that mean that past and future exist together ? Can we say that this apple, which is falling, is somewhere in the Universe, still on the tree ? To avoid the conundrum and all the paradoxes that it entails, the solution is to acknowledge that, if there is a unique reality, actually the reality which is scientifically accessible, because it enables experiments and measures, is specific : it depends on the observer. This does not mean that it would be wrong to represent the reality in its entirety, as it can be done with charts, frames or other abstract mathematical objects. They are necessary to give a consistent picture, and more bluntly, to give a picture that is accessible to our mind. But we cannot identify this abstract representation, common to everybody, with the world as it is, because the one in which I can move, act and measure, is my world. This is one of the reasons that motivate the introduction of Geometry in this paper through GR : it is common to introduce subtle concepts such as location and velocity through a frame, which is evoked in passing, as if it was obvious, standing somewhere at the disposition of the public. There is nothing like this. I can build my frame, my charts, and from there conceive that it can be extended, and compared to what other Physicists have done. But comparison requires first dissociation, and this is more easily done in a context to which we are less used to, by years of schematic representations.

1.2 The tangent vector space

At each point of a smooth manifold M one can define a set which has the structure of a vector space, with the same dimension as M . The best way to see it is to differentiate the map φ_M with respect to the coordinates (this is close to the mathematical construct). To any vector $u \in \mathbb{R}^4$ is associated the vector $u_m = \sum_{\alpha=0}^3 u^\alpha \partial_\alpha \varphi_M (\xi^0, \xi^1, \xi^2, \xi^3)$ which is denoted $u_m = \sum_{\alpha=0}^3 u^\alpha \partial \xi_\alpha$.

The basis $(\partial \xi_\alpha)_{\alpha=0}^3$ associated to a chart, called a **holonomic basis**, depends on the chart, but the vector space at m denoted $T_m M$ does not depend on the chart. With this vector space structure one can define a dual space $T_m M^*$ and holonomic dual bases denoted $d\xi^\alpha$ with : $d\xi^\alpha (\partial \xi_\beta) = \delta_\beta^\alpha$, and any other tensorial structure (see Math.16).

As one can see in the definition of the holonomic basis, the tangent space is

generated by small displacements along one coordinate, around a point m . So, physically, locally the manifold is close to an affine space with a chosen origin m , and locally GR and SR look the same. This is similar to what we see on Earth : locally it looks flat.

So there are essential distinctions between :

- a point on the manifold, which is a geometric object (it does not depend on coordinates, even if it can be represented by coordinates in a chart) but has no vectorial structure attached (the linear combination of points has no meaning);
- a vector in the tangent bundle, which is also a geometric quantity (it exists independently of its measure by components in a basis) but has an algebraic structure : the linear combination of vectors is well defined.

Some physical properties of objects can be represented by vectors, others cannot, and the distinction comes from the fundamental assumptions of the theory. It is enshrined in the theory itself.

The vector spaces $T_m M$ depend on m , and there is no canonical (meaning independent of the choice of a specific tool) procedure to compare vectors belonging to the tangent spaces at two different points. These vectors u_m can be considered as a couple of a location m and a vector u , which can be defined in a holonomic basis or not, and all together they constitute the tangent bundle TM .

However because the manifolds are actually affine spaces, in SR and Galilean Geometry the tangent spaces at different points share the same structure (which is the underlying tangent vector space), and only in these cases they can be assimilated to \mathbb{R}^4 . This is the origin of much confusion on the subject, and the motivation to start in the GR context where the concepts are clearly differentiated.

1.3 Vector fields

A vector field on M is a map : $V : M \rightarrow TM :: V(m) = \sum_{\alpha=0}^3 v^\alpha(m) \partial \xi_\alpha$ which associates to any point m a vector of the tangent space $T_m M$. The vector does not depend on the choice of a basis or a chart, so its components change in a change of chart as (Math.16.1.2) :

$$v^\alpha(m) \rightarrow \tilde{v}^\alpha(m) = \sum_{\beta=0}^3 [J(m)]_\beta^\alpha v^\beta(m)$$

where $[J(m)] = \left[\frac{\partial \eta^\alpha}{\partial \xi^\beta}(m) \right]$ is the 4x4 matrix called the jacobian

Similarly a one form on M is a map $\varpi : M \rightarrow TM^* :: \varpi(m) = \sum_{\alpha=0}^3 \varpi_\alpha(m) d\xi^\alpha$ and the components change as :

$$\varpi_\alpha(m) \rightarrow \tilde{\varpi}_\alpha(m) = \sum_{\beta=0}^3 [K(m)]_\alpha^\beta \varpi_\beta(m) \text{ and } [K(m)] = [J(m)]^{-1}$$

The sets of vector fields, denoted $\mathfrak{X}(TM)$, and of one forms, denoted $\mathfrak{X}(TM^*)$ or $\Lambda_1(M; \mathbb{R})$ are infinite dimensional vector spaces (with pointwise operations).

A **curve** on a manifold is a one dimensional submanifold : this is a geometric structure, and there is a vector space associated to each point of the curve, which is a one dimensional vector subspace of $T_m M$.

A **path** on a manifold is a map : $p : \mathbb{R} \rightarrow M :: m = p(\tau)$ where p is a differentiable map such that $p'(\tau) \neq 0$. Its image is a curve L_p , and p defines

a bijection between \mathbb{R} (or any interval of \mathbb{R}) and the curve (this is a chart of the curve), the curve is a 1 dimensional submanifold embedded in M . The same curve can be defined by different paths. The tangent is the map : $p'(t) : \mathbb{R} \rightarrow T_{p(t)}M :: \frac{dp}{d\tau} \in T_{p(\tau)}L_p$. In a change of parameter in the path : $\tilde{\tau} = f(\tau)$ (which is a change of chart) for the same point : $m = \tilde{p}(\tilde{\tau}) = p(f(\tau))$ the new tangent is proportional to the previous one : $\frac{dm}{d\tau} = \frac{d\tilde{p}}{d\tilde{\tau}} \frac{d\tilde{\tau}}{d\tau} \Leftrightarrow \frac{dm}{d\tilde{\tau}} = \frac{1}{f'} \frac{dm}{d\tau}$

For any smooth vector field there is a collection of smooth paths (the **integrals** of the field) such that the tangent at any point of the curve is the vector field. There is a unique **integral line** which goes through a given point. The **flow** of a vector field V is the map (Math.14.3.5):

$\Phi_V : \mathbb{R} \times M \rightarrow M :: \Phi_V(\tau, a)$ such that $\Phi_V(., a) : \mathbb{R} \rightarrow M :: m = \Phi_V(\tau, a)$ is the integral path going through a :

$$\forall \theta \in \mathbb{R} : \frac{\partial}{\partial \tau} \Phi_V(\tau, a) |_{\tau=\theta} = V(\Phi_V(\theta, a)) \quad (1)$$

and $\Phi_V(., a)$ is a local diffeomorphism

$$\forall \tau, \tau' \in \mathbb{R} : \Phi_V(\tau + \tau', a) = \Phi_V(\tau, \Phi_V(\tau', a)) \quad (2)$$

$$\Phi_V(0, a) = a \quad (3)$$

$$\forall \tau \in \mathbb{R} : \Phi_V(-\tau, \Phi_V(\tau, a)) = a \quad (4)$$

For a given vector field, the parameter τ is defined up to a constant, so *it is uniquely defined* with the condition $\Phi_V(0, a) = a$.

In general the flow is defined only for an interval of the parameter, but this restriction does not exist if Ω is relatively compact.

A map $f : C \rightarrow E$ from a curve to a Banach vector space E can be extended to a map $F : \Omega \rightarrow E$ (Maths.1467). So any smooth path can be considered as the integral of some vector field (not uniquely defined), and it is convenient to express a path as the flow of a vector field.

1.4 Fundamental symmetry breakdown

The four coordinates to locate a point in the physical Universe are not equivalent : the measure of the time ξ^0 cannot be done with the same procedures as the other coordinates, and one cannot move along in time. This is the fundamental symmetry breakdown.

One assumes that *a given observer* can tell if two events A, B occur in his present time (they are simultaneous), and that the relation “two events are simultaneous” is a relation of equivalence between events. Then the observer can label each class of equivalence of events by the time of his clock. Which can be expressed by telling that for each observer, there is a function : $f_o : M \rightarrow \mathbb{R} :: f_o(m) = t$ which assigns a time t , *with respect to the clock of the observer*, at any point of the universe (or at least Ω). The points : $\Omega(t) = \{m = f_o(t), m \in \Omega\}$

correspond to the **present** of the observer. No assumption is made about the clock, and different clocks can be used, with the condition that, as for any chart, it is possible for a given observer to convert the time given by a clock to the time given by another clock.

In Galilean Geometry instantaneous communication is possible, so it is possible to define a universal time, to which any observer can refer to locate his position, and the present does not depend on the observer. The manifold M can be assimilated to the product $\mathbb{R} \times \mathbb{R}^3$. The usual representation of material bodies moving in the same affine space is a bit misleading, actually one should say that this affine space $\mathbb{R}^3(t)$ changes continuously, in the same way, for everybody. Told this way we see that Galilean Geometry relies on a huge assumption about the physical universe.

In Relativist Geometry instantaneous communication is impossible, so it is impossible to synchronize all the clocks. However a given observer can synchronize the clocks which correspond to his present, this is the meaning of the function f_o . Different procedures have been proposed for this purpose, the simplest uses electromagnetic signals which are bounced by the target. But this process is specific to each observer, and there is a priori no way to compare the time of clocks synchronized to two different observers, and located at two different spatial locations. We will see how this can be done.

Whenever there is, on a manifold, a map such that f_o , with $f'_o(m) \neq 0$, it defines on M a foliation : there is a collection of hypersurfaces (3 dimensional submanifolds) $\Omega_3(t)$, and the vectors u of the tangent spaces on $\Omega_3(t)$ are such that $f'_o(m)u = 0$, meanwhile the vectors which are transversal to $\Omega_3(t)$ (corresponding to paths which cross the hypersurface only once) are such that $f'_o(m)u > 0$ for any path with t increasing. So there are two faces on $\Omega_3(t)$: one for the incoming paths, and the other one for the outgoing paths. The hypersurfaces $\Omega_3(t)$ are diffeomorphic : they can be deduced from each other by a differentiable bijection, which is the flow of a vector field. Conversely if there is such a foliation one can define a unique function f_o with these properties (Maths.1502). The successions of present “spaces” for any observer is such a foliation, so our representation is consistent. And we state :

Proposition 2 *For any observer there is a function*

$$f_o : M \rightarrow \mathbb{R} :: f_o(m) = t \text{ with } f'_o(m) \neq 0 \quad (5)$$

which defines in the area Ω of the Universe a foliation by hypersurfaces

$$\Omega_3(t) = \{m = f_o(t), m \in \Omega\} \quad (6)$$

which represents the location of the events occurring at a given time t on his clock.

An observer can then define a chart of M , by taking the time on his clock, and the coordinates of a point x in the 3 dimensional hypersurfaces $\Omega_3(t)$. However we need a way to build consistently these spatial coordinates (how they are related from one hypersurface $\Omega_3(t)$ to another $\Omega_3(t')$).

1.5 Trajectories of material bodies

The Universe is a container where physical objects live, and the manifold provides a way to measure a location. This is a 4 dimensional manifold which includes the time, but that does not mean that everything is frozen on the manifold : *the universe does not change, but its content changes*. As bodies move in the universe, their representation are paths on the manifold. And the fundamental symmetry breakdown gives a special meaning to the coordinate with respect to which the changes are measured. *Time is not only a parameter to locate an event, it is also a variable which defines the rates of change in the present of an observer*.

1.5.1 Material bodies and particles

The common definition of a material body in Physics is that of a set of material points which are related. A **material point** is assumed to have a location corresponding to a point of the manifold. According to the relations between material points of the same body we have rigid solids (the distance between two points is constant), deformable solids (the deformation tensor is locally given by the matrix of the transformation of a frame), fluids (the speed of material points are given by a vector field). These relations are formulated by phenomenological laws, they are essential in practical applications, but not in a theoretical study. So we will consider material bodies which have no internal structures, or whose internal structure can be neglected at the scale of the study, that we will call **particles**. A particle then can be an electron, a nucleus, a molecule, or even a star system, according to the scale of the study. As in Mechanics a particle is a material point, and its location can be assimilated to a point from a geometrical point of view. We will see later how one can extend the concept of solid bodies to the relativist context.

1.5.2 World line and proper time

As required in any scientific theory a particle must be defined by its properties, and the first is that it occupies a precise location at any time. The successive locations of the material body define a curve and the particle travels on this curve according to a specific path called its **world line**. Any path can be defined by the flow of a vector such that the derivative with respect to the parameter is the tangent to the curve. The parameter called the **proper time** is then defined uniquely, up to the choice of an origin. The derivative with respect to the proper time is called the **velocity**. By definition *this is a vector*, defined at each point of the curve, and belonging to the tangent space to M . So the velocity has a definition which is independent of any basis.

Remark : For brevity I will call velocity the 4-vector, also usually called 4-velocity, and spatial speed the common 3 vector.

Observers are assumed to have similarly a world line and a proper time (they have other properties, notably they define a basis).

To sum up :

Definition 3 Any particle or observer travels in the universe on a curve according to a specific path, $p : \mathbb{R} \rightarrow M :: m = p(\tau)$ called the world line, parametrized by the proper time τ , defined uniquely up to an origin. The derivative of the world line with respect to the proper time is a vector, the velocity, u . So that :

$$u(\theta) = \left. \frac{dp}{d\tau} \right|_{\tau=\theta} \in T_{p(\theta)}M \quad (7)$$

$$p(\tau) = \Phi_u(\tau, a) \text{ with } a = \Phi_u(0, a) = p(0) \quad (8)$$

Observers are assumed to have clocks, that they use to measure their temporal location with respect to some starting point. The basic assumption is the following :

Proposition 4 For any observer his proper time is the time on his clock.

So the proper time of a particle can be seen formally as the time on the clock of an observer who would be attached to the particle.

We will strive to denote t the time of an observer (specific to an observer) and τ any other proper time. So for a given observer :

$$t = \tau$$

$$p_o : \mathbb{R} \rightarrow M :: m = p_o(t)$$

$$u(\theta) = \left. \frac{dp}{dt} \right|_{t=\theta} \in T_{p(\theta)}M$$

$$p_o(t) = \Phi_u(\tau, a) \text{ with } a = \Phi_u(0, a) = p(0)$$

The observer uses the time on his clock to locate temporally any event : this is the purpose of the function f_o and of the foliation $\Omega_3(t)$. The curve on which any particle travels meets only once each hypersurface $\Omega_3(t)$: it is seen only once. This happens at a time t :

$$f_o(p(\tau)) = t = f_o(\Phi_u(\tau, a))$$

So there is some relation between t and the proper time τ . It is specific, both to the observer and to the particle. It is bijective and both increases simultaneously, so that : $\frac{d\tau}{dt} > 0$.

The travel of the particle on the curve can be represented by the time of an observer. We will call then this path a **trajectory**.

A clock measures the elapsed time. It seems legitimate to assume that, in the procedure, one chooses clocks which run at the same rate. But, to do this, one needs some way to compare this rate. The most natural is to use a scalar measure of the velocity $\frac{d}{d\tau}p_o(\tau)$, and to assume that it is the same : material bodies would travel along their world lines at the same speed. But, as velocities are 4 dimensional vectors, one needs a special scalar product.

1.6 Metric on the manifold

1.6.1 Lorentz metric

A scalar product is defined by a bilinear symmetric form g acting on *vectors* of the tangent space, at each point of the manifold, thus by a tensor field called a **metric**. In a holonomic basis g reads :

$$g(m) = \sum_{\alpha\beta=0}^3 g_{\alpha\beta}(m) d\xi^\alpha \otimes d\xi^\beta \text{ with } g_{\alpha\beta} = g_{\beta\alpha} \quad (9)$$

The matrix of g is symmetric and invertible, if we assume that the scalar product is not degenerate. It is diagonalizable, and its eigen values are real. One wants to account for the symmetry breakdown, so these eigen values cannot have all the same sign (a direction is privileged). One knows that the hypersurface $\Omega_3(t)$ are Riemannian : there is a definite positive scalar product (acting on the 3 dimensional vector space tangent to $\Omega_3(t)$), and that transversal vectors correspond to the velocities of material bodies. So there are only two solutions for the signs of the eigen values of $[g(m)]$: either $(-,+,+,+)$ or $(+,-,-,-)$ which provides both a **Lorentz metric**. The scalar product, in an orthonormal basis $(\varepsilon_i)_{i=0}^3$ at m reads :

$$\text{signature } (3, 1) : \langle u, v \rangle = u^1v^1 + u^2v^2 + u^3v^3 - u^0v^0 \quad (10)$$

$$\text{signature } (1, 3) : \langle u, v \rangle = -u^1v^1 - u^2v^2 - u^3v^3 + u^0v^0 \quad (11)$$

Such a scalar product defines by restriction on each hypersurface $\Omega_3(t)$ a positive or a negative definite metric, which applies to spatial vectors (tangent to $\Omega_3(t)$) and provides, up to sign, the usual euclidean metric. So that both signatures are acceptable.

Which leads to :

Proposition 5 *The manifold M representing the Universe is endowed with a non degenerate metric, called the **Lorentz metric**, with signature either $(3,1)$ or $(1,3)$ defined at each point.*

This reasoning is a legitimate assumption, which is consistent with all the other concepts and assumptions, this is not the proof of the existence of such a metric. Such a proof comes from the formula in a change of frames between observers, which can be checked experimentally.

Notice that on a finite dimensional, connected, Hausdorff manifold, there is always a definite positive metric (Maths.1385). There is no relation between this metric and a Lorentz metric. Not all manifolds can have a Lorentz metric, the conditions are technical (see Giachetta p.224 for more) but one can safely assume that they are met in a limited region Ω .

1.6.2 Gauge group

The existence of a metric implies that, at any point, there are orthonormal bases $(\varepsilon_i)_{i=0}^3$ with the property :

Definition 6 $\langle \varepsilon_i, \varepsilon_j \rangle = \eta_{ij}$ for the signature $(3,1)$ and $\langle \varepsilon_i, \varepsilon_j \rangle = -\eta_{ij}$ for the signature $(1,3)$

with the matrix :

$$\text{Notation 7 } [\eta] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so that the vector ε_0 is the time vector.

An orthonormal basis, at each point, is a **gauge**. The choice of an orthonormal basis depends on the observer : he has freedom of gauge. One goes from one gauge to another by a linear map χ which preserves the scalar product. They constitute a group, called the **gauge group**. In any basis these maps are represented by a matrix $[\chi]$ such that :

$$[\chi]^t [\eta] [\chi] = [\eta] \quad (12)$$

The group denoted equivalently $O(3,1)$ or $O(1,3)$, does not depend on the signature (replace $[\eta]$ by $-[\eta]$).

$O(3,1)$ is a 6 dimensional Lie group with Lie algebra $o(3,1)$ whose matrices $[h]$ are such that : $[h]^t [\eta] + [\eta] [h] = 0$. (Maths.24.5.3). The Lie algebra is a vector space and we will use the basis :

$$\begin{aligned} [\kappa_1] &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; [\kappa_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}; [\kappa_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ [\kappa_4] &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; [\kappa_5] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; [\kappa_6] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

so that any matrix of $o(3,1)$ can be written :

$$[\kappa] = [J(r)] + [K(w)] \text{ with two vectors } r, w \text{ of } \mathbb{R}^3$$

$$[J(r)] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r_3 & r_2 \\ 0 & r_3 & 0 & -r_1 \\ 0 & -r_2 & r_1 & 0 \end{bmatrix}; [K(w)] = \begin{bmatrix} 0 & w_1 & w_2 & w_3 \\ w_1 & 0 & 0 & 0 \\ w_2 & 0 & 0 & 0 \\ w_3 & 0 & 0 & 0 \end{bmatrix}$$

The exponential of these matrices read (Maths.493) :

$$\exp [K(w)] = I_4 + \frac{\sinh \sqrt{w^t w}}{\sqrt{w^t w}} K(w) + \frac{\cosh \sqrt{w^t w} - 1}{w^t w} K(w)K(w)$$

$$\exp [K(w)] = \begin{bmatrix} \cosh \sqrt{w^t w} & w^t \frac{\sinh \sqrt{w^t w}}{\sqrt{w^t w}} \\ w \frac{\sinh \sqrt{w^t w}}{\sqrt{w^t w}} & I_3 + \frac{\cosh \sqrt{w^t w} - 1}{w^t w} w w^t \end{bmatrix}$$

$$\exp [J(r)] = I_4 + \frac{\sin \sqrt{r^t r}}{\sqrt{r^t r}} J(r) + \frac{1 - \cos \sqrt{r^t r}}{r^t r} J(r)J(r) = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$$

where R a 3×3 matrix of $O(3)$.

The group $O(3)$ has two connected components : the subgroup $SO(3)$ with determinant = 1, and the subset $O_1(3)$ with determinant -1.

$O(3, 1)$ has four connected components which can be distinguished according to the sign of the determinant and their projection under the compact subgroup $SO(3) \times \{I\}$.

Any matrix of $SO(3, 1)$ can be written as the product : $[\chi] = \exp [K(w)] \exp [J(r)]$ (or equivalently $\exp [J(r')] \exp [K(w')]$). So we have the 4 cases :

- $SO_0(3, 1)$: with determinant 1: $[\chi] = \exp K(w) \times \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$
- $SO_1(3, 1)$: with determinant 1: $[\chi] = \exp K(w) \times \begin{bmatrix} -1 & 0 \\ 0 & -R \end{bmatrix}$
- $SO_2(3, 1)$ with determinant = -1: $[\chi] = \exp K(w) \times \begin{bmatrix} -1 & 0 \\ 0 & R \end{bmatrix}$
- $SO_3(3, 1)$ with determinant = -1: $[\chi] = \exp K(w) \times \begin{bmatrix} 1 & 0 \\ 0 & -R \end{bmatrix}$

where R a 3×3 matrix of $SO(3)$, so that $-R \in O_1(3)$

$SO_k(3, 1)$, $k = 1, 2, 3$ are generated by the product of any element of $SO_0(3, 1)$ by either :

the time reversal matrix : $T = \begin{bmatrix} -1 & 0 \\ 0 & I_3 \end{bmatrix}$

or the space reversal matrix : $S = \begin{bmatrix} 1 & 0 \\ 0 & -I_3 \end{bmatrix}$

1.6.3 Orientation

Is the universe orientable ? In a universe where all observers have the same time, the simple existence of stereoisomers which do not have the same chemical properties suffices to answer positively. In a space-time universe one needs a process with an outcome which discriminates an orientation. All chemical reactions starting with a balanced mix of stereoisomers produce an equally balanced mix (stereoisomers have the same level of energy). However there are experiments involving the weak interactions which show the required property. So we can state that the 4 dimensional universe is orientable, and then we can distinguish orientation preserving gauge transformations. The right group to consider is $SO(3, 1)$.

The relativist universe is no longer isotropic : all directions are not equivalent. At any point m one can discriminate the vectors $v \in T_m M$ according to the value of the scalar product $\langle v, v \rangle$.

Definition 8 *Time like* vectors are vectors v such that $\langle v, v \rangle < 0$ with the signature $(3, 1)$ and $\langle v, v \rangle > 0$ with the signature $(1, 3)$

Space like vectors are vectors v such that $\langle v, v \rangle > 0$ with the signature $(3, 1)$ and $\langle v, v \rangle < 0$ with the signature $(1, 3)$

Moreover the subset of time like vectors has two disconnected components (this is no longer true in universes with more than one “time component” (Maths.307)). This has two important consequences.

A change of gauge, physically, implies some transport of the frame (one does not jump from one point to another) : we have a map : $\chi : I \rightarrow SO(3, 1)$ such that at each point of the path $p_o : I \rightarrow M$ defined on a interval I of \mathbb{R} , $\chi(t)$ is an isometry. The path which is followed matters. In particular it is connected. The frame $(\varepsilon_i)_{i=0}^3$ is transported by : $\tilde{\varepsilon}_i(\tau) = \chi(t)\varepsilon_i(0)$. So $\{[\chi(\tau)], t \in I\}$, image of the connected interval I by a continuous map is a connected subset of $SO(3, 1)$, and because $\chi(0) = Id$ it must be the component of the identity. So the right group to consider is the **connected component of the identity** $SO_0(3, 1)$.

Because the subset of time like vectors has two disconnected components one can discriminate these components and, in accordance with the assumptions about the velocity of material bodies, it is logical to consider that their velocity is **future oriented**. And one can distinguish gauge transformations which preserve this time orientation.

Definition 9 *We will assume that the future orientation is given in a gauge by the vector ε_0 . So a vector u is time like and future oriented if :*

$$\begin{aligned} \langle u, u \rangle < 0, \langle u, \varepsilon_0 \rangle < 0 \text{ with the signature } (3, 1) \\ \langle u, u \rangle > 0, \langle u, \varepsilon_0 \rangle > 0 \text{ with the signature } (1, 3) \end{aligned}$$

A matrix $[\chi]$ of $SO_0(3, 1)$ preserves the time orientation iff $[\chi]_0^0 > 0$ and this will always happen if $[\chi] = \exp[K(w)] \exp[J(r)]$ that is if $[\chi] \in SO_0(3, 1)$.

A gauge transformation which preserves both the time orientation, and the global orientation must preserve also the spatial orientation.

1.7 Velocities have a constant Lorentz norm

The velocity $\frac{dp_o}{dt}$ is a vector which is defined independently of any basis, for any observer it is transversal to $\Omega_3(t)$. It is legitimate to say that it is future oriented, and so it must be time-like. One of the basic assumptions of Relativity is that it has a constant length, as measured by the metric, identical for all observers. So it is possible to use the norm of the velocity to define a standard rate at which the clocks run.

Because the proper time of any material body can be defined as the time on the clock of an observer attached to the body this proposition is extended to any particle.

The time is not measured with the same unit as the lengths, used for the spatial components of the velocity. The ratio ξ^i/t has the dimension of a spatial speed. So we make the general assumption that for any observer or particle the velocity is such that $\left\langle \frac{dp}{d\tau}, \frac{dp}{d\tau} \right\rangle = -c^2$ where τ is the proper time. Notice that c is a *constant, with no specific value*.

And we sum up :

Proposition 10 *The velocity $\frac{dp}{d\tau}$ of any particle or observer is a time like, future oriented vector with Lorentz norm*

$$\left\langle \frac{dp}{d\tau}, \frac{dp}{d\tau} \right\rangle = -c^2 \quad (13)$$

(with signature (3,1) or c^2 with signature (1,3)) where c is a fundamental constant.

1.8 Standard chart of an observer

As a consequence :

Theorem 11 *For any observer there is a vector field $\varepsilon_0 \in \mathfrak{X}(TM)$ which is future oriented, with length $\langle \varepsilon_0(m), \varepsilon_0(m) \rangle = -1$, normal to $\Omega_3(t)$ and such that : $\varepsilon_0(p_0(t)) = \frac{1}{c} \frac{dp_0}{dt}$ where $\frac{dp_0}{dt}$ is the velocity of the observer at each point of his world line.*

Proof. For an observer the function $f : \Omega \rightarrow \mathbb{R}$ has for derivative a one form $f'(m) \neq 0$ such that $\forall v \in T_m \Omega_3(t) : f'(m)v = 0$. Using the metric, it is possible to associate to $f'(m)$ a vector : $\varepsilon_0(m) = \text{grad}f : \langle \varepsilon_0(m), v \rangle = f'(m)v$. Thus $\varepsilon_0(m)$ is normal to $\Omega_3(t)$. Along the world line of the observer $\varepsilon_0(m)$ is in the direction of the velocity of the observer. And it is always possible to choose $\varepsilon_0(m)$ such that it is future oriented and with length $\langle \varepsilon_0(m), \varepsilon_0(m) \rangle = -1$ ■

As a consequence :

Theorem 12 *$\Omega_3(t)$ are space like hypersurfaces, with unitary normal $\varepsilon_0 \in \mathfrak{X}(TM)$*

Using the vector field ε_0 , and any any chart φ_Ω of $\Omega(0)$ there is a standard chart associated to an observer.

Definition 13 *The **standard chart** on M of any observer is defined as :*

$$\begin{aligned} \varphi_o : \mathbb{R}^4 \rightarrow \Omega :: \varphi_o(\xi^0, \xi^1, \xi^2, \xi^3) &= \Phi_{\varepsilon_0}(ct, x) \\ \xi^0 = ct, \varphi_\Omega(\xi^1, \xi^2, \xi^3) &= x \text{ in any chart of } \Omega(0) \end{aligned}$$

c is required in $\Phi_{\varepsilon_0}(ct, x)$ so that :

$$\xi^0 = ct \quad (14)$$

which makes all the coordinates homogeneous in units [Length].

The holonomic basis associated to this chart is such that :

$$\partial\xi_0 = \frac{\partial\varphi_o}{\partial\xi^0} = \frac{1}{c} \frac{\partial}{\partial t} \Phi_{\varepsilon_0}(ct, x) = \varepsilon_0 \quad (15)$$

For any point $m = \varphi_o(\xi^0, \xi^1, \xi^2, \xi^3) \in \Phi_{\varepsilon_0}(ct, x)$ the point x is the point where the integral curve of ε_0 passing by m crosses $\Omega_3(0)$.

So the main characteristic of an observer can be summed in the vector field ε_0 (which is equivalently deduced from the function f_o).

1.9 Trajectory and speed of a particle

A particle follows a world line $q(\tau)$, parametrized by its proper time. Any observer sees only one instance of the particle, located at the point where the world line crosses the hypersurface $\Omega_3(t)$ so we have a relation between τ and t . This relation identifies the respective location of the observer and the particle on their own world lines. With the standard chart of the observer it is possible to measure the velocity of the particle at any location, and of course at the location where it belongs to $\Omega_3(t)$.

The trajectory (parametrized by t) of any particle in the standard chart of an observer is :

$$q(t) = \Phi_{\varepsilon_0}(ct, x(t)) = \varphi_o(ct, \xi^1(t), \xi^2(t), \xi^3(t))$$

By differentiation with respect to t :

$$\begin{aligned} \frac{dq}{dt} &= c\varepsilon_0(q(t)) + \frac{\partial}{\partial x}\Phi_{\varepsilon_0}(ct, x(t)) \frac{\partial x}{\partial t} \\ \frac{\partial}{\partial x}\Phi_{\varepsilon_0}(ct, x(t)) \frac{\partial x}{\partial t} &= \sum_{\alpha=1}^3 \frac{d\xi_\alpha}{dt} \partial\xi_\alpha \in T_m\Omega_3(t) \text{ so is orthogonal to } \varepsilon_0(q(t)) \end{aligned}$$

Definition 14 The *spatial speed* of a particle on its trajectory with respect to an observer is the vector of $T_{q(t)}\Omega_3(t)$:

$$\vec{v} = \frac{\partial}{\partial x}\Phi_{\varepsilon_0}(ct, x(t)) \frac{\partial x}{\partial t} = \sum_{\alpha=1}^3 \frac{d\xi_\alpha}{dt} \partial\xi_\alpha$$

Thus for any particle in the standard chart of an observer :

$$V(t) = \frac{dq}{dt} = c\varepsilon_0(q(t)) + \vec{v} \quad (16)$$

For the observer in the standard chart we had :

$$\frac{dp_0}{dt} = c\varepsilon_0(p_0(t)) \Leftrightarrow \vec{v} = 0$$

Notice that the velocity, and the spatial speed, are measured *in the chart of the observer at the point $q(t)$ where is the particle*. Because we have defined a standard chart it is possible to measure the speed of a particle located at a point $q(t)$ which is different from the location of the observer. And we can express the relation between τ and t .

Theorem 15 The proper time τ of any particle and the corresponding time of any observer t are related by :

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}} \quad (17)$$

where \vec{v} is the spatial speed of the particle, with respect to the observer and measured in his standard chart.

The velocity of the particle is :

$$\frac{dp}{d\tau} = \frac{1}{\sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}}} (\vec{v} + c\varepsilon_0(m)) \quad (18)$$

Proof. i) Let be a particle A with world line :

$$p : \mathbb{R} \rightarrow M :: m = p(\tau) = \Phi_u(\tau, a) \text{ with } a = \Phi_u(0, a) = p(0)$$

In the standard chart $\Phi_{\varepsilon_0}(ct, x)$ of the observer O its trajectory is :

$$q : \mathbb{R} \rightarrow M :: m = q(t) = \Phi_{\varepsilon_0}(ct, x(t))$$

So there is a relation between t, τ :

$$m = p(\tau) = \Phi_u(\tau, a) = q(t) = \Phi_{\varepsilon_0}(ct, x(t))$$

By differentiation with respect to t :

$$\frac{d}{dt}q(t) = c\varepsilon_0(p_A(t)) + \vec{v}$$

$$\frac{dq}{dt} = \vec{v} + c\varepsilon_0(m)$$

$$\frac{dq}{dt} = \frac{dp}{d\tau} \frac{d\tau}{dt}$$

$$\left\langle \frac{dp}{d\tau}, \frac{dp}{d\tau} \right\rangle = -c^2$$

$$\left\langle \frac{dq}{dt}, \frac{dq}{dt} \right\rangle = -c^2 \left(\frac{d\tau}{dt} \right)^2$$

$$\left\langle \frac{dq}{dt}, \frac{dq}{dt} \right\rangle = \langle \vec{v}, \vec{v} \rangle_3 - c^2 \text{ because } \varepsilon_0(m) \perp \Omega_3(t)$$

$$\|\vec{v}\|^2 - c^2 = -c^2 \left(\frac{d\tau}{dt} \right)^2$$

$$\text{and because } \frac{d\tau}{dt} > 0 : \frac{d\tau}{dt} = \sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}}$$

ii) The velocity of the particle is :

$$\frac{dp}{d\tau} = \frac{dq}{dt} \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}}} (\vec{v} + c\varepsilon_0(m)) \blacksquare$$

As a consequence :

$$\|\vec{v}\| < c \tag{19}$$

$V(t) = \frac{dp}{dt}$ is the measure of the motion of the particle with respect to the observer : it can be seen as the relative velocity of the particle with respect to the observer. It involves \vec{v} which has the same meaning as usual, but we see that in Relativity one goes from the 4 velocity $u = \frac{dp}{d\tau}$ (which has an absolute meaning)

to the relative velocity $V(t) = \frac{dp}{dt} = \frac{dp}{d\tau} \frac{d\tau}{dt} = u \sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}}$ by a scalar. If we have two particles A,B, with their path $q_A(\tau_A), q_B(\tau_B)$ can we define their relative motion, for instance of B relative to A ? The simplest way to do it in relativity is

to consider A as an observer, then $V_{B/A}(\tau_A) = \frac{dq_B}{d\tau_A} = u_B \sqrt{1 - \frac{\|\vec{v}_{B/A}\|^2}{c^2}}$ which is defined *in the chart* associated to the observer A.

1.10 Fiber bundles

As said before, the location of a particle is absolute : this is the point in the physical Universe that it occupies at some time. But the measure of this location is relative to the observer, starting with the time at which the particle is at a given place. Similarly the velocity of a particle or an observer is absolute : in its definition there is no reference to a chart or a frame. This is an essential point in Relativity. State that the velocity of a particle is absolute confers to the variable a specific status : it is a geometric vector. The status - vector or not - of a variable is not arbitrary : it is part of the assumptions of the theory. Velocity is an intrinsic property of material bodies and particles, the measure of this velocity depends on the observer : it is relative.

This remark extends to all measures. A physical measure in itself has no meaning if one does not know how it has been done. The label “done by the observer O” is necessary. So we cannot contend ourselves with maps $X : M \rightarrow E$. We need a way to attach a tag, identifying the way the measure has been done, to the value of the variable. The mathematical tool to achieve that is the fiber bundle formalism. This is more than a sophisticated mathematical theory, it embodies the relation between measure (the value) and conditions of the measure (the gauge).

(see Math.Part VI)

1.10.1 General fiber bundle

A fiber bundle, denoted $P(M, F, \pi_P)$, is a manifold P, which is locally the product of two manifolds, the base M and the standard fiber F, with a projection : $\pi_P : P \rightarrow M$ which is a surjective submersion. The subset of P : $\pi_P^{-1}(m)$ is the fiber over m . It is usually defined over a collection of open subsets of M, patched together, but we will assume that on the area Ω there is only one component (the fiber bundles are assumed to be trivial). A **trivialization** is a map :

$$\varphi_P : M \times F \rightarrow P :: p = \varphi_P(m, v)$$

and any element of P is projected on M : $\forall v \in F : \pi_P(\varphi_P(m, v)) = m$. So it is similar to a chart, but the arguments are points of the manifolds.

A **section p** on P is defined by a map : $v : M \rightarrow F$ and $\mathbf{p} = \varphi_P(m, v(m))$. The set of sections is denoted $\mathfrak{X}(P)$.

A fiber bundle can be defined by different trivializations. In a **change of trivialization** the *same* element p is defined by a different map φ_P : this is very similar to the charts for manifold.

$$p = \varphi_P(m, v) = \tilde{\varphi}_P(m, \tilde{v})$$

and there is a necessary relation between v and \tilde{v} (m stays always the same) depending on the kind of fiber bundle.

1.10.2 Principal bundle

If $F = G$ is a Lie group then P is a **principal bundle** : its elements are a value $g(m)$ of G localized at a point m . There is a right action of G on P :

$$p = \varphi_P(m, g) \rightarrow p \cdot g' = \varphi_P(m, g \cdot g')$$

p will usually define the basis used to measure vectors, so p is commonly called a gauge. There is a special gauge which can be defined at any point (it will usually be the gauge of the observer) : the **standard gauge**, the element of the fiber bundle such that : $\mathbf{p}(m) = \varphi_P(m, 1)$ where 1 denotes the unit element of G .

A change of trivialization is induced by a map : $\chi : M \rightarrow G$ that is by a section $\chi \in \mathfrak{X}(P)$ and :

$$p = \varphi_P(m, g) = \tilde{\varphi}_P(m, \chi(m) \cdot g) = \tilde{\varphi}_P(m, \tilde{g}) \Leftrightarrow \tilde{g} = \chi(m) \cdot g \quad (\chi(m) \text{ acts on the left})$$

$\chi(m)$ can be identical over M (the change is said to be global) or depends on m (the change is local).

The expression of the elements of a section changes as :

$$\sigma \in \mathfrak{X}(P) :: \sigma = \varphi_P(m, \sigma(m)) = \tilde{\varphi}_P(m, \tilde{\sigma}(m)) \Leftrightarrow \tilde{\sigma}(m) = \chi(m) \cdot \sigma(m)$$

$$\sigma(m) = \varphi_P(m, \sigma(m)) = \tilde{\varphi}_P(m, \chi(m) \cdot \sigma(m)) \quad (20)$$

A change of trivialization induces a **change of standard gauge** :

$$\mathbf{p}(m) = \varphi_P(m, 1) = \tilde{\varphi}_P(m, \chi(m))$$

$$\rightarrow \tilde{\mathbf{p}}(m) = \tilde{\varphi}_P(m, 1) = \tilde{\varphi}_P(m, \chi(m) \cdot \chi(m)^{-1}) = \mathbf{p}(m) \cdot \chi(m)^{-1}$$

$$\mathbf{p}(m) = \varphi_P(m, 1) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \quad (21)$$

$$\sigma(m) = \varphi_P(m, \sigma(m)) = \tilde{\varphi}_P(m, \chi(m) \cdot \sigma(m)) \quad (22)$$

1.10.3 Vector bundle

If $F = V$ is a vector space then P is a **vector bundle** and it has at each point the structure of a vector space :

$$w_m = \varphi_P(m, w), w'_m = \varphi_P(m, w'), \alpha, \beta \in \mathbb{R} :$$

$$\alpha w_m + \beta w'_m = \varphi_P(m, \alpha w + \beta w')$$

A holonomic basis is defined by a basis $(\varepsilon_i)_{i \in I}$ of V and : $\varepsilon_i(m) = \varphi_P(m, \varepsilon_i)$.

Usually vector bundles are defined as associated vector bundles. The principal bundle defines locally a standard with respect to which the measure is done. The result belong to a fixed set, but its value is labeled by the standard which is used and related to a point of a manifold.

1.10.4 Associated fiber bundle

Whenever there is a manifold F , a left action λ of G on F , one can built an **associated fiber bundle** denoted $P[F, \lambda]$ comprised of couples :

$$(p, v) \in P \times F \text{ with the equivalence relation : } (p, v) \sim (p \cdot g, \lambda(g^{-1}, v))$$

It is convenient to define these couples by using the standard gauge on P :

$$(\mathbf{p}(m), v) = (\varphi_P(m, 1), v) \sim (\varphi_P(m, g), \lambda(g^{-1}, v)) \quad (23)$$

A standard gauge is nothing more than the use of an arbitrary standard, represented by 1, with respect to which the measure is done. This is not a section : the standard gauge is the embodiment of the free will of the observer, who can choose the way he proceeds to the measure, and is not fixed by any physical law. A change of standard gauge $\mathbf{p}(m) = \varphi_P(m, 1) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1}$ in the principal bundle impacts all associated fiber bundles (this is similar to the change of units) :

$$v_p = (\mathbf{p}(m), v) = \left(\mathbf{p}(m) \cdot \chi(m)^{-1}, \lambda(\chi(m), v) \right) \quad (24)$$

Similarly for the components of a section :

$$\mathbf{v} \in \mathfrak{X}(P[V, \lambda]) :: \mathbf{v}(m) = (\mathbf{p}(m), v(m)) = \left(\mathbf{p}(m) \cdot \chi(m)^{-1}, \lambda(\chi(m), v) \right)$$

If \mathbf{F} is a vector space \mathbf{V} and $[V, \rho]$ a representation of the group \mathbf{G} then we have an **associated vector bundle** $P[V, \rho]$ which has locally the structure of a vector space. It is convenient to define a **holonomic basis** $(\varepsilon_i(m))_{i=1}^n$ from a basis $(\varepsilon_i)_{i=1}^n$ of \mathbf{V} by : $\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$ then any vector of $P[V, \rho]$ reads :

$$v_m = (\mathbf{p}(m), v) = \left(\mathbf{p}(m), \sum_{i=1}^n v^i \varepsilon_i \right) = \sum_{i=1}^n v^i \varepsilon_i(m) \quad (25)$$

A change of standard gauge $\mathbf{p}(m) = \varphi_P(m, 1) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1}$ in the principal bundle impacts all associated vector bundles.

The holonomic basis of a vector bundle changes as :

$$\begin{aligned} \varepsilon_i(m) &= (\mathbf{p}(m), \varepsilon_i) \rightarrow \\ \tilde{\varepsilon}_i(m) &= (\tilde{\mathbf{p}}(m), \varepsilon_i) = \left(\mathbf{p}(m) \cdot \chi(m)^{-1}, \varepsilon_i \right) \\ &\sim \left(\left(\mathbf{p}(m) \cdot \chi(m)^{-1} \right) \cdot \chi(m), \rho \left(\chi(m)^{-1} \right) \varepsilon_i \right) \\ &= \left(\mathbf{p}(m), \rho \left(\chi(m)^{-1} \right) (\varepsilon_i) \right) = \rho \left(\chi(m)^{-1} \right) \varepsilon_i(m) \end{aligned}$$

$$\mathbf{p}(m) = \varphi_P(m, 1) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \quad (26)$$

$$\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i) \rightarrow \tilde{\varepsilon}_i(m) = \rho(\chi(m))^{-1} \varepsilon_i(m) \quad (27)$$

so that the components of a vector in the holonomic basis change as :

$$\begin{aligned} v_m &= \sum_{i=1}^n v^i \varepsilon_i(m) = \sum_{i=1}^n \tilde{v}^i \tilde{\varepsilon}_i(m) = \sum_{i=1}^n \tilde{v}^i \rho(\chi(m))^{-1} \varepsilon_i(m) \\ &\Rightarrow \tilde{v}^i = \sum_j [\rho(\chi(m))]_j^i v^j \end{aligned}$$

$$\mathbf{p}(m) = \varphi_P(m, 1) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \quad (28)$$

$$v^i \rightarrow \tilde{v}^i = \sum_j [\rho(\chi(m))]_j^i v^j \quad (29)$$

The set of sections on $P[V, \rho]$, denoted $\mathfrak{X}(P[V, \rho])$, is an infinite dimensional vector space. In a change of standard gauge the components of a section change as :

$$v \in \mathfrak{X}(P[V, \rho]) :: v(m) = \sum_{i=1}^n v^i(m) \varepsilon_i(m) = \sum_{i=1}^n \tilde{v}^i(m) \tilde{\varepsilon}_i(m)$$

$$\Leftrightarrow \tilde{v}^i(m) = \sum_j \left[\rho \left(\chi(m)^{-1} \right) \right]_j^i v^j(m)$$

so that $(\mathfrak{X}(P[V, \rho]), \rho)$ is an infinite dimensional representations of the group G.

I have given with great precision the rules in a change of gauge, as they will be used quite often (and are a source of constant mistakes !). They are necessary to ensure that a quantity is intrinsic : if it is geometric, its measure must change according to the rules. And conversely if it changes according to the rules, then one can say that it is intrinsic (this is similar to tensors). Because this is a source of confusion, I will try to stick to these precise terms :

- a section = a point of a fiber bundle whose value is defined for each $m \in M$, this is a geometric object

- a gauge = a point of the principal bundle of P, this is a geometric object, which does not depend on a trivialization

- a standard gauge = a specific element of P, whose definition depends of the trivialization. This is not a section.

- a change of trivialization does not change the points of P, the gauge or the sections, but change the standard gauge and the way the points of P are defined with respect to the standard gauge

- it is equivalent to define a change of trivialization by the change of maps $\varphi_P \rightarrow \tilde{\varphi}_P$ or by the change of standard gauge : $\mathbf{p}(m) = \varphi_P(m, 1) \rightarrow \tilde{\mathbf{p}}(m) = \tilde{\varphi}_P(m, 1)$

Notice that the elements of a section stay the same, but their definition changes, meanwhile the holonomic bases are defined by different elements. This is very similar to what we have in any vector space in a change of basis : the vectors of the basis change, the other vectors stay the same, but their components change.

1.10.5 Scalar product and norm

Whenever there is a scalar product (bilinear symmetric or Hermitian two form) $\langle \rangle$ on a vector space V, so that (V, ρ) is a *unitary* representation of the group G : $\langle \rho(g)v, \rho(g)v' \rangle = \langle v, v' \rangle$, the scalar product can be extended on the associated vector bundle $P[V, \rho]$:

$$\langle (\mathbf{p}(m), v), (\mathbf{p}(m), v') \rangle_{P[V, \rho]} = \langle v, v' \rangle_W \quad (30)$$

If this scalar product is definite positive, with any measure μ on the manifold M (usually the Lebesgue measure associated to a volume form as in the relativist context), one can define the spaces of integrable sections $L^q(M, \mu, P[V, \rho])$ of $P[V, \rho]$ (by taking the integral of the norm pointwise). For $q = 2$ they are Hilbert spaces, and unitary representation of the group G. Notice that the signature of the scalar product is that of the product defined on $P[V, \rho]$, *the metric on M is not involved*.

If there is a **norm** on V , that is a map :

$$\|\cdot\| : V \rightarrow \mathbb{R}_+$$

such that :

$$\|X\| \geq 0$$

$$\|X\| = 0 \Leftrightarrow X = 0$$

$$\|kX\| = |k| \|X\|$$

$$\|X + X'\| \leq \|X\| + \|X'\|$$

which does not depend on ρ :

$$\forall g \in G : \|\rho(g) X\| = \|X\|$$

then one can define a norm pointwise on $P[V, \rho]$:

$$\|(p(m), v)\| = \|v\|$$

$$(p(m), v) \sim (p(m) \rho(g^{-1}), \rho(g) v)$$

$$\|(p(m) \rho(g^{-1}), \rho(g) v)\| = \|\rho(g) v\| = \|v\|$$

and the space of integrable maps :

$$L^1(\mathfrak{X}(P[V, \rho])) = \{X \in \mathfrak{X}(P[V, \rho]), \int_{\Omega} \|X\| \mu < \infty\}$$

is a separable Fréchet space if Ω is a compact subset.

We have several fiber bundles in the Geometry of the Universe that we have defined. The simplest is the usual tangent bundle TM over M , which is a vector bundle associated to the choice of an invertible map at each point (the gauge group is $SL(\mathbb{R}, 4)$). But we have another one through the standard chart of an observer ;

Definition 16 For any observer there is a **fiber bundle structure** $M_o(\mathbb{R}, \Omega(0), \pi_o)$ on M with base \mathbb{R} and :

$$\text{projection} : \pi_o(m) = f_o(m)$$

$$\text{trivialization} : \Phi_{\varepsilon_o} : \mathbb{R} \times \Omega(0) \rightarrow \Omega :: \Phi_{\varepsilon_o}(ct, x) = m$$

1.11 Standard gauges associated to an observer

Following the Principle of Locality any physical map, used to measure the *components of a vector* at a point m in M , must be done at m , that is in a local frame. One property of the observers is that they have freedom of gauge : along their travel on their world line $p_o(t)$ they can choose a gauge, by choosing 4 orthogonal vectors.

For the time vector the observer has actually no choice : this is necessarily the vector ε_0 in the direction of his velocity $\frac{dp_o}{dt}$. And this vector has been extended as the vector field ε_0 orthogonal to $\Omega_3(t)$. The 3 other vectors of an orthonormal basis, corresponding to the space, belong to the space tangent at $\Omega_3(t)$, they are orthogonal to ε_0 and we assume that the observer can define these 3 vectors at any point of his present $\Omega_3(t)$. This can be achieved by a system of communication (not instantaneous) between observers who would be located at each point.

This is equivalent to assume that, for each observer, there is a principal bundle $P_o(M, SO_0(3, 1), \pi_p)$, a gauge $\mathbf{p}(m) = \varphi_P(m, 1)$ and an associated vector bundle $P_o[\mathbb{R}^4, \iota]$ where (\mathbb{R}^4, ι) is the standard representation of $SO_0(3, 1)$. It

defines at each point an holonomic orthonormal basis : $\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$. To sum up :

Proposition 17 For each observer there is :

a principal **fiber bundle structure** $\mathbf{P}_o(M, SO_0(3,1), \pi_p)$ on M with fiber the connected component of unity $SO_0(3,1)$, which defines at each point a **standard gauge** : $\mathbf{p}(m) = \varphi_P(m, 1)$

an **associated vector bundle** structure $P_o[\mathbb{R}^4, \iota]$ where (\mathbb{R}^4, ι) is the standard representation of $SO_0(3,1)$, which defines at any point $m \in \Omega$ the **standard basis** $\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i), i=0..3$ where $\varepsilon_0(m)$ is orthogonal to the hypersurfaces $\Omega_3(t)$ to which m belongs.

Notice that these structures depend on the observer. Starting with the principal bundle P_o , a change of gauge can be defined at any point by a section $\chi \in \mathfrak{X}(P_o)$ as seen above, with an impact on any associated bundle.

A standard basis is such that its time vector is $\varepsilon_0(m)$, so at the location of the observer it is in the direction of his velocity. Standard bases are not unique : their time vector is the same, but their space vectors can be rotated in $\Omega_3(t)$. Because they constitute an euclidean orthonormal basis the rotation is given by a matrix of $SO(3)$.

1.12 Formulas for a change of observer

Theorem 18 For any two observers O, A the components of the vectors of the standard orthonormal basis of A , expressed in the standard basis of O are expressed by the matrix of $SO_0(3,1)$, where \vec{v} is the instantaneous spatial speed of A with respect to O and R a matrix of $SO(3)$:

$$[\chi] = \begin{bmatrix} \frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}} & \frac{\frac{v^t}{c}}{\sqrt{1-\frac{\|v\|^2}{c^2}}} \\ \frac{\frac{v}{c}}{\sqrt{1-\frac{\|v\|^2}{c^2}}} & \left(\frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}} - 1 \right) \frac{vv^t}{\|v\|^2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \quad (31)$$

Proof. Let be :

O be an observer (this will be main observer) with associated vector field ε_0 , proper time t and world line $p_o(t)$

A be another observer with associated vector field ε'_0 , proper time τ

Both observers use their standard chart φ_o, φ_A and their standard orthonormal basis, whose time vector is in the direction of their velocity. The location of A on his world line is the point m such that A belongs to the hypersurface $\Omega_3(t)$

The velocity of A at m :

$\frac{dp_A}{d\tau} = c\varepsilon'_0(m)$ by definition of the standard basis of A

$\frac{dp_A}{d\tau} = \frac{1}{\sqrt{1-\frac{\|\vec{v}\|^2}{c^2}}} (\vec{v} + c\varepsilon_0(m))$ as measured in the standard basis of O

The matrix $[\chi]$ to go from the orthonormal basis $(\varepsilon_i(m))_{i=0}^3$ to $(\varepsilon'_i(m))_{i=0}^3$ belongs to $SO_0(3, 1)$. It reads :

$$[\chi(t)] = \begin{bmatrix} \cosh \sqrt{w^t w} & w^t \frac{\sinh \sqrt{w^t w}}{\sqrt{w^t w}} \\ w \frac{\sinh \sqrt{w^t w}}{\sqrt{w^t w}} & I_3 + \frac{\cosh \sqrt{w^t w} - 1}{w^t w} w w^t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$$

for some $w \in \mathbb{R}^3, R \in SO(3)$

The elements of the first column of $[\chi(t)]$ are the components of $\varepsilon'_0(m)$, that is of $\frac{1}{c} \frac{d\mu_A}{d\tau}$ expressed in the basis of O :

$$\cosh \sqrt{w^t w} = \frac{1}{\sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}}}$$

$$w \frac{\sinh \sqrt{w^t w}}{\sqrt{w^t w}} = \frac{\vec{v}}{c} \frac{1}{\sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}}}$$

$$w = k \vec{v} \Rightarrow w^t w = k^2 \|\vec{v}\|^2$$

which leads to the classic formula with

$$w = \frac{v}{\|\vec{v}\|} \operatorname{argtanh} \left\| \frac{v}{c} \right\| = \frac{1}{2} \frac{v}{\|\vec{v}\|} \ln \left(\frac{c + \|\vec{v}\|}{c - \|\vec{v}\|} \right) \sim \frac{1}{2} \frac{v}{\|\vec{v}\|} \ln \left(1 + 2 \frac{\|\vec{v}\|}{c} \right) \sim \frac{v}{c} \blacksquare$$

Some key points to understand these formulas :

- They hold for *any observers* O, A, who use their standard orthonormal basis (the time vector is oriented in the direction of their velocity). There is no condition such as inertial frames.

- The points of M where O and A are located can be different, but they belong both to the same hypersurface $\Omega_3(t)$. The spatial speed \vec{v} is a vector belonging to the space tangent at $\Omega_3(t)$ at the location m of A (and not at the location of O at t).

- The formulas are related to the standard orthonormal bases $(\varepsilon_i(m))_{i=0}^3$ of O and $(\varepsilon'_i(m))_{i=0}^3$ located at the point m of $\Omega_3(t)$ where A is located.

- These formulas apply to the components of vectors in the standard orthonormal bases.

These formulas have been verified with a great accuracy, and *the experiments show that c is the speed of light*. This is an example of a theory which is checked by the consequences that can be drawn from its basic assumptions.

We will see below how these formulas apply in Special Relativity.

If we take $\frac{v}{c} \rightarrow 0$ we get $[\chi] = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$, that is a rotation of the usual space.

The Galilean Geometry is an approximation of SR when the speeds are small with respect to c . Then the velocities are $\frac{d\mu_A}{d\tau} = (\vec{v} + c\varepsilon_0)$ with a common vector ε_0 .

1.13 The Tetrad

1.13.1 The principal fiber bundle

So far we have defined a chart φ_o and a fiber bundle structure P_o for an observer : the construct is based on the motion of the observer, and his capability to extend his frame over the hypersurfaces $\Omega_3(t)$. With the formulas above we see how one can go from one observer to another, and thus relate the different fiber

bundles P_o . The computations in a change of frame can be done with measures done by the observers, and have been checked experimentally. So it is legitimate to assume that there is a more general structure of principal bundle, denoted $\mathbf{P}_G(M, SO_0(3, 1), \pi_G)$, over M . In this representation the bases used by any observer is just a choice of specific trivialization, or equivalently of standard gauge, and one goes from one trivialization to another with the matrix $[\chi]$.

If it is easy to define such a mathematical structure, it is necessary to understand its physical meaning. There is no way to relate an element $p \in P_G$ at a point m of M to a physical phenomenon : a trivialization is, like a chart, arbitrary, but the relations in a change of trivialization can be measured by their consequences on the bases of associated bundle (as can be checked the relations between components in a change of chart). In particular a gauge, denoted $\varphi_G(m, 1)$ is arbitrary, but it will be convenient to keep the notation in this paper, and to precise its meaning : we state that the standard gauge $\mathbf{p}(m) = \varphi_G(m, 1)$ is the gauge such that the associated basis $\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$ is the standard basis chosen by an observer. So the standard gauge is actually defined by the standard basis. Such a basis, for any observer, is orthonormal and its time vector is time like, future oriented, and defined by the velocity of the observer. Any observer can choose another orthonormal basis, which is deduced by an action of $SO_0(3, 1)$, and thus by an element of P_G . So a change of trivialization, or standard gauge, is equivalent to a change of standard basis. If the change of basis involves a change of the time vector $\varepsilon_0(m)$ we assume that this is a change of observer.

Proposition 19 *There is a unique structure of principal bundle*

$\mathbf{P}_G(M, SO_0(3, 1), \pi_G)$ with base M , standard fiber $SO_0(3, 1)$. A change of observer is given by a change of trivialization on P_G .

The standard gauge $\mathbf{p}(m) = \varphi_G(m, 1)$ is, for any observer, associated to his standard basis $\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$.

As a consequence the change of observer is a change of gauge, given by a section χ (global or not) of \mathbf{P}_G , the vectors of the standard basis transform according to the matrix $[\chi]$. The transformation holds at any point. Moreover *the operation is associative : the combination of relative motions is represented by the product of the matrices*. This is convenient, and a big change with what is done in SR with Cartesian frames as we will see.

1.13.2 Tetrad

The vectors of a standard basis (the **tetrad**) can be expressed in the holonomic basis of any chart (of an observer or not).

$$\varepsilon_i(m) = \sum_{\alpha=0}^3 P_i^\alpha(m) \partial \xi_\alpha \Leftrightarrow \partial \xi_\alpha = \sum_{i=0}^3 P_\alpha^i(m) \varepsilon_i(m) \quad (32)$$

where $[P]$ is a real invertible matrix (which has no other specific property, it does not belong to $SO(3,1)$) and we denote

Notation 20 $[P'] = [P]^{-1} = [P'_\alpha]$.

The dual of $(\partial\xi_\alpha)_{\alpha=0}^3$ is $(d\xi^\alpha)_{\alpha=0}^3$ with the defining relation :
 $d\xi^\alpha (\partial\xi_\beta) = \delta_\beta^\alpha$.

The dual $(\varepsilon^i(m))_{i=0}^3$ is :

$$\varepsilon^i(m) = \sum_{\alpha=0}^3 P'_\alpha{}^i d\xi^\alpha \Leftrightarrow d\xi^\alpha = \sum_{i=0}^3 P_i^\alpha \varepsilon^i(m) \quad (33)$$

$$\varepsilon^i(m) (\varepsilon_j(m)) = \sum_{\alpha\beta=0}^3 P'_\alpha{}^i P_j^\beta d\xi^\alpha (\partial\xi_\beta) = \sum_{\alpha=0}^3 P'_\alpha{}^i P_j^\alpha = \delta_j^i$$

Notice that, if it is usual to associate to a vector $u \in T_m M$ a covector : $u^* = \sum_{\alpha\beta} g_{\alpha\beta} u^\beta d\xi^\alpha$, then $u^*(v) = \langle u, v \rangle$ so that $(\varepsilon_i)^* \neq \varepsilon^i$: $\varepsilon_i^*(v) = \eta_{ii} v^i \neq \varepsilon^i(v) = v^i$.

In the fiber bundle representation the vectors of the tetrad are variables which are vectors $\varepsilon_i \in \mathfrak{X}(TM)$ or covectors $\varepsilon^i \in \mathfrak{X}(TM^*)$. The quantities $(P_i^\alpha(m))_{i=1}^3$ (called vierbein) and $(P'_\alpha{}^i(m))_{i=1}^3$ are the components of the vectors $\varepsilon_i(m)$ or the covectors $\varepsilon^i(m)$ in any chart. They can be measured, if one has a chart. They depend on the observer, change with the location m and in a change of chart as the components of a vector or a covector. The quantities $\varepsilon_i(m)$ are geometric quantities. They are one of the variables in any model in GR : as such they replace the metric g . However it is obvious that $[P]$ is defined, in any chart, up to a matrix of $SO(3,1)$, so there is some freedom in the choice of the gauge, and this has consequences in the specification of a lagrangian (the derivative $\partial_\beta P_i^\alpha$ cannot appear explicitly).

In a change of gauge on the principal bundle $P_G : \mathbf{p}(m) = \varphi_P(m, 1) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1}$ the holonomic basis becomes with $[\chi(m)] \in SO_0(3,1)$

$$\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i) \rightarrow \tilde{\varepsilon}_i(m) = [\chi(m)]^{-1} \varepsilon_i(m)$$

$$\sum_{\alpha=0}^3 \tilde{P}_i^\alpha(m) \partial\xi_\alpha = \left[\chi(m)^{-1} \right]_i^j \sum_{\alpha=0}^3 P_j^\alpha(m) \partial\xi_\alpha$$

$$\mathbf{p}(m) = \varphi_P(m, 1) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : [P] \rightarrow [\tilde{P}] = [\chi(m)]^{-1} [P] \quad (34)$$

With respect to the standard chart of the observer :

$$\varepsilon_0(p_o(t)) = \partial\xi_0 \Rightarrow P_0^i = \delta_0^i$$

$$\alpha = 1, 2, 3 : \frac{\partial}{\partial\xi^\alpha} \varphi_o(\xi^0, \xi^1, \xi^2, \xi^3) = \partial\xi_\alpha = \frac{\partial}{\partial x} \Phi_{\varepsilon_0}(ct, x) \frac{\partial x}{\partial\xi^\alpha} \in T_m \Omega_3(t) \Rightarrow$$

$$P_\alpha^0 = 0$$

$$\text{so } [P'] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & P'_{11} & P'_{12} & P'_{13} \\ 0 & P'_{21} & P'_{22} & P'_{23} \\ 0 & P'_{31} & P'_{32} & P'_{33} \end{bmatrix} ; [P] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & P_{11} & P_{12} & P_{13} \\ 0 & P_{21} & P_{22} & P_{23} \\ 0 & P_{31} & P_{32} & P_{33} \end{bmatrix}$$

1.13.3 Metric

The scalar product can be computed from the components of the tetrad. By definition :

$$g_{\alpha\beta}(m) = \langle \partial\xi_\alpha, \partial\xi_\beta \rangle = \sum_{ij=0}^3 \eta_{ij} [P']_\alpha^i [P']_\beta^j$$

The induced metric on the cotangent bundle (Maths.1608) is denoted with upper indexes :

$$g^* = \sum_{\alpha\beta} g^{\alpha\beta} \partial\xi_\alpha \otimes \partial\xi_\beta$$

and its matrix is $[g]^{-1}$:

$$g^{\alpha\beta}(m) = \sum_{ij=0}^3 \eta^{ij} [P']_i^\alpha [P']_j^\beta$$

$$[g]^{-1} = [P] [\eta] [P]^t \Leftrightarrow [g] = [P']^t [\eta] [P'] \quad (35)$$

It does not depend on the gauge on P_G :

$$[\tilde{g}] = [\tilde{P}']^t [\eta] [\tilde{P}'] = [P']^t [\chi(m)^{-1}]^t [\eta] [\chi(m)^{-1}] [P'] = [P']^t [\eta] [P']$$

So in the standard chart of the observer : $g^{00} = -1$.

$$[g] = [P']^t [\eta] [P'] = \begin{bmatrix} -1 & 0 \\ 0 & [g]_3 \end{bmatrix}$$

and $[g]_3$ is definite positive.

The metric defines a **volume form** on M (Maths.1609). Its expression in any chart is, by definition :

$$\varpi_4(m) = \varepsilon_0 \wedge \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 = \sqrt{|\det[g]|} d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$$

$$[g] = [P']^t [\eta] [P'] \Rightarrow \det[g] = -(\det[P'])^2 \Rightarrow \sqrt{|\det[g]|} = \det[P']$$

assuming that the standard basis of $P_G[\mathbb{R}^4, i]$ is direct.

$$\varpi_4 = \det[P'] d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \quad (36)$$

1.13.4 Induced metric

The metric on M induces a metric on any submanifold but it is not necessarily non degenerated (Maths.19.3.1).

On hypersurfaces the metric is non degenerated if the unitary normal n is such that $\langle n, n \rangle \neq 0$ (Maths.1642). The induced volume form is (Maths.1644) :

$$\mu_3 = i_n \varpi_4 = \det[P'] d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3 (n)$$

For $\Omega_3(t)$ the unitary normal n is ε_0 , the induced metric is Riemannian and the volume form ϖ_3 is :

$$\varpi_3 = i_{\varepsilon_0} \varpi_4 = \det[P'] d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3 (\varepsilon_0)$$

$$= \det[P'] d\xi^0 (\varepsilon_0) \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$$

$$= \det[P'] d\xi^1 \wedge d\xi^2 \wedge d\xi^3$$

$$\varpi_3 = \det[P'] d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \quad (37)$$

and conversely :

$\varpi_4 = \varepsilon_0 \wedge \varpi_3 = \det [P'] d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$
 ϖ_3 is defined with respect to the coordinates ξ^1, ξ^2, ξ^3 but the measure depends on $\xi^0 = t$.

For a curve C , represented by any path : $p : \mathbb{R} \rightarrow C :: m = p(\theta)$ the condition is $\left\langle \frac{dp}{d\theta}, \frac{dp}{d\theta} \right\rangle \neq 0$. The volume form on any curve defined by a path : $q : \mathbb{R} \rightarrow M$ with tangent $V = \frac{dq}{d\theta}$ is $\sqrt{|\langle V, V \rangle|} d\theta$. So on the trajectory $q(t)$ of a particle it is

$$\varpi_1(t) = \sqrt{-\langle V, V \rangle} dt \quad (38)$$

For a particle there is the privileged parametrization by the proper time, and as $\left\langle \frac{dp}{d\tau}, \frac{dp}{d\tau} \right\rangle = -c^2$ the length between two points A,B is :

$$\ell_p = \int_{\tau_A}^{\tau_B} \sqrt{-\left\langle \frac{dp}{d\tau}, \frac{dp}{d\tau} \right\rangle} d\tau = \int_{\tau_A}^{\tau_B} c d\tau = c(\tau_B - \tau_A)$$

This is an illustration of the idea that all world lines correspond to a travel at the same speed.

1.14 From particles to material bodies

In Mechanics a material body is comprised of “material points”, that is elements of matter whose location is a single geometric point, and change with time in a consistent way : their trajectories do not cross, and the body keeps some cohesion, which is represented by a deformation tensor for deformable solid bodies. In Relativity the material points, particles in our terminology, follow independent world lines, which do not cross, and thus can be represented by a field of vectors u , future oriented with length $\langle u, u \rangle = -c^2$, such that, at some time 0, the particles are all together in a compact subset $\varpi(0)$ of a 3 dimensional space like submanifold. Then the location of any particle of the material body is given by $\Phi_u(\tau, a)$ where τ is its proper time and $a \in \varpi(0)$ its location at $\tau = 0$. The area swept by the body is $\widehat{\omega} = \{\Phi_u(\tau, a) : \tau \in \mathbb{R}, a \in \varpi(0)\}$ and we have the function : $f : \widehat{\omega} \rightarrow \mathbb{R} : f(m) = \tau : \exists a \in \varpi(0) : \Phi_u(\tau, a) = m$. The function f defines a foliation in diffeomorphic compact 3 dimensional hypersurfaces $\omega(\tau)$ which can be seen as the state of the material body at τ (Maths.1503). So $\Phi_u(\tau, a)$ can be seen as a chart of $\widehat{\omega}$, and the material body has a unique proper time τ . We can then give a definition of a material body which is independent of any observer.

Definition 21 *A material body is defined by a field of vectors u , future oriented with length $\langle u, u \rangle = -c^2$, and a compact subset $\varpi(0)$ of a 3 dimensional space like submanifold. The body is located at its proper time τ on the set $\omega(\tau)$ diffeomorphic to $\varpi(0)$.*

The vector field $u \in \mathfrak{X}(TM)$ does not depend on a chart, but for any observer O the trajectories of the material points of the body will follow a vector field $V \in$

$\mathfrak{X}(TM)$: the curves do not change, but they are traveled with the parameter t and not τ .

A material point can be labeled by its position $y \in \omega(0)$ then its location p is

$\Phi_u(\tau, a)$ at the proper time τ along the world line going through y

$\Phi_{\varepsilon_0}(t, x(t)) = \varphi_O(t, x(t))$ at the time t of the observer in the chart of

the observer given by his vector field ε_0

and $\tau, t, y, x(t)$ are related by :

$$u = \frac{dp}{d\tau}$$

$$V(t) = \frac{dp}{dt} = u \frac{\sqrt{-(V, V)}}{c} = c\varepsilon_0 + \vec{v}$$

$$\vec{v} = \frac{dx}{dt}$$

$$x(0) = a$$

At $t = 0$ for the observer the set of points of the material body is $\widehat{\omega}(0) = \widehat{\omega} \cap \Omega(0)$

At $t > 0$ for the observer the set of points of the material body is $\widehat{\omega}(t) = \widehat{\omega} \cap \Omega(t)$

$$\widehat{\omega}(t) = \{\varphi_O(t, x(t)), x(0) \in \widehat{\omega}(0)\}$$

$$= \{\Phi_V(t, x(0)), x(0) \in \widehat{\omega}(0)\}$$

$$= \{\Phi_u(\tau(t), a), a \in \widehat{\omega}(0)\}$$

$$\text{and } \omega(\tau(t)) = \{\Phi_u(\tau(t), a), a \in \omega(0)\}$$

So *the characterization of a material body is observer dependant* : they do not see the same body.

However we will assume :

Proposition 22 *For any material body there are observers with proper time t such that, at $t = 0$ they observe the entire material body : $\omega(0) \subset \Omega(0)$*

Then at any given time t : $\widehat{\omega}(t) = \omega(\tau(t))$. This is a legitimate assumption, which will be mainly used to compute the characteristics of material bodies.

In Mechanics a solid is a material body such that the distance between any two of its points is constant. $\omega(\tau)$ is a Riemannian manifold, with the metric g_τ induced by the metric g on M . g_τ defines the length of a curve on $\omega(\tau)$ and the distance between two of its points is then the minimum of the length of all the lines which join the points. Because $\omega(\tau)$ is compact such a minimum exists, however the metric g_τ itself depends on the point and τ , so this concept of solid cannot be extended in Relativity.

In Galilean Geometry the local deformation of a material body is studied through the change of a frame attached to each point, with respect to a fixed common frame. In the previous framework the local deformation can be measured through the derivative of the flow Φ_u . Choose a chart of $\varpi(0)$: $a = \phi(\eta_1, \eta_2, \eta_3)$ then the derivatives $u = \frac{\partial \Phi_u}{\partial \tau}$, $u_i = \frac{\partial \Phi_u}{\partial \eta_i}$, $i = 1, 2, 3$ provides a basis at each point of $\omega(\tau)$ and the components of these vectors with respect to any chart of M provide a deformation tensor. We will see in the following how one can improve these points.

1.15 Special Relativity

All the results of this chapter hold in Special Relativity. This theory, which is still the geometric framework of QTF and Quantum Physics, adds two assumptions : the Universe M can be represented as an affine space, and the metric does not depend on the location (these assumptions are independent). As consequences :

- the underlying vector space \vec{M} (the Minkovski space) is common to all observers : the vectors of all tangent spaces to M belong to \vec{M}

- one can define orthonormal bases which can be freely transported and compared from a location to another

- because the scalar product of vectors does not depend on the location, at each point one can define time-like and space-like vectors, and a future orientation (this condition relates the mathematical and the physical representations, and \vec{M} is not simply \mathbb{R}^4)

- there are fixed charts $(O, (\varepsilon_i)_{i=0}^3)$, called frames, comprised of an origin (a location O in M : a point) and an orthonormal basis $(\varepsilon_i)_{i=0}^3$. There is necessarily one vector such that $\langle \varepsilon_i, \varepsilon_i \rangle = -1$. It is possible to define, non unique, orthonormal bases such that ε_0 is timelike and future oriented.

- the coordinates of a point m , in any frame $(O, (\varepsilon_i)_{i=0}^3)$, are the components of the vector OM . The transition maps which give the coordinates of m in another frame $(A, (\tilde{\varepsilon}_i)_{i=0}^3)$ are then given by the formulas :

$$\begin{aligned} OM &= \sum_{i=0}^3 x_i \varepsilon_i \\ AM &= \sum_{i=0}^3 \tilde{x}_i \tilde{\varepsilon}_i \\ OM &= OA + AM = \sum_{i=0}^3 L_i \varepsilon_i + \sum_{i=0}^3 \tilde{x}_i \tilde{\varepsilon}_i \\ \tilde{\varepsilon}_i &= \sum_{j=0}^3 [\chi]_i^j \varepsilon_j, [\chi] \in SO(3, 1) \end{aligned}$$

However one needs to go from this abstract representation to a physical definition of frames.

Observers can label points which are in their present with their proper time. The role of the function $f(m) = t$ is crucial, because it defines the 3 dimensional hypersurfaces $\Omega(t)$. They are not necessarily hyperplanes, but they must be space like and do not cross each other : a point m cannot belong to 2 different hypersurfaces. These hypersurfaces define the vector field $\varepsilon_0(m)$ to which belongs the velocity of the observer (up to c). In SR one can compare vectors at different points, and usually the vectors $\varepsilon_0(m)$ are different from one location to another. They are identical only if $\Omega(t)$ are hyperplanes normal to a vector ε_0 , which implies that the world line of the observer is a straight line, and because the proper time is the parameter of the flow, if the motion of the observer is a translation at a constant spatial speed. These observers are called **inertial**. Notice that this definition is purely geometric and does not involve gravitation or inertia : the motion of an observer is absolute, and inertial observers are such that their velocity is a constant vector.

Observers can define a standard chart $\varphi_o(\xi^0, \xi^1, \xi^2, \xi^3)$ with $\xi^0 = ct, \partial\xi_0 = \varepsilon_0(m)$ with the flow of $\varepsilon_0(m)$ and a chart φ_Ω of $\Omega(0)$ which provides the coordinates ξ^1, ξ^2, ξ^3 of the point x where the integral curves of ε_0 passing through m crosses $\Omega(0)$. The general results hold and such a chart can always be defined. However this chart is usually not defined by a frame $(O, (\varepsilon_i)_{i=0}^3)$: the vectors of the basis must be constant, and notably ε_0 so this is possible only if the observer is inertial : *a frame can be associated to an observer only if this is an inertial observer.*

For inertial observers the integral curves are straight lines parallel to ε_0 . Any spatial basis $(\varepsilon_i)_{i=1}^3$ of $\Omega(0)$ can be transported on $\Omega(t)$. The standard chart is then similar to a frame in the 4 dimensional affine space $(O(0), (\varepsilon_i)_{i=0}^3)$ with origin $O(0)$, the 3 spatial vectors $(\varepsilon_i)_{i=1}^3$ and the time vector ε_0 . The coordinates of a point $m \in \Omega_3(t)$ are :

$$\overrightarrow{O(0)m} = ct\varepsilon_0 + \sum_{i=1}^3 \xi^i \varepsilon_i \text{ where } \overrightarrow{O(t)m} = \sum_{i=1}^3 \xi^i \varepsilon_i$$

and the velocity of a particle with trajectory $q(t)$, as measured by O, is :

$$V = c\varepsilon_0 + \vec{v} \text{ with } \vec{v} = \frac{d}{dt} \overrightarrow{O(t)q(t)} = \sum_{i=1}^3 \frac{d\xi^i}{dt} \varepsilon_i \text{ because } \frac{d\varepsilon_i}{dt} = 0$$

If there is another inertial observer with standard chart defined by a frame $(A(0), (\tilde{\varepsilon}_i)_{i=0}^3)$ the coordinates of $A(t)$, as seen by $O(t)$, are :

$$\overrightarrow{O(0)A(t)} = ct\varepsilon_0 + \sum_{i=1}^3 \xi^i(t) \varepsilon_i \text{ where } \overrightarrow{O(t)A(t)} = \sum_{i=1}^3 \xi^i(t) \varepsilon_i$$

$$\text{The spatial speed of A is : } \vec{v} = \frac{d}{dt} \overrightarrow{O(t)A(t)} = \sum_{i=1}^3 \frac{d\xi^i}{dt} \varepsilon_i$$

We can then implement the general results for the change of basis : $\varepsilon_i \rightarrow \tilde{\varepsilon}_i$.

As for the change of coordinates we have :

$$\overrightarrow{O(0)m} = ct\varepsilon_0 + \sum_{i=1}^3 \xi^i \varepsilon_i$$

$$\overrightarrow{A(0)m} = c\tau\tilde{\varepsilon}_0 + \sum_{i=1}^3 \tilde{\xi}^i \tilde{\varepsilon}_i$$

$$\begin{bmatrix} \tilde{\xi}^i \end{bmatrix} = [\chi]^{-1} \begin{bmatrix} \xi^i \end{bmatrix}$$

$$\overrightarrow{O(0)m} = \sum_{i=0}^3 \xi^i \varepsilon_i = \overrightarrow{O(0)A(0)} + \overrightarrow{A(0)m} = L + \sum_{i=0}^3 \tilde{\xi}^i \tilde{\varepsilon}_i$$

with a constant vector $\overrightarrow{O(0)A(0)} = L$.

So the transformation of the coordinates is given by the product of a fixed translation and a fixed rotation in the Minkovski space. The set of such transformations is a group, called the Poincaré's group.

This result holds only for two inertial observers, and we need a physical mean to tell what are these observers. The usual answer is that they do not feel a change in the inertial forces to which they are submitted. This is similar to the Galilean observers of Classic Mechanics. For non inertial observers the general formulas hold, but the charts cannot be defined through frames as in an affine space.

The concept of material body presented above holds. But if $\varpi(0)$ belongs to a hyperplane then the $\varpi(\tau)$ will be hyperplanes only if they are all orthonormal to a common vector, that is if the vector field which defines the material body is a constant vector : the body must be in a uniform translation (and not rotate on itself).

The formulas of the Lorentz transformations have a tremendous importance in all aspects of Relativist Physics, they are of a constant use, as well as the Poincaré's group which is the starting point in the identification of particles. However any demonstration based on frames, as it is usually done, holds only for inertial observers. A physical theory which is valid only in the study of bodies in uniform translation would be of little interest. As we have proven in this chapter Relativist Geometry can be explained, in a rigorous and quite simple way, without the need of inertial observers. And these are required only for the use of frames. It would be a pity to loose the deep import of Relativity in order to keep a familiar, but not essential, mathematical tool. As a consequence the role assigned to the Poincaré's group must be revisited.

1.16 Some issues about relativity

It is useful to review here some issues which arise frequently about Relativity.

1.16.1 Preferred frames

Relativity is often expressed as “all inertial frames are equivalent for the Physical Laws”. We have seen above that actually inertial frames are required only to define coordinates in affine space : this is a non issue in GR, and in SR it is possible to achieve the usual results with the use of standard charts which are not given by orthogonal frames. But, beyond this point, this statement is misleading.

The Theory of Relativity is more specific than the Principle of Relativity (which can be stated as ”Physical laws do not depend on the observer”), it involves inertia and gravitation but this is at first a Theory about the Geometry of the Universe, and it shows that the geometric measures (of lengths and time) are specific to each observer. The Universe which is Scientifically accessible - meaning by the way of measures, data and figures - depends on the observer. We can represent the Universe with 4 dimensions, conceive of a 4 dimensional manifold which extends over the past and the future, but we must cope with the fact that we are stuck into our present, and it is different for each of us. The reintegration of the observer in Physics is one of the most important feature of Relativity, and the true meaning of the celebrated formulas for a change of frames. An observer is an object in Physics, and as such some properties are attached to it, among them the free will : the possibility to choose the way he proceeds to an experiment, without being himself included in the experiment. But as a consequence the measures are related to his choice.

Mathematics give powerful tools to represent manifolds, in any dimensions. And it seems easy to formulate any model using any chart as it is commonly done. This is wrong from a physical point of view. There is no banal chart or frame : it is always linked to an observer, there is a preferred chart, and so a preferred frame for an observer. It is not related to inertia : it is a matter of geometry, and a consequence of the fundamental symmetry breakdown. The observer has no choice in the selection of the time vector of his orthonormal basis, if he wants to change the vector, he has to change his velocity, and this is why the formulas in a change of frames are between two different observers moving with respect to each other. And not any change is possible : an observer cannot travel in the past, or faster than light. These features are clear when one sticks to a chart of an observer, as we will do in this paper. Not only they facilitate the computations, they are a reminder of the physical meaning of the chart. This precision is specially important in the fiber bundle formalism, which is, from this point of view, a wise precaution as compared to the usual formalism using undifferentiated charts.

1.16.2 Time travel

The distinction between future and past oriented vectors come from the existence of the Lorentz metric. As it is defined everywhere, it exists everywhere, and along any path. It is not difficult to see that the border between the two kinds of vectors is for null vectors $\langle u, u \rangle = 0$. So a particle which would have a path such that its velocity is past oriented should, at some point, have a null velocity, and, with regard to another observer located at the same point, travel at the speed of light. Afterwards his velocity would be space like ($\langle u, u \rangle > 0$) before being back time like but past oriented. So the first issue lies in the possibility of travel at the speed of light. Clearly this would be a discontinuity on the path and "Scotty engages the drive" from Star Trek has some truth.

But the main issue with time travel lies in the fact that, if ever we would be able to come back to the location where we have been in the past (meaning a point of the universe located in our past), we would not find our old self. The idea that we exist in the past assumes that we exist at any time along our world line, as a frozen copy of ourselves. This possibility is sometimes invoked, but it raises another one : what makes us feel that each instant of time is different ? If we do not travel physically along our world line, what does move ? And of course this assumption raises many other issues in Physics...

1.16.3 Twins paradox

The paradox is well known : one of the twins embarks in a rocket and travels for some time, then comes back and finds that he is younger than his twin who has stayed on Earth. This paradox is true (and has been checked with particles) and comes from two relativist features : the Universe is 4 dimensional, and the definition of the proper time of an observer.

Because the Universe is 4 dimensional, to go from a point A to a point B there are several curves. Each curve can be travelled according to different paths. We have assumed that observers move along a curve according to a specific path, their world line, and then :

$$\ell_{AB} = c(\tau_B - \tau_A)$$

Because the curves are different, the elapsed proper time is usually different.

The proper time is the time measured by a clock attached to the observer, it is his biological time. Assuming that all observers travel along their world lines with a velocity such that at $\left\langle \frac{dp_o}{d\tau}, \frac{dp_o}{d\tau} \right\rangle = -c^2$ is equivalent to say that, with respect to their clock, they age at the same rate. So if they travel along different curves there is no reason that the total duration of their travel would be the same.

Whom of the two twins would have aged the most ? It is not easy to do the computation in GR, but simpler in the SR context.

We can define a fixed frame $(O, (\varepsilon_i)_{i=0}^3)$ with origin O at the time $t = 0$, A is spatially immobile with respect to this frame, moves along the time axis and his coordinates are then : $OA : p_A(\tau_A) = c\tau_A\varepsilon_0$

The twin B moves in the direction of the first axis. His coordinates are then : $OB : p_B(\tau_B) = c\tau_B\varepsilon_0 + x_B(\tau_B)\varepsilon_1$

The spatial speed of B with respect to A is : $\frac{dOB}{d\tau_A} = V(\tau_B)\varepsilon_1$

The velocity of B is : $u_B = \frac{dOB}{d\tau_B} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(V\varepsilon_1 + c\varepsilon_0)$

To be realistic we must assume that B travels at a constant acceleration, but needs to brake before reaching first his turning point, then A. In the first phase we have for instance :

$$V = \gamma c\tau_B \text{ with } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$p_B(\tau_B) = \int_0^{\tau_B} \frac{c}{\sqrt{1-(\gamma t)^2}} (\gamma t\varepsilon_1 + \varepsilon_0) dt = \frac{c}{\gamma} \left[\sqrt{1-y^2}\varepsilon_1 + \varepsilon_0 \arcsin y \right]_0^{\gamma\tau_B}$$

A full computation gives : $\frac{\tau_A}{\tau_B} = \frac{\arcsin v_M}{v_M}$ where v_M is the maximum speed in the travel, which gives for $v_M = c$: $\frac{\tau_A}{\tau_B} = 1.57$ that is less than what is commonly assumed.

The Sagnac effect, used in accelerometers, is based on the same idea : two laser beams are sent in a loop in opposite direction : their 4 dimensional paths are not the same, and the difference in the 4 dimensional lengths can be measured by interferometry.

1.16.4 The expansion of the Universe

A manifold by itself can have some topological properties. It can be compact. It can have holes, defined through homotopy (Maths.10.4.1) : there is a hole if there are curves in M which cannot be continuously deformed to be reduced to a point. A hole does not imply some catastrophic feature : a doughnut has a hole. Thus it does not imply that the charts become singular. But there are only few purely topological features which can be defined on a manifold, and they are one of the topic of Differential Geometry. In particular a manifold has no shape to speak of.

The metric on M is an addition to the structure of the Universe. It is a mathematical feature from which more features can be defined on M, such that curvature. It is also a physical feature and in GR the metric, and so the curvature of M at a point, depends on the distribution of matter. It is customary (see Wald) to define singularities in the Universe by singularities of geodesics, but geodesics are curves whose definition depends on the metric. A singularity for the metric, as Black holes or Bing Bang, is not necessarily a singular point for the manifold itself.

GR has open the possibility to build cosmological models, representing the totality of the Universe. It is clear that Cosmology requires some revision of the usual concepts of Physics (what is an independent observer ?) and even of the premises of our epistemology (do we have free will ?). A requirement which is rarely fulfilled in the common Cosmological Theories. From some general reasoning and Astronomical observations, it is generally assumed that the Universe has the structure of a fiber bundle with base \mathbb{R} (a warped Universe) which can be seen as the generalization of M_o , that we have defined above for an observer. Thus there is some universal time (the projection from M to \mathbb{R}) and

a foliation of M in hypersurfaces similar to $\Omega_3(t)$, which represent the present for the observers who are located on them (see Wald and Peebles for more on this topic). This is what we have defined as a material body : the part of the universe on which stands all matter would be a single body moving together since the Big Bang (the image of an inflating balloon). So there would not be any physical content before or after this $\Omega_3(t)$ (inside the balloon), but nothing can support this interpretation, or the converse, and probably it will never be.

The Riemannian metric $\varpi_3(t)$ on each $\Omega_3(t)$ is induced by the metric on M , and therefore depends on the universal time t . In the most popular models it comes that the distance between two points on $\Omega_3(t)$, measured by the Riemannian metric, increases with t , and this is the foundation of the narrative about an expanding universe, which is supported by astronomical observations. But, assuming that these models are correct, this needs to be well understood. The change of the metric on $\Omega_3(t)$ makes that the volume form $\varpi_3(t)$ increases, but the hypersurfaces $\Omega_3(t)$ belong to the same manifold M , which does not change with time. The physical universe would be a deformable body, whose volume increases inside the unchanged container. Moreover it is generally assumed that material points, belonging to the same material body but traveling on their own world lines, stick together : they are not affected by this dilation, only the vacuum which separates material bodies.

2 KINEMATICS

Kinematics is the next step after Geometry. We go further in the physical world, and try to understand what are the relations between the motion of material bodies and the forces which are applied to them. All material bodies manifest some resistance to a change of their motion, either in direction or in speed. This feature is the inertia, and is measured by different quantities which incorporate, in one way or another, the mass of the material body. The mass m is a characteristic of the material body : it does not change with the motion, the forces or the observer. And from motion and mass are defined key quantities : the momenta.

The Newton's law : $\vec{F} = m \vec{\gamma}$ is expressed, more appropriately by : $\vec{F} = \frac{d\vec{p}}{dt}$ where \vec{p} is the momentum. The inertial forces are, by construct, the opposite of the forces which are necessary to change the momenta of a material body. So, Kinematics is, in many ways, the Physics of inertia. The issue of the origin of these inertial forces, which appear everywhere and with a great strength, is the gravitational field, as we will see. Let us see now how one goes from Geometry, that is motion, to Kinematics, that is inertia.

The study of rotations and rotational momenta in the 4 dimensional Universe will lead to a new representation of the momenta, based on Spinors, which are vectors in a 4 dimensional abstract space, to the introduction of antiparticles and of spin.

2.1 Translational Momentum in the relativist context

In Newtonian Physics the bridge between Geometry and Kinetics is hold by momenta. And, because motion of material bodies involves both a translational motion and a rotational motion, there are linear momenta and rotational momenta.

In Galilean Geometry the linear momentum of a particle is simply : $\vec{p} = \mu \vec{v}$ with a constant scalar μ which is the inertial mass. Its natural extension in the relativist context is the quadri-vector : $P = \mu u$ where u is the velocity. This generalization has two consequences :

- the quadri-vector P is intrinsic : its definition as well as the value of the scalar product $\langle P, P \rangle = -\mu^2 c^2$ do not depend on an observer

- but its measure depends on the observer. In his standard basis it reads :

$$P = \mu \frac{1}{\sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}}} (\vec{v} + c\varepsilon_0(m))$$

In the relativist context location and motion are absolute. If the Universe has a physical meaning, then each of its points is singular, and this is clearly represented by a manifold. The proper time, and the derivative of the location with respect to the proper time, are defined without any reference to a frame, so the vector velocity u is absolute, and this property has been used to compute the rules in a change of frames. If motion is absolute, *its measure is relative*, depends on the observer and its measure changes according to geometric rules, because they are geometric quantities. The spatial speed appears when an observer

has been chosen. The definition of the momentum by $P = \mu u$ is consistent with the idea that the kinematic features of a particle are intrinsic, and can be represented by a quantity which does not depend on an observer (even if its measure depends on it).

This is a big change from the Newtonian definitions : the momentum $\vec{p} = \mu \vec{v}$ as well as the kinetic energy $\frac{1}{2}\mu \|\vec{v}\|^2$ are relative and depend on the observer.

If we keep the concept of Energy as measured by [Momentum]×[Speed] then the energy of the particle $\langle P, u \rangle = -\mu c^2$ is constant and, for an observer, is split between a part related to the spatial speed $\mu \frac{\|\vec{v}\|^2}{1 - \frac{\|\vec{v}\|^2}{c^2}}$, corresponding to a kinetic energy, and a part which is stored in the particle $-\mu \frac{c^2}{1 - \frac{\|\vec{v}\|^2}{c^2}}$. But, if one wants to keep the principle of conservation of energy, one has to accept that mass itself can be transformed into energy, according to the famous relation $E = \mu c^2$.

However there are several interpretations of these concepts. Physicists like to keep a concept of momentum linked to the spatial velocity and, with a fixed mass, define the linear momentum as : $\vec{p}_r = \mu \frac{\vec{v}}{\sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}}}$, that is the spatial

part of P. Then they define the Energy E of the particle by : $E^2 = c^2 \|\vec{p}_r\|^2 + \mu^2 c^4$ that is one part corresponding to a kinetic energy, and another one to an energy at rest. This sums up to define the energy by rewriting Pc with the two components :

$$Pc = (c\vec{p}_r, E) \Rightarrow \langle Pc, Pc \rangle = -\mu^2 c^4 = c^2 \|\vec{p}_r\|^2 - E^2$$

The introduction of c in $c\vec{p}_r$ is necessary in order to have the same unit Energy in both parts. In this formulation Pc is a 4 vector, and its components change according to the Lorentz formula, so E depends on the observer. The advantages of this expression is that for small speed it gives :

$$\begin{aligned} E &\simeq \sqrt{c^2 \mu^2 \|\vec{v}\|^2 \left(1 + \frac{1}{2} \frac{\|\vec{v}\|^2}{c^2}\right) + \mu^2 c^4} \\ &= \mu c^2 \sqrt{\left(1 + \frac{\|\vec{v}\|^2}{c^2}\right) \left(1 + \frac{1}{2} \frac{\|\vec{v}\|^2}{c^2}\right)} \simeq \mu c^2 \sqrt{1 + \frac{\|\vec{v}\|^2}{c^2}} \simeq \frac{1}{2} \mu \|\vec{v}\|^2 + \mu c^2 \end{aligned}$$

and it can be adapted to massless particles such as photons.

So if one can say, with Feynman (in *Lectures in Physics*), that the conservation of energy is a law without exception, the trouble is that in the Relativist context the definition of energy itself depends on the observer and is fairly subtle... We will see that, if it is possible to give a satisfying definition of the energy of a particle, this is more complicated for fields, and further more for bosons.

The only physical quantity which has a clear meaning, is independent of an observer, and is characteristic of a particle, is Pc , which has the dimension of energy, and that we will call **relativist momentum**. The mass at rest, independent from an observer, is $\mu = \frac{1}{c} \sqrt{-\langle Pc, Pc \rangle}$. The usual energy is computed by taking the component of Pc along the direction of $\varepsilon_0(m)$. So it depends on

the observer. These quantities are individualized and localized : they are linked to the particle and its position.

Moreover material bodies have a rotation, and a rotational momentum \vec{J} which depends on the mass and the shape of the body. Momenta are characteristics features of material bodies, and the bases for the definition of forces. According to the Newton's law: $\vec{F} = \frac{d\vec{p}}{dt}$ and $\vec{\gamma} = \frac{d\vec{J}}{dt}$ for a torque $\vec{\gamma}$. Rotation and rotational momentum are topics which are more complicated than it seems, in the relativist context in particular. Moreover at the atomic scale particles show properties which look like rotation, and specific features, which have lead to the concepts of spins and spinors.

2.2 The issues of the concept of rotation

2.2.1 Rotation in Galilean Geometry

The concept of rotation is well defined in Mathematics : this is the operation which transforms the *orthonormal basis* of a vector space into another, and in Galilean Geometry is represented by a matrix of the group $SO(3)$. This is a compact, 3 dimensional Lie group, of matrices such that $R^t R = I$. Then an instantaneous rotation, that is the derivative of a rotation with respect to time, is represented by an element of its Lie algebra $so(3)$ which is the vector space of 3×3 real antisymmetric matrices. If we take the following matrices as basis of $so(3)$:

$$\kappa_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}; \kappa_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}; \kappa_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then any matrix of $so(3)$ reads :

$\sum_{i=1}^3 r^i [\kappa_i] = [j(r)]$ with the operator

$$j : \mathbb{R}^3 \rightarrow L(\mathbb{R}, 3) :: [j(r)] = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \quad (39)$$

The operator j is very convenient to represent quantities which are rotated¹ and has many nice algebraic properties (see formulas in the annex) and we will use it often in this paper.

For any vector $u : \sum_{ij=1}^3 [j(r)]_j^i u^j \varepsilon_i = \vec{r} \times \vec{u}$ with the cross product \times .

Because $SO(3)$ is a compact Lie group the exponential is surjective :

$$\forall g \in SO(3), \exists \kappa = \sum_{i=1}^3 r^i [\kappa_i] \in so(3) : g = \exp \kappa$$

It is easy to show that :

$$[g] = \exp [j(r)] = I_3 + [j(r)] \frac{\sin \sqrt{r^t r}}{\sqrt{r^t r}} + [j(r)] [j(r)] \frac{1 - \cos \sqrt{r^t r}}{r^t r}$$

¹It is similar to the Levi-Civita tensor ϵ but, in my opinion, much easier to use.

The representation of an instantaneous rotation by a vector $\vec{r} \in \mathbb{R}^3$ is further supported by two facts.

The axis of rotation, which is by definition the unique eigen vector of $[g]$ with eigen value 1 and norm 1 in the standard representation of $SO(3)$, has for

components $\begin{bmatrix} r^1 \\ r^2 \\ r^3 \end{bmatrix} / \sqrt{r^t r}$

Similarly one can define the angle θ of the rotation resulting from a given matrix, and $\theta = \sqrt{r^t r}$, which is also the instantaneous rotational speed with the period $T = 2\pi / \sqrt{r^t r}$

Proof. For any vector u of norm 1 : $\langle u, [g] u \rangle = \cos \theta$ where θ is an angle which depends on u and $[g] = \exp [j(r)]$. With the formula above, and using $[j(r)] [j(r)] = [r] [r]^t - \langle r, r \rangle I$ and $\langle u, [j(r)] u \rangle = 0$ we get :

$$\langle u, [g] u \rangle = 1 + \left(\langle u, r \rangle^2 - \langle r, r \rangle \right) \frac{1 - \cos \sqrt{r^t r}}{r^t r}$$

which is minimum for $\langle u, r \rangle = 0$ that is for the vectors orthogonal to the axis, and :

$$\cos \theta = \cos \sqrt{r^t r}$$

For a rotation at constant speed $\varpi : \theta(t) = \varpi t = t \sqrt{r^t r}$ so $T = 2\pi / \sqrt{r^t r}$ ■

So we have a very satisfying representation of geometric rotations : a rotation $R(t)$ can be defined by a single vector, which can be easily related to the motion as it can be perceived. However this mathematical representation is not faithful. The same rotation can be defined equally by the opposite axis, and the opposite angle. This is related to the mathematical fact that $SO(3)$ is not the only group which has $so(3)$ as Lie algebra. The more general group is the Spin group $Spin(3)$ which has also for elements the scalars + 1 and - 1, so that $R(t)$, corresponding to $(r, \sqrt{r^t r})$ and $-R(t)$, corresponding to $(-r, -\sqrt{r^t r})$ can represent the same physical rotation. So, actually, the group which should be used to represent rotations in Galilean Geometry is $Spin(3)$, which makes the distinction between the two rotations, and not $SO(3)$. This is not a problem in Mathematics, but in Physics the distinction matters : in the real world one goes from one point to another along a path, by a continuous transformation which preserves the orientation of a vector, thus the orientation of \vec{r} matters ². A single vector of \mathbb{R}^3 cannot by itself properly identify a physical rotation, one needs an additional parameter which is ± 1 (to tell which one of the two orientations of \vec{r} is chosen, with respect to a direction, the spatial speed on the path).

But to represent rotation of material bodies by geometric rotations, as above, raises several issues.

We could expect that the total rotational momentum of a body is the sum of the rotational momentum of its components, as it happens with the translational momentum. But material points have no attached frame in Mechanics. So actually the rotational momentum is defined only at the level of the body, through a geometric rotation, and it has a meaning only for material bodies

²In his book "The road to reality" Penrose gives a nice, simple trick with a belt and book to show this fact.

which keep some shape, represented by fixed relations between the positions of the material points. So this is doable for solids, but even for them there is a problem. If the solid has a cylindrical symmetry, by definition it is impossible to measure a geometric rotation around the axis, however the physical rotation can be measured by a rotational inertia. And some force fields, such as magnetism, can exercise a pointwise action, represented by a torque, so that the implementation of the Newton's law to the rotational momentum becomes muddled for particles without structure, like the atoms.

So, even if in Mechanics it is convenient to assimilate physical rotation with geometric rotation, they are not the same and the kinematics of rotating bodies is more complicated. Notice also that the representation of rotation by a single vector in Galilean Geometry relies on a fortuitous fact : the lie algebra $so(3)$ is a 3 dimensional vector space, as the space \mathbb{R}^3 , so that it is easy to set up an isomorphism j of vector spaces.

To represent the motion (translation + rotation) of a solid body we have another discrepancy between the physical and the mathematical definitions. In Classic Mechanics the motions of solids are defined by two vectors (\vec{v}, \vec{r}) and 6 parameters : the spatial speed \vec{v} of the center of mass G and the instantaneous rotation \vec{r} , assimilated to 2 vectors of \mathbb{R}^3 . In Mathematics the transformation between two frames is represented by the group of displacements, which is the semi-product of $SO(3)$ with the group of translations. Using the isomorphism j between $SO(3)$ and \mathbb{R}^3 a displacement can be represented by two vectors (\vec{L}, \vec{r}) but \vec{L} is the translation of the origins of the frames : $\vec{L} = \vec{OG}$ and not the speed $\vec{v} = \frac{d\vec{OG}}{dt}$.

2.2.2 The group of displacements in Relativist Geometry

In Relativist Geometry the focus is usually put on the Poincaré's group, the semi product of the group $SO(3, 1)$ of rotations and of the 4 dimensional translations. This is the simple generalization of the group of displacements of Galilean Geometry. However this raises several issues.

The Poincaré's group represents the transformations between coordinates in Cartesian frames. So its use is valid only in SR, and for inertial observers. There has been attempts to extend the concept to the group of isometries (that is maps $f : M \rightarrow M$ such that the derivative $f'(m)$ preserves the scalar product) but this is difficult. And there is a more fundamental objection : the physical comparison between bases located at different points should involve a transport of the vectors (one does not jump from a point to another), but in SR or GR the universe is no longer isotropic, so the path which is followed for the transport matters (which is obvious in the Lorentz formulas, which involve \vec{v}).

The concept of solid is not generalizable in Relativity, thus one cannot expect to represent the motion of a body by the transformation to go from the frame of an observer to a frame which would be attached to the solid.

And actually, even in Galilean Geometry the point of view of the Physicists

(who represent a motion by (\vec{v}, \vec{r})) and of the mathematicians (who define the displacement by (\vec{L}, \vec{r})) are not the same : even if they both use two vectors and the same number of parameters, they have not the same meaning. In Relativity the discrepancy is worse : an element of the Poincaré's group is defined by 10 parameters. Geometric rotations in the 4 dimensional space have a very different meaning than rotations in the 3 dimensional space. This is obvious with a look at the matrices of $SO(3, 1)$ which read : $[\chi] = \exp [K (w)] \exp [J (r)]$. The second term has the meaning of a rotation in the space, but the first term (usually called the boost) involves the translational motion, as shown in the formula : $w = \frac{v}{\|v\|} \arg \tanh \left\| \frac{v}{c} \right\|$.

For a physicist the motion of a material body is related to the instantaneous change of its location and disposition (as it is done in Galilean Geometry with (\vec{v}, \vec{r})) and not to the transformation between fixed frames (the vector \vec{L} has no physical interest in the matter). So the Lorentz matrix (defined by two 3 dimensional vectors (w, r) related to (\vec{r}, \vec{v})) only is significant from this point of view, and w can be clearly (even if it is in a complicated way) related to \vec{v} . Moreover we have seen that the formulas hold for any observer.

Our purpose here is to find a way to represent kinematic characteristics of material bodies, by vectorial quantities. This is not to find the formulas in a change of coordinates (that we have), so the Poincaré's group is of no use. But the Lorentz group is essential because it gives the rules for the transformation of the components of vectors.

2.3 Momenta in the fiber bundle representation

As we have seen fiber bundles provide an efficient representation of the geometry of the Universe, notably in the GR picture. So it is legitimate to look at what it can provide on this issue. The aim is to represent the kinematic characteristics of material bodies, incorporating both their geometric motion (translation and rotation) and inertial features, in a single quantity, which can be implemented for particles, that is without the need to involve a fixed structure of the body. This quantity should be intrinsic, meaning that its value, measured in a frame attached to the particle, should stay constant along its world line, in accordance with the assumption that the motion is absolute and the mass is constant . Meanwhile its value, as measured by an observer, would change according to the observer, as a geometric quantity (with the same gauge group). The gauge group should be the spin group $Spin(3, 1)$ (or $Spin(1, 3)$, they are isomorphic, we will denote both Spin), as we noticed before.

Let us denote the principal bundle $P_G (M, Spin, \pi_G)$, the standard fiber a manifold E , and γ the action of the Spin group. Then the quantity would be an element \mathbf{S} of $P_G [E, \gamma]$.

\mathbf{S} , for a given particle, should stay constant on its world line. The world line is some map $p : \mathbb{R} \rightarrow M :: m = p(\tau)$ and the trajectory for an observer $q : \mathbb{R} \rightarrow M :: m = q(t) = p(\tau(t))$.

If S represents an intrinsic kinematic characteristic of the particle, it should

stay constant along its worldline $p(\tau)$: there is a privileged frame $g(\tau) \in Spin$ such that $\mathbf{S} = (\varphi_G(p(\tau), g(\tau)), S)$ with $S = Ct$

The value measured by the observer in his gauge $\varphi_G(q(t), \mathbf{1})$ is then :

$$(\varphi_G(q(t), \mathbf{1}), S(t))$$

$$\sim \varphi_G(q(t), g(t(\tau))), \gamma(g^{-1}, S(t)) = (\varphi_G(p(\tau), g(\tau)), S)$$

which is equivalent to say that $S(t) = \gamma(g(\tau(t)), S)$ with S constant.

The relation $S(t) = \gamma(g(\tau(t)), S)$ is just the consequence of our very general assumptions. The measure of $S(t)$ varies locally according to the observer (the measure of motion is relative) but its intrinsic value does not change (motion is absolute). But to get a full profit of this representation we have to adopt an entirely new point of view. We cannot any longer view the particle as living in M and spinning in its tangent space. Actually the particle lives in E , which happens to be associated to P_G . Its trajectory is a curve in E , which projects on a curve in M . E can be seen as the physical world (at least a part of it), that we can represent through networks of frames in M . So $S(t)$ cannot be seen properly as a motion, it is only a characteristic of the particle (such as mass and charge). Experience shows that it can be measured through geometric frames but this does not imply the existence of a real spinning motion of the particle. In some way this is what physicists do, intuitively, in Galilean geometry : the rotation, the rotational moment, are not represented as elements of the group or the Lie algebra, but as vectors (it happens that it is the same vector space as M , but this is fortuitous).

The issue that we face is then to precise E and γ . We will make the following, reasonable, assumptions :

i) E is some vector space, so that we have an associated vector bundle $P_G[E, \gamma]$

ii) it implies that (E, γ) is a representation of $Spin$.

iii) this representation should be finite dimensional (we consider here the value of $S(t)$ at some point).

iv) if (E, γ) is a representation of $Spin$, then $(E, \gamma'(1))$ is a representation of its Lie algebra, both are subsets of the Clifford algebra $Cl(3, 1)$ so that, if γ is a linear map, then $\gamma'(1) = \gamma$ and this is not a big leap forward to assume that (E, γ) is a representation of the Clifford algebra itself.

Then the quantity S is a vector of E , called a **spinor**. Spinors have been first introduced by Dirac in an equation, obtained by some magic, but truly pure intuition, which is still the basis of most Quantum Electrodynamics and Quantum Theory of Fields.

3 CLIFFORD ALGEBRAS AND SPINORS

Spinors, as well as the spin, cannot be properly understood without a look at their mathematical berth, which is Clifford algebra. This is a fascinating algebraic structure on vector spaces which is seen in details in Maths.Part 9. The results which will be used in this book are summarized in this section, the proofs are given in the Annex.

3.1 Clifford algebra and Spin groups

3.1.1 Clifford Algebras

A Clifford algebra $Cl(F, \langle \rangle)$ is an algebraic structure, which can be defined on any vector space $(F, \langle \rangle)$ on a field K (\mathbb{R} or \mathbb{C}) endowed with a bilinear *symmetric* form $\langle \rangle$. The set $Cl(F, \langle \rangle)$ is defined from K and F and a product, denoted \cdot , with the property that for any two *vectors* u, v :

$$\forall u, v \in F : u \cdot v + v \cdot u = 2 \langle u, v \rangle \quad (40)$$

A Clifford algebra is then a set which is larger than F : it includes *all vectors* of F , plus scalars, and any linear combinations of products of vectors of F . A Clifford algebra on a n dimensional vector space is a 2^n dimensional vector space on K , and an algebra with \cdot . Clifford algebras built on vector spaces on the same field, with same dimension and bilinear form with same signature are isomorphic. On a 4 dimensional real vector space $(F, \langle \rangle)$ endowed with a Lorentz metric there are two structures of Clifford Algebra, denoted $Cl(3, 1)$ and $Cl(1, 3)$, depending on the signature of the metric, and they are not isomorphic. In the following we will state the results for $Cl(3, 1)$, and for $Cl(1, 3)$ only when they are different.

The easiest way to work with a Clifford algebra is to use an orthonormal basis of F . On any 4 dimensional real vector space $(F, \langle \rangle)$ with a bilinear symmetric form of signature (3,1) or (1,3) we will denote :

Notation 23 $(\varepsilon_i)_{i=0}^3$ is an orthonormal basis with scalar product : $\langle \varepsilon_i, \varepsilon_i \rangle = \eta_{ii}$

So we have the relation :

$$\varepsilon_i \cdot \varepsilon_j + \varepsilon_j \cdot \varepsilon_i = 2\eta_{ij} \quad (41)$$

Then a basis of the Clifford algebra is a set comprised of 1 and all ordered products of $\varepsilon_i, i = 0...3$.

In any orthonormal basis there is a fourth vector which is such that $\varepsilon_i \cdot \varepsilon_i = -1$ (for the signature (3,1)) or $+1$ (for the signature (1,3)). We will label this vector ε_0 .

3.1.2 Spin group

The group $Pin(3, 1)$ is the subset of the Clifford algebra $Cl(3, 1)$:

$$Pin(3, 1) = \{u_1 \cdot u_2 \dots \cdot u_k, \langle u_p, u_p \rangle = \pm 1, u_p \in F\}. Pin(3, 1) \text{ is a Lie group,}$$

Spin(3, 1) is its subgroup where we have an even number of vectors :

$$Spin(3, 1) = \{u_1 \cdot u_2 \dots \cdot u_{2k}, \langle u_p, u_p \rangle = \pm 1, u_p \in F\}$$

and similarly for $Pin(1, 3)$ and $Spin(1, 3)$.

Notice that the *scalars* ± 1 belong to the groups. The identity element is the scalar 1.

$Pin(3, 1)$ and $Pin(1, 3)$ are not isomorphic. $Spin(3, 1)$ and $Spin(1, 3)$ are isomorphic.

3.1.3 Adjoint map

For any $s \in Pin(3, 1)$, the map, called the **adjoint map** :

$$\mathbf{Ad}_s : Cl(3, 1) \rightarrow Cl(3, 1) :: \mathbf{Ad}_s w = s \cdot w \cdot s^{-1} \quad (42)$$

is such that

$$\forall w \in F : \mathbf{Ad}_s w \in F \quad (43)$$

and it preserves the scalar product on F :

$$\forall u, v \in F, s \in Pin(3, 1) : \langle \mathbf{Ad}_s u, \mathbf{Ad}_s v \rangle_F = \langle u, v \rangle_F \quad (44)$$

Moreover :

$$\forall s, s' \in Pin(3, 1) : \mathbf{Ad}_s \circ \mathbf{Ad}_{s'} = \mathbf{Ad}_{s \cdot s'} \quad (45)$$

Because the action \mathbf{Ad}_s of $Spin(3, 1)$ on F gives another vector of F and preserves the scalar product, it can be represented by a 4×4 orthogonal matrix. Using any orthonormal basis $(\varepsilon_i)_{i=0}^3$ of F , then \mathbf{Ad}_s is represented by a matrix $\Pi(\mathbf{Ad}_s) = [h(s)] \in SO(3, 1)$. To two elements $\pm s \in Spin(3, 1)$ correspond a single matrix $[h(s)]$. $Spin(3, 1)$ is the double cover (as manifold) of $SO(3, 1)$. $Spin(3, 1)$ has two connected components (which contains either +1 or -1) and its connected component, that we will denote for brevity also $Spin(3, 1)$, is simply connected and is the universal cover group of $SO_0(3, 1)$. So with the Spin group one can define two physical rotations, corresponding to opposite signs.

3.1.4 Lie algebra of the Spin group

Theorem 24 *The elements of the Lie algebra $T_1 Spin(3, 1)$ belong to the Clifford algebra and can be written as the linear combination of elements $\varepsilon_i \cdot \varepsilon_j$:*

As any algebra $Cl(F, \langle \rangle)$ is a Lie algebra with the bracket :

$$\forall w, w' \in Cl(F, \langle \rangle) : [w, w'] = w \cdot w' - w' \cdot w$$

and the Lie algebra $T_1 Spin(3, 1)$ of $Spin(3, 1)$ is a subset of $Cl(3, 1)$ (Maths.532).

The derivative $\Pi'(1) : T_1 Spin(3, 1) \rightarrow so(3, 1)$ is an isomorphism of Lie algebras. The inverse map : $\Pi'(1)^{-1} : so(3, 1) \rightarrow T_1 Spin(3, 1)$ is an isomorphism

of Lie algebras which reads (Maths.534) with any orthonormal basis $(\varepsilon_i)_{i=0}^3$ of F :

$\Pi'(1)^{-1} : so(3,1) \rightarrow T_1Spin(3,1) :: \Pi'(1)^{-1}([\kappa]) = \frac{1}{4} \sum_{i,j=0}^3 ([\kappa][\eta])_j^i \varepsilon_i \cdot \varepsilon_j$
and any element of $T_1Spin(3,1)$ is such expressed in the basis of $Cl(F, \langle \rangle)$
: it is the linear combinations of the ordered products of all the four vectors of a basis.

With any orthonormal basis and the following choices of basis $(\vec{\kappa}_a)_{a=1}^6$ of $T_1Spin(3,1)$ then $\Pi'(1)^{-1}$ takes a simple form :

$$\Pi'(1)^{-1}([\kappa_1]) = \vec{\kappa}_1 = \frac{1}{2} \varepsilon_3 \cdot \varepsilon_2,$$

$$\Pi'(1)^{-1}([\kappa_2]) = \vec{\kappa}_2 = \frac{1}{2} \varepsilon_1 \cdot \varepsilon_3,$$

$$\Pi'(1)^{-1}([\kappa_3]) = \vec{\kappa}_3 = \frac{1}{2} \varepsilon_2 \cdot \varepsilon_1,$$

$$\Pi'(1)^{-1}([\kappa_4]) = \vec{\kappa}_4 = \frac{1}{2} \varepsilon_0 \cdot \varepsilon_1,$$

$$\Pi'(1)^{-1}([\kappa_5]) = \vec{\kappa}_5 = \frac{1}{2} \varepsilon_0 \cdot \varepsilon_2,$$

$$\Pi'(1)^{-1}([\kappa_6]) = \vec{\kappa}_6 = \frac{1}{2} \varepsilon_0 \cdot \varepsilon_3$$

where $([\kappa_a])_{a=1}^6$ is the basis of $so(3,1)$ already noticed such that : $[\kappa] = K(w) + J(r) = \sum_{a=1}^3 r^a [\kappa_a] + w^a [\kappa_{a+3}]$
 $a = 1, 2, 3 : \vec{\kappa}_a = -\frac{1}{2} \varepsilon(a, i, j) \varepsilon_i \cdot \varepsilon_j, a = 4, 5, 6 : \vec{\kappa}_a = \frac{1}{2} \varepsilon_0 \cdot \varepsilon_{a-3}$ is a basis of $T_1Spin(3,1)$

We will use extensively the convenient (the order of the indices matters) :

Notation 25 for both $Cl(3,1), Cl(1,3)$:

$$v(r, w) = \frac{1}{2} (w^1 \varepsilon_0 \cdot \varepsilon_1 + w^2 \varepsilon_0 \cdot \varepsilon_2 + w^3 \varepsilon_0 \cdot \varepsilon_3 + r^3 \varepsilon_2 \cdot \varepsilon_1 + r^2 \varepsilon_1 \cdot \varepsilon_3 + r^1 \varepsilon_3 \cdot \varepsilon_2) \quad (46)$$

With this notation, whatever the orthonormal basis $(\varepsilon_i)_{i=0}^3$, any element X of the Lie algebras $T_1Spin(3,1)$ or $T_1Spin(1,3)$ reads :

$$X = v(r, w) = \sum_{a=1}^3 r^a \vec{\kappa}_a + w^a \vec{\kappa}_{a+3} \quad (47)$$

with $(r, w) \in \mathbb{R}^3 \times \mathbb{R}^3$ then $X = v(r, w)$ is the image of :

$$\Pi'(1)(v(r, w)) = K(w) + J(r) \in so(3,1) \text{ if } X \in T_1Spin(3,1)$$

$$\Pi'(1)(v(r, w)) = -(K(w) + J(r)) \in so(1,3) \text{ if } X \in T_1Spin(1,3)$$

$(Cl(3,1), \mathbf{Ad})$ and $(T_1Spin(3,1), \mathbf{Ad})$ are representations of $Spin(3,1)$. One property that we will use often is the following. A change of orthonormal basis : $\varepsilon_i \rightarrow \tilde{\varepsilon}_i$ can be expressed by an element $s \in Spin(3,1)$: $\varepsilon_i \rightarrow \tilde{\varepsilon}_i = \mathbf{Ad}_s \varepsilon_i$. Then the vectors $v \in F$ stay the same, but their components in the new basis change according to a matrix $[h(s)]$ of $SO(3,1)$:

$$v = \sum_{i=1}^4 v^i \varepsilon_i = \sum_{i=1}^4 \tilde{v}^i \tilde{\varepsilon}_i = \sum_{i=1}^4 \tilde{v}^i \mathbf{Ad}_s \varepsilon_i = \sum_{i=1}^4 \tilde{v}^i [h(s)]_i^j \varepsilon_j$$

The change impacts also the elements of the Lie algebra $T_1Spin(3,1)$:

$$X = \sum_a X_a \vec{\kappa}_a = \sum_a \tilde{X}_a \vec{\kappa}_a$$

$$\vec{\kappa}_a = \tilde{\varepsilon}_i \cdot \tilde{\varepsilon}_j = s \cdot \varepsilon_i \cdot s^{-1} \cdot s \cdot \varepsilon_j \cdot s^{-1} = \mathbf{Ad}_s(\varepsilon_i \cdot \varepsilon_j) = \mathbf{Ad}_s(\vec{\kappa}_a)$$

$\tilde{X}_a = \sum_{b=1}^6 [\mathbf{Ad}_s]_a^b X_b$ where $[\mathbf{Ad}_s]$ is a 6×6 matrix (not the same as the 4×4 matrix $h(s)$ of $SO(3,1)$).

With signature (3,1), with the operator j introduced previously and :

Notation 26 $\varepsilon_5 = \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$

$$\begin{aligned} \varepsilon_5 \cdot \varepsilon_5 &= -1 \\ v(r', w') \cdot v(r, w) &= \frac{1}{4}(w^t w' - r^t r') \\ &+ \frac{1}{2}v(-j(r)r' + j(w)w', -j(w)r' - j(r)w') - \frac{1}{4}(w^t r' + r^t w') \varepsilon_5 \end{aligned}$$

the bracket on the Lie algebra:

$$\begin{aligned} [v(r, w), v(r', w')] &= v(r, w) \cdot v(r', w') - v(r', w') \cdot v(r, w) \\ [v(r, w), v(r', w')] &= v(j(r)r' - j(w)w', j(w)r' + j(r)w') \\ \varepsilon_5 \cdot v(r, w) &= v(r, w) \cdot \varepsilon_5 = v(r, -w) \end{aligned}$$

With signature (1,3) :

$$\begin{aligned} v(r, w) \cdot v(r', w') &= \frac{1}{4}(w^t w' - r^t r') - \frac{1}{2}v(-j(r)r' + j(w)w', j(w)r' + j(r)w') \\ &- \frac{1}{4}(w^t r' + r^t w') \varepsilon_5 \\ [v(r, w), v(r', w')] &= -v(j(r)r' - j(w)w', j(w)r' + j(r)w') \\ \varepsilon_5 \cdot v(r, w) &= v(r, w) \cdot \varepsilon_5 = v(w, r) \end{aligned}$$

3.1.5 Expression of elements of the spin group

Theorem 27 *The elements of the Spin groups read (see annex) in both signatures :*

$$s = a + v(r, w) + b\varepsilon_5 \quad (48)$$

$a, (w^j, r^j)_{j=1}^3, b$ are real scalar which are related. We have the necessary identities :

$$a^2 - b^2 = 1 + \frac{1}{4}(w^t w - r^t r) \quad (49)$$

$$ab = -\frac{1}{4}r^t w \quad (50)$$

The inverse is :

$$(a + v(r, w) + b\varepsilon_5)^{-1} = a - v(r, w) + b\varepsilon_5 \quad (51)$$

The exponential is not surjective on $so(3,1)$ or $T_1 Spin(3,1)$: for each $v(r, w) \in T_1 Spin(3,1)$ there are two elements $\pm \exp v(r, w) \in Spin(3,1)$:

$\exp tv(R, W) = \pm \sigma_w(t) \cdot \sigma_r(t)$ with opposite sign :

$$\sigma_w(t) = \sqrt{1 + \frac{1}{4}W^t W} \sinh^2 \frac{1}{2}t\sqrt{W^t W} + \sinh \frac{1}{2}t\sqrt{W^t W} v(0, W)$$

$$\sigma_r(t) = \sqrt{1 - \frac{1}{4}R^t R} \sin^2 t \frac{1}{2}\sqrt{R^t R} + \sin t \frac{1}{2}\sqrt{R^t R} v(R, 0)$$

And we have the identity (Maths.1768) :

$$\forall v(r, w) \in T_1 Spin(3, 1) : \quad (52)$$

$$\exp \mathbf{Ad}_g v(r, w) = \mathbf{Ad}_g \exp v(r, w) = g \cdot \exp v(r, w) \cdot g^{-1} \quad (53)$$

The product $s \cdot s'$ reads :

$$s \cdot s' = (a + v(r, w) + b\varepsilon_5) \cdot (a' + v(r', w') + b'\varepsilon_5) = a'' + v(r'', w'') + b''\varepsilon_5$$

with :

$$a'' = aa' - b'b + \frac{1}{4}(w^t w' - r^t r')$$

$$b'' = ab' + ba' - \frac{1}{4}(w^t r' + r^t w')$$

and in $Spin(3, 1)$:

$$r'' = \frac{1}{2}(j(r)r' - j(w)w') + a'r + ar' - b'w - bw'$$

$$w'' = \frac{1}{2}(j(w)r' + j(r)w') + a'w + aw' + b'r + br'$$

and in $Spin(1, 3)$:

$$r'' = \frac{1}{2}(j(r)r' - j(w)w') + a'r + ar' + b'w + bw'$$

$$w'' = -\frac{1}{2}(j(w)r' + j(r)w') + a'w + aw' + b'r + br'$$

3.2 Scalar product and Norm

3.2.1 Scalar product on the Clifford algebra

There is a scalar product on $Cl(F, \langle \rangle)$ defined by :

$$\langle u_{i_1} \cdot u_{i_2} \cdot \dots \cdot u_{i_n}, v_{j_1} \cdot v_{j_2} \cdot \dots \cdot v_{j_n} \rangle = \langle u_{i_1}, v_{j_1} \rangle \langle u_{i_2}, v_{j_2} \rangle \dots \langle u_{i_n}, v_{j_n} \rangle$$

It does not depend on the choice of a basis, and any orthonormal basis defined as above is orthonormal :

$\langle \varepsilon_{i_1} \cdot \varepsilon_{i_2} \cdot \dots \cdot \varepsilon_{i_n}, \varepsilon_{j_1} \cdot \varepsilon_{j_2} \cdot \dots \cdot \varepsilon_{j_n} \rangle = \eta_{i_1 j_1} \dots \eta_{i_n j_n} \epsilon(i_1, \dots, i_n, j_1, \dots, j_n)$ the latter term is the signature of the permutation $(i_1, \dots, i_n, j_1, \dots, j_n)$ and 0 if any two indices are equal.

This scalar product on $Cl(3, 1), Cl(3, 1)$ has the signature (8, 8) : it is non degenerate but neither definite positive or negative. It is invariant by \mathbf{Ad} .

$$\forall w, w' \in Cl(F, \langle \rangle) : \langle \mathbf{Ad}_s w, \mathbf{Ad}_s w' \rangle_{Cl(F, \langle \rangle)} = \langle w, w' \rangle_{Cl(F, \langle \rangle)} \quad (54)$$

$(Cl(3, 1), \mathbf{Ad})$ is a unitary representation of $Spin(3, 1)$ and $(Cl(1, 3), \mathbf{Ad})$ a unitary representation of $Spin(1, 3)$.

It reads for elements of $T_1 Spin(3, 1)$:

$$\langle v(r, w), v(r', w') \rangle_{Cl} = \frac{1}{4}(r^t r' - w^t w') \quad (55)$$

3.2.2 Norm on the Lie algebra

Theorem 28 *There is a norm on the Lie algebra $T_1 Spin(3, 1)$ which is preserved by the adjoint map*

i) There is a bilinear symmetric form on $T_1Spin(3,1)$, called the Killing form B which, in the basis above, has the same expression as in $so(3,1)$ (Maths.1669) :

$$B(v(r,w), v(r',w')) = 4(r^t r' - w^t w') = 16 \langle v(r,w), v(r',w') \rangle_{Cl} \quad (56)$$

It is invariant by \mathbf{Ad} as the scalar product.

ii) The Lie algebras $so(3,1), T_1Spin(3,1)$ are semi-simple. They have a Cartan involution, that is a map : $\theta : T_1Spin(3,1) \rightarrow T_1Spin(3,1)$ such that : $\theta^2 = Id$ and the bilinear form $B_\theta(X,Y) = -B(X,\theta Y)$ is definite positive.

θ has two eigen values ± 1 , by taking L_0 the eigen space for -1 and P_0 the eigen space for +1, the Lie algebra is the sum of two, orthogonal (for B so for $\langle \rangle_{Cl}$), vector subspaces (called a Cartan decomposition Maths.1742) such that :

$$T_1Spin(3,1) = L_0 \oplus P_0$$

$$L_0 \text{ is a Lie subalgebra, } [L_0, L_0] \subset L_0, [L_0, P_0] \subset P_0, [P_0, P_0] \subset L_0$$

The Killing form and this decomposition do not depend on the choice of a basis in $T_1Spin(3,1)$ (these are general properties of Lie algebras).

The Cartan involutions are defined in $T_1Spin(3,1)$ by :

$$\theta(\varepsilon_0)(X) = \varepsilon_0 \cdot X \cdot \varepsilon_0$$

with any vector $\varepsilon_0 : \varepsilon_0 \cdot \varepsilon_0 = -1$. We have one involution for each orthonormal basis.

It is easy to see that for the basis built with ε_0 :

$$\forall a = 1, 2, 3 : \varepsilon_0 \cdot \vec{\kappa}_a \cdot \varepsilon_0 = -\vec{\kappa}_a$$

$$\forall a = 4, 5, 6 : \varepsilon_0 \cdot \vec{\kappa}_a \cdot \varepsilon_0 = \vec{\kappa}_a$$

$$\theta(\varepsilon_0)v(r,w) = v(-r,w)$$

$$B(v(r,w), \theta(\varepsilon_0)v(r,w)) = -4(w^t w + r^t r')$$

Thus :

$$L_0 = \{v(r,0), r \in \mathbb{R}^3\} = \{X \in T_1Spin(3,1) : \theta(\varepsilon_0)(X) = -X\}$$

$$P_0 = \{v(0,w), w \in \mathbb{R}^3\} = \{X \in T_1Spin(3,1) : \theta(\varepsilon_0)(X) = X\}$$

iii) Let us check that the decomposition does not depend on the choice of a basis. In a change of basis on $T_1Spin(3,1)$:

$$\varepsilon_i \rightarrow \tilde{\varepsilon}_i = Ad_s \varepsilon_i$$

$$\vec{\kappa}_a \rightarrow \widetilde{\vec{\kappa}}_a = Ad_s \vec{\kappa}_a$$

$$\theta(\tilde{\varepsilon}_0)\left(\widetilde{\vec{\kappa}}_a\right) = Ad_s \varepsilon_0 \cdot Ad_s \vec{\kappa}_a \cdot Ad_s \varepsilon_0 = Ad_s(\varepsilon_0 \cdot \vec{\kappa}_a \cdot \varepsilon_0) = Ad_s(\theta(\varepsilon_0)(\vec{\kappa}_a))$$

So :

$$\forall a = 1, 2, 3 : \theta(\tilde{\varepsilon}_0)\left(\widetilde{\vec{\kappa}}_a\right) = -Ad_s(\vec{\kappa}_a) = -\widetilde{\vec{\kappa}}_a$$

$$\forall a = 4, 5, 6 : \theta(\tilde{\varepsilon}_0)\left(\widetilde{\vec{\kappa}}_a\right) = Ad_s(\vec{\kappa}_a) = \widetilde{\vec{\kappa}}_a$$

$X \in L_0$:

$$X = \sum_{a=1}^3 X^a \vec{\kappa}_a = \sum_{a=1}^3 \tilde{X}^a \widetilde{\vec{\kappa}}_a \Rightarrow \theta(\tilde{\varepsilon}_0)(X) = -\sum_{a=1}^3 \tilde{X}^a \widetilde{\vec{\kappa}}_a = -X$$

$X \in P_0$:

$$X = \sum_{a=4}^6 X^a \vec{\kappa}_a = \sum_{a=4}^6 \tilde{X}^a \widetilde{\vec{\kappa}}_a \Rightarrow \theta(\tilde{\varepsilon}_0)(X) = \sum_{a=4}^6 \tilde{X}^a \widetilde{\vec{\kappa}}_a = X$$

iv) Because the vector spaces L_0, P_0 do not depend on the choice of an orthonormal basis we can define uniquely the projections :

$$\begin{aligned}\pi_L : T_1Spin(3,1) \rightarrow L_0 &:: \pi_L(X) = \frac{1}{2}(X - \theta(\varepsilon_0)(X)) = \frac{1}{2}(X - \varepsilon_0 \cdot X \cdot \varepsilon_0) \\ \pi_P : T_1Spin(3,1) \rightarrow P_0 &:: \pi_P(X) = \frac{1}{2}(X + \theta(\varepsilon_0)(X)) = \frac{1}{2}(X + \varepsilon_0 \cdot X \cdot \varepsilon_0) \\ X &= \pi_L(X) + \pi_P(X)\end{aligned}$$

L_0, P_0 are orthogonal for the scalar product, thus :

$$\langle X, X \rangle_{Cl} = \langle \pi_L(X), \pi_L(X) \rangle_{Cl} + \langle \pi_P(X), \pi_P(X) \rangle_{Cl}$$

The bilinear form :

$$\begin{aligned}\langle X, \pi_L X - \pi_P X \rangle_{Cl} &= \langle \pi_L(X) + \pi_P(X), \pi_L X - \pi_P X \rangle_{Cl} \\ &= \langle \pi_L(X), \pi_L(X) \rangle_{Cl} - \langle \pi_P(X), \pi_P(X) \rangle_{Cl}\end{aligned}$$

is definite positive and invariant by the adjoint map.

$$\|X\| = \sqrt{\langle X, \pi_L X - \pi_P X \rangle_{Cl}} \text{ is a norm on } T_1Spin(3,1)$$

$$\begin{aligned}\|X + X'\|^2 &= \langle X + X', \pi_L(X + X') - \pi_P(X + X') \rangle_{Cl} \\ &= \langle \pi_L(X + X'), \pi_L(X + X') \rangle_{Cl} - \langle \pi_P(X + X'), \pi_P(X + X') \rangle_{Cl} \\ \langle \pi_L(X + X'), \pi_L(X + X') \rangle_{Cl} &\leq \langle \pi_L(X), \pi_L(X) \rangle_{Cl} + \langle \pi_L(X'), \pi_L(X') \rangle_{Cl} \\ - \langle \pi_P(X + X'), \pi_P(X + X') \rangle_{Cl} &\leq - \langle \pi_P(X), \pi_P(X) \rangle_{Cl} - \langle \pi_P(X'), \pi_P(X') \rangle_{Cl}\end{aligned}$$

It reads :

$$\|v(r, w)\| = \frac{1}{2} \sqrt{r^t r + w^t w} = \frac{1}{2} \sqrt{\langle \pi_L X - \pi_P X, \pi_L X - \pi_P X \rangle_{Cl}} \quad (57)$$

and does not depend on a basis on $T_1Spin(3,1)$ and is invariant by the adjoint map.

3.3 Symmetry breakdown

3.3.1 Clifford algebra $Cl(3)$

The elements of $SO(3,1)$ are the product of spatial rotations (represented by $\exp J(r)$) and boosts, linked to the speed and represented by $\exp K(w)$. We have similarly a decomposition of the elements of $Spin(3,1)$. But to understand this topic, from both a mathematical and a physical point of view, we need to distinguish the abstract algebraic structure and the sets on which the structures have been defined.

From a vector space $(F, \langle \rangle)$ one can built only one Clifford algebra with the structure $Cl(3,1)$: as a set $Cl(3,1)$ must comprise all the vectors of F . But from any vector subspace of F one can built Clifford algebras : their algebraic structure depends on the dimension of the vector space, and on the signature of the metric induced on the vector subspace. To have a Clifford algebra structure $Cl(3)$ on F one needs a 3 dimensional vector subspace on which the scalar product is definite positive, so it cannot include any vector such that $\langle u, u \rangle < 0$ (and conversely for the signature $(1,3)$: the scalar product must be definite negative). The subsets of F which are a 3 dimensional vector subspace and do not contain any vector such that $\langle u, u \rangle < 0$ are no unique. So we have different subsets of $Cl(3,1)$ with the structure of a Clifford algebra $Cl(3)$, all isomorphic but which do not contain the same vectors. Because the Spin Groups are built from elements of the Clifford algebra, we have similarly isomorphic Spin groups $Spin(3)$, but with different elements.

The simplest way to deal with these issues is to fix an orthonormal basis. Any orthonormal basis of F contains one vector such that $\langle \varepsilon_i, \varepsilon_i \rangle = -1$ (or $+1$ with the signature $(1, 3)$). If we exclude this vector we can generate a vector subspace and then a Clifford algebra $Cl(3)$. So the identification of a specific set with the structure of $Cl(3)$ sums up to single out such a vector, that we will denote as ε_0 .

3.3.2 Decomposition of the elements of the Spin group

Theorem 29 *For a given vector ε_0 any element $s \in Spin(3, 1)$ can be written uniquely (up to sign) : $s = \epsilon(a_w + v(0, w)) \cdot \epsilon(a_w + v(r, 0))$ with $\epsilon = \pm 1$*

Let us choose an orthonormal basis $(\varepsilon_i)_{i=0}^3$. Then there is a unique vector subspace F^\perp orthogonal to ε_0 , where the scalar product is definite positive, and from $(F^\perp, \langle \rangle)$ one can build a Clifford algebra with structure $Cl(3)$. Its spin group has the structure $Spin(3)$ which has for Lie algebra $T_1Spin(3)$.

In the basis :

an element of $T_1Spin(3)$ reads : $X = v(r, 0) = \sum_{a=1}^3 r^a \vec{\kappa}_a$ and is associated to a 3×3 matrix $j(r)$

any element of $Spin(3)$ reads : $s_r = a_r + v(r, 0)$ with $a_r^2 = 1 - \frac{1}{4}r^t r$

$Spin(3)$ has 2 connected components. The connected component of the identity is comprised of elements with $a_r > 0$.

$Spin(3), T_1Spin(3)$ are subsets of $Cl(3, 1)$: they are comprised of specific combinations of vectors of F .

The **quotient space** $Spin(3, 1)/Spin(3)$ is not a group but a 3 dimensional manifold, called a homogeneous space (Maths.22.4.3). It is characterized by the equivalence relation :

$$\forall s, s' \in Spin(3, 1) : s \sim s' \Leftrightarrow \exists s_r \in Spin(3) : s' = s \cdot s_r$$

The projection : $\pi_w : Spin(3, 1) \rightarrow Spin(3, 1)/Spin(3)$ is a submersion, its derivative $\pi'_w(s)$ is surjective.

In each class of $Spin(3, 1)/Spin(3)$ there are only two elements (which belong to $Spin(3, 1)$) which can be written as : $s_w = a_w + v(0, w)$ and they have opposite sign : $\pm s_w$ belong to the same class (see Annex for the proofs).

Any element $s \in Spin(3, 1)$ can be written : $s = s_w \cdot s_r$ with $s_w \in Spin(3, 1)/Spin(3), s_r \in Spin(3)$:

$$s = \epsilon(a_w + v(0, w)) \cdot \epsilon(a_w + v(r, 0)) \quad \text{with } \epsilon = \pm 1 \quad (58)$$

For each s there are two decompositions, unique up to sign. $\pm s_w$ belong to the class of equivalence of s in $Spin(3, 1)/Spin(3)$. They are specific representatives of the projection of s on the homogeneous space. *The decomposition depends on the choice of ε_0 .*

Theorem 30 *$Spin(3, 1)$ is a principal fiber bundle $Spin(3, 1)(Spin(3, 1)/Spin(3), Spin(3), \pi_w)$.*

and there is a left action of $Spin(3,1)$ on $Spin(3,1)/Spin(3)$:
 $\lambda : Spin(3,1) \times Spin(3,1)/Spin(3) \rightarrow Spin(3,1)/Spin(3)$:
 $\lambda(s, s_w) = \pi_w(s \cdot s_w)$

Proof. Indeed we have :

$$\begin{aligned} \lambda(s, s_w) &= \pi_w(s \cdot s_w) \Leftrightarrow \exists s_r : s \cdot s_w = \pi_w(s \cdot s_w) \cdot s_r \Leftrightarrow \lambda(s, s_w) = s \cdot s_w \cdot s_r^{-1} \\ \lambda(s, \lambda(s', s_w)) &= \lambda(s, s' \cdot s_w \cdot s_r'^{-1}) = s \cdot s' \cdot s_w \cdot s_r'^{-1} \cdot s_r^{-1} \\ \lambda(s \cdot s', s_w) &= s \cdot s' \cdot s \cdot s_w \cdot s_r'' \\ s \cdot s' \cdot s \cdot s_w \cdot s_r'' &\sim s \cdot s' \cdot s_w \cdot s_r'^{-1} \cdot s_r^{-1} \quad \blacksquare \end{aligned}$$

The subset of $Spin(3,1)$ of the elements s_r which commute with ε_0 :

$$\mathbf{Ad}_{s_r} \varepsilon_0 = s_r \cdot \varepsilon_0 \cdot s_r^{-1} = \varepsilon_0$$

read :

$$s_r = \epsilon \left(\sqrt{1 - \frac{1}{4} r^t r} + v(r, 0) \right) \quad \text{with } \epsilon = \pm 1, r^t r \leq 4 \quad (59)$$

They belong to the realization of $Spin(3)$ which is specified by ε_0 . They are generated by vectors belonging to the subspace spanned by the vectors $(\varepsilon_i)_{i=1}^3$. They have a special physical meaning : they are the spatial rotations for an observer *with a velocity in the direction of ε_0 . The vector r , with 3 components, has the same meaning as in Galilean Geometry.* This is just a different representation of the Lie algebra $o(3)$.

The elements of $Spin(3,1)/Spin(3)$ are coordinated by w , and the matrix $[K(w)]$ corresponds to a gauge transformation for an observer moving with a spatial speed \vec{v} parallel to w , without spatial rotation.

Similarly the elements of the Lie algebra $T_1 Spin(3)$ are characterized by $X \in T_1 Spin(3) \Leftrightarrow X \cdot \varepsilon_0 - \varepsilon_0 \cdot X = 0 \Leftrightarrow w = 0$

In the tangent space $T_m M$ of the manifold M all rotations (given by $Spin(3,1)$) are on the same footing. But, because of our assumptions about the motion of observers (along time like lines), any observer introduces a breakdown of symmetry : some rotations are privileged. Indeed the spatial rotations are special, in that they are the ones for which the axis belongs to the physical space.

To sum up :

$(\varepsilon_i)_{i=1}^3$ are the generators in $Cl(F, \langle \rangle)$ of a set with the structure of $Spin(3)$ iff $\langle \varepsilon_i, \varepsilon_i \rangle = +1$ (then the 4th vector $\langle \varepsilon_0, \varepsilon_0 \rangle = -1$)

Similarly any element of the Lie algebra $T_1 Spin(3,1)$ read:

$$X = v(r, w) = \sum_{a=1}^3 r^a \vec{\kappa}_a + w^a \vec{\kappa}_{a+3}$$

and $\sum_{a=1}^3 r^a \vec{\kappa}_a$ identifies a specific Lie algebra $T_1 Spin(3)$ if $\langle \varepsilon_i, \varepsilon_i \rangle = +1$.

In the following of this paper we will always assume that the orthonormal basis is such that ε_0 is the 4th vector : $\langle \varepsilon_0, \varepsilon_0 \rangle = -1$ with signature (3,1) and $\langle \varepsilon_0, \varepsilon_0 \rangle = +1$ with signature (1,3).

3.4 Representation of Clifford algebras

3.4.1 Complexification of real Clifford algebras

A real Clifford algebra can be complexified, using the same procedure as for any finite dimensional vector space. The complexified of both $Cl(3, 1)$ and $Cl(1, 3)$ is $Cl(\mathbb{C}, 4)$, the Clifford algebra on \mathbb{C}^4 with the bilinear *symmetric* form of signature $(++++)$.

There are real algebra morphisms (injective but not surjective) from the real Clifford algebras to $Cl(\mathbb{C}, 4)$.

Let $\tilde{C} : (\mathbb{R}^4, \langle \cdot, \cdot \rangle_L) \rightarrow Cl(\mathbb{C}, 4)$ be the real linear map defined by : $\tilde{C}(\varepsilon_j) = \varepsilon_j, j = 1, 2, 3; \tilde{C}(\varepsilon_0) = i\varepsilon_0$ and the scalar product $\langle \cdot, \cdot \rangle_L$ with signature $(-+++)$.

It is easy to check that : $\tilde{C}(u) \cdot \tilde{C}(v) + \tilde{C}(v) \cdot \tilde{C}(u) = 2\langle u, v \rangle_L$ so, by the universal property of Clifford algebras, there is a unique real algebra morphism $C : Cl(3, 1) \rightarrow Cl(\mathbb{C}, 4)$ such that $\tilde{C} = C \circ j$ where j is the canonical injection $(\mathbb{R}^4, \langle \cdot, \cdot \rangle_L) \rightarrow Cl(3, 1)$. We will denote for simplicity $\tilde{C} = C$. The image $C(Cl(3, 1))$ is a real subalgebra of $Cl(\mathbb{C}, 4)$.

Similarly with $\tilde{C}'(\varepsilon_j) = i\varepsilon_j, j = 1, 2, 3; \tilde{C}'(\varepsilon_0) = \varepsilon_0$ we have a real algebra morphism $C' : Cl(1, 3) \rightarrow Cl(\mathbb{C}, 4)$ and $C'(Cl(1, 3))$ is a real subalgebra of $Cl(\mathbb{C}, 4)$. Moreover $C'(\varepsilon_j) = -i\eta_{jj}C(\varepsilon_j)$ (η always correspond to the signature $-+++$).

3.4.2 Algebraic and geometric representations

An *algebraic* representation of a Clifford algebra is a map γ which associates to each element w of the Clifford algebra a matrix $[\gamma(w)]$ and such that γ is a isomorphism of algebra : all the operations in the Clifford algebra (multiplication by a scalar, sum, Clifford product) are reproduced on the matrices. A representation is fully defined by the family of matrices $(\gamma_i)_{i=0}^3$ representing each vector $(\varepsilon_i)_{i=0}^3$ of an orthonormal basis. The choice of these matrices is not unique : the only condition is that $[\gamma_i][\gamma_j] + [\gamma_j][\gamma_i] = 2\eta_{ij}[I]$ and any family of matrices deduced by conjugation with a fixed matrix gives an equivalent algebraic representation. An element of the Clifford algebra is then represented by a linear combination of generators :

$$\gamma(w) = \gamma\left(\sum_{\{i_1 \dots i_r\}} a^{i_1 \dots i_r} \varepsilon_{i_1} \cdot \dots \cdot \varepsilon_{i_r}\right) = \sum_{\{i_1 \dots i_r\}} a^{i_1 \dots i_r} \gamma_{i_1} \dots \gamma_{i_r}$$

A *geometric* representation (E, γ) of a Clifford algebra is an isomorphism $\gamma : Cl \rightarrow L(E; E)$ in which $[\gamma(w)]$ is the matrix of an endomorphism of E , represented in some basis. From an algebraic representation one can deduce a geometric representation, and they are equivalent up to the choice of a basis.

We look for a geometric representation : the quantity S that we are looking for is represented, not by γ matrices, but by vectors S of the space E , which are called spinors. Higher orders spinors are tensorial products of vectors of E .

A Clifford algebra has, up to isomorphism, a unique faithful algebraic irreducible representation in an algebra of matrices (γ is a bijection). As can be expected the representations depend on the signature :

For $Cl(3, 1)$ this is $\mathbb{R}(4)$ the 4×4 real matrices (the corresponding spinors are the Majorana spinors)

For $Cl(1, 3)$ this is $H(2)$ the 2×2 matrices with quaternionic elements

In both cases an element of the Clifford algebra is characterized by $2^4 = 16$ real parameters.

The geometry of the universe is based upon real structures. Thus we should consider representations of $Cl(3, 1)$ or $Cl(1, 3)$, which raises the issue of the signature. However it happens, from experience ³, that the vector space E must be complex.

The irreducible representation of $Cl(\mathbb{C}, 4)$ is by 4×4 matrices on complex numbers which must meet the condition : $\gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{jk} I_4$.

If (E, γ) is a complex representation of $Cl(\mathbb{C}, 4)$ then $(E, \gamma \circ C)$ is a real geometric representation of $Cl(3, 1)$ on the complex vector space E : the map $\gamma \circ C : Cl(3, 1) \rightarrow L(E; E)$ is a real morphism of algebras, and the maps $\gamma \circ C(w)$ are complex linear. The matrices of the real representation are $i\gamma_0, \gamma_j, j = 1, 2, 3, i\gamma_0$. Similarly $(E, \gamma \circ C')$ is a real geometric representation of $Cl(1, 3)$ with matrices $\gamma_0, i\gamma_j, j = 1, 2, 3$.

Using this trick we see that we are fortunate, in that we have the same representation (E, γ) for both signatures, and a complex vector space E . Moreover it is easy to specify the representation through additional features of E (such as chirality as we will see). A spinor has 8 real components (vs 16 real components for elements of the real Clifford algebras) thus a spinor carries more information than a simple vector of \mathbb{R}^4 and this solves part of the issue of the number of parameters needed to represent the motion (both translation and rotation).

3.4.3 Chirality

Any Clifford algebra Cl is the direct sum of one Clifford subalgebra Cl_0 comprised of elements which are the sum of products of an even number of vectors, and a vector subspace Cl_1 comprised of elements which are the sum of products of an odd number of vectors. Moreover some Clifford algebras present a specific feature : they are the direct sum of two subalgebras which can be seen as algebras of left handed and right handed elements. This property depends on the existence of an element ϖ such that $\varpi \cdot \varpi = 1$. This element exists in any complex algebra, but not in $Cl(1, 3), Cl(3, 1)$. As chirality is a defining feature of particles, this is an additional argument for using $Cl(\mathbb{C}, 4)$.

In this subsection and the next one the vectors $(\varepsilon_j)_{j=0}^3$ denote the canonical basis of \mathbb{C}^4 endowed with the bilinear symmetric form of signature $++++$. In $Cl(\mathbb{C}, 4)$ the special element is : $\varpi = \pm \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \in Spin(\mathbb{C}, 4)$. Thus there is a choice and we will use : $\varpi = \varepsilon_5 = \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$.

The Clifford algebra splits in two subalgebras :

$$\begin{aligned} Cl(\mathbb{C}, 4) &= Cl^R(\mathbb{C}, 4) \oplus Cl^L(\mathbb{C}, 4) : \\ Cl^R(\mathbb{C}, 4) &= \{w \in Cl(\mathbb{C}, 4) : \varepsilon_5 \cdot w = w\}, \end{aligned}$$

³We will see that this is necessary to account for the Electromagnetic field.

$$Cl^L(\mathbb{C}, 4) = \{w \in Cl(\mathbb{C}, 4) : \varepsilon_5 \cdot w = -w\}$$

and any element of $Cl(\mathbb{C}, 4)$ can be uniquely written as : $w = w_R + w_L$

The projections from $Cl(\mathbb{C}, 4)$ on each subalgebra are the maps

$$p_R = \frac{1}{2}(1 + \varepsilon_5), p_L = \frac{1}{2}(1 - \varepsilon_5) :$$

$$w_R = p_R \cdot w, w_L = p_L \cdot w$$

$$p_R \cdot p_L = p_L \cdot p_R = 0, p_R^2 = p_R, p_L^2 = p_L, p_R + p_L = 1$$

We have similarly : $E = E^R \oplus E^L$ with

$$E^R = \gamma_R(E), E^L = \gamma_L(E), \gamma_R = \gamma(p_R), \gamma_L = \gamma(p_L) \Rightarrow \gamma(\varepsilon_5) = \gamma_R - \gamma_L$$

$$u \in E : u = u_R + u_L :$$

$$u_R = \gamma_R(u) = \frac{1}{2}(u + \gamma(\varepsilon_5)u) ;$$

$$u_L = \gamma_L(u) = \frac{1}{2}(u - \gamma(\varepsilon_5)u)$$

For any homogeneous element $w = v_1 \cdot v_2 \dots \cdot v_k, v_j \in \mathbb{C}^4$ we have $\varepsilon_5 \cdot w =$

$$(-1)^k w \cdot \varepsilon_5$$

$$\forall w \in Cl(\mathbb{C}, 4), u \in E :$$

$$\gamma_R(\gamma(w)u_R) = \frac{1}{2}(1 + (-1)^k)\gamma(w)u_R$$

$$k \text{ even} : \gamma_R(\gamma(w)u_R) = \gamma(w)u_R$$

$$k \text{ odd} : \gamma_R(\gamma(w)u_R) = 0$$

For k even : $\gamma(w)$ preserves both E^R, E^L (as vector subspaces)

For k odd : $\gamma(w)$ exchanges E^R, E^L

In particular the elements of the images $C(\text{Spin}(3, 1))$ and $C'(\text{Spin}(1, 3))$ by γ preserve both E^R, E^L . So we have reducible representations of these groups.

3.4.4 The choice of the representation

An algebraic representation is defined by the choice of its generators γ_i , and any set of generators conjugate by a fixed matrix gives an equivalent representation. We can specify the generators by the choice of a basis $(e_i)_{i=1}^4$ of E . The previous result leads to a natural choice : take $(e_i)_{i=1}^2$ as basis of E^R and $(e_i)_{i=3}^4$ as basis of E^L , then :

$$\gamma_R = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \gamma_L = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}, \gamma_5 = \gamma(\varepsilon_5) = \gamma_R - \gamma_L = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$$

$$\text{Denote : } \gamma_j = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} \text{ with four } 2 \times 2 \text{ complex matrices.}$$

ε_5 belongs to the Spin group $\text{Spin}(\mathbb{C}, 4)$, commutes with any element of $Cl_0(\mathbb{C}, 4)$ and anticommutes with any vector, thus $\gamma_5 \gamma_j = -\gamma_j \gamma_5$ which imposes the condition :

$$\begin{bmatrix} A_j & -B_j \\ C_j & -D_j \end{bmatrix} = - \begin{bmatrix} A_j & B_j \\ -C_j & -D_j \end{bmatrix} \Rightarrow \gamma_j = \begin{bmatrix} 0 & B_j \\ C_j & 0 \end{bmatrix}$$

The defining relations : $\gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{jk}I_4$ lead to :

$$\begin{bmatrix} B_j C_k + B_k C_j & 0 \\ 0 & C_j B_k + C_k B_j \end{bmatrix} = 2\delta_{jk}I_4$$

$$j \neq k : B_j C_k + B_k C_j = C_j B_k + C_k B_j = 0$$

$$j = k : B_j C_j = C_j B_j = I_2 \Leftrightarrow C_j = B_j^{-1}$$

thus $(\gamma_i)_{i=0}^3$ is fully defined by a set $(B_i)_{i=0}^3$ of 2×2 complex matrices

$$\gamma_j = \begin{bmatrix} 0 & B_j \\ B_j^{-1} & 0 \end{bmatrix}$$

meeting : $j \neq k : B_j B_k^{-1} + B_k B_j^{-1} = B_j^{-1} B_k + B_k^{-1} B_j = 0$

which reads :

$$B_j B_k^{-1} = - (B_j B_k^{-1})^{-1} \Leftrightarrow (B_j B_k^{-1})^2 = -I_2$$

$$B_j^{-1} B_k = - (B_j^{-1} B_k)^{-1} \Leftrightarrow (B_j^{-1} B_k)^2 = -I_2$$

Let us define : $k = 1, 2, 3 : M_k = -i B_k B_0^{-1}$

The matrices $(M_k)_{k=1}^3$ are such that :

$$M_k^2 = - (B_j B_0^{-1})^2 = -I_2$$

$$M_j M_k + M_k M_j = -B_j B_0^{-1} B_k B_0^{-1} - B_k B_0^{-1} B_j B_0^{-1}$$

$$= - (B_j B_k^{-1} B_0 - B_k B_j^{-1} B_0) B_0^{-1}$$

$$= B_j B_k^{-1} + B_k B_j^{-1} = 0$$

that is $k = 1, 2, 3 : M_j M_k + M_k M_j = 2\delta_{jk} I_2$

Moreover : $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \Rightarrow$

$$B_0 B_1^{-1} B_2 B_3^{-1} = I_2$$

$$B_0^{-1} B_1 B_2^{-1} B_3 = -I_2$$

with $B_k = i M_k B_0, B_k^{-1} = -i B_0^{-1} M_k^{-1}$

$$B_0 (-i B_0^{-1} M_1^{-1}) (i M_2 B_0) (-i B_0^{-1} M_3^{-1}) = I_2 = -i M_1^{-1} M_2 M_3^{-1}$$

$$B_0^{-1} (i M_1 B_0) (-i B_0^{-1} M_2^{-1}) (i M_3 B_0) = -I_2 = i B_0^{-1} M_1 M_2^{-1} M_3 B_0$$

which reads :

$$i M_2 = -M_1 M_3 = M_3 M_1$$

$$-M_1^{-1} M_3^{-1} = i M_2^{-1} \Leftrightarrow i M_2 = M_3 M_1$$

$$M_2 M_3 + M_3 M_2 = 0 = i M_1 M_3 M_3 + M_3 M_2 \Leftrightarrow i M_1 = -M_3 M_2 = M_2 M_3$$

$$M_1 M_2 + M_2 M_1 = 0 = i M_3 M_2 M_2 + M_2 M_1 \Rightarrow i M_3 = -M_2 M_1 = M_1 M_2$$

The set of 3 matrices $(M_k)_{k=1}^3$ has the multiplication table :

$$\begin{bmatrix} 1 \setminus 2 & M_1 & M_2 & M_3 \\ M_1 & I & i M_3 & -i M_2 \\ M_2 & -i M_3 & I & i M_1 \\ M_3 & i M_2 & -i M_1 & I \end{bmatrix}$$

which is the same as the set of Pauli's matrices :

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (60)$$

$$\sigma_i^2 = \sigma_0; \text{ For } j \neq k : \sigma_j \sigma_k = \epsilon(j, k, l) i \sigma_l \quad (61)$$

Notation 31 $\epsilon(j, k, l)$ = the signature of the permutation of the 3 different integers i, j, k , 0 if two integers are equal

There is still some freedom in the choice of the γ_i matrices by the choice of B_0 and the simplest is : $B_0 = -iI_2 \Rightarrow B_k = \sigma_k$

Moreover, because scalars belong to Clifford algebras, one must have the identity matrix I_4 and $\gamma(z) = zI_4$

Thus :

$$\gamma_0 = \begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix}; \gamma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}; \gamma_2 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}; \gamma_3 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix}; \quad (62)$$

The matrices γ_j are then unitary and Hermitian :

$$\gamma_j = \gamma_j^* = \gamma_j^{-1} \quad (63)$$

which is extremely convenient.

We will use the following (see the annex for more formulas) :

Notation 32 $j = 1, 2, 3 : \tilde{\gamma}_j = \begin{bmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{bmatrix}$

$$j \neq k, l = 1, 2, 3 : \gamma_j \gamma_k = -\gamma_k \gamma_j = i\epsilon(j, k, l) \tilde{\gamma}_l$$

$$j = 1, 2, 3 : \gamma_j \gamma_0 = -\gamma_0 \gamma_j = i \begin{bmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{bmatrix} = i\gamma_5 \tilde{\gamma}_j$$

Notice that the choice of the matrices is done in $Cl(\mathbb{C}, 4)$, so it is independent of the choice of signature. However we have the representations of the real algebras by the matrices $\gamma C(\varepsilon_j)$ and $\gamma C'(\varepsilon_j)$

$$Cl(3, 1) : \gamma C(\varepsilon_j) = \gamma_j, j = 1, 2, 3; \gamma C(\varepsilon_0) = i\gamma_0; \gamma C(\varepsilon_5) = i\gamma_5 \quad (64)$$

$$Cl(1, 3) : \gamma C'(\varepsilon_j) = i\gamma_j, j = 1, 2, 3; \gamma C'(\varepsilon_0) = \gamma_0; \gamma C'(\varepsilon_5) = \gamma_5 \quad (65)$$

The representation that we have chosen here is not unique and others, equivalent, would hold. However from my point of view this is the most convenient because of the nice properties of the γ matrices. The choice of $\varpi = -\varepsilon_5 = -\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$ would have lead to take $\tilde{\gamma}_j = -\gamma_j$. In the Standard Model we have a representation of $Cl(1, 3)$ by the matrices : $\tilde{\gamma}_0 = i\gamma_0, \tilde{\gamma}_j = \gamma_j, j = 1, 2, 3$ and $\tilde{\gamma}_5 = -i\tilde{\gamma}_0\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3$

3.4.5 Expression of the matrices for the Lie algebra and the Spin groups

The matrices $\gamma C(v(r, w)), \gamma C'(v(r, w))$ are of constant use.

In $Cl(\mathbf{R},3,1)$:

$$\begin{aligned}\gamma C(v(r, w)) &= -i\frac{1}{2}\sum_{a=1}^3(w^a\gamma_a\gamma_0 + r^a\tilde{\gamma}_a) \\ &= \frac{1}{2}\sum_{a=1}^3\begin{bmatrix}(w_a - ir_a)\sigma_a & 0 \\ 0 & -(w_a + ir_a)\sigma_a\end{bmatrix}\end{aligned}\quad (66)$$

In $Cl(\mathbf{R},1,3)$:

$$\begin{aligned}\gamma C'(v(r, w)) &= -i\frac{1}{2}\sum_{a=1}^3(w^a\gamma_a\gamma_0 - r^a\tilde{\gamma}_a) \\ &= \frac{1}{2}\sum_{a=1}^3\begin{bmatrix}(w_a + ir_a)\sigma_a & 0 \\ 0 & -(w_a - ir_a)\sigma_a\end{bmatrix}\end{aligned}\quad (67)$$

so one goes from one signature to the other by changing the sign of r.

The 2×2 matrices $\frac{1}{2}\sum_{a=1}^3(w_a - ir_a)\sigma_a$ and $\frac{1}{2}\sum_{a=1}^3(w_a + ir_a)\sigma_a$ belong to $SU(2)$

The elements of the spin groups are represented by the matrices :

In $Cl(\mathbf{R},3,1)$:

$$\gamma C(a + v(r, w) + b\varepsilon_5) = aI - i\frac{1}{2}\sum_{a=1}^3(w_a\gamma_a\gamma_0 + r_a\tilde{\gamma}_a) + b\gamma_5 \quad (68)$$

In $Cl(\mathbf{R},1,3)$:

$$\gamma C'(a + v(r, w) + b\varepsilon_5) = aI - i\frac{1}{2}\sum_{a=1}^3(w_a\gamma_a\gamma_0 - r_a\tilde{\gamma}_a) + b\gamma_5 \quad (69)$$

3.5 Scalar product of Spinors

We need a scalar product on E, preserved by a gauge transformation, that is by both $Spin(3, 1)$ and $Spin(1, 3)$.

Theorem 33 *The only scalar products on E, preserved by $\{\gamma C(\sigma), \sigma \in Spin(3, 1)\}$ are $G = \begin{bmatrix} 0 & k\sigma_0 \\ \bar{k}\sigma_0 & 0 \end{bmatrix}$ with $k \in \mathbb{C}$*

Proof. It is represented in the basis of E by a 4×4 Hermitian matrix G such that : $G = G^*$

$$\forall s \in Spin(3, 1) : [\gamma \circ C(s)]^* G [\gamma \circ C(s)] = G$$

$$\text{or } \forall s \in Spin(1, 3) : [\gamma \circ C'(s)]^* G [\gamma \circ C'(s)] =$$

$$[\gamma \circ C(s)]^* G = G [\gamma \circ C(s)]^{-1} = G [\gamma \circ C(s^{-1})]$$

$$\gamma C(s) = \begin{bmatrix} (a + ib)\sigma_0 + \frac{1}{2}\sum_{a=1}^3(w_a - ir_a)\sigma_a & 0 \\ 0 & (a - ib)\sigma_0 - \frac{1}{2}\sum_{a=1}^3(w_a + ir_a)\sigma_a \end{bmatrix}$$

$$\gamma C(s)^* = \begin{bmatrix} (a - ib)\sigma_0 + \frac{1}{2}\sum_{a=1}^3(w_a + ir_a)\sigma_a & 0 \\ 0 & (a + ib)\sigma_0 - \frac{1}{2}\sum_{a=1}^3(w_a - ir_a)\sigma_a \end{bmatrix}$$

$$G = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}, \text{ with } A = A^*, C = C^*$$

$$[\gamma \circ C(s)]^* G$$

$$\begin{aligned}
&= \begin{bmatrix} (a-ib)A + \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a) \sigma_a A & (a-ib)B + \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a) \sigma_a B \\ (a+ib)B^* - \frac{1}{2} \sum_{a=1}^3 (w_a - ir_a) \sigma_a B^* & (a+ib)C - \frac{1}{2} \sum_{a=1}^3 (w_a - ir_a) \sigma_a C \end{bmatrix} \\
G [\gamma \circ C (s^{-1})] \\
&= \begin{bmatrix} (a+ib)A - \frac{1}{2} \sum_{a=1}^3 (w_a - ir_a) A \sigma_a & (a-ib)B + \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a) B \sigma_a \\ (a+ib)B^* - \frac{1}{2} \sum_{a=1}^3 (w_a - ir_a) B^* \sigma_a & (a-ib)C + \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a) C \sigma_a \end{bmatrix} \\
&(a-ib)A + \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a) \sigma_a A = (a+ib)A - \frac{1}{2} \sum_{a=1}^3 (w_a - ir_a) A \sigma_a \\
&(a-ib)B + \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a) \sigma_a B = (a-ib)B + \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a) B \sigma_a \\
&(a+ib)B^* - \frac{1}{2} \sum_{a=1}^3 (w_a - ir_a) \sigma_a B^* = (a+ib)B^* - \frac{1}{2} \sum_{a=1}^3 (w_a - ir_a) B^* \sigma_a \\
&(a+ib)C - \frac{1}{2} \sum_{a=1}^3 (w_a - ir_a) \sigma_a C = (a-ib)C + \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a) C \sigma_a \\
&(w_a + ir_a) \sigma_a B = \sum_{a=1}^3 (w_a + ir_a) B \sigma_a \\
&\sum_{a=1}^3 (w_a - ir_a) \sigma_a B^* = \sum_{a=1}^3 (w_a - ir_a) B^* \sigma_a \\
2ibA \\
&= \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a) \sigma_a A + (w_a - ir_a) A \sigma_a \\
&= \frac{1}{2} \sum_{a=1}^3 w_a (\sigma_a A + A \sigma_a) + ir_a ((\sigma_a A - A \sigma_a)) \\
2ibC \\
&= \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a) C \sigma_a + (w_a - ir_a) \sigma_a C \\
&= \frac{1}{2} \sum_{a=1}^3 w_a (\sigma_a C + C \sigma_a) + ir_a ((C \sigma_a - \sigma_a C)) \\
&\text{By taking the adjoint on the two last equations :} \\
-2ibA \\
&= \frac{1}{2} \sum_{a=1}^3 w_a (A \sigma_a + \sigma_a A) - ir_a ((A \sigma_a - \sigma_a A)) \\
&= -\frac{1}{2} \sum_{a=1}^3 w_a (\sigma_a A + A \sigma_a) + ir_a ((\sigma_a A - A \sigma_a)) \Rightarrow A = 0 \\
-2ibC \\
&= \frac{1}{2} \sum_{a=1}^3 w_a (\sigma_a C + C \sigma_a) - ir_a ((C \sigma_a - \sigma_a C)) \\
&= -\frac{1}{2} \sum_{a=1}^3 w_a (\sigma_a C + C \sigma_a) + ir_a ((C \sigma_a - \sigma_a C)) \Rightarrow C = 0 \\
&\text{We are left with :} \\
\forall w, r : \sum_{a=1}^3 (w_a + ir_a) \sigma_a B = \sum_{a=1}^3 (w_a + ir_a) B \sigma_a \\
&\text{which implies that B commutes with all the Dirac matrices, which happens} \\
&\text{only for the scalar matrices : } B = k\sigma_0. \\
G = \begin{bmatrix} 0 & k\sigma_0 \\ \bar{k}\sigma_0 & 0 \end{bmatrix} \blacksquare
\end{aligned}$$

The scalar product will never be definite positive, so we can take $k = -i$ that is $G = \gamma_0$. And it is easy to check that it works also for the signature (1,3).

Any vector of E reads :

$$u = \sum_{i=1}^4 u^i e_i = u_R + u_L \text{ with } u_R = \sum_{i=1}^2 u^i e_i, u_L = \sum_{i=3}^4 u^i e_i$$

The scalar product of two vectors u, v of E is then:

$$\left\langle \sum_{i=1}^4 u^i e_i, \sum_{i=1}^4 v^i e_i \right\rangle_E = [u]^* [\gamma_0] [v] = i (u_L^* v_R - u_R^* v_L) \quad (70)$$

It is not definite positive. It is preserved both by $Spin(3,1)$ and $Spin(1,3)$.

It is definite positive on E_R and definite negative on E_L .

The basis $(e_i)_{i=1}^4$ of E is not orthonormal : $\langle e_j, e_k \rangle = i$ $\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

3.6 Norm on the space E of spinors

The scalar product on E is not definite, but we can define a norm, as we have done for the Lie algebra. We have introduced previously the two chiral operators p_R, p_L on $Cl(\mathbb{C}, 4)$. They do not belong to the images $\gamma C(Cl(3, 1)), \gamma C(Cl(1, 3))$ however they define two subspaces $E = E_R \oplus E_L$ and the elements of the images $C(Spin(3, 1))$ and $C'(Spin(1, 3))$ by γ preserve both E^R, E^L . So we have operators $\gamma_R : E \rightarrow E_R, \gamma_L : E \rightarrow E_L$ such that :

$$\begin{aligned} \gamma_R &= \gamma_R \cdot \gamma_R; \gamma_L = \gamma_L \cdot \gamma_L \\ \gamma_R + \gamma_L &= Id \\ \forall \sigma \in Spin(3, 1), Spin(1, 3) : \gamma C(\sigma) \circ \gamma_R &= \gamma_R \circ \gamma C(\sigma); \gamma C(\sigma) \circ \gamma_L = \\ \gamma_L \circ \gamma C(\sigma) \end{aligned}$$

$\gamma_R = \gamma(p_R), \gamma_L = \gamma(p_L)$ are complex linear maps, as images of the complex linear maps p_R, p_L by the complex linear map γ . So they preserve any real structure on E : $\gamma_R(\text{Re } u + i \text{Im } u) = \gamma(p_R)(\text{Re } u + i \text{Im } u) = \gamma(p_R) \text{Re } u + i \gamma(p_R)(\text{Im } u)$ and $\gamma_R = \overline{\gamma_R}, \gamma_L = \overline{\gamma_L}$ (Maths.357).

In the basis $(e_i)_{i=1}^4$:

$$\begin{aligned} \gamma_R &= \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix} = \gamma_R^* \\ \gamma_L &= \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0 \end{bmatrix} = \gamma_L^* \end{aligned}$$

There is no definite scalar product on E , but there is a norm, that is a map :

$$\begin{aligned} ||| & : E \times E \rightarrow \mathbb{R} \\ \text{such that :} \\ ||S|| & \geq 0 \\ ||S|| = 0 & \Rightarrow S = 0 \\ ||kS|| & = |k| ||S|| \\ ||S + S'|| & \leq ||S|| + ||S'|| \end{aligned}$$

Theorem 34 *The vector space E is a normed vector space with the norm, invariant by $Spin(3, 1), Spin(1, 3)$:*

$$||S||_E = \sqrt{\langle \gamma_R S, \gamma_R S \rangle_E - \langle \gamma_L S, \gamma_L S \rangle_E}$$

Proof. E_R, E_L are two 2 dimensional complex vector spaces, they can be endowed with a norm which is invariant by $Spin(3, 1), Spin(1, 3)$:

$$\begin{aligned} S \in E_R : ||S||_{E_R}^2 &= \langle S, S \rangle_E \\ S \in E_L : ||S||_{E_L}^2 &= -\langle S, S \rangle_E \\ \text{The norm are invariant by } \gamma_R, \gamma_L : \\ S \in E_R &\Leftrightarrow \exists S' \in E : S = \gamma_R(S') \end{aligned}$$

$$\begin{aligned}
& \|\gamma_R(S)\|_{E_R} = \|\gamma_R^2(S')\|_{E_R} = \|\gamma_R(S')\|_{E_R} = \|S\|_{E_R} \\
& \text{Define : } \|\cdot\|_E : E \times E \rightarrow \mathbb{R} :: \|S\|_E = \sqrt{\|\gamma_R S\|_{E_R}^2 + \|\gamma_L S\|_{E_L}^2} \\
& \|S\|_E = 0 \Rightarrow \|\gamma_R S\|_{E_R} = 0; \|\gamma_L S\|_{E_L} = 0 \Rightarrow \gamma_R S = 0; \gamma_L S = 0 \Rightarrow \\
& (\gamma_R + \gamma_L)[S] = 0 = S \\
& \|S + S'\|_E^2 = \|\gamma_R(S + S')\|_{E_R}^2 + \|\gamma_L(S + S')\|_{E_L}^2 = \|S + S'\|_{E_R}^2 + \|S + S'\|_{E_L}^2 \\
& \leq \|S\|_{E_R}^2 + \|S'\|_{E_R}^2 + \|S\|_{E_L}^2 + \|S'\|_{E_L}^2 = \|S\|_E^2 + \|S'\|_E^2 \\
& \text{This norm is invariant by } Spin(3, 1), Spin(1, 3): \\
& \|\gamma C(\sigma) S\|_E = \sqrt{\|\gamma C(\sigma) \gamma_R S\|_{E_R}^2 + \|\gamma C(\sigma) \gamma_L S\|_{E_L}^2} \\
& = \sqrt{\|\gamma_R \gamma C(\sigma) S\|_{E_R}^2 + \|\gamma_L \gamma C(\sigma) S\|_{E_L}^2} \\
& = \sqrt{\|\gamma C(\sigma) S\|_{E_R}^2 + \|\gamma C(\sigma) S\|_{E_L}^2} = \sqrt{\|S\|_{E_R}^2 + \|S\|_{E_L}^2} = \|S\|_E \quad \blacksquare
\end{aligned}$$

4 THE SPINOR MODEL OF KINEMATICS

We have now the mathematical tools to enter the representation of kinematics of material bodies in General Relativity. First we will make some adjustment to the fiber bundles used so far to represent the geometry, to account for the introduction of the Spin group.

4.1 Description of the fiber bundles

4.1.1 The geometric fiber bundles

The geometric model is similar to the previous one, with the replacement of $SO(3,1)$ by the Spin group.

Definition 35 *The **principal bundle** $P_G(M, Spin_0(3,1), \pi_G)$ has for fiber the connected component of the unity of the Spin group, for trivialization the map :*

$$\begin{aligned} \varphi_G : M \times Spin_0(3,1) &\rightarrow P_G :: p = \varphi_G(m, s). \\ \text{The **standard gauge** used by observers is } \mathbf{p}(m) &= \varphi_G(m, \mathbf{1}) \end{aligned}$$

A section $\sigma \in \mathfrak{X}(P_G)$ is defined by a map: $\sigma : M \rightarrow Spin(3,1)$ such that : $\sigma(m) = \varphi_G(m, \sigma(m))$ and in a change of gauge :

$$\mathbf{p}(m) = \varphi_G(m, \mathbf{1}) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \quad (71)$$

$$\sigma(m) = \varphi_G(m, \sigma(m)) = \tilde{\varphi}_G(m, \chi(m) \cdot \sigma(m)) \quad (72)$$

Definition 36 *The **vectors on the fiber bundle** TM are represented in the associated vector bundle $P_G[\mathbb{R}^4, \mathbf{Ad}]$ defined through the holonomic orthonormal basis :*

$$\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$$

So $\varepsilon_0(m) = (\mathbf{p}(m), \varepsilon_0)$ is the 4th vector both in the Clifford algebra and in the tangent space $T_m M$. It corresponds to the velocity of the observer.

With the equivalence relation : $(\mathbf{p}(m), v) \sim (\varphi_G(m, g), \mathbf{Ad}_{g^{-1}}v)$

In a change of gauge on P_G the holonomic basis becomes :

$$\mathbf{p}(m) = \varphi_G(m, \mathbf{1}) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \quad (73)$$

$$\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i) \rightarrow \tilde{\varepsilon}_i(m) = \mathbf{Ad}_{\chi(m)^{-1}}\varepsilon_i(m) \quad (74)$$

For a given observer $\varepsilon_0(p_o(t)) = \frac{1}{c} \frac{dp_o}{dt}$ is fixed along his world line.

The Lorentz scalar product on \mathbb{R}^4 is preserved by \mathbf{Ad} thus it can be extended to $P_G[\mathbb{R}^4, \mathbf{Ad}]$.

Definition 37 *The **Clifford bundle** $Cl(TM)$ is the associated vector bundle $P_G[Cl(3,1), \mathbf{Ad}]$ defined through the basis $(\varepsilon_i(m))_{i=0}^3$.*

In a change of gauge on P_G the elements of $\text{Cl}(m)$ transforms as :

$$\mathbf{p}(m) = \varphi_G(m, 1) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \quad (75)$$

$$\sigma(m) = \varphi_P(m, \sigma(m)) = \tilde{\varphi}_P(m, \chi(m) \cdot \sigma(m)) \quad (76)$$

$$\mathbf{w}(m) = (\mathbf{p}(m), w) \rightarrow \mathbf{Ad}_{\chi(m)^{-1}} \mathbf{w}(m) \quad (77)$$

4.1.2 The kinematic bundle

In addition to the previous bundles we define the associated bundle in which the spinors live :

Proposition 38 *The relativist momentum of particles are represented by Spinors, which are, at each point of the world line of the particle, vectors of the associated vector bundle $P_G[E, \gamma C]$. They are measured by observers in the standard gauge defined through the holonomic basis : $\mathbf{e}_i(m) = (\mathbf{p}(m), e_i)$*

With the equivalence relation : $(\mathbf{p}(m), S) \sim (\varphi_G(m, g), \gamma C(g^{-1})S)$ so that in a change of gauge the holonomic basis becomes :

$$\mathbf{p}(m) = \varphi_G(m, 1) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \quad (78)$$

$$\mathbf{e}_i(m) = (\mathbf{p}(m), e_i) \rightarrow \tilde{\mathbf{e}}_i(m) = \gamma C(\chi(m)^{-1}) \mathbf{e}_i(m) \quad (79)$$

and the components of a section $S \in \mathfrak{X}(P_G[E, \gamma C])$ change as :

$$\mathbf{p}(m) = \varphi_G(m, 1) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \quad (80)$$

$$\mathbf{S}(m) = (\mathbf{p}(m), S_m) = \left(\mathbf{p}(m) \cdot \chi(m)^{-1}, \gamma C(\chi(m)) S_m \right) \quad (81)$$

From a mathematical point of view the holonomic basis $(\varepsilon_i(m))_{i=0}^3$, $(\mathbf{e}_i(m))_{i=1}^4$, are defined through the same standard gauge $\mathbf{p}(m)$ chosen by the observer. This gauge is arbitrary. For the tetrad the vectors $\varepsilon_i(m)$ can be measured in the holonomic basis of any chart through P . We have nothing similar for $\mathbf{e}_i(m)$, and actually the vectors e_i of \mathbf{E} themselves are abstract. However we will see in the following how the basis $\mathbf{e}_i(m)$ used by an observer can be related to physical phenomena (inertial observers).

The scalar product on \mathbf{E} is preserved by γC thus it can be extended to $P_G[E, \gamma C]$ and to the space of sections $\mathfrak{X}(P_G[E, \gamma C])$ by :

$$\langle \mathbf{S}, \mathbf{S}' \rangle = \int_{\Omega} \langle \mathbf{S}(m), \mathbf{S}'(m) \rangle_E \varpi_4(m)$$

Moreover we have the following :

Theorem 39 *The set of integrable sections :*

$$L^1(\mathfrak{X}(P_G[E, \gamma C]), \varpi_4) = \left\{ \int_{\Omega} \|S\| \varpi_4 < \infty \right\}$$

with the norm on E is a separable, infinite dimensional Fréchet space.

Proof. Consider the vector space : $\mathfrak{X}(P_G[E, \gamma C])$ endowed with the norm :

$$\|\mathbf{S}\| = \int_{\Omega} \|S(m)\|_E \varpi_4(m) \text{ and the norm}$$

$$\|S(m)\|_E = \sqrt{\langle \gamma_R S, \gamma_R S \rangle_E - \langle \gamma_L S, \gamma_L S \rangle_E}$$

Restrict this space to $L^1(M, P_G[E, \gamma C], \varpi_4)$

$$= \{\mathbf{S} \in \mathfrak{X}(P_G[E, \gamma C]) \mid \int_{\Omega} \|S(m)\|_E \varpi_4(m) < \infty\}$$

This is a Fréchet space (Maths.2276). Moreover it is separable, because Ω is relatively compact and the smooth compactly supported maps are a countable basis in L^1 (see Lieb).

Because the norm is invariant by the Spin group this space does not depend on the choice of trivialization. ■

The result still holds if we impose that the sections are differentiable.

4.1.3 Fundamental symmetry breakdown

The observer uses the frame $(O, (\varepsilon_i)_{i=0}^3)$ to measure the components of vectors of TM, and the holonomic maps $(\mathbf{e}_i(m))_{i=0}^3$ to measure the spinors. The breakdown, specific to each observer, comes from the distinction of his present, and is materialized in his standard basis by the vector $\varepsilon_0(m)$. This choice leads to a split of the Spin group between the spatial rotations, represented by $Spin(3)$, and the homogeneous space $Spin(3, 1)/Spin(3)$.

We have an associated fiber bundle :

$$P_W = P_G[Spin(3, 1)/Spin(3), \lambda] :$$

$$(\mathbf{p}(m), s_w) = (\varphi_G(m, 1), s_w) \sim (\varphi_G(m, s), \lambda(s^{-1}, s_w))$$

with the left action :

$$\lambda : Spin(3, 1) \times Spin(3, 1)/Spin(3) \rightarrow Spin(3, 1)/Spin(3) : \lambda(s, s_w) = \pi_w(s \cdot s_w)$$

On the manifold P_G there is a structure of principal fiber bundle

$$P_R(P_W, Spin(3), \pi_R) \text{ with trivialization :}$$

$$\varphi_R : P_W \times Spin(3) \rightarrow P_G ::$$

$$\varphi_R((\mathbf{p}(m), s_w), s_r) = \varphi_G(m, s_w \cdot s_r) = \varphi_R((\varphi_G(m, s), \lambda(s^{-1}, s_w)), s_r)$$

As the latest trivialization shows, for a given s , s_r depends on s_w in that it is a part of $s \in Spin(3, 1)$.

It sums up to define the local basis in two steps : first by choosing s_w second by choosing s_r

Any section $\sigma \in \mathfrak{X}(P_G)$ can be decomposed, for a given vector field ε_0 and a fixed $\epsilon = \pm 1$, in two sections :

$$\epsilon \sigma_w \in \mathfrak{X}(P_W), \epsilon \sigma_r \in \mathfrak{X}(P_R) \text{ with } \sigma(m) = \epsilon \sigma_w(m) \cdot \epsilon \sigma_r(m)$$

The set of vectors of $T_m M$ used to build $Spin(3)$ is defined by $\varepsilon_0(m)$.

Our objective is now to understand the relation between sections

$\mathbf{S} \in \mathbf{P}_G[E, \gamma C]$ of the vector bundle and the motion of a particle, and to precise the possible values of S . We will proceed in several steps. First we will focus on the trajectories and we will show that they can be matched with sections of P_W .

4.2 Trajectories and the Spin Group

A change of observer is a change of gauge $\mathbf{p}(m) = \varphi_G(m, \mathbf{1})$ (or trivialization) on the principal bundle $P_G(M, Spin_0(3, 1), \pi_G)$. We see now how the formulas given previously can be expressed in the formalism of Clifford algebras.

Theorem 40 Any section $\sigma \in \mathfrak{X}(P_G)$ defines, for any positive function $f \in C_\infty(\Omega; \mathbb{R}_+)$ and observer, two vector fields $V \in \mathfrak{X}(P_G[\mathbb{R}^4, \mathbf{Ad}])$ by :

$$V(m) = f(m) \mathbf{Ad}_{\sigma(m)} \varepsilon_0(m) = f(m) \left((2a_w^2 - 1) \varepsilon_0 + \epsilon a_w (w^1 \varepsilon_1 + w^2 \varepsilon_2 + w^3 \varepsilon_3) \right) \quad (82)$$

where $\sigma_w(m) = \epsilon(a_w + v(0, w))$ is the projection of σ on P_W along ε_0

Then V is time like, future oriented and $\langle V(m), V(m) \rangle = -f^2(m)$ and is invariant in a change of gauge on P_G

Conversely, for any time like, future oriented vector field $V \in \mathfrak{X}(P_G[\mathbb{R}^4, \mathbf{Ad}])$ there are two sections $\sigma_w \in \mathfrak{X}(P_G)$ such that :

$$\frac{V}{\sqrt{-\langle V, V \rangle}} = u = \mathbf{Ad}_{\sigma(m)} \varepsilon_0(m) : \sigma_w = \epsilon \left(\sqrt{\frac{1}{2}(u_0 + 1)} + \frac{1}{\sqrt{\frac{1}{2}(u_0 + 1)}} v(0, u_i) \right)$$

Proof. i) $\sigma(m) = \epsilon \sigma_w(m) \cdot \epsilon \sigma_r(m)$

$\sigma_w(m) = \epsilon(a_w + v(0, w))$ so let be $a_w > 0$ which defines ϵ

ii) Define

$u \in \mathfrak{X}(P_G[\mathbb{R}^4, \mathbf{Ad}]) : u(m) = \mathbf{Ad}_{\sigma_w(m)} \varepsilon_0(m) = (\mathbf{p}(m), \mathbf{Ad}_{\sigma_w} \varepsilon_0) = (\mathbf{p}(m), u)$

$$\begin{aligned} u &= \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} = (\epsilon a_w + v(0, w)) \cdot \varepsilon_0 \cdot (\epsilon a_w - v(0, w)) \\ &= (\epsilon a_w + v(0, w)) \cdot (\epsilon a_w \varepsilon_0 - \varepsilon_0 \cdot v(0, w)) \\ &= a_w^2 \varepsilon_0 + \epsilon a_w (-\varepsilon_0 \cdot v(0, w) + v(0, w) \cdot \varepsilon_0) - v(0, w) \cdot \varepsilon_0 \cdot v(0, w) \\ &= a_w^2 \varepsilon_0 + \frac{1}{2} \epsilon a_w (-\varepsilon_0 \cdot \varepsilon_0 \cdot (w^1 \varepsilon_1 + w^2 \varepsilon_2 + w^3 \varepsilon_3) - (w^1 \varepsilon_1 + w^2 \varepsilon_2 + w^3 \varepsilon_3) \cdot \varepsilon_0 \cdot \varepsilon_0) \\ &\quad - \frac{1}{4} \varepsilon_0 \cdot (w^1 \varepsilon_1 + w^2 \varepsilon_2 + w^3 \varepsilon_3) \cdot \varepsilon_0 \cdot \varepsilon_0 \cdot (w^1 \varepsilon_1 + w^2 \varepsilon_2 + w^3 \varepsilon_3) \\ &= a_w^2 \varepsilon_0 + \frac{1}{2} \epsilon a_w (w + w) + \frac{1}{4} \varepsilon_0 \cdot w \cdot w \\ &= a_w^2 \varepsilon_0 + a_w \epsilon w + \frac{1}{4} \varepsilon_0 \cdot \langle w, w \rangle \\ &= \left(a_w^2 + \frac{1}{4} w^t w \right) \varepsilon_0 + \epsilon a_w w \end{aligned}$$

$$u = (2a_w^2 - 1) \varepsilon_0 + \epsilon a_w (w^1 \varepsilon_1 + w^2 \varepsilon_2 + w^3 \varepsilon_3)$$

$$\text{iii) } \langle u, \varepsilon_0 \rangle = - (2a_w^2 - 1) = 1 - 2 \left(1 + \frac{1}{4} w^t w \right) = -1 - \frac{1}{2} w^t w < 0$$

$$\langle u, u \rangle = a_w^2 w^t w - (2a_w^2 - 1)^2 = a_w^2 (4(a_w^2 - 1)) - (2a_w^2 - 1)^2 = -1$$

$$\text{iv) } V = f(m) u \Rightarrow$$

$$\langle V, \varepsilon_0 \rangle = f(m) \langle u, \varepsilon_0 \rangle < 0$$

$$\langle V, V \rangle = -f^2 < 0$$

v) In a change of gauge on P_G :

$$\mathbf{p}(m) = \varphi_G(m, \mathbf{1}) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} :$$

$$\sigma(m) \rightarrow \tilde{\sigma}(m) = \chi(m)^{-1} \cdot \sigma(m)$$

$$u(m) = (\mathbf{p}(m), \mathbf{Ad}_{\sigma_w} \varepsilon_0) \sim \left(\mathbf{p}(m) \cdot \chi(m)^{-1}, \mathbf{Ad}_\chi \mathbf{Ad}_{\sigma_w} \varepsilon_0 \right) = (\tilde{\mathbf{p}}(m), \mathbf{Ad}_{\tilde{\sigma}_w} \varepsilon_0) = \tilde{u}(m)$$

$$\text{vi) Let } u = \frac{V}{\sqrt{-\langle V, V \rangle}} \text{ then } u \text{ is time like, } \langle u, u \rangle = -1, \langle u, \varepsilon_0 \rangle < 0$$

$$\begin{aligned}
u &= ((2a_w^2 - 1) \varepsilon_0 + \epsilon a_w (w^1 \varepsilon_1 + w^2 \varepsilon_2 + w^3 \varepsilon_3)) = \sum_{i=0}^3 u_i \varepsilon_i \\
i=1,2,3 : w_i &= \epsilon u_i / a_w \\
w^t w &= \frac{\sum_{i=1}^3 u_i^2}{a_w^2} \\
\langle u, u \rangle &= -1 = \sum_{i=1}^3 u_i^2 - u_0^2 \\
u_0^2 &= 1 + \sum_{i=1}^3 u_i^2 \\
\langle u, \varepsilon_0 \rangle &= u_0 < 0 \\
u_0 &= -\sqrt{1 + \sum_{i=1}^3 u_i^2} < -1 \\
1 + \frac{1}{4} w^t w &= 1 + \frac{1}{4} \frac{\sum_{i=1}^3 u_i^2}{a_w^2} = \frac{1+4 \sum_{i=1}^3 u_i^2}{4a_w^2} = \frac{1+4(u_0^2-1)}{4a_w^2} = \frac{-3+4u_0^2}{4a_w^2} > \frac{-3+4}{4a_w^2} = \frac{1}{a_w^2} \\
\text{So we can define :} \\
u_0 &= 2a_w^2 - 1 \\
a_w &= \epsilon \sqrt{\frac{1}{2}(u_0 + 1)} \\
w_i &= u_i / a_w = \epsilon u_i / \sqrt{\frac{1}{2}(u_0 + 1)} \\
\sigma_w &= \epsilon \left(\sqrt{\frac{1}{2}(u_0 + 1)} + \frac{1}{\sqrt{\frac{1}{2}(u_0 + 1)}} v(0, u_i) \right) = a_w + v(0, w) \\
\text{with } w_i &= \frac{u_i}{\sqrt{\frac{1}{2}(u_0 + 1)}} = \frac{V_i}{\sqrt{\frac{1}{2} \left(\frac{V_0}{\sqrt{-\langle V, V \rangle}} + 1 \right)}} \frac{1}{\sqrt{-\langle V, V \rangle}} = \frac{V_i}{\sqrt{-\frac{1}{2} (\langle V, V \rangle - V_0 \sqrt{-\langle V, V \rangle})}}
\end{aligned}$$

■

If we take $f(m) = c$ any section $\sigma \in \mathfrak{X}(P_G)$ defines two fields of world lines, with opposite spatial speed :

$$u = \frac{dp}{d\tau} = \frac{1}{\sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}}} (\vec{v} + c\varepsilon_0(m)) = c \left((2a_w^2 - 1) \varepsilon_0 + \epsilon a_w \sum_{i=1}^3 w_i \varepsilon_i \right) \quad (83)$$

If we take $f(m) = \frac{c}{2a_w^2 - 1}$ any section $\sigma \in \mathfrak{X}(P_G)$ defines two fields of trajectories, with opposite spatial speed :

$$V = \frac{dp}{dt} = \vec{v} + c\varepsilon_0(m) = c \left(\varepsilon_0 + \epsilon \frac{a_w}{2a_w^2 - 1} \sum_{i=1}^3 w_i \varepsilon_i \right) \quad (84)$$

$$\begin{aligned}
\vec{v} &= \epsilon c \frac{a_w}{2a_w^2 - 1} \sum_{i=1}^3 w_i \varepsilon_i \\
a_w &= \sqrt{\frac{1}{2} \left(1 + \sqrt{1 + \frac{1}{4 \left(1 - \frac{v^2}{c^2} \right)}} \right)} \simeq 0.7 + \frac{1}{16} \frac{v^2}{c^2} \\
w &= \frac{\sqrt{1 + \frac{1}{4 \left(1 - \frac{v^2}{c^2} \right)}}}{\sqrt{1 + \sqrt{1 + \frac{1}{4 \left(1 - \frac{v^2}{c^2} \right)}}}} \frac{\vec{v}}{c} \simeq 0.8 \frac{\vec{v}}{c}
\end{aligned}$$

Remarks :

- i) All this is defined with respect to an observer, who fixes $\varepsilon_0(m)$
- ii) $V \in \mathfrak{X}(P_G[\mathbb{R}^4, \mathbf{Ad}])$ so can be equivalently defined as a section of TM :
 $V = \sum_{\alpha=0}^3 V^\alpha \partial \xi_\alpha = \sum_{\alpha=0}^3 \sum_{i=0}^3 V^i P_i^\alpha \partial \xi_\alpha$

iii) If V is past oriented ($u_0 < 0$) or null ($\langle V, V \rangle = 0$) there is no solution :
 $2a_w^2 - 1 = \frac{1}{2}(u_0 - 1) < -\frac{1}{2} \Rightarrow a_w^2 < 1$ and $a_w^2 \neq 1 + \frac{1}{4}w^t w$

This gives a strong physical meaning to the representation of world lines by section of P_W .

iv) Any map $\sigma : \mathbb{R} \rightarrow P_G$ is projected on M as a curve, which is not necessarily time like or defines a world line.

4.3 Spatial spinor

A section $\sigma \in \mathfrak{X}(P_G)$ defines at each point an element of $Spin(3, 1)$, which can uniquely (up to sign) be decomposed in $\sigma = \epsilon\sigma_w \cdot \epsilon\sigma_r$ with respect to a given observer.

The first component $\sigma_w = \pm(a_w + v(0, w)) \in Spin(3, 1)$ defines, in the standard basis of the observer, a vector field of world lines, and a trajectory. w is aligned in the direction of the spatial speed or the opposite.

The second part $\sigma_r = a_r + v(0, r)$ belonging to one of the two connected components of $Spin(3)$ (according to the sign of a_r , it is + for the component of the identity) leaves invariant $\varepsilon_0(m)$ and defines a spatial rotation, in the hyperplane orthogonal to $\varepsilon_0(m)$.

So with a single σ we have both a translational motion (along a worldline) and a spatial rotation, at the same point.

Notice that this is not a rotation around a point (like an orbit), but a *rotation at a point*. The action \mathbf{Ad}_{σ_r} on any vector of the tangent space $T_m M$ rotates the vector, but leaves invariant $\varepsilon_0(m)$, so this is an action on the physical space $\Omega_3(t)$. In the Spinor $S = \gamma C(\sigma(m)) S_0$ this action is done on vectors of E , and not on vector of $T_m M$.

The decomposition $\sigma = \epsilon\sigma_w \cdot \epsilon\sigma_r$ and the identification of the vectors of $Spin(3)$ request a choice of $\varepsilon_0(m)$: it is observer-dependant.

Our basic hypothesis is that the spinor S is a kinematic characteristic of the state of the particle : it represents the relativist momentum. With a section $\sigma \in \mathfrak{X}(P_G)$ and fixed vector $S_0 \in E$ we have a representation of the momenta, both translational and rotational.

Definition 41 We can then define the *spatial spinor* as :

$$\mathbf{S}_r(t) = \gamma C(\sigma_w^{-1}) \mathbf{S}(t) = \gamma C(\sigma_r(t)) S_0 \quad (85)$$

The spatial spinor is the representation of a rotational momentum. For a given trajectory and observer there are two possible, opposite, values of the spatial spinor : $S_r(t) = \pm \gamma C(\sigma_r(t)) S_0$. In all cases $S(t) = \gamma C(\sigma(t)) S_0$: the total spinor stays the same, the distinction between σ_w, σ_r and the opposite values is the consequence of the breakdown of symmetry induced by the observer. The sign \pm is related to a trajectory (the orientation of w with respect to the spatial speed) so one can speak of **spin up or down** with respect to the trajectory. This feature is entirely linked to the Relativist picture, and has nothing to do with QM. The name spin is used freely in Quantum Physics, and

this is sometimes confusing. The usual spin, as rotational momentum, is the Spatial Spinor and we will give a precise definition later. And to be clear I will call the present feature (spin up or down) **Relativist Spin** which takes the values of $\epsilon = \pm 1$.

If we assume that the spatial spinor is, by itself, an intrinsic feature of the particle, then one must assume that the map : $\sigma_r : \mathbb{R} \rightarrow P_R$ is continuous, thus σ_r must belong and stay in one of the two connected components of P_R . Normally the decomposition $\sigma = \epsilon\sigma_w \cdot \epsilon\sigma_r$ is continuous, and the passage to the opposite sign is, for the spatial spinor, a discontinuity, and also for the relativist spin.

The issue now is to precise what can be S_0 , that we will call **inertial spinor**. The 4 dimensional relativist momentum $P = \mu u$, which is a geometric quantity, has a constant scalar product : $\langle P, P \rangle = -\mu^2 c^2$ where μ is, by definition, the mass at rest. The scalar product $\langle S, S \rangle = \langle S_0, S_0 \rangle$ is preserved on the world line, so we will look at vectors S such that : $S \neq 0 \Rightarrow \langle S_0, S_0 \rangle \neq 0$.

4.4 Inertial spinor

We have seen that the unique scalar product on E is non degenerate, but not definite positive. So it is logical to require that S_0 belongs to some vector subspace E_0 of E , over which the scalar product is definite, either positive or negative. Moreover a change of spatial frame should change only S_r , thus E_0 should be invariant under the action of $Spin(3)$.

- So there should be some vector subspace E_0 of E such that :
- it is invariant by $\gamma C(\sigma_r)$ for $\sigma_r \in Spin(3) : \forall S_0 \in E_0, \sigma_r \in Spin(3) : \gamma C(\sigma_r) S_0 \in E_0$
 - on which the scalar product is either definite positive or definite negative : $\forall S_0 \in E_0 : \langle S_0, S_0 \rangle_E = 0 \Rightarrow S_0 = 0$

Theorem 42 *The only vector subspace of E invariant by γC on $Spin(3)$ and over which the scalar product is definite*

- positive is $E_0 = \left\{ S = \begin{bmatrix} S_R \\ S_L \end{bmatrix} = \begin{bmatrix} v \\ iv \end{bmatrix}, v \in \mathbb{C}^2 \right\}$
- negative is $E'_0 = \left\{ S = \begin{bmatrix} S_R \\ S_L \end{bmatrix} = \begin{bmatrix} v \\ -iv \end{bmatrix}, v \in \mathbb{C}^2 \right\}$

Proof. i) The scalar product on E (which does not depend on the signature) reads :

$$u = \begin{bmatrix} u_R \\ u_L \end{bmatrix} \in E : \begin{bmatrix} u_R^* & u_L^* \end{bmatrix} \begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix} \begin{bmatrix} u_R \\ u_L \end{bmatrix} = i(u_L^* u_R - u_R^* u_L) = i(\overline{u}_L^t u_R - \overline{u}_R^t u_L)$$

$$u_L = v_L + iw_L \text{ with } v_L, w_L \in \mathbb{R}^2$$

$$u_R = v_R + iw_R \text{ with } v_R, w_R \in \mathbb{R}^2$$

$$\langle S, S \rangle = i((v_L^t - iw_L^t)(v_R + iw_R) - (v_R^t - iw_R^t)(v_L + iw_L))$$

$$= i(v_L^t v_R + iv_L^t w_R - iw_L^t v_R + w_L^t w_R - v_R^t v_L - iv_R^t w_L + iw_R^t v_L - w_R^t w_L)$$

$$= -2(v_L^t w_R - w_L^t v_R)$$

$$\langle S, S \rangle = 0 \Leftrightarrow v_L^t w_R = w_L^t v_R$$

So it is definite for any u such that :

$$v_L = -\epsilon w_R, w_L = \epsilon v_R \Rightarrow$$

$$u_L = -\epsilon w_R + \epsilon i v_R = \epsilon i (v_R + i w_R) = \epsilon i u_R$$

$$u = \begin{bmatrix} u_R \\ \epsilon i u_R \end{bmatrix}$$

$$\langle S, S \rangle = 2\epsilon (w_R^t w_R + v_R^t v_R) = 2\epsilon u_R^* u_R$$

It is definite positive for $\epsilon = +1$ and definite negative for $\epsilon = -1$

ii) The vector subspace must be invariant by $\gamma C(s_r)$. Which is equivalent to $S_L = \epsilon i S_R$

For any $S_0 \in E_0, E'_0, s \in Spin(3, 1)$

$$\gamma C(a + v(r, w) + b\varepsilon_5) S_0 =$$

$$\begin{bmatrix} (a + ib)\sigma_0 + \frac{1}{2} \sum_a (w_a - ir_a)\sigma_a & 0 \\ 0 & (a - ib)\sigma_0 - \frac{1}{2} \sum_a (w_a + ir_a)\sigma_a \end{bmatrix} \begin{bmatrix} v \\ \epsilon i v \end{bmatrix}$$

$$= \begin{bmatrix} S_R \\ S_L \end{bmatrix}$$

$$S_R = \left((a + ib)\sigma_0 + \frac{1}{2} \sum_{a=1}^3 (w_a - ir_a)\sigma_a \right) v$$

$$S_L = \epsilon \left((a - ib)\sigma_0 - \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a)\sigma_a \right) i v$$

and $S \in E_0 \Leftrightarrow S_L = \epsilon i S_R$

$$\Leftrightarrow \epsilon i \left((a - ib)\sigma_0 - \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a)\sigma_a \right) v$$

$$= \epsilon i \left((a + ib)\sigma_0 + \frac{1}{2} \sum_{a=1}^3 (w_a - ir_a)\sigma_a \right) v$$

$$\Leftrightarrow \left(-ib\sigma_0 - \frac{1}{2} \sum_{a=1}^3 w_a \sigma_a \right) v = \left(ib\sigma_0 + \frac{1}{2} \sum_{a=1}^3 w_a \sigma_a \right) v$$

$$\Leftrightarrow \left(ib\sigma_0 + \frac{1}{2} \sum_{a=1}^3 w_a \sigma_a \right) v = 0$$

This condition is met for $w = 0$ that is $s \in Spin(3)$.

iii) It is easy to see that the result does not depend on the signature:

$$\gamma C'(s) \begin{bmatrix} v \\ \epsilon i v \end{bmatrix} = \begin{bmatrix} S_R \\ S_L \end{bmatrix} \Rightarrow$$

$$S_R = \left((a + b)\sigma_0 + \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a)\sigma_a \right) v$$

$$S_L = \epsilon i \left((a - b)\sigma_0 + \frac{1}{2} \sum_{a=1}^3 (-w_a + ir_a)\sigma_a \right) v$$

$$\left(2b\sigma_0 + \sum_{a=1}^3 w_a \sigma_a \right) v = 0 \quad \blacksquare$$

So particles have both a left S_L and a right S_R part, which are linked but not equal. We have one of the known features of particles : chirality.

Because $E_0 \cap E'_0 = \{0\}$, $(E_0, \gamma C), (E'_0, \gamma C)$ are two, non equivalent, irreducible representations of $Spin(3)$. So they can be seen as corresponding to two kinds of particles according to ϵ .

The inertial spinor is defined, from the components of the two complex vectors of S_R , by 4 real scalars.

The quantity :

$$\langle S_0, S_0 \rangle_E = \epsilon 2S_R^* S_R \quad (86)$$

(with the same meaning of ϵ as above) is a scalar, which is conserved along the trajectory. Similarly to the 4 vector $P = \mu u$, we can define a mass at rest M_p by : $\langle S_0, S_0 \rangle = M_p^2 c^4$, but because $\langle S_0, S_0 \rangle$ can be negative we have to consider :

$$\langle S_0, S_0 \rangle = \epsilon M_p^2 c^4 \quad (87)$$

where ϵ is a characteristic of the particle. We retrieve a celebrated Dirac's result from his equation. So we define :

Definition 43 *particles are such that $S_L = iS_R$.*

Their mass is $M_p = \frac{1}{c^2} \sqrt{\langle S_0, S_0 \rangle_E} = \frac{1}{c^2} \sqrt{2S_R^ S_R}$*

antiparticles are such that $S_L = -iS_R$

Their mass is $M_p = \frac{1}{c^2} \sqrt{-\langle S_0, S_0 \rangle_E} = \frac{1}{c^2} \sqrt{-2S_R^ S_R}$*

Do antiparticles have negative mass ? The idea of a negative mass is still controversial. Dirac considered that antiparticles move backwards in time and indeed a negative mass combined with the first Newton's law seems to have this effect. But here the world line of the particle is defined by σ_w , and there is no doubt about the behavior of an antiparticle : it moves towards the future. The mass at rest M_p is somewhat conventional, the defining relation is $\langle S_0, S_0 \rangle = \epsilon M_p^2 c^2$ so we can choose any sign for M_p , and it seems more appropriate to take $M_p > 0$ both for particles and antiparticles.

Then S_R reads :

$$S_R = \frac{M_p}{c^2 \sqrt{2}} \begin{bmatrix} a \\ b \end{bmatrix} \text{ and } 1 = (|a|^2 + |b|^2)$$

It is customary to represent the polarization of the plane wave of an electric field by two complex quantities (the Jones vector) :

$$E_x = E_{0x} e^{i\alpha_x}$$

$$E_y = E_{0y} e^{i\alpha_y}$$

where (E_{0x}, E_{0y}) are the components of a vector E_0 along the axes x,y.

So we can write similarly :

$$S_R = \frac{M_p c^2}{\sqrt{2}} \begin{bmatrix} e^{i\alpha_1} \cos \alpha_0 \\ e^{i\alpha_2} \sin \alpha_0 \end{bmatrix} \quad (88)$$

Particles :

$$S_0 = \frac{M_p c^2}{\sqrt{2}} \begin{bmatrix} e^{i\alpha_1} \cos \alpha_0 \\ e^{i\alpha_2} \sin \alpha_0 \\ ie^{i\alpha_1} \cos \alpha_0 \\ ie^{i\alpha_2} \sin \alpha_0 \end{bmatrix}$$

Antiparticles :

$$S_0 = \frac{M_p c^2}{\sqrt{2}} \begin{bmatrix} e^{i\alpha_1} \cos \alpha_0 \\ e^{i\alpha_2} \sin \alpha_0 \\ -ie^{i\alpha_1} \cos \alpha_0 \\ -ie^{i\alpha_2} \sin \alpha_0 \end{bmatrix}$$

To each particle corresponds an antiparticle with the same mass. And particles show polarization characteristics similar to waves. This is how the inertial spinor is seen in Quantum Physics : the 3 real variables $\alpha_0, \alpha_1, \alpha_2$ define a polarization of the particle when it behaves as a wave. Each kind of elementary particles is characterized by a vector of S_0 , and it is associated to its charge with respect to the EM field, and the 3 variables $\alpha_0, \alpha_1, \alpha_2$ define the magnetic moment as we will see. But for neutral material bodies usually its value can be taken equal to one of the vectors of the orthonormal basis :

$$\mathcal{E}_1 = \frac{1}{\sqrt{2}}(e_1 + ie_3), \mathcal{E}_2 = \frac{1}{\sqrt{2}}(e_2 + ie_4)$$

The space E_0, E'_0 are orthogonal, so :

$$\forall S_0 \in E_0, S'_0 \in E'_0 : \langle \gamma C(\sigma) S_0, \gamma C(\sigma) S'_0 \rangle = 0$$

The definition of E_0, E'_0 does not depend on the observer.

The spatial spinor S_r belongs to E_0, E'_0 by construct. The action of $Spin(3)$ on E_0, E'_0 is proper, continuous and free, thus (Maths.1793) the orbits have a unique structure of manifold of dimension : $\dim E_0 - \dim Spin(3) = 1$. For any value $S_0 \in E_0, E'_0$ when $\sigma_r \in Spin(3)$ then $S_r = \gamma C(\sigma_r) S_0$ stays on a curve on E_0, E'_0 , and conversely each vector S of E_0, E'_0 belongs to a unique such curve. $Spin(3)$ is compact, so this curve is compact, $Spin(3)$ has two connected components, so the curve is formed of two, compact, connected components.

The demonstration above is actually the equivalent - expressed in the formalism of fiber bundles and spinors - of the classic Wigner's classification of particles (see for instance Weinberg), done through the analysis of equivariance of the relativist momentum by the Poincaré's group. We could, in the same way, consider also the null spinors (assimilated to bosons), which is the vector subspace of E :

$$\langle S, S \rangle = 0 \Leftrightarrow v_L^t w_R = w_L^t v_R$$

4.5 Space and time reversal

Space reversal and time reversal are changes of basis in \mathbb{R}^4 , so they depend on the basis.

A change of orthonormal basis in \mathbb{R}^4 is represented by an orthogonal matrix, and in the Clifford algebra by the action \mathbf{Ad}_s for some element s of the Pin group (it is not necessarily represented by an element of the connected component of the Spin group) :

$$w \rightarrow \tilde{w} = \mathbf{Ad}_s w = s \cdot w \cdot s^{-1}$$

The impact on a representation is :

$$\gamma(w) \rightarrow \gamma(\tilde{w}) = [\gamma(s)] [\gamma(w)] [\gamma(s)^{-1}]$$

In a change of basis in E represented by a matrix Q the components of a vector $u \in E$ change according to : $[u] \rightarrow [\tilde{u}] = Q^{-1}[u]$ and the matrices γ representing endomorphisms change as : $\gamma \rightarrow \tilde{\gamma} = Q^{-1}\gamma Q$. So the change of basis in \mathbb{R}^4 corresponds to a change of basis represented by the matrix $Q = [\gamma(s)]^{-1}$ in E , and the components of a vector S of E change as : $[u] \rightarrow [\tilde{u}] = [\gamma(s)][u]$.

4.5.1 Time reversal

Time reversal is the operation :

$$u = u^0\varepsilon_0 + u^1\varepsilon_1 + u^2\varepsilon_2 + u^3\varepsilon_3 \rightarrow -u^0\varepsilon_0 + u^1\varepsilon_1 + u^2\varepsilon_2 + u^3\varepsilon_3$$

corresponding to $s = \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$, with $s^{-1} = \varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_1$ in $Cl(3, 1)$, $s^{-1} = \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$ in $Cl(1, 3)$

$Cl(3, 1)$:

$$[\gamma C(\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3)] = \gamma_3 \gamma_2 \gamma_1 = i \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix}$$

$$i \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \epsilon iv \end{bmatrix} = \begin{bmatrix} -\epsilon v \\ iv \end{bmatrix} = \begin{bmatrix} v' \\ -\epsilon iv' \end{bmatrix}$$

$Cl(1, 3)$:

$$[\gamma C(\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3)] = -i \gamma_1 \gamma_2 \gamma_3 = \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \epsilon iv \end{bmatrix} = \begin{bmatrix} \epsilon iv \\ v \end{bmatrix} = \begin{bmatrix} v' \\ -\epsilon iv' \end{bmatrix}$$

So with both signatures particles and antiparticles are exchanged.

4.5.2 Space reversal :

Space reversal is the operation :

$$u = u^0\varepsilon_0 + u^1\varepsilon_1 + u^2\varepsilon_2 + u^3\varepsilon_3 \rightarrow u^0\varepsilon_0 - u^1\varepsilon_1 - u^2\varepsilon_2 - u^3\varepsilon_3$$

corresponding to $s = \varepsilon_0$, $s^{-1} = -\varepsilon_0$ in $Cl(3, 1)$, $s^{-1} = \varepsilon_0$ in $Cl(1, 3)$

$Cl(3, 1)$:

$$[\gamma C(\varepsilon_0)] = i \gamma_0 = \begin{bmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \epsilon iv \end{bmatrix} = \begin{bmatrix} \epsilon iv \\ -v \end{bmatrix} = \begin{bmatrix} v' \\ \epsilon iv' \end{bmatrix}$$

$Cl(1, 3)$:

$$[\gamma C(\varepsilon_0)] = \gamma_0 = \begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \epsilon iv \end{bmatrix} = \begin{bmatrix} \epsilon v \\ iv \end{bmatrix} = \begin{bmatrix} v' \\ \epsilon iv' \end{bmatrix}$$

So with both signatures particle and antiparticles stay in the same category.

These results are consistent with what is checked in Particles Physics, and the Standard Model. However the latter does not consider both signatures. Here we see that this feature does not allow to distinguish one signature as more physical than the other.

4.6 Total Spinor

The relation $S_L = \epsilon i S_R$ does not hold any more at the level of the total spinor, however we have still $\langle S, S \rangle = \langle S_0, S_0 \rangle_E$ which is positive for particles, and negative for anti-particles, so the distinction holds. The total spinor $S = \gamma C(\sigma) S_0 = \gamma C(\sigma_w) S_r$ belongs to a subset \hat{E}_0 of E larger than E_0 .

Definition 44 $\hat{E}_0 = \{ \gamma C(\sigma_w) S_0, \sigma_w \in Spin(3, 1) / Spin(3), S_0 \in E_0 \}$
 $= \left\{ a_w S_0 - i \frac{1}{2} \sum_{a=1}^3 w_a \gamma_a \gamma_0 S_0, S_0 \in E_0 \right\}$

with a similar set \hat{E}'_0 for antiparticles.

The expression is :

$$\gamma C(a + v(r, w) + b\epsilon_5) \begin{bmatrix} S_R \\ \epsilon i S_R \end{bmatrix} = \begin{bmatrix} ((a+b) + \frac{1}{2} \sum_a (w_a - ir_a) \sigma_a) S_R \\ \epsilon i ((a-b) - \frac{1}{2} \sum_a (w_a + ir_a) \sigma_a) S_R \end{bmatrix}$$

thus with $r = 0$ we have :

$$\gamma C(a + v(0, w)) \begin{bmatrix} S_R \\ \epsilon i S_R \end{bmatrix} = \begin{bmatrix} (a + \frac{1}{2} \sum_a w_a \sigma_a) S_R \\ \epsilon i (a - \frac{1}{2} \sum_a w_a \sigma_a) S_R \end{bmatrix}$$

\hat{E}_0, \hat{E}'_0 are not vector spaces ($a_w = \epsilon \sqrt{1 + \frac{1}{4} w^t w}$) but real manifolds, embedded in E , with real dimension 7 (4 parameters for S_0 , 3 for w). By definition they are invariant by $Spin(3, 1)$.

Moreover we have the following :

Theorem 45 For a given value of the inertial spinor S_0 , and a measured value $S \in \hat{E}_0$ of the spinor S , there is a unique element $\sigma \in Spin(3, 1)$ such that $\gamma C(\sigma) S_0 = S$

Proof. i) The action of $Spin(3, 1)$ on E_0, E'_0 is free :

$$\forall S_0 \in E_0, E'_0 : \gamma C(s) S_0 = S_0 \Leftrightarrow \sigma = 1$$

$$S_0 = \begin{bmatrix} v \\ \epsilon i v \end{bmatrix}$$

$$\gamma C(s) S_0 = S_0 \Leftrightarrow$$

$$S_R = \left((a + ib) \sigma_0 + \frac{1}{2} \sum_{a=1}^3 (w_a - ir_a) \sigma_a \right) v = v$$

$$S_L = \left((a - ib) \sigma_0 - \frac{1}{2} \sum_{a=1}^3 (w_a + ir_a) \sigma_a \right) \epsilon i v = \epsilon i v$$

$$\Rightarrow \left(2a\sigma_0 - i \sum_{a=1}^3 r_a \sigma_a \right) v = 2v$$

$$\left(2ib\sigma_0 + \sum_{a=1}^3 w_a \sigma_a \right) v = 0$$

$$\left(\sum_{a=1}^3 r_a \sigma_a \right) v = 2i(1-a)v$$

$$\left(\sum_{a=1}^3 w_a \sigma_a \right) v = -2ibv$$

$$\Rightarrow \sum_{a=1}^3 r_a v^* \sigma_a v = 2i(1-a)v^* v$$

$$\sum_{a=1}^3 w_a v^* \sigma_a v = -2ibv^* v$$

The scalars $v^* \sigma_a v$ are real because the Dirac matrices are Hermitian, as is $v^* v$, so

$$\Rightarrow b = 0, a = 1$$

$$\Rightarrow r = w = 0$$

and the only solution is $\sigma = 1$.

$$\text{ii) } \gamma C(\sigma) S_0 = \gamma C(\sigma') S_0 \Rightarrow S_0 = \gamma C(\sigma^{-1}) \gamma C(\sigma') S_0 \Rightarrow \sigma^{-1} \cdot \sigma' = 1 \quad \blacksquare$$

We can assume that S_0 depends only on the type of particle, then with the knowledge of S_0 the measure of the spinor S defines uniquely the motion (translation and rotation) with respect to the observer. As $S = \gamma C(\sigma) S_0$ and σ can itself be uniquely, up to spin, decomposed in $\sigma = \sigma_w \cdot \sigma_r$, we have the correspondence with the formulas in the transition between observers where $[\chi] = \exp[K(w)] \exp[J(r)]$: σ_w corresponds to the boost, and we have seen how it can be computed from the spatial speed, and σ_r corresponds to $\exp[J(r)]$, the vector r is in both cases a Lie algebra representative of a spatial rotation. So conversely, knowing S_0 the spinor S can be computed from familiar data.

5 SPINOR FIELDS

5.1 Definition

The great interest of Spinors is that they sum up the kinematics of a particle in one single, geometric quantity which has a value at any point in a fiber bundle. It is then possible to conceive fields of particles whose world lines are defined by the same vector field, which is an usual case in Physics.

We have defined previously the associated vector bundle $P_G[E, \gamma C]$, defined through the holonomic basis : $\mathbf{e}_i(m) = (\mathbf{p}(m), e_i)$ with the equivalence relation : $(\mathbf{p}(m), S) \sim (\mathbf{p}(m) \cdot g, \gamma C(g^{-1}) S)$.

Definition 46 We will call **Spinor field** a section $\mathbf{S} \in \mathfrak{X}(P_G[E, \gamma C])$ which represents the relativist momentum of a particle or an antiparticle, such that $\int_{\Omega} \|\mathbf{S}(m)\| \varpi_4 < \infty$

Equivalently a spinor field, denoted $\mathfrak{X}(S_0)$ is defined by a vector $S_0 \in E_0$ and a section $\sigma \in \mathfrak{X}(P_G)$ such that $\mathbf{S}(m) = \gamma C(\sigma(m)) S_0$.

Let $\mathbf{S} \in \mathfrak{X}(P_G[E, \gamma C])$. Then $\langle S(m), S(m) \rangle = i(S_L^* S_R - S_R^* S_L) = y(m)$ defines a function on M. If \mathbf{S} represents the relativist momentum of a particle, then $\langle S(m), S(m) \rangle = \langle S_0, S_0 \rangle$. A necessary condition for a section of $P_G[E, \gamma C]$ to represent the relativist momentum of a particle is that $\langle S(m), S(m) \rangle = y$ has a fixed, positive, value. Then the set $E(y)$ of vectors $S_0 \in E_0$ such that $\exists \sigma \in \mathfrak{X}(P_G) : S = \gamma C(\sigma) S_0$ is given by :

$$E(y) = \left\{ S_0 = \begin{bmatrix} v \\ iv \end{bmatrix}, v^* v = \frac{1}{2} y, v \in \mathbb{C}^2 \right\}$$

And for a given vector $S_0 \in E(y)$, at each point m there is a unique $\sigma(m) \in Spin(3, 1)$ such that : $S(m) = \gamma C(\sigma(m)) S_0$. It defines a section $\sigma \in \mathfrak{X}(P_G)$, and at each point m , for each value ± 1 of the relativist spin and for a given observer, a vector field which is the tangent to the world line, and a spatial spin.

And similarly for antiparticles.

A spinor field represents particles which have the same inertial behavior. If, in a model, we have several particles, interacting with each others or with force fields, each particle can be assigned to a spinor field, which represents a general solution of the problem. One can also associate a density to spinor fields.

For elementary particles the vector S_0 is one of its fundamental characteristic.

For other material bodies S_0 is a kinematic characteristic which, for deformable solids, can be computed, as we will see below.

A Spinor field is defined without any reference to an observer : it has an intrinsic meaning, as it was expected for the momentum of a particle. And the decomposition in translational momentum on one hand, and rotational momentum on the other hand, is relative to each observer. So we have here a new, significant, feature of the relativist momentum.

Two comments :

i) A section of P_G can be seen as belonging to the Clifford bundle $Cl(M)$, and its components are (a, b, r, w) so we could hope to define the section through some fiber bundle with only (r, w) . Unfortunately the formulas for the components of an element of a $Cl(M)$ in a change of gauge are complicated and non linear. So, even if the use of r, w is useful, one cannot hope to define simply an consistently σ only through two vector fields $r(m), w(m)$.

ii) In a spinor field, a particle follows, with respect to an observer, a trajectory which is given by a vector field V deduced from a_w . So the value of the spinor at a point incorporates also the instantaneous velocity : at a given time $t + \delta t$ the particle is located at a point $\Phi_o(t + \delta t, x(t + \delta t))$ and $x(t + \delta t)$ is given by \vec{v} , which is part of V and defined by a_w , and thus by the value of the spinor at t .

We will see now how the usual spin enters the picture, but for this we need some more mathematics on group representations.

5.2 More on the theory of the representations of groups

5.2.1 Functional Representations

Functional representations are representations on vector spaces of functions or maps. Any locally compact topological group has at least one unitary faithful representation (usually infinite dimensional) of this kind, and they are common in Physics. The principles are the following (Maths.23.2.2).

Let H be a Banach vector space of maps $\varphi : E \rightarrow F$ from a topological space E to a vector space F , G a topological group with a continuous left action λ on $E : \lambda : G \times E \rightarrow E :: \lambda(g, x)$ such that $\lambda(g \cdot g', x) = \lambda(g, \lambda(g', x)), \lambda(1, x) = x$

Define the left action Λ of G on $H : \Lambda : G \times H \rightarrow H :: \Lambda(g, \varphi)(x) = \varphi(\lambda(g^{-1}, x))$

Then (H, Λ) is a representation of G . Thus G acts *on the argument* of φ .

If H is a Hilbert space and G has a Haar measure μ (a measure on G , all the groups that we will encounter have one Maths.22.5) then the representation is unitary with the scalar product :

$$\langle \varphi_1, \varphi_2 \rangle = \int_G \langle \Lambda(g, \varphi_1), \Lambda(g, \varphi_2) \rangle_H \mu(g)$$

If G is a Lie group and the maps of H and λ are differentiable (which implies that E is a manifold) then $(H, \Lambda'_g(1, \cdot))$ is a representation of the Lie algebra T_1G where $X \in T_1G$ acts by a differential operator :

$$\Lambda'_g(1, \varphi)(X)(x) = -\varphi'(x) \lambda'_g(1, x) X = \frac{d}{dt} \varphi(\lambda(\exp(-tX), x))|_{t=0}$$

For a right action $\rho : E \times G \rightarrow E :: \rho(g, x)$ we have similar results, with

$$P : H \times G \rightarrow H :: P(\varphi, g)(x) = \varphi(\rho(x, g))$$

$$P'_g(\varphi, 1)(X)(x) = -\varphi'(x) \rho'_g(x, 1) X = \frac{d}{dt} \varphi(\rho(x, \exp(-tX)))|_{t=0}$$

H can be a vector space of sections on a vector bundle. In a functional representation each function is a vector of the representation, so it is usually infinite dimensional. However the representation can be finite dimensional, by taking polynomials as functions, but this is not always possible : the set of polynomials must be algebraically closed under the action of the group.

5.2.2 Isomorphisms of groups

Most of the groups that are encountered in Physics are related to the group $SL(\mathbb{C}, 2)$ of 2×2 complex matrices with determinant 1 (Maths.24).

Any matrix of the Lie algebra $sl(\mathbb{C}, 2)$ reads with $Z = (z_1, z_2, z_3) \in \mathbb{C}^3$

$$f(Z) = \begin{bmatrix} iz_3 & z_2 + iz_1 \\ -z_2 + iz_1 & -iz_3 \end{bmatrix} \Rightarrow \text{Tr} f(Z) = 0$$

which is equivalent to take as basis the Dirac matrices.

The exponential is not surjective on $sl(\mathbb{C}, 2)$ and any matrix of $SL(\mathbb{C}, 2)$ reads :

$$I \cosh D + \frac{\sinh D}{D} f(Z) \text{ with } D^2 = -\det f(Z) = -(z_1^2 + z_2^2 + z_3^2)$$

The group $SU(2)$ of 2×2 unitary matrices ($NN^* = I$) is a compact real subgroup of $SL(\mathbb{C}, 2)$. Its Lie algebra is comprised of matrices $f(r)$ with $r \in \mathbb{R}^3$.

The exponential is surjective on $SU(2)$: $\exp f(r) = I \cos \sqrt{r^t r} + \frac{\sin \sqrt{r^t r}}{\sqrt{r^t r}} f(r)$

$T_1 Spin(3, 1)$ is isomorphic to $sl(\mathbb{C}, 2)$ (Math.1959) : $v(r, w) \rightarrow f(r + iw)$

$Spin(3, 1)$ is isomorphic to $SL(\mathbb{C}, 2)$: $a + v(r, w) + b\varepsilon_5 \rightarrow \exp f(r + iw)$

$T_1 Spin(3)$ is isomorphic to $su(2)$: $v(r, 0) \rightarrow f(r)$ and $so(3)$: $v(r, 0) \rightarrow j(r)$

$Spin(3)$ is isomorphic to $SU(2)$:

$$a_r + v(r, 0) \rightarrow \exp f(r) = I \cos \sqrt{r^t r} + \frac{\sin \sqrt{r^t r}}{\sqrt{r^t r}} f(r)$$

5.2.3 Representations of Spin(3,1), Spin(3) and SO(3)

$SL(\mathbb{C}, 2)$ and $Spin(3, 1)$ have the same representations which are (up to equivalence) :

- a unique, non unitary, irreducible representation of dimension n (Maths.1953), which can be seen as the tensorial product of two finite dimensional representations ($P^j \otimes P^k, D_j \times D_k$) of $SU(2) \times SU(2)$ (see below).

- the only unitary representations are over spaces of complex functions : they are infinite dimensional and each irreducible representation is parametrized by 2 scalars $z \in \mathbb{Z}, k \in \mathbb{R}$ (Maths.1955).

$SU(2)$ as $Spin(3)$ are compact groups, so their unitary representations are reducible (Math.1960) in a sum of orthogonal, finite dimensional, unitary representations. The only irreducible, finite dimensional, unitary, representations, denoted (P^j, D^j) are on the space P^j of degree $2j$ homogeneous polynomials with 2 complex variables z_1, z_2 , where conventionally j is an integer or half an integer. P^j is $2j + 1$ dimensional and the elements of an orthonormal basis are denoted :

$|j, m\rangle = \frac{1}{\sqrt{(j-m)!(j+m)!}} z_1^{j+m} z_2^{j-m}$ with $-j \leq m \leq +j$. And D^j is defined by

$$g \in U(2) : D^j(g) P \left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right) = P \left([g]^{-1} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right)$$

Thus the functions read : $\varphi(z_1, z_2) = \sum_{j \in \frac{1}{2}\mathbb{Z}} \sum_{m=-j}^{m=+j} \varphi^{jm} |j, m\rangle$ with complex constants φ^{jm}

It induces a representation (P^j, d^j) of the Lie algebras where d^j is a differential operator acting on the polynomials P :

$$X \in su(2) : d_j(X)(P)(z_1, z_2) = \frac{d}{dt} P \left([\exp(-tX)] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right) \Big|_{t=0}$$

which gives, for polynomials, another polynomial.

$d_j(X)$ is a linear map on P^j , which is also linear with respect to X , thus it is convenient to define d_j by the action $d_j(\kappa_a)$ of a basis $(\kappa_a)_{a=1}^3$ of the Lie algebra and the three operators are denoted L_x, L_y, L_z . They are expressed in the orthonormal basis $|j, m\rangle$ by square $2j+1$ matrices (depending on the conventions to represent the Lie algebra). The usage is to denote $L_z |j, m\rangle = m |j, m\rangle$.

The only irreducible, unitary, representations of $SO(3)$ are given by (P^j, D^j) with j integer.

5.2.4 Casimir element

The **universal enveloping algebra** U of a Lie algebra is actually a vector space, built from *tensorial powers* of the Lie algebra, and whose basis is given by ordered products of elements of the basis $(\kappa_i)_{i \in I}$ of the Lie algebra (Maths.1692). Universal enveloping algebras are necessary when interacting systems are considered (such as in Chemistry), because their representation involve the tensorial product of the variables.

Any representation (E, f) of the Lie algebra can be extended to a representation (E, F) of its universal enveloping algebra (Maths.1891) where the action is :

$$F \left(\kappa_{i_1}^{n_1} \dots \kappa_{i_p}^{n_p} \right) = f(\kappa_{i_1})^{n_1} \circ \dots \circ f(\kappa_{i_p})^{n_p}$$

When the representation (E, f) comes from a functional representation, in the induced representation on U the action of F is represented by differential operators, of the same order than $n_1 + n_2 + \dots + n_p$.

In the representation of $T_1 Spin(3, 1)$ by matrices of $so(3, 1)$ the universal enveloping algebra is actually an algebra of matrices (see Annex) where the operator j plays a key role.

The **Casimir element** is a special element Ω of U, defined through the Killing form (Maths.1698). In an irreducible representation (E, f) of a semi simple Lie algebra, as $Spin(3, 1)$, the image of the Casimir element acts by a non zero fixed scalar $F(\Omega)u = ku$. In functional representations it acts by a differential operator of second order : $F(\Omega)\varphi(x) = D_2\varphi(x) = k\varphi(x)$: φ is an eigen vector of D_2 . As a consequence, if there is a scalar product on E : $\langle F(\Omega)u, F(\Omega)u \rangle = \langle ku, ku \rangle = k^2 \langle u, u \rangle$. If $(E_1, f_1), (E_2, f_2)$ are two equivalent representations of the same algebra A :

$$\begin{aligned} \exists \phi : E_1 &\rightarrow E_2 \text{ such that :} \\ \forall \kappa \in A : f_1(\kappa) &= \phi^{-1} \circ f_2(\kappa) \circ \phi \end{aligned}$$

$$\begin{aligned}
F_1 \left(\kappa_{i_1}^{n_1} \dots \kappa_{i_p}^{n_p} \right) &= f_1 \left(\kappa_{i_1} \right)^{n_1} \circ \dots \circ f_1 \left(\kappa_{i_p} \right)^{n_p} = \left(\phi^{-1} \circ f_2 \left(\kappa_{i_1} \right) \circ \phi \right)^{n_1} \circ \dots \circ \\
\left(\phi^{-1} \circ f_2 \left(\kappa_{i_p} \right) \circ \phi \right)^{n_p} &= \phi^{-n_1 - \dots - n_p} \circ F_2 \left(\kappa_{i_1}^{n_1} \dots \kappa_{i_p}^{n_p} \right) \circ \phi^{n_1 + \dots + n_p} \\
F_1 \left(\Omega \right) &= \phi^{-n_1 - \dots - n_p} \circ F_2 \left(\kappa_{i_1}^{n_1} \dots \kappa_{i_p}^{n_p} \right) \circ \phi^{n_1 + \dots + n_p} \left(u \right) = \phi^{-n_1 - \dots - n_p} \circ \left(k_2 \phi^{n_1 + \dots + n_p} \left(u \right) \right) = \\
k_2 u &= k_1 u
\end{aligned}$$

Thus the Casimir element acts with the same scalar in all equivalent representations.

The Killing form on $T_1 Spin(3, 1)$ is :

$$B(v(r, w), v(r', w')) = 4(w^t w' - r^t r')$$

thus the elements

$$\begin{aligned}
\kappa_1 &= -\frac{1}{8}\varepsilon_3 \cdot \varepsilon_2, \kappa_2 = -\frac{1}{8}\varepsilon_1 \cdot \varepsilon_3, \kappa_3 = -\frac{1}{8}\varepsilon_2 \cdot \varepsilon_1, \\
\kappa_4 &= \frac{1}{8}\varepsilon_0 \cdot \varepsilon_1, \kappa_5 = \frac{1}{8}\varepsilon_0 \cdot \varepsilon_2, \kappa_6 = \frac{1}{8}\varepsilon_0 \cdot \varepsilon_3
\end{aligned}$$

constitute an orthonormal basis for B and the Casimir element of $U(T_1 Spin(3, 1))$

is :

$$\Omega = \left(\sum_{i=4}^6 (\kappa_i)^2 - \sum_{i=1}^3 (\kappa_i)^2 \right)$$

The action of the Casimir element in the representation $(E, \gamma C)$ of $Spin(3, 1)$

is :

$$F_E(\Omega)u = \left(\sum_{i=4}^6 (\gamma C(\kappa_i))^2 - \sum_{i=1}^3 (\gamma C(\kappa_i))^2 \right) u = \frac{3}{2}u$$

In the representation (P^j, d^j) of $T_1 Spin(3)$, if we denote $L_x = f(\kappa_1), L_y = f(\kappa_2), L_z = f(\kappa_3)$ with 3 arbitrary orthogonal axes :

$$\begin{aligned}
F(\Omega)|j, m\rangle &= L^2|j, m\rangle = (L_x^2 + L_y^2 + L_z^2)|j, m\rangle = j(j+1)|j, m\rangle \\
d^j(\kappa_i) \left(\sum_{m=-j}^{m=+j} X^m |j, m\rangle \right) &= \sum_{m=-j}^{m=+j} X^m d^j(\kappa_i)|j, m\rangle
\end{aligned}$$

5.3 The Spin of a particle

5.3.1 Definition

The space $\mathfrak{X}(P_G[E, \gamma C])$ of sections is a functional representation of $Spin(3, 1)$ with the global action γC and the argument σ . The subspace $\mathfrak{X}(S_0)$ is invariant by the right or left global actions of $Spin(3, 1)$: $\gamma C(\sigma(m))S_0 \rightarrow \gamma C(s)\gamma C(\sigma(m))S_0$ or $\gamma C(\sigma(m))S_0 \rightarrow \gamma C(\sigma((m)))\gamma C(s)S_0$. In particular it is invariant by the action of $Spin(3)$:

$$\rho : \mathfrak{X}(S_0) \times Spin(3) \rightarrow \rho(S(m), s_r) = \gamma C(\sigma(m) \cdot s_r) S_0$$

Moreover the value of $Y(m) = \langle S(m), S(m) \rangle$ is invariant by $Spin(3, 1)$.

The spinor fields $S \in \mathfrak{X}(S_0)$ can equivalently be defined by a couple (S_0, σ) where $\sigma \in \mathfrak{X}(P_G)$. For a given observer each $\sigma(m)$ has two decompositions : $\sigma(m) = \epsilon\sigma_w(m) \cdot \epsilon\sigma_r(m)$ so the couple (S_0, σ) defines precisely two Spatial Spinor fields : $S_r(m) = \gamma C(\sigma_r(m))S_0$.

Conversely one can define Spatial Spinor Fields by a couple : (S_0, σ_r) where $\sigma_r \in \mathfrak{X}(P_R)$ and they constitute a set $\mathfrak{X}_r(S_0)$ which is invariant by $Spin(3)$ (but not by $Spin(3, 1)$).

Let us denote : $\pi_\epsilon : \mathfrak{X}(S_0) \rightarrow \mathfrak{X}_r(S_0)$ the maps which associates, for a given observer, to each Spinor field the Spatial Spinor field with $\epsilon = \pm 1$

On the set : $\mathfrak{X}(S_0)$ we can define the equivalence relation :

$$S \sim S' \Leftrightarrow \pi_\epsilon(S) = \pi_\epsilon(S')$$

Each class of equivalence is the set of spinor fields which have, for the observer, the same kinematic behavior with regard to a rotation by $Spin(3)$. The value of $\pi_\epsilon(S)$ for a given spinor field is **the Spin of the particle**, in its usual meaning. So to any given spinor field corresponds, for an observer, two Spins, with the Spin up or down. And conversely for a given Spin there can be infinitely many spinor fields, defined by a section of the associated bundle $\sigma_w \in \mathfrak{X}(P_W)$.

The projection π_ϵ depends on the choice of a vector ϵ_0 , so the Spin depends on the observer and its measure depends on the spatial basis he has chosen. The spin can be seen as a rotational momentum.

Similarly we have the projection : $\pi_w : Spin(3,1) \rightarrow Spin(3,1)/Spin(3)$ and we can define the equivalence relation in $\mathfrak{X}(S_0)$:

$$S = \gamma C(\sigma) S_0 \sim S' = \gamma C(\sigma') S_0 \Leftrightarrow \pi_w(\sigma) = \pi_w(\sigma')$$

The class of equivalence represents the particles which have the same trajectories. And we define the **translational momentum** by :

$$S_w = \pi_w(\sigma) S_0$$

This is also a geometric quantity, invariant by $Spin(3,1)$, but observer dependant.

5.3.2 Quantization of the Spinor

Theorem 47 *The set $L^1(S_0) = L^1(M, P_G[E, \gamma C], \varpi_4) \cap \mathfrak{X}(S_0)$ of integrable spinor fields associated to a particle is characterized by 2 scalars : $k \in \mathbb{R}, z \in \mathbb{Z}$.*

The Spin, up or down, associated to each section by an observer is characterized by a scalar $j \in \frac{1}{2}\mathbb{N}$ and belongs to a $2j+1$ dimensional vector space : $S_r(m) = \sum_{p=-j}^{+j} S_r^p |j, p\rangle$ with the constant components S_r^p and an orthonormal basis $|j, p\rangle$

Proof. i) The space $L^1(M, P_G[E, \gamma C], \varpi_4)$ is a Fréchet space. The Theorem 2 of [QMR] applies and there are a Hilbert space H and an isometry $\Upsilon : L^1 \rightarrow H :: \psi = \Upsilon(S)$. Moreover $(L^1, \gamma C)$ is an infinitely dimensional representation of $Spin(3,1)$ (the scalar product, thus the norm, is invariant by $Spin(3,1)$, and L^1 is invariant by $Spin(3,1)$). We can apply the theorem 22 of [QMR] : $(H, \hat{\gamma})$ is a unitary representation of $Spin(3,1)$ with $\hat{\gamma}(\sigma) = \Upsilon \circ \gamma C(\sigma) \circ \Upsilon^{-1}$.

ii) Consider the function : $Y : L^1 \rightarrow \mathbb{C} :: Y = \langle S, S \rangle$. For a given section, Y has a value at each point of M and Y is invariant by $Spin(3,1)$. We can implement the theorem 24 of [QMR] to each value y of Y is associated a class of equivalence in L^1 and in H .

If we fix $Y = \langle S_0, S_0 \rangle = Ct$ we have two subsets

$$L^1(S_0) = L^1 \cap \mathfrak{X}(S_0) \text{ in } L^1 \text{ and } H(S_0) \text{ in } H.$$

$$H(S_0) \text{ is invariant by } \hat{\gamma}$$

iii) The unitary representations of $Spin(3,1)$ read : $H = \oplus_{z,k} H_{z,k} \oplus H_c$ where $H_{z,k}$ are unitary irreducible representations, defined by the parameters

$z \in \mathbb{Z}, k \in \mathbb{R}$, and H_c does not contain any irreducible representation, so H_c is not invariant under the action of $Spin(3, 1)$ (Maths.1914).

As a consequence $H(S_0)$ is isomorphic to a subset of one of the irreducible representations $H_{z,k}$ and the spinor field is characterized by two scalars $k \in \mathbb{R}, z \in \mathbb{Z}$ linked to S_0 .

iv) In L^1 , for each section S and a given observer, the Spatial Spinor S_r is a representation of $Spin(3)$. Moreover for S_0, ϵ fixed it belongs to one of the irreducible representations of $Spin(3)$. It is isomorphic to one of the representations (P^j, D^j) with $j \in \frac{1}{2}\mathbb{N}$. These representations are finite dimensional, so S_r belongs to a $2j + 1$ dimensional vector space : $S_r(m) = \sum_{p=-j}^{+j} S_r^p |j, p\rangle$ with the constant components S_r^p . ■

Assume that we study a system comprising of unknown particles $p = 1 \dots N$. The modeling of their kinematic characteristics leads naturally to assume that these particles belong to some spinor field : $S_p \in \mathfrak{X}(P_G[E, \gamma C])$ with unknown inertial spinor. Because no value of S_0 is imposed we have a vector space and we can implement the theorem 2 of [QMR].

What the theorem above tells us is that the solutions must be found in maps : $S_p : \Omega \rightarrow E$ which can be sorted out :

- by the value of $\langle S_0, S_0 \rangle$, that is their mass
- by the value of some integer $z \in \mathbb{Z}$
- and their spin by a half integer $j \in \frac{1}{2}\mathbb{N}$

They correspond to particles which have the same behavior when submitted to a force field (they have the same world lines and spatial spinor for any observer). In other words the spinor is not the only characteristic which determines the behavior of a particle, and these others characteristics can be labeled by a signed integer. This is the starting point to the representation of charged particles.

For elementary particles it is experimentally seen that $j = \frac{1}{2}$, and this is the origin of the name “particles of spin $\frac{1}{2}$ ”. For composite particles or nuclei the spin can be higher.

Even if the set $\mathfrak{X}(S_0)$ is not a vector space, it is a manifold which is embedded in a vector space, so that each of its points (a map S_p) can be written as a fixed linear combination of vectors of a basis. The vector space is always infinite dimensional for the translational momentum, but each spin belongs to a finite dimensional vector space, which is isomorphic to some (P^j, D^j) : $S_r(m) = \sum_{p=-j}^{+j} S_r^p |j, p\rangle$ where S_r^p are fixed scalars and $|j, p\rangle$ are, for a given system, fixed maps $|j, p\rangle : \Omega \rightarrow E_0$, images of vectors of the basis of P^j by some isometry. Each vector $|j, p\rangle$ is assimilated to a state of the particle, and j, p are the quantum numbers labeling the state. The maps $|j, p\rangle$ are not polynomials (as in P^j), they are used only to define the algebraic structure of the space $H(S_0)$, however they have an interpretation for models of atoms (see below). Under the action of $Spin(3)$ the vectors $S_r(m)$ transform according to the same matrices as in D^j :

$\gamma C(\sigma(m) \cdot s_r) S_0 = \sum_{p=-j}^{+j} S_r^p [D^j(f(s_r))] |j, p\rangle$ where $f(s_r)$ is the image of σ in $SU(2)$.

By itself the theorem does not provide a solution : a vector of a basis of the vector spaces is itself some map $E_i : \Omega \rightarrow E$. But it shows that the solution cannot take any value, even before we implement any physical law relating the fields and the kinematic characteristics. In a given system the solutions that appear follow the same pattern, whatever the initial conditions, or the value of the other variables (notably the fields).

There is one important difference in the behavior of the spin, according to the value of j . The Spin is invariant by a rotation by $Spin(3)$, and the scalars $\pm 1 \in Spin(3)$. The actions of $+s$ and $-s$ give opposite results. $Spin(3)$ is the double cover of $SO(3)$: to the same element g of $SO(3)$ are associated two elements $\pm s$ of $Spin(3)$. The representations (P^j, D_j) with $j \in \mathbb{N}$ are also representations of $SO(3)$. It implies that the vector spaces are invariant by $\pm s$. The fact that j is an integer means that *the particle has a physical specific symmetry* : the rotations $\pm s$ give the same result. And equivalently, if j is half an integer the rotations by $\pm s$ give opposite results.

5.3.3 Measure of the spatial spin of a particle

A particle has, whatever the scale, by definition, no internal structure, so it is impossible to observe its geometric rotation. However it has a spin, its spatial spinor S_r is a variable which can be represented in a finite dimensional space : S_r is an observable. The measure of the spatial spinor, similar to a rotational momentum, is done by observing the behavior of the particle when it is submitted to a force field which acts differently according to the value of the spinor. This is similar to the measure of the rotation of a perfectly symmetric ball by observing its trajectory when it is submitted to a dissymmetric initial impulsions (golfers will understand).

Most particles have a magnetic moment, linked to their spin. So the usual way to measure the latter is to submit the particles to a non homogeneous magnetic field. This is the Stern-Gerlach analyzer described in all handbooks, where particles have different trajectories according to their magnetic moment. MRI uses a method based on the same principle with oscillating fields whose variation is measured. The process can be modelled as follows.

The spinors of the particles are represented by some section $\mathcal{S} \in \mathfrak{X}(S_0)$. The device operates only on the spin : $S_r(m) = \gamma C(\sigma_r(m)) S_0$ and is parametrized by a spatial rotation $s_r \in Spin(3)$, and usually by a vector $\rho \in \mathbb{R}^3$, corresponding to a rotation s_r .

The first effect is a breakdown of symmetry : s_r has not the same impact for the particles with spin up or down. This manifests by two separate beams in the Stern-Gerlach experiment.

An observable $\Phi(S_r)$ of S_r is a projection on some finite dimensional vector space of maps (see [QMR]). Because of the quantization, this vector spaces has for vectors $|j, p\rangle$ which are fixed maps, image of the vectors of basis of P^j which are eigen vectors of the observable. The action of the device can be modelled as an operator $L(\rho)$ acting on this space, and the matrices to go from one orientation ρ_1 to another ρ_2 are the same as in (P^j, d^j) . It reads :

$$L(\rho) \Phi(S_r) = \sum_{p=-j}^{+j} S_r^p [d^j(\rho)] |j, p\rangle$$

For a given beam we have a breakdown of the measures, corresponding to each of the states labelled by p.

Arbitrary axes x, y, z are chosen for the device, which provide 3 measures $L_x(S_r), L_y(S_r), L_z(S_r)$, such that $L_z(S_r) |j, m\rangle = m |j, m\rangle$.

The Casimir operator Ω is such that $L^2\Phi(S_r) = (L_x^2 + L_y^2 + L_z^2)(S_r) = j(j+1)\Phi(S_r)$

5.3.4 Atoms and electrons

QM has been developed from the study of atoms, with a basic model (Bohr's atom) in which electrons move around the nucleus. Even if this idea still holds, and this is how atoms are commonly viewed, it had been quickly obvious that a classic model does not work. However using what has been developed previously, we can have another representation.

Let us consider a system comprised of one electron moving around a nucleus. If we consider the atom as a particle, that is without considering its internal structure, its relativist momentum can be represented by a spinor S , and its rotational momentum by a spin S_r . The previous results hold and the spin can be represented in a finite dimensional vector space isomorphic to P_j . However j , which belongs to $\frac{1}{2}\mathbb{N}$, is not necessarily equal to $\frac{1}{2}$.

As noticed before, the polynomials P^j have no physical meaning. However in this case it is usual to provide one. By a purely mathematical computation it is possible to show that the representation (P^j, D^j) is equivalent to a representation on square integrable functions $f(x)$ on \mathbb{R}^3 , and from there on spherical harmonic polynomials (Maths.1958). It is then assumed that the arguments of the function $f(x)$ are related to the coordinates (in an euclidean frame) of the electron. This is a legacy of the first models of atoms. Actually there is no need for such an assumption to build a consistent model, which would be useless in the GR context, and the image of electrons rotating around a nucleus and spinning has no physical support.

For atoms with several electrons, the model must involve the tensorial products of each spinor. The previous representations of $SU(2)$ are then extended to the tensorial products of P^j , and their derivative to representations of the universal enveloping algebra. It is often possible to rearrange these representations, by combinations using Clebsch-Jordan coefficients (Maths.1960), and in this endeavour the spherical harmonic polynomials are useful because they provide many identities. This is one major application of QM in Chemistry.

The same kind of model is used for composite particles in Quantum Theory of Fields.

5.4 Material bodies and spinors

5.4.1 Representation of a material body by sections of P_G

We have seen in the previous chapter that a material body B can be defined, from a geometric point of view, by a vector field u whose integral curves are the world lines of its particles. Then the flow $\Phi_u(\tau, a)$ defines the body B itself at each proper time τ as a compact subset $\omega(\tau)$ of a 3 dimensional hypersurface. And there are privileged observers B for whom $\omega(0) \subset \Omega(0)$.

So a material body can be defined with respect to these observers, up to a constant ± 1 by a section σ_w of P_W or, up to a spatial spinor, by a section of P_G and a compact, space like hypersurface $\omega(0)$. Then σ_w provides u , Φ_u and $\omega(0)$ defines $\omega(\tau)$. The section σ_w can be seen as the general definition of B , which can be fitted to any initial conditions $\omega(0)$. This is the most efficient way to define geometrically a material body in physical models.

5.4.2 Spinors representing a solid

We have seen that the usual concepts of motion of a body over itself (usually a rotation of the body) cannot be easily represented in relativist geometry. This is the main motivation for the introduction of spinors, and any material body whose internal structure can be neglected (at the scale of the study) can be represented, from the kinematic point of view, by a spinor which accounts for its rotation (through the spin). If the location of the material body can be represented by a geometric point, then the kinematic representation of B is given by a map : $S_B : \mathbb{R} \rightarrow P_G[E, \gamma C]$, such that $S_B(t) = \gamma C(\sigma(t)) S_0$. We do not need more : S_B provides everything, including the rotational momentum. Thus, even if no internal structure or rotation of the body is assumed, eventually it can be accounted for.

However this representation assumes that S_0 is known. As in Classic Mechanics for the inertial tensor, the computation of the inertial spinor S_0 is, for a solid, a separate issue. It can be done through the aggregation of material points (particles) with a specific law giving the shape and the density of the body. And the inertial spinor is not necessarily constant : we can consider deformable solids. Actually we can define a *rigid solid as a material body such that S_0 is constant*.

Proposition 48 *A deformable solid body can be represented by a map :*

$$S : \mathbb{R} \rightarrow P_G[E, \gamma C] \text{ such that } \langle S(\tau), S(\tau) \rangle > 0 \text{ or } \langle S(\tau), S(\tau) \rangle < 0$$

A rigid solid body can be represented by a map :

$$S : \mathbb{R} \rightarrow P_G[E, \gamma C] :: S(\tau) = \gamma C(\sigma(\tau)) S_0 \text{ for a fixed } S_0 \in E_0 \text{ or } S_0 \in E'_0$$

where τ is the proper time of the body

To assume that the material points behave in a coherent way in a solid assumes that there are forces which assure this cohesion. And indeed a material body can be deformed or broken. So we can say that the fact, assumed and which can be checked, that a material body can be represented by a unique spinor incorporates the existence of these internal forces. And ultimately the break of a material body can result in several spinors. So in modelling the evolution of a material body we should include additional assumptions about the laws (which are similar to the phenomenological laws for deformable solids) for the change of S_0 . And in a discontinuous process add the laws which rules the splitting in different spinors.

5.4.3 Aggregating matter fields

With these definitions we can consider the task to compute the spinor that we will denote S_B , for a deformable solid, by aggregating material points. This is similar to the computation of the inertial tensor in Classic Mechanics : this is a specific endeavour, done in a separate model, using specific assumptions (about the shape, density, motion of the particles) and the result is then used in a more general model (for instance to compute the motion of different bodies). The single spinor corresponding to the whole body is assigned, in the more general model, to any point : all the material points have then the same location.

The first issue is the definition of the motion of the material points with respect to the body. We need a chart to do it, which is given by an observer B , such that at his proper time $t = 0$ the set $\omega(0)$ is in his present $\Omega(0)$. Then at any given time t the set of particles constituting the solid stays in his present. B uses his standard chart :

$\varphi_B(t, \eta^1, \eta^2, \eta^3) = \Phi_{\varepsilon_0}(t, x(\eta^1, \eta^2, \eta^3))$ where $x(\eta^1, \eta^2, \eta^3)$ is a chart on $\omega(0)$ and ε_0 his time like vector field

The particles follow the trajectories given by a vector field V and their location at t is $\Phi_V(t, a) = \Phi_{\varepsilon_0}(t, x(t)) = \varphi_B(t, x(t))$ with $x(0) = a$.

$\omega(t) = \{\Phi_V(t, a), a \in \omega(0)\} = \{\Phi_{\varepsilon_0}(t, x(t)), x(0) \in \omega(0)\}$ represents the location of the body at t and $\Omega = \{\Omega(t), t \in [0, T]\}$

The material points are represented by a section $S(t, x) = \gamma C(\sigma(t, x)) S_0 \in \mathfrak{X}(S_0)$. The choice of S_0 can be arbitrary. S is a geometric quantity which does not depend on a chart, however σ_w provides a vector field of world lines u for the material points with respect to ε_0 .

We assume that the observer defines a tetrad $(\varepsilon_i(m))$ from which the metric and the volume form are deduced in the usual way.

The density $\mu(t, x)$ is defined over Ω with respect to the volume form ϖ_4 . Because $\omega(t) \subset \Omega(t)$ the unitary, future oriented, normal to $\omega(t)$ is $\varepsilon_0 = \partial\xi_0$ and μ induces a density $\mu_3(t, x)$ over $\omega(t)$ with respect to the volume form ϖ_3 :

$\mu_3(t, x) \varpi_3(t, x) = i_V(\mu(t, x) \varpi_4(t, x))$ which is the flux of matter going through $\omega(t)$.

As noticed before the holonomic basis $e_i(m) = (\mathbf{p}(m), \varepsilon_i)$ of $P_G[E, \gamma C]$ is

arbitrary, in that there is no physical reference for the choice of the vectors e_i . We can assume that, for a deformable solid, there is a common basis associated to the chart Φ_{ε_0} .

$$S(t, x) = \gamma C(\sigma(t, x)) S_0$$

Then the integral :

$$\int_{\omega(t)} \gamma C(\sigma(t, x)) \mu_3(t, x) S_0 \varpi_3(t, x) \\ = \left[\gamma C \left(\int_{\omega(t)} (\sigma(t, x)) \mu_3(\tau, x) \varpi_3(t, x) \right) \right] S_0$$

is well defined on the fixed vector space \mathbb{E} .

$$\sigma(t, x) = a(t, x) + v(r(t, x), w(t, x)) + b(t, x) \varepsilon_5$$

with the identities :

$$a(t, x)^2 - b(t, x)^2 = 1 + \frac{1}{4}(w^t w - r^t r)$$

$$a(t, x) b(t, x) = -\frac{1}{4} w^t r$$

Denote

$$\hat{r}(t) = \int_{\omega(t)} \mu_3(x, t) r(t, x) \varpi_3(t, x), \hat{w}(t) = \int_{\omega(t)} \mu_3(x, t) w(t, x) \varpi_3(t, x)$$

$$\hat{a}(t) = \int_{\omega(t)} a(t, x) \mu_3(t, x) \varpi_3(t, x), \hat{b}(t) = \int_{\omega(t)} b(t, x) \mu_3(t, x) \varpi_3(t, x)$$

$$\int_{\omega(t)} \mu_3(x, t) \sigma(t, x) \varpi_3(t, x) = \hat{a}(t) + v(\hat{r}(t), \hat{w}(t)) + \varepsilon_5 \hat{b}(t)$$

We impose, for a deformable solid, that :

$$\exists N(t) \in \mathbb{R}, R(t), W(t) \in \mathbb{R}^3 :$$

$$\hat{a}(t) + v(\hat{r}(t), \hat{w}(t)) + \varepsilon_5 \hat{b}(t) = N(t) (A(t) + v(R(t), W(t)) + B(t) \varepsilon_5) \quad (89)$$

such that $\sigma_B(t) = A(t) + v(R(t), W(t)) + B(t) \varepsilon_5 \in Spin(3, 1)$ which requires :

$$A^2 - B^2 = 1 + \frac{1}{4}(W^t W - R^t R)$$

$$AB = -\frac{1}{4} W^t R$$

and implies :

$$R(t) = \frac{1}{N} \hat{r}(t), W(t) = \frac{1}{N} \hat{w}(t)$$

$$A(t) = \frac{1}{N} \hat{a}(t), B(t) = \frac{1}{N} \hat{b}(t)$$

Which sums up to the two conditions :

$$\hat{a}(t) \hat{b}(t) = -\frac{1}{4} \hat{w}^t \hat{r} \quad (90)$$

$$\hat{a}^2 - \hat{b}^2 = N^2 + \frac{1}{4} (\hat{w}^t \hat{w} - \hat{r}^t \hat{r}) \quad (91)$$

$$\Rightarrow N^2 = \hat{a}^2 - \hat{b}^2 - \frac{1}{4} (\hat{w}^t \hat{w} - \hat{r}^t \hat{r}) > 0$$

Then the Spinor of the body is : $S_B(t) = N(t) \gamma C(\sigma_B(t)) S_0$

The conditions can be seen as resulting from the forces which keep the cohesion of the body.

The mass of the solid is proportional to

$$\langle S_B(t), S_B(t) \rangle = N^2(t) \langle S_0, S_0 \rangle = \left(\hat{a}^2 - \hat{b}^2 - \frac{1}{4} (\hat{w}^t \hat{w} - \hat{r}^t \hat{r}) \right) \langle S_0, S_0 \rangle$$

and is not necessarily constant. So we may impose the additional condition

:

$$\frac{d}{dt} \left(\widehat{a}^2 - \widehat{b}^2 - \frac{1}{4} (\widehat{w}^t \widehat{w} - \widehat{r}^t \widehat{r}) \right) = 0 \Leftrightarrow \widehat{a} \frac{d\widehat{a}}{dt} - \widehat{b} \frac{d\widehat{b}}{dt} - \frac{1}{4} (\widehat{w}^t \frac{d\widehat{w}}{dt} - \widehat{r}^t \frac{d\widehat{r}}{dt}) = 0$$

See below continuity equation.

In this aggregation the section σ represents the individual motion of the constituting material points, with respect to a gauge attached to the solid. The element $\sigma_B(t)$ represents the average motion of these points with respect to the gauge of the observer B in the computation of σ_B . The motion of the solid itself, with respect to the gauge of an observer O (in a different, more general model), is represented by an element $\sigma_o \in P_G$. The total motion (solid + solid on itself) is defined by a change of gauge in P_G and the resulting spinor (as it would be used in a model representing the solid) is then :

$$S(t) = N(t) \gamma C(\sigma_o(t)) \gamma C(\sigma_B(t)) S_0$$

Which sums up to replace the fixed inertial spinor S_0 by the variable spinor $S_B(t) = N(t) \gamma C(\sigma_B(t)) S_0$.

The physical meaning of σ_o must be understood with respect to the way the solid is defined : for instance if $\sigma(t, x)$ represents a rotation around an axis, then σ_o will be a rotation of this axis. The vector $r \in \mathbb{R}^3$ in σ_r , which has no geometric meaning for a particle, gets one for a solid, similar to the usual.

5.4.4 Continuity equation

The conservation of the mass of the body means, for the observer B, that :

$$\mathcal{M}(t) = \int_{\omega(t)} \mu_3(t, x) \varpi_3 = Ct = \int_{\omega(t)} i_V(\mu \varpi_4)$$

Consider the manifold $\omega([t_1, t_2])$ with borders $\omega(t_1), \omega(t_2)$:

$$\mathcal{M}(t_2) - \mathcal{M}(t_1) = \int_{\partial\omega([t_1, t_2])} i_V(\mu \varpi_4) = \int_{\omega([t_1, t_2])} d(i_V \mu \varpi_4)$$

$$d(i_V \mu \varpi_4) = \mathcal{L}_V(\mu \varpi_4) - i_V d(\mu \varpi_4) = \mathcal{L}_V(\mu \varpi_4) - i_V(d\mu \wedge \varpi_4) - i_V \mu d\varpi_4 = \mathcal{L}_V(\mu \varpi_4)$$

$$\mathcal{E}(t_2) - \mathcal{E}(t_1) = \int_{\omega([t_1, t_2])} \mathcal{L}_V(\mu \varpi_4)$$

with the Lie derivative \mathcal{L} (Maths.1517,1587)

The conservation of the mass is equivalent to the condition $\mathcal{L}_V(\mu \varpi_4) = 0$.

$$\begin{aligned} & \mathcal{L}_V \mu \varpi_4 \\ &= \frac{d\mu}{dt} \varpi_4 + \mu \mathcal{L}_V \varpi_4 \\ &= \frac{d\mu}{dt} \varpi_4 + \mu (\text{div} V) \varpi_4 \\ &= \frac{d\mu}{dt} + \mu (\text{div} V) \varpi_4 \end{aligned}$$

and we retrieve the usual continuity equation :

$$\frac{d\mu}{dt} + \mu \text{div} V = 0 \tag{92}$$

Then $N(t) = \int_{\omega(t)} \mu_3(t, x) \varpi_3$

If $\mu = Ct$ (incompressible solid) the condition becomes : $\text{div} V = 0$

5.4.5 Symmetries of a solid

By symmetries we mean symmetries of the whole body B : under a geometric transformation the body looks the same for an observer. So they are transformations occuring in each $\omega(t)$ and for a privileged observer who can see the

whole body. It is equivalent to consider either the transformation of the body or the transformation of the observer (as long as he keeps the same vector field ε_0), thus symmetries can be represented as a global change of observer with an element $s(t)$ belonging to a subgroup of $Spin(3)$. And B has a symmetry if the section $S(t, x) = \gamma C(\sigma(t, x)) S_0 \in \mathfrak{X}(S_0)$ is such that $\sigma(t, x) = \sigma_w(t, x) \cdot s(t)$. Then :

$$\begin{aligned} \sigma(t, x) &= \sigma_w(t, x) \cdot s(t) \\ &= (a_w(t, x) + v(0, w(t, x))) \cdot (a_r(t) + v(r(t), 0)) \\ &= a_w a_r + v(a_w r, \frac{1}{2} j(w) r + a_r w) - \frac{1}{4} (w^t r) \varepsilon_5 \\ &\text{and } \sigma_B(t) = N(t) \sigma_{Bw}(t) \cdot s(t) \\ &\text{with } \sigma_{Bw}(t) = \frac{1}{N(t)} \int_{\omega(t)} \sigma_w(t, x) \mu_3(t, x) \varpi_3(t, x) \text{ and the condition } \widehat{a}^2 - \\ \widehat{b}^2 &= N^2 + \frac{1}{4} (\widehat{w}^t \widehat{w} - \widehat{r}^t \widehat{r}). \end{aligned}$$

This approach can be useful in Astrophysics, where trajectories of stars systems or galaxies are studied. The spinor S_B can account for the rotational momentum of the bodies, which is significant and contributes to the total kinetic energy of the system.

5.5 Relativist Momentum, Spin and Energy

To sum up :

The Spinor S is the relativist momentum of the particle. Encompassing both the translational and the rotational motions, it is intrinsic and does not depend on the choice of a basis, and its measure $S = \gamma C(\sigma) S_0$ depends on the observer. This is the equivalent of the 4 vector $P = \mu u$, but it adds the rotational component.

The quantity $M_p = \frac{1}{c^2} \sqrt{|\langle S_0, S_0 \rangle_E|} = \frac{1}{c^2} \sqrt{|\langle S, S \rangle_E|}$ is the mass at rest of the particle. It does not depend on the motion, the gauge or the chart.

The decomposition of the relativist momentum by an observer shows :

- the relativist momentum $\epsilon = \pm 1$, spin up or down, with respect to the trajectory

- the translational momentum $\gamma C(\sigma_w) S_0$ with $\sigma_w = a_w + v(0, w)$. With respect to a basis the vector w can be identified with a spatial speed \vec{v} by the formulas 18. And the usual translational momentum is : $\vec{p}_r = M_p \frac{\vec{v}}{\sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}}}$

- the spin $\gamma C(\sigma_r) S_0$ with $\sigma_r = a_r + v(r, 0)$. In a basis the vector r can be assimilated with a rotation of axis $\vec{r} = \sum_{i=1}^3 r_i \varepsilon_i(m)$ and rotational speed $\sqrt{\langle \vec{r}, \vec{r} \rangle_3}$. But this assimilation is formal. A particle, by definition, has no internal structure, so it should look the same after any rotation around its position. The Spin is not a geometric rotation, and because of that in the measure of the Spin the choice of the axis x, y, z does not matter, even if the measures change as if it was a rotation.

The **kinematic energy of a particle** can be defined as $\frac{1}{i} \langle S, \frac{dS}{dt} \rangle$ (the $\frac{1}{i}$ factor accounts for the fact that $\langle S, \frac{dS}{dt} \rangle$ is imaginary because $\frac{d}{dt} \langle S, S \rangle = 0$). Its measure depends on the observer, through the choice of t . The intrinsic quantity would be $\frac{1}{i} \langle S, \frac{dS}{d\tau} \rangle$ with the proper time τ , and $\frac{1}{i} \langle S, \frac{dS}{dt} \rangle = \left(\sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}} \right) \frac{1}{i} \langle S, \frac{dS}{d\tau} \rangle$. It can be expressed with (r, w) and their derivatives⁴ and we will see that the result is :

$$\frac{1}{i} \left\langle S, \frac{dS}{dt} \right\rangle = k^t X$$

where

X is the vector :

$$X = [A_r(r)] \frac{dr}{d\tau} + [A_w(r, w)] \frac{dw}{d\tau}$$

with $[A_r(r)], [A_w(r, w)]$ matrices depending on r, w ,

and k is a vector, similar to the inertia tensor, which encompasses both the translational and the rotational motions, and acts also as a magnetic moment.

Its value is :

$$k = -\epsilon \frac{1}{4} M_p^2 c^4 \begin{bmatrix} (\sin 2\alpha_0) \cos(\alpha_2 - \alpha_1) \\ (\sin 2\alpha_0) \sin(\alpha_2 - \alpha_1) \\ 2 \cos 2\alpha_0 \end{bmatrix}$$

with

$$S_R = \frac{M_p c^2}{\sqrt{2}} \begin{bmatrix} e^{i\alpha_1} \cos \alpha_0 \\ e^{i\alpha_2} \sin \alpha_0 \end{bmatrix} \text{ and } \epsilon = -1 \text{ for antiparticles.}$$

The representation of the kinematic characteristics of a deformable solid has been seen before. For an object which has an internal structure this is done by tensorial products of spinors.

⁴We can expect that the kinematic energy has the dimension $[\text{Mass}][\text{Speed}]^2$. And we must remember that r, w have themselves the dimension of speed. So the definition is consistent.

6 GRAVITATIONAL FIELD

The concept of fields has appeared in the XIX^o century, in the wake of the electromagnetism theory, to replace the picture of action at a distance between particles. A force field is one object of Physics, which has distinctive properties :

i) Because particles are localized, a field must be able to act anywhere, that is to be present everywhere. So the first feature of force fields, as opposed to particles, is that, a priori, they are defined all over the universe, even if their action can decrease quickly with the distance.

ii) A force field propagates : the value of the field depends on the location, this propagation occurs when there is no particle, thus it is assumed that it results from the interaction of the force fields with themselves.

iii) Force fields interact with particles, which are themselves seen as the source of the fields. This interaction depend on charges which are carried by the particles.

iv) The interactions, of the fields with themselves or with particles are, in continuous processes, represented in the lagrangian according to the Principle of Least Action.

v) In some cases the force fields can act in discontinuous processes, in which they can be represented as particles (bosons and gravitons).

Thus we need a representation of the charges and of the fields. The situation is totally different for the gravitational field and the other fields.

The only, generally accepted, model of the gravitational field is part of General Relativity, actually distinct from its theory on Geometry that we have seen previously. Its main assumption is the Principle of Equivalence, which states that the gravitational charge is equal to the inertial mass. Using the Levy-Civita connection deduced from the metric, its key variable is the scalar curvature, from which, with the Principle of Least Action, can be deduced the Einstein equation, which is still the keystone in the study of gravitation. There has been many variants of the model, but the focus on the metric and the curvature has eclipsed the study of the gravitational field itself, and there is even no clear consensus about if the gravitational field can be distinguished from the geometry.

The Standard Model, built in the SR framework, encompasses all force fields other than gravitation. It is based on the formalism of gauge fields : the key component is a connection on a principal bundle, which acts on the state of particles through a covariant derivative. The state of particles is itself represented in an associated vector bundle, whose fiber is the tensor product $E \otimes F$ where E represents the spinor and F the charges. It should suffice to discriminate the behavior of particles submitted to a field : we have seen that spinor fields are characterized by a scalar z , which corresponds to a charge. Actually there is no unit for the charge in the Standard Model : elementary particles, distinguished according to their flavor and charges with the 3 forces (electromagnetic, weak

and strong interactions), constitute a basis of F , and the behavior of a particle is compared to the behavior of known particles. The model introduces another group (the direct product $U(1) \times SU(2) \times SU(3)$), which is a compact real Lie group, the vector space F is a unitary representation of the group, and from there are built a principal bundle, where lives the connection, and an associated vector bundle for $E \otimes F$.

There has been many attempts to incorporate all force fields in a single model : this is the Holy Graal of contemporary Physics. It would be impossible to list all of them. Anyway the purpose of this paper is not to propose another theory, but study how the formalism of fiber bundle and gauge fields can be used, in the context of the geometry of GR, to represent the gravitational field. We will see first how the concept of spinor can be matched with the principle of equivalence. Then we will introduce the general concept of connection for the gravitational field and see its relation the Levy-Civita connection. The propagation of fields is studied through a new variable, the strength of the field, and we will see how it relates to the Riemann tensor and the scalar curvature. It is the main tool to understand and quantize the gravitational field, notably to define its spin.

6.1 The Law of Equivalence

In the Newton's law of gravitation $F = G \frac{MM'}{r^2}$ and his law of Mechanics : $F = \mu\gamma$ the scalars M, μ represent respectively the gravitational charge and the inertial mass, and there is no reason why they should be equal. However this fact has been verified with great accuracy (two bodies fall in the vacuum at the same speed). This has lead Einstein to state the fundamental Law of Equivalence "Gravitational charge and inertial mass are identical". From which he built the Theory of General Relativity.

We have defined the relativist momentum of a particle as the spinor, extended to a deformable solid. This Law leads us to take as gravitational charge of particles the inertial spinor S_0 .

Proposition 49 *The Gravitational charge of a particle is represented by its inertial spinor S_0*

So, if we stay only with the gravitational field, the space E and the representation $(E, \gamma C)$ suffice to represent the state of particles. The kinematic characteristics of particles of the same flavor (quarks, leptons) are not differentiated according to their other charges.

The gravitational charge is no longer a single scalar, but can be expressed in the same unit as the mass. It is consistent with the fact that the spinor encompasses the rotational momentum.

However this assumption has another consequence. In the previous chapter we assumed that :

- there is, along the world line of a particle, a privileged frame $\varphi_G(m, \sigma(m))$ such that the spinor of the particle is $(\varphi_G(m, \sigma(m)), S_0)$ with $S_0 = Ct$

- the observer measures the spinor $S(m)$ in his gauge : $\varphi_G(m, 1)$ and $(\varphi_G(m, \sigma(m)), S_0) \sim (\varphi_G(m, 1), \gamma C(\sigma(m)) S_0) = (\varphi_G(m, 1), S(m))$ thus : $S(m) = \gamma C(\sigma(m)) S_0$

We have now to consider an interpretation which is mathematically equivalent, but physically different :

- the observer measures the spinor $S(m)$ with $(\varphi_G(m, 1), \gamma C(\sigma(m)) S_0)$
- in presence of gravity this spinor is equivalent to : $(\varphi_G(m, \sigma(m)), S_0)$

The privileged gauge (for the particle) is provided by the gravitational field. And *the action of the motion, that is of the inertial forces, is equivalent to the action of gravity on the state of the particle*, which is the meaning of the Law of Equivalence. This is a key point to understand the gravitational and the other fields : *particles have intrinsic properties, that they keep all over their travel on their world lines but, because of the existence of the field, their measure by an observer is distinct from this intrinsic value.* This leads to see the fields as the value of the element of the group ($\sigma \in Spin(3, 1)$ for the gravitational field) but, as we will see, the action of the field goes through a special derivative because it manifests itself in the motion of the particle.

6.2 Representation of the charges for the other fields

For the other fields, in line with the Standard Model, the assumption are the following :

Proposition 50 *There is a compact, connected, real Lie group U which characterizes the force fields other than gravitation.*

There is a n dimensional complex vector space F , endowed with a scalar product denoted $\langle \rangle_F$ and (F, ϱ) is a unitary representation of U

The states of particles are vectors of the tensorial product $E \otimes F$

The intrinsic characteristics of each type of particles are represented by a fundamental state $\psi_0 \in E \otimes F$, and all particles sharing the same characteristics behave identically under the actions of all the fields.

In the Standard Model $U = SU(3) \times SU(2) \times U(1)$. There is a principal fiber bundle Q with fiber $Spin(3, 1) \times U$ and the state of a particle is represented by a tensor ψ of the associated vector bundle $Q[E \otimes F, \gamma C \times \varrho]$. We will not develop this model here (for more see [MP]), however it is useful to see how it works for the electromagnetic-field (EM).

It is represented by the group $U(1)$, the set of complex numbers with module 1 ($uu^* = 1$). This is a compact abelian group. Its irreducible representations are unidimensional, that is multiple of a given vector.

For any given arbitrary vector f there are 3 possible irreducible non equivalent representations :

- the standard one : $(F, \varrho) : \varrho(e^{i\phi}) f = e^{i\phi} f$ and $F = \{e^{i\phi} f, \phi \in \mathbb{R}\}$

- the contragredient : $(F, \bar{\varrho}) : \bar{\varrho}(e^{i\phi}) f = e^{-i\phi} f$ and $F = \{e^{i\phi} f, \phi \in \mathbb{R}\}$

(Maths.23.1.2)

- the trivial representation : $(F, \varrho) : \varrho(e^{i\phi})f = f$ and $F = \{f\}$

The standard representation corresponds to negative charge, the contragredient representation to positive charge and the trivial one to neutral charge (the only known are the neutrinos). The choice positive / negative is arbitrary.

The EM field interacts similarly with the left and right part of a spinor, so the space of states of the particles is the sum of tensorial products : $S \otimes f$. The theory can be fully expressed this way. However it is legitimate to choose the vectors f in E , which is a 4 dimensional complex vector space⁵. For elementary particles, then :

i) Let $\{S_p \in E_0, p = 1...N\}$ be N vectors representing inertial spinors of particles. Then for each of them their states are represented by $\{e^{i\phi}S_p, \phi \in \mathbb{R}\}$ with the standard representation;

ii) Let $\{S'_p \in E'_0, p = 1...N\}$ be N vectors representing inertial spinors of the antiparticles associated to S_p . Because E_0, E'_0 are orthogonal these vectors are orthogonal to the S_p . Then for each of them their states are represented by $\{e^{i\phi}S'_p, \phi \in \mathbb{R}\}$ with the contragredient representation;

iii) Neutral particles $\{S_q \in E_0 \oplus E'_0, q = 1...N'\}$ correspond to the trivial representation : their states is just one vector $S_q \in E_0$ or E'_0 .

Particles have a charge opposite to their anti-particle. Notice that the maps $\varrho, \bar{\varrho}$ are distinct from $\gamma C(\sigma)$ for any $\sigma \in Spin, T_1Spin$, so the state of a particle can change only by the action of $U(1)$.

A basis of E , for elementary particles, is then $\{(S_p, S'_p)_{p=1}^N, (S_q)_{q=1}^{N'}\}$. Each vector defines the mass, the charge and the type of the particle. In this picture there is no unit for the electric charge.

In the case of the EM field the structure brought by the charges is then built in the space E_0 or E'_0 . The state of an elementary particle is then :

$$\psi = e^{i\phi}S \tag{93}$$

where ϕ is a phase factor which must be considered as variable for charged particles, and $\phi = 0$ for neutral particles. Usually this phase can be ignored : particles whose states differ by a phase factor have the same behavior, with regard to the electromagnetic field and they have the same mass : $\langle (\exp i\phi)S_0, (\exp i\phi')S_0 \rangle = \langle S_0, S_0 \rangle$ and the same EM charge.

This is the origin of the introduction of rays in QM. Two particles such that their states differ by a phase factor $e^{i\phi}$ behave the same way, for the gravitational field or the EM field, so they can be deemed representing the same state.

When only the EM and the gravitational fields are involved the states of elementary particles can be represented in E . Moreover, using a classic theorem of QM, the state of a composite particle, comprised of several particles, should be represented by the tensorial product of the states, thus in some space $E \otimes E \dots \otimes E$. Mathematically the tensorial product of non equivalent representations is well defined. The action of $U(1)$ on the tensor $S_1 \otimes S_2$ is $e^{i\phi(\epsilon_1 + \epsilon_2)}S_1 \otimes S_2$ with $\epsilon_k = \pm 1$ depending on the representation. However a basis of the tensorial

⁵This is one of the motivation to choose E as a complex vector space to represent spinors.

product is comprised of tensorial products of all the vectors of the basis, which would not have the same behavior under the action of $U(1)$ (the basis of E has positive, negative and neutral particles). So actually the only combinations which are acceptable are made of particles of the same kind (positive, negative or neutral) and the action is then $e^{iq\phi\epsilon}$ where q is the number of particles, where q can be a positive, a negative or a null integer. As a consequence :

- when only the EM field is involved composite particles are comprised of particles with the same type of charge (this does not hold when the weak and strong interactions are considered)

- the electric charge of particles must be an integer multiple of an elementary charge.

Such tensorial products of spinors can be used for nuclei, atoms or molecules. The associated EM charge is an integer multiple of the elementary charge. Of course this does not matter for neutral particles as they do not interact with the EM field.

When only the gravitational field is involved the states of particles or material bodies can be represented either by a tensorial product of spinors, or by a single spinor in E .

7 CONNECTIONS

The action of a field on a particle goes through the derivative of a momentum. Because of the anisotropy of the universe, the value of the derivative will depend on the direction on M , represented by a vector, so we are looking for a map : $M \rightarrow \Lambda_1(T_1Spin(3,1))$, that is a one form valued in the Lie Algebra. This derivative is the **covariant derivative**. The derivative $\frac{d\vec{p}}{dt} = \vec{F}$ is replaced by the covariant derivative $\nabla_V S$ of the relativist momentum along the direction V .

Covariant derivatives are built from more general mathematical objects, called connections.

7.1 Connections on Fiber bundles

A fiber bundle $P(M, V, \pi)$ is a manifold, and its tangent space is split in two parts, related to its two manifolds components. By differentiation of the trivialization :

$$\begin{aligned} \varphi : M \times V &\rightarrow P :: p = \varphi(m, u) \\ \varphi' : T_m M \times T_u V &\rightarrow T_p P :: v_p = \varphi'_m(m, g) v_m + \varphi'_u(m, g) v_u \\ \pi(p) = m &\Rightarrow \pi'(p) v_p = v_m \text{ and the vector subspace } V_p P = \{\pi'(p) v_p = 0\} \end{aligned}$$

of $T_p P$ called the vertical space does not depend on the trivialization. It is isomorphic to the tangent space of V .

Our purpose is to look for a way to define a derivative of p , and the decomposition of the vector v_p shows that it requires two components : one linked to a motion in M , and another to a change in V . However, even if $\pi'(p) v_p = v_m$, this is not sufficient to define a decomposition which would be independent on the choice of a trivialization. A **connection** is just this tool : it is a projection of v_p on the vertical space $V_p P$. It is a one form on P valued in the vertical bundle VP . So it enables us to distinguish in a variation of p what can be imputed to a change of m and what can be imputed to a change of u . A section of P depends only on m : $\mathbf{p}(m) = \varphi(m, u(m))$ so by differentiation with respect to m this is a map from TM to TP and the value of a connection at each $\mathbf{p}(m)$ is a one form over M , valued in VP , called the **covariant derivative**. So it meets our purpose. Moreover because the vertical space is isomorphic to the tangent space on V , the value of the connection can be expressed in a simpler vector space. A connection and a covariant derivative are defined through a specific map, called the potential of the connection, which is the key variable.

All this holds for any kind of fiber bundle (Maths.27), but the connection takes different forms according to the kind of fiber bundle. Moreover a connection on a principle bundle defines a connection on any associated fiber bundle.

7.2 The connection of the gravitational field

7.2.1 Potential

The vertical bundle VP_G of the principal bundle $P_G(M, Spin(3,1), \pi_G)$ is isomorphic to the Lie algebra $T_1Spin(3,1)$.

The **potential** G of a principal connection \mathbf{G} on P_G is a map : $G \in \Lambda_1(M; T_1Spin(3, 1))$.

Using the Clifford algebra to represent the Lie algebra, G reads :

$$G : TM \rightarrow T_1Spin(3, 1) :: G(m) = \sum_{\alpha=0}^3 \sum_{a=1}^6 G_{\alpha}^a(m) \vec{k}_a \otimes d\xi^{\alpha} \quad (94)$$

or equivalently :

$$G(m) = \sum_{\alpha=0}^3 v(G_{r\alpha}(m), G_{w\alpha}(m)) d\xi^{\alpha}$$

$G_{r\alpha}(m), G_{w\alpha}(m)$ are two vectors $\in \mathbb{R}^3$. So the *gravitational field has a transversal ($G_{w\alpha}$) and a rotational ($G_{r\alpha}$) component*. This is the unavoidable consequence of the gauge group.

The potential is not a section of a fiber bundle, but a map. In a change of gauge the potential transforms by an affine map⁶ :

$$\mathbf{p}(m) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1} : \quad (95)$$

$$G(m) \rightarrow \tilde{G}(m) = \mathbf{Ad}_{\chi} \left(G(m) - L'_{\chi^{-1}}(\chi) \chi'(m) \right) \quad (96)$$

$$\mathbf{Ad}_{\chi} v(G_{r\alpha} - X_{r\alpha}, G_{w\alpha} - X_{w\alpha}) = v(\tilde{G}_r(m), \tilde{G}_w(m))$$

(see Annex for the values of X, Y)

We introduce the convenient notation that will be used in the following :

Notation 51 $v(\hat{G}_r(\tau), \hat{G}_w(\tau))$ is the value of the potential of the gravitational field along the integral curve $m(\tau) = \Phi_V(\tau, x)$ of any vector field V

$$v(\hat{G}_r(\tau), \hat{G}_w(\tau)) = \sum_{\alpha=0}^3 V^{\alpha} v(G_{r\alpha}(\tau), G_{w\alpha}(\tau))$$

$$\text{with : } V^{\alpha} = \sum_{i=0}^3 P_i^{\alpha} V^i$$

$$v(\hat{G}_r(\tau), \hat{G}_w(\tau)) = v([G_r][P][V]^{\alpha}, [G_w][P][V]^{\alpha})$$

$$v(\hat{G}_r(\tau), \hat{G}_w(\tau)) \text{ can be extended to } v(G_r(m), G_w(m)) \text{ on } \Omega \text{ (Maths.1467).}$$

There are several covariant derivatives deduced from this connection.

⁶Similarly the Christofel coefficients are not tensor and transform in a complicated way in a change of charts.

7.2.2 Covariant derivative on P_G

The connection acts on sections of the principal bundle, and the covariant derivative of $\sigma = \varphi_G(m, \sigma(m)) \in \mathfrak{X}(P_G)$ is :

$$\nabla^G : \mathfrak{X}(P_G) \rightarrow \Lambda_1(M; T_1 Spin) :: \nabla^G \sigma = \sum_{\alpha=0}^3 (\sigma^{-1} \cdot \partial_\alpha \sigma + \mathbf{Ad}_{\sigma^{-1}} G_\alpha) d\xi^\alpha \quad (97)$$

The covariant derivative is invariant in a change of gauge.

Proof. $\varphi_G(m, g) = \tilde{\varphi}_G(m, \chi(m)g)$

$$\begin{aligned} G(m) &\rightarrow \tilde{G}(m) = \text{Ad}_\chi \left(G(m) - L'_{\chi^{-1}}(\chi) \chi'(m) \right) \\ \sigma(m) &\rightarrow \tilde{\sigma}(m) = \chi(m) \cdot \sigma(m) \\ \nabla^G \sigma &\rightarrow \nabla^{\tilde{G}} \tilde{\sigma} = \tilde{\sigma}^{-1} \cdot \tilde{\sigma}' + \mathbf{Ad}_{\tilde{\sigma}^{-1}} \tilde{G} \\ &= \sigma^{-1} \cdot \chi^{-1} \cdot (\chi' \cdot \sigma + \chi \cdot \sigma') + \mathbf{Ad}_{\sigma^{-1}} \mathbf{Ad}_{\chi^{-1}} (\text{Ad}_\chi (G - \chi^{-1} \cdot \chi')) \\ &= \sigma^{-1} \cdot \chi^{-1} \cdot \chi' \cdot \sigma + \sigma^{-1} \cdot \chi^{-1} \cdot \chi \cdot \sigma' + \mathbf{Ad}_{\sigma^{-1}} (G - \chi^{-1} \cdot \chi') \\ &= \sigma^{-1} \cdot \chi^{-1} \cdot \chi' \cdot \sigma + \sigma^{-1} \cdot \sigma' + \mathbf{Ad}_{\sigma^{-1}} G - \sigma^{-1} \cdot \chi^{-1} \cdot \chi' \cdot \sigma \\ &= \sigma^{-1} \cdot \sigma' + \mathbf{Ad}_{\sigma^{-1}} G = \nabla^G \sigma \quad \blacksquare \end{aligned}$$

7.2.3 Covariant derivative for spinors

A connection on a principal bundle induces a linear connection on any associated vector bundle.

The covariant derivative reads for a section $\mathbf{S} \in \mathfrak{X}(P_G[E, \gamma C])$:

$$\nabla^S S = \sum_{\alpha=0}^3 (\partial_\alpha S + \gamma C(G_\alpha) S) d\xi^\alpha = \sum_{\alpha=0}^3 (\partial_\alpha S + \gamma C(v(G_{r\alpha}, G_{w\alpha})) S) d\xi^\alpha \quad (98)$$

With the signature (3,1) :

$$\gamma C(v(G_{r\alpha}, G_{w\alpha})) = -i \frac{1}{2} \sum_{a=1}^3 (G_{w\alpha} \gamma_a \gamma_0 + G_{r\alpha} \tilde{\gamma}_a) \quad (99)$$

With the signature (1,3) :

$$\gamma C'(v(G_{r\alpha}, G_{w\alpha})) = -i \frac{1}{2} \sum_{a=1}^3 (G_{w\alpha} \gamma_a \gamma_0 - G_{r\alpha} \tilde{\gamma}_a) \quad (100)$$

So we go from the signature (3,1) to (1,3) by a change of the sign of $G_{r\alpha}$.

G_α being valued in $T_1 Spin(3,1)$ and γC being a representation of the Clifford algebra the expression makes sense. Its coordinates expression is with right and left chiral parts:

$$\nabla^S S = \sum_{\alpha=0}^3 \left[\begin{array}{l} \partial_\alpha S_R + \frac{1}{2} \sum_{g=1}^3 (G_{w\alpha}^g - i G_{r\alpha}^g) \sigma_g S_R \\ \partial_\alpha S_L - \frac{1}{2} \sum_{g=1}^3 (G_{w\alpha}^g + i G_{r\alpha}^g) \sigma_g S_L \end{array} \right] d\xi^\alpha$$

It preserves the chirality.

In a change of gauge :

$$\mathbf{p}(m) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1}$$

a section on $\mathfrak{X}(P_G[E, \gamma C])$ transforms as :

$$\mathbf{S}(m) = (\mathbf{p}(m), S(m)) = (\tilde{\mathbf{p}}(m), \tilde{S}(m)) \sim (\mathbf{p}(m), \gamma C(\chi(m)^{-1}) \tilde{S}(m))$$

$$\Rightarrow \tilde{S}(m) = \gamma C(\chi(m)) S(m)$$

The covariant derivative transforms as a section of $P_G[E, \gamma C]$:

$$\begin{aligned} \text{Proof. } \nabla^S S &\rightarrow \widetilde{\nabla^S S} = \sum_{\alpha=0}^3 \left(\partial_\alpha \tilde{S} + \gamma C(\tilde{G}_\alpha) \tilde{S} \right) d\xi^\alpha \\ &= \sum_{\alpha=0}^3 \left(\gamma C(\partial_\alpha \chi) S + \gamma C(\chi) \partial_\alpha S + \gamma C(Ad_\chi(G - \chi^{-1} \partial_\alpha \chi)) \gamma C(\chi) S \right) d\xi^\alpha \\ &= \sum_{\alpha=0}^3 \left(\gamma C(\partial_\alpha \chi) S + \gamma C(\chi) \partial_\alpha S + \gamma C(\chi(G - \chi^{-1} \partial_\alpha \chi) \chi^{-1}) \gamma C(\chi) S \right) d\xi^\alpha \\ &= \sum_{\alpha=0}^3 \left(\gamma C(\partial_\alpha \chi) S + \gamma C(\chi) \partial_\alpha S + \gamma C(\chi) \gamma C(G) S - \gamma C(\partial_\alpha \chi) S \right) d\xi^\alpha \\ &= \sum_{\alpha=0}^3 \gamma C(\chi) (\partial_\alpha S + \gamma C(G) S) d\xi^\alpha = \gamma C(\chi) \nabla^S S \quad \blacksquare \end{aligned}$$

so the operator reads: $\nabla^S : \mathfrak{X}(P_G[E, \gamma C]) \rightarrow *_1(M; \mathfrak{X}(P_G[E, \gamma C]))$

The action, that is the 4 dimensional force exercised on a particle with velocity V by the gravitational field is then given by :

$$S \rightarrow \nabla_V S = \frac{dS}{dt} + \left[\gamma C(\hat{G}) \right] [S] \text{ which is the equivalent of the Newton's}$$

law $F = \frac{dP}{dt} = V \nabla P$ with the operator $\nabla = \partial_\alpha$. It depends linearly on the potentials, and on the state of the particle and its derivative.

7.2.4 Covariant derivatives for vector fields on M

The connection on P_G induces a linear connection ∇^M on the associated vector bundle $P_G[\mathbb{R}^4, \mathbf{Ad}]$, which is TM with orthonormal bases, with **Christoffel symbols** :

$$\Gamma_M(m) = (\mathbf{Ad}_s)_{s=1}'(G(m))$$

with the product of vectors in $Cl(m)$:

$$v = \sum_{j=0}^3 v^j \varepsilon_j(m) \rightarrow$$

$$\sum_{i,j=0}^3 [\Gamma_M(m)]_i^j v^i \varepsilon_j(m) = v(G_{r\alpha}, G_{w\alpha}) \cdot v - v \cdot v(G_{r\alpha}, G_{w\alpha})$$

It is then more convenient to use the representation of $T_1 Spin(3, 1)$ by matrices of $so(3, 1)$:

$$[\Gamma_{M\alpha}] = \sum_{a=1}^6 G_\alpha^a [\kappa_a] = \begin{bmatrix} 0 & G_{w\alpha}^1 & G_{w\alpha}^2 & G_{w\alpha}^3 \\ G_{w\alpha}^1 & 0 & -G_{r\alpha}^3 & G_{r\alpha}^2 \\ G_{w\alpha}^2 & G_{r\alpha}^3 & 0 & -G_{r\alpha}^1 \\ G_{w\alpha}^3 & -G_{r\alpha}^2 & G_{r\alpha}^1 & 0 \end{bmatrix}$$

In a change of gauge :

$$G(m) \rightarrow \tilde{G}(m) = Ad_\chi \left(G(m) - L'_{\chi^{-1}}(\chi) \chi'(m) \right)$$

$$[\tilde{\Gamma}_{M\alpha}] = [h(s)] ([\Gamma_{M\alpha}] - [h(s^{-1})] [h(s')])$$

The covariant derivative of a section $V \in \mathfrak{X}(P_G[\mathbb{R}^4, \mathbf{Ad}])$ is then :

$$\nabla^M V = \sum_{\alpha i=0}^3 \left(\partial_\alpha V^i + \sum_{j=0}^3 [\Gamma_{M\alpha}(m)]_j^i V^j \right) \varepsilon_i(m) \otimes d\xi^\alpha \quad (101)$$

For any vector field $W : \nabla_W^M : \mathfrak{X}(TM) \rightarrow \mathfrak{X}(TM)$ is a linear map which preserves the scalar product of vectors (Maths.2205):

$$\forall W \in \mathfrak{X}(TM) : \langle \nabla_W^M U, \nabla_W^M V \rangle = \langle U, V \rangle$$

The isomorphism $so(3, 1) \rightarrow T_1 Spin(3, 1) \subset Cl(\mathbb{R}, 3, 1)$ reads :
 $[J(r) + K(w)] \rightarrow v(r, w) = \frac{1}{4} \sum_{i=0}^3 ([J(r) + K(w)] [\eta])_j^i \varepsilon_i \cdot \varepsilon_j$
thus in matrix form the Christoffel coefficient of the connection on $P_G[E, \gamma C]$ reads (Maths.9.2.4) :

$$\begin{aligned} [\Gamma_\alpha(m)] &= [\gamma C(G_\alpha)] \\ &= \frac{1}{4} \sum_{ijpq=0}^3 ([J(G_{r\alpha}) + K(G_{w\alpha})] [\eta])_j^i ([\gamma C(\varepsilon_i)] [\gamma C(\varepsilon_j)])_q^p \varepsilon_p(m) \otimes \varepsilon_q(m) \end{aligned}$$

But on the other hand the Christoffel coefficient of the connection on $P_G[\mathbb{R}^4, \mathbf{Ad}]$ is:

$$[\Gamma_{M\alpha}(m)] = \sum_{ij=0}^3 [K(G_{w\alpha}) + J(G_{r\alpha})]_j^i \varepsilon_i(m) \otimes \varepsilon_j(m)$$

thus :

$$[\Gamma_\alpha(m)] = \frac{1}{4} \sum_{ij=0}^3 ([\Gamma_{M\alpha}(m)] [\eta])_j^i [\gamma C(\varepsilon_i)] [\gamma C(\varepsilon_j)]$$

$P_G[E, \gamma C]$ is a spin bundle, and we have the identity between the derivatives

:

$$\forall V \in \mathfrak{X}(P_G[\mathbb{R}^4, \mathbf{Ad}]), S \in \mathfrak{X}(P_G[E, \gamma C]) :$$

$$\nabla(\gamma C(V)S) = \gamma C(\nabla^M V)S + \gamma C(V)\nabla S$$

which makes of \mathbf{G} a Clifford connection (Maths.2207).

7.3 Kinetic and potential energy

7.3.1 Kinetic energy

We have an important result :

Theorem 52 *The scalar product $\langle S, \nabla_\alpha S \rangle$ is purely imaginary : $\langle S, \nabla_\alpha S \rangle = i \operatorname{Im} \langle S, \nabla_\alpha S \rangle$*

Proof. $\nabla_\alpha S = \partial_\alpha S - \frac{i}{2} \sum_{a=1}^3 G_{w\alpha}^a \gamma_a \gamma_0 S + G_{r\alpha} \tilde{\gamma}_a S$
 $\langle S, \nabla_\alpha S \rangle = S^* \gamma_0 \partial_\alpha S - \frac{i}{2} \sum_{a=1}^3 G_{w\alpha}^a S^* \gamma_0 \gamma_a \gamma_0 S + G_{r\alpha} S^* \gamma_0 \tilde{\gamma}_a S$
 $[S]^* \gamma_a [S], [S]^* \gamma_0 \tilde{\gamma}_a [S]$ are real, $[S]^* \gamma_0 [\partial_\alpha S]$ are imaginary :
 $\frac{\langle S^* \gamma_a [S] \rangle}{\langle S^* \gamma_0 \tilde{\gamma}_a S \rangle} = (S^* \gamma_a S)^* = S^* \gamma_a S$
 $\frac{\langle S^* \gamma_0 \tilde{\gamma}_a S \rangle}{\langle S^* \gamma_0 \tilde{\gamma}_a S \rangle} = S^* \gamma_0 \tilde{\gamma}_a S^* = S^* (\gamma_0 \tilde{\gamma}_a)^* S = S^* \gamma_0 \tilde{\gamma}_a S$
 $\langle S, S \rangle = \langle S_0, S_0 \rangle \Rightarrow \langle S, \partial_\alpha S \rangle + \langle \partial_\alpha S, S \rangle = 0 \blacksquare$

$$\operatorname{Im} \langle S, \nabla_\alpha S \rangle = \frac{1}{i} (\langle S, \partial_\alpha S \rangle + \langle S, [\gamma C(G_\alpha)] S \rangle) \quad (102)$$

Along the trajectory, defined by a vector ε_0 and the time t of an observer :

$$\operatorname{Im} \langle S, \nabla_V S \rangle = \frac{1}{i} \left(\langle S, \frac{dS}{dt} \rangle + \langle S, \gamma C(\hat{G}) S \rangle \right)$$

$\frac{1}{i} \langle S, \frac{dS}{dt} \rangle = \frac{1}{i} \sum_{\alpha=0}^3 V_\alpha \langle S, \partial_\alpha S \rangle$ can be seen as the **kinetic energy** of the particle with respect to the observer (defined by t)

$\frac{1}{i} \langle S, \gamma C(\widehat{G}) S \rangle$ can be seen as the **potential energy** of the particle in the gravitational field along its trajectory

$\text{Im} \langle S, \nabla_V S \rangle$ can be seen as the **systemic energy** of the particle, as part of a system which includes the gravitational field. The energy depends on the choice of an observer : V is defined by σ_w with respect to ε_0 .

7.3.2 Inertial tensor

We can give a more precise expression of the covariant derivative, using the inertial spinor S_0 :

$$\begin{aligned} & \nabla_\alpha S \\ &= [\gamma C(\partial_\alpha \sigma)] [S_0] + [\gamma C(G_\alpha)] [\gamma C(\sigma)] [S_0] \\ &= [\gamma C(\sigma)] \left([\gamma C(\sigma^{-1} \cdot \partial_\alpha \sigma)] [S_0] + [\gamma C(\sigma^{-1} \cdot G_\alpha \cdot \sigma)] [S_0] \right) \\ &= [\gamma C(\sigma)] \left([\gamma C(\sigma^{-1} \cdot \partial_\alpha \sigma + \mathbf{Ad}_{\sigma^{-1}} G_\alpha)] \right) [\psi_0] \end{aligned}$$

By combination with the previous result :

$$\begin{aligned} & \text{Im} \langle \psi, \nabla_\alpha S \rangle \\ &= \frac{1}{i} \langle \gamma C(\sigma) S_0, \gamma C(\sigma) [\gamma C(\sigma^{-1} \partial_\alpha \sigma + \mathbf{Ad}_{\sigma^{-1}} G_\alpha)] [S_0] \rangle \\ &= \frac{1}{i} \langle S_0, [\gamma C(\sigma^{-1} \partial_\alpha \sigma + \mathbf{Ad}_{\sigma^{-1}} G_\alpha)] [S_0] \rangle \end{aligned}$$

Let us denote $[S_0] = \begin{bmatrix} S_R \\ \epsilon i S_R \end{bmatrix}$ with S_R a 2 vector and $\epsilon = +1$ for particles, and $\epsilon = -1$ for antiparticles.

$$\begin{aligned} & \sigma^{-1} \partial_\alpha \sigma + \mathbf{Ad}_{\sigma^{-1}} G_\alpha = v(X_\alpha, Y_\alpha) \in T_1 Spin(3, 1) \\ & \gamma C(v(X_\alpha, Y_\alpha)) = \frac{1}{2} \begin{bmatrix} \sum_{a=1}^3 (Y_\alpha^a - iX_\alpha^a) \sigma_a & 0 \\ 0 & -\sum_{a=1}^3 (Y_\alpha^a + iX_\alpha^a) \sigma_a \end{bmatrix} \\ & \gamma C(v(X_\alpha, Y_\alpha)) [S_0] = \frac{1}{2} \begin{bmatrix} \sum_{a=1}^3 (Y_\alpha^a - iX_\alpha^a) \sigma_a S_R \\ -\epsilon i \sum_{a=1}^3 (Y_\alpha^a + iX_\alpha^a) \sigma_a S_R \end{bmatrix} \\ & \langle S_0, \gamma C(\sigma^{-1} \partial_\alpha \sigma + \mathbf{Ad}_{\sigma^{-1}} G_\alpha) S_0 \rangle \\ &= \frac{1}{2} \left(S_R^* \left(-\epsilon \sum_{a=1}^3 (Y_\alpha^a + iX_\alpha^a) \sigma_a S_R \right) - \epsilon i S_R^* \left(i \sum_{a=1}^3 (Y_\alpha^a - iX_\alpha^a) \sigma_a S_R \right) \right) \\ &= \epsilon \frac{1}{2} \left(-\sum_{a=1}^3 (Y_\alpha^a + iX_\alpha^a) (S_R^* \sigma_a S_R) + \sum_{a=1}^3 (Y_\alpha^a - iX_\alpha^a) (S_R^* \sigma_a S_R) \right) \\ &= \frac{1}{2} \epsilon \sum_{a=1}^3 \left(-(Y_\alpha^a + iX_\alpha^a) + (Y_\alpha^a - iX_\alpha^a) \right) (S_R^* \sigma_a S_R) \\ &= -\epsilon i \sum_{a=1}^3 X_\alpha^a (S_R^* \sigma_a S_R) \\ &= i k^t X_\alpha \end{aligned}$$

The quantities $S_R^* \sigma_a S_R$ are 3 fixed scalars which define a vector $k = -\epsilon (S_R^* \sigma_a S_R)_{a=1}^3 \in \mathbb{R}^3$. Thus :

$$\text{Im} \langle \psi, \nabla_\alpha S \rangle = k^t X_\alpha \quad (103)$$

With : $\langle S_0, S_0 \rangle = \epsilon M_p^2 c^4$

$$S_R = \frac{M_p c^2}{\sqrt{2}} \begin{bmatrix} e^{i\alpha_1} \cos \alpha_0 \\ e^{i\alpha_2} \sin \alpha_0 \end{bmatrix}$$

$$k = -\epsilon \frac{1}{2} M_p^2 c^4 k_0 = -\epsilon \frac{1}{2} M_p^2 c^4 \begin{bmatrix} (\sin 2\alpha_0) \cos(\alpha_2 - \alpha_1) \\ (\sin 2\alpha_0) \sin(\alpha_2 - \alpha_1) \\ \cos 2\alpha_0 \end{bmatrix}; k_0^t k_0 = 1$$

The vector k is similar to the inertia tensor, but here this is a vector, and it encompasses both the translational and the rotational motions. And of course it holds for a particle without internal structure but can be computed for a solid body using the aggregation method.

Using $\sigma = \sigma_w \cdot \sigma_r$ one can compute X_α (see the formulas in Annex). It reads :

$$\begin{aligned}
\sigma &= \sigma_w \cdot \sigma_r \\
\sigma_w &= a_w + v(0, w) \\
\sigma_r &= a_r + v(r, 0) \\
a_w^2 &= 1 + \frac{1}{4}w^t w \\
a_r^2 &= 1 - \frac{1}{4}r^t r \\
v(X_\alpha, Y_\alpha) & \\
&= \sigma^{-1} \partial_\alpha \sigma + \mathbf{Ad}_{\sigma^{-1}} G_\alpha \\
&= \sigma_r^{-1} \cdot \sigma_w^{-1} \cdot \partial_\alpha \sigma_w \cdot \sigma_r + \sigma_r^{-1} \cdot \sigma_w^{-1} \cdot \sigma_w \cdot \partial_\alpha \sigma_r + \mathbf{Ad}_{\sigma_r^{-1}} \mathbf{Ad}_{\sigma_w^{-1}} G_\alpha \\
&= \sigma_r^{-1} \cdot (\sigma_w^{-1} \cdot \partial_\alpha \sigma_w \cdot \sigma_r + \partial_\alpha \sigma_r) + \mathbf{Ad}_{\sigma_r^{-1}} \mathbf{Ad}_{\sigma_w^{-1}} G_\alpha \\
&= \sigma_r^{-1} \cdot (\sigma_w^{-1} \cdot \partial_\alpha \sigma_w + \partial_\alpha \sigma_r \cdot \sigma_r^{-1}) \cdot \sigma_r + \mathbf{Ad}_{\sigma_r^{-1}} \mathbf{Ad}_{\sigma_w^{-1}} G_\alpha \\
&= \mathbf{Ad}_{\sigma_r^{-1}} \left(\sigma_w^{-1} \cdot \partial_\alpha \sigma_w + \partial_\alpha \sigma_r \cdot \sigma_r^{-1} + \mathbf{Ad}_{\sigma_w^{-1}} G_\alpha \right) \\
\partial_\alpha a_w &= \frac{1}{4a_w} w^t \partial_\alpha w \\
\partial_\alpha a_r &= -\frac{1}{4a_r} r^t \partial_\alpha r \\
\sigma_w^{-1} \cdot \partial_\alpha \sigma_w & \\
&= v \left(\frac{1}{2} j(w) \partial_\alpha w, -w \partial_\alpha a_w + a_w \partial_\alpha w \right) \\
&= v \left(\frac{1}{2} j(w) \partial_\alpha w, -\frac{1}{4a_w} w w^t \partial_\alpha w + a_w \partial_\alpha w \right) \\
&= v \left(\frac{1}{2} j(w) \partial_\alpha w, -\frac{1}{4a_w} (j(w) j(w) + w^t w) \partial_\alpha w + a_w \partial_\alpha w \right) \\
&= v \left(\frac{1}{2} j(w) \partial_\alpha w, \left(-\frac{1}{4a_w} j(w) j(w) - 4\frac{1}{4a_w} (a_w^2 - 1) + a_w \right) \partial_\alpha w \right) \\
&= v \left(\frac{1}{2} j(w) \partial_\alpha w, \left(\frac{1}{a_w} - \frac{1}{4a_w} j(w) j(w) \right) \partial_\alpha w \right) \\
\partial_\alpha \sigma_r \cdot \sigma_r^{-1} & \\
&= v \left(-\frac{1}{2} j(r) \partial_\alpha r - r \partial_\alpha a_r + a_r \partial_\alpha r, 0 \right) \\
&= v \left(-\frac{1}{2} j(r) \partial_\alpha r + \frac{1}{4a_r} r r^t \partial_\alpha r + a_r \partial_\alpha r, 0 \right) \\
&= v \left(\left(-\frac{1}{2} j(r) + \frac{1}{4a_r} (j(r) j(r) + r^t r) + a_r \right) \partial_\alpha r, 0 \right) \\
&= v \left(\left(-\frac{1}{2} j(r) + \frac{1}{4a_r} j(r) j(r) + \frac{4(1-a_r^2)}{4a_r} + a_r \right) \partial_\alpha r, 0 \right) \\
&= v \left(\left(\frac{1}{a_r} - \frac{1}{2} j(r) + \frac{1}{4a_r} j(r) j(r) \right) \partial_\alpha r, 0 \right) \\
\mathbf{Ad}_{\sigma_w^{-1}} G_\alpha & \\
&= v \left([a_w^2 - \frac{1}{2} j(w) j(w)] G_{r\alpha} + a_w j(w) G_{w\alpha}, \right. \\
&\quad \left. [a_w^2 - 1 - a_w j(w) + \frac{1}{4} j(w) j(w)] G_{r\alpha} + [1 - \frac{3}{4} j(w) j(w)] G_{w\alpha} \right) \\
\sigma_w^{-1} \cdot \partial_\alpha \sigma_w + \partial_\alpha \sigma_r \cdot \sigma_r^{-1} + \mathbf{Ad}_{\sigma_w^{-1}} G_\alpha & \\
&= v \left(\frac{1}{2} j(w) \partial_\alpha w + \left(\frac{1}{a_r} - \frac{1}{2} j(r) + \frac{1}{4a_r} j(r) j(r) \right) \partial_\alpha r + [a_w^2 - \frac{1}{2} j(w) j(w)] G_{r\alpha} + \right. \\
&\quad \left. a_w j(w) G_{w\alpha}, \right)
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{1}{a_w} - \frac{1}{4a_w} j(w) j(w) \right) \partial_\alpha w + [a_w^2 - 1 - a_w j(w) + \frac{1}{4} j(w) j(w)] G_{r\alpha} + [1 - \frac{3}{4} j(w) j(w)] G_{w\alpha} \\
v(X_\alpha, Y_\alpha) &= Ad_{\sigma_r^{-1}} v(x, y) \\
& \text{with} \\
X_\alpha &= [1 - a_r j(r) + \frac{3}{4} j(r) j(r)] x + [1 - a_r^2 + \frac{1}{4} j(r) j(r)] y \\
Y_\alpha &= [a_r^2 - a_r j(r) + \frac{1}{2} j(r) j(r)] y \\
x &= \left(\frac{1}{a_r} - \frac{1}{2} j(r) + \frac{1}{4a_r} j(r) j(r) \right) \partial_\alpha r + \frac{1}{2} j(w) \partial_\alpha w + (a_w^2 - \frac{1}{2} j(w) j(w)) G_{r\alpha} + \\
& a_w j(w) G_{w\alpha} \\
y &= \frac{1}{a_w} \left(1 - \frac{1}{4} j(w) j(w) \right) \partial_\alpha w + [a_w^2 - 1 - a_w j(w) + \frac{1}{4} j(w) j(w)] G_{r\alpha} + [1 - \frac{3}{4} j(w) j(w)] G_{w\alpha}
\end{aligned}$$

$$X_\alpha = [A_r(r)] \partial_\alpha r + [A_w(r, w)] \partial_\alpha w + [D_r(r, w)] G_{r\alpha} + [D_w(r, w)] G_{w\alpha} \quad (104)$$

The matrices $[A_r], [A_w], [D_r], [D_w]$ are polynomials of $j(r), j(r) j(r), j(w), j(w) j(w)$:

$$\begin{aligned}
[A_r(r)] &= [B_r(r)] \left[\frac{1}{a_r} - \frac{1}{2} j(r) + \frac{1}{4a_r} j(r) j(r) \right] = \left[\frac{1}{a_r} + \frac{2-5a_r^2}{2} j(r) + \frac{5a_r^2+1}{4a_r} j(r) j(r) \right] \\
[A_w(r, w)] &= [B_r(r)] [B_w(w)] + [C_r(r)] [C_w(w)] \\
[D_r(r, w)] &= [B_r(r)] [B_{wr}(w)] + [C_r(r)] [C_{wr}(w)] \\
[D_w(r, w)] &= [B_r(r)] [B_{ww}(w)] + [C_r(r)] [C_{ww}(w)]
\end{aligned}$$

with :

$$\begin{aligned}
[B_r(r)] &= [1 - a_r j(r) + \frac{3}{4} j(r) j(r)] \\
[B_w(w)] &= \frac{1}{2} j(w) \\
[C_r(r)] &= [1 - a_r^2 + \frac{1}{4} j(r) j(r)] \\
[C_w(w)] &= \frac{1}{a_w} [1 - \frac{1}{4} j(w) j(w)] \\
[B_{wr}(w)] &= [a_w^2 - \frac{1}{2} j(w) j(w)] \\
[B_{ww}(w)] &= a_w [j(w)] \\
[C_{wr}(w)] &= [a_w^2 - 1 - a_w j(w) + \frac{1}{4} j(w) j(w)] \\
[C_{ww}(w)] &= [1 - \frac{3}{4} j(w) j(w)]
\end{aligned}$$

The kinetic energy of the particle has the simple expression :

$$\frac{1}{i} \left\langle S, \frac{dS}{dt} \right\rangle = k^t \left([A_r] \frac{dr}{dt} + [A_w] \frac{dw}{dt} \right) \quad (105)$$

7.3.3 The electromagnetic field

The EM field can be represented in the gauge field formalism. The Lie algebra of $U(1)$ is \mathbb{R} . So the potential \dot{A} of the connection is a real valued one form on M : $\dot{A} = \sum_{\alpha=0}^3 \dot{A}_\alpha d\xi^\alpha \in \Lambda_1(M; \mathbb{R})$ which is usually represented as a vector field and not a form.

The action of $U(1)$ depends on the representation, thus on the charge of the particle :

- negative charge : $\varrho(\varkappa) S = (\exp i\varkappa) S$
- positive charge : $\varrho(\varkappa) S = (\exp(-i\varkappa)) S$
- neutral : $\varrho(\varkappa) S = S$

The covariant derivative reads : $\nabla_{\alpha}^F \psi = \partial_{\alpha} \psi + qi \hat{A}_{\alpha} \psi$ where q is the charge (for composite bodies) expressed as a signed integer multiple of the negative elementary charge, and $q = 0$ for neutral particles.

The systemic energy is :

$$\begin{aligned} \frac{1}{i} \langle S, \nabla_V S \rangle &= k^t \hat{X} + q \hat{A} \langle S_0, S_0 \rangle = -\epsilon \frac{1}{2} M_p^2 c^4 k_0^t \hat{X} + q \epsilon \hat{A} M_p^2 c^4 \\ \frac{1}{i} \langle S, \nabla_V S \rangle &= \epsilon M_p^2 c^4 \left(\frac{1}{2} k_0^t \hat{X} + q \hat{A}_{\alpha} \right) \end{aligned} \quad (106)$$

It gives to the vector k the physical meaning of a magnetic momentum.

7.4 Geodesics

There are several definitions of Geodesics, which, in different formulations, mean the curves of minimum length between two points. In Euclidean Geometry they are straight lines, in GR they are usually curves, and they play an important role because free particles move along geodesics. Moreover there is a unique geodesic passing through a point with a given tangent vector.

A connection enables to define the **parallel transport** of a vector (or a basis) along a curve (or a vector field).

Let C be a curve defined by a path $p : \mathbb{R} \rightarrow M : p(\tau)$ with $p(0) = a$, and a vector $v \in T_a M$. The parallel transported vector is given by a map :

$$V : \mathbb{R} \rightarrow T_{p(\tau)} M : V(\tau) \text{ such that : } \nabla_{\frac{dp}{d\tau}}^M V(\tau) = 0, V(0) = v$$

thus we have the differential equation with $V(\tau) = \sum_{i=0}^3 V^i(\tau) \varepsilon_i(p(\tau))$

$$\nabla_{\frac{dp}{d\tau}}^M V(\tau) = \sum_{\alpha i=0}^3 \left(\partial_{\alpha} V^i + \sum_{j=0}^3 \Gamma_M(p(\tau))_{\alpha j}^i V^j \right) \left(\frac{dp}{d\tau} \right)^{\alpha} \varepsilon_i(p(\tau)) = 0$$

$$\frac{dV^i}{d\tau} + \sum_{\alpha j=0}^3 \Gamma_M(p(\tau))_{\alpha j}^i V^j \left(\frac{dp}{d\tau} \right)^{\alpha} = 0$$

A **geodesic** is a path such that its tangent is parallel transported by the connection :

$$p : \mathbb{R} \rightarrow M : p(\tau) \text{ with } p(0) = a$$

$$V(\tau) = \frac{dp}{d\tau} = \sum_{i=0}^3 V^i(\tau) \varepsilon_i(p(\tau)) = \sum_{k\alpha=0}^3 V^k(\tau) P_k^{\alpha}(p(\tau)) \partial \xi_{\alpha}$$

$$\frac{dV^i}{d\tau} + \sum_{\alpha j k=0}^3 \Gamma_M(p(\tau))_{\alpha j}^i V^j(\tau) V^k(\tau) P_k^{\alpha}(p(\tau)) = 0$$

or in matrix form :

$$\left[\frac{dV}{d\tau} \right] + \sum_{\alpha} ([\Gamma_{M\alpha}] [V]) ([P'] [V])^{\alpha} = 0$$

The scalar product $\langle V, V \rangle$ is constant :

$$\frac{d}{d\tau} \langle V, V \rangle = \frac{d}{d\tau} \left([V]^t [\eta] [V] \right)$$

$$= - \sum_{\alpha} ([P'] [V])^{\alpha} [V]^t [\Gamma_{M\alpha}]^t [\eta] [V] - \sum_{\alpha} ([P'] [V])^{\alpha} [V]^t [\eta] ([\Gamma_{M\alpha}] [V])$$

$$= - \sum_{\alpha} ([P'] [V])^{\alpha} [V]^t \left([\Gamma_{M\alpha}]^t [\eta] + [\eta] [\Gamma_{M\alpha}] \right) [V] = 0$$

A **field of geodesics** is a vector field U such that it is parallel transported along its integral curves $p(\tau) = \Phi_U(\tau, x)$.

As $\langle U, U \rangle$ is constant, for a time like geodesic field we can take $\langle U, U \rangle = -1$ and, for a given observer, associate a section $\sigma_w \in P_W$:

$$U = \left((2a_w^2 - 1) \varepsilon_0(m) + a_w \sum_{j=1}^3 w_j \varepsilon_j(m) \right)$$

$$\text{and } \mathbf{U}(m) = \mathbf{Ad}_{\sigma_w} \varepsilon_0(m)$$

The formalism of vector bundles enables us to give a useful description of these geodesics, through the value of σ_w with respect to G .

Theorem 53 For a given observer geodesics are represented by sections $\sigma_w \in \mathfrak{X}(P_G)$ such that $\nabla_U^G \sigma_w \in T_1 \text{Spin}(3)$.

They are solutions of the differential equation :

$$a_w \frac{dw}{d\tau} =$$

$$\left((1 - a_w^2) a_w^2 + a_w j(w) - \frac{1}{4} a_w^2 j(w) j(w) \right) \widehat{G}_r + \left(3 - 4a_w^2 + \frac{1}{2} j(w) j(w) \right) \widehat{G}_w$$

where $v(\widehat{G}_r, \widehat{G}_w)$ is the value of the potential of the gravitational field along the geodesic

Proof. i) In the standard basis and with the Clifford algebra formalism :

$$\nabla_U^M U = \frac{dU}{d\tau} + \sum_{\alpha=0}^3 (V^\alpha (v(G_{r\alpha}, G_{w\alpha}) \cdot U - U \cdot v(G_{r\alpha}, G_{w\alpha})))$$

$$= \frac{d}{d\tau} \mathbf{Ad}_{\sigma_w} \varepsilon_0 + v(\widehat{G}_r, \widehat{G}_w) \cdot \mathbf{Ad}_{\sigma_w} \varepsilon_0 - \mathbf{Ad}_{\sigma_w} \varepsilon_0 \cdot v(\widehat{G}_r, \widehat{G}_w)$$

$$\text{with } V^\alpha = \sum_i P_i^\alpha U^i, \widehat{G}_r = \sum_{\alpha=0}^3 G_{r\alpha} V^\alpha; \widehat{G}_w = \sum_{\alpha=0}^3 G_{w\alpha} V^\alpha, \mathbf{U}(m) =$$

$$\mathbf{Ad}_{\sigma_w} \varepsilon_0(m)$$

$$\frac{d}{d\tau} \mathbf{Ad}_{\sigma_w} \varepsilon_0 = \frac{d\sigma_w}{d\tau} \cdot \varepsilon_0 \cdot \sigma_w^{-1} - \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} \cdot \frac{d\sigma_w}{d\tau} \cdot \sigma_w^{-1}$$

$$= \left(\sigma_w \cdot \nabla_U^G \sigma_w - v(\widehat{G}_r, \widehat{G}_w) \cdot \sigma_w \right) \cdot \varepsilon_0 \cdot \sigma_w^{-1} - \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} \cdot \left(\sigma_w \cdot \nabla_U^G \sigma_w - v(\widehat{G}_r, \widehat{G}_w) \cdot \sigma_w \right) \cdot \sigma_w^{-1}$$

$$\sigma_w^{-1}$$

$$= \sigma_w \cdot \nabla_U^G \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} - v(\widehat{G}_r, \widehat{G}_w) \cdot \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} - \sigma_w \cdot \varepsilon_0 \cdot \nabla_U^G \sigma_w \cdot \sigma_w^{-1} +$$

$$\sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} \cdot v(\widehat{G}_r, \widehat{G}_w)$$

$$\text{with } \frac{d\sigma_w}{d\tau} = \sigma_w \cdot \nabla_U^G \sigma_w - v(\widehat{G}_r, \widehat{G}_w) \cdot \sigma_w$$

$$\nabla_U^M U = \sigma_w \cdot \nabla_U^G \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} - \sigma_w \cdot \varepsilon_0 \cdot \nabla_U^G \sigma_w \cdot \sigma_w^{-1}$$

$$- v(\widehat{G}_r, \widehat{G}_w) \cdot \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} + \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} \cdot v(\widehat{G}_r, \widehat{G}_w)$$

$$+ v(\widehat{G}_r, \widehat{G}_w) \cdot \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} - \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} \cdot v(\widehat{G}_r, \widehat{G}_w)$$

$$= \sigma_w \cdot (\nabla_U^G \sigma_w \cdot \varepsilon_0 - \varepsilon_0 \cdot \nabla_U^G \sigma_w) \cdot \sigma_w^{-1}$$

So, with the covariant derivative on the principal bundle P_G . We have a geodesic iff :

$$\nabla_U^G \sigma_w \cdot \varepsilon_0 - \varepsilon_0 \cdot \nabla_U^G \sigma_w = 0$$

that is iff $\nabla_U^G \sigma_w$ commutes with ε_0 .

For any element $v(r, w)$ of $T_1 \text{Spin}(3, 1)$ we have the identity :

$$v(r, w) \cdot \varepsilon_0 - \varepsilon_0 \cdot v(r, w) = w \text{ (see Annex for the proof)}$$

$$\text{So : } v(r, w) \in T_1 \text{Spin}(3) \Leftrightarrow v(r, w) \cdot \varepsilon_0 - \varepsilon_0 \cdot v(r, w) = 0 \Leftrightarrow w = 0$$

And the geodesics are represented by sections such that $\nabla_U^G \sigma_w \in T_1 \text{Spin}(3)$.

ii) The sections

$$\nabla_U^G \sigma_w = \sigma_w^{-1} \cdot (\partial_\alpha \sigma_w + v(G_{r\alpha}, G_{w\alpha}) \cdot \sigma_w) \text{ has been computed :}$$

$$\begin{aligned}
\nabla_\alpha^G \sigma_w &= v(-\frac{1}{2}j(w) \partial_\alpha w + (a_w^2 + \frac{1}{4} - 2j(w)j(w)) G_{r\alpha} + a_w j(w) G_{w\alpha}, \\
&-w \partial_\alpha a_w + a_w \partial_\alpha w + (a_w^2 - 1 - a_w j(w) + \frac{1}{4}j(w)j(w)) G_{r\alpha} + (1 - \frac{3}{4}j(w)j(w)) G_{w\alpha}) \\
\text{So geodesic fields are associated to the sections such that :} \\
\sum_\alpha V^\alpha (-w \partial_\alpha a_w + a_w \partial_\alpha w + (a_w^2 - 1 - a_w j(w) + \frac{1}{4}j(w)j(w)) G_{r\alpha} \\
+ (1 - \frac{3}{4}j(w)j(w)) G_{w\alpha}) &= 0 \\
w \frac{da_w}{d\tau} - a_w \frac{dw}{d\tau} &= (a_w^2 - 1 - a_w j(w) + \frac{1}{4}j(w)j(w)) \widehat{G}_r + (1 - \frac{3}{4}j(w)j(w)) \widehat{G}_w \\
\text{By left multiplication with } w^t : \\
w^t w \frac{da_w}{d\tau} - a_w w^t \frac{dw}{d\tau} &= (a_w^2 - 1) w^t \widehat{G}_r + w^t \widehat{G}_w \\
a_w^2 &= 1 + \frac{1}{4}w^t w \Rightarrow (w^t \frac{dw}{dt}) = 4a_w \frac{da_w}{d\tau} \\
w^t w \frac{da_w}{d\tau} - 4a_w^2 \frac{da_w}{d\tau} &= (a_w^2 - 1) w^t \widehat{G}_r + w^t \widehat{G}_w \\
4(a_w^2 - 1 - a_w^2) \frac{da_w}{d\tau} &= (a_w^2 - 1) w^t \widehat{G}_r + w^t \widehat{G}_w \\
\frac{da_w}{d\tau} &= \frac{1}{4}(1 - a_w^2) w^t \widehat{G}_r - \frac{1}{4}w^t \widehat{G}_w \\
\frac{1}{4}(1 - a_w^2) w w^t \widehat{G}_r - \frac{1}{4}w w^t \widehat{G}_w - a_w \frac{dw}{d\tau} \\
&= (a_w^2 - 1 - a_w j(w) + \frac{1}{4}j(w)j(w)) \widehat{G}_r + (1 - \frac{3}{4}j(w)j(w)) \widehat{G}_w \\
&(- (a_w^2 - 1) a_w^2 + a_w j(w) - \frac{1}{4}a_w^2 j(w)j(w)) \widehat{G}_r \\
&- (4(a_w^2 - 1) + 1 - \frac{1}{2}j(w)j(w)) \widehat{G}_w = a_w \frac{dw}{d\tau} \blacksquare
\end{aligned}$$

The interpretation of this result is clear : along the geodesic the forces, gravitational or inertial, are such that they induce spatial rotations, without change in the direction of the transversal motion.

There is nothing equivalent for the null curves, such that their tangent vector u has a null scalar product : $\langle u, u \rangle = 0$. But the definition of the flow of a vector field, which does not involve the metric, still holds.

7.5 The Levi-Civita connection

In Differential Geometry one defines affine connections (Maths.1537), which are bilinear operators acting on vector fields (sections of the tangent bundle) $\nabla \in \mathcal{L}^2(\mathfrak{X}(TM), \mathfrak{X}(TM); \mathfrak{X}(TM))$ such that:

$$\begin{aligned}
\forall f \in C_1(M; \mathbb{R}) : \\
\nabla_{fV} W &= f \nabla_V W \\
\nabla_V fW &= f \nabla_V W + (i_V df) W \\
\text{They read in holonomic basis of a chart :} \\
\nabla_\alpha V &= \sum_\beta (\partial_\beta V^\alpha + \sum_\gamma \Gamma_{\beta\gamma}^\alpha V^\gamma) \partial^{\xi^\beta} \otimes d\xi_\alpha
\end{aligned}$$

with Christoffel symbols $\Gamma_{\beta\gamma}^\alpha(m)$ which change in a change of chart in a complicated way. So an affine connection is a covariant derivative, defined in the tangent bundle, and acting on sections of the tangent bundle, which are vector fields, or tensors. There can be many different affine connections.

An affine connection is said to be symmetric if $\Gamma_{\beta\gamma}^\alpha = \Gamma_{\gamma\beta}^\alpha$

When there is a metric (Riemannian or not) defined by a tensor g on a manifold, an affine connection is said to be metric if $\nabla_\alpha g = 0$: it preserves the scalar product of two vectors. With a metric, one can define a unique, metric, symmetric connection, called the Levi-Civita connection. It reads (Maths.1626)

:

$$\Gamma_{\beta\gamma}^{\alpha} = \frac{1}{2} \sum_{\eta} g^{\alpha\eta} (\partial_{\beta} g_{\gamma\eta} + \partial_{\gamma} g_{\beta\eta} - \partial_{\eta} g_{\beta\gamma})$$

And this has been the bread and butter of workers on GR for decenniums, in a formalism where the metric is at the core of the model.

With a principal bundle, and a principal connection, one can define covariant derivatives in any associated vector bundle, including of course the tangent bundle to M . And it has all the properties of the usual covariant derivative of affine connections. Connections on fiber bundles are a more general tool than usual affine connections which are strictly limited to the tangent bundle. We have seen that the connection \mathbf{G} on P_G induces a linear connection on P_G $[\mathbb{R}^4, \mathbf{Ad}]$, which is nothing more than TM with an orthonormal basis, and a covariant derivative ∇^M with Christoffel symbol Γ_M . By translating the orthonormal basis $(\varepsilon_i)_{i=0}^3$ into the holonomic basis $(\partial\xi_{\alpha})_{\alpha=0}^3$ of any chart using the tetrad, a straightforward computation (Maths.2005) gives the Christoffel coefficients $\widehat{\Gamma}_{\alpha\beta}^{\gamma}$ of the affine connection Γ_M , expressed in the basis of the chart :

$$\widehat{\Gamma}_{\alpha\beta}^{\gamma} = P_i^{\gamma} \left(\partial_{\alpha} P_{\beta}^i + \Gamma_{M\alpha j}^i P_{\beta}^j \right)$$

In matrix form :

$$\widehat{\Gamma}_{\alpha\beta}^{\gamma} = \left[\widehat{\Gamma}_{\alpha} \right]_{\beta}^{\gamma}, \Gamma_{M\alpha j}^i = [\Gamma_{M\alpha}]_j^i,$$

$$[\Gamma_{M\alpha}] = \sum_{a=1}^6 G_{a\alpha} [\kappa_a]$$

$$\left[\widehat{\Gamma}_{\alpha} \right] = [P] \left([\partial_{\alpha} P'] + [\Gamma_{M\alpha}] [P'] \right) \Leftrightarrow [\Gamma_{M\alpha}] = \left([P'] \left[\widehat{\Gamma}_{\alpha} \right] - [\partial_{\alpha} P'] \right) [P]$$

with :

$$a=1,2,3 : [\kappa_a]_q^p = \sum_{bc=1}^3 \epsilon(a,b,c) \delta_c^p \delta_q^b$$

$$a=4,5,6 : ([\kappa_a])_q^p = \delta_0^p \delta_q^{a-3} + \delta_{a-3}^p \delta_q^0$$

$$\left[\widehat{\Gamma}_{\alpha} \right]_{\beta}^{\gamma} = ([P] [\partial_{\alpha} P'])_{\beta}^{\gamma}$$

$$+ \sum_{a=1}^3 \left(\sum_{bc=1}^3 \epsilon(a,b,c) G_{r\alpha}^a [P]_c^{\gamma} [P']_{\beta}^b + G_{w\alpha}^a \left([P]_0^{\gamma} [P']_{\beta}^a + [P]_a^{\gamma} [P']_{\beta}^0 \right) \right)$$

Any affine connection deduced this way from a principal connection is necessarily metric, but it is not necessarily symmetric.

To sum up :

- affine connections are defined in the strict framework of the tangent bundle, and the Levi-Civita connection is one of these connections, with specific properties (it is metric and symmetric); the covariant derivative which is deduced acts only on vectors fields (or tensors) of the tangent bundle.

- connections on principal bundle define connections on any associated vector bundle and act on sections of these bundles. So one can compute a covariant derivative acting on vectors fields of the tangent bundle, which is necessarily metric but not necessarily symmetric.

So, using the formalism of fiber bundles we do not miss anything, we can get the usual results, but in a more elegant and simple way. One can require from the principal connection \mathbf{G} on P_G that the induced connection on TM is symmetric, which will then be identical to the Levi-Civita connection. This requests :

$$\forall \alpha, \beta, \gamma :$$

$$\left[\widehat{\Gamma}_\alpha\right]_\beta^\gamma = ([P]([\partial_\alpha P'] + [\Gamma_{M\alpha}][P']])_\beta^\gamma = \left[\widehat{\Gamma}_\beta\right]_\alpha^\gamma = ([P]([\partial_\beta P'] + [\Gamma_{M\beta}][P']])_\alpha^\gamma$$

which has no obvious meaning for Γ_M .

Actually the Levi-Civita connection is traditionally used because it is the natural mathematical choice when one starts from the metric. Moreover it is assumed that the gravitational field (whose action goes through the connection) acts symmetrically, in the meaning that it has no torsion (or no torque). But actually this assumption has not been verified (which is difficult), and different theories have been proposed, notably by Einstein and Cartan, which consider connections with torsion, that is connections other than the Levi-Civita connection. However, when starting from the metric, they lead mostly to more complicated computations, in what is already a dreadful endeavour. In the fiber bundle framework there is no such problem and actually it would be the requirement of symmetry, always possible at any point, which would introduce a complication. Moreover the introduction of spinors and the distinction of the components G_r, G_w of the connection, are a more efficient way to deal with rotation and torque, so it is justified that we keep the more general connection. An additional argument is that the Levi-Civita connection does not make any distinction between the bases, which can be induced by any chart. But, as we have seen, there is always a privileged chart, that of the observer, and the use of an orthogonal basis, in the fiber bundle formalism, is a useful reminder of this feature.

7.6 The inertial observer

The states of the particles and the fields are linked, so to measure one we have to know the other : to measure a charge one uses a known field, and to measure a field one uses a known particle. This process requires actually two measures, involving the motion of the particle, it is done locally and is represented by the standard gauges : $\mathbf{p}_G(m) = \varphi_G(m, 1)$ and the related holonomic bases $\mathbf{e}_i(m) = (\mathbf{p}_G(m), e_i)$. The measures are done with respect to the standards (represented by 1), which are arbitrary. For this reason the gauges and the holonomic bases are not sections, just a specific choice done by the observer. This is consistent with the principle of locality (the measures are done locally) and the free will of the observer (he is not submitted himself to the laws of the system).

However one can consider another kind of gauges, such that they do not change on the travel of an observer on his world line. This seems more physical, and in accordance with the common understanding of inertial observers : they keep their gauges, in which they proceed to their measures, “constant”. However the implementation then must account for the existence of external fields : the observer keeps his free will (which is asserted by the fact that he adjusts his gauge), but he is also submitted to the action of the fields, and notably to the gravitational field.

The action of the fields is given by the covariant derivative of sections, so *the gauges of these observers are sections*, and they change with the fields according

to :

$$p_G \in \mathfrak{X}(P_G) : p_G(m) = \varphi_G(m, \sigma(m)) : \nabla_{\varepsilon_0}^G p_G = 0$$

The world line of the observer is defined by $\varepsilon_0(m)$. The bases of the associated vector bundles $P_G[\mathbb{R}^4, \mathbf{Ad}]$, $P_G[E, \gamma C]$ change :

$$\varepsilon_i(m) \rightarrow \tilde{\varepsilon}_i(m) = Ad_{\sigma^{-1}} \varepsilon_i(m)$$

$$e_i(m) \rightarrow \tilde{e}_i(m) = \gamma C(\sigma) e_i(m)$$

$$\nabla_{\varepsilon_0}^G p_G = 0 \Leftrightarrow \sigma^{-1} \cdot \frac{d\sigma}{dt} + \mathbf{Ad}_{\sigma^{-1}} v(\hat{G}_r, \hat{G}_w) = 0 \Leftrightarrow \frac{d\sigma}{dt} = -v(\hat{G}_r, \hat{G}_w) \cdot \sigma$$

So the inertial gauges depend on the existing fields along the trajectory. In the absence of any field one retrieves : $\sigma = Ct$ and the standard gauge. This is the usual meaning of inertial observers. As for now one cannot escape gravity, inertial observers should follow trajectories such that their motion (through the inertial forces) balance the gravitational field. As it changes slowly with location it will require a constant acceleration, contrary to the common understanding of the inertial observer⁷.

A great consideration is given, both in Newtonian Mechanics and Relativity, to the inertial observers. The main motivation is that they allow the use of “fixed” frames in affine space, and we have seen that this is the only way to do it in SR. The formalism of fiber bundle seems abstract, but avoids this issue : the concept of standard gauge relies on the fact that the measures of the states of particles or of fields require the comparison with a known quantity (the standard), in similar conditions. And actually, as often in Physics, it is quite impossible to enshrine in the theories the conditions of the experiments. Doing this would bring more confusion than rigor.

The same remark applies to a procedure common in electromagnetism or linearized gravity. When facing a complicated mathematical relation it is tempting to reduce it to a simpler form by calling to what is called gauge freedom. Actually this procedure uses the fact that the same quantity is expressed in different forms according to the gauge (in the fiber bundle definition). So one can replace one by another, which is equivalent, and better looking. To have any meaning this procedure shall follow the requirements of the change of gauge, clearly stated in the fiber bundle formalism. But we have to keep in mind that a change of gauge has a physical meaning, and an implication on the observer who does the measures. A change of gauge can be physically unacceptable by the constraints which would be imposed to the observer, and any experimental proof which would ignore these requirements in its protocols would be non valid.

⁷An observer in the International Space Station can be considered as inertial, but obviously he is submitted to an acceleration which balances Earth gravity.

8 THE PROPAGATION OF FIELDS

We have noticed that Relativity obliges to dissociate the abstract representation of the reality in its entirety, from the representation of the reality which can be scientifically accessible to an observer : this is the true motivation for the use of fiber bundles. It is more necessary when addressing the propagation of fields. Fields are assumed to be defined everywhere, this is one of their key properties. Have the fields a defined value everywhere, including in the future ? What is the true meaning of their propagation, in a 4 dimensional Universe ? We have to remind that the manifold M is just a container, and its physical content is not frozen, it changes. So the propagation of a field is no more than the variation of its value along any future oriented path. But if this image is clear for particles, which have a definite path, it is less so for fields. Are there privileged paths for their propagation ? There is no simple answer to this question. We need to refer to one of our basic assumptions : the representation of the physical world depends on the observer. The observer has a privileged path, given by his velocity and represented by the vector $\varepsilon_0(m)$. This vector plays a fundamental role in the understanding of the propagation of fields. For any observer his own vector $\varepsilon_0(m)$ is the direction of the propagation of the fields. There is no other way to define it, practically, that is in a way which is accessible to measures. So *the propagation of fields is observer dependant*. As a consequence one can prove (MP 7.1.2) that in the vacuum the field propagates at the speed of light⁸, whatever the observer.

The variation of the field is measured by a derivative, and as the fields are characterized by their potential, we need a derivative of the potentials G . This is the strength of the field, a 2 form \mathcal{F} on M , which is logical because the potentials are themselves one form. However \mathcal{F} as well as the potential is valued in the Lie algebra. If we want to underline the role played by the observer, both in the definition of ε_0 and of holonomic bases in the Lie algebras, it is necessary to put in concordance the mathematical definitions. From this point of view the potential is not the good choice : it is only a map, which changes according to a strange rule in a change of gauge. So the key role in the propagation of the fields will be given to \mathcal{F} .

8.1 The strength of the connection

8.1.1 Definition

The potential is a one form over M , so its derivative will be a two form. There are several mathematical objects which can be considered, related to the curvature of the connection (Maths.27.1.4), but for principal connections the **strength of the connection** is the most pertinent. This is a 2-form \mathcal{F} on M valued in the Lie algebra (Maths.2194), which can be seen as the exterior covariant derivative of the potential and is a good estimate of its rate of change.

⁸In the vacuum. The propagation of light in any medium is a process which involves the interaction of the field with the particles of the medium.

The strength of the connection is a two form on M valued in the Lie algebra $T_1Spin(3, 1)$ which reads with the basis $(\vec{\kappa}_a)_{a=1}^6$:

$$\mathcal{F}_G = \sum_{a=1}^6 \left(dG^a + \sum_{\alpha\beta=0}^3 [G_\alpha, G_\beta]^a d\xi^\alpha \wedge d\xi^\beta \right) \otimes \vec{\kappa}_a$$

where d is the exterior differential on TM and $[]$ is the bracket in $T_1Spin(3, 1)$.⁹

$$\mathcal{F}_G = \sum_{a=1}^6 \sum_{\alpha,\beta=0}^3 \mathcal{F}_{G_{\alpha\beta}}^a d\xi^\alpha \wedge d\xi^\beta \otimes \vec{\kappa}_a \quad (107)$$

$$\mathcal{F}_{G_{\alpha\beta}}^a = \sum_{a=1}^6 \sum_{\alpha,\beta=0}^3 \left(\partial_\alpha G_\beta^a - \partial_\beta G_\alpha^a + [G_\alpha, G_\beta]^a \right) d\xi^\alpha \wedge d\xi^\beta \otimes \vec{\kappa}_a$$

Or equivalently :

$$\mathcal{F}_G = d \left(\sum_{\alpha=0}^3 v(G_{r\alpha}, G_{w\alpha}) d\xi^\alpha \right) + \sum_{\alpha\beta=0}^3 [v(G_{r\alpha}, G_{w\alpha}), v(G_{r\beta}, G_{w\beta})] d\xi^\alpha \wedge d\xi^\beta$$

We can distinguish the two parts, $\mathcal{F}_r, \mathcal{F}_w$:

$$\mathcal{F}_G = \sum_{\alpha,\beta} v(\mathcal{F}_{r\alpha\beta}, \mathcal{F}_{w\alpha\beta}) d\xi^\alpha \wedge d\xi^\beta \quad (108)$$

and we have :

$$a = 1,2,3 : \mathcal{F}_{G_{\alpha\beta}}^a = \mathcal{F}_{r\alpha\beta}^a$$

$$a = 4,5,6 : \mathcal{F}_{G_{\alpha\beta}}^a = \mathcal{F}_{w\alpha\beta}^a$$

with the signature (3,1) :

$$[v(r, w), v(r', w')] = v(j(r) r' - j(w) w', j(w) r' + j(r) w')$$

$$[G_\alpha, G_\beta] = [v(G_{r\alpha}, G_{w\alpha}), v(G_{r\beta}, G_{w\beta})]$$

$$= v(j(G_{r\alpha}) G_{r\beta} - j(G_{w\alpha}) G_{w\beta}, j(G_{w\alpha}) G_{r\beta} + j(G_{r\alpha}) G_{w\beta})$$

$$\mathcal{F}_{r\alpha\beta} = v \left(\frac{\partial G_{r\beta}}{\partial \xi^\alpha} - \frac{\partial G_{r\alpha}}{\partial \xi^\beta} + j(G_{r\alpha}) G_{r\beta} - j(G_{w\alpha}) G_{w\beta}, 0 \right) \quad (109)$$

$$\mathcal{F}_{w\alpha\beta} = v \left(0, \frac{\partial G_{w\beta}}{\partial \xi^\alpha} - \frac{\partial G_{w\alpha}}{\partial \xi^\beta} + j(G_{w\alpha}) G_{r\beta} + j(G_{r\alpha}) G_{w\beta} \right) \quad (110)$$

With the signature (1,3):

$$[v(r, w), v(r', w')] = -v(j(r) r' - j(w) w', j(w) r' + j(r) w')$$

$$[G_\alpha, G_\beta]$$

$$= -v(j(G_{r\alpha}) G_{r\beta} - j(G_{w\alpha}) G_{w\beta}, j(G_{w\alpha}) G_{r\beta} + j(G_{r\alpha}) G_{w\beta})$$

$$\mathcal{F}_{r\alpha\beta} = -v \left(\frac{\partial G_{r\beta}}{\partial \xi^\alpha} - \frac{\partial G_{r\alpha}}{\partial \xi^\beta} + j(G_{r\alpha}) G_{r\beta} - j(G_{w\alpha}) G_{w\beta}, 0 \right) \quad (111)$$

$$\mathcal{F}_{w\alpha\beta} = -v \left(0, \frac{\partial G_{w\beta}}{\partial \xi^\alpha} - \frac{\partial G_{w\alpha}}{\partial \xi^\beta} + j(G_{w\alpha}) G_{r\beta} + j(G_{r\alpha}) G_{w\beta} \right) \quad (112)$$

⁹The notations and conventions for r forms vary according to the authors and if the indices are ordered or not. On this see Maths.1525,1529.

Notice that the indices α, β are not ordered, that it involves only the principal bundle, and not the associated vector bundles, and is valued in a fixed vector space.

\mathcal{F}_G can be written in the matrix form of $so(3, 1)$, using $\vec{\kappa}_a \rightarrow [\kappa_a]$:

$$[\mathcal{F}_{G\alpha\beta}] = \sum_{a=1}^6 \mathcal{F}_{G\alpha\beta}^a [\kappa_a] = [K(\mathcal{F}_{w\alpha\beta})] + [J(\mathcal{F}_{r\alpha\beta})] \quad (113)$$

$$[\mathcal{F}_{\alpha\beta}] = \begin{bmatrix} 0 & \mathcal{F}_{w\alpha\beta}^1 & \mathcal{F}_{w\alpha\beta}^2 & \mathcal{F}_{w\alpha\beta}^3 \\ \mathcal{F}_{w\alpha\beta}^1 & 0 & -\mathcal{F}_{r\alpha\beta}^3 & \mathcal{F}_{r\alpha\beta}^2 \\ \mathcal{F}_{w\alpha\beta}^2 & \mathcal{F}_{r\alpha\beta}^3 & 0 & -\mathcal{F}_{\alpha\beta}^1 \\ \mathcal{F}_{w\alpha\beta}^3 & -\mathcal{F}_{r\alpha\beta}^2 & \mathcal{F}_{r\alpha\beta}^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{F}_{w\alpha\beta} \\ \mathcal{F}_{w\alpha\beta} & j(\mathcal{F}_{r\alpha\beta}) \end{bmatrix}$$

which underlines the rotational feature of the component \mathcal{F}_r , and the transversal aspect of the component \mathcal{F}_w .

With :

$$\begin{aligned} \Gamma_{M\alpha} &= [K(G_{w\alpha})] + [J(G_{r\alpha})] = \begin{bmatrix} 0 & G_{w\alpha}^1 & G_{w\alpha}^2 & G_{w\alpha}^3 \\ G_{w\alpha}^1 & 0 & -G_{r\alpha}^3 & G_{r\alpha}^2 \\ G_{w\alpha}^2 & G_{r\alpha}^3 & 0 & -G_{\alpha}^1 \\ G_{w\alpha}^3 & -G_{r\alpha}^2 & G_{r\alpha}^1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & G_{w\alpha} \\ G_{w\alpha} & j(G_{r\alpha}) \end{bmatrix} \end{aligned}$$

$$[\mathcal{F}_{\alpha\beta}] = [\partial_\alpha \Gamma_{M\beta}] - [\partial_\beta \Gamma_{M\alpha}] + [\Gamma_{M\alpha}] [\Gamma_{M\beta}] - [\Gamma_{M\beta}] [\Gamma_{M\alpha}] \quad (114)$$

8.1.2 Adjoint bundle

The strength of the connection is a map valued in the Lie algebra, that is a fixed vector space. We have seen above how the potential changes in a change of gauge. In a change of gauge on the principal bundle the strength changes as :

$$\mathbf{p}_G(m) = \varphi_G(m, 1) \rightarrow \tilde{\mathbf{p}}_G(m) = \mathbf{p}_G(m) \cdot s(m)^{-1} : \quad (115)$$

$$\mathcal{F}_{G\alpha\beta} \rightarrow \tilde{\mathcal{F}}_{G\alpha\beta}(m) = \mathbf{Ad}_{s(m)} \mathcal{F}_{G\alpha\beta} \quad (116)$$

$$v(\tilde{\mathcal{F}}_{r\alpha\beta}, \tilde{\mathcal{F}}_{w\alpha\beta}) = \mathbf{Ad}_{s(m)} v(\mathcal{F}_{r\alpha\beta}, \mathcal{F}_{w\alpha\beta})$$

This feature allows to consider the strength as section of the **adjoint bundle**, which is defined as the associated vector bundle

$P_G [T_1 Spin(3, 1), \mathbf{Ad}]$ using the representation of the groups on their Lie algebra through the adjoint map (Ad on $T_1 Spin(3, 1)$ is identical to \mathbf{Ad}). This gives a more geometrical meaning to the concept, and these relations are crucial in the definition of the lagrangian.

Electromagnetic field The strength of the electromagnetic field is a 2 form valued in $\mathbb{R} : \mathcal{F}_A \in \Lambda_2(M; \mathbb{R})$.

Because the Lie algebra is abelian the bracket is null and : $\mathcal{F}_A = d\dot{A}$ which gives the first Maxwell's law : $d\mathcal{F}_A = 0$.

In a change of gauge : $\mathcal{F}_{A\alpha\beta} \rightarrow \tilde{\mathcal{F}}_{A\alpha\beta}(m) = Ad_{\mathcal{z}(m)}\mathcal{F}_{A\alpha\beta} = \mathcal{F}_{A\alpha\beta}$. The strength of the EM field is invariant in a change of gauge.

8.2 Scalar curvature

The strength of the potential is a general object, which is related to the different concepts of curvature used in the theory of connections. It involves only the principal bundle, and not the associated vector bundles. However in GR another definition of curvature is commonly used, and it is necessary to see how these concepts are related.

8.2.1 Riemann curvature of a principal connection

A connection on a principal bundle leads to the definition of a quantity, called **Riemann curvature**, on any associated vector bundle, through a path which involves exterior covariant derivatives (Maths.2203). The result is a two-form on M, valued in the endomorphisms on the vector space, which in the case of $P_G[\mathbb{R}^4, \mathbf{Ad}]$ is expressed by the 4 order tensor :

$$\tilde{R} = \sum_{\{\alpha\beta\}ij} R_{\alpha\beta j}^i d\xi^\alpha \wedge d\xi^\beta \otimes \varepsilon_i(m) \otimes \varepsilon^j(m) \text{ with the dual basis } \varepsilon^j(m)$$

$$R_{\alpha\beta j}^i = \partial_\alpha \Gamma_{M\beta j}^i - \partial_\beta \Gamma_{M\alpha j}^i + \sum_{k=0}^3 \left(\Gamma_{M\alpha k}^i \Gamma_{M\beta j}^k - \Gamma_{M\beta k}^i \Gamma_{M\alpha j}^k \right)$$

In matrix form (see above) :

$$[R_{\alpha\beta}]_j^i = ([\partial_\alpha \Gamma_{M\beta}] - [\partial_\beta \Gamma_{M\alpha}] + [\Gamma_{M\alpha}][\Gamma_{M\beta}] - [\Gamma_{M\beta}][\Gamma_{M\alpha}])_j^i = [\mathcal{F}_{\alpha\beta}]_j^i$$

$$R = \sum_{\{\alpha\beta\}ij} [\mathcal{F}_{\alpha\beta}]_j^i d\xi^\alpha \wedge d\xi^\beta \otimes \varepsilon_i(m) \otimes \varepsilon^j(m)$$

The Riemann curvature is the image of the strength of the field on $P_G[\mathbb{R}^4, \mathbf{Ad}]$. This is the same quantity, but in the representation of $T_1 Spin(3, 1)$ in the matrix algebra $so(3, 1)$.

By construct this quantity is covariant (in a change of chart on M) and equivariant (in a change of gauge on P_G) :

In a change of gauge :

$$\mathbf{p}_G(m) = \varphi_G(m, 1) \rightarrow \tilde{\mathbf{p}}_G(m) = \mathbf{p}_G(m) \cdot s(m)^{-1} :$$

$$[\mathcal{F}_{\alpha\beta}]_j^i \rightarrow \sum_{kl=0}^3 [h(s)]_k^i [\mathcal{F}_{\alpha\beta}]_l^k [h(s^{-1})]_j^l$$

with $[h(s)]$ the matrix of $SO(3, 1)$ associated to $s \in Spin(3, 1)$

$$R \rightarrow \tilde{R} = \sum_{\{\alpha\beta\}ij} [\tilde{\mathcal{F}}_{\alpha\beta}]_j^i d\xi^\alpha \wedge d\xi^\beta \otimes \tilde{\varepsilon}_i(m) \otimes \tilde{\varepsilon}^j(m)$$

$$= \sum_{\{\alpha\beta\}ij} [h(s)]_k^i [\mathcal{F}_{\alpha\beta}]_l^k [h(s^{-1})]_j^l d\xi^\alpha \wedge d\xi^\beta \otimes [h(s^{-1})]_i^p \varepsilon_p(m) \otimes [h(s)]_q^j \varepsilon^q(m)$$

$$= \sum_{\{\alpha\beta\}ij} [\mathcal{F}_{\alpha\beta}]_q^p d\xi^\alpha \wedge d\xi^\beta \otimes \varepsilon_p(m) \otimes \varepsilon^q(m)$$

so $\tilde{R} = R$

It can be expressed in the holonomic basis of any chart on M using the tetrad

:

$$\varepsilon_i(m) = \sum_{\gamma=0}^3 P_i^\gamma d\xi_\gamma$$

$$\varepsilon^j(m) = \sum_{\eta=0}^3 P_\eta^j d\xi^\eta$$

$$R = \sum_{\{\alpha\beta\}\gamma\eta} ([P] [\mathcal{F}_{G\alpha\beta}] [P'])^\gamma d\xi^\alpha \wedge d\xi^\beta \otimes \partial\xi_\gamma \otimes d\xi^\eta$$

So we have the steps :

Principal connection $\mathbf{G} \rightarrow$ Riemann curvature R on $P_G [\mathbb{R}^4, \mathbf{Ad}] \rightarrow$ Riemann curvature R on TM in any chart

and the Riemann curvature R on TM is the same object as the strength of the connection \mathcal{F} , but expressed in matrix form in any holonomic basis of a chart.

8.2.2 Riemann tensor of an affine connection

With a common affine connection $\widehat{\Gamma}_\alpha$ on TM one can also define similarly a **Riemann tensor** (Maths.1543) :

$$\widehat{R} = \sum_{\{\alpha\beta\}} \sum_{\gamma\eta} \widehat{R}_{\alpha\beta\eta}^\gamma d\xi^\alpha \wedge d\xi^\beta \otimes \partial\xi_\gamma \otimes d\xi^\eta$$

which, expressed in matrix form with : $\widehat{R}_{\alpha\beta\eta}^\gamma = \left[\widehat{R}_{\alpha\beta} \right]_\eta^\gamma$, reads :

$$\left[\widehat{R}_{\alpha\beta} \right] = \left[\partial_\alpha \widehat{\Gamma}_\beta \right] - \left[\partial_\beta \widehat{\Gamma}_\alpha \right] + \left[\widehat{\Gamma}_\alpha \right] \left[\widehat{\Gamma}_\beta \right] - \left[\widehat{\Gamma}_\beta \right] \left[\widehat{\Gamma}_\alpha \right]$$

$$\widehat{R} = \sum_{\{\alpha\beta\}} \sum_{\gamma\eta} \widehat{R}_{\alpha\beta\eta}^\gamma d\xi^\alpha \wedge d\xi^\beta \otimes \partial\xi_\gamma \otimes d\xi^\eta$$

When we take as affine connection the one which is deduced from \mathbf{G} :

$$\widehat{\Gamma}_{\alpha\beta}^\gamma = \left[\widehat{\Gamma}_\alpha \right]_\beta^\gamma = ([P] ([\partial_\alpha P'] + [\Gamma_{M\alpha}] [P']))_\beta^\gamma$$

we get the same result :

$$\left[\widehat{R}_{\alpha\beta} \right] = [R_{\alpha\beta}] = [P] [\mathcal{F}_{G\alpha\beta}] [P'] \Leftrightarrow [\mathcal{F}_{G\alpha\beta}] = [P'] [R_{\alpha\beta}] [P] \quad (117)$$

Proof. $\left[\widehat{R}_{\alpha\beta} \right]$

$$\begin{aligned} &= [\partial_\alpha P] [\partial_\beta P'] + [\partial_\alpha P] [\Gamma_{M\beta}] [P'] + [P] \left[\partial_{\beta\alpha}^2 P' \right] + [P] [\partial_\alpha \Gamma_{M\beta}] [P'] \\ &+ [P] [\Gamma_{M\beta}] [\partial_\alpha P'] - [\partial_\beta P] [\partial_\alpha P'] - [\partial_\beta P] [\Gamma_{M\alpha}] [P'] - [P] \left[\partial_{\alpha\beta}^2 P' \right] \\ &- [P] [\partial_\beta \Gamma_{M\alpha}] [P'] - [P] [\Gamma_{M\alpha}] [\partial_\beta P'] + [P] [\partial_\alpha P'] [P] [\partial_\beta P'] \\ &+ [P] [\Gamma_{M\alpha}] [P'] [P] [\partial_\beta P'] + [P] [\partial_\alpha P'] [P] [\Gamma_{M\beta}] [P'] \\ &+ [P] [\Gamma_{M\alpha}] [P'] [P] [\Gamma_{M\beta}] [P'] - [P] [\partial_\beta P'] [P] [\partial_\alpha P'] \\ &- [P] [\Gamma_{M\beta}] [P'] [P] [\partial_\alpha P'] - [P] [\partial_\beta P'] [P] [\Gamma_{M\alpha}] [P'] \\ &- [P] [\Gamma_{M\beta}] [P'] [P] [\Gamma_{M\alpha}] [P'] \\ &= + [P] ([\partial_\alpha \Gamma_{M\beta}] - [\partial_\beta \Gamma_{M\alpha}] + [\Gamma_{G\alpha}] [\Gamma_{M\beta}] - [\Gamma_{M\beta}] [\Gamma_{M\alpha}]) [P'] \\ &+ [\partial_\alpha P] [\partial_\beta P'] - [\partial_\beta P] [\partial_\alpha P'] + [P] [\partial_\alpha P'] [P] [\partial_\beta P'] \\ &- [P] [\partial_\beta P'] [P] [\partial_\alpha P'] + [\partial_\alpha P] [\Gamma_{M\beta}] [P'] - [\partial_\beta P] [\Gamma_{M\alpha}] [P'] \\ &+ [P] [\Gamma_{M\beta}] [\partial_\alpha P'] - [P] [\Gamma_{M\alpha}] [\partial_\beta P'] + [P] [\Gamma_{M\alpha}] [\partial_\beta P'] \\ &- [P] [\Gamma_{M\beta}] [\partial_\alpha P'] + [P] [\partial_\alpha P'] [P] [\Gamma_{M\beta}] [P'] - [P] [\partial_\beta P'] [P] [\Gamma_{M\alpha}] [P'] \\ &= [P] [\mathcal{F}_{G\alpha\beta}] [P'] + [\partial_\alpha P] [\partial_\beta P'] - [\partial_\beta P] [\partial_\alpha P'] \\ &- [\partial_\alpha P] [P'] [P] [\partial_\beta P'] + [\partial_\beta P] [P'] [P] [\partial_\alpha P'] + [\partial_\alpha P] [\Gamma_{M\beta}] [P'] \\ &- [\partial_\beta P] [\Gamma_{M\alpha}] [P'] + [P] [\Gamma_{M\beta}] [\partial_\alpha P'] - [P] [\Gamma_{M\alpha}] [\partial_\beta P'] + [P] [\Gamma_{M\alpha}] [\partial_\beta P'] \\ &- [P] [\Gamma_{M\beta}] [\partial_\alpha P'] - [\partial_\alpha P] [P'] [P] [\Gamma_{M\beta}] [P'] + [\partial_\beta P] [P'] [P] [\Gamma_{M\alpha}] [P'] \\ &= [P] [\mathcal{F}_{G\alpha\beta}] [P'] \\ &\text{with } [P] [\partial_\alpha P'] + [\partial_\alpha P] [P'] = 0 \quad \blacksquare \end{aligned}$$

So the Riemann tensor is the Riemann curvature of the principal connection, expressed in the holonomic basis of a chart, and it is the same object as the strength of the connection.

The Riemann tensor can be computed with any affine connection, as well as with any principal connection. In the usual RG formalism the Riemann tensor is computed with a special connection : the Levy-Civita connection.

The Riemann tensor is antisymmetric, in the meaning :

$$\begin{aligned} \widehat{R}_{\alpha\beta\gamma\eta} &= -\widehat{R}_{\alpha\beta\eta\gamma} \text{ with } \widehat{R}_{\alpha\beta\gamma\eta} = \sum_{\lambda} \widehat{R}_{\alpha\beta\gamma}^{\lambda} g_{\lambda\eta} \\ [\mathcal{F}_{G\alpha\beta}] &\in so(3,1) \text{ so } [\eta] [\mathcal{F}_{G\alpha\beta}] + [\mathcal{F}_{G\alpha\beta}]^t [\eta] = 0 \text{ and} \\ \widehat{R}_{\alpha\beta\gamma\eta} &= \sum_{\lambda} \widehat{R}_{\alpha\beta\gamma}^{\lambda} g_{\lambda\eta} = \sum_{\lambda} ([P] [\mathcal{F}_{G\alpha\beta}] [P'])_{\gamma}^{\lambda} g_{\lambda\eta} = \left([P']^t [\eta] [\mathcal{F}_{G\alpha\beta}] [P'] \right)_{\gamma}^{\eta} \\ &= \left(\left([P']^t [\eta] [\mathcal{F}_{G\alpha\beta}] [P'] \right)_{\eta}^t \right)_{\gamma}^{\eta} = \left([P']^t [\mathcal{F}_{G\alpha\beta}]^t [\eta] [P'] \right)_{\eta}^{\gamma} \\ &= - \left([P']^t [\eta] [\mathcal{F}_{G\alpha\beta}] [P'] \right)_{\eta}^{\gamma} = -\widehat{R}_{\alpha\beta\eta\gamma} \end{aligned}$$

Thus this symmetry is not specific to the Lévi-Civita connection as it is usually assumed (Wald p.39).

8.2.3 Ricci tensor and scalar curvature

The Riemann tensor \widehat{R} , coming from any connection, is a 2 form but can be expressed as an antisymmetric tensor with non ordered indices :

$$\widehat{R} = \sum_{\alpha\beta\gamma\eta} \left[\widehat{R}_{\alpha\beta} \right]_{\eta}^{\gamma} d\xi^{\alpha} \otimes d\xi^{\beta} \otimes \partial\xi_{\gamma} \otimes d\xi^{\eta}$$

and we can contract the covariant index α, β or η with the contravariant index γ . The result does not depend on a basis : it is covariant (Maths.385).

The different solutions give :

$$\alpha : \sum_{\beta\eta} \left(\sum_{\alpha} \left[\widehat{R}_{\alpha\beta} \right]_{\eta}^{\alpha} \right) d\xi^{\beta} \otimes d\xi^{\eta}$$

$$\beta : \sum_{\alpha\eta} \left(\sum_{\beta} \left[\widehat{R}_{\alpha\beta} \right]_{\eta}^{\beta} \right) d\xi^{\alpha} \otimes d\xi^{\eta}$$

$$\eta : \sum_{\alpha\eta} \left(\sum_{\gamma} \left[\widehat{R}_{\alpha\beta} \right]_{\gamma}^{\eta} \right) d\xi^{\alpha} \otimes d\xi^{\beta}$$

The last solution has no interest because :

$$Tr ([P] [\mathcal{F}_{G\alpha\beta}] [P']) = Tr ([\mathcal{F}_{G\alpha\beta}] [P'] [P]) = Tr ([\mathcal{F}_{G\alpha\beta}]) = 0$$

The first two read :

$$\sum_{\beta\gamma} [P]_k^{\alpha} [\mathcal{F}_{G\alpha\beta}]_l^k [P']_{\eta}^l [P]_i^{\beta} \varepsilon^i \otimes [P]_j^{\eta} \varepsilon^j = \sum_{\beta\gamma} [P]_k^{\alpha} [\mathcal{F}_{G\alpha\beta}]_j^k [P]_i^{\beta} \varepsilon^i \otimes \varepsilon^j = \sum_{\alpha\beta j} ([P] [\mathcal{F}_{G\alpha\beta}])_j^{\alpha} d\xi^{\beta} \otimes \varepsilon^j$$

$$\sum_{\alpha\gamma} [P]_k^{\beta} [\mathcal{F}_{G\alpha\beta}]_l^k [P']_{\eta}^l [P]_i^{\alpha} \varepsilon^i \otimes [P]_j^{\eta} \varepsilon^j = \sum_{\alpha\gamma} [P]_k^{\beta} [\mathcal{F}_{G\alpha\beta}]_j^k [P]_i^{\alpha} \varepsilon^i \otimes \varepsilon^j = \sum_{\beta\gamma} ([P] [\mathcal{F}_{G\alpha\beta}])_j^{\beta} d\xi^{\alpha} \otimes \varepsilon^j$$

The **Ricci tensor** is the contraction on the two indices γ, β of \widehat{R} :

$$Ric = \sum_{\alpha\eta} Ric_{\alpha\eta} d\xi^{\alpha} \otimes d\xi^{\eta} = \sum_{\alpha\eta} \left(\sum_{\beta} \left[\widehat{R}_{\alpha\beta} \right]_{\eta}^{\beta} \right) d\xi^{\alpha} \otimes d\xi^{\eta}$$

This is a tensor, from which one can compute another tensor by lowering the last index:

$$\sum_{\lambda} g^{\eta\lambda} Ric_{\alpha\eta} d\xi^{\alpha} \otimes d\xi^{\eta} = \sum_{\alpha\lambda} Ric_{\alpha}^{\lambda} d\xi^{\alpha} \otimes \partial\xi_{\lambda}$$

and the contraction (called the trace of this tensor) provides the **scalar curvature** :

$$\mathbf{R} = \sum_{\alpha} Ric_{\alpha}^{\alpha} = \sum_{\alpha\beta\eta} g^{\alpha\eta} \left[\widehat{R}_{\alpha\beta} \right]_{\eta}^{\beta}$$

The same procedure applied to the contraction on the two indices γ, α of \widehat{R} gives the opposite scalar :

$$\mathbf{R} = \sum_{\alpha\beta\eta} g^{\beta\eta} \left[\widehat{R}_{\alpha\beta} \right]_{\eta}^{\alpha} = - \sum_{\alpha\beta\eta} g^{\alpha\eta} \left[\widehat{R}_{\beta\alpha} \right]_{\eta}^{\beta} = - \sum_{\alpha\beta\eta} g^{\alpha\eta} \left[\widehat{R}_{\alpha\beta} \right]_{\eta}^{\beta}$$

This manipulation is mathematically valid, and provides a unique scalar, which does not depend on a chart, and can be used in a lagrangian. However its physical justification (see Wald) is weak.

In the usual GR formalism the scalar curvature is computed with the Riemann tensor deduced from the Levy-Civita connection but, as we can see, it can be computed with any principal connection. Starting from $[P] [\mathcal{F}_{G\alpha\beta}] [P'] = \left[\widehat{\mathcal{F}}_{G\alpha\beta} \right]$ one gets :

$$\mathbf{R} = \sum_{\alpha\beta\eta} g^{\alpha\eta} ([P] [\mathcal{F}_{G\alpha\beta}] [P'])_{\eta}^{\beta} \text{ and with } [g]^{-1} = [P] [\eta] [P]^t$$

$$\mathbf{R} = \sum_{\alpha\beta\eta} ([P] [\mathcal{F}_{G\alpha\beta}] [P'])_{\eta}^{\beta} [g^{-1}]_{\alpha}^{\eta} = \sum_{\alpha\beta} \left([P] [\mathcal{F}_{G\alpha\beta}] [P'] [P] [\eta] [P]^t \right)_{\alpha}^{\beta}$$

$$\mathbf{R} = \sum_{\alpha\beta} \left([P] [\mathcal{F}_{G\alpha\beta}] [\eta] [P]^t \right)_{\alpha}^{\beta} \quad (118)$$

The computation of the scalar curvature gives ::

$$\mathbf{R} = \sum_{a=1}^6 \sum_{\alpha\beta} \mathcal{F}_{G\alpha\beta}^a \left([P] [\kappa_a] [\eta] [P]^t \right)_{\alpha}^{\beta}$$

For $a = 1, 2, 3$: $[\kappa_a] [\eta] = [\kappa_a]$

$$[\kappa_a]_q^p = \sum_{bc=1}^3 \epsilon(a, b, c) \delta_c^p \delta_q^b$$

$$\left([P] [\kappa_a] [\eta] [P]^t \right)_{\alpha}^{\beta} = \sum_{p,q=1}^3 \epsilon(a, q, p) [P]_p^{\beta} [P]_q^{\alpha}$$

$$\sum_{a=1}^3 \sum_{\alpha\beta} \mathcal{F}_{G\alpha\beta}^a \left([P] [\kappa_a] [\eta] [P]^t \right)_{\alpha}^{\beta} = \sum_{a=1}^3 \sum_{\alpha\beta} \mathcal{F}_{r\alpha\beta}^a \sum_{p,q=1}^3 \epsilon(a, q, p) [P]_p^{\beta} [P]_q^{\alpha}$$

For $a = 4, 5, 6$:

$$([\kappa_a] [\eta])_q^p = \delta_0^p \delta_q^{a-3} - \delta_{a-3}^p \delta_q^0$$

$$\left([P] [\kappa_a] [\eta] [P]^t \right)_{\alpha}^{\beta} = [P]_0^{\beta} [P]_{a-3}^{\alpha} - [P]_{a-3}^{\beta} [P]_0^{\alpha}$$

$$\sum_{a=4}^6 \sum_{\alpha\beta} \mathcal{F}_{G\alpha\beta}^a \left([P] [\kappa_a] [\eta] [P]^t \right)_{\alpha}^{\beta} = \sum_{a=1}^3 \sum_{\alpha\beta} \mathcal{F}_{w\alpha\beta}^a \left([P]_0^{\beta} [P]_a^{\alpha} - [P]_a^{\beta} [P]_0^{\alpha} \right)$$

Thus :

$$\mathbf{R} = \sum_{a=1}^3 \sum_{\alpha\beta} \left(\mathcal{F}_{w\alpha\beta}^a \left([P]_0^{\beta} [P]_a^{\alpha} - [P]_a^{\beta} [P]_0^{\alpha} \right) + \mathcal{F}_{r\alpha\beta}^a \sum_{p,q=1}^3 \epsilon(a, q, p) [P]_p^{\beta} [P]_q^{\alpha} \right) \quad (119)$$

which reads in a geometric way :

$$\mathbf{R} = \sum_{a=1}^3 \left(\mathcal{F}_w^a (\varepsilon_a, \varepsilon_0) + \sum_{p,q=1}^3 \epsilon (a, q, p) \mathcal{F}_r^a (\varepsilon_q, \varepsilon_p) \right)$$

In the standard chart with : $[P]_0^\alpha = \delta_0^\alpha$

$$\begin{aligned} \mathbf{R} &= \sum_{a=1}^3 \sum_{\alpha\beta} \left(\mathcal{F}_{w\alpha\beta}^a \left(\delta_0^\beta [P]_a^\alpha - [P]_a^\beta \delta_0^\alpha \right) + \mathcal{F}_{r\alpha\beta}^a \sum_{p,q=1}^3 \epsilon (a, q, p) [P]_p^\beta [P]_q^\alpha \right) \\ &= 2 \sum_{\alpha < \beta} \left\{ \sum_{a=1}^3 \mathcal{F}_{G\alpha\beta}^a \sum_{p,q=1}^3 \epsilon (a, q, p) [P]_p^\beta [P]_q^\alpha \right. \\ &\quad \left. + \sum_{a=4}^6 \mathcal{F}_{G\alpha\beta}^a \left([P]_0^\beta [P]_{a-3}^\alpha - [P]_{a-3}^\beta [P]_0^\alpha \right) \right\} \end{aligned}$$

This expression has two important features :

- the scalar curvature is linear with respect to the strength of the field.
In the implementation of the Principle of Least Action it provides equations which are linear with respect to \mathcal{F}_G , which is a big improvement from the usual computations.

- it shows that *the scalar curvature has a transversal component and a rotational component*. This happens for any scalar curvature, but is just masked in the usual expression through the metric. This feature is not without significance, as it is related to the distinction between the space and the time Universe.

To sum up, with the fiber bundle and connections formalism it is possible to compute, more easily, a scalar curvature which has the usual meaning. And by imposing symmetry to the affine connection we get exactly the same quantity. However, as we have seen before, the symmetry of the connection has no obvious physical meaning, and similarly for the scalar curvature. So, in the following, we will stay with the strength of the connection, which gives a good representation of the propagation of the field, and puts gravitation in the same footing as the other fields.

9 THE RELATIVIST MOMENTUM OF THE GRAVITATIONAL FIELD

9.1 Polarization of the fields

Polarization of light is a concept familiar to Physicists, and can be easily observed. It has a deeper meaning, related to the fact that the vector $\partial\xi_0$ is the only privileged direction for the propagation of the field with respect to the observer.

With singling out $d\xi^0$, any two form read :

$$\mathcal{F} = \mathcal{F}_R + \mathcal{F}_W$$

with :

$$\mathcal{F}_R = 2 (\mathcal{F}_{32}d\xi^3 \wedge d\xi^2 + \mathcal{F}_{13}d\xi^1 \wedge d\xi^3 + \mathcal{F}_{21}d\xi^2 \wedge d\xi^1)$$

$$\mathcal{F}_W = 2 (\mathcal{F}_{01}d\xi^0 \wedge d\xi^1 + \mathcal{F}_{02}d\xi^0 \wedge d\xi^2 + \mathcal{F}_{03}d\xi^0 \wedge d\xi^3)$$

(the 2 accounting for the symmetric part)

The action of these two forms on vectors is :

$$\mathcal{F}_W \left(\sum_{\alpha=0}^3 u^\alpha \partial\xi_\alpha \right) = -2u^0 (\mathcal{F}_{01}d\xi^1 + \mathcal{F}_{02}d\xi^2 + \mathcal{F}_{03}d\xi^3)$$

$$+ 2 (\mathcal{F}_{01}u^1 + \mathcal{F}_{02}u^2 + \mathcal{F}_{03}u^3) d\xi^0$$

$$\mathcal{F}_R (u^0 \partial\xi_0) = 0$$

$$\mathcal{F}_R \left(\sum_{\alpha=0}^3 u^\alpha \partial\xi_\alpha \right) = - \sum_{\alpha=1}^3 [j (\mathcal{F}_{32}d\xi^1 + \mathcal{F}_{13}d\xi^2 + \mathcal{F}_{21}d\xi^3) u]_\alpha d\xi^\alpha$$

so the action of \mathcal{F}_R^a can be seen as a rotation in the physical space by the vector \mathcal{F}_R .

This decomposition seems a bit formal, however for the EM field this is exactly the decomposition in electric field (\mathcal{F}_W) and magnetic field (\mathcal{F}_R). It characterizes the **polarization of the field** in the direction given by the vector $\varepsilon_0 = \partial\xi_0$ which characterizes the observer. And we see that it can be extended to any field, separately for each component \mathcal{F}^a . The decomposition into the components $\mathcal{F}_R, \mathcal{F}_W$ depends on the chart. For instance the decomposition of the EM field in electric and magnetic field depends on the observer. But we can go further in exploring this decomposition.

\mathcal{F} is formally defined as a 2 form, that is with respect to any chart, in one hand, and is valued in the adjoint bundle, that is with respect to a gauge provided by P_G on the other hand. The privileged direction $\partial\xi_0 = \varepsilon_0$ is masked in the anonymity of the labels in a banalized chart. But there is no such banal chart in Relativity : a chart is always linked to an observer, and each observer has a privileged orientation, that of its future. To put the matter right it is necessary to express \mathcal{F} in a basis which varies with the observer in a consistent and clear way. It is possible to do this, first by expressing the two forms in the orthonormal basis, then by expressing it in a unique vector bundle.

9.2 From the holonomic basis of a chart to the orthonormal basis

Any scalar two form on M can be expressed in the dual basis $(\varepsilon^i(m))_{i=0}^3$:

$$\mathcal{F} = \sum_{\alpha\beta=0}^3 \mathcal{F}_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta = \sum_{ij=0}^3 F_{ij} \varepsilon^i(m) \wedge \varepsilon^j(m)$$

This is a classic change of basis with :

$$\varepsilon^i(m) = \sum_{\alpha=0}^3 P_\alpha^i d\xi^\alpha \Leftrightarrow d\xi^\alpha = \sum_{i=0}^3 P_i^\alpha \varepsilon^i(m)$$

$$F_{ij} = \sum_{\alpha\beta=0}^3 \mathcal{F}_{\alpha\beta} P_i^\alpha P_j^\beta \Leftrightarrow \mathcal{F}_{\alpha\beta} = \sum_{ij=0}^3 F_{ij} P_\alpha^i P_\beta^j \quad (120)$$

In the Special Relativity context the components F_{ij} are the components of the field, measured in an inertial orthonormal frame, so all usual formulas in this context can be transposed in curved space time by using the frame $(\varepsilon^i(m))_{i=0}^3$.

9.3 The dual Clifford bundle

The Clifford bundle $Cl(TM)$ is defined through the basis $(\varepsilon_i(m))_{i=0}^3$. In a change of gauge on P_G :

$$\mathbf{p}(m) = \varphi_G(m, 1) \rightarrow \tilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1}$$

The holonomic basis of $P_G[\mathbb{R}^4, \mathbf{Ad}]$ changes as :

$$\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i) \rightarrow \tilde{\varepsilon}_i(m) = \chi(m)^{-1} \cdot \varepsilon_i(m) \cdot \chi(m)$$

and the Clifford product of vectors as :

$$\varepsilon_{i_1}(m) \cdot \varepsilon_{i_2}(m) \cdot \dots \cdot \varepsilon_{i_p}(m) \rightarrow \tilde{\varepsilon}_{i_1}(m) \cdot \dots \cdot \tilde{\varepsilon}_{i_p}(m) = \chi(m)^{-1} \cdot \varepsilon_{i_1}(m) \cdot \varepsilon_{i_2}(m) \cdot \dots \cdot \varepsilon_{i_p}(m) \cdot \chi(m)$$

that is :

$$\tilde{\varepsilon}_i(m) \cdot \dots \cdot \tilde{\varepsilon}_{i_p}(m) = \mathbf{Ad}_{\chi(m)^{-1}} \varepsilon_{i_1}(m) \cdot \dots \cdot \varepsilon_{i_p}(m)$$

and the components of an element of $Cl(TM)$ in the holonomic basis $(\varepsilon_i(m))_{i=0}^3$

as

$$\tilde{w}^{i_1 \dots i_p} = [\mathbf{Ad}]_{j_1}^{i_1} \dots [\mathbf{Ad}]_{j_p}^{i_p} w^{j_1 \dots j_p}$$

$$\text{so } Cl(TM) \equiv P_G[Cl(3, 1), \mathbf{Ad}]$$

The Clifford algebra built on covectors : ε^i in \mathbb{R}^4 with the bisymmetric linear form of same signature is isomorphic to the Clifford algebras that we have denoted $Cl(3, 1)$ or $Cl(1, 3)$: this is just the replacement of ε_i by the dual ε^i . Notice that $\varepsilon^i(\varepsilon_j) = \delta_j^i$ and not η_{ij} . We will denote them $Cl(3, 1)^*$ or $Cl(1, 3)^*$.

The Clifford bundle $Cl(TM^*)$ is the associated vector bundle defined through the basis $(\varepsilon^i(m))_{i=0}^3$. In a change of gauge on P_G the holonomic basis changes as :

$$\varepsilon^i(m) = (\mathbf{p}(m), \varepsilon^i) \rightarrow \tilde{\varepsilon}^i(m) = \chi(m) \cdot \varepsilon^i(m) \cdot \chi(m)^{-1}$$

the elements of $Cl(TM^*)$ transform as :

$$\varepsilon^{i_1}(m) \dots \cdot \varepsilon^{i_p}(m) \rightarrow \tilde{\varepsilon}^{i_1}(m) \cdot \dots \cdot \tilde{\varepsilon}^{i_p}(m) = \mathbf{Ad}_{\chi(m)} \varepsilon^{i_1}(m) \dots \cdot \varepsilon^{i_p}(m)$$

and the components of an element of $Cl(TM^*)$ in the holonomic basis

$(\varepsilon^i(m))_{i=0}^3$ as

$$\tilde{w}_{i_1 \dots i_p} = [\mathbf{Ad}]_{i_1}^{j_1} \dots [\mathbf{Ad}]_{i_p}^{j_p} w_{j_1 \dots j_p}$$

So we will denote this associated vector bundle $P_G[Cl(3, 1)^*, \mathbf{Ad}^{-1}]$.

There are a Spin group $Spin(3,1)^*$ and its Spin algebra $T_1Spin(3,1)^*$ defined in $P_G [Cl(3,1)^*, \mathbf{Ad}^{-1}]$ and we denote the basis of $T_1Spin(3,1)^*$ ¹⁰ :

$$\begin{aligned}\vec{\kappa}^1 &= \frac{1}{2}\varepsilon^3 \cdot \varepsilon^2, \vec{\kappa}^2 = \frac{1}{2}\varepsilon^1 \cdot \varepsilon^3, \vec{\kappa}^3 = \frac{1}{2}\varepsilon^2 \cdot \varepsilon^1, \\ \vec{\kappa}^4 &= \frac{1}{2}\varepsilon^0 \cdot \varepsilon^1, \vec{\kappa}^5 = \frac{1}{2}\varepsilon^0 \cdot \varepsilon^2, \vec{\kappa}^6 = \frac{1}{2}\varepsilon^0 \cdot \varepsilon^3\end{aligned}$$

9.4 2 forms expressed in the Clifford bundle

The algebra $\oplus_{r=0}^4 \Lambda_r TM^*$ of forms on M is isomorphic (as vector space, the isomorphism does not extend to the product) to the Clifford algebra $Cl(TM^*)$. This isomorphism is defined through any orthonormal basis.

$$j : \Lambda TM^* \rightarrow Cl(TM^*) : j(\varepsilon^{i_1} \wedge \dots \wedge \varepsilon^{i_p}) = \varepsilon^{i_1} \cdot \dots \cdot \varepsilon^{i_p} \quad (121)$$

Then we can associate, by the isomorphism j the 2 form $\sum_{ij=0}^3 F_{ij} \varepsilon^i(m) \wedge \varepsilon^j(m)$ to the element of the dual Clifford bundle $Cl(TM^*)$:

$$j(\mathcal{F}) = \sum_{ij=0}^3 F_{ij} \varepsilon^i(m) \cdot \varepsilon^j(m) \quad (122)$$

It reads :

$$j(\mathcal{F}) = 2(F_{32}\varepsilon^3 \cdot \varepsilon^2 + F_{13}\varepsilon^1 \cdot \varepsilon^3 + F_{21}\varepsilon^2 \cdot \varepsilon^1 + F_{01}\varepsilon^0 \cdot \varepsilon^1 + F_{02}\varepsilon^0 \cdot \varepsilon^2 + F_{03}\varepsilon^0 \cdot \varepsilon^3)$$

With the decomposition :

$$\mathcal{F}_R = 2(F_{32}\varepsilon^3 \wedge \varepsilon^2 + F_{13}\varepsilon^1 \wedge \varepsilon^3 + F_{21}\varepsilon^2 \wedge \varepsilon^1)$$

$$\mathcal{F}_W = 2(F_{01}\varepsilon^0 \wedge \varepsilon^1 + F_{02}\varepsilon^0 \wedge \varepsilon^2 + F_{03}\varepsilon^0 \wedge \varepsilon^3)$$

and $v^*(\mathcal{F}_R, \mathcal{F}_W)$

$$= \frac{1}{2}(F_{32}\varepsilon^3 \cdot \varepsilon^2 + F_{13}\varepsilon^1 \cdot \varepsilon^3 + F_{21}\varepsilon^2 \cdot \varepsilon^1 + F_{01}\varepsilon^0 \cdot \varepsilon^1 + F_{02}\varepsilon^0 \cdot \varepsilon^2 + F_{03}\varepsilon^0 \cdot \varepsilon^3)$$

$$j(\mathcal{F}) = 4v^*((F_{32}, F_{13}, F_{21}), (F_{01}, F_{02}, F_{03})) = 4v^*(\mathcal{F}_R, \mathcal{F}_W) \quad (123)$$

In the basis $\vec{\kappa}^a$: $j(\mathcal{F}) = \sum_{a=1}^6 [j(\mathcal{F})]_a \vec{\kappa}^a$

That we can sum in the formula :

$$\{[j(\mathcal{F})]_a\}_{a=1}^6 = 4 \sum_{\alpha\beta=0}^3 \mathcal{F}_{\alpha\beta} \left\{ P_3^\alpha P_2^\beta, P_1^\alpha P_3^\beta, P_2^\alpha P_1^\beta, P_0^\alpha P_1^\beta, P_0^\alpha P_2^\beta, P_0^\alpha P_3^\beta \right\}$$

The isomorphism uses the bases of $T_m M^*$ and $P_G[\mathbb{R}^4, \mathbf{Ad}]$. We need to check how $j(\mathcal{F})$ behaves in a change of gauge or chart.

In a change of chart on M, the variables change as :

$$P_i^\alpha \rightarrow \tilde{P}_i^\alpha = \sum_\lambda J_\lambda^\alpha P_i^\lambda$$

$$\mathcal{F}_{\alpha\beta} \rightarrow \tilde{\mathcal{F}}_{\alpha\beta} = \sum_{\gamma\eta} K_\alpha^\gamma K_\beta^\eta \mathcal{F}_{\gamma\eta}$$

$$\text{with the jacobian : } J = \left[J_\beta^\alpha \right] = \left[\frac{\partial \tilde{\xi}^\alpha}{\partial \xi^\beta} \right] \text{ and } K = J^{-1}$$

$$\widetilde{[j(\mathcal{F})]}_a = \sum_{\alpha\beta=0}^3 \tilde{\mathcal{F}}_{\alpha\beta} \left\{ \tilde{P}_3^\alpha \tilde{P}_2^\beta, \tilde{P}_1^\alpha \tilde{P}_3^\beta, \tilde{P}_2^\alpha \tilde{P}_1^\beta, \tilde{P}_0^\alpha \tilde{P}_1^\beta, \tilde{P}_0^\alpha \tilde{P}_2^\beta, \tilde{P}_0^\alpha \tilde{P}_3^\beta \right\}$$

$$\widetilde{[j(\mathcal{F})]}_a = \sum_{\alpha\beta=0}^3 \sum_{\gamma\eta} K_\alpha^\gamma K_\beta^\eta \mathcal{F}_{\gamma\eta} \sum_{\lambda\mu} J_\lambda^\alpha J_\mu^\beta \left\{ P_3^\lambda P_2^\mu, P_1^\lambda P_3^\mu, P_2^\lambda P_1^\mu, P_0^\lambda P_1^\mu, P_0^\lambda P_2^\mu, P_0^\lambda P_3^\mu \right\}$$

¹⁰Notice that this is the basis of the subset $T_1Spin(3,1)^*$ of $Cl(M^*)$ and not the basis of the vector space dual of $T_1Spin(3,1)$.

$$\widetilde{[j(\mathcal{F})]}_a = \sum_{\lambda\mu=0}^3 \mathcal{F}_{\lambda\mu} \{P_3^\lambda P_2^\mu, P_1^\lambda P_3^\mu, P_2^\lambda P_1^\mu, P_0^\lambda P_1^\mu, P_0^\lambda P_2^\mu, P_0^\lambda P_3^\mu\}$$

$$\widetilde{[j(\mathcal{F})]}_a = [j(\mathcal{F})]_a$$

So $j(\mathcal{F})$ is invariant in a change of chart on M.

9.5 Strength of the gravitational field in the Clifford algebras

The field is valued in the Lie algebras. The corresponding quantities read :

$$\begin{aligned} j(\mathcal{F}_G) &= \sum_{a=1}^6 \sum_{\alpha\beta=0}^3 \mathcal{F}_{G\alpha\beta}^a P_i^\alpha P_j^\beta \varepsilon^i \otimes \varepsilon^j \otimes \vec{\kappa}_a \\ &= \sum_{a=1}^6 \sum_{\alpha\beta=0}^3 v(\mathcal{F}_{r\alpha\beta}, \mathcal{F}_{w\alpha\beta}) P_i^\alpha P_j^\beta \varepsilon^i \otimes \varepsilon^j \end{aligned}$$

$$j(\mathcal{F}_G)(m) = \sum_{a,b=1}^6 [j(\mathcal{F}_G)]_b^a \vec{\kappa}^b(m) \otimes \vec{\kappa}_a(m) \quad (124)$$

$$F_{rij}^a = \sum_{\alpha\beta=0}^3 \mathcal{F}_{r\alpha\beta}^a P_i^\alpha P_j^\beta$$

$$F_{wij}^a = \sum_{\alpha\beta=0}^3 \mathcal{F}_{w\alpha\beta}^a P_i^\alpha P_j^\beta$$

$$\begin{aligned} j(\mathcal{F}_G) &= 2\{v(F_{r32}, F_{w32}) \varepsilon^3 \cdot \varepsilon^2 + v(F_{r13}, F_{w13}) \varepsilon^1 \cdot \varepsilon^3 + v(F_{r21}, F_{w21}) \varepsilon^2 \cdot \varepsilon^1 \\ &+ v(F_{r01}, F_{w01}) \varepsilon^0 \cdot \varepsilon^1 + v(F_{r02}, F_{w02}) \varepsilon^0 \cdot \varepsilon^2 + v(F_{r03}, F_{w03}) \varepsilon^0 \cdot \varepsilon^3\} \end{aligned}$$

Expressed with the 6×6 matrix $[j(\mathcal{F}_G^a)]$

$$\begin{aligned} a=1,2,3 : [j(\mathcal{F}_G)]_1^a &= 4F_{r32}^a; [j(\mathcal{F}_G)]_2^a = 4F_{r13}^a; [j(\mathcal{F}_G)]_3^a = 4F_{r21}^a; [j(\mathcal{F}_G)]_4^a = \\ &4F_{r01}^a; [j(\mathcal{F}_G)]_5^a = 4F_{r02}^a; [j(\mathcal{F}_G)]_6^a = 4F_{r03}^a \\ a=4,5,6 : [j(\mathcal{F}_G)]_1^a &= 4F_{w32}^a; [j(\mathcal{F}_G)]_2^a = 4F_{w13}^a; [j(\mathcal{F}_G)]_3^a = 4F_{w21}^a; [j(\mathcal{F}_G)]_4^a = \\ &4F_{w01}^a; [j(\mathcal{F}_G)]_5^a = 4F_{w02}^a; [j(\mathcal{F}_G)]_6^a = 4F_{w03}^a \end{aligned}$$

In a change of gauge on the principal bundle $P_G : \mathbf{p}_G(m) = \varphi_G(m, 1) \rightarrow \tilde{\mathbf{p}}_G(m) = \mathbf{p}_G(m) \cdot \chi(m)^{-1}$ the holonomic basis becomes with $\chi(m) \in Spin(3, 1)$

$\vec{\kappa}^b \rightarrow \widetilde{\vec{\kappa}}^b = \mathbf{Ad}_\chi \vec{\kappa}^b$ notice that this is a 6×6 matrix for $\mathbf{Ad}_\chi : T_1 Spin(3, 1) \rightarrow T_1 Spin(3, 1)$

$$\vec{\kappa}_a \rightarrow \mathbf{Ad}_{\chi^{-1}} \vec{\kappa}_a$$

$$j(\mathcal{F}_G)(m) = \sum_{a,b=1}^6 [j(\mathcal{F}_G)]_b^a \vec{\kappa}^b \otimes \vec{\kappa}_a = \sum_{a,b=1}^6 [\widetilde{[j(\mathcal{F}_G)]}]_b^a \widetilde{\vec{\kappa}}^b \otimes \vec{\kappa}_a$$

$$= \sum_{a,b=1}^6 [j(\mathcal{F}_G)]_b^a \mathbf{Ad}_\chi \vec{\kappa}^b \otimes \mathbf{Ad}_{\chi^{-1}} \vec{\kappa}_a$$

$$= \sum_{a,bcd=1}^6 [j(\mathcal{F}_G)]_b^a [\mathbf{Ad}_\chi]_d^b \vec{\kappa}^d \otimes [\mathbf{Ad}_{\chi^{-1}}]_a^c \vec{\kappa}_c$$

$$[j(\mathcal{F}_G)] = [\mathbf{Ad}_{\chi^{-1}}] [j(\mathcal{F}_G)] [\mathbf{Ad}_\chi]$$

$$[\widetilde{[j(\mathcal{F}_G)]}] = [\mathbf{Ad}_\chi] [j(\mathcal{F}_G)] [\mathbf{Ad}_{\chi^{-1}}] \quad (125)$$

$(T_1 Spin(3, 1)^* \otimes T_1 Spin(3, 1), \mathbf{Ad}^{-1} \times \mathbf{Ad})$ is a representation of $Spin(3, 1)$
 $P_G [T_1 Spin(3, 1)^* \otimes T_1 Spin(3, 1), \mathbf{Ad}^{-1} \times \mathbf{Ad}]$ is an associated vector bundle

The fields are defined everywhere, and we can associate to \mathcal{F}_G the section $j(\mathcal{F}_G)$ of the associated bundle :

$$(\mathbf{p}(m), j(\mathcal{F}_G)) \sim (\mathbf{p}(m) \cdot \chi^{-1}, [\mathbf{Ad}_\chi] [j(\mathcal{F}_G)] [\mathbf{Ad}_{\chi^{-1}}])$$

$j(\mathcal{F}_G)$ is a tensor : it has an intrinsic existence, independent of any chart or gauge, and its measure changes, in a change of gauge on P_G , in a clear and unified way, with respect to the observer. It involves only the strength of the field, as expected, and as the potentials cannot figure explicitly in a lagrangian, this is consistent. And we state :

Proposition 54 *The relativist momentum density of the gravitational field is represented as a section of the associated vector bundle :*

$$j(\mathcal{F}_G) \in \mathfrak{X}(P_G [T_1Spin(3,1)^* \otimes T_1Spin(3,1), \mathbf{Ad}^{-1} \times \mathbf{Ad}])$$

10 ENERGY OF THE GRAVITATIONAL FIELD

It is intuitive that Force fields carry energy. Because fields are present everywhere, it will be a density, valued at any point and related to the volume form ϖ_4 . It should be a quantity which changes in a consistent way with the observer. And, as for particles, it would be logical if it involves a scalar product of the relativist momentum. We could do it from the expression above, but it is more illuminating to proceed from the expression in any chart, as it leads to introduce useful tools. There are two vector spaces involved : the tangent space of 2 forms with the holonomic basis of a chart, and the Lie algebra. So we need to address successively both spaces.

10.1 Scalar product of forms over M

There is a scalar product G_r on the space $\Lambda_r M$ of scalar r forms computed with the metric g (Maths.1611). This is a bilinear symmetric form, which does not depend on a chart, is non degenerate and definite positive if g is Riemannian.

The **Hodge dual** $*\lambda$ of a r form λ is a $4 - r$ form (Maths.1613) such that :
 $\forall \mu \in \Lambda_r(M) : \mu \wedge *\lambda = G_r(\mu, \lambda) \varpi_4$

The Hodge dual $*\mathcal{F}$ of a scalar 2-form $\mathcal{F} \in \Lambda_2 M$ is a 2 form whose expression, with the Lorentz metric, is simple when a specific ordering is used, which have been used before about the polarization of the fields :

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_R + \mathcal{F}_W \text{ with} \\ \mathcal{F}_R &= 2 (\mathcal{F}_{32} d\xi^3 \wedge d\xi^2 + \mathcal{F}_{13} d\xi^1 \wedge d\xi^3 + \mathcal{F}_{21} d\xi^2 \wedge d\xi^1) \\ \mathcal{F}_W &= 2 (\mathcal{F}_{01} d\xi^0 \wedge d\xi^1 + \mathcal{F}_{02} d\xi^0 \wedge d\xi^2 + \mathcal{F}_{03} d\xi^0 \wedge d\xi^3) \\ &\text{(the 2 accounting for the symmetric part)} \end{aligned}$$

By raising the indices with g we get :

$$\begin{aligned} \mathcal{F}^{\alpha\beta} &= \sum_{\lambda\mu} g^{\alpha\lambda} g^{\beta\mu} \mathcal{F}_{\lambda\mu} \\ *\mathcal{F} &= *\mathcal{F}_R + *\mathcal{F}_W \\ *\mathcal{F}_R &= 2 (\mathcal{F}^{01} d\xi^3 \wedge d\xi^2 + \mathcal{F}^{02} d\xi^1 \wedge d\xi^3 + \mathcal{F}^{03} d\xi^2 \wedge d\xi^1) \det P' \\ *\mathcal{F}_W &= 2 (\mathcal{F}^{32} d\xi^0 \wedge d\xi^1 + \mathcal{F}^{13} d\xi^0 \wedge d\xi^2 + \mathcal{F}^{21} d\xi^0 \wedge d\xi^3) \det P' \end{aligned}$$

thus the components of the parts are exchanged and the indices are raised with the metric g. Notice that the Hodge dual is a 2 form : even if the notation uses raised indexes, they refer to the basis $d\xi^\alpha \wedge d\xi^\beta$.

Take any two scalar 2 forms \mathcal{F}, K and their decomposition as above, a straightforward computation gives :

$$\begin{aligned} *\mathcal{F}_W \wedge K_W &= 0 \\ *\mathcal{F}_W \wedge K_R &= -4 (\mathcal{F}^{32} K_{32} + \mathcal{F}^{13} K_{13} + \mathcal{F}^{21} K_{21}) \varpi_4 \\ *\mathcal{F}_R \wedge K_W &= -4 (\mathcal{F}^{01} K_{01} + \mathcal{F}^{02} K_{02} + \mathcal{F}^{03} K_{03}) \varpi_4 \\ *\mathcal{F}_R \wedge K_R &= 0 \\ G_2(\mathcal{F}_W, K_W) &= G_2(\mathcal{F}_R, K_R) = 0 \\ G_2(\mathcal{F}_W, K_R) &= -4 (\mathcal{F}^{32} K_{32} + \mathcal{F}^{13} K_{13} + \mathcal{F}^{21} K_{21}) \\ G_2(\mathcal{F}_R, K_W) &= -4 (\mathcal{F}^{01} K_{01} + \mathcal{F}^{02} K_{02} + \mathcal{F}^{03} K_{03}) \\ \text{From there, because } G_2 &\text{ is bilinear :} \\ G_2(\mathcal{F}, K) &= G_2(\mathcal{F}_W + \mathcal{F}_R, K_R + K_W) \end{aligned}$$

$$= -4 (\mathcal{F}^{32} K_{32} + \mathcal{F}^{13} K_{13} + \mathcal{F}^{21} K_{21} + \mathcal{F}^{01} K_{01} + \mathcal{F}^{02} K_{02} + \mathcal{F}^{03} K_{03})$$

$$G_2(\mathcal{F}, K) = -2 \sum_{\alpha\beta} \mathcal{F}^{\alpha\beta} K_{\alpha\beta} \quad (126)$$

These quantities can be expressed in the orthonormal basis $(\varepsilon^i(m))_{i=0}^3$.

$$\begin{aligned} G_2(\mathcal{F}, K) &= -2 \sum_{\alpha\beta} \mathcal{F}^{\alpha\beta} K_{\alpha\beta} = -2 \sum_{\alpha\beta} g^{\alpha\lambda} g^{\beta\mu} \sum_{ijpq=0}^3 F_{ij} P_\lambda^i P_\mu^j K_{pq} P_\alpha^p P_\beta^q \\ &= -2 \sum_{ijpq} \eta^{pi} \eta^{qj} F_{ij} K_{pq} = -2 \sum_{ij=0}^3 \eta^{ii} \eta^{jj} F_{ij} K_{ij} \\ &= -2 \left(\sum_{j=1}^3 \eta^{00} \eta^{jj} F_{0j} K_{0j} + \sum_{i=0}^3 \eta^{ii} \eta^{00} F_{i0} K_{i0} + \sum_{ij=1}^3 \eta^{ii} \eta^{jj} F_{ij} K_{ij} \right) \\ &= -2 \left(\sum_{ij=1}^3 F_{ij} K_{ij} - 2 \sum_{j=1}^3 F_{0j} K_{0j} \right) \end{aligned}$$

$$G_2(\mathcal{F}, \mathcal{F}) = 4 \left(\sum_{j=1}^3 (F_{0j})^2 - \sum_{\{ij\}=1}^3 (F_{ij})^2 \right) \quad (127)$$

G_2 is not definite positive or negative. The decomposition emphasizes the role played by the component along ε_0 : the spatial components (F_{ij}) and the temporal components (F_{0j}) contribute to the scalar product with definite opposite signs.

This scalar product can easily be expressed for $J(\mathcal{F})$ using the scalar product on the Clifford algebra $Cl(3, 1)^*$:

$$\begin{aligned} \langle v(r, w), v(r', w') \rangle_{Cl(3,1)^*} &= \frac{1}{4} (r^t r' - w^t w') \\ J(\mathcal{F}) &= 4 \sum_{a=1}^6 \sum_{b=1}^m v^* ((F_{32}, F_{13}, F_{21}), (F_{01}, F_{02}, F_{03})) \in T_1 Spin(3, 1)^* \\ \langle J(\mathcal{F}), J(\mathcal{F}) \rangle_{Cl(3,1)^*} &= \frac{1}{4} 16 \left((F_{32})^2 + (F_{13})^2 + (F_{21})^2 - \left((F_{01})^2 + (F_{02})^2 + (F_{03})^2 \right) \right) \\ \langle J(\mathcal{F}), J(\mathcal{F}) \rangle_{Cl(3,1)^*} &= 4 (F_R^t F_R - F_W^t F_W) = -G_2(\mathcal{F}, \mathcal{F}) \end{aligned}$$

$$\langle J(\mathcal{F}), J(\mathcal{F}) \rangle_{Cl(3,1)^*} = 4 \left(\sum_{\{ij\}=1}^3 (F_{ij})^2 - \sum_{j=1}^3 (F_{0j})^2 \right) = -G_2(\mathcal{F}, \mathcal{F}) \quad (128)$$

So, up to the sign, we have the same quantity.

10.2 Scalar products on the Lie algebras

The strength can be seen as a section of the associated vector bundle $P_G [T_1 Spin(3, 1), Ad]$ and then the scalar product must be preserved by the adjoint map Ad . There are not too many possibilities. It can be shown that, for simple groups of matrices, the only scalar products on their Lie algebra which are invariant by the adjoint map are of the kind : $\langle [X], [Y] \rangle = k Tr([X]^* [Y])$ which sums up, in our case, to use the Killing form. This is a bilinear form (Maths.1609) which

is preserved by any automorphism of the Lie algebra (thus in any representation). However it is negative definite if and only if the group is compact and semi-simple (Maths.1847).

We have seen in previous Sections that the scalar product on $T_1 Spin(3,1)$, induced by the scalar product on the Clifford algebra, is, up to a constant, the Killing form :

$$\begin{aligned}
\langle v(r, w), v(r', w') \rangle_{Cl(3,1)} &= \frac{1}{4} (r^t r' - w^t w') \\
a=1,2,3 : \mathcal{F}_{G\alpha\beta}^a &= \mathcal{F}_{r\alpha\beta}^a \\
a=4,5,6 : \mathcal{F}_{G\alpha\beta}^a &= \mathcal{F}_{w\alpha\beta}^a \\
\langle \mathcal{F}_{G\alpha\beta}(m), \mathcal{F}'_{G\lambda\mu}(m) \rangle_{Cl} &= \left\langle v(\mathcal{F}_{r\alpha\beta}, \mathcal{F}_{w\alpha\beta}), v(\mathcal{F}'_{r\lambda\mu}, \mathcal{F}'_{w\lambda\mu}) \right\rangle_{Cl} \\
&= \frac{1}{4} \left(\mathcal{F}_{r\alpha\beta}^t \mathcal{F}'_{r\lambda\mu} - \mathcal{F}_{w\alpha\beta}^t \mathcal{F}'_{w\lambda\mu} \right) \\
\langle \mathcal{F}_{G\alpha\beta}(m), \mathcal{F}'_{G\lambda\mu}(m) \rangle_{Cl} &= \frac{1}{4} \left(\sum_{a=1}^3 \mathcal{F}_{G\alpha\beta}^a \mathcal{F}'_{G\lambda\mu}^a - \sum_{a=4}^6 \mathcal{F}_{G\alpha\beta}^a \mathcal{F}'_{G\lambda\mu}^a \right) \quad (129)
\end{aligned}$$

The result does not depend on the signature. This scalar product is invariant in a change of gauge, non degenerate but not definite positive.

10.3 Scalar product for the strength of the field

We have to combine both scalar products.

The scalar product on $T_1 Spin(3,1)^* \otimes T_1 Spin(3,1)$ is expressed by :

$$\begin{aligned}
&\left\langle \sum_{ab=1}^6 [X]_a^b \kappa^a \otimes \kappa_b, \sum_{cd=1}^6 [Y]_c^d \kappa^c \otimes \kappa_d \right\rangle \\
&= \left\langle \sum_{b=1}^6 \left(\sum_{a=1}^6 [X]_a^b \kappa^a \right) \otimes \kappa_b, \sum_{d=1}^6 \left(\sum_{c=1}^6 [Y]_c^d \kappa^c \right) \otimes \kappa_d \right\rangle \\
&= \frac{1}{4} \left(\sum_{b=1}^3 \left\langle \sum_{a=1}^6 [X]_a^b \kappa^a, \sum_{c=1}^6 [Y]_c^b \kappa^c \right\rangle - \sum_{b=4}^6 \left\langle \sum_{a=1}^6 [X]_a^b \kappa^a, \sum_{c=1}^6 [Y]_c^b \kappa^c \right\rangle \right) \\
&= \frac{1}{4} \sum_{b=1}^3 \frac{1}{4} \left(\sum_{a=1}^3 [X]_a^b [Y]_a^b - \sum_{a=4}^6 [X]_a^b [Y]_a^b \right) \\
&\quad - \frac{1}{4} \sum_{b=4}^6 \left(\frac{1}{4} \left(\sum_{a=1}^3 [X]_a^b [Y]_a^b - \sum_{a=4}^6 [X]_a^b [Y]_a^b \right) \right) \\
&= \frac{1}{16} \sum_{a=1}^3 \left(\sum_{b=1}^3 [X]_a^b [Y]_a^b - \sum_{b=4}^6 [X]_a^b [Y]_a^b \right) \\
&\quad - \sum_{a=4}^6 \left(\sum_{b=1}^3 [X]_a^b [Y]_a^b - \sum_{b=4}^6 [X]_a^b [Y]_a^b \right) \\
&\langle J(\mathcal{F}_G), J(\mathcal{F}_G) \rangle \\
&= \sum_{b=1}^3 \left([F_{r32}^b]^2 + [F_{r13}^b]^2 + [F_{r21}^b]^2 \right) - \left([F_{w32}^b]^2 + [F_{w13}^b]^2 + [F_{w21}^b]^2 \right) \\
&\quad - \left([F_{r01}^b]^2 + [F_{r02}^b]^2 + [F_{r03}^b]^2 \right) + \left([F_{w01}^b]^2 + [F_{w02}^b]^2 + [F_{w03}^b]^2 \right) \\
&= \sum_{b=1}^3 \left([F_{r32}^b]^2 + [F_{r13}^b]^2 + [F_{r21}^b]^2 \right) - \left([F_{r01}^b]^2 + [F_{r02}^b]^2 + [F_{r03}^b]^2 \right) \\
&\quad - \left(\left([F_{w32}^b]^2 + [F_{w13}^b]^2 + [F_{w21}^b]^2 \right) - \left([F_{w01}^b]^2 + [F_{w02}^b]^2 + [F_{w03}^b]^2 \right) \right) \\
&= \sum_{b=1}^3 -\frac{1}{4} G_2(\mathcal{F}_r, \mathcal{F}_r) + \frac{1}{4} G_2(\mathcal{F}_w, \mathcal{F}_w)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{b=1}^3 -\frac{1}{4} \left(-2 \sum_{\alpha\beta} \mathcal{F}_r^{b\alpha\beta} \mathcal{F}_{r\alpha\beta}^b \right) + \frac{1}{4} \left(-2 \sum_{\alpha\beta} \mathcal{F}_w^{b\alpha\beta} \mathcal{F}_{w\alpha\beta}^b \right) \\
&= \frac{1}{2} \sum_{\alpha\beta=0}^3 \left((\mathcal{F}_r^{\alpha\beta})^t \mathcal{F}_{r\alpha\beta} - (\mathcal{F}_w^{\alpha\beta})^t \mathcal{F}_{w\alpha\beta} \right) \\
\langle j(\mathcal{F}_G), j(\mathcal{F}_G) \rangle &= \frac{1}{2} \sum_{\alpha\beta=0}^3 \left((\mathcal{F}_r^{\alpha\beta})^t \mathcal{F}_{r\alpha\beta} - (\mathcal{F}_w^{\alpha\beta})^t \mathcal{F}_{w\alpha\beta} \right) \quad (130)
\end{aligned}$$

This scalar product is invariant in a change of charts or in a change of gauge. However its expression depends on the decomposition between the transversal and rotational components, characteristic of the observer, as can be seen in the orthonormal basis. It does not depend on the signature of the metric but is not definite positive. All these quantities are, of course, estimated up to constants depending on the units.

10.4 Energy of the field

And we can state :

Proposition 55 *The energy density of the gravitational field is the scalar product $\langle j(\mathcal{F}_G), j(\mathcal{F}_G) \rangle$*

The question of the energy of the gravitational field has been a topic of discussion for years and it is generally accepted that it has no precise value (see Wald). We have here another, consistent answer : there is a density of relativist momentum and energy, which have precise definitions, and whose measure depends on the observer. Notice that these quantities can be computed in the usual frame work : it then involves the Riemann tensor, it is simply more obvious in the fiber bundle presentation. Moreover the gravitational field has a distinctive feature : the energy of the gravitational field has two components which have opposite signs. This can explain the fact that the gravitational field, as we measure it, is exceptionally weak : the electromagnetic force is some 39 orders of magnitude greater than the force of gravity.

The energy of the gravitational field does not depend on the chart or the observer, it is similar to $\langle S_0, S_0 \rangle$ for particles. This quantity is not necessarily positive. However, incorporated in a lagrangian along with the definition of the energy for particles, it provides the expected solutions. We must also remember that the similar scalar product $\langle S_0, S_0 \rangle$ can be negative for antiparticles.

In Thermodynamics, only the variation of the energy of a system has a physical meaning, and it was a source of discussion when Relativity introduced the idea of an absolute energy for particles with $c^2 \langle P, P \rangle$. This absolute energy is linked to the existence of a fundamental state (here represented by S_0). And one can wonder if something similar does exist for the fields. Indeed QTF considers the “energy of the vacuum”. This vacuum is intended as an area where there is no particle or boson (so where is no field), but it has quite an animated life

with virtual particles. So it is worth to consider the existence of a fundamental state for the fields.

The quantity for which it could be defined must be geometric, so it would apply to $J(\mathcal{F}_G)$ which would change with the action $\mathbf{Ad}^{-1} \times \mathbf{Ad}$ of the group. But then we would have to introduce explicitly the value $s(m)$ of elements of $Spin(3,1)$:

$$J(\mathcal{F}_G)(m) = [\mathbf{Ad}_{s(m)}] [J(\mathcal{F}_G)_0] [\mathbf{Ad}_{s^{-1}(m)}]$$

Formally this is not much trouble for $Spin(3,1)$ but less so for U , at least for the fields other than the EM field. Physically we would expect that similarly the state of a particle would be defined on $E \otimes E$ with the action $\gamma C \times \gamma C^{-1}$

$$\psi(m) = [\gamma C(s(m))] [\psi_0] \left[\gamma C(s(m)^{-1}) \right]$$

Then the key variables of any model would be a section $s \in P_G$. But we will not explore this venue.

10.4.1 Identity

We have a useful property which is more general, and holds for all the fields:

Theorem 56 *On the Lie algebra T_1U of a Lie group U , endowed with a symmetric scalar product $\langle \rangle_{T_1U}$ which is preserved by the adjoint map :*

$$\forall X, Y, Z \in T_1U : \langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle \quad (131)$$

Proof. $\forall g \in U : \langle Ad_g X, Ad_g Y \rangle = \langle X, Y \rangle$

take the derivative with respect to g at $g = 1$ for $Z \in T_1U$:

$$(Ad_g X)'(Z) = ad(Z)(X) = [Z, X]$$

$$\langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle = 0 \Leftrightarrow \langle X, [Y, Z] \rangle = \langle [Z, X], Y \rangle$$

exchange X, Z :

$$\Rightarrow \langle Z, [Y, X] \rangle = \langle [X, Z], Y \rangle = -\langle [Z, X], Y \rangle = -\langle X, [Y, Z] \rangle = -\langle Z, [X, Y] \rangle$$

■

10.5 Norm on the spaces of the relativist momentum of the fields

As usual a norm is required for the application of QM theorems.

For the gravitational field :

$$J(\mathcal{F}_G) \in \mathfrak{X}(P_G [T_1Spin(3,1)^* \otimes T_1Spin(3,1), \mathbf{Ad}^{-1} \times \mathbf{Ad}])$$

With the Cartan decomposition in $Cl(3,1)^* \otimes Cl(3,1)$:

$$T_1Spin(3,1)^* \otimes T_1Spin(3,1) = (L_0^* \oplus P_0^*) \otimes (L_0 \oplus P_0) = (L_0^* \otimes L_0) \oplus (L_0^* \otimes P_0) \oplus (P_0^* \otimes L_0) \oplus (P_0^* \otimes P_0)$$

The projections :

$$\pi_{ll} : T_1Spin(3,1)^* \otimes T_1Spin(3,1) \rightarrow L_0^* \otimes L_0$$

$$\pi_{lp} : T_1Spin(3,1)^* \otimes T_1Spin(3,1) \rightarrow L_0^* \otimes P_0, \dots$$

are well defined

$$\forall T \in T_1Spin(3,1)^* \otimes T_1Spin(3,1) = \pi_{ll}(T) + \pi_{lp}(T) + \pi_{pl}(T) + \pi_{pp}(T)$$

The scalar product is extended to the tensorial product by :

$\forall X, X' \in l_0, Y, Y' \in p_0 : \langle X \otimes Y, X' \otimes Y' \rangle = \langle X, X' \rangle_{l_0} \langle Y, Y' \rangle_{p_0}$
and similarly for the others. The norm on $T_1Spin(3, 1)^* \otimes T_1Spin(3, 1)$ is

then :

$$\|T\| = \sqrt{\langle \pi_{uu}(T), \pi_{uu}(T) \rangle + \langle \pi_{lp}(T), \pi_{lp}(T) \rangle + \langle \pi_{pl}(T), \pi_{pl}(T) \rangle + \langle \pi_{pp}(T), \pi_{pp}(T) \rangle}$$

We define the norm on the space $\Lambda_2(M; T_1Spin(3, 1))$ by :

$$\|\mathcal{F}_G\| = \|J(\mathcal{F}_G)\|$$

This norms is invariant in a change of chart or in a change of gauge.

With these norms the set :

$$\begin{aligned} L_G^1 &= L^1(M, \mathfrak{X}(P_G[T_1Spin(3, 1)^* \otimes T_1Spin(3, 1), \mathbf{Ad}^{-1} \times \mathbf{Ad}]), \varpi_4) = \\ &= \{J(\mathcal{F}_G) \in \mathfrak{X}(P_G[T_1Spin(3, 1)^* \otimes T_1Spin(3, 1), \mathbf{Ad}^{-1} \times \mathbf{Ad}]) : \int_{\Omega} \|\mathcal{F}_G\| \varpi_4 < \infty\} \end{aligned}$$

is a separable Fréchet space.

11 STRUCTURE OF THE GRAVITATIONAL FIELD

Force Fields present several specific features, which can be easily identified in the representation with fiber bundles.

11.1 Quantization of the gravitational field

We consider as variables sections of the vector bundle as defined above. We can implement the theorem (QMR Th.22). There are a Hilbert space H_G , and an isometry Υ_G which associate a vector to each map $j(\mathcal{F}_G)$:

$$\begin{aligned} \Upsilon_G &: L_G^1 \rightarrow H_G \\ \widehat{U}_G &= \Upsilon_G \circ \mathbf{Ad}^{-1} \times \mathbf{Ad} \circ \Upsilon_G^{-1} \\ &\left(H_G, \widehat{U}_G \right) \text{ is a unitary representation of } Spin(3,1), \end{aligned}$$

Moreover we can also implement the theorem (QMR Th.24) by adding the variable $\langle j(\mathcal{F}_G), j(\mathcal{F}_G) \rangle = Y_G$. To each level of energy of the fields in the system is associated a subset $H_G(Y_G)$ invariant by \widehat{U}_G , that is belonging to one of the irreducible representations of $\left(H_G, \widehat{U}_G \right)$.

The only unitary representations of $Spin(3,1)$ are infinite dimensional, and parametrized by two scalars $z \in \mathbb{Z}, k \in \mathbb{R}$.

As a consequence, for a given level of energy, the gravitational field is characterized by two scalars $z \in \mathbb{Z}, k \in \mathbb{R}$. The real k can be linked to Y_G and we will see below that z is linked to the spin of the field.

11.2 Spin of the Gravitational Field

With the Cartan decomposition in $Cl(3,1)^* \otimes Cl(3,1)$:

$$T_1 Spin(3,1)^* \otimes T_1 Spin(3,1) = (L_0^* \oplus P_0^*) \otimes (L_0 \oplus P_0) = (L_0^* \otimes L_0) \oplus (L_0^* \otimes P_0) \oplus (P_0^* \otimes L_0) \oplus (P_0^* \otimes P_0)$$

The projection :

$\pi : T_1 Spin(3,1)^* \otimes T_1 Spin(3,1) \rightarrow L_0^* \otimes L_0$ is well defined and $L_0^* \otimes L_0$ has the structure $T_1 Spin(3)^* \otimes T_1 Spin(3)$

$$\begin{aligned} \pi(j(\mathcal{F}_G)) &= \sum_{a,b=1}^3 [j(\mathcal{F}_G)]_b^a \vec{\kappa}^b(m) \otimes \vec{\kappa}_a(m) \\ &= 4 \sum_{a=1}^3 (F_{r32}^a \vec{\kappa}^1 + F_{r13}^a \vec{\kappa}^2 + F_{r21}^a \vec{\kappa}^3) \otimes \vec{\kappa}_a \end{aligned}$$

$$\pi(j(\mathcal{F}_G)) = 4 \sum_{a=1}^3 \sum_{\alpha\beta=0}^3 \mathcal{F}_{r\alpha\beta}^a \left(P_3^\alpha P_2^\beta \vec{\kappa}^1 + P_3^\alpha P_2^\beta \vec{\kappa}^2 + P_3^\alpha P_2^\beta \vec{\kappa}^3 \right) \otimes \vec{\kappa}_a \quad (132)$$

We can define a relation of equivalence on

$$\mathfrak{X}(P_G [T_1 Spin(3,1)^* \otimes T_1 Spin(3,1), \mathbf{Ad}^{-1} \times \mathbf{Ad}]) :$$

$$j(\mathcal{F}_G) \sim j(\mathcal{F}'_G) \Leftrightarrow \pi(j(\mathcal{F}_G)) = \pi(j(\mathcal{F}'_G))$$

and the class of equivalence of $j(\mathcal{F}_G)$ does not depend on a spatial basis. $\pi(j(\mathcal{F}_G))$ can be defined as the **spin of the gravitational field**. It is quantized

: $\pi(j(\mathcal{F}_G))$ belongs to a $2j+1$ dimensional vector space, isomorphic to (P^j, D^j) with $j \in \mathbb{N}$. *The gravitational field has an integer Spin.* The spin depends only on the components $\mathcal{F}_{r\alpha\beta}^a$.

An elementary representation of the gravitational field, on a given $\Omega_3(t)$, involves only $\pi(j(\mathcal{F}_G))$: a tensor defined in an euclidean, spatial basis. The vacuum, on a hypersurface $\Omega_3(t)$, is invariant by the action of $SO(3)$. For t fixed the maps $\pi(j(\mathcal{F}_G))(\varphi_o(t, x))$ are sum of harmonic polynomials, and clearly one can identify the arguments of the functions, at least locally, with coordinates of points in $\Omega_3(t)$. Harmonic polynomials have been introduced originally as solutions of the gravitational field, and so we are lead to identify $\pi(j(\mathcal{F}_G))$ with the spatial component of the gravitational field, as it is usually understood. Which gives a physical meaning to the decomposition of \mathcal{F}_G in $\mathcal{F}_r, \mathcal{F}_w$.

11.3 Scalar curvature

The tensor $j(\mathcal{F}_G)$ reads, with the dual basis

$$j(\mathcal{F}_G) = 4\{\sum_{a=1}^6 F_{G32}^a \vec{\kappa}^1 \otimes \vec{\kappa}_a + F_{G13}^a \vec{\kappa}^2 \otimes \vec{\kappa}_a + F_{G21}^a \vec{\kappa}^3 \otimes \vec{\kappa}_a + F_{G01}^a \vec{\kappa}^4 \otimes \vec{\kappa}_a + F_{G02}^a \vec{\kappa}^5 \otimes \vec{\kappa}_a + F_{G03}^a \vec{\kappa}^6 \otimes \vec{\kappa}_a\}$$

Thus we can contract the tensor, and the scalar :

$$\begin{aligned} R' &= 4(F_{G32}^1 + F_{G13}^2 + F_{G21}^3 + F_{G01}^4 + F_{G02}^5 + F_{G03}^6) \\ &= 4\sum_{\alpha\beta=0}^3 \mathcal{F}_{r\alpha\beta}^1 P_3^\alpha P_2^\beta + \mathcal{F}_{r\alpha\beta}^2 P_3^\alpha P_2^\beta + \mathcal{F}_{r\alpha\beta}^3 P_3^\alpha P_2^\beta + \mathcal{F}_{w\alpha\beta}^1 P_0^\alpha P_1^\beta + \mathcal{F}_{w\alpha\beta}^2 P_0^\alpha P_2^\beta + \\ &\quad \mathcal{F}_{w\alpha\beta}^3 P_0^\alpha P_3^\beta \end{aligned}$$

$$R' = 2\left(-\sum_{aij=1}^3 \sum_{\alpha\beta} \epsilon(a, i, j) \mathcal{F}_{r\alpha\beta}^a P_i^\alpha P_j^\beta + 2\sum_{a=1}^3 \sum_{\alpha=1}^3 \mathcal{F}_{w0\alpha}^a P_a^\alpha\right)$$

is invariant in a change of gauge or chart. And we retrieve the usual scalar curvature, up to a constant :

$$\begin{aligned} \mathbf{R} &= \sum_{a=1}^3 2\sum_{\alpha=1}^3 \mathcal{F}_{w\alpha 0}^a [P]_a^\alpha + \sum_{\alpha\beta} \mathcal{F}_{r\alpha\beta}^a \sum_{p,q=1}^3 \epsilon(a, q, p) [P]_p^\beta [P]_q^\alpha \\ R' &= -2R \end{aligned}$$

From the quantization of $j(\mathcal{F}_G)$ one can deduce that *the scalar curvature should be related to the level of energy and to the spin.*

The curvature is comprised of a translational and a rotational part. The latter is fully defined by the spin :

$$R'_R = 4(F_{G32}^1 + F_{G13}^2 + F_{G21}^3)$$

It is invariant by a change of spatial gauge, and its value is quantized.

The decomposition of \mathcal{F}_G in its transversal and spatial components seems physically significant. \mathcal{F}_G is equivalent to the Riemann tensor, for which the decomposition is less obvious. The physical meaning of the torsion has perhaps been wrongly understood. So this remark suggests that the right variable to study the gravitational field is (G_r, G_w) with 24 components, and not the Levi-Civita connection.

11.3.1 Symmetry of the force field

It is common to assume that a field is symmetric : a geometric transformation in the space $\Omega(t)$ of an observer is such that the field looks the same. This

problem is simple in the fiber bundle representation : a symmetry of the field is equivalent to a global change of observer given by an element $s(t) \in Spin(3)$.

$$[j(\mathcal{F}_G)] = [\mathbf{Ad}_s] [j(\mathcal{F}_G)] [\mathbf{Ad}_{s^{-1}}] = [j(\mathcal{F}_G)] \Leftrightarrow [\mathbf{Ad}_s] [j(\mathcal{F}_G)] = [j(\mathcal{F}_G)] [\mathbf{Ad}_s]$$

with the 6×6 matrix :

$$[j(\mathcal{F}_G)] = \begin{bmatrix} F_r^r & F_w^r \\ F_r^w & F_w^w \end{bmatrix}$$

and $s(t) = a_r(t) + v(r(t), 0)$ it sums up to the matricial equations :

$$F_r^r = ([1 + a_r j(r) + \frac{3}{4} j(r) j(r)] F_r^r + [1 - a_r^2 + \frac{1}{4} j(r) j(r)] F_w^w) [1 - a_r j(r) + \frac{3}{4} j(r) j(r)]$$

$$F_r^w = [a_r^2 + a_r j(r) + \frac{1}{2} j(r) j(r)] F_r^w [1 - a_r j(r) + \frac{3}{4} j(r) j(r)] =$$

$$F_w^r = \{ ([1 + a_r j(r) + \frac{3}{4} j(r) j(r)] F_r^r + [1 - a_r^2 + \frac{1}{4} j(r) j(r)] F_w^w) [1 - a_r^2 + \frac{1}{4} j(r) j(r)] \\ + ([1 + a_r j(r) + \frac{3}{4} j(r) j(r)] F_w^r + [1 - a_r^2 + \frac{1}{4} j(r) j(r)] F_w^w) [a_r^2 - a_r j(r) + \frac{1}{2} j(r) j(r)] \}$$

$$F_w^w = [a_r^2 + a_r j(r) + \frac{1}{2} j(r) j(r)] F_r^w [1 - a_r^2 + \frac{1}{4} j(r) j(r)]$$

$$+ [a_r^2 + a_r j(r) + \frac{1}{2} j(r) j(r)] F_w^w [a_r^2 - a_r j(r) + \frac{1}{2} j(r) j(r)] =$$

The set of polynomials of 3×3 matrices $A + B j(r) + C j(r) j(r)$ is a commutative algebra, so the solution is :

$$[j(\mathcal{F}_G)] = \begin{bmatrix} [A_r^r + B_r^r j(r) + C_r^r j(r) j(r)] & [A_w^r + B_w^r j(r) + C_w^r j(r) j(r)] \\ [A_r^w + B_r^w j(r) + C_r^w j(r) j(r)] & [A_w^w + B_w^w j(r) + C_w^w j(r) j(r)] \end{bmatrix}$$

and the equations give 3 potential solutions :

i) $r = 0$: this is the spherical solution

ii) $a_r^2 = \frac{3}{5}; A_r^w = 0;$

$$[-\frac{2}{5} - \frac{1}{4} j(r) j(r)] F_w^r + [\frac{2}{5} + \frac{1}{4} j(r) j(r)] (F_r^r + \frac{3}{5} F_w^w) = 0$$

iii) $F_r^w = 0;$

$$[a_r^2 - 1 + (\frac{5}{4} a_r^2 - 1) j(r) j(r)] F_w^r + [1 - a_r^2 + \frac{1}{4} j(r) j(r)] (F_r^r + a_r^2 F_w^w) = 0$$

12 CONCLUSION

In the first part of this paper we have shown that the geometry of GR can be set up in a comprehensive and consistent framework, in the second part we have shown that the kinematic of particles can be represented in a formalism which is consistent with QTF and GR, and in the third part that gauge theories can be extended to the gravitational field. Moreover the tools which have been defined enable to deal, more easily and in a general way, with the usual problems. In particular we have proposed the definition of a deformable solid, which can be used in Astro-Physics.

Of course all the material presented here can be put at work. The natural successive step is to introduce a lagrangian. Its perturbative formulation is :

$$\int_{\Omega} \{ \sum_{\alpha\beta} C_G \langle \mathcal{F}_G, \mathcal{F}_G \rangle + C_I \mu^{\frac{1}{i}} \langle S, \nabla_V S \rangle \} \varpi_4$$

which leads to first order differential equations :

$$A_1(r) \frac{dr}{dt} + A_2(r, w) \frac{dw}{dt} + A_3(r, w) \widehat{G}_r + A_4(r, w) \widehat{G}_w = rK_r$$

$$B_1(r, w) \frac{dr}{dt} + B_2(r, w) \frac{dw}{dt} + B_3(r, w) \widehat{G}_r + B_4(r, w) \widehat{G}_w = -\frac{c\alpha_w}{2\alpha_w^2 - 1} \sum_{\alpha, i=1}^3 P_i^\alpha \left(k^t X_\alpha + \dot{A}_\alpha \right) \varepsilon_i +$$

wK_w

$$k^t \widehat{X} = 0$$

for the spinor field, where $A_1(r), \dots$ are matrices, K_r, K_w are scalars, and to the equations :

$$\forall a, \forall \alpha = 0..3 : \sum_{\beta=0}^3 \partial_\beta \left(\mathcal{F}_G^{\alpha\beta} \det P' \right) = 0$$

$$\frac{C_I}{4C_G} \mu V \otimes v^* \left(k^t [D_r(r, w)], -k^t [D_w(r)] \right) = \sum_{\beta} \left[\mathcal{F}_G^{\alpha\beta}, G_\beta \right] \otimes \partial \xi_\alpha$$

for the gravitational field. The energy of the system is conserved, and we have the additional equation :

$$\forall \alpha, \beta : C_I \mu^{\frac{1}{i}} V^\beta \langle S, \partial_\alpha S \rangle + 4C_G \sum_{\gamma=0}^3 \left\langle \mathcal{F}_G^{\beta\gamma}, \partial_\alpha G_\gamma - \partial_\gamma G_\alpha \right\rangle$$

$$= \delta_\beta^\alpha \left(\frac{1}{4} C_I \mu^{\frac{1}{i}} \langle S, \frac{dS}{dt} \rangle + 2C_G \sum_{\lambda\mu=0}^3 \left\langle \mathcal{F}_G^{\lambda\mu}, \partial_\lambda G_\mu \right\rangle \right)$$

Moreover the formalism helps to have a glimpse at the gravitons.

All these additions are seen in the book “Mathematics in Physics”.

The representation of the gravitational field by connections on one hand, and of the gravitational charges by spinors on the other hand, shows striking similarities with the EM field : indeed they are the only fields which have an infinite range, the EM charge can be incorporated in the gravitational charge, and their propagation equations are similar. This similitude has been remarked by many authors, Heaviside, Negut, Jefimenko, Tajmar, de Matos,...and it has been developed in a full Theory, which has sometimes be opposed to GR. We find here that these similitudes exist in the frame of a GR theory which allows for a more general connection and the use of the Riemann tensor, so it seems more promising to explore this avenue than to fight against GR. As we have seen the gravitational field shows in all its aspects two components. The “magnetic” component can be assimilated to the usual gravity : this is the one which acts in the 3 dimensional space. The “electric” component acts in

the time dimension, and it seems logical to give it a cosmological interpretation : it would be the engine which moves matter on its world line. Both components have opposite effects, and there is no compelling reason that it should always be attractive. The representation of the gravitational field in the bundle $P_G [T_1Spin(3,1)^* \otimes T_1Spin(3,1), \mathbf{Ad}^{-1} \times \mathbf{Ad}]$ enables to explore more efficiently the structure of the field than the usual Petrov-Pirani-Penrose classification based on the Weyl's tensor.

13 BIBLIOGRAPHY

- R.D.Anderson *Some open questions in infinite dimensional topology* Proceeding of the 3d Prague symposium Praha (1972)
- R.Coquereaux *Clifford algebra, spinors and fundamental interactions : twenty years after* arXiv:math-ph/0509040v1 (16 sep 2005)
- J.C.Dutailly *Mathematics for theoretical physics* arXiv:1209-5665v2 [math-ph] (4 feb 2014)
- J.C.Dutailly *Mathematics in physics* Cnrs hal-01169985 (30 jun 2015)
- J.C.Dutailly *Quantum Mechanics revisited* Cnrs hal-00770220 (1 july 2015)
- G.Giachetta,L.Mangiarotti, G.Sardanashvily *Advanced classical field theory* World Scientific (2009)
- M.Guidry *Gauge fields theories* Wiley (1991)
- A.W.Knapp *Lie groups beyond an introduction* Birkhäuser (2005)
- I.Kolar, P.Michor, J.Slovak *Natural operations in differential geometry* Springer-Verlag (1991)
- S.Kopeikin, B.Mashloin *Gravitomagnetic effects in the propagation of electromagnetic waves in variable gravitational fields of arbitrary moving and spinning bodies* arXiv / gr-qc / 0110101 v2 (25 oct 2001)
- E.H.Lieb, M.Loss *Analysis* American Mathematical Society (2000)
- A.Garrett Lisi *An explicit embedding of gravity and the standard model* arXiv: gr-qc 1006498v1 (25 june 2010)
- A.Garrett Lisi *An exceptionnaly simple theory of everything* arXiv:0711.0770v1 [hep-th] (6 nov 2007)
- A.Garrett Lisi *Clifford bundle formulation of BF gravity generalized to the standard model* arXiv:gr-qc 0511120v2 (21 nov 2005)
- M.A.Naimark *Linear representations of the Lorentz group* Pergamon (1964)
- E.Negut *On intrinsic properties of relativistic motions* Revue Roumaine des Sciences Techniques (1990).
- L.Nielsen *A Maxwell Analog Gravitation Theory* Niels Bohr Institute, Copenhagen, Gamma No. 9 (1972).
- Y.Ohnuki *Unitary representations of the Poincaré group and relativistic wave equations* World Scientific (1988)
- P.J.E.Peebles *Principles of physical cosmology* Princeton series in Physics (1993)
- E.Poisson *An introduction to the Lorentz-Dirac equation* arXiv:gr-qc/ 9912045v1 (10 dec 1999)
- S.Pokorski *Gauge fields theories* Cambridge (2000)
- T.C.Quinn *Axiomatic approach to the radiation reaction of scalar point particles in curved space-time* arXiv:gr-qc/0005030v1/ (10 may 2000)
- Mesgun Sehbatu *The standard model and beyond* (1992)
- M.Tajmar, C.J. de Matos *Coupling of Gravitation and Electromagnetism in the Weak Field Approximation* arXiv:gr-qc/0003011 (2003)
- R.M.Wald *General Relativity* Univ.Chicago (1984)
- S.Weinberg *The quantum theory of fields* Cambridge University Press (1995)
- S.Weinberg *Dreams of a Final Theory* Pantheon Books (1992)

14 ANNEX 1 : CLIFFORD ALGEBRAS

This annex gives proofs of some results presented in the core of the paper.

14.1 Products in the Clifford algebra

Many results are consequences of the computation of products in the Clifford algebra. The computations are straightforward but the results precious. In the following $\langle \varepsilon_0, \varepsilon_0 \rangle = -1$ with the signature (3,1) and +1 with the signature (1,3). The operator j is reminded in the Formulas at the end of this Annex.

14.1.1 Product $v(r, w) \cdot v(r', w')$

$$\begin{aligned} v(r, w) &= \frac{1}{2} (w^1 \varepsilon_0 \cdot \varepsilon_1 + w^2 \varepsilon_0 \cdot \varepsilon_2 + w^3 \varepsilon_0 \cdot \varepsilon_3 + r^3 \varepsilon_2 \cdot \varepsilon_1 + r^2 \varepsilon_1 \cdot \varepsilon_3 + r^1 \varepsilon_3 \cdot \varepsilon_2) \\ v(r', w') &= \frac{1}{2} (w'^1 \varepsilon_0 \cdot \varepsilon_1 + w'^2 \varepsilon_0 \cdot \varepsilon_2 + w'^3 \varepsilon_0 \cdot \varepsilon_3 + r'^3 \varepsilon_2 \cdot \varepsilon_1 + r'^2 \varepsilon_1 \cdot \varepsilon_3 + r'^1 \varepsilon_3 \cdot \varepsilon_2) \\ &\text{With signature (3,1) :} \end{aligned}$$

$$\begin{aligned} v(r, w) \cdot v(r', w') &= \frac{1}{4} (w^t w' - r^t r') + \frac{1}{2} v(j(r) r' - j(w) w', j(w) r' + j(r) w') - \\ &\frac{1}{4} (w^t r' + r^t w') \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \end{aligned}$$

From there the bracket on the Lie algebra :

$$[v(r, w), v(r', w')] = v(r, w) \cdot v(r', w') - v(r', w') \cdot v(r, w)$$

$$[v(r, w), v(r', w')] = v(j(r) r' - j(w) w', j(w) r' + j(r) w') \quad (133)$$

With signature (1,3) :

$$\begin{aligned} v(r, w) \cdot v(r', w') &= \frac{1}{4} (w^t w' - r^t r') - \frac{1}{2} v(-j(r) r' + j(w) w', j(w) r' + j(r) w') - \\ &\frac{1}{4} (w^t r' + r^t w') \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \end{aligned}$$

From there the bracket on the Lie algebra :

$$[v(r, w), v(r', w')] = -v(j(r) r' - j(w) w', j(w) r' + j(r) w') \quad (134)$$

In both signatures the basis of the Lie algebra is denoted :

$$\begin{aligned} \vec{K}_1 &= \frac{1}{2} \varepsilon_3 \cdot \varepsilon_2, \\ \vec{K}_2 &= \frac{1}{2} \varepsilon_1 \cdot \varepsilon_3, \\ \vec{K}_3 &= \frac{1}{2} \varepsilon_2 \cdot \varepsilon_1, \\ \vec{K}_4 &= \frac{1}{2} \varepsilon_0 \cdot \varepsilon_1, \\ \vec{K}_5 &= \frac{1}{2} \varepsilon_0 \cdot \varepsilon_2, \\ \vec{K}_6 &= \frac{1}{2} \varepsilon_0 \cdot \varepsilon_3 \\ a, b, c &= 1, 2, 3 \\ [\vec{K}_a, \vec{K}_b] &= \epsilon(c, a, b) \vec{K}_c \\ [\vec{K}_a, \vec{K}_{3+b}] &= \epsilon(c, a, b) \vec{K}_{3+c} \\ [\vec{K}_{3+a}, \vec{K}_{3+b}] &= \epsilon(c, a, b) \vec{K}_{3+c} \end{aligned}$$

14.1.2 Product on $Spin(3,1)$

Because they belong to $Cl_0(3,1)$ the elements of $Spin(3,1)$ can be written :

$$s = a + \frac{1}{2} (w^1 \varepsilon_0 \cdot \varepsilon_1 + w^2 \varepsilon_0 \cdot \varepsilon_2 + w^3 \varepsilon_0 \cdot \varepsilon_3 + r^3 \varepsilon_2 \cdot \varepsilon_1 + r^2 \varepsilon_1 \cdot \varepsilon_3 + r^1 \varepsilon_3 \cdot \varepsilon_2) + b \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$$

where $a, (w^j, r^j)_{j=1}^3, b$ are real scalar which are related. That we will write with

$$\varepsilon_5 = \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \quad (135)$$

$$s = a + v(r, w) + b \varepsilon_5 \quad (136)$$

And similarly in $Cl(1,3)$

$$s = a + v(r, w) + b \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$$

The product of two elements of the spin group expressed as :

$$s = a + v(r, w) + b \varepsilon_5$$

$$s' = a' + v(r', w') + b' \varepsilon_5$$

can be computed with the previous formulas.

i) With signature (3,1)

$$v(r, w) \varepsilon_5 = v(r, -w)$$

$$\varepsilon_5 v(r', w') = v(r', -w')$$

$$\varepsilon_5 \cdot \varepsilon_5 = -1$$

$$s \cdot s' = a'' + v(r'', w'') + b'' \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$$

with :

$$a'' = aa' - b'b + \frac{1}{4} (w^t w' - r^t r')$$

$$b'' = ab' + ba' - \frac{1}{4} (w^t r' + r^t w')$$

$$r'' = \frac{1}{2} (j(r) r' - j(w) w') + a' r + a r' - b' w - b w'$$

$$w'' = \frac{1}{2} (j(w) r' + j(r) w') + a' w + a w' + b' r + b r'$$

So we have in particular :

$$(a + v(0, w)) \cdot (a' + v(0, w')) = aa' + \frac{1}{4} w^t w' + v(-\frac{1}{2} (j(w) w', a' w + a w'))$$

$$(a + v(r, 0)) \cdot (a' + v(r', 0)) = aa' - \frac{1}{4} r^t r' + v(\frac{1}{2} j(r) r' + (a' r + a r'), 0)$$

$$(a_w + v(0, w)) \cdot (a_r + v(r, 0)) = a_w a_r + v(a_w r, a_r w) - \frac{1}{2} (w^t r) \varepsilon_5$$

ii) With signature (1,3)

$$s = a + v(r, w) + b \varepsilon_5$$

$$s' = a' + v(r', w') + b' \varepsilon_5$$

$$v(r, w) \cdot \varepsilon_5 = v(w, r)$$

$$\varepsilon_5 v(r', w') = v(w', r')$$

$$\varepsilon_5 \cdot \varepsilon_5 = -1$$

$$s \cdot s' = a'' + v(r'', w'') + b'' \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$$

with :

$$a'' = aa' - b'b + \frac{1}{4} (w^t w' - r^t r')$$

$$b'' = ab' + ba' - \frac{1}{4} (w^t r' + r^t w')$$

$$r'' = \frac{1}{2} (j(r) r' - j(w) w') + a' r + a r' + b' w + b w'$$

$$w'' = -\frac{1}{2} (j(w)r' + j(r)w') + a'w + aw' + b'r + br'$$

14.2 Characterization of the elements of the Spin group

14.2.1 Inverse

The elements of $Spin(3, 1)$ are the product of an even number of vectors of norm ± 1 . Consequently we have :

$$s \cdot s^t = (v_1 \cdot \dots \cdot v_{2p}) \cdot (v_{2p} \cdot \dots \cdot v_1) = 1$$

The transposition is an involution on the Clifford algebra, thus :

$$(a + v(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3) \cdot (a + v(r, w)^t + b\varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot \varepsilon_0) = 1$$

$$(a + v(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3) \cdot (a - v(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3) = 1$$

$$\Leftrightarrow (a + v(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3)^{-1} = (a - v(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3)$$

and we have the same result in $Cl(1, 3)$

$$(a + v(r, w) + b\varepsilon_5)^{-1} = a - v(r, w) + b\varepsilon_5 \quad (137)$$

14.2.2 Relation between a,b, r, w

By a straightforward computation this identity gives the following relation between a,b,r,w :

1. With signature (3,1)

$$(a + v(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3) \cdot (a - v(r, w) + b\varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot \varepsilon_0) = 1$$

$$= a'' + v(r'', w'') + b''\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$$

with :

$$a'' = a^2 - b^2 + \frac{1}{4}(-w^t w + r^t r) = 1$$

$$b'' = ab + ba - \frac{1}{4}(-w^t r - r^t w) = 0$$

$$r'' = \frac{1}{2}(-j(r)r + j(w)w) + ar - ar - bw + bw = 0$$

$$w'' = \frac{1}{2}(-j(w)r - j(r)w) + aw - aw + br - br = 0$$

$$a^2 - b^2 = 1 + \frac{1}{4}(w^t w - r^t r)$$

So, for any element : $a + v(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$

we have :

$$a^2 - b^2 = 1 + \frac{1}{4}(w^t w - r^t r) \quad (138)$$

$$ab = -\frac{1}{4}r^t w \quad (139)$$

and we keep only 6 free parameters. a, b are defined from r, w , up to sign, with the conditions:

$$\text{i) } r^t w \neq 0 : b = -\frac{1}{4a}r^t w$$

$$a^2 = \frac{1}{2} \left(\left(1 + \frac{1}{4}(w^t w - r^t r) \right) + \sqrt{\left(1 + \frac{1}{4}(w^t w - r^t r) \right)^2 + \frac{1}{4}(r^t w)^2} \right)$$

$$\text{ii) } r^t w = 0 :$$

$$(w^t w - r^t r) \geq -4 : a = \epsilon \sqrt{1 + \frac{1}{4}(w^t w - r^t r)}; b = 0$$

$$(w^t w - r^t r) \leq -4 : b = \epsilon \sqrt{-\left(1 + \frac{1}{4}(w^t w - r^t r)\right)}; a = 0$$

So :

$$\text{if } r = 0 \text{ then } : s = \epsilon \sqrt{1 + \frac{1}{4}w^t w} + v(0, w)$$

if $w = 0$ then

$$r^t r \leq 4 : s = \epsilon \sqrt{1 - \frac{1}{4}r^t r} + v(r, 0)$$

$$r^t r \geq 4 : s = v(r, 0) + \epsilon \sqrt{\frac{1}{4}r^t r - 1} \epsilon_5$$

2. With signature (1,3)

$$(a - v(r, w) + b \epsilon_0 \cdot \epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3) \cdot (a + v(r, w) + b \epsilon_3 \cdot \epsilon_2 \cdot \epsilon_1 \cdot \epsilon_0) = 1$$

$$= a'' + v(r'', w'') + b'' \epsilon_0 \cdot \epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3$$

with :

$$r'' = \frac{1}{2}(-j(r)r + j(w)w) + ar - ar + bw - bw = 0$$

$$w'' = -\frac{1}{2}(-j(w)r - j(r)w) + aw - aw + br - br = 0$$

$$a'' = a^2 - b^2 + \frac{1}{4}(-w^t w + r^t r)$$

$$b'' = ab + ba - \frac{1}{4}(-w^t r - r^t w)$$

we get the same relations.

14.3 Homogeneous Space

The Clifford Algebras as well as the corresponding Spin groups, for any vector space E of the same dimension and bilinear form g of the same signature are isomorphic. The Clifford Algebra $Cl(3)$ is a subalgebra of $Cl(3, 1)$ and $Spin(3)$ a subgroup of $Spin(3, 1)$.

The Clifford algebras and Spin Group structures are built from the product of vectors. The structure $Cl(3)$ can be defined from a set of vectors only if their scalar product is always definite positive. So, in a given vector space (E, g) with Clifford Algebra isomorphic to $Cl(3, 1)$ the set isomorphic to $Cl(3)$ is not unique : there is one set for each choice of a vector $\epsilon_0 \in E$ such that $g(\epsilon_0, \epsilon_0) = -1$.

14.3.1 The sets isomorphic to $Cl(3)$

In E let be E^\perp the orthogonal complement to $\epsilon_0 : E^\perp = \{u \in E : g(\epsilon_0, u) = 0\}$. This is a 3 dimensional vector space. The scalar product g^\perp induced on E^\perp by g is definite positive : in a basis of E^\perp the matrix of g^\perp has 3 positive eigen values, otherwise with ϵ_0 we would have another signature. The Clifford Algebra $Cl(E^\perp, g^\perp)$ generated by (E^\perp, g^\perp) is a subset of $Cl(E, g)$, Clifford isomorphic to $Cl(3)$. There is a morphism of Clifford algebras : $\pi : Cl(E, g) \rightarrow Cl(E^\perp, g^\perp)$ such that : $\forall u \in E : u = \pi(u) + u^0 \epsilon_0, g(\pi(u), \pi(u)) > 0$ iff $u \neq 0$.

The Spin Group $Spin(E, g)$ isomorphic to $Spin(3, 1)$ is projected into a subgroup $Sp(E^\perp, g^\perp)$ isomorphic to $Spin(3)$.

$Spin(E^\perp, g^\perp)$ is characterized by the following theorem, which can be proven with any orthonormal basis of E containing ϵ_0 :

Theorem 57 *The subset of $Spin(3, 1)$ of the elements which commute with ε_0 is a subgroup of $Spin(3, 1)$. They leave ε_0 unchanged : $\mathbf{Ad}_{s_r}\varepsilon_0 = s_r \cdot \varepsilon_0 \cdot s_r^{-1} = \varepsilon_0 = [g]_0^i \varepsilon_i$. They read : $s_r = a + v(r, 0)$*

Proof. $s_r = a + v(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$

$$s_r \cdot \varepsilon_0 = \varepsilon_0 \cdot s_r$$

In $Cl(3, 1)$:

$$s \cdot \varepsilon_0 = a\varepsilon_0 + v(r, w)\varepsilon_0 - b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 = \varepsilon_0 \cdot s = a\varepsilon_0 + \varepsilon_0 v(r, w) + b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$$

$$v(r, w)\varepsilon_0 =$$

$$\frac{1}{2} (w^1 \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_0 + w^2 \varepsilon_0 \cdot \varepsilon_2 \cdot \varepsilon_0 + w^3 \varepsilon_0 \cdot \varepsilon_3 \cdot \varepsilon_0 + r^3 \varepsilon_2 \cdot \varepsilon_1 \cdot \varepsilon_0 + r^2 \varepsilon_1 \cdot \varepsilon_3 \cdot \varepsilon_0 + r^1 \varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_0)$$

$$= \frac{1}{2} (w^1 \varepsilon_1 + w^2 \varepsilon_2 + w^3 \varepsilon_3 - r^3 \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 + r^2 \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_3 - r^1 \varepsilon_0 \cdot \varepsilon_2 \cdot \varepsilon_3)$$

$$\varepsilon_0 v(r, w)$$

$$= \frac{1}{2} (w^1 \varepsilon_0 \varepsilon_0 \cdot \varepsilon_1 + w^2 \varepsilon_0 \varepsilon_0 \cdot \varepsilon_2 + w^3 \varepsilon_0 \varepsilon_0 \cdot \varepsilon_3 + r^3 \varepsilon_0 \varepsilon_2 \cdot \varepsilon_1 + r^2 \varepsilon_0 \varepsilon_1 \cdot \varepsilon_3 + r^1 \varepsilon_0 \varepsilon_3 \cdot \varepsilon_2)$$

$$= \frac{1}{2} (-w^1 \varepsilon_1 - w^2 \varepsilon_2 - w^3 \varepsilon_3 - r^3 \varepsilon_0 \varepsilon_1 \cdot \varepsilon_2 + r^2 \varepsilon_0 \varepsilon_1 \cdot \varepsilon_3 - r^1 \varepsilon_0 \varepsilon_2 \cdot \varepsilon_3)$$

$$a\varepsilon_0 + \frac{1}{2} (w^1 \varepsilon_1 + w^2 \varepsilon_2 + w^3 \varepsilon_3 - r^3 \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 + r^2 \varepsilon_0 \varepsilon_1 \cdot \varepsilon_3 - r^1 \varepsilon_0 \varepsilon_2 \cdot \varepsilon_3) - b\varepsilon_1 \cdot$$

$$\varepsilon_2 \cdot \varepsilon_3$$

$$= a\varepsilon_0 + \frac{1}{2} (-w^1 \varepsilon_1 - w^2 \varepsilon_2 - w^3 \varepsilon_3 - r^3 \varepsilon_0 \varepsilon_1 \cdot \varepsilon_2 + r^2 \varepsilon_0 \varepsilon_1 \cdot \varepsilon_3 - r^1 \varepsilon_0 \varepsilon_2 \cdot \varepsilon_3) +$$

$$b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$$

$$\Rightarrow w = 0, b = 0$$

In $Cl(1, 3)$:

$$s \cdot \varepsilon_0 = a\varepsilon_0 - v(g)\varepsilon_0 - b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 = \varepsilon_0 \cdot s = a\varepsilon_0 - \varepsilon_0 v(g) + b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \Rightarrow b = 0$$

$$v(g)\varepsilon_0 =$$

$$\frac{1}{2} (w^1 \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_0 + w^2 \varepsilon_0 \cdot \varepsilon_2 \cdot \varepsilon_0 + w^3 \varepsilon_0 \cdot \varepsilon_3 \cdot \varepsilon_0 + r^3 \varepsilon_2 \cdot \varepsilon_1 \cdot \varepsilon_0 + r^2 \varepsilon_1 \cdot \varepsilon_3 \cdot \varepsilon_0 + r^1 \varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_0)$$

$$= \frac{1}{2} (-w^4 \varepsilon_1 - w^2 \varepsilon_2 - w^3 \varepsilon_3 - r^3 \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 + r^2 \varepsilon_0 \varepsilon_1 \cdot \varepsilon_3 - r^1 \varepsilon_0 \varepsilon_2 \cdot \varepsilon_3)$$

$$\varepsilon_0 v(g) = \frac{1}{2} (w^1 \varepsilon_0 \varepsilon_0 \cdot \varepsilon_1 + w^2 \varepsilon_0 \varepsilon_0 \cdot \varepsilon_2 + w^3 \varepsilon_0 \varepsilon_0 \cdot \varepsilon_3 + r^3 \varepsilon_0 \varepsilon_2 \cdot \varepsilon_1 + r^2 \varepsilon_0 \varepsilon_1 \cdot \varepsilon_3 + r^1 \varepsilon_0 \varepsilon_3 \cdot \varepsilon_2)$$

$$= \frac{1}{2} (w^4 \varepsilon_1 + w^2 \varepsilon_2 + w^3 \varepsilon_3 - r^3 \varepsilon_0 \varepsilon_1 \cdot \varepsilon_2 + r^2 \varepsilon_0 \varepsilon_1 \cdot \varepsilon_3 - r^1 \varepsilon_0 \varepsilon_2 \cdot \varepsilon_3)$$

$$\Rightarrow w = 0 \quad \blacksquare$$

So the elements such that $s = v(r, 0) + \epsilon \sqrt{\frac{1}{4}r^t r} - 1\varepsilon_5$ are excluded and we are left with

$$s = \epsilon \left(\sqrt{1 - \frac{1}{4}r^t r} + v(r, 0) \right) \quad (140)$$

These elements constitute a subgroup, as it can easily be checked with the formula for the product. They are generated by vectors belonging to the subspace spanned by the vectors $(\varepsilon_i)_{i=1}^3$ so they belong to $Cl(E^\perp, g^\perp)$ and can be identified with $Spin(E^\perp, g^\perp)$, isomorphic to $Sp(3)$.

The scalars $\epsilon = \pm 1$ belong to the group. The group is not connected. The elements $s = \sqrt{1 - \frac{1}{4}r^t r} + v(r, 0)$ constitute the component of the identity.

14.3.2 Decomposition of the Lie algebra

The Lie algebra $T_1 Spin(3, 1)$ is the direct sum of the two vector vector subspaces :

$$X \in T_1 Spin(3, 1) :: X = v(r, w) = v(r, 0) + v(0, w)$$

The Lie bracket reads :

$$[v(r, w), v(r', w')] = v(j(r)r' - j(w)w', j(w)r' + j(r)w')$$

thus :

$$[v(r, 0), v(r', 0)] = v(j(r)r', 0)$$

$$[v(0, w), v(0, w')] = v(-j(w)w', 0)$$

The vector subspace $v(r, 0)$ can be identified with $T_1Spin(3)$. It is generated by the vectors $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ if $\langle \varepsilon_0, \varepsilon_0 \rangle = -1$. And similarly in $Cl(1, 3)$.

For any element $v(r, w)$ of $T_1Spin(3, 1)$ we have the identity :

$$v(r, w) \cdot \varepsilon_0 - \varepsilon_0 \cdot v(r, w) = w$$

$$\text{Thus } v(r, w) \in T_1Spin(3) \Leftrightarrow v(r, w) \cdot \varepsilon_0 - \varepsilon_0 \cdot v(r, w) = 0 \Leftrightarrow w = 0$$

14.3.3 Homogeneous space

The quotient space $Spin(3, 1)/Spin(3)$ (called a homogeneous space) is not a group but a 3 dimensional manifold. It is characterized by the equivalence relation :

$$s = a + v(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \sim s' = a' + v(r', w') + b'\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$$

$$\Leftrightarrow \exists s_r \in Spin(3) : s' = s \cdot s_r$$

As any quotient space its elements are *subsets* of $Spin(3, 1)$.

Theorem 58 *In each class of the homogeneous space there are two elements, defined up to sign, which read : $s_w = \pm(a_w + v(0, w))$*

Proof. Each coset $[s]$ is in bijjective correspondence with $Spin(3)$.

$$\text{So } [s] = \left\{ s' = s \cdot \left(\epsilon \sqrt{1 - \frac{1}{4}\rho^t \rho} + v(\rho, 0) \right), \rho^t \rho \leq 4 \right\}$$

In $Spin(3, 1)$:

$$s = a + v(r, w) + b\varepsilon_5$$

$$s' = a' + v(r', w') + b'\varepsilon_5$$

$$a' = a\epsilon \sqrt{1 - \frac{1}{4}\rho^t \rho} - \frac{1}{4}r^t \rho$$

$$b' = b\epsilon \sqrt{1 - \frac{1}{4}\rho^t \rho} - \frac{1}{4}w^t \rho$$

$$r' = \frac{1}{2}j(r)\rho + r\epsilon \sqrt{1 - \frac{1}{4}\rho^t \rho} + a\rho$$

$$w' = \frac{1}{2}j(w)\rho + w\epsilon \sqrt{1 - \frac{1}{4}\rho^t \rho} + b\rho$$

$$a^2 - b^2 = 1 + \frac{1}{4}(w^t w - r^t r)$$

$$ab = -\frac{1}{4}r^t w$$

We can always choose in the class an element s' such that : $r' = 0$. It requires

$$: \left(\frac{1}{2}j(r) + aI \right) \rho = -r\epsilon \sqrt{1 - \frac{1}{4}\rho^t \rho}$$

This linear equation in ρ has always a unique solution :

$$\rho = -\epsilon \sqrt{1 - \frac{1}{4}\rho^t \rho} \frac{1}{a(4a^2 + r^t r)} \left((4a^2 + r^t r) I - 2aj(r) + j(r)j(r) \right) r$$

$$= -\epsilon \sqrt{1 - \frac{1}{4}\rho^t \rho} \frac{1}{a(4a^2 + r^t r)} (4a^2 + r^t r) r$$

$$= -\epsilon \sqrt{1 - \frac{1}{4}\rho^t \rho} \frac{1}{a} r$$

$$\rho^t \rho = \left(1 - \frac{1}{4}\rho^t \rho \right) \frac{1}{a^2} (r^t r) \Rightarrow$$

$$\begin{aligned}
& (a^2 + \frac{1}{4}(r^t r)) \rho^t \rho = (r^t r) \\
& \rho^t \rho = \frac{4(r^t r)}{4a^2 + (r^t r)} \leq 4 \\
& \sqrt{1 - \frac{1}{4}\rho^t \rho} = \sqrt{\frac{4a^2}{4a^2 + (r^t r)}} = \frac{2a}{\sqrt{4a^2 + (r^t r)}} \\
& \rho = -\epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} r \\
& a' = a\epsilon \sqrt{1 - \frac{1}{4}\rho^t \rho} - \frac{1}{4}r^t \rho = \frac{1}{2} \frac{\epsilon}{\sqrt{4a^2 + (r^t r)}} (4a^2 + r^t r) = \frac{1}{2}\epsilon \sqrt{4a^2 + r^t r} \\
& w' = \frac{1}{2}j(w) \rho + w\epsilon \sqrt{1 - \frac{1}{4}\rho^t \rho} + b\rho = \epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} (\frac{1}{2}j(r)w + aw - br) \\
& b' = b\epsilon \sqrt{1 - \frac{1}{4}\rho^t \rho} - \frac{1}{4}w^t \rho = \epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} (ab + \frac{1}{4}w^t r) = 0 \\
& s = (\epsilon \sqrt{4a^2 + r^t r}) \times \\
& \left(\frac{1}{2} + v \left(0, \epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} (\frac{1}{2}j(r)w + aw - br) \right) \right) \cdot \left(\epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} \right) (a + v(r, 0))
\end{aligned}$$

So any element of $Spin(3, 1)$ can be written uniquely (up to sign) :

$$s = a + v(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 = s_w \cdot s_r = (a_w + v(0, w_w)) \cdot (a_r + v(0, r_r))$$

and :

$$(a_w + v(0, w_w)) \cdot (a_r + v(0, r_r)) = a_w a_r + v(a_w r_r, a_r w_w) - \frac{1}{2}(w_w^t r_r) \varepsilon_5$$

In $Cl(1, 3)$ we have the same decomposition with the same components.

$$\begin{aligned}
s &= a + v(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 = \\
r'' &= a_w r_r = \frac{1}{2}\epsilon \sqrt{4a^2 + r^t r} \epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} r = r
\end{aligned}$$

$$\begin{aligned}
w'' &= \frac{1}{2}j(w_w) r_r + a_r w_w \\
&= \frac{1}{2}j \left((\epsilon \sqrt{4a^2 + r^t r}) \epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} (\frac{1}{2}j(r)w + aw - br) \right) \left(\epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} \right) r \\
&+ \left(\epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} \right) a (\epsilon \sqrt{4a^2 + r^t r}) \epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} (\frac{1}{2}j(r)w + aw - br) \\
&= 2j \left(\epsilon \frac{1}{\sqrt{4a^2 + (r^t r)}} (\frac{1}{2}j(r)w + aw - br) \right) r + a\epsilon \frac{4}{\sqrt{4a^2 + (r^t r)}} (\frac{1}{2}j(r)w + aw - br) \\
&= \left(\epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} \right) (j \left((\frac{1}{2}j(r)w + aw - br) \right) r + a2 (\frac{1}{2}j(r)w + aw - br)) \\
&= \left(\epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} \right) (\frac{1}{2}j(j(r)w)r - aj(w)r + aj(r)w + 2a^2w - 2abr) \\
&= \left(\epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} \right) (\frac{1}{2}(wr^t - rw^t)r + 2a^2w + \frac{1}{2}(r^t w)r) \\
&= \left(\epsilon \frac{2}{\sqrt{4a^2 + (r^t r)}} \right) (\frac{1}{2}w(r^t r) - \frac{1}{2}r(w^t r) + 2a^2w + \frac{1}{2}(r^t w)r) \\
&= \left(\epsilon \frac{1}{\sqrt{4a^2 + (r^t r)}} \right) ((4a^2 + (r^t r))w) = w \blacksquare
\end{aligned}$$

Remark : the elements $\pm s_w$ are equivalent :

$$(a_w + v(0, w_w)) \sim -(a_w + v(0, w_w))$$

Take $s_r = -1 \in Spin(3)$: $-s_w = s_w \cdot s_r$

So $\pm s_w$ belong to the same class of equivalence. In the decomposition :

$$s = \epsilon s_w \cdot \epsilon s_r, \epsilon s_w \text{ is a specific projection of } s \text{ on the homogenous space.}$$

14.4 Adjoint map

The translations on $Spin(3, 1)$ are :

$$L_g h = g \cdot h, R_g h = h \cdot g$$

and their derivatives :

$$L'_g h : T_h Spin(3, 1) \rightarrow T_{g \cdot h} Spin(3, 1) :: L'_g h(X_h) = g \cdot X_h$$

$$R'_g h : T_h Spin(3, 1) \rightarrow T_{h \cdot g} Spin(3, 1) :: R'_g h(X_h) = X_h \cdot g$$

Their inverse are, as in any Lie groups :

$$(L'_g h)^{-1} = L'_{g^{-1}}(g \cdot h); (R'_g h)^{-1} = R'_{g^{-1}}(h \cdot g)$$

$T_g Spin(3, 1) \subset Cl(3, 1)$ and there are two linear maps :

$$L'_{g^{-1}} g : T_g Spin(3, 1) \rightarrow T_1 Spin(3, 1) :: L'_{g^{-1}} g(Z_g) = g^{-1} \cdot Z_g$$

$$R'_{g^{-1}} g : T_g Spin(3, 1) \rightarrow T_1 Spin(3, 1) :: R'_{g^{-1}} g(Z_g) = Z_g \cdot g^{-1}$$

And the adjoint map:

$$Ad_g : T_1 Spin(3, 1) \rightarrow T_1 Spin(3, 1) :: Ad_g = L'_g g^{-1} \circ R'_{g^{-1}} 1 = R'_{g^{-1}} g \circ L'_g 1$$

$$Ad_g Z = L'_g g^{-1} \circ R'_{g^{-1}} 1(Z) = L'_g g^{-1}(Z \cdot g^{-1}) = g \cdot Z \cdot g^{-1}$$

$$Ad_g Z = (\mathbf{Ad}_g Z)'_{x=1} = \mathbf{Ad}_g Z$$

With

$$g = a + v(r, w) + b\varepsilon_5$$

$$Z = v(x, y)$$

$$Ad_g X = (a + v(r, w) + b\varepsilon_5) \cdot v(x, y) \cdot (a - v(r, w) + b\varepsilon_5)$$

A straightforward computation gives :

$$\mathbf{Ad}_g v(x, y) = v(X, Y)$$

with

$$X =$$

$$\left[(a+b)^2 - ab + \frac{1}{4}r^t r + (a+b)j(r) + \frac{1}{4}(j(w)j(r) - 2j(w)j(w) + 3j(r)j(r)) \right] x$$

$$+ \left[ab + \frac{1}{4}r^t r + (b-a)j(w) - \frac{1}{4}(3j(w)j(r) + 2j(r)j(w) - j(r)j(r)) \right] y$$

$$Y =$$

$$\left[\frac{1}{4}w^t w - ab + (a+b)j(w) + \frac{1}{4}(j(w)j(w) + 2j(w)j(r) + 3j(r)j(w)) \right] x$$

$$+ \left[(a-b)^2 - \frac{1}{4}w^t w - ab + (a-b)j(r) - \frac{1}{4}(3j(w)j(w) - j(r)j(w) - 2j(r)j(r)) \right] y$$

14.5 Derivatives

Let $g : M \rightarrow Spin(3, 1) :: g(m) = a(m) + v(r(m), w(m)) + b(m)\varepsilon_5$

$$g'(m) : T_m M \rightarrow T_g Spin(3, 1) ::$$

$$g'(m)u_m = a'(m)u_m + v(r'(m)u_m, w'(m)u_m) + b'(m)u_m\varepsilon_5$$

$$\text{where } u_m \in T_m M, a'(m)u_m, b'(m)u_m \in \mathbb{R}, r'(m)u_m, w'(m)u_m \in \mathbb{R}^3$$

Thus :

$$g^{-1} \cdot (a'(m)u_m + v(r'(m)u_m, w'(m)u_m) + b'(m)u_m\varepsilon_5) \in T_1 Spin(3, 1)$$

$$(a'(m)u_m + v(r'(m)u_m, w'(m)u_m) + b'(m)u_m\varepsilon_5) \cdot g^{-1} \in T_1 Spin(3, 1)$$

$$L'_{g^{-1}} g(g'(m)u_m) =$$

$$v\left(-\frac{1}{2}(j(r)r' - j(w)w') - a'r + ar' + b'w - bw',\right.$$

$$\left. -\frac{1}{2}(j(w)r' + j(r)w') - a'w + aw' - b'r + br'\right)$$

$$R'_{g^{-1}} g(g'(m)u_m) =$$

$$v\left(-\frac{1}{2}(j(r)r' - j(w)w') - a'r + ar' + b'w - bw',\right.$$

$$\left. -\frac{1}{2}(j(w)r' + j(r)w') - a'w + aw' - b'r + br'\right)$$

14.6 Exponential on T_1Spin

The exponential on a Lie algebra is the flow of left invariant vector fields (Maths.22.2.6).

i) Left invariant vector fields on $Spin(3, 1)$

As $Spin(3, 1) \subset Cl(3, 1)$ which is a vector space, a vector field $X \in \mathfrak{X}(TSpin(3, 1))$ reads $X(\sigma) \in Cl(3, 1)$ with the relation :

$$L'_g \sigma (X(\sigma)) = X(L_g \sigma) = g \cdot X(\sigma) = X(g \cdot \sigma)$$

Thus the left invariant vector fields read :

$$X(\sigma) = \sigma \cdot v(R, W) \text{ with } v(R, W) \in T_1Spin(3, 1)$$

ii) The flow of $X = \sigma \cdot v(R, W) \in \mathfrak{X}(TSpin(3, 1))$ reads:

$$\Phi_X(t, 1) = a(t) + v(r(t), w(t)) + b(t)\varepsilon_5 \in Spin(3, 1)$$

$$\Phi_X(t, 1) = \exp tX = \exp v(tR, tW)$$

$$\exp v(tR, tW) = a(t) + v(r(t), w(t)) + b(t)\varepsilon_5$$

$$\frac{d}{dt} \exp v(tR, tW) |_{t=\theta} = \exp v(\theta R, \theta W) \cdot v(R, W)$$

$$\frac{d}{dt} (a(t) + v(r(t), w(t)) + b(t)\varepsilon_5) |_{t=\theta} = (a(\theta) + v(r(\theta), w(\theta)) + b(\theta)\varepsilon_5) \cdot v(R, W)$$

with :

$$a^2 - b^2 = 1 + \frac{1}{4}(w^t w - r^t r)$$

$$ab = -\frac{1}{4}r^t w$$

$$1 = a(0) + v(r(0), w(0)) + b(0)\varepsilon_5$$

The derivations give :

$$\frac{\partial a}{\partial t} + v\left(\frac{\partial r}{\partial t}, \frac{\partial w}{\partial t}\right) + \frac{\partial b}{\partial t}\varepsilon_5 |_{t=\theta} = (a(\theta) + v(r(\theta), w(\theta)) + b(\theta)\varepsilon_5) \cdot v(R, W)$$

$$= a(\theta)v(R, W) + b(\theta)v(R, -W) + v(r(\theta), w(\theta)) \cdot v(R, W)$$

$$= v((a(\theta) + b(\theta))R, (a(\theta) - b(\theta))W)$$

$$+ \frac{1}{4}(W^t w - R^t r) + \frac{1}{2}v(-j(R)r + j(W)w, -j(W)r - j(R)w) - \frac{1}{4}(W^t r + R^t w)\varepsilon_5$$

$$\frac{\partial a}{\partial t} |_{t=\theta} = \frac{1}{4}(W^t w - R^t r)$$

$$\frac{\partial b}{\partial t} |_{t=\theta} = -\frac{1}{4}(W^t r + R^t w)$$

$$\frac{\partial r}{\partial t} |_{t=\theta} = (a(\theta) + b(\theta))R + \frac{1}{2}(-j(R)r + j(W)w)$$

$$\frac{\partial w}{\partial t} |_{t=\theta} = (a(\theta) - b(\theta))W + \frac{1}{2}(-j(W)r - j(R)w)$$

$$a \frac{\partial a}{\partial t} |_{t=\theta} - b \frac{\partial b}{\partial t} |_{t=\theta} = \frac{1}{4}\left(w^t \frac{\partial w}{\partial t} - r^t \frac{\partial r}{\partial t}\right)$$

$$\frac{\partial a}{\partial t} b + a \frac{\partial b}{\partial t} = -\frac{1}{4}r^t \frac{\partial w}{\partial t} - \frac{1}{4}w^t \frac{\partial r}{\partial t}$$

The last two equations give :

$$b(W + R)^t (w + r) = 0$$

$$b(W + R)^t (w - r) = 0$$

iii) We have the morphism :

$$\Pi : Spin(3, 1) \rightarrow SO(3, 1) :: \Pi(\pm\sigma) = [h(\sigma)] \text{ such that :}$$

$$\forall u \in \mathbb{R}^4 : \mathbf{Ad}_\sigma u = \sigma \cdot u \cdot \sigma^{-1} = [h(\sigma)]u = \Pi(\pm\sigma)u$$

Take a vector field $X(\sigma) = \sigma \cdot v(R, W) \in \mathfrak{X}(TSpin(3, 1))$ then (Maths.1460)

:

$$\Pi \circ \Phi_X = \Phi_{\Pi_* X} \circ \Pi$$

$$\Pi_* X([h(\sigma)]) = \Pi'(\sigma)X(\sigma)$$

$$\Pi'(1)X(1) = K(W) + J(R)$$

$$\Pi(\exp tv(R, W)) = \Phi_{\Pi'(1)v(R, W)}(t, \Pi(1)) = \exp t(K(W) + J(R)) = \exp tK(W) \exp tJ(R)$$

$$\Pi(\exp tv(R, 0)) = \exp tJ(R)$$

$\Pi(\exp tv(0, W)) = \exp tK(W)$
 $\Pi(\exp tv(R, W)) = \Pi(\exp tv(0, W)) \Pi(\exp tv(R, 0))$
 and because this is a morphism :

$\exp tv(R, W) = \exp tv(0, W) \cdot \exp tv(R, 0)$
 iv) Coming back to the previous equations :

For $\exp v(0, tW)$:

$$\begin{aligned}
 \frac{\partial a}{\partial t} \Big|_{t=\theta} &= \frac{1}{4} W^t w \\
 \frac{\partial b}{\partial t} \Big|_{t=\theta} &= -\frac{1}{4} (W^t r) \\
 \frac{\partial r}{\partial t} \Big|_{t=\theta} &= \frac{1}{2} j(W) w \\
 \frac{\partial w}{\partial t} \Big|_{t=\theta} &= (a(\theta) - b(\theta)) W - \frac{1}{2} j(W) r \\
 bW^t(w+r) &= 0 \\
 bW^t(w-r) &= 0 \\
 \text{if } b \neq 0 : \\
 W^t w = -W^t r = W^t r &= 0 \\
 b \frac{\partial b}{\partial t} \Big|_{t=\theta} = -\frac{1}{4} b(W^t r) = 0 &\Rightarrow b^2 = Ct \Rightarrow b = Ct \Rightarrow W^t r = 0, \\
 \Rightarrow W^t w = 0 &\Rightarrow a = Ct \\
 \Rightarrow r, w = Ct &
 \end{aligned}$$

Thus $b = 0 \Rightarrow W^t r = 0$

$$\frac{d^2 w}{dt^2} = \frac{1}{4} (W^t w) W - \frac{1}{4} j(W) j(W) w = \frac{1}{4} (W^t w) W - \frac{1}{4} (W W^t w - (W^t W) w) = \frac{1}{4} (W^t W) w$$

$$w(t) = w_1 \exp \frac{1}{2} t \sqrt{W^t W} + w_2 \exp \left(-\frac{1}{2} t \sqrt{W^t W} \right)$$

$$w(0) = 0 = w_1 + w_2$$

$$\frac{dw}{dt}(0) = W = w_1 - w_2$$

$$w(t) = \frac{1}{2} \left(\exp \frac{1}{2} t \sqrt{W^t W} - \exp \left(-\frac{1}{2} t \sqrt{W^t W} \right) \right) W = W \sinh \frac{1}{2} t \sqrt{W^t W}$$

$$\frac{\partial r}{\partial t} \Big|_{t=\theta} = \frac{1}{2} j(W) W \sinh \frac{1}{2} t \sqrt{W^t W} = 0$$

$$r(0) = R = 0 \Rightarrow r(t) = 0$$

$$w^t w = W^t W \sinh^2 \frac{1}{2} t \sqrt{W^t W}$$

$$a^2 - b^2 = a^2 = 1 + \frac{1}{4} \left(W^t W \sinh^2 \frac{1}{2} t \sqrt{W^t W} \right)$$

$$\exp v(0, tW) = \sqrt{1 + \frac{1}{4} \left(W^t W \sinh^2 \frac{1}{2} t \sqrt{W^t W} \right)} + \sinh \frac{1}{2} t \sqrt{W^t W} v(0, W)$$

For $\exp v(tR, 0)$

$$\frac{\partial a}{\partial t} \Big|_{t=\theta} = -\frac{1}{4} R^t r$$

$$\frac{\partial b}{\partial t} \Big|_{t=\theta} = -\frac{1}{4} R^t w$$

$$\frac{\partial r}{\partial t} \Big|_{t=\theta} = (a(\theta) + b(\theta)) R - \frac{1}{2} j(R) r$$

$$\frac{\partial w}{\partial t} \Big|_{t=\theta} = -\frac{1}{2} j(R) w$$

$$bR^t(w+r) = 0$$

$$bR^t(w-r) = 0$$

$$\Rightarrow b = 0, R^t w = 0$$

$$\frac{d^2 r}{dt^2} = -\frac{1}{4} (R^t r) R - \frac{1}{2} j(R) (aR - \frac{1}{2} j(R) r) = -\frac{1}{4} (R^t r) R - \frac{1}{2} a j(R) R + \frac{1}{4} j(R) j(R) r$$

$$\frac{d^2 r}{dt^2} = -\frac{1}{4} (R^t r) R + \frac{1}{4} ((R^t r) R - R^t R r)$$

$$\frac{d^2 r}{dt^2} = -\frac{1}{4} (R^t R) r$$

$$\begin{aligned}
r(t) &= r_1 \exp it \frac{1}{2} \sqrt{R^t \bar{R}} + r_1 \exp \left(-it \frac{1}{2} \sqrt{R^t \bar{R}} \right) \\
r(0) &= 0 = r_1 + r_2 \\
\frac{dr}{dt}(0) &= R = r_1 - r_2 \\
r(t) &= R \sin t \frac{1}{2} \sqrt{R^t \bar{R}} \\
r^t r &= R^r R \sin^2 t \frac{1}{2} \sqrt{R^t \bar{R}} \\
a^2 - b^2 &= a^2 = 1 - \frac{1}{4} R^r R \sin^2 t \frac{1}{2} \sqrt{R^t \bar{R}} \\
\exp t v(R, 0) &= \sqrt{1 - \frac{1}{4} R^r R \sin^2 t \frac{1}{2} \sqrt{R^t \bar{R}}} + \sin t \frac{1}{2} \sqrt{R^t \bar{R}} v(R, 0) \\
\exp t v(R, W) &= \left(\sqrt{1 + \frac{1}{4} \left(W^t W \sinh^2 \frac{1}{2} t \sqrt{W^t \bar{W}} \right)} + \sinh \frac{1}{2} t \sqrt{W^t \bar{W}} v(0, W) \right) \cdot \\
&\left(\sqrt{1 - \frac{1}{4} R^r R \sin^2 t \frac{1}{2} \sqrt{R^t \bar{R}}} + \sin t \frac{1}{2} \sqrt{R^t \bar{R}} v(R, 0) \right)
\end{aligned}$$

15 ANNEX 2 : FORMULAS

15.1 Relativist Geometry

Divergence of a vector field V :

$$divV = \frac{1}{\det P'} \sum_{\alpha=0}^3 \frac{d}{d\xi^\alpha} (V^\alpha \det P') = \sum_{\alpha=0}^3 \frac{dV^\alpha}{d\xi^\alpha} + \frac{1}{\det P'} \sum_{\alpha=0}^3 V^\alpha \frac{d \det P'}{d\xi^\alpha}$$

For a motion on an integral curve of the vector field V : $\frac{1}{\det P'} \sum_{\alpha=0}^3 V^\alpha \frac{d \det P'}{d\xi^\alpha} = \frac{1}{\det P'} \frac{d \det P'}{d\tau}$

Between the proper time τ of a particle and the time t of an observer :

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}} = \frac{1}{c} \sqrt{-\langle V, V \rangle} = \frac{c}{u^0} = \frac{1}{2a_w^2 - 1}$$

Between the velocity u of a particle and the speed V as measured by an observer :

$$u = \frac{dp}{d\tau} = V \frac{c}{\sqrt{-\langle V, V \rangle}} = \frac{1}{\sqrt{1 - \frac{\|\vec{v}\|^2}{c^2}}} (\vec{v} + c\varepsilon_0(m))$$

$$= c \left((2a_w^2 - 1) \varepsilon_0 + \epsilon a_w \sum_{i=1}^3 w_i \varepsilon_i \right)$$

$$V = \frac{dp}{dt} = u \frac{\sqrt{-\langle V, V \rangle}}{c} = \vec{v} + c\varepsilon_0(m) = c \left(\varepsilon_0 + \epsilon \frac{a_w}{2a_w^2 - 1} \sum_{i=1}^3 w_i \varepsilon_i \right)$$

$$V(t) = \frac{dq}{dt} = c\varepsilon_0(q(t)) + \vec{v}$$

15.2 Operator j

Let $r \in \mathbb{C}^3, w \in \mathbb{C}^3$:

$$[j(r)]w = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} r_2 w_3 - r_3 w_2 \\ -r_1 w_3 + r_3 w_1 \\ r_1 w_2 - r_2 w_1 \end{bmatrix}$$

$$[j(r)w]^a = \sum_{b,c=1}^3 \epsilon(a,b,c) r_b w_c$$

$$j(r)^t = -j(r) = j(-r)$$

$$j(x)y = -j(y)x$$

$$y^t j(x) = -x^t j(y)$$

$$j(x)y = 0 \Leftrightarrow \exists k \in R : y = kx$$

15.2.1 eigenvectors:

$$r = \sqrt{r^t r}$$

$$0 : \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$ir : \begin{bmatrix} -(-r_1 r_2 + ir_3 r) \\ -(r_1^2 + r_3^2) \\ r_2 r_3 + ir_1 r \end{bmatrix}$$

$$-ir : \begin{bmatrix} -(r_1 r_2 + ir_3 r) \\ (r_1^2 + r_3^2) \\ -r_2 r_3 + r_1 ir \end{bmatrix}$$

15.2.2 Identities

$$\begin{aligned}
j(x)j(y) &= yx^t - (y^t x)I \\
j(x)j(x)j(x) &= j(x)(xx^t - (x^t x)I) = -(x^t x)j(x) \\
j(j(x)y) &= yx^t - xy^t = j(x)j(y) - j(y)j(x) \\
j(j(x)j(x)y) &= (y^t x)j(x) - (x^t x)j(y) \\
j(x)j(y)j(x) &= -(y^t x)j(x) \\
j(x)j(x)j(y) &= j(x)yx^t - (y^t x)j(x) = -j(y)xx^t - (y^t x)j(x) = -j(y)(j(x)j(x) + x^t x) - \\
&\quad (y^t x)j(x) \\
x^t j(r)j(s)y &= x^t(sr^t - r^t sI)y = (x^t s)(r^t y) - (x^t y)(r^t s) \\
\|j(x)y\|^2 &= (x^t x)(y^t y) - (x^t y)^2 \\
M \in L(3) : M^t j(Mx)M &= (\det M)j(x) \\
M \in O(3) : j(Mx)My &= Mj(x)y \Leftrightarrow Mx \times My = M(x \times y) \\
k > 0 : j(r)^{2k} &= (-r^t r)^{k-1}(rr^t - (r^t r)I) = (-r^t r)^{k-1}j(r)j(r) \\
k \geq 0 : J(r)^{2k+1} &= (-r^t r)^k j(r) \\
\exp[j(r)] &= I_3 + [j(r)] \frac{\sin \sqrt{r^t r}}{\sqrt{r^t r}} + [j(r)][j(r)] \frac{1 - \cos \sqrt{r^t r}}{r^t r}
\end{aligned}$$

15.2.3 Polynomials

The set of polynomials $aI + bj(z) + cj(z)j(z)$ where $z \in \mathbb{C}^3$ is fixed, $a, b, c \in \mathbb{C}$ is a commutative ring.

$$\begin{aligned}
&(a + bj(z) + cj(z)j(z))(a' + b'j(z) + c'j(z)j(z)) \\
&= aa' + (ab' + a'b - (z^t z)(c'b + b'c))j(z) + (ac' + b'b + a'c - (z^t z)c'c)j(z)j(z)
\end{aligned}$$

The inverse are :

$$\begin{aligned}
[aI + bj(z) + cj(z)j(z)]^{-1} &= \frac{1}{a}I - \frac{bc^2}{(a-cz^2)^2 + b^2z^2}j(z) + \frac{(b^2+c^2z^2-a)}{a((a-cz^2)^2 + cb^2z^2)}j(z)j(z) \\
\det(a + bj(r)j(r) + cj(r)j(k) + dj(k)j(r)) &= \\
cd^2(r^2k^2z - z^3) + ad^2z^2 + c^2d(r^2k^2z - z^3) + bcd(r^4k^2 - r^2z^2) + acd(-k^2r^2 + 3z^2) + \\
2abdr^2z - 2a^2dz + 2abcr^2z - 2a^2cz - 2a^2cz &+ \\
+ ab^2r^4 - 2a^2br^2 + a^3 & \\
\text{with } z = r^t k, r^2 = r^t r, k^2 = k^t k &
\end{aligned}$$

15.3 Dirac's matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \sigma_0$$

$$\sigma_i^2 = \sigma_0$$

$$j \neq k, l = 1, 2, 3 : \sigma_j \sigma_k = \epsilon(j, k, l) i \sigma_l$$

$$\sigma_1 \sigma_2 \sigma_3 = i \sigma_0$$

$$\left(\sum_{a=1}^3 U_a \sigma_a \right) \left(\sum_{b=1}^3 V_b \sigma_b \right) = \sum_{a < b=1}^3 (j(U)V)^a i \sigma_a + \sum_{a=1}^3 U_a V_a \sigma_0$$

$$\sum_{a=1}^3 z_a \sigma_a = \begin{bmatrix} z_3 & z_1 - iz_2 \\ z_1 + iz_2 & -z_3 \end{bmatrix}$$

$$\left(\sum_{a=1}^3 z_a \sigma_a \right)^{-1} = \frac{1}{z_1^2 + z_2^2 + z_3^2} \sum_{a=1}^3 z_a \sigma_a$$

15.4 γ matrices

$$\gamma_0 = \begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix}; \gamma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}; \gamma_2 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}; \gamma_3 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix};$$

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} I_4$$

$$\gamma_i = \gamma_i^* = \gamma_i^{-1}$$

$$\gamma_1 \gamma_2 \gamma_3 = i \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix}$$

$$\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{bmatrix}$$

$$\gamma_5 \gamma_a = -\gamma_a \gamma_5$$

$$Cl(3, 1) : \gamma C(\varepsilon_j) = \gamma_j, j = 1, 2, 3; \gamma C(\varepsilon_0) = i\gamma_0; \gamma C(\varepsilon_5) = i\gamma_5$$

$$Cl(1, 3) : \gamma C'(\varepsilon_j) = i\gamma_j, j = 1, 2, 3; \gamma C'(\varepsilon_j) = \gamma_0; \gamma C'(\varepsilon_5) = \gamma_5$$

$$j = 1, 2, 3 : \tilde{\gamma}_j = \begin{bmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{bmatrix}$$

$$j \neq k, l = 1, 2, 3 : \gamma_j \gamma_k = -\gamma_k \gamma_j = i\epsilon(j, k, l) \tilde{\gamma}_l$$

$$a = 1, 2, 3 : \gamma C(\vec{\kappa}_a) = -\frac{1}{2} i \tilde{\gamma}_a$$

$$a = 4, 5, 6 : \gamma C(\vec{\kappa}_a) = \frac{1}{2} \begin{bmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{bmatrix}$$

$$\gamma_0 \tilde{\gamma}_j = \tilde{\gamma}_j \gamma_0 = i \begin{bmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{bmatrix}$$

$$\tilde{\gamma}_a \tilde{\gamma}_a = I_4$$

$$\tilde{\gamma}_a \tilde{\gamma}_b = i\epsilon(a, b, c) \tilde{\gamma}_c$$

$$\tilde{\gamma}_a \tilde{\gamma}_b + \tilde{\gamma}_b \tilde{\gamma}_a = 2\delta_{ab} I_4$$

$$\gamma_a \tilde{\gamma}_b = i\epsilon(a, b, c) \gamma_c$$

$$\gamma_a \tilde{\gamma}_a = \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix}$$

$$\gamma_0 \tilde{\gamma}_a = \begin{bmatrix} 0 & -i\sigma_a \\ i\sigma_a & 0 \end{bmatrix} = (\gamma_0 \tilde{\gamma}_a)^* = \begin{bmatrix} 0 & -i\sigma_a^* \\ i\sigma_a^* & 0 \end{bmatrix}$$

$$j = 1, 2, 3 : \gamma_j \gamma_0 = -\gamma_0 \gamma_j = i \begin{bmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{bmatrix} = i\gamma_5 \tilde{\gamma}_j$$

$$\gamma_1 \gamma_2 = -\gamma_2 \gamma_1 = i \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}; \gamma_2 \gamma_3 = -\gamma_3 \gamma_2 = i \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix};$$

$$\gamma_3 \gamma_1 = -\gamma_1 \gamma_3 = i \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

$$\gamma_1 \gamma_0 = -\gamma_0 \gamma_1 = i \begin{bmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix}; \gamma_2 \gamma_0 = -\gamma_0 \gamma_2 = i \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix};$$

$$\gamma_3 \gamma_0 = -\gamma_0 \gamma_3 = i \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}$$

$$\gamma C(a + v(r, w) + b\varepsilon_5) = aI + ib\gamma_5 - i\frac{1}{2} \sum_{a=1}^3 (w_a \gamma_a \gamma_0 + r_a \tilde{\gamma}_a)$$

In $Cl(3, 1)$:

$$\gamma C(v(r, w)) = -i\frac{1}{2} \sum_{a=1}^3 (w_a \gamma_a \gamma_0 + r_a \tilde{\gamma}_a)$$

$$= \frac{1}{2} \begin{bmatrix} \sum_{a=1}^3 (w_a - ir_a) \sigma_a & 0 \\ 0 & -\sum_{a=1}^3 (w_a + ir_a) \sigma_a \end{bmatrix}$$

In $Cl(1, 3)$:

$$\gamma C'(v(r, w)) = -i\frac{1}{2}\sum_{a=1}^3 (w_a \gamma_a \gamma_0 - r_a \tilde{\gamma}_a)$$

15.5 Clifford algebra

$$\begin{aligned}\varepsilon_i \cdot \varepsilon_j + \varepsilon_j \cdot \varepsilon_i &= 2\eta_{ij} \\ \varepsilon_5 \cdot \varepsilon_5 &= -1\end{aligned}$$

15.5.1 Lie Algebras

$$\begin{aligned}v(r, w) &= \frac{1}{2}(w^1 \varepsilon_0 \cdot \varepsilon_1 + w^2 \varepsilon_0 \cdot \varepsilon_2 + w^3 \varepsilon_0 \cdot \varepsilon_3 + r^3 \varepsilon_2 \cdot \varepsilon_1 + r^2 \varepsilon_1 \cdot \varepsilon_3 + r^1 \varepsilon_3 \cdot \varepsilon_2) \\ \vec{K}_1 &= v((1, 0, 0), (0, 0, 0)) = \frac{1}{2} \varepsilon_3 \cdot \varepsilon_2, \\ \vec{K}_2 &= v((0, 1, 0), (0, 0, 0)) = \frac{1}{2} \varepsilon_1 \cdot \varepsilon_3, \\ \vec{K}_3 &= v((0, 0, 1), (0, 0, 0)) = \frac{1}{2} \varepsilon_2 \cdot \varepsilon_1, \\ \vec{K}_4 &= v((0, 0, 0), (1, 0, 0)) = \frac{1}{2} \varepsilon_0 \cdot \varepsilon_1, \\ \vec{K}_5 &= v((0, 0, 0), (0, 1, 0)) = \frac{1}{2} \varepsilon_0 \cdot \varepsilon_2, \\ \vec{K}_6 &= v((0, 0, 0), (0, 0, 1)) = \frac{1}{2} \varepsilon_0 \cdot \varepsilon_3 \\ v(r, w) \cdot \varepsilon_0 &= \frac{1}{2}(w^1 \varepsilon_1 + w^2 \varepsilon_2 + w^3 \varepsilon_3 + r^3 \varepsilon_2 \cdot \varepsilon_1 \cdot \varepsilon_0 + r^2 \varepsilon_1 \cdot \varepsilon_3 \cdot \varepsilon_0 + r^1 \varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_0) \\ \varepsilon_0 \cdot v(r, w) &= \frac{1}{2}(-w^1 \varepsilon_1 - w^2 \varepsilon_2 + w^3 \varepsilon_3 + r^3 \varepsilon_2 \cdot \varepsilon_1 \cdot \varepsilon_0 + r^2 \varepsilon_1 \cdot \varepsilon_3 \cdot \varepsilon_0 + r^1 \varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_0) \\ v(r, w) \cdot \varepsilon_0 - \varepsilon_0 \cdot v(r, w) &= w \\ v(r, w) \cdot \varepsilon_0 + \varepsilon_0 \cdot v(r, w) &= 2v(r, 0) \cdot \varepsilon_0\end{aligned}$$

In $Cl(3, 1)$:

$$\begin{aligned}v(r, w) \cdot \varepsilon_5 &= \varepsilon_5 \cdot v(r, w) = v(r, -w) \\ v(r', w') \cdot v(r, w) &= \frac{1}{4}(w^t w' - r^t r') + \frac{1}{2}v(-j(r) r' + j(w) w', -j(w) r' - j(r) w') - \frac{1}{4}(w^t r' + r^t w') \varepsilon_5 \\ [v(r, w), v(r', w')] &= v(j(r) r' - j(w) w', j(w) r' + j(r) w')\end{aligned}$$

$$\mathbf{Ad}_s v(x, y) = v(X, Y)$$

with

$$s = a + v(r, w) + b\varepsilon_5$$

$$\begin{aligned}X &= \left[(a+b)^2 - ab + \frac{1}{4}r^t r + (a+b)j(r) + \frac{1}{4}(j(w)j(r) - 2j(w)j(w) + 3j(r)j(r)) \right] x \\ &+ \left[ab + \frac{1}{4}r^t r + (b-a)j(w) - \frac{1}{4}(3j(w)j(r) + 2j(r)j(w) - j(r)j(r)) \right] y \\ Y &= \left[\frac{1}{4}w^t w - ab + (a+b)j(w) + \frac{1}{4}(j(w)j(w) + 2j(w)j(r) + 3j(r)j(w)) \right] x \\ &+ \left[(a-b)^2 - \frac{1}{4}w^t w - ab + (a-b)j(r) - \frac{1}{4}(3j(w)j(w) - j(r)j(w) - 2j(r)j(r)) \right] y\end{aligned}$$

In $Cl(1, 3)$:

$$\begin{aligned}v(r, w) \cdot \varepsilon_5 &= \varepsilon_5 \cdot v(r, w) = v(w, r) \\ v(r, w) \cdot v(r', w') &= \frac{1}{4}(w^t w' - r^t r') - \frac{1}{2}v(-j(r) r' + j(w) w', j(w) r' + j(r) w') - \frac{1}{4}(w^t r' + r^t w') \varepsilon_5 \\ [v(r, w), v(r', w')] &= -v(j(r) r' - j(w) w', j(w) r' + j(r) w')\end{aligned}$$

15.5.2 Spin groups

$$\begin{aligned}s &= a + v(r, w) + b\varepsilon_5 \\ a^2 - b^2 &= 1 + \frac{1}{4}(w^t w - r^t r) \\ ab &= -\frac{1}{4}r^t w\end{aligned}$$

$$\text{if } r = 0 \text{ then } a = \epsilon \sqrt{1 + \frac{1}{4} w^t w}; b = 0$$

if $w = 0$ then

$$r^t r \leq 4 : a = \epsilon \sqrt{1 - \frac{1}{4} r^t r}; b = 0$$

$$r^t r \geq 4 : b = \epsilon \sqrt{-1 + \frac{1}{4} r^t r}; a = 0$$

Product :

$$(a + v(r, w) + b\varepsilon_5)^{-1} = a - v(r, w) + b\varepsilon_5$$

$$s \cdot s' = a'' + v(r'', w'') + b'' \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$$

with :

$$a'' = aa' - b'b + \frac{1}{4}(w^t w' - r^t r')$$

$$b'' = ab' + ba' - \frac{1}{4}(w^t r' + r^t w')$$

i) In $Cl(3, 1)$:

$$r'' = \frac{1}{2}(j(r)r' - j(w)w') + a'r + ar' - b'w - bw'$$

$$w'' = \frac{1}{2}(j(w)r' + j(r)w') + a'w + aw' + b'r + br'$$

$$(a + v(0, w)) \cdot (a' + v(0, w')) = aa' + \frac{1}{4}w^t w' + v(-\frac{1}{2}(j(w)w', a'w + aw'))$$

$$(a + v(r, 0)) \cdot (a' + v(r', 0)) = aa' - \frac{1}{4}r^t r' + v(\frac{1}{2}j(r)r' + (a'r + ar'), 0)$$

$$(a_w + v(0, w)) \cdot (a_r + v(r, 0)) = a_w a_r + v(a_w r, \frac{1}{2}j(w)r + a_r w) - \frac{1}{4}(w^t r) \varepsilon_5$$

ii) In $Cl(1, 3)$:

$$r'' = \frac{1}{2}(j(r)r' - j(w)w') + a'r + ar' + b'w + bw'$$

$$w'' = -\frac{1}{2}(j(w)r' + j(r)w') + a'w + aw' + b'r + br'$$