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Abstract

In this monography, we review the theory and establish new and general results regarding spreading properties for heterogeneous reaction-diffusion equations:

$$\partial_t u - \sum_{i,j=1}^{N} a_{i,j}(t,x)\partial_{ij} u - \sum_{i=1}^{N} q_i(t,x)\partial_i u = f(t,x,u).$$

These are concerned with the dynamics of the solution starting from initial data with compact support. The nonlinearity f is of Fisher-KPP type, and admits 0 as an unstable steady state and 1 as a globally attractive one (or, more generally, admits entire solutions $p^{\pm}(t,x)$, where p^- is unstable and p^+ is globally attractive). Here, the coefficients $a_{i,j}, q_i, f$ are only assumed to be uniformly elliptic, continuous and bounded in (t,x). To describe the spreading dynamics, we construct two non-empty star-shaped compact sets $\underline{\mathcal{S}} \subset \overline{\mathcal{S}} \subset \mathbb{R}^N$ such that for all compact set $K \subset \text{int}(\underline{\mathcal{S}})$ (resp. all closed set $F \subset \mathbb{R}^N \backslash \overline{\mathcal{S}}$), one has $\lim_{t\to+\infty} \sup_{x\in tK} |u(t,x) - 1| = 0$ (resp. $\lim_{t\to+\infty} \sup_{x\in tF} |u(t,x)| = 0$).

The characterizations of these sets involve two new notions of generalized principal eigenvalues for linear parabolic operators in unbounded domains. In particular, it allows us to show that $\overline{\mathcal{S}} = \underline{\mathcal{S}}$ and to establish an *exact* asymptotic speed of propagation in various frameworks. These include: almost periodic, asymptotically almost periodic, uniquely ergodic, slowly varying, radially periodic and random stationary ergodic equations. In dimension N, if the coefficients converge in radial segments, again we show that $\overline{\mathcal{S}} = \underline{\mathcal{S}}$ and this set is characterized using some geometric optics minimization problem. Lastly, we construct an explicit example of non-convex expansion sets.

Key-words: Reaction-diffusion equations, Heterogeneous reaction-diffusion equations, Propagation and spreading properties, Principal eigenvalues, Linear parabolic operator, Hamilton-Jacobi equations, Homogenization, Almost periodicity, Unique ergodicity, Slowly oscillating media.

AMS classification. Primary: 35B40, 35B27, 35K57. Secondary: 35B50, 35K10, 35P05, 47B65, 49L25.

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n.321186 - ReaDi, Reaction-Diffusion Equations, Propagation and modeling held by Henri Berestycki.

The authors are grateful to two anonymous referees for their insight on the paper, the questions they raised, and their useful suggestions.

Contents

1 Introduction

The classical reaction-diffusion equation

$$\partial_t u - d\Delta u = f(u) \quad \text{for} \quad x \in \mathbb{R}^N$$

arises as a basic model in several different contexts. In particular it plays a central role in modelling in biology and ecology. Having in mind population dynamics, one can think of u as a density of a certain biological species and one is interested in the invasion of a territory where this population is not present initially ($u = 0$) whereby the population reaches a maximum level, say $u = 1$, as time goes to infinity. For instance, one chooses normalized variables so that $u = 1$ corresponds to the maximum carrying capacity of the environment. This equation describes the instantaneous time change $\partial_t u$ of $u(t, x)$ at time t and location

x as resulting from *diffusion*, encapsulated in the term Δu (d is a diffusion coefficient) and *reaction*, represented by the nonlinear term $f(u)$.

The equation above was introduced independently by Fisher [39] and by Kolmogorov, Petrovsky and Piskunov (KPP) [57] in 1937. The original motivation stemmed from population genetics and aimed at representing how a genetic trait spreads in space in a given population. A typical example of nonlinearity in this context is of the form $f(u) = u(1-u)$. This equation is often refered to as the F-KPP or KPP equation. At about the same time, and independenly, Zeldovich and Frank-Kamenetskii [108] introduced the same equation, but with a different non-linearity, as the simplest model to describe flame propagation.

In 1951, Skellam [96] had the idea to use this equation to study biological invasions. He was motivated by the invasion of a territory in central Europe by muskrats, for which precise data are available. The model proved to yield a good description, in agreement with the observations. The term $f(u)$ is derived from the logistic law of population growth: $f(u) = ru(1 - u/K)$, of KPP type. Here r is the *intrinsic growth rate* and K is the *carrying capacity*. This type of equation also arises in other phase transition phenomena and involves several types of non-linearities depending on the context. Since these pioneering works, this type of equation and systems and their generalizations are ubiquitous in mathematical biology and ecology.

There is a large literature devoted to this equation which along with its generalizations is still the object of much study. There is a variety of approaches, ranging from PDE's to probability theory to statistical physics and to asymptotic methods. The fundamental results concern the existence of traveling fronts and spreading properties. The former are special solutions of the form $u(t,x) = \phi(x \cdot e - ct)$ where e is a unit vector representing the direction of propagation, c is the velocity of the front and $\phi : \mathbb{R} \to \mathbb{R}$ is its profile. Basic results are due to KPP [57] and Aronson-Weinberger [4].

Spreading properties on the other hand refer to identifying conditions under which invasion (or spreading) occurs and to understand its dynamics. A fundamental resut for this aspect is the following which we state first in the framework of the nonlinearity $f(u) = ru(1-u)$. It concerns solutions stemming from an initial condition $u(0,x) = u_0(x)$ where $u_0 \geq 0, u_0 \not\equiv 0$ and u_0 has compact support. Then, the spreading is described by the following properties:

$$\begin{cases} \lim\limits_{t\to+\infty} \sup\limits_{|x|\leq wt} |u(t,x) - 1| = 0 & \text{if } 0 \leq w < 2\sqrt{dr}, \\ \lim\limits_{t\to+\infty} \sup\limits_{|x|\geq wt} u(t,x) = 0 & \text{if } w > 2\sqrt{dr}. \end{cases}$$

We summarize this result by saying that $w^* := 2\sqrt{dr}$ is the *asymptotic speed of spreading in every direction* for solutions with compactly supported initial data. This result is due to Aronson and Weinberger [4] and is essentially already contained in the original work of KPP [57] in dimension one. For a general presentation of all these results regarding traveling fronts and homogeneous spreading, we refer the reader to [8]

Several authors have refined this spreading property by studying the exact location of the front. The first such study is due to Bramson [29] who showed, by large deviations methods, that there is a logarithmic correction to the position w^*t. Recently, the paper [46] proposed a PDE method for this. Following these articles some recent works were able to establish further terms in the expansion of the location of the front for large t (see the papers [25, 26, 32, 46]).

We describe the equation above as being *homogeneous*. By this, we mean that the equation is isotropic, and with coefficients and nonlinear term that do not depend on the location in space x nor on time t. Another element that enters its qualification as homogeneous is that it is set in all of space $\mathbb{R}^N$. In particular there are no spatial obstacles to propagation.

The present study is devoted to understanding spreading properties for Fisher-KPP type equations in non-homogeneous settings. More precisely we consider very general operators. First, the diffusion, of the form $\mathrm{Tr}[(a_{ij}(t,x))D^2u]$, is no longer assumed isotropic and involves coefficients that depend on t,x. Then, the operator may include a transport term $q(t,x)\cdot\nabla u$. And lastly the reaction term $f = f(t,x,u)$ varies in space and time.

Thus, this monography is devoted to large time behavior of the solutions of the Cauchy problem:

$$\left\{\begin{array}{rcll} \partial_t u - \sum_{i,j=1}^N a_{i,j}(t,x)\partial_{ij}u - \sum_{i=1}^N q_i(t,x)\partial_i u &=& f(t,x,u) & \text{in } (0,\infty)\times\mathbb{R}^N, \\ u(0,x) &=& u_0(x) & \text{for all} \quad x\in\mathbb{R}^N. \end{array}\right. \tag{1}$$

where the coefficients $(a_{i,j})_{i,j}$, $(q_i)_i$ and f are only assumed to be uniformly continuous, bounded in (t,x) and the matrix field $(a_{i,j})_{i,j}$ is uniformly elliptic. In the sequel we will often write operators with the usual summation convention over repeated indices.

The reaction term f is supposed to be monostable and of KPP type, meaning that it admits two steady states 0 and 1, 0 being unstable and 1 being globally attractive, and that it is below its tangent at the unstable steady state 0. We will write more precise assumptions later in a general framework. A typical example of such nonlinearity that generalizes the homogeneous situation is provided by $f(t,x,s) = b(t,x)s(1-s)$ with b bounded and $\inf_{\mathbb{R}\times\mathbb{R}^N} b > 0$. Lastly, we consider compactly supported initial data u_0 with $0 \le u_0 \le 1$. We will see that this framework, up to a change of variables, also includes the more general situation when f admits entire solutions $p^\pm(t,x)$, with p^- unstable and p^+ globally attractive (rather than 0 and 1 respectively).

The goal of this manuscript is to undersand *spreading properties* for this problem in this general setting. To this end, we want to characterize as sharply as possible two non-empty compact sets $\underline{\mathcal{S}} \subset \overline{\mathcal{S}} \subset \mathbb{R}^N$ so that

$$\left\{\begin{array}{rrcl} \text{for all compact set } K\subset \mathrm{int}\underline{\mathcal{S}}, & \lim_{t\to+\infty}\left\{\sup_{x\in tK}|u(t,x)-1|\right\} &=& 0, \\ \text{for all closed set } F\subset\mathbb{R}^N\backslash\overline{\mathcal{S}}, & \lim_{t\to+\infty}\left\{\sup_{x\in tF}|u(t,x)|\right\} &=& 0. \end{array}\right. \tag{2}$$

There is of course a link between such sets and the notion of spreading speeds. Let $e\in\mathbb{S}^{N-1}$ and take $\underline{w},\overline{w}>0$ such that $\underline{w}e\in\underline{\mathcal{S}}$ and $\overline{w}e\in\overline{\mathcal{S}}$. Then the definitions of $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$ yield

$$\lim_{t\to+\infty} u(t,\underline{w}te) = 1 \quad \text{and} \quad \lim_{t\to+\infty} u(t,\overline{w}te) = 0.$$

In other words, if one consider a function $t\mapsto X(t)$ such that $u\big(t,X(t)e\big) = 1/2$, then

$$\underline{w} \le \liminf_{t\to+\infty}\frac{X(t)}{t} \le \limsup_{t\to+\infty}\frac{X(t)}{t} \le \overline{w}.$$

Thus the transition between the unstable steady state $u \equiv 0$ and the attractive on $u \equiv 1$ is located between $\underline{w}t$ and $\overline{w}t$ along direction e. In particular, if one is able to show that $\underline{w} = \overline{w}$, the above inequalities turn into equalities and provide an exact approximation for $X(t)$. This is why we say in this case that there exists an *exact asymptotic spreading speed*.

1.1 A review of the state of the art

Before going any further on the precise statements, let us first recall some known results in the homogeneous, periodic and random stationary ergodic cases. By synthesizing these earlier results, we have naturally derived in our earlier one-dimensional paper [19] two spreading speeds associated with the solutions of the general heterogeneous Fisher-KPP equation. Our approach is similar in the present manuscript, but we have to carry a much deeper investigation of these earlier works.

Homogeneous equation

Let first recall some well-known results more generally in the case where the coefficients do not depend on (t, x). In this case, equation (1) indeed reduces to the classical homogeneous equation

$$\partial_t u - \Delta u = f(u), \tag{3}$$

where $f(0) = f(1) = 0$ and $f(s) > 0$ if $s \in (0, 1)$. This case has been widely studied. When $\liminf_{s\to 0^+} f(s)/s^{1+2/N} > 0$, a classical result due to Aronson and Weinberger [4] yields that there is invasion, namely that $u(t, x) \to 1$ as $t \to \infty$, everywhere in x. Furthermore, there exists $w^* > 0$ such that the solution u of the Cauchy problem associated with a given non-null compactly supported initial datum satisfies

$$\begin{cases} \liminf\limits_{t\to+\infty} \inf\limits_{|x|\leq wt} u(t, x) = 1 & \text{if } 0 \leq w < w^*, \\ \lim\limits_{t\to+\infty} \sup\limits_{|x|\geq wt} u(t, x) = 0 & \text{if } w > w^*. \end{cases} \tag{4}$$

In other words $\underline{\mathcal{S}} = \overline{\mathcal{S}} = \{x,\ |x| \leq w^*\}$. The spreading speed w^* is also characterized as the minimal speed of traveling fronts solutions, defined in [4, 57, 8]. Moreover, this speed is exactly $w^* = 2\sqrt{f'(0)}$ for KPP nonlinearities, that is, for nonlinearities f satisfying $f(s) \leq f'(0)s$ for all $s \geq 0$ (see [4]).

The main aim of the present manuscript is to extend spreading properties to general heterogeneous equations in the full space (1). The classical example of a non-homogeneous framework is that of periodic heterogeneous coefficients. This case is completely understood. Let us start by describing the results in this framework.

Periodic media

Let us consider the case where all the coefficients $a_{i,j}$, q_i and f are space-time periodic. A function $h = h(t, x)$ is called space-time periodic if there exist some positive constants $T, L_1, ..., L_N$ so that

$$h(t, x) = h(t, x + L_i\varepsilon_i) = h(t + T, x)$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, where $(\varepsilon_i)_i$ is a given orthonormal basis of $\mathbb{R}^N$. The periods $T, L_1, ..., L_N$ will be fixed in the sequel. Periodicity is understood to mean the same period(s) for all the terms.

The spreading properties in space periodic media have first been proved using probabilistic tools by Freidlin and Gärtner [42] in 1979 and Freidlin [41] in 1984, when the coefficients only depend on x. These properties have been extended to space-time periodic media by Weinberger in 2002 [102], using a rather elaborate discrete abstract formalism. The authors of the present paper, together with Hamel have given two alternative proofs of spreading properties in multidimensional space-time periodic media in [13] (see also [74, 79]). These methods both use accurate properties of the periodic principal eigenvalues associated with the linearized equation at 0. Lastly, Majda and Souganidis [69] proved homogenization results that are close to, but different from, spreading properties in the space-time periodic setting. Here, we will make this connection precise in Section 5.1.

In periodic media, the asymptotic spreading speed depends on the direction of propagation. Thus, the property proved in [13, 41, 42, 102] is the existence of an asymptotic directional spreading speed $w^*(e) > 0$ in each direction $e \in \mathbb{S}^{N-1}$, so that for any initial datum $u_0 \not\equiv 0$, $0 \leq u_0 \leq 1$ with compact support, one has

$$\begin{cases} \liminf\limits_{t\to+\infty} u(t, x + wte) = 1 & \text{if } 0 \leq w < w^*(e), \\ \lim\limits_{t\to+\infty} u(t, x + wte) = 0 & \text{if } w > w^*(e), \end{cases} \tag{5}$$

locally in $x \in \mathbb{R}^N$. It is possible to characterize $w^*(e)$ in terms of periodic principal eigenvalues in the KPP case, that is, when $f(t,x,s) \leq f_u'(t,x,0)s$ for all $(t,x,s) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^+$. Namely, let $\mathcal{L}$ the parabolic operator associated with the linearized equation near 0:

$$\mathcal{L}\phi := -\partial_t\phi + a_{i,j}(t,x)\partial_{ij}\phi + q_i(t,x)\partial_i\phi + f_u'(t,x,0)\phi,$$

and let $L_p\phi := e^{-p\cdot x}\mathcal{L}(e^{p\cdot x}\phi)$ for all $p \in \mathbb{R}^N$. We know from the Krein-Rutman theory that the operator L_p admits a unique periodic principal eigenvalue k_p^{per}, that is, an eigenvalue associated with a periodic and positive eigenfunction. Then the characterization proved by Freidlin and Gärtner [41, 42] in the space periodic framework and extended to space-time periodic frameworks in [13, 102] reads

$$w^*(e) = \min_{p\cdot e>0} \frac{k_{-p}^{per}}{p \cdot e}. \tag{6}$$

This quantity can also be written using the minimal speed of existence of pulsating traveling fronts (defined and investigated in [9, 15, 17, 74, 79, 102]), which is indeed the appropriate characterization when f is not of KPP type [102].

Lastly, Weinberger [102] proved that the convergence (5) is uniform in all directions, meaning that

$$\begin{cases} \text{for all compact set } K \subset \text{int}\mathcal{S}, & \lim_{t\to+\infty} \big\{ \sup_{x\in tK} |u(t,x) - 1| \big\} = 0, \\ \text{for all closed set } F \subset \mathbb{R}^N \backslash \mathcal{S}, & \lim_{t\to+\infty} \big\{ \sup_{x\in tF} |u(t,x)| \big\} = 0, \end{cases} \tag{7}$$

with

$$\mathcal{S} = \{x,\ \forall p \in \mathbb{R}^N, k_{-p}^{per} \geq p \cdot x\}. \tag{8}$$

Of course, as for all $e \in \mathbb{S}^{N-1}$ and $w > 0$, $we \in \mathcal{S}$ if and only if $w < w^*(e)$, we recover (5) as a corollary of (7).

By analogy with crystallography the set $\mathcal{S}$ is sometimes called the *Wulff shape* of equation (1). Indeed, in [103], Wulff proved that for a given crystal volume $|B|$ and for a given surface tension σ, the set B that minimizes the surface energy $\int_{\partial B} \sigma(n(x))dx$, where n is the normal vector to ∂B, is $\mathcal{W} = \{x, x \cdot e \leq \sigma(e) \text{ for all } e \in \mathbb{S}^{N-1}\}$, up to rescaling and translation. Here, the analogy is that $\mathcal{S}$ has a similar definition, with $p \mapsto k_{-p}^{per}$ playing the role of a surface tension.

The exact location of the front could be derived and involves a logarithmic correction as in homogeneous media [47, 89].

Random stationary ergodic media

The first proof of the existence of an exact spreading speed in random stationary ergodic media goes back to the pioneering papers of Freidlin and Gärtner [42] and Freidlin [41], who considered time-independent reaction terms in dimension 1 using large deviation techniques. In multi-dimensional media, the existence of an exact spreading speed has been proved by Nolen and Xin for space-time heterogeneous advection terms and homogeneous reaction terms [81, 82, 83]. As they claimed in [81], their approach should work when the diffusion term is also random stationary ergodic, but it does not fit heterogeneous reaction terms.

In these cases, the exact asymptotic spreading speed is characterized through some Lyapounov exponents associated with the underlying Brownian process. Similar quantities appear in related problems such as homogenization of reaction-diffusion equations (see [67] and the references therein). The connections between these various approaches will be discussed in details in Section 5.1.

We underline that all these earlier papers made some *stationarity hypothesis* on the random heterogeneity, which means that the statistical properties of the medium do not depend on time and space. Many classes of deterministic coefficients could indeed be turned into a random stationary ergodic setting so that the orginal deterministic media is a given event. This a well-known fact for periodic, almost periodic (see [84]) and uniquely ergodic deterministic coefficients. In such cases, one could thus derive spreading properties for almost every event.

However, the *given* original deterministic equation, for which we want to prove a spreading property, might not be in the set of events with probability 1 for which spreading properties are derived with this method. Thus the probabilistic approach does not cover this given equation. For example, consider the simple case of deterministic coefficients having a compactly supported heterogeneity (see (11) below for a precise definition). Then, the homogeneous equation associated with translations at infinity is an event with probability 1 for the standard probability measure, while the original equation has probability 0. Hence, the probabilistic approach misses the original equation in that case. Thus, even if one can transform deterministic heterogeneous equations into random stationary ergodic ones, it may not provide any result on spreading properties for a given deterministic equation. One can even construct more complex examples for which it is not even possible to determine explicitly the set of probability 1 for which there exists a spreading property (see for example the discussion in Section 8 on uniquely ergodic coefficients).

1.2 The general heterogeneous case: setting of the problem

The main purpose of the present manuscript is to prove spreading properties in general heterogeneous media. Heterogeneity can arise for different reasons, owing to the geometry or to the coefficients in the equation. Regarding geometry, the first author together with Hamel and Nadirashvili [16] have studied spreading properties for the homogeneous equation in general unbounded domains (these include spirals, complementaries of infinite combs, cusps, etc.) with Neumann boundary conditions. In these geometries, linear spreading speeds do not always exist. Furthermore, several examples are constructed in [16] where the spreading speed is either infinite or null.

The present manuscript deals with heterogeneous media for problems set in $\mathbb{R}^N$ but in which the terms in the equation are allowed to depend on space and time in a fairly general fashion. As in [16], given any compactly supported initial datum u_0 and the corresponding solution u of (1), we introduce two speeds:

$$\begin{aligned} w_*(e) &:= \sup\Big\{ \quad w \geq 0, \ \ \lim_{t\to+\infty} \big\{ \inf_{w'\in[0,w]} \quad u(t, x + w'te)\big\} = 1 \quad \text{loc. } x \in \mathbb{R}^N \Big\}, \\ w^*(e) &:= \inf\Big\{ \quad w \geq 0, \ \ \lim_{t\to+\infty} \big\{ \sup_{w'\geq w} \quad u(t, x + w'te)\big\} = 0 \quad \text{loc. } x \in \mathbb{R}^N \Big\}. \end{aligned} \tag{9}$$

We could reformulate the goal of this manuscript in the following way: we want to get accurate estimates on $w_*(e)$ and $w^*(e)$ and to try to identify classes of equations for which $w^*(e) = w_*(e)$ (and is independent of u_0). This last equality does not always hold, which justifies the introduction of two speeds rather than a single one. Indeed, Garnier, Giletti and the second author [44] exhibited an example of space heterogeneous equation in dimension 1 for which there exists a range of speeds w such that the $\omega-$limit set of $t \mapsto u(t, wt)$ is $[0, 1]$. In this case the location of the transition between 0 and 1 oscillates within the interval (w_*t, w^*t) at large time t.

Together with Hamel, the authors have proved in a previous paper [13] that under a natural positivity assumption, but otherwise in a general framework, there is at least a positive linear spreading speed, which means with the above definition that $w_*(e) > 0$ for any $e \in \mathbb{S}^{N-1}$. More precisely, we proved[1] in [13] that if $q(t, x) = \nabla \cdot A(t, x)$, where $A(t, x) = \big(a_{i,j}(t, x)\big)_{i,j}$ (hence we assume a divergence form operator), and $f'_u(t, x, 0) > 0$ uniformly when $|x|$ is large, the following inequality holds:

$$w_*(e) \geq w_0 := 2\sqrt{\liminf_{|x|\to+\infty} \inf_{t\in\mathbb{R}^+} \gamma(t, x) f'_u(t, x, 0)}, \tag{10}$$

[1] Actually, the result we obtained in [13] is a little more accurate and the hypotheses are somewhat more general, we refer the reader to [13] for the precise assumptions.

where $\gamma(t,x)$ is the smallest eigenvalue of the matrix $A(t,x)$. We also established upper estimates on $w^*(e)$, which ensure that $\sup_{e\in\mathbb{S}^{N-1}} w^*(e) < +\infty$, under mild hypotheses on A, q and f.

We point out a corollary of this result. Consider a compactly supported heterogeneity, that is, assume $q \equiv 0$ and

$$f(t,x,s) = (b_0 - b(x))s(1-s) \tag{11}$$

with $b_0 > 0, b \geq 0$ and $b = b(x)$ as well as $A(x) - I_N$ are smooth compactly supported perturbations of the homogeneous equation. Then the result of [13] gives $w_*(e) \geq w_0 = 2\sqrt{b_0}$. It is also easy to check that $w^*(e) \leq 2\sqrt{b_0}$ since $f(t,x,s) \leq b_0 s(1-s)$. Thus, in this case

$$w^*(e) = w_*(e) = 2\sqrt{b_0}.$$

This result was also derived by Kong and Shen in [58], who considered other types of dispersion rules as well. This simple observation shows that, in a sense, only what happens at infinity plays a role in the computation of $w^*(e)$ and $w_*(e)$.

On the other hand, when the coefficients are space-time periodic, the expansion set could be characterized through periodic principal eigenvalues [13, 42, 102]. In this framework, estimate (10) is not optimal in general: one needs to take into account the whole structure of equation (1) through the periodic principal eigenvalues of the linearized equation in the neighborhood of $u = 0$ to get an accurate result.

Summarizing the indications from periodic and compactly supported heterogeneities, to estimate $w^*(e)$ and $w_*(e)$, we see that we need to take into account:

- the behavior of the operator when $|t| \to +\infty$ and $|x| \to +\infty$, and
- some notion of "principal eigenvalue" of the linearized parabolic operator near $u = 0$.

Therefore, we are led to extend the notion of principal eigenvalues to linear parabolic operators in unbounded domains. We will define these generalized principal eigenvalues through the existence of sub or supersolutions of the linear equation (see the definitions in Section 2.2 below). This definition is similar, but different from, the definition of the generalized principal eigenvalue of an elliptic operator introduced by Berestycki, Nirenberg and Varadhan [21] for bounded domains and extended to unbounded ones by Berestycki, Hamel and Rossi [18]. Some important properties of classical principal eigenvalues are not satisfied by generalized principal eigenvalues and thus the classical techniques that have been used to prove spreading properties in periodic media in [13, 41, 42, 102] are no longer available here. This is why we use homogenization techniques. In Section 5.1, we describe the link between homogenization problems and asymptotic spreading.

1.3 The link between traveling waves and spreading properties

Let us conclude this Introduction with a few words about traveling waves. We have recalled above that in homogeneous and periodic media, there is an explicit link between the asymptotic spreading speed and the minimal speed of existence of traveling waves. For example, these two quantities are equal in dimension 1. This is why most of the papers address propagation problems using both notions indistinctly.

In general heterogeneous media, the first author and Hamel [10, 11] and Matano [70] have introduced two generalizations of the notion of traveling wave. Several recent papers [10, 11, 12, 72, 73, 80, 93, 106] investigated the existence, uniqueness and stability of such waves in the case when the nonlinearity is bistable or of ignition type and in dimension 1. In higher dimensions, for the same types of nonlinearities, Zlatoš has proved that such waves might not exist (see [107] and references therein).

When the nonlinearity is monostable and time-heterogenous, the existence of generalized transition waves has been proved by the second author and Rossi [75] (see also [76, 87]). It is not true in general that such waves exist for space-heterogeneous monostable equations. In fact, Nolen, Roquejoffre, Ryzhik and Zlatoš [78] constructed a counter-example for a compactly supported heterogeneity. Zlatoš further provided conditions in this framework ensuring the existence of generalized transition waves in dimension 1 [105], for example when only the diffusion term is heterogeneous.

Hence, for some classes of heterogeneities, there exists an exact asymptotic spreading speed but generalized transition waves do not exist. This emphasizes that one needs to be careful and to distinguish between the two approaches in general heterogeneous media.

2 A general formula for the expansion sets

2.1 Notations and hypotheses

We will use the following notations in the whole manuscript. We denote the Euclidean norm in $\mathbb{R}^N$ by $|\cdot|$, that is, for all $x\in\mathbb{R}^N$, $|x|^2 := \sum_{i=1}^N x_i^2$. The set $\mathcal{C}(\mathbb{R}\times\mathbb{R}^N)$ is the set of the continuous functions over $\mathbb{R}\times\mathbb{R}^N$ equipped with the topology of locally uniform convergence. For all $\delta\in(0,1)$, the set $\mathcal{C}^{\delta/2,\delta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ is the set of functions g such that for all compact set $K\subset\mathbb{R}\times\mathbb{R}^N$, there exists a constant $C=C(g,K)>0$ such that

$$\forall (t,x)\in K, (s,y)\in K, \quad |g(s,y)-g(t,x)|\leq C(|s-t|^{\delta/2}+|y-x|^{\delta}).$$

We shall require some regularity assumptions on f, A, q throughout the manuscript. First, we assume that A, q and $f(\cdot,\cdot,s)$ are uniformly continuous and uniformly bounded with respect to $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, uniformly with respect to $s\in[0,1]$. The function $f:\mathbb{R}\times\mathbb{R}^N\times[0,1]\to\mathbb{R}$ is assumed to be of class $C^{\frac{\delta}{2},\delta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ in (t,x), locally in s, for a given $0<\delta<1$. We also assume that f is locally Lipschitz-continuous in s and of class $\mathcal{C}^{1+\gamma}$ in s for $s\in[0,\beta]$ uniformly with respect to $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ with $\beta>0$ and $0<\gamma<1$. We assume that for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$:

$$f(t,x,0)=f(t,x,1)=0 \quad \text{and} \quad \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} f(t,x,s)>0 \text{ if } s\in(0,1), \tag{12}$$

and that f is of KPP type, that is,

$$f(t,x,s)\leq f'_u(t,x,0)s \quad \text{for all } (t,x,s)\in\mathbb{R}\times\mathbb{R}^N\times[0,1]. \tag{13}$$

The matrix field $A = (a_{i,j})_{i,j} : \mathbb{R} \times \mathbb{R}^N \to S_N(\mathbb{R})$ belongs to $\mathcal{C}^{\frac{\delta}{2},\delta}_{loc}(\mathbb{R} \times \mathbb{R}^N)$. We assume furthermore that A is a uniformly elliptic and continuous matrix field: there exist some positive constants γ and Γ such that for all $\xi \in \mathbb{R}^N, (t,x) \in \mathbb{R} \times \mathbb{R}^N$, one has:

$$\gamma|\xi|^2 \leq \sum_{1\leq i,j\leq N} a_{i,j}(t,x)\xi_i\xi_j \leq \Gamma|\xi|^2. \tag{14}$$

The drift term $q : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is in $\mathcal{C}^{\frac{\delta}{2},\delta}_{loc}(\mathbb{R} \times \mathbb{R}^N)$.

Lastly, we make the following instability hypothesis on the steady state 0:

$$\begin{array}{c}\text{for any } u_0 \not\equiv 0 \text{ such that } 0 \leq u_0 \leq 1, \text{ there exists } w > 0 \text{ such that} \\ \text{the solution } u \text{ of (1) satisfies } \lim_{t\to+\infty} \sup_{|x|\leq wt} |u(t,x) - 1| = 0.\end{array} \tag{15}$$

In other words, $w_*(e) \geq w > 0$ for all $e \in \mathbb{S}^{N-1}$.

In order to sum up the heuristical meaning of these hypotheses:

- we consider smooth coefficients and the diffusion term is elliptic (14),
- hypotheses (12) and (15) mean that 0 and 1 are two steady states and that 1 is globally attractive (and thus 0 is unstable),
- the nonlinearity is of KPP-type (13): it is below its tangent at $u = 0$.

A typical equation satisfying our hypotheses is:

$$\partial_t u = \nabla \cdot \big(A(t,x)\nabla u\big) + c(t,x)u(1-u) \quad \text{ in } (0,\infty) \times \mathbb{R}^N,$$

where A is an elliptic matrix field and c, A and ∇A are uniformly positive, bounded and uniformly continuous with respect to (t,x). Indeed, it has been proved in [18, 13] that if

$$\sup_{R>0} \inf_{t>R, |x|>R} \Big(4f'_u(t,x,0) \min_{e\in\mathbb{S}^{N-1}} (eA(t,x)e) - |q(t,x) + \nabla \cdot A(t,x)|^2\Big) > 0, \tag{16}$$

then (15) is satisfied.

Lastly, let us mention the case where one considers two time global heterogeneous solutions of (1), $p_- = p_-(t,x)$ and $p_+ = p_+(t,x)$ instead of 0 and 1. Then as soon as $\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \big(p_+ - p_-\big)(t,x) > 0$ and $p_+ - p_-$ is bounded, one could perform the change of variables $\widetilde{u}(t,x) = \big(u(t,x) - p_-(t,x)\big)/\big(p_+(t,x) - p_-(t,x)\big)$ in order to turn (1) into an equation with steady states 0 and 1. Thus there is no loss of generality in assuming $p_- \equiv 0$ and $p_+ \equiv 1$ as soon as $\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} \big(p_+ - p_-\big)(t,x) > 0$ and $p_+ - p_-$ is bounded, as already noticed in [76].

2.2 The main tool: generalized principal eigenvalues

In this Section we define the notion of generalized principal eigenvalues that will be needed in the statement of spreading properties. Consider the parabolic operator defined for all $\phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ by

$$\begin{array}{rcl}\mathcal{L}\phi & = & -\partial_t\phi + a_{i,j}(t,x)\partial_{ij}\phi + q_i(t,x)\partial_i\phi + f'_u(t,x,0)\phi, \\ & = & -\partial_t\phi + tr(A(t,x)\nabla^2\phi) + q(t,x)\cdot\nabla\phi + f'_u(t,x,0)\phi.\end{array} \tag{17}$$

Definition 2.1 *The* **generalized principal eigenvalues** *associated with operator $\mathcal{L}$ in a smooth open set $Q \subset \mathbb{R} \times \mathbb{R}^N$ are:*

$$\underline{\lambda_1}(\mathcal{L}, Q) := \sup\{\lambda \mid \exists \phi \in \mathcal{C}^{1,2}(Q) \cap W^{1,\infty}(Q),\ \inf_Q \phi > 0 \text{ and } \mathcal{L}\phi \geq \lambda\phi \text{ in } Q\}. \tag{18}$$

$$\overline{\lambda_1}(\mathcal{L}, Q) := \inf\{\lambda \mid \exists \phi \in \mathcal{C}^{1,2}(Q) \cap W^{1,\infty}(Q),\ \inf_Q \phi > 0 \text{ and } \mathcal{L}\phi \leq \lambda\phi \text{ in } Q\}. \tag{19}$$

Actually, this definition is the first instance where generalized principal eigenvalues are defined for linear parabolic operators with general space-time heterogeneous coefficients.

For elliptic operators, similar quantities have been introduced by Berestycki, Nirenberg and Varadhan [21] for bounded domains with a non-smooth boundary and by Berestycki, Hamel and Rossi in [18] in unbounded domains (see also [24]). These quantities are involved in the statement of many properties of parabolic and elliptic equations in unbounded domains, such as maximum principles, existence and uniqueness results. The main difference with [18, 21, 24] is that here we both impose $\inf_Q \phi > 0$ and $\phi \in W^{1,\infty}(Q)$. As already observed in [19, 24], the conditions we require on the test-functions in the definitions of generalized principal eigenvalues are very important and might give very different quantities.

In our previous work [19] dealing with dimension 1, we required different conditions on the test-functions. Namely, we just imposed $\lim_{x\to+\infty} \frac{1}{x} \ln \phi(x) = 0$ instead of the boundedness and the uniform positivity of ϕ. This milder condition enabled us to prove that $\underline{\lambda_1} = \overline{\lambda_1}$ almost surely when the coefficients are random stationary ergodic in $x \in \mathbb{R}$. In the present manuscript, we explain after the statement of Proposition 4.2 below what was the difficulty we were not able to overcome in order to consider such mild conditions on the test-functions. Indeed, we had to require the test-functions ϕ involved in the definitions of the generalized principal eigenvalues to be bounded and uniformly positive, and we cannot hope to prove that the two generalized principal eigenvalues are equal in multidimensional random stationary ergodic media under such conditions on the test-functions. The expected asymptotic behavior for test-functions in such media is the subexponential, but unbounded, growth. We will be able to handle such behaviors of the test-functions only when the coefficients do not depend on t (see Theorem 47 below).

We will prove in Section 4 several properties of these generalized principal eigenvalues. If the operator $\mathcal{L}$ admits a classical eigenvalue associated with an eigenfunction lying in the appropriate class of test-functions, that is, if there exist $\lambda \in \mathbb{R}$ and $\phi \in \mathcal{C}^{1,2}(Q) \cap W^{1,\infty}(Q)$, with $\inf_Q \phi > 0$, such that $\mathcal{L}\phi = \lambda\phi$ over Q, where Q is an open set containing balls of arbitrary radii, then $\underline{\lambda_1}(\mathcal{L}, Q) = \overline{\lambda_1}(\mathcal{L}, Q) = \lambda$. In other words, if there exists a classical eigenvalue, then the two generalized eigenvalues equal this classical eigenvalue in such domains. This ensures that our generalization is meaningful. We will also prove that when the coefficients are almost periodic or uniquely ergodic in (t, x), then $\underline{\lambda_1} = \overline{\lambda_1}$, although almost periodic operators do not always admit a classical eigenvalue. When the coefficients do not depend on space, it is possible to compute explicitly these quantities. Lastly, we give, in a general framework, some comparison and continuity results for $\underline{\lambda_1}$ and $\overline{\lambda_1}$.

2.3 Statement of the results in dimension 1

We first consider the case $N = 1$. The definitions of our speeds $\underline{w}$ and $\overline{w}$ is much simpler in dimension 1 and is a useful first step in order to understand the multidimensional framework. When the coefficients do not depend on t, this case has been considered and fully described in our earlier paper [19].

When $N = 1$, equation (1) reads

$$\begin{cases} \partial_t u - a(t,x)\partial_{xx}u - q(t,x)\partial_x u = f(t,x,u) \text{ in } \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x) \quad \text{for all} \quad x \in \mathbb{R}. \end{cases} \tag{20}$$

For all $p \in \mathbb{R}$, let

$$\overline{H}^+(p) := \inf_{R>0} \overline{\lambda_1}(L_p, (R,\infty)^2) \quad \text{and} \quad \underline{H}^+(p) := \sup_{R>0} \underline{\lambda_1}(L_p, (R,\infty)^2), \tag{21}$$

where we define for all $\phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R})$ and $p \in \mathbb{R}$:

$$L_p\phi := -\partial_t\phi + a(t,x)\partial_{xx}\phi + (q(t,x) + 2pa(t,x))\partial_x\phi + (f_s'(t,x,0) + pq(t,x) + p^2 a(t,x))\phi. \tag{22}$$

These quantities will play the role of Hamiltonians in our proof. We thus need to check that it satisfy some basic properties in order to apply the classical theory of Hamilton-Jacobi equations. This will be done later in the general multidimensional framework in Proposition 2.2.

We are now in position to define our speeds $\underline{w}$ and $\overline{w}$:

$$\underline{w} := \min_{p>0} \frac{\underline{H}^+(-p)}{p} \quad \text{and} \quad \overline{w} := \min_{p>0} \frac{\overline{H}^+(-p)}{p}. \tag{23}$$

In dimension $N = 1$, our main result reads:

Theorem 1 *Assume that $N = 1$. Take u_0 a measurable and compactly supported function such that $0 \leq u_0 \leq 1$ and $u_0 \not\equiv 0$ and let u the solution of the associated Cauchy problem (20). Then if $u(t,x) \to 1$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}$, one has*

- *for all $w \in [0, \underline{w})$, $\lim_{t\to+\infty} \inf_{0\leq x\leq wt} u(t,x) = 1$,*
- *for all $w > \overline{w}$, $\lim_{t\to+\infty} \sup_{x\geq wt} u(t,x) = 0$.*

In other words, one has $\underline{w} \leq w_*(1) \leq w^*(1) \leq \overline{w}$. We underline that the speeds $\underline{w}$ and $\overline{w}$ are not necessarily equal as proved later in Proposition 13.1. It is already known that $\underline{w} = \overline{w}$ in homogeneous or space-time periodic media (see the Introduction). In order to check that our constructions of $\overline{w}$ and $\underline{w}$ are nearly optimal, we prove in Section 3 that $\underline{w} = \overline{w}$ in various types of media.

Note that the present result is less accurate than the main result of [19] since we consider bounded and uniformly positive test-functions in the definitions of the generalized principal eigenvalues, whereas sub-exponential test-functions were considered in [19]. On the other hand, here we consider coefficients depending on t and not only on x as in [19].

2.4 Statement of the results in dimension N

We are now in position to state a general spreading result in dimension N. Our aim is to state a general abstract result in the most general framework we can handle, for fully general heterogeneous coefficients only satisfying boundedness and uniform continuity assumptions (see Section 2.1). We will then show in section 3 that this result applies and provides exact asymptotic spreading speeds in various settings.

In general heterogeneous media, we know from earlier works [13] on compactly supported heterogeneities that only what happens when t and x are large should play a role in the construction of $\underline{w}(e)$ and $\overline{w}(e)$. In dimension 1, we thus only considered the generalized eigenvalues in the half-spaces $(R,\infty)\times(R,\infty)$, with R large. In multi-dimensional media, we need to take into account the direction of the propagation and the situation becomes much more involved. We will indeed restrict ourselves to the cones of angle α in the direction of propagation e and to $t>R$ and $|x|>R$, where α will be small and R will be large:

$$C_{R,\alpha}(e) := \Big\{(t,x)\in\mathbb{R}\times\mathbb{R}^N, \quad t>R, \quad |x|>R, \quad \Big|\frac{x}{|x|}-e\Big|<\alpha\Big\}. \tag{24}$$

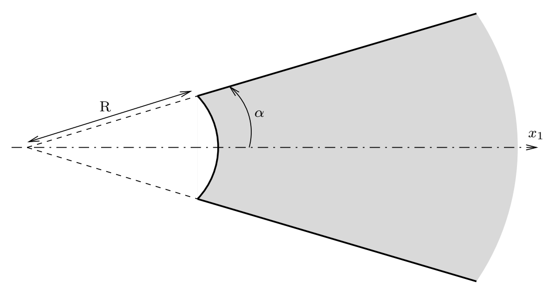

Figure 1: The projection of the set $C_{R,\alpha}(e_1)$ on the x-plane.

Let us introduce the operators L_p associated with exponential solutions of the linearized equation near $u\equiv 0$, defined for all $p\in\mathbb{R}^N$ and $\phi\in\mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ by $L_p\phi := e^{-p\cdot x}\mathcal{L}\big(e^{p\cdot x}\phi\big)$. More explicitly:

$$L_p\phi := -\partial_t\phi + tr(A(t,x)\nabla^2\phi) + (q(t,x)+2A(t,x)p)\cdot\nabla\phi + (f'_u(t,x,0)+p\cdot q(t,x)+pA(t,x)p)\phi. \tag{25}$$

For all $p\in\mathbb{R}^N$ and $e\in\mathbb{S}^{N-1}$, we let

$$\overline{H}(e,p) := \inf_{R>0,\alpha\in(0,1)} \overline{\lambda_1}(L_p, C_{R,\alpha}(e)) \text{ and } \underline{H}(e,p) := \sup_{R>0,\alpha\in(0,1)} \underline{\lambda_1}(L_p, C_{R,\alpha}(e)). \tag{26}$$

It is easy to see that $\overline{\lambda_1}(L_p, C_{R,\alpha}(e))$ is nonincreasing in R and nondecreasing in α and that $\underline{\lambda_1}(L_p, C_{R,\alpha}(e))$ is nondecreasing in R and nonincreasing in α. Thus, the infimum and the supremum in (26) can be replaced by limits as $R\to+\infty$ and $\alpha\to 0$.

The properties of these Hamiltonians are given in the following Proposition:

Proposition 2.2 *1. The functions $p\to\overline{H}(e,p)$ and $p\to\underline{H}(e,p)$ are locally Lipschitz-continuous, uniformly with respect to $e\in\mathbb{S}^{N-1}$, and $p\mapsto\overline{H}(e,p)$ is convex for all $e\in\mathbb{S}^{N-1}$.*

2. *For all* $p \in \mathbb{R}^N$, $e \mapsto \underline{H}(e,p)$ *is lower semicontinuous and* $e \mapsto \overline{H}(e,p)$ *is upper semicontinuous.*

3. *There exist* $C \geq c > 0$ *such that for all* $(e,p) \in \mathbb{S}^{N-1} \times \mathbb{R}^N$*:*

$$c(|p|^2 - 1) \leq \underline{H}(e,p) \leq \overline{H}(e,p) \leq C(1+|p|^2).$$

We underline that the Hamiltonians $\underline{H}$ and $\overline{H}$ are not continuous with respect to e in general (see the example of Proposition 3.13 below). This is the source of serious difficulties.

Using these Hamiltonians, we will now define two functions from which we derive the expansion sets. Define the convex conjugates with respect to p:

$$\underline{H}^\star(e,q) := \sup_{p \in \mathbb{R}^N} \big(p \cdot q - \underline{H}(e,p)\big) \quad \text{and} \quad \overline{H}^\star(e,q) := \sup_{p \in \mathbb{R}^N} \big(p \cdot q - \overline{H}(e,p)\big),$$

which are well-defined thanks to Proposition 2.2. Let

$$\begin{array}{ll} \underline{U}(x) & := \inf_{\gamma \in \mathcal{A}} \max_{t \in [0,1]} \Big\{ \int_t^1 \underline{H}^\star\big(\frac{\gamma(s)}{|\gamma(s)|}, -\gamma'(s)\big) ds \Big\}, \\ \overline{U}(x) & := \inf_{\gamma \in \mathcal{A}} \max_{t \in [0,1]} \Big\{ \int_t^1 \overline{H}^\star\big(\frac{\gamma(s)}{|\gamma(s)|}, -\gamma'(s)\big) ds \Big\}, \\ \gamma \in \mathcal{A} & := \big\{\gamma \in H^1([0,1]),\ \gamma(0) = 0,\ \gamma(1) = x,\ \forall s \in (0,1),\ \gamma(s) \neq 0\big\}. \end{array} \tag{27}$$

We will show in Lemma 5.7 below that, as $e \mapsto \overline{H}(e,p)$ is upper semicontinuous, $\overline{U}$ is indeed a minimum, in other words, for all x, there exists an admissible path γ from 0 to x minimizing the maximum over $t \in [0,1]$ of the integral.

We define our expansion sets in general heterogeneous media as

$$\underline{\mathcal{S}} := \text{cl}\{\underline{U} = 0\} \quad \text{and} \quad \overline{\mathcal{S}} := \{\overline{U} = 0\}. \tag{28}$$

The reader might recognize here representations formulas for the solutions of Hamilton-Jacobi equations. Indeed, the sets $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$ are related to the zero sets of the solutions of such equations. Such representations formulas are well-known for Hamilton-Jacobi equations with continuous coefficients (see for example [38, 69]). This link will be described in Section 5 below. Our Hamiltonians are not continuous here, but we will make use of these formulas in order to derive properties of the expansion sets.

We are now in position to state our main result.

Theorem 2 *Take* u_0 *a measurable and compactly supported function such that* $0 \leq u_0 \leq 1$ *and* $u_0 \not\equiv 0$ *and let* u *the solution of the associated Cauchy problem* (1)*. One has*

$$\left\{ \begin{array}{lll} \textit{for all compact set } K \subset \text{int}\underline{\mathcal{S}}, & \lim_{t\to+\infty} \big\{ \sup_{x \in tK} |u(t,x) - 1| \big\} & = 0, \\ \textit{for all closed set } F \subset \mathbb{R}^N \backslash \overline{\mathcal{S}}, & \lim_{t\to+\infty} \big\{ \sup_{x \in tF} |u(t,x)| \big\} & = 0. \end{array} \right. \tag{29}$$

In order to state this result in terms of speeds, define for all $e \in \mathbb{S}^{N-1}$:

$$\underline{w}(e) = \sup\{w > 0,\ we \in \underline{\mathcal{S}}\} \quad \text{and} \quad \overline{w}(e) = \sup\{w > 0,\ we \in \overline{\mathcal{S}}\}. \tag{30}$$

Then it follows from Theorem 2 that

$$\underline{w}(e) \leq w_*(e) \leq w^*(e) \leq \overline{w}(e).$$

In dimension 1, one could check that the path γ involved in the definition of $\overline{U}$ is necessarily $\gamma(s) = sx$. We thus recover the results of Section 2.3: $\underline{w}(e_1) = \min_{p>0} \underline{H}(e_1, -p)/p$ and $\overline{w}(e_1) = \min_{p>0} \overline{H}(e_1, -p)/p$ in dimension 1 This is quite similar to the so-called Wulff-type characterization (33), where the expansion set could be written as the polar set of the eigenvalues. We will indeed prove that such a Wulff-type characterization holds for recurrent media (which include periodic and almost periodic media).

Such a characterization could not hold for general heterogeneous multi-dimensional equations. Indeed, in multidimensional media, the population might propagate faster by changing its direction of propagation at some point, that is, the minimizing path γ in the definition of $\overline{U}$ is not necessarily a line. Several examples will be provided in Section 3.9. Hence, the integral characterizations (27) are much more accurate than Wulff-type ones since they enable multidimensional propagation strategies for the solution of the Cauchy problem.

2.5 Geometry of the expansion sets

When the expansion set is of Wulff-type (33), it immediately follows from this characterization that it is convex. In more general frameworks, the convexity of the expansion sets is a difficult problem. Indeed, the expansion sets could be non-convex, as shown in Proposition 3.15. However, when the Hamiltonian $\underline{H}$ is assumed to be quasiconcave w.r.t $x \in \mathbb{R}^N$, then the lower expansion set is convex.

Proposition 2.3 *Assume that the function $x \in \mathbb{R}^N \backslash \{0\} \mapsto \underline{H}(x/|x|, p)$, extended to 0 by $\underline{H}(0,p) := \sup_{e \in \mathbb{S}^{N-1}} \underline{H}(e,p)$, is quasiconcave over $\mathbb{R}^N$ for all $p \in \mathbb{R}^N$. Then the set $\underline{\mathcal{S}}$ is convex*

Here, a function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be quasiconcave if $\{f \geq \alpha\}$ is a convex set for all $\alpha \in \mathbb{R}$.

This Proposition is certainly not optimal: one could construct Hamiltonians that are not quasiconcave which give rise to convex expansion sets, as in Proposition 3.15 below. However, we believe that it is optimal if one does not require any further conditions on the coefficients, such as comparison between the Hamiltonians in their different level sets.

If $\underline{H}$ is concave with respect to x, then we are led to a Hamilton-Jacobi equation with a Hamiltonian which is concave in x. It is well-known that for such equations, the solutions associated with concave initial data are concave with respect to x [1, 50]. However, as the function $x \mapsto \underline{H}(x/|x|, p)$ is clearly $1-$homogeneous with respect to x, if it were concave then it would be constant. Moreover, we will exhibit several examples with discontinuous Hamiltonians, for which the concavity is of course excluded. This is why the quasiconcavity hypothesis is relevant for our problem.

The only works we know on Hamilton-Jacobi equations that are quasiconcave are [51, 52]. In these papers, Imbert and Monneau considered Hamiltonians that are quasiconcave with respect to p, not x, and thus the issues they faced are different from ours.

Without any quasiconcavity assumption on the Hamiltonians, one can still prove that the expansion sets are star-shaped, compact and have a smooth boundary, under some mild additional hypothesis.

Proposition 2.4 *Assume that there exists a constant $c > 0$ such that $\underline{H}(resp.\overline{H})(e,p) \geq c|p|^2$ for all $(p,e) \in \mathbb{R}^N \times \mathbb{S}^{N-1}$. Then the sets $\underline{\mathcal{S}}$ (resp. $\overline{\mathcal{S}}$) is compact and star-shaped with respect to 0. If, furthermore, the stronger growth assumption $\underline{H}(resp.\overline{H})(e,p) \geq c(|p|^2+1)$ holds for all $(p,e) \in \mathbb{R}^N \times \mathbb{S}^{N-1}$, then $\underline{\mathcal{S}}$ (resp. $\overline{\mathcal{S}}$) contains an open ball centered at 0 and has a Lipschitz-continuous boundary.*

We do not know what is the range of sets that can be obtained as expansion sets for some appropriately chosen coefficients. For example, is it possible to obtain any set satisfying the properties of Proposition 2.4 as an upper or lower expansion set? We leave it as an open problem.[2]

As we have already mentioned, under hypothesis (16) on the coefficients, there exists a positive lower expansion speed and thus (15) is satisfied. This assumption also ensures that the hypotheses of Proposition 2.4 are satisfied. For future reference, we state this fact here.

Lemma 2.5 *If (16) is satisfied, then $\underline{H}(e,p) \geq c(|p|^2+1)$ for all $(p,e) \in \mathbb{R}^N \times \mathbb{S}^{N-1}$.*

3 Exact asymptotic spreading speed in different frameworks

3.1 Homogeneous, periodic and homogeneous at infinity coefficients

The cases of homogeneous, periodic and compactly supported coefficients are already known to admit an exact asymptotic spreading speed. These results have been recalled in Section 1.1. Our construction is optimal in these frameworks.

Proposition 3.1 *1. Assume that A and $f'_u(\cdot,\cdot,0)$ are constant with respect to (t,x), and that $q \equiv 0$, then one has $\overline{\lambda_1}(L_p, \mathbb{R}\times\mathbb{R}^N) = \underline{\lambda_1}(L_p, \mathbb{R}\times\mathbb{R}^N) = f'_u(0) + pAp$ for all $p \in \mathbb{R}^N$ and*

$$\underline{w}(e) = \overline{w}(e) = 2\sqrt{eAef'_u(0)} \quad \text{for all } e \in \mathbb{S}^{N-1}.$$

[2] We thank an anonymous referee for raising this point.

2. *Assume that A, q and $f'_u(\cdot,\cdot,0)$ are periodic in (t,x) (in the same meaning as in Section 1.1). Define k_p^{per} as in Section 1.1. Then one has $\overline{\lambda_1}(L_p,\mathbb{R}\times\mathbb{R}^N)=\underline{\lambda_1}(L_p,\mathbb{R}\times\mathbb{R}^N)=k_p^{per}$ for all $p\in\mathbb{R}^N$ and*

$$\underline{w}(e)=\overline{w}(e)=\min_{p\cdot e>0}\frac{k_{-p}^{per}}{p\cdot e}\qquad \textit{for all } e\in\mathbb{S}^{N-1}.$$

3. *Assume that there exist a positive matrix $A^*\in\mathcal{S}_N(\mathbb{R})$, a vector $q^*\in\mathbb{R}^N$ and a constant $c^*\in\mathbb{R}$ such that*

$$\lim_{R\to+\infty}\sup_{t\geq R,|x|\geq R}\left(|A(t,x)-A^*|+|q(t,x)-q^*|+|f'_u(t,x,0)-c^*|\right)=0. \tag{31}$$

Then $\underline{H}(e,p)=\overline{H}(e,p)=pA^*p+q^*\cdot p+c^*$ for all $p\in\mathbb{R}^N$ and

$$\overline{w}(e)=\underline{w}(e)=2\sqrt{eA^*ec^*}+q^*\cdot e\qquad \textit{for all } e\in\mathbb{S}^{N-1}. \tag{32}$$

We now investigate classes of heterogeneities for which no spreading properties was know before.

3.2 Recurrent media

When the coefficients are recurrent, our definitions of the expansion sets simplify to Wulff-type constructions, as in periodic media. We will consider in the next section an important class of recurrent coefficients: almost periodic ones. However, even if the characterizations of the expansion sets simplify, these sets might not be equal in recurrent media, and we provide an example for which $\underline{\mathcal{S}}\neq\overline{\mathcal{S}}$.

Definition 3.2 *A uniformly continuous and bounded function $g:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is* recurrent *with respect to $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ if for any sequence $(t_n,x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\mathbb{R}^N$ such that $g^*(t,x)=\lim_{n\to+\infty}g(t_n+t,x_n+x)$ exists locally uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, there exists a sequence $(s_n,y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\mathbb{R}^N$ such that $\lim_{n\to+\infty}g^*(t-s_n,x-y_n)=g(t,x)$ locally uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}^N$.*

The heuristic meaning of this definition is that the patterns of the heterogeneities repeat at infinity. It is easy to check that homogeneous, periodic and almost periodic functions are recurrent. We thus expect similar phenomena as in periodic media to arise, even if the recurrence property is much milder than periodicity. Indeed, some functions might be recurrent without being almost periodic, such as the function (see [101])

$$g(x)=\frac{\sin x+\sin\sqrt{2}x}{|1+e^{ix}+e^{i\sqrt{2}x}|}.$$

Proposition 3.3 *Assume that A, q and $f'_u(\cdot,\cdot,0)$ are recurrent with respect to $(t,x)\in\mathbb{R}\times\mathbb{R}^N$. Then*

$$\underline{\mathcal{S}}=\{x,\ \forall p\in\mathbb{R}^N,\underline{\lambda_1}(L_{-p},\mathbb{R}\times\mathbb{R}^N)\geq p\cdot x\}\quad \textit{and}\quad \overline{\mathcal{S}}=\{x,\ \forall p\in\mathbb{R}^N,\overline{\lambda_1}(L_{-p},\mathbb{R}\times\mathbb{R}^N)\geq p\cdot x\}. \tag{33}$$

Note that such a Wulff-type characterization of the expansion sets immediately implies for all $e \in \mathbb{S}^{N-1}$:

$$\overline{w}(e) := \min_{p\cdot e>0} \frac{\overline{\lambda_1}(L_{-p}, \mathbb{R}\times\mathbb{R}^N)}{p\cdot e} \quad \text{and} \quad \underline{w}(e) := \min_{p\cdot e>0} \frac{\underline{\lambda_1}(L_{-p}, \mathbb{R}\times\mathbb{R}^N)}{p\cdot e}, \tag{34}$$

that is:

$$\forall w \in \big[0, \underline{w}(e)\big), \quad \lim_{t\to+\infty} u(t, x+wte) = 1 \quad \text{and} \quad \forall w > \overline{w}(e), \quad \lim_{t\to+\infty} u(t, x+wte) = 0,$$

locally uniformly with respect to $x \in \mathbb{R}^N$. Hence, this result exactly means that the transition between 0 and 1, that is, the level sets of $u(t,\cdot)$ are contained in $\big[\underline{w}(e)t, \overline{w}(e)t\big]$ along direction e at sufficiently large time t. Such a characterization of the spreading speeds is very close to the one holding in periodic media (see (6) below).

We have constructed the two expansion sets $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$ as precisely as possible. However, these two sets might be different, that is, there does not necessarily exist an exact spreading speed in recurrent media. For instance, in Example 2 of Section 13 we exhibit a situation where the advection term is recurrent with respect to time and for which there exists a range of speeds (w_*, w^*) such that for all $w \in (w_*, w^*)$, if u is the solution of the Cauchy problem (20) associated with a compactly supported initial datum, then for all $e \in \mathbb{S}^{N-1}$, the ω-limit set of the function $t \mapsto u(t, wte)$ is the full interval $[0,1]$. From this one sees that one cannot expect to describe the invasion by a single expansion set, hence the introduction here of two expansion sets $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$.

3.3 Almost periodic media

An important class of recurrent coefficients is that of almost periodic functions, for which we will show that $\underline{\mathcal{S}} = \overline{\mathcal{S}}$. We will use Bochner's definition of almost periodic functions:

Definition 3.4 [28] *A function $g : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is* almost periodic *with respect to $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ if from any sequence $(t_n, x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\mathbb{R}^N$ one can extract a subsequence $(t_{n_k}, x_{n_k})_{k\in\mathbb{N}}$ such that $g(t_{n_k}+t, x_{n_k}+x)$ converges uniformly in $(t,x) \in \mathbb{R}\times\mathbb{R}^N$.*

Theorem 3 *Assume that A, q and $f'_u(\cdot,\cdot,0)$ are almost periodic with respect to $(t,x) \in \mathbb{R}\times\mathbb{R}^N$. Then $\underline{\mathcal{S}} = \overline{\mathcal{S}}$ and*

$$\overline{w}(e) = \underline{w}(e) = \min_{p\cdot e>0} \frac{\overline{\lambda_1}(L_{-p}, \mathbb{R}\times\mathbb{R}^N)}{p\cdot e} = \min_{p\cdot e>0} \frac{\underline{\lambda_1}(L_{-p}, \mathbb{R}\times\mathbb{R}^N)}{p\cdot e}. \tag{35}$$

Let us also mention here the works of Shen, who proved these spreading properties in the particular case $q \equiv 0$, $A = A(x)$ is periodic in x and f is limit periodic in t and periodic in x (Theorem 4.1 in [94]). Limit periodic functions, that is, uniform limits over $\mathbb{R}$ of periodic functions, are a sub-class of almost periodic functions.

This Theorem is an immediate corollary of Proposition 3.3 and the following result, which is new and of independent interest. We will thus leave the proof of Theorem 3 to the reader.

Theorem 4 *Assume that A, q and c are almost periodic, where $c \in \mathcal{C}^{\delta/2,\delta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ is a given uniformly continuous function. Let $\mathcal{L} = -\partial_t + tr(A\nabla^2) + q\cdot\nabla + c$. Then one has $\overline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N) = \underline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N)$.*

As almost periodic functions are uniquely ergodic ones, these results could be derived from that of Section 8 below. However, we state these independently since we will indeed provide direct proofs in the almost periodic framework.

It is well-known that elliptic operators with almost periodic coefficients do not always admit almost periodic eigenfunctions. Indeed, consider the operator defined for all $\phi \in \mathcal{C}^2(\mathbb{R})$ by $\mathcal{L}\phi := \phi'' + c(x)\phi$. Bjerklov [27] showed that, if $c(x) = K\big(\cos(2\pi x) + \cos(2\pi\alpha x)\big)$ with $\alpha \notin \mathbb{Q}$ and K large enough, the Lyapounov exponent of $\mathcal{L}$ is strictly positive, which implies, through Ruelle-Oseledec's theorem, that any eigenfunction should either blow up or decay to zero exponentially, contradicting a possible almost periodicity (see also [98]). Hence, Theorem 4 is an example where classical eigenvalues do not exist while generalized principal eigenvalues are equal.

On the other hand, if K is small enough and α satisfies the diophantine condition

$$\forall (n,m)\in\mathbb{Z}^2,\quad |n+m\alpha| \geq k(|n|+|m|)^{-\sigma} \quad \text{for some } \ k,\sigma>0,$$

then Kozlov [61] proved the existence of an almost periodic eigenfunction.

In the almost periodic framework, in dimension 1, the existence of generalized transition waves has been proved by the second author and Rossi [77] under the assumption that the linearized operator near $u\equiv 0$ admits an almost periodic eigenfunction. The existence of generalized transition waves remains an open problem when there does not exist such an eigenfunction.

3.4 Asymptotically almost periodic media

An exact asymptotic spreading speed still exists when the coefficients converge to almost periodic functions at infinity thanks to Theorem 2.

Proposition 3.5 *Assume that there exist space-time almost periodic functions A^*, q^* and c^* such that*

$$\lim_{R\to+\infty} \sup_{t\geq R, |x|\geq R} \big(|A(t,x)-A^*(t,x)| + |q(t,x)-q^*(t,x)| + |f_u'(t,x,0)-c^*(t,x)|\big) = 0. \quad (36)$$

Then $\underline{H}(e,p) = \overline{H}(e,p) = \overline{\lambda_1}(L_p^*,\mathbb{R}\times\mathbb{R}^N)$ for all $p\in\mathbb{R}^N$ and

$$\overline{w}(e) = \underline{w}(e) = \min_{p\cdot e>0} \frac{\overline{\lambda_1}(L_{-p}^*,\mathbb{R}\times\mathbb{R}^N)}{p\cdot e} = \min_{p\cdot e>0} \frac{\underline{\lambda_1}(L_{-p}^*,\mathbb{R}\times\mathbb{R}^N)}{p\cdot e}. \quad (37)$$

where $\mathcal{L}^* = -\partial_t + tr(A^*(t,x)\nabla^2) + q^*(t,x)\cdot\nabla + c^*(t,x)$ and $L_p^*\phi = e^{-p\cdot x}\mathcal{L}^*(e^{p\cdot x}\phi)$.

The proof of this Proposition is similar to that of Proposition 2.6 of our previous work [19]. We will thus omit its proof.

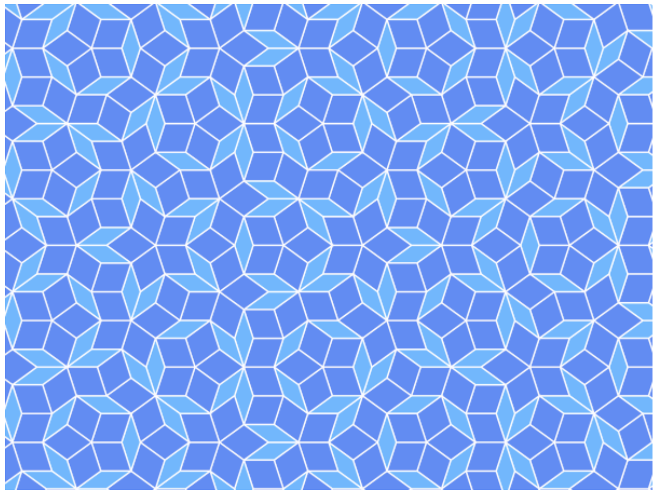

Figure 2: A representation of the Penrose tiling

3.5 Uniquely ergodic media

We now consider uniquely ergodic coefficients.

Definition 3.6 *A uniformly continuous and bounded function $f : \mathbb{R}^N \to \mathbb{R}^m$ is called* uniquely ergodic *if there exists a unique invariant probability measure $\mathbb{P}$ on its hull $\mathcal{H}_f := cl\{\tau_a f, a \in \mathbb{R}^N\}$, where the closure is understood with respect to the locally uniform convergence, and where the invariance is understood with respect to the translations $\tau_a f(x) := f(x+a)$ for all $x \in \mathbb{R}^N$.*

Periodic, almost periodic and compactly supported functions are particular sub classes of the uniquely ergodic one. A classic example of uniquely ergodic function is constructed from the Penrose tiling. We refer to [85] for a definition of it. If one defines on each tile a compactly supported function, the function thus obtained on $\mathbb{R}^N$ is uniquely ergodic [71, 85]. However, it is not almost periodic. The class of ergodic functions is therefore wider than that of almost periodic functions.

The notion of unique ergodicity is commonly used in dynamical system theory since it provides uniformity of the convergence in the Birkhoff ergodic theorem. This yields the following equivalent characterization (which is proved for example in Proposition 2.7 of [71]).

Proposition 3.7 *[71] Let $f : \mathbb{R}^N \to \mathbb{R}^m$ a uniformly continuous and bounded function. The following assertions are equivalent:*

- *f is uniquely ergodic*

- *for any continuous function $\Psi : \mathcal{H}_f \to \mathbb{R}$, the following limit exists uniformly with respect to $a \in \mathbb{R}^N$:*

$$\lim_{R\to+\infty} \frac{1}{|B_R(a)|}\int_{B_R(a)} \Psi(\tau_y f)dy.$$

Indeed, this limit is equal to $\mathbb{P}(\Psi)$.

The interest for reaction-diffusion equations with uniquely ergodic coefficients has raised since the 2000's, when the case of periodic ones was completely understood. Shen has investigated the existence of generalized transition wave solutions of Fisher-KPP equations with time uniquely ergodic coefficients [95] (see also [75]). Matano conjectured the existence of generalized transition waves (see Section 1.3 below and [10, 70]) and of spreading properties in Fisher-KPP equations with space uniquely ergodic coefficients in several conferences.

In the present manuscript, we show the existence of spreading properties for Fisher-KPP equations with space uniquely ergodic coefficients.

Theorem 5 *Assume that A, q and $f_u'(\cdot,0)$ only depend on x and are uniquely ergodic with respect to $x \in \mathbb{R}^N$. Then $\underline{\mathcal{S}} = \overline{\mathcal{S}}$ and*

$$\overline{w}(e) = \underline{w}(e) = \min_{p\cdot e>0} \frac{\overline{\lambda_1}(L_{-p}, \mathbb{R}\times\mathbb{R}^N)}{p\cdot e} = \min_{p\cdot e>0} \frac{\underline{\lambda_1}(L_{-p}, \mathbb{R}\times\mathbb{R}^N)}{p\cdot e}. \tag{38}$$

Theorem 5 is an immediate corollary of Theorem 2 and the next result on the equality generalized principal eigenvalues for elliptic operators with uniquely ergodic coefficients. We will thus omit its proof and only prove Theorem 6, which is of independent interest.

Theorem 6 *Assume that A, q and c only depend on x and are uniquely ergodic, where $c \in \mathcal{C}^\delta_{loc}(\mathbb{R}^N)$ is a given uniformly continuous and bounded function. Define the elliptic operator: $\mathcal{L} = tr(A\nabla^2) + q\cdot\nabla + c$. Then one has:*

$$\underline{\lambda_1}(\mathcal{L}, \mathbb{R}^N) = \overline{\lambda_1}(\mathcal{L}, \mathbb{R}^N).$$

Uniquely ergodic coefficients could be viewed as random stationary ergodic ones, for which the existence of spreading properties for almost every events is known. However, as far as we know, in multi-dimensional media, spreading properties have only been derived for random stationary ergodic advection terms (and homogeneous reaction terms) by Nolen and Xin in [81], and serious difficulties arise when the reaction term is heterogeneous. Moreover, it is not clear how to recover spreading properties for the given set of coefficients (A, q, f) through this observation, as already explained in [19]. For example, in the case of the Penrose tiling, the set of events is the closure under local convergence of the set of translations of the coefficients, and the probability measure is the mean value. For a given tiling, we could thus derive from the probabilistic approach that there exists an exact spreading speed for almost every translation (including translations at infinity) of the given original tiling. But it is almost impossible to determine the set of probability 0 for which we do not know whether a spreading property holds or not, and the original tiling might be in this set. We prove in the present manuscript that an exact spreading speed does exist not only for almost every but for that tiling. Lastly, the characterization in terms of generalized principal eigenvalues (38) we derive in the present manuscript is quite different from the characterizations of the spreading speeds in random stationary ergodic media, which involves Lyapounov exponents (see [81] for instance).

3.6 Radially periodic media

We now consider coefficients that are periodic with respect to the radial coordinate $r = |x|$. As far as we know, this class of heterogeneity has never been investigated before.

Proposition 3.8 *Assume that one can write*

$$A(t,x) = a_{per}(|x|)I_N, \quad q(t,x) = 0 \quad \textit{and} \quad f'_u(t,x,0) = c_{per}(|x|)$$

where a_{per} and c_{per} are periodic with respect to $r = |x|$: there exists $L > 0$ such that for all $r \in (0,\infty)$:

$$a_{per}(r+L) = a_{per}(r) \quad \textit{and} \quad c_{per}(r+L) = c_{per}(r).$$

For all $p \in \mathbb{R}$, let:

$$L_p^{per}\phi := a_{per}(r)\phi'' + 2pa_{per}(r)\phi' + \big(p^2 a_{per}(r) + c_{per}(r)\big)\phi$$

and $\lambda_1^{per}(L_p^{per})$ the periodic principal eigenvalue associated with this operator.

Then $\underline{w}(e)$ and $\overline{w}(e)$ do not depend on e and

$$\underline{w}(e) = \overline{w}(e) = \min_{p>0} \frac{\lambda_1^{per}(L_{-p}^{per})}{p}.$$

The proof of this result is non-trivial since classical eigenvalues do not exist in this framework. Hence, one more time the notions of generalized principal eigenvalues will be useful. Moreover, the fact that only the heterogeneity of the coefficients in the truncated cones $C_{R,\alpha}(e)$ matters in the computation of these eigenvalues will also be needed.

3.7 Spatially independent media

When the coefficients only depend on t, the formulas for $\overline{w}(e)$ and $\underline{w}(e)$ are simpler. For example, if the coefficients are periodic in t, then the spreading speed is that associated with the average coefficients over the period. Our aim is to extend this property to general time-heterogeneous coefficients.

Proposition 3.9 *Assume that $A = I_N$, $q \equiv 0$ and $f'_u(\cdot,0)$ do not depend on x. Then for all $e \in \mathbb{S}^{N-1}$,*

$$\underline{w}(e) = \liminf_{t\to+\infty} \inf_{s>0} 2\sqrt{\frac{1}{t}\int_s^{s+t} f'_u(s',0)ds'} \tag{39}$$

$$\overline{w}(e) = \limsup_{t\to+\infty} \sup_{s>0} 2\sqrt{\frac{1}{t}\int_s^{s+t} f'_u(s',0)ds'}. \tag{40}$$

The reader might easily check that the proof is also available when only q or A depends on t.

The existence of generalized transition waves in such media has been proved, under similar hypotheses as in the present manuscript, by the second author and Rossi [75]. The speed of these fronts are determined through some upper and lower means of the coefficients that are very similar to the average involved in the definitions of $\underline{w}(e)$ and $\overline{w}(e)$.

When the coefficients are periodic in T, we recover that $\underline{w}(e) = \overline{w}(e)$ is the spreading speed associated with the average reaction term. For general time-heterogeneous coefficients, it is not always true that $\underline{w}(e) = \overline{w}(e)$. This is because one can consider several ways of averaging. Indeed, our result is not optimal and it might be due to our choice of averaging (see Section 13 below).

However, when the coefficients admits a uniform mean value over $\mathbb{R}$, then a variant of our result gives $\underline{w}(e) = \overline{w}(e)$ for all e. We can thus handle uniquely ergodic coefficients for example. No such result exists in the literature as far as we know.

Proposition 3.10 *Assume that A, q and f do not depend on x and that there exists $\langle A\rangle \in \mathcal{S}_N(\mathbb{R})$, $\langle q\rangle \in \mathbb{R}^N$ and $\langle c\rangle \in \mathbb{R}$ such that*

$$\lim_{t\to+\infty} \frac{1}{t}\int_a^{a+t} A(s)ds = \langle A\rangle, \ \lim_{t\to+\infty} \frac{1}{t}\int_a^{a+t} q(s)ds = \langle q\rangle \ \text{and} \ \lim_{t\to+\infty} \frac{1}{t}\int_a^{a+t} f'_u(s,0)ds = \langle c\rangle \tag{41}$$

uniformly with respect to $a > 0$. Then for all $e \in \mathbb{S}^{N-1}$,

$$w^*(e) = w_*(e) = \underline{w}(e) = \overline{w}(e) = 2\sqrt{e\langle A\rangle e\langle c\rangle} - \langle q\rangle.$$

3.8 Space periodic and time-heterogeneous media

We assume in this section that $A(t,\cdot)$, $q(t,\cdot)$ and $f'_u(t,\cdot,0)$ are periodic in x for all $t \in \mathbb{R}$, with a general dependence with respect to t. In this case, one could derive a more explicit characterization of the generalized principal eigenvalues.

Lemma 3.11 (Lemma 3.1 in [76]) *For all $p \in \mathbb{R}^N$, the equation $L_p\eta = 0$ admits a space-periodic, time-global solution η_p. Moreover, η_p is unique up to a multiplicative constant and there exists a constant $C > 0$ such that for all $T > 0$, $(x,t) \in \mathbb{R}^{N+1}$, one has:*

$$\frac{1}{C}\|\eta_p(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}e^{-CT} \le \eta_p(x,t+T) \le C\|\eta_p(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}e^{CT}. \tag{42}$$

The existence and uniqueness of a positive time-global solution of the linear parabolic equation has also been proved in [48] for Dirichlet and Neumann boundary conditions and in [49] for coefficients having limits when $|x| \to +\infty$. If the coefficients are also assumed to be periodic in time, then $\eta_p(t,x)e^{-\lambda_{per}(L_p)t}$ is the space-time periodic principal eigenvalue. If the coefficients do not depend on x, then $\eta_p = e^{\int_0^t f'_u(s,0)ds + p^2 t}$.

For Dirichlet boundary conditions, we further this study in joint work with Rossi [20]. Following the ideas of [20] and using the time-global solution given by Proposition 3.10, one could prove the following characterization of the generalized principal eigenvalues.

Proposition 3.12 *[20] One has*

$$\underline{\lambda_1}(L_p, Q) = \lim_{t\to+\infty}\left(\inf_{s>0}\frac{\ln\|\eta_p(s+t,\cdot)\|_{L^\infty(\mathbb{R}^N)} - \ln\|\eta_p(s,\cdot)\|_{L^\infty(\mathbb{R}^N)}}{t}\right)$$

$$\overline{\lambda_1}(L_p, Q) = \lim_{t\to+\infty}\left(\sup_{s>0}\frac{\ln\|\eta_p(s+t,\cdot)\|_{L^\infty(\mathbb{R}^N)} - \ln\|\eta_p(s,\cdot)\|_{L^\infty(\mathbb{R}^N)}}{t}\right)$$

We will not provide a proof of this result in the present paper, since it is very similar to [20]. This paper further investigates various notions of generalized principal eigenvalues for parabolic operators.

If the coefficients are periodic in t, then we recover the classical Floquet-type characterization of the periodic principal eigenvalue.

Furthermore, if the coefficients do not depend on x but are general in t, then we recover Proposition 10.1 below. Indeed, the proof of Proposition 3.12 relies on the same idea as that of Proposition 10.1, except that the additional dependence in x makes it more technical.

We do not know whether it is possible to derive such a characterization for more general classes of dependence, such as for almost periodic or uniquely ergodic coefficients in x. We leave this as an open problem.

3.9 Directionally homogeneous media

We investigate in this Section the case where the coefficients converge in radial segments of $\mathbb{R}^2$. These types of heterogeneities give rise to very rich phenomena, such as non-convex expansion sets.

We start with the case where the diffusion term converges in the half-spaces $\{x_1 < 0\}$ and $\{x_1 > 0\}$

Proposition 3.13 *Assume that $N = 2$, $q \equiv 0$, f does not depend on (t,x) and $A(x_1,x_2) = a(x_1)I_2$ is a smooth function such that $\lim_{x_1\to\pm\infty} a(x_1) = a_\pm$, with $a_+ > a_- > 0$. Then $\overline{\mathcal{S}} = \underline{\mathcal{S}}$ and this set is the convex envelope of*

$$\{x\in\mathbb{R}^2, |x| \le 2\sqrt{f'(0)a_+}, x_1 \ge 0\} \cup \{x\in\mathbb{R}^2, |x| \le 2\sqrt{f'(0)a_-}, x_1 \le 0\}.$$

It is easy to compute that

$$\overline{H}(e,p) = \underline{H}(e,p) = \begin{cases} a_+p^2 + f'(0) & \text{if} \quad e_1 > 0, \\ a_-p^2 + f'(0) & \text{if} \quad e_1 < 0. \end{cases}$$

Thus, when $e_1 < 0$ and $e_1 \neq -1$, the spreading speed $w^*(e) = w_*(e)$ is not equal to

$$v(e) = \min_{p\cdot e>0}\frac{\overline{H}(e,-p)}{p\cdot e} = 2\sqrt{f'(0)a_-}$$

and the expansion set is not obtained through a Wulff-type construction like (33). In other words, the spreading speed in direction e does not only depend on what happens in direction

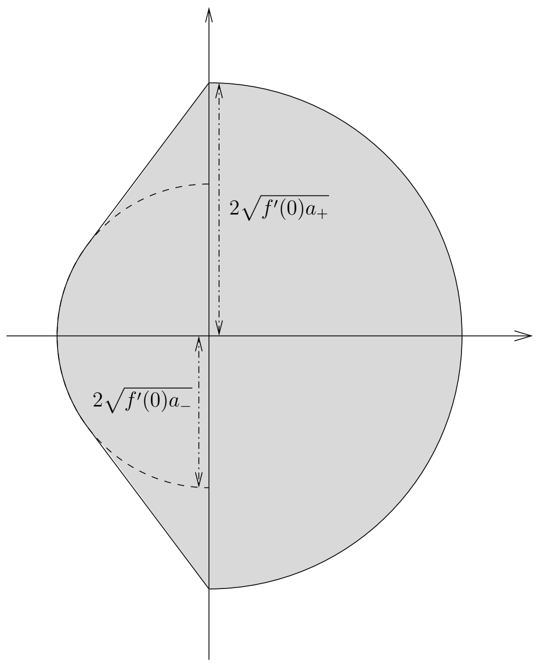

Figure 3: The expansion set $\overline{\mathcal{S}} = \underline{\mathcal{S}}$ given by Proposition 3.13 for $N = 2$.

e. Heuristically, in the present example, in order to go as far as possible during a given time t, an individual has to first go in direction e_2 at speed $2\sqrt{f'(0)a_+}$ and then to get into the left medium at speed $2\sqrt{f'(0)a_-}$. The notion hidden beyond this heuristic remark is that of geodesics with respect to the riemannian metric associated with the speeds $2\sqrt{f'(0)a_+}$ and $2\sqrt{f'(0)a_-}$.

This shows that there is a strong link between geometric optics and reaction-diffusion equations, as already noticed by Freidlin [40, 41] and Evans and Souganidis [38]. Indeed, Freidlin investigated in [40] the asymptotic behavior as $\varepsilon \to 0$ of the equation

$$\begin{cases} \partial_t v_\varepsilon = \varepsilon a(x)\Delta v_\varepsilon + \frac{1}{\varepsilon} f(v_\varepsilon) \text{ in } (0,\infty)\times\mathbb{R}^N, \\ v_\varepsilon(0,x) = v_0(x) \text{ for all } x \in \mathbb{R}^N, \end{cases} \tag{43}$$

where $(a_{ij})_{i,j}$ and f are smooth and v_0 is a compactly supported function which does not depend on ε. He proved that

$$\lim_{\varepsilon\to 0} v_\varepsilon(t,x) = \begin{cases} 1 & \text{if} \quad V(t,x) > 0, \\ 0 & \text{if} \quad V(t,x) < 0, \end{cases} \quad \text{locally in } (t,x) \in (0,\infty)\times\mathbb{R}^N, \tag{44}$$

where $V(t,x) = 4f'(0)t - d^2(x,G_0)/t$, G_0 is the support of v_0 and d is the riemannian metric associated with $dx_i dx_j/a(x)$. As we will see later along the proof of our main result, our problem is almost equivalent to (43), but with coefficients depending on ε: $a(x/\varepsilon)$ and $v_0(x/\varepsilon)$ instead of $a(x)$ and $v_0(x)$. Indeed, the particular dependence of the diffusion term in Proposition 3.13 yields that $a(x/\varepsilon)$ is close to a_+ if $x_1 > 0$ and to a_- if $x_1 < 0$ when ε is small. This shrinked diffusion term is discontinuous and, more important, the rescaled initial datum $v_0(x/\varepsilon)$ becomes very singular when $\varepsilon \to 0$, unlike the smooth one in Freidlin's problem (43). Thus we could not directly apply Freidlin's result. However, we will find at an intermediate step a characterization of the expansion set which is close to Freidlin's (44), which is not surprising. We will then explicitly compute the geodesics, which makes another difference with earlier papers on the link between geometric optics and Hamilton-Jacobi equations. Computing these geodesics, we will recover some Snell-Descartes law (see the Remark below the proof of Proposition 3.13).

Next, let consider the same framework but with f depending on x_1 instead of a.

Proposition 3.14 *Assume that $N = 2$, $q \equiv 0$, $A = I_2$ and $f(t,x,s) = c(x_1)s(1-s)$, where c is a smooth function such that $\lim_{x_1 \to \pm\infty} c(x_1) = \mu_\pm$, with $\mu_+ > \mu_- > 0$.*
Then $\overline{\mathcal{S}} = \underline{\mathcal{S}}$ and this set is the convex envelope of

$$\{x \in \mathbb{R}^2, |x| \leq 2\sqrt{\mu_+}, x_1 \geq 0\} \cup \{x \in \mathbb{R}^2, |x| \leq 2\sqrt{\mu_-}, x_1 \leq 0\}.$$

Surprisingly, the functions $\overline{U}$ and $\underline{U}$ are quite different from the ones arising along the proof of Proposition 3.13. However, their level-sets $\overline{\mathcal{S}} = \{\overline{U} = 0\}$ and $\underline{\mathcal{S}} = \mathrm{cl}\{\underline{U} = 0\}$ are very similar to that of Proposition 3.13 and we find the same type of picture as Figure 3.9.

If $A(t,x) = a(x_1)I_N$ and if there exist two periodic functions $x_1 \mapsto a_+(x_1)$ and $x_1 \mapsto a_-(x_1)$ such that $a(x_1) - a_\pm(x_1) \to 0$ as $x_1 \to \pm\infty$, then it does not seem possible to write the expansion set as the convex hull of two half-circles as in Proposition 3.13 holds in general. Indeed, the proof of Proposition 3.13 relies on the particular structure of the Hamiltons $\overline{H}(e,p)$ and $\underline{H}(e,p)$, which are quadratic polynomials with respect to p for all e.

We also mention here the recent work of Roquejoffre, Rossi and the first author [22] on a coupled reaction-diffusion equation modeling the diffusion of a species along a line. Computing their expansion set, the authors faced similar problems but found a picture quite different from Figure 3.9.

3.10 A non-convex expansion set

If a converges to a_- in a smaller part of $\mathbb{R}^2$ than a half-space, then the expansion set is not as in Proposition 3.13.

Proposition 3.15 *Assume that $N = 2$, $q \equiv 0$, f does not depend on (t,x) and $A(x) = a(x)I_2$ is a smooth function such that*

$$\lim_{x_1 \to +\infty} a(x_1, \alpha x_1) = \begin{cases} a_+ & \text{if} \quad |\alpha| < r_0 \\ a_- & \text{if} \quad |\alpha| > r_0 \end{cases}$$

where $a_+ > a_- > 0$ and $0 < r_0 < r := \sqrt{\frac{a_-}{a_+ - a_-}}$ Then $\overline{\mathcal{S}} = \underline{\mathcal{S}}$ and this set is:

$$\Big\{|x| < 2\sqrt{f'(0)a_+}, |x_2| \geq r_0 x_1\Big\} \cup \Big\{x_1 < \frac{1 - r_0 r}{r_0 + r}|x_2| + \frac{2\sqrt{f'(0)a_+(1+r_0^2)}}{1 + r_0/r}, |x_2| \leq r_0 x_1\Big\}.$$

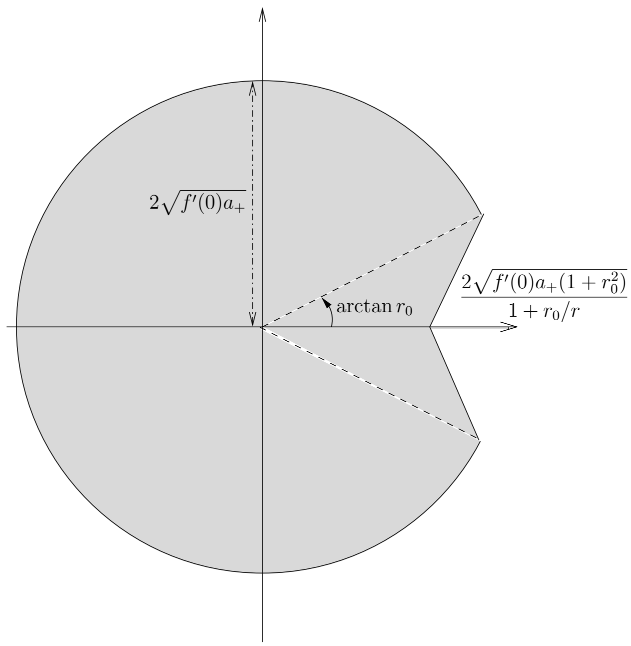

Figure 4: The non-convex expansion set $\overline{\mathcal{S}} = \underline{\mathcal{S}}$ given by Proposition 3.15.

This expansion set is non-convex if $r_0 r < 1$, as displayed in the Figure illustrating Proposition 3.15.

This is the first time, as far as we know, that a reaction-diffusion giving rise to a non-convex expansion set is exhibited. Indeed, for all the classes of heterogeneities previously investigated in the literature, the expansion sets were characterized through a Wulff-type construction (8), which is clearly convex. Thus the investigation of more general types of heterogeneities was needed in order to find non-convex expansion sets.

As a conclusion, if $N = 2$, $q \equiv 0$, f does not depend on (t, x) and $A(x) = a(x)I_N$, where a converges to some limit function $a_\infty(x)$ in a finite number of radial segments, then Proposition 11.1 below yields that $\overline{\mathcal{S}} = \underline{\mathcal{S}}$. Hence, if in addition a_∞ is assumed to be quasiconcave, then the reader can check that Proposition 2.3 yields that $\overline{\mathcal{S}}$ is convex. However, this result is not optimal since, for example, under the assumptions of Proposition 3.15, one would obtain the function $a_\infty(x) = a_+$ if $|x_2| > r_0 x_1$, $a_\infty(x) = a_+$ if $|x_2| < r_0 x_1$, which is not quasiconcave since $r_0 > 0$, however the expansion set is convex if $r_0 r \geq 1$.

We mention here, in the continuity of [22], R. Ducasse's work on a so-called fast-line model with a conical field, exhibiting similar non-convex level-sets [36].

3.11 An alternative definition of the expansion set and applications to random and slowly varying media

We conclude this section with an alternative definition of the expansion set, involving another notion of generalized principal eigenvalues, which allows us to prove the existence of an exact asymptotic spreading speed in random stationary ergodic and slowly varying media.

We need in this Section the following additional assumption:

$$A, q \text{ and } f'_s(\cdot,0) \text{ do not depend on } t. \tag{45}$$

Our alternative definition involves another set of test-functions:

$$\mathcal{B} := \Big\{\phi \in \mathcal{C}^2(\mathbb{R}^N),\ \phi > 0,\ \nabla\phi/\phi \in L^\infty(\mathbb{R}^N),\ \lim_{|x|\to+\infty} \frac{\ln\phi(x)}{|x|} = 0\Big\}$$

For any open set $\mathcal{O} \subset \mathbb{R}^N$, we define two generalized principal eigenvalues associated with such test-functions:

$$\begin{array}{rcll} \underline{\eta_1}(\mathcal{L},\mathcal{O}) & := & \sup\{\ \eta \mid \exists\phi \in \mathcal{B}, & \mathcal{L}\phi \geq \eta\phi \quad \text{in } \mathcal{O}\}, \\ \overline{\eta_1}(\mathcal{L},\mathcal{O}) & := & \inf\{\ \eta \mid \exists\phi \in \mathcal{B}, & \mathcal{L}\phi \leq \eta\phi \quad \text{in } \mathcal{O}\}. \end{array} \tag{46}$$

It is immediate that $\underline{\eta_1} \geq \underline{\lambda_1}$ and $\overline{\eta_1} \leq \overline{\lambda_1}$ since bounded functions with a positive infimum belong to $\mathcal{B}$.

When $\mathcal{O} = C_{R,\alpha}(e)$, one has $\underline{\eta_1} \leq \overline{\eta_1}$. But we do not know if such a comparison holds in sets containing balls of arbitrary radii (see Proposition 4.2 below).

Lemma 3.16 *One has $\overline{\eta_1}(C_{R,\alpha}(e)) \geq \underline{\eta_1}(C_{R,\alpha}(e))$ for all $R > 0$, $\alpha > 0$ and $e \in \mathbb{S}^{N-1}$.*

Of course, if $\mathcal{O}$ contains a truncated cone $C_{R,\alpha}(e)$ for some $R > 0, \alpha > 0$ and $e \in \mathbb{S}^{N-1}$, then as $\overline{\eta_1}(\mathcal{O}) \geq \overline{\eta_1}(C_{R,\alpha}(e))$ and $\underline{\eta_1}(C_{R,\alpha}(e)) \leq \underline{\eta_1}(\mathcal{O})$, one gets $\overline{\eta_1}(\mathcal{O}) \geq \underline{\eta_1}(\mathcal{O})$ as well.

We are now in position to define similar quantities as in Section 2.4 with these new notions of generalized principal eigenvalues. Let:

$$\overline{J}(e,p) := \inf_{R>0,\alpha\in(0,1)} \overline{\eta_1}(L_p, C_{R,\alpha}(e)) \quad \text{and} \quad \underline{J}(e,p) := \sup_{R>0,\alpha\in(0,1)} \underline{\eta_1}(L_p, C_{R,\alpha}(e)),$$

$$\underline{J}^\star(e,q) := \sup_{p\in\mathbb{R}^N} \big(p\cdot q - \underline{J}(e,p)\big) \quad \text{and} \quad \overline{J}^\star(e,q) := \sup_{p\in\mathbb{R}^N} \big(p\cdot q - \overline{J}(e,p)\big),$$

$$\begin{array}{ll} \underline{V}(x) := \inf \max_{t\in[0,1]} & \Big\{ \int_t^1 \underline{J}^\star\big(\frac{\gamma(s)}{|\gamma(s)|}, -\gamma'(s)\big)ds, \quad \gamma \in H^1([0,1]),\ \gamma(0) = 0,\ \gamma(1) = x, \\ & \hfill \forall s \in (0,1),\ \gamma(s) \neq 0\Big\}, \\ \overline{V}(x) := \inf \max_{t\in[0,1]} & \Big\{ \int_t^1 \overline{J}^\star\big(\frac{\gamma(s)}{|\gamma(s)|}), -\gamma'(s)\big)ds, \quad \gamma \in H^1([0,1]),\ \gamma(0) = 0,\ \gamma(1) = x, \\ & \hfill \forall s \in (0,1),\ \gamma(s) \neq 0\Big\}. \end{array}$$

$$\underline{\mathcal{T}} := \mathrm{cl}\{\underline{V} = 0\} \quad \text{and} \quad \overline{\mathcal{T}} := \{\overline{V} = 0\}.$$

One could easily check that the Hamiltonians $\overline{J}$ and $\underline{J}$ satisfy similar properties as that of $\overline{H}$ and $\underline{H}$ stated in Proposition 2.2.

One can show that a spreading property also holds with this alternative definition of the expansion sets.

Theorem 7 *Under the hypotheses of Section 2.1 and (45), if $u_0 \not\equiv 0$ is a measurable and compactly supported function such that $0 \leq u_0 \leq 1$ and u is the associated solution of the Cauchy problem* (1)*, one has*

$$\left\{ \begin{array}{lll} \textit{for all compact set } K \subset \mathrm{int}\underline{\mathcal{T}}, & \lim_{t\to+\infty} \big\{ \sup_{x\in tK} |u(t,x) - 1| \big\} & = 0, \\ \textit{for all closed set } F \subset \mathbb{R}^N\backslash\overline{\mathcal{T}}, & \lim_{t\to+\infty} \big\{ \sup_{x\in tF} |u(t,x)| \big\} & = 0. \end{array} \right. \tag{47}$$

Application: Random stationary ergodic coefficients

Consider a probability space $(\Omega, \mathbb{P}, \mathcal{F})$ and assume that the reaction rate $f : (x, \omega, s) \in \mathbb{R}^N \times \Omega \times [0, 1] \to \mathbb{R}$, the advection term $q : (x, \omega) \in \mathbb{R}^N \times \Omega \to \mathbb{R}^N$ and the diffusion term $A : (x, \omega) \in \mathbb{R}^N \times \Omega \to \mathcal{M}_N(\mathbb{R})$ are random variables. We suppose that the hypotheses stated in Section 2.1 are satisfied for almost every $\omega \in \Omega$.

The functions $f'_s(\cdot, \cdot, 0)$, q and A are assumed to be random stationary ergodic. The stationarity hypothesis means that there exists a group $(\pi_x)_{x \in \mathbb{R}^N}$ of measure-preserving transformations such that $A(x + y, \omega) = A(x, \pi_y \omega)$, $q(x + y, \omega) = q(x, \pi_y \omega)$ and $f'_u(x + y, \omega, 0) = f'_u(x, \pi_y \omega, 0)$ for all $(x, y, \omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega$. This hypothesis heuristically means that the statistical properties of the medium does not depend on the place where one observes it. The ergodicity hypothesis means that if $\pi_x A = A$ for all $x \in \mathbb{R}^N$ and for a given $A \in \mathcal{F}$, then $\mathbb{P}(A) = 0$ or 1.

We expect to compute the speeds $\underline{w}$ and $\overline{w}$ for almost every $\omega \in \Omega$. Such a result is already known in dimension $N = 1$ when the full nonlinearity f (and not only its derivative near $u = 0$) is a random stationary ergodic function since the pioneering work of Freidlin and Gartner [42]. They proved that for almost every $\omega \in \Omega$, one has $w^* = w_*$ and that this exact spreading speed can be computed using a family of Lyapounov exponents associated with the linearization of the equation near $u = 0$. This result has been generalized by Nolen and Xin for various types of space-time random stationary ergodic advection terms [81, 82, 83] in dimension N.

Our aim is to check that it is possible to derive $\underline{w} = \overline{w}$ almost surely from Theorem 7 and to find a characterization of the exact spreading speed that involves the generalized principal eigenvalues. The linearized operator now depends on the event ω and we write for all $\omega \in \Omega$, $p \in \mathbb{R}$ and $\phi \in \mathcal{C}^2(\mathbb{R})$:

$$L_p^\omega \phi := tr(A(x, \omega)\nabla^2 \phi) + (q(x, \omega) + 2A(x, \omega)p) \cdot \nabla \phi + (f'_u(x, \omega, 0) + p \cdot q(x, \omega) + pA(x, \omega)p)\phi. \tag{48}$$

The following Proposition is an immediate corollary of [31].

Proposition 3.17 *Assume that Ω is a Polish space, $\mathcal{F}$ is the Borel $\sigma-$field on Ω and $\mathbb{P}$ is a Borel probability measure. Then, if A, q and f do not depend on t, one has*

$$\underline{\eta_1}(L_p^\omega) = \overline{\eta_1}(L_p^\omega)$$

for all $p \in \mathbb{R}^N$ for almost every $\omega \in \Omega$.

Hence, for all $\omega \in \Omega_0$ and $e \in \mathbb{S}^{N-1}$:

$$\overline{w}^\omega(e) = \min_{p \cdot e > 0} \frac{\overline{\eta_1}(L_{-p}^\omega, \mathbb{R})}{p \cdot e} = \underline{w}^\omega(e) = \min_{p \cdot e > 0} \frac{\underline{\eta_1}(L_{-p}^\omega, \mathbb{R})}{p \cdot e} \tag{49}$$

and this quantity does not depend on $\omega \in \Omega_0$.

We have proved this result in dimension 1 without assuming Ω to be a Polish set [19]. We thus naturally conjecture that this assumption could be dropped.

Proposition 3.17 shows that the identity $\underline{w}^\omega = \overline{w}^\omega$, which was already known in particular frameworks [41, 42], can be derived from Theorem 7. Moreover, we obtain a new characterization of this exact spreading speed involving generalized principal eigenvalues instead the Lyapounov exponents used in [41, 42].

The definition of the set of admissible test-functions $\mathcal{B}$ is important here. If one considers another set of admissible test-functions, such as bounded test-functions with a positive infimum as in our earlier definitions of generalized principal eigenvalues (18) and (19), then the associated generalized principal eigenvalues are not equal in general. Hence, the class of random stationary ergodic coefficients emphasizes that it might be relevant to use the milder assumption $\lim_{|x|\to+\infty} \frac{1}{x} \ln \phi(x) = 0$ in the definition of the set of admissible test-functions.

Application: Slowly varying media

Consider now $A = I_N$, $q \equiv 0$ and a reaction term f such that there exist $c_0 \in \mathcal{C}^0(\mathbb{R})$ and a length function $L \in \mathcal{C}^2(\mathbb{R})$ satisfying:

$$\begin{cases} f_s'(x,0) = c_0\big(x/L(|x|)\big) \text{ for all } x \in \mathbb{R}^N, \\ 0 < \min_{[0,1]} c_0 < \max_{[0,1]} c_0 \quad \text{and} \quad c_0 \text{ is 1-periodic}, \\ \lim_{z\to+\infty} \dfrac{L(z)}{z} = 0, \quad \lim\limits_{z\to+\infty} \dfrac{L'(z)z}{L(z)} = 0 \;\text{ and }\; \lim\limits_{z\to+\infty} \dfrac{L''(z)z}{L(z)} = 0. \end{cases} \tag{50}$$

Typical length functions L satisfying these hypotheses are

- $L(z) = z/(\ln z)^\alpha$, with $\alpha > 1$,
- $L(z) = z^\alpha, \alpha \in (0,1)$,
- $L(z) = (\ln z)^\alpha, \alpha > 0$.

Such a reaction term is said to be *slowly varying* and has been considered by the second author, together with Garnier and Giletti, in dimension $N = 1$ [44]. Applying the results of our earlier one-dimensional paper [19], it was proved by these authors that there exists an exact asymptotic spreading speed, which could be characterized.

We generalize here this result to dimension N.

Proposition 3.18 *Under hypotheses (50), one has for all $p \in \mathbb{R}^N$:*

$$\lim_{R\to+\infty} \underline{\eta_1}(L_p, \mathbb{R}^N \backslash B_R) = \lim_{R\to+\infty} \overline{\eta_1}(L_p, \mathbb{R}^N \backslash B_R) = H(p),$$

where $H(p)$ is defined in Proposition 3.19 below.

Hence, for all $e \in \mathbb{S}^{N-1}$:

$$\overline{w}(e) = \underline{w}(e) = \min_{p\cdot e>0} \frac{H(-p)}{p \cdot e}. \tag{51}$$

The Hamiltonians $H(p)$ is defined in the next Proposition. The quantities $H(p)$ could be viewed as the limits of periodic principal eigenvalues when the given period of the coefficients tends to $+\infty$.

Proposition 3.19 *[64] For all $p \in \mathbb{R}$, there exists a unique real number $H(p)$ such that there exists a continuous periodic viscosity solution v_p of*

$$|\nabla v_p(y) + p|^2 + c_0(y) = H(p) \quad \text{over } \mathbb{R}. \tag{52}$$

Note that if the length function increases two slowly, for example if $L(z) = z/(\ln z)^\alpha$ with $\alpha < 1$, then there might not exist an exact asymptotic spreading speed and one might get $w_* = 2\sqrt{\min_{[0,1]} c_0}$ and $w^* = 2\sqrt{\max_{[0,1]} c_0}$ [44]. This is why we need hypotheses on the length function such as (50).

4 Properties of the generalized principal eigenvalues

The aim of this Section is to state some basic properties of the generalized principal eigenvalues and to prove Proposition 2.2. In all the Section, we consider an operator $\mathcal{L}$ defined for all $\phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ by

$$\mathcal{L}\phi = -\partial_t \phi + a_{i,j}(t,x)\partial_{ij}\phi + q_i(t,x)\partial_i \phi + c(t,x)\phi,$$

where A and q satisfy the hypotheses of Section 2.1 and $c \in \mathcal{C}^{\delta/2,\delta}_{loc}(\mathbb{R} \times \mathbb{R}^N) \cap L^\infty(\mathbb{R} \times \mathbb{R}^N)$ is a given uniformly continuous function. Recall that, for all $p \in \mathbb{R}^N$,

$$\begin{array}{rcl} L_p\phi = e^{-p\cdot x}\mathcal{L}(e^{p\cdot x}\phi) & = & -\partial_t\phi + tr(A(t,x)\nabla^2\phi) + 2pA(t,x)\nabla\phi + q(t,x)\cdot\nabla\phi \\ & & +(pA(t,x)p + q(t,x)\cdot p + c(t,x))\phi. \end{array} \tag{53}$$

Therefore, by proving some properties for $\overline{\lambda_1}(\mathcal{L}, Q)$ and $\underline{\lambda_1}(\mathcal{L}, Q)$ with general A, q and c, we immediately derive properties regarding $\overline{\lambda_1}(L_p, Q)$ and $\underline{\lambda_1}(L_p, Q)$.

4.1 Earlier notions of generalized principal eigenvalues

Generalized eigenvalues for elliptic operators

Consider first an elliptic operator $\mathcal{L}$ defined for all $\phi \in \mathcal{C}^2(\mathbb{R}^N)$ by

$$\mathcal{L}\phi = a_{i,j}(x)\partial_{ij}\phi + q_i(x)\partial_i\phi + c(x)\phi,$$

where $c \in \mathcal{C}^\delta_{loc}(\mathbb{R}^N)$ is a uniformly continuous and bounded function.

For such operators, a first notion of generalized principal eigenvalues was introduced by the first author, together with Hamel and Rossi [3] [18]:

$$\begin{array}{l} \underline{\mu_1}(\mathcal{L}, \mathbb{R}^N) := \sup\{\lambda \mid \exists \phi \in \mathcal{C}^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ s.t. } \mathcal{L}\phi \geq \lambda\phi \text{ in } \mathbb{R}^N\}, \\ \overline{\mu_1}(\mathcal{L}, \mathbb{R}^N) := \inf\{\lambda \mid \exists \phi \in \mathcal{C}^2(\mathbb{R}^N),\ \inf_{\mathbb{R}\times\mathbb{R}^N}\phi > 0 \text{ and } \mathcal{L}\phi \leq \lambda\phi \text{ in } \mathbb{R}^N\}, \\ \mu_1(\mathcal{L}, \mathbb{R}^N) := \inf\{\lambda \mid \exists \phi \in \mathcal{C}^2(\mathbb{R}^N),\ \text{ s.t. } \mathcal{L}\phi \leq \lambda\phi \text{ in } \mathbb{R}^N\}. \end{array} \tag{54}$$

[3] Indeed, with the notations of [18], $\underline{\mu_1}(\mathcal{L}, \mathbb{R}^N) := -\lambda_1'(-\mathcal{L}, \mathbb{R}^N)$, $\overline{\mu_1}(\mathcal{L}, \mathbb{R}^N) := -\lambda_1''(-\mathcal{L}, \mathbb{R}^N)$ and $\mu_1(\mathcal{L}, \mathbb{R}^N) := -\lambda_1(-\mathcal{L}, \mathbb{R}^N)$

These quantities are defined in [18, 24] for more general unbounded domains than $\mathbb{R}^N$, under additional assumptions on the behavior of the test-functions on their boundaries, and under more general hypotheses on the coefficients of the operator.

The reader should notice that the main difference with the definitions (18) and (19) of generalized principal eigenvalues $\underline{\lambda_1}$ and $\overline{\lambda_1}$ we use in the present paper lays in the class of test-functions. In order to define $\underline{\mu_1}$, one only requires the test-functions ψ to be bounded, while we require it to be bounded and have a positive infimum in the definition of $\underline{\lambda_1}$. Similarly, $\overline{\mu_1}$ is define through test-functions ψ with a positive infimum, while $\underline{\lambda_1}$ involve test-functions which are both bounded with a positive infimum. This slight difference gives rise to different quantities, as we will make it clearer later.

The main properties of these eigenvalues were derived in [24]:

- $\mu_1(\mathcal{L}, \mathbb{R}^N) \leq \underline{\mu_1}(\mathcal{L}, \mathbb{R}^N) \leq \overline{\mu_1}(\mathcal{L}, \mathbb{R}^N)$,
- $\mu_1(\mathcal{L}, \mathbb{R}^N)$ is the limit of the Dirichlet principal eigenvalues associated with $\mathcal{L}$ on any increasing sequence of bounded smooth domains Ω_n such that $\cup_{n\in\mathbb{N}}\Omega_n = \mathbb{R}^N$,
- if $\mathcal{L}$ is self-adjoint (that is, $q_i(x) = \sum_{j=1}^N \partial_i a_{i,j}(x)$), then $\mu_1(\mathcal{L}, \mathbb{R}^N) = \underline{\mu_1}(\mathcal{L}, \mathbb{R}^N) = \overline{\mu_1}(\mathcal{L}, \mathbb{R}^N)$,
- if $\overline{\mu_1}(\mathcal{L}, \mathbb{R}^N) < 0$, then the operator $-\mathcal{L}$ satisfies a maximum principle, while it does not if $\underline{\mu_1}(\mathcal{L}, \mathbb{R}^N) \geq 0$ (see [24] for a precise definition of this property).

It was also conjectured in these earlier papers that $\overline{\mu_1}(\mathcal{L}, \mathbb{R}^N) = \underline{\mu_1}(\mathcal{L}, \mathbb{R}^N)$ even when the operator $\mathcal{L}$ is not self-adjoint.

Generalized eigenvalues for parabolic operators

We now come back to a parabolic operator $\mathcal{L}$, as introduced in (17). If this operator is defined over $\mathbb{R} \times \Omega$, where Ω is a bounded and smooth domain, with Dirichlet boundary conditions on $\partial\Omega$, then Huska, Polacik and Safonov [48] introduced a notion of principal Floquet bundle. Roughly speaking, there exists a unique (up to multiplication) time-global positive solution ϕ of $\mathcal{L}\phi = 0$ in $\mathbb{R} \times \Omega$, $\phi = 0$ over $\mathbb{R} \times \Omega$, and this solution attracts, in a sense, all the solutions of this equation at large time. These authors further extended in [49] this notion to the unbounded case $\Omega = \mathbb{R}^N$. For this, they needed the assumption that the zero-order term is uniformly nonpositive at infinity, which ensures that the time-global solution ϕ is exponentially decreasing in x at infinity.

We investigated, together with Rossi [20], the links between this notion and that of generalized principal eigenvalues. We do not enter into details here and refer to the article in preparation [20].

If $\Omega = \mathbb{R}^N$, principal Floquet bundles do not exist in general. We thus introduce:

$$\underline{\mu_1}(\mathcal{L}, \mathbb{R} \times \mathbb{R}^N) = \sup\{\lambda \mid \exists \phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N),\ \phi \in L^\infty(\mathbb{R} \times \mathbb{R}^N) \text{ and } \mathcal{L}\phi \geq \lambda\phi \text{ in } \mathbb{R} \times \mathbb{R}^N\}$$
$$\overline{\mu_1}(\mathcal{L}, \mathbb{R} \times \mathbb{R}^N) = \inf\{\lambda \mid \exists \phi \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N),\ \inf_{\mathbb{R}\times\mathbb{R}^N} \phi > 0 \text{ and } \mathcal{L}\phi \leq \lambda\phi \text{ in } \mathbb{R} \times \mathbb{R}^N\}.$$

We use the same notations as in (54) because one can prove [20] that, when the coefficients do not depend on t, the parabolic and elliptic definitions of generalized principal eigenvalues coincide. Note that it is not clear how to define an analogous quantity μ_1 for parabolic operators.

One has the following comparison between these various notions of principal eigenvalues, that will be useful in the sequel.

Lemma 4.1 *One has*

$$\overline{\lambda_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N) \geq \overline{\mu_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N) \geq \underline{\mu_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N) \geq \underline{\lambda_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N).$$

Proof. Assume that $\overline{\mu_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N) < \underline{\mu_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N)$. Take μ', μ'' such that

$$\underline{\mu_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N) > \mu' > \mu'' > \overline{\mu_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N).$$

There exist $\phi, \psi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ such that $\phi \in W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N)$, $\inf_{\mathbb{R}\times\mathbb{R}^N}\psi > 0$, $\mathcal{L}\phi \geq \mu'\phi$ and $\mathcal{L}\psi \leq \mu''\psi$ in $\mathbb{R}\times\mathbb{R}^N$. Let $\gamma := \inf_{\mathbb{R}\times\mathbb{R}^N}\frac{\psi}{\phi}$ and $z := \psi - \gamma\phi$. The function z is nonnegative and $\inf_{\mathbb{R}\times\mathbb{R}^N} z = 0$. Moreover, it satisfies

$$\mathcal{L}z \leq \mu''\psi - \gamma\mu'\phi = \mu' z + (\mu'' - \mu')\psi \text{ in } \mathbb{R}\times\mathbb{R}^N.$$

Let $\varepsilon = (\mu' - \mu'')\inf_{\mathbb{R}\times\mathbb{R}^N}\psi > 0$, then

$$-(\mathcal{L} - \mu')z \geq \varepsilon \text{ in } \mathbb{R}\times\mathbb{R}^N \text{ in the sense of viscosity solutions.}$$

It now follows from the strong maximum principle for parabolic operators in unbounded domains proved in Lemma 3.4 of [13] that $\inf_{\mathbb{R}\times\mathbb{R}^N} z > 0$, which contradicts the definition of z. Thus,

$$\overline{\mu_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N) \geq \underline{\mu_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N).$$

Obviously, $\underline{\lambda_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N) \leq \underline{\mu_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N)$ and $\overline{\mu_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N) \leq \overline{\lambda_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N)$. □

4.2 Comparison between the generalized principal eigenvalues

We are now in position to state an inequality between $\overline{\lambda_1}$ and $\underline{\lambda_1}$ in more general domains than $\mathbb{R}\times\mathbb{R}^N$.

Proposition 4.2 *Consider an open set $Q \subset \mathbb{R}\times\mathbb{R}^N$ that contains balls of arbitrary radii. Then*

$$\overline{\lambda_1}(\mathcal{L}, Q) \geq \underline{\lambda_1}(\mathcal{L}, Q).$$

Remark: By "Q contains balls of arbitrary radii", we mean that for all $R > 0$, there exists $(t_R, x_R) \in \mathbb{R}\times\mathbb{R}^N$ such that $\{(t,x) \in \mathbb{R}\times\mathbb{R}^N, |t - t_R| < R, |x - x_R| < R\} \subset Q$. When this property is not satisfied, for example when Q is bounded, then the inequality of Proposition 4.2 may fail (see Proposition 4.5 below).

This is where we need a stronger hypothesis on the behavior of the test-functions at infinity than in [19]. In this previous paper investigating space heterogeneous one-dimensional

Fisher-KPP equations, we defined the generalized principal eigenvalues by requiring the test-functions to be positive and smooth enough over (R,∞) and sub-exponential at infinity (that is, $\lim_{x\to+\infty}\frac{1}{x}\ln\phi(x)=0$). The tricky part in the proof of the comparison between the two generalized principal eigenvalues was that we did not prescribe any given behavior at the boundary $x=R$. However, we managed to overcome this difficulty through one-dimensional arguments.

In the present paper, the boundary of $C_{R,\alpha}(e)$ is quite larger and we do not know if such a comparison holds. We thus impose a stronger hypothesis on the test-functions: boundedness and uniform positivity. By proving some comparison between the eigenvalues over Q and over $\mathbb{R}\times\mathbb{R}^N$, we will be able to assume that $Q=\mathbb{R}\times\mathbb{R}^N$, which has no boundary.

Proof of Proposition 4.2. Assume that $\underline{\lambda_1}(\mathcal{L},Q)>\overline{\lambda_1}(\mathcal{L},Q)$ and take

$$\underline{\lambda_1}(\mathcal{L},Q)>\lambda'>\lambda''>\overline{\lambda_1}(\mathcal{L},Q).$$

There exists $\phi\in\mathcal{C}^{1,2}(Q)\times W^{1,\infty}(Q)$ such that $\inf_Q\phi>0$ and $\mathcal{L}\phi\geq\lambda'\phi$ in Q. Take $(t_R,x_R)_{R>0}$ as in the Remark below Proposition 4.2 and let $\phi_R(t,x)=\phi(t+t_R,x+x_R)$. The family $(\phi_R)_R$ is equicontinuous and uniformly bounded since $\phi\in W^{1,\infty}(Q)$. By the Ascoli theorem, there exist a sequence $R_n\to+\infty$ as $n\to+\infty$ and $\phi_\infty\in W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N)$ such that $\phi_{R_n}\to\phi_\infty$ as $n\to+\infty$ locally uniformly in $\mathbb{R}\times\mathbb{R}^N$. One has $\inf_{\mathbb{R}\times\mathbb{R}^N}\phi_\infty\geq\inf_Q\phi$ and $\sup_{\mathbb{R}\times\mathbb{R}^N}\phi_\infty\leq\sup_Q\phi$. Similarly, as the coefficients A, q and c are uniformly continuous and bounded, one can assume, up to extraction, that there exist A_∞, q_∞ and c_∞ such that $A(t+t_{R_n},x+x_{R_n})\to A_\infty(t,x)$, $q(t+t_{R_n},x+x_{R_n})\to q_\infty(t,x)$ and $c(t+t_{R_n},x+x_{R_n})\to c_\infty(t,x)$ as $n\to+\infty$ locally uniformly in $\mathbb{R}\times\mathbb{R}^N$. Define

$$\mathcal{L}^*=-\partial_t+tr(A_\infty(t,x)\nabla^2)+q_\infty(t,x)\cdot\nabla+c_\infty(t,x).$$

Then the stability theorem for Hamilton-Jacobi equations (see Remark 6.2 in [33]) gives $\mathcal{L}^*\phi_\infty\geq\lambda'\phi_\infty$ in $\mathbb{R}\times\mathbb{R}^N$ in the sense of viscosity solutions. Even if it means decreasing λ' slightly, we can assume, using a convolution argument, that $\phi_\infty\in\mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ and that the inequation holds in the classical sense.

Similarly, as $\lambda''>\overline{\lambda_1}(\mathcal{L},Q)$, one can construct a function $\psi_\infty\in\mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N)$ such that $\inf_{\mathbb{R}\times\mathbb{R}^N}\psi_\infty>0$ and, up to one more extraction, $\mathcal{L}^*\psi_\infty\leq\lambda''\psi_\infty$ in $\mathbb{R}\times\mathbb{R}^N$.

The definitions of $\underline{\mu_1}(\mathcal{L}^*,\mathbb{R}\times\mathbb{R}^N)$ and $\overline{\mu_1}(\mathcal{L}^*,\mathbb{R}\times\mathbb{R}^N)$ in Lemma 4.1 above yield

$$\underline{\mu_1}(\mathcal{L}^*,\mathbb{R}\times\mathbb{R}^N)\geq\lambda' \text{ and } \overline{\mu_1}(\mathcal{L}^*,\mathbb{R}\times\mathbb{R}^N)\leq\lambda''.$$

But Lemma 4.1 gives $\underline{\mu_1}(\mathcal{L}^*,\mathbb{R}\times\mathbb{R}^N)\leq\overline{\mu_1}(\mathcal{L}^*,\mathbb{R}\times\mathbb{R}^N)$, which contradicts $\lambda''<\lambda'$. □

4.3 Continuity with respect to the coefficients and properties of the Hamiltonians

We will require in the sequel the continuity of the generalized principal eigenvalues associated with L_p with respect to p. This smoothness will indeed be derived from the continuity of the eigenvalues associated with $\mathcal{L}$ with respect to the first order term q and the zero order term

c. The uniform Lipschitz-continuity with respect to c is easy to derive from the maximum principle. The continuity in q is indeed trickier and is stated in the next Proposition. It is an open problem to prove the continuity with respect to the diffusion term A.

Proposition 4.3 *Consider two operators $\mathcal{L}$ and $\mathcal{L}'$ defined for all $\phi \in \mathcal{C}^{1,2}$ by*

$$\begin{array}{rcl} \mathcal{L}\phi & = & -\partial_t\phi + a_{i,j}(t,x)\partial_{ij}\phi + q_i(t,x)\partial_i\phi + c(t,x)\phi, \\ \mathcal{L}'\phi & = & -\partial_t\phi + a_{i,j}(t,x)\partial_{ij}\phi + r_i(t,x)\partial_i\phi + d(t,x)\phi, \end{array}$$

where $c, d \in \mathcal{C}^{\delta/2,\delta}_{loc}(\mathbb{R}\times\mathbb{R}^N) \cap L^\infty(\mathbb{R}\times\mathbb{R}^N)$ and A, q and r satisfy the hypotheses of Section 2.1. Then, for all open set $Q \subset \mathbb{R}\times\mathbb{R}^N$,

$$\begin{array}{rl} & |\overline{\lambda_1}(\mathcal{L}',Q) - \overline{\lambda_1}(\mathcal{L},Q)| \leq C\|q-r\|_\infty + \|c-d\|_\infty + \frac{1}{4\gamma}\|q-r\|_\infty^2 \\ \textit{and} & |\underline{\lambda_1}(\mathcal{L}',Q) - \underline{\lambda_1}(\mathcal{L},Q)| \leq C\|q-r\|_\infty + \|c-d\|_\infty + \frac{1}{4\gamma}\|q-r\|_\infty^2, \end{array}$$

where γ is given by (14) and $C = \frac{1}{\sqrt{\gamma}}\max\left\{\sqrt{\|c\|_\infty}, \sqrt{\|d\|_\infty}\right\}$.

Proof. We use the same type of arguments as in our previous paper [19]. Let $\delta = \|q-r\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}$ and $\varepsilon = \|c-d\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}$. For all constant M, one has $\underline{\lambda_1}(\mathcal{L}+M,Q) = \underline{\lambda_1}(\mathcal{L},Q) + M$. Thus, adding a sufficiently large M, one can assume that c and d are positive functions and that $\underline{\lambda_1}(\mathcal{L},Q) > 0$ and $\underline{\lambda_1}(\mathcal{L}',Q) > 0$.

For all $\kappa > 0$, there exists a function $\phi \in \mathcal{C}^{1,2}(Q) \cap W^{1,\infty}(Q)$ such that $\inf_Q \phi > 0$ and

$$\mathcal{L}\phi = -\partial_t\phi + tr(A(t,x)\nabla^2\phi) + q(t,x)\cdot\nabla\phi + c(t,x)\phi \geq (\underline{\lambda_1}(\mathcal{L},Q) - \kappa)\phi \text{ in } Q.$$

Consider any $\alpha > 1$ and define $\psi = \phi^\alpha$. On the set Q, this function satisfies

$$\begin{array}{l} -\mathcal{L}'\psi = \partial_t\psi - a_{i,j}(t,x)\partial_{ij}\psi - r_i(t,x)\partial_i\psi - d(t,x)\psi \\ = \alpha\phi^{\alpha-1}\big(\partial_t\phi - a_{i,j}(t,x)\partial_{ij}\phi - r_i(t,x)\partial_i\phi\big) - d(t,x)\phi^\alpha - \alpha(\alpha-1)\phi^{\alpha-2}\nabla\phi A(t,x)\nabla\phi \\ \leq \alpha\delta\phi^{\alpha-1}|\nabla\phi| + (\alpha c(t,x) - d(t,x))\phi^\alpha - (\underline{\lambda_1}(\mathcal{L},Q)-\kappa)\alpha\phi^\alpha - \alpha(\alpha-1)\gamma\phi^{\alpha-2}|\nabla\phi|^2 \\ \leq \frac{\alpha}{4(\alpha-1)\gamma}\delta^2\psi + (\alpha-1)\|c\|_\infty\psi + \varepsilon\psi - (\underline{\lambda_1}(\mathcal{L},Q)-\kappa)\alpha\psi. \end{array}$$

Thus for all $\alpha > 1$, $\kappa > 0$ so that $\underline{\lambda_1}(\mathcal{L},Q) - \kappa > 0$, one has:

$$\underline{\lambda_1}(\mathcal{L}',Q) \geq \underline{\lambda_1}(\mathcal{L},Q) - \kappa - \frac{\alpha}{4(\alpha-1)\gamma}\delta^2 - (\alpha-1)\|c\|_\infty - \varepsilon.$$

Take $\alpha = 1 + \frac{\delta}{2\sqrt{\|c\|_\infty\gamma}}$. Letting $\kappa \to 0$, this gives

$$\underline{\lambda_1}(\mathcal{L}',Q) \geq \underline{\lambda_1}(\mathcal{L},Q) - \delta\sqrt{\frac{\|c\|_\infty}{\gamma}} - \varepsilon - \frac{\delta^2}{4\gamma}.$$

A symmetry argument gives

$$|\underline{\lambda_1}(\mathcal{L}', Q) - \underline{\lambda_1}(\mathcal{L}, Q)| \leq \delta \max\Big\{\sqrt{\frac{\|c\|_\infty}{\gamma}}, \sqrt{\frac{\|d\|_\infty}{\gamma}}\Big\} + \varepsilon + \frac{\delta^2}{4\gamma}.$$

A similar argument, with $0 < \alpha < 1$, gives the Lipschitz-continuity of $\overline{\lambda_1}$. □

Proof of Proposition 2.2. The convexity and the upper and lower bounds on $\underline{H}$ and $\overline{H}$ follow from the same arguments as that of Proposition 2.3 in [19], that we recall here for sake of completeness.

Indeed, using the same proof as that of Proposition 3.6 in [13], the reader easily gets that the function $p \mapsto \overline{\lambda_1}(L_p, Q)$ is convex for all open set $Q \subset \mathbb{R} \times \mathbb{R}^N$. Thus $p \mapsto \overline{H}(e, p)$ is convex for all $e \in \mathbb{S}^{N-1}$. Proposition 4.3 and (53) give the local Lipschitz-continuity of $\underline{H}$ and $\overline{H}$ with respect to p. Proposition 4.2 gives $\overline{H}(e, p) \geq \underline{H}(e, p)$ for all $(e, p) \in \mathbb{S}^{N-1} \times \mathbb{R}^N$.

For all $p \in \mathbb{R}^N$, $\alpha \in (0, 1)$, $e \in \mathbb{S}^{N-1}$ and $R > 0$, the infimum of the zero-order term of L_p over $C_{R,\alpha}(e)$ is bounded from below by $\inf_{t>R,|x|>R} \big(pA(t, x)p + q(t, x) \cdot p + f_s'(t, x, 0)\big)$. Thus, taking a constant test-function in the definition of $\underline{\lambda_1}$, one gets

$$\underline{\lambda_1}(L_p, C_{R,\alpha}(e)) \geq \inf_{t>R,|x|>R} \big(pA(t, x)p + q(t, x) \cdot p + f_s'(t, x, 0)\big). \tag{55}$$

Using (14), we obtain

$$\underline{H}(e, p) \geq \gamma|p|^2 - \|q\|_\infty|p| + \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} f_s'(t, x, 0).$$

Hence, there exists a constant $c > 0$ such that for all $p \in \mathbb{R}^N$ and $e \in \mathbb{S}^{N-1}$,

$$\underline{H}(e, p) \geq c(|p|^2 - 1).$$

The other inequality is obtained in a similar way.

We now check the upper semicontinuity of $\underline{H}$ (the proof for $\overline{H}$ being similar, we will omit it). Let $e \in \mathbb{S}^{N-1}$, $p \in \mathbb{R}^N$, $\alpha > 0$ and $R > 0$. Consider some $e' \in \mathbb{S}^{N-1}$ close to e. The geometry of $C_{R,\alpha}(e)$ yields that for $|e' - e| < \alpha$, $C_{R,\alpha'}(e') \subset C_{R,\alpha}(e)$, with $\alpha' = \alpha - |e - e|$. Hence, a test-function ϕ associated with $\underline{\lambda_1}(L_p, C_{R,\alpha}(e))$ through (18) is admissible as a test-function for $\underline{\lambda_1}(L_p, C_{R,\alpha'}(e'))$, and it easily follows from the definition of $\underline{\lambda}$ that

$$\underline{\lambda_1}(L_p, C_{R,\alpha}(e)) \leq \underline{\lambda_1}(L_p, C_{R,\alpha'}(e')) \leq \underline{H}(e', p) \quad \text{if } |e - e'| < \alpha.$$

The definition of $\underline{H}$ yields that for all $\varepsilon > 0$, there exist $\alpha_0 > 0$ and $R_0 > 0$ such that $\underline{H}(e, p) \leq \underline{\lambda_1}(L_p, C_{R,\alpha}(e)) + \varepsilon$ for all $\alpha \in (0, \alpha_0]$ and $R \geq R_0$. We conclude that $\underline{H}(e, p) \leq \underline{H}(e', p) + \varepsilon$ if $|e - e'| < \alpha_0$, which concludes the proof. □

Proof of Lemma 2.5. Taking the minimum over $p \in \mathbb{R}^N$ in (55), one gets

$$\begin{array}{rcl} \inf_{p\in\mathbb{R}^N} \underline{\lambda_1}(L_p, C_{R,\alpha}(e)) & \geq & \inf_{t>R,|x|>R} \inf_{p\in\mathbb{R}^N} \big(pA(t, x)p + q(t, x) \cdot p + f_s'(t, x, 0)\big) \\ & \geq & \inf_{t>R,|x|>R} \big(f_s'(t, x, 0) - \frac{1}{4} q(t, x)A^{-1}(t, x)q(t, x)\big). \end{array} \tag{56}$$

Finally, using (16), one gets

$$\inf_{p\in\mathbb{R}^N} \underline{H}(e,p) \geq \sup_{R>0} \inf_{t>R,|x|>R} \big(c(t,x) - \frac{1}{4}q(t,x)A^{-1}(t,x)q(t,x)\big) > 0.$$

Similarly, combining (55) and (14), we obtain

$$\underline{H}(e,p) \geq \gamma|p|^2 - \|q\|_\infty|p| + \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} f_s'(t,x,0).$$

Hence, there exists a constant $c>0$ such that for all $p\in\mathbb{R}^N$ and $e\in\mathbb{S}^{N-1}$,

$$\underline{H}(e,p) \geq c(1+|p|^2).$$

The other inequality is obtained in a similar way. □

4.4 Comparisons with earlier notions of eigenvalues

We conclude this Section with some comparisons with classical notions of principal eigenvalues. These results help to understand the notion of generalized principal eigenvalue and to compare our results with earlier works.

The case where there exists a classical eigenvalue

First, when the coefficients are periodic, then $\underline{\lambda_1} = \overline{\lambda_1}$ equals the classical notion of periodic principal eigenvalue. More generally, when there exists an exact eigenfunction which is $W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N)$ and uniformly positive, then the associated eigenvalue equals the generalized principal eigenvalues.

Proposition 4.4 *Consider an open set $Q\subset\mathbb{R}\times\mathbb{R}^N$ that contains balls of arbitrary radii. Assume that there exist $\lambda\in\mathbb{R}$ and $\phi\in\mathcal{C}^{1,2}(Q)$ such that $\inf_Q\phi>0$, $\phi\in W^{1,\infty}(Q)$ and $\mathcal{L}\phi=\lambda\phi$ in Q. Then*

$$\lambda = \underline{\lambda_1}(\mathcal{L},Q) = \overline{\lambda_1}(\mathcal{L},Q).$$

In particular, if the coefficients are space-time periodic, using the same notations as in Section 1.1, one has

$$k_0^{per} = \underline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N) = \overline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N).$$

Remark. The converse assertion is not necessarily true: it may happen that $\underline{\lambda_1} = \overline{\lambda_1}$ while there exists no classical eigenvalue. For example, the two generalized principal eigenvalues are equal if the coefficients are almost periodic in (t,x) (see Theorem 4 below) but it is well-known that almost periodic operators do not admit classical eigenvalues in general [86].

Proof. Using ϕ as a test-function in the definitions (18) of $\underline{\lambda_1}(\mathcal{L},Q)$ and (19) of $\overline{\lambda_1}(\mathcal{L},Q)$, one gets $\underline{\lambda_1}(\mathcal{L},Q)\geq\lambda$ and $\overline{\lambda_1}(\mathcal{L},Q)\leq\lambda$. As $\underline{\lambda_1}(\mathcal{L},Q)\leq\overline{\lambda_1}(\mathcal{L},Q)$ from Proposition 4.2, this gives the conclusion.

If the coefficients are periodic, then there exists a space-time periodic principal eigenfunction ϕ such that $\mathcal{L}\phi = k_0^{per}(\mathcal{L})\phi$ and $\phi>0$. As ϕ is periodic, it is bounded and $\inf_{\mathbb{R}\times\mathbb{R}^N}\phi>0$. Thus $k_0^{per}(\mathcal{L}) = \overline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N) = \underline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N)$. □

The case of bounded domains

When the coefficients do not depend on t and $Q = \mathbb{R} \times \omega$, with ω bounded and smooth, then $\underline{\lambda_1}(\mathcal{L}, \mathbb{R} \times \omega)$ is infinite and $\overline{\lambda_1}(\mathcal{L}, \mathbb{R} \times \omega)$ is the classical Dirichlet principal eigenvalue $\lambda_D(\mathcal{L}, \omega)$, defined by the existence of some $\phi_D \in \mathcal{C}^2(\omega) \cap \mathcal{C}^0(\overline{\omega})$ such that

$$\begin{cases} \mathcal{L}\phi_D = \lambda_D(\mathcal{L}, \omega)\phi_D \text{ in } \omega, \\ \phi_D > 0 \text{ in } \omega, \\ \phi_D = 0 \text{ over } \partial\omega. \end{cases} \tag{57}$$

Hence, $\underline{\lambda_1}(\mathcal{L}, \mathbb{R} \times \omega) \leq \overline{\lambda_1}(\mathcal{L}, \mathbb{R} \times \omega)$ is not true anymore if ω is bounded and smooth.

Proposition 4.5 *Assume that A, q and c do not depend on t and that $Q = \mathbb{R} \times \omega$, with ω bounded and smooth. Then*

$$\overline{\lambda_1}(\mathcal{L}, \mathbb{R} \times \omega) = \lambda_D(\mathcal{L}, \omega) \quad \textit{and} \quad \underline{\lambda_1}(\mathcal{L}, \mathbb{R} \times \omega) = +\infty.$$

Proof. For all $\varepsilon > 0$, we define $\omega_\varepsilon = \{x \in \mathbb{R}^N, d(x, \omega) < \varepsilon\}$ and χ_ε the principal eigenfunction associated with $\lambda_\varepsilon = \lambda_D(\mathcal{L}, \omega_\varepsilon)$. It is well-known (see [21] for example) that $\lambda_\varepsilon \searrow \lambda_D(\mathcal{L}, \omega)$.

On one hand, as $\inf_\omega \chi_\varepsilon > 0$ for all $\varepsilon > 0$, one can take χ_ε as a test-function in the definition of $\overline{\lambda_1}(\mathcal{L}, \mathbb{R} \times \omega)$, which gives $\overline{\lambda_1}(\mathcal{L}, \mathbb{R} \times \omega) \leq \lambda_\varepsilon$ for all $\varepsilon > 0$. Thus, $\overline{\lambda_1}(\mathcal{L}, \mathbb{R} \times \omega) \leq \lambda_D(\mathcal{L}, \omega)$.

On the other hand, assume that this inequality is strict and take λ' such that

$$\overline{\lambda_1}(\mathcal{L}, \mathbb{R} \times \omega) < \lambda' < \lambda_D(\mathcal{L}, \omega).$$

There exists $\psi \in \mathcal{C}^{1,2}(\mathbb{R} \times \omega) \cap W^{1,\infty}(\mathbb{R} \times \omega)$ such that $\inf_{\mathbb{R} \times \omega} \psi > 0$ and $\mathcal{L}\psi \leq \lambda'\psi$. Let $\kappa = \inf_{(t,x) \in \mathbb{R} \times \omega} \frac{\psi(t,x)}{\phi_D(x)} < \infty$ and $z = \psi - \kappa\phi_D$. Then $\inf_{\mathbb{R} \times \omega} z = 0$ and

$$\mathcal{L}z \leq (\lambda' - \lambda_D)\psi + \lambda_D(\mathcal{L}, \omega)z.$$

Thus, there exists $\varepsilon > 0$ such that $-(\mathcal{L} - \lambda_D(\mathcal{L}, \omega))z \geq \varepsilon$. Lemma 3.4 of [13] then gives $\inf_{\mathbb{R} \times \omega} z > 0$, which is the required contradiction. Hence $\overline{\lambda_1}(\mathcal{L}, \mathbb{R} \times \omega) \geq \lambda_D(\mathcal{L}, \omega)$.

Lastly, for all $\kappa \in \mathbb{R}$, let $\psi_\kappa(t, x) := e^{\kappa x_1}$. As ω is bounded, $\inf_{\mathbb{R} \times \omega} \psi_\kappa > 0$. A straightforward computation gives $\inf_{\mathbb{R} \times \omega} \frac{\mathcal{L}\psi_\kappa}{\psi_\kappa} \to +\infty$ as $\kappa \to +\infty$. Thus $\underline{\lambda_1}(\mathcal{L}, \mathbb{R} \times \omega) = +\infty$. $\square$

A relation with the earlier notions of generalized principal eigenvalues

We conclude this discussion with a result providing a link with the earlier notions of generalized principal eigenvalues used in [18, 24] $\underline{\mu_1}$ and $\overline{\mu_1}$ in the full space $\mathbb{R} \times \mathbb{R}^N$. This Theorem will not be used in the sequel but is of independent interest. We do not know if it holds for more general domains than $\mathbb{R} \times \mathbb{R}^N$, such as the truncated cones $C_{R,\alpha}(e)$ for example.

We start with the definition of limit operators.

Definition 4.6 *We say that* $\mathcal{L}^* := -\partial_t + a^*_{i,j}\partial_{ij} + q^*_i\partial_i + c^*$ *is a* **limit operator** *of* $\mathcal{L}$ *if there exists a sequence* $\big((t_n,x_n)\big)_n$ *in* $\mathbb{R}\times\mathbb{R}^N$ *such that* $\big(a_{i,j}(\cdot+t_n,\cdot+x_n)\big)_n$*,* $\big(q_i(\cdot+t_n,\cdot+x_n)\big)_n$ *and* $\big(c(\cdot+t_n,\cdot+x_n)\big)_n$ *converge respectively to* $a^*_{i,j}$*,* q^*_i *and* c^* *in* $\mathcal{C}^0_{loc}$ *as* $n\to+\infty$*.*

We denote by $Hull(\mathcal{L})$ *the set of all the limit operators of* $\mathcal{L}$*.*

Theorem 8 *One has*

$$\overline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N) = \max_{\mathcal{L}^*\in Hull(\mathcal{L})} \overline{\mu_1}(\mathcal{L}^*,\mathbb{R}\times\mathbb{R}^N) \quad \text{and} \quad \underline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N) = \min_{\mathcal{L}^*\in Hull(\mathcal{L})} \underline{\mu_1}(\mathcal{L}^*,\mathbb{R}\times\mathbb{R}^N).$$

Moreover, if $\mathcal{L}^*$ *is the limit operator maximizing* $\overline{\mu_1}(\mathcal{L}^*,\mathbb{R}\times\mathbb{R}^N)$ *over* $Hull(\mathcal{L})$ *(or the one minimizing* $\underline{\mu_1}(\mathcal{L}^*,\mathbb{R}\times\mathbb{R}^N)$*), then*

$$\underline{\mu_1}(\mathcal{L}^*,\mathbb{R}\times\mathbb{R}^N) = \overline{\mu_1}(\mathcal{L}^*,\mathbb{R}\times\mathbb{R}^N).$$

Before going into the proof of this result, note that it makes it easy to construct various examples for which $\overline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N) > \overline{\mu_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N)$ or $\underline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N) < \underline{\mu_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N)$, without contradicting the conjecture $\overline{\mu_1}(\mathcal{L}) = \underline{\mu_1}(\mathcal{L})$ stated in [18, 24] for elliptic operators. This conjecture thus remains open.

Proof of Theorem 8. As all eigenvalues are defined on $\mathbb{R}\times\mathbb{R}^N$, we will just use the notations $\overline{\lambda_1}(\mathcal{L})$, $\overline{\mu_1}(\mathcal{L}^*)$, $\underline{\lambda_1}(\mathcal{L})$ and $\underline{\mu_1}(\mathcal{L}^*)$ along the proof with no ambiguity.

1. First, for all $\varepsilon>0$, there exists a solution $\phi_\varepsilon\in\mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ of

$$-\mathcal{L}\phi_\varepsilon = (-\varepsilon\ln\phi_\varepsilon)\phi_\varepsilon, \quad \text{in } \mathbb{R}\times\mathbb{R}^N, \tag{58}$$

with $\phi_\varepsilon\in L^\infty(\mathbb{R}\times\mathbb{R}^N)$ and $\inf_{\mathbb{R}\times\mathbb{R}^N}\phi_\varepsilon>0$. This construction is known and we just remind to the reader the main arguments. Take $m>0$ such that $\exp\big(-\|c\|_\infty/\varepsilon\big)\geq m$. Then

$$-\mathcal{L}m = -c(x)m \leq \|c\|_\infty m \leq (-\varepsilon\ln m)m \quad \text{in } \mathbb{R}\times\mathbb{R}^N.$$

Similarly, $M\geq\exp\big(\|c\|_\infty/\varepsilon\big)$ is a supersolution of this equation. Hence, there exists a solution ϕ_ε of (58), with $m\leq\phi_\varepsilon\leq M$.

2. Consider next an arbitrary sequence $(\varepsilon_n)_n$ such that $\lim_{n\to+\infty}\varepsilon_n=0$ and the limit

$$\overline{\ell} = \lim_{n\to+\infty}\big(\varepsilon_n\sup_{\mathbb{R}\times\mathbb{R}^N}\ln\phi_{\varepsilon_n}\big)$$

exists. Consider a sequence $(t_n,x_n)_n$ such that $(1-1/n)\sup_{\mathbb{R}\times\mathbb{R}^N}\phi_{\varepsilon_n}\leq\phi_{\varepsilon_n}(t_n,x_n)$ for all n. Call

$$\varphi_n(x) := \frac{\phi_{\varepsilon_n}(t+t_n,x+x_n)}{\phi_{\varepsilon_n}(t_n,x_n)}.$$

This function satisfies

$$\begin{aligned}&\partial_t\varphi_n - a_{i,j}(t+t_n,x+x_n)\partial_{ij}\varphi_n - q_i(t+t_n,x+x_n)\partial_i\varphi_n - c(t+t_n,x+x_n)\varphi_n\\ &= \Big(-\varepsilon_n\ln\big(\phi_{\varepsilon_n}(t_n,x_n)\varphi_n\big)\Big)\varphi_n \quad \text{in } \mathbb{R}\times\mathbb{R}^N.\end{aligned}$$

Moreover, the construction of ϕ_ε ensures that $\|\varepsilon \ln \phi_\varepsilon\|_\infty \leq \|c\|_\infty$ for all $\varepsilon > 0$. Hence, the Harnack inequality applies: for all $R > 0$ and $T > \tau > 0$, φ_n is bounded by a positive constant on the set $[-T, -\tau] \times B_R$, uniformly with respect to n. Similarly, parabolic regularity estimates apply with constants independent of n and thus the sequence $(\varphi_n)_n$ is uniformly bounded in $W^{1,p/2;2,p}([-T,T] \times B_R)$ for all $p \in (1, \infty)$. The Ascoli theorem yields that the sequence $(\varphi_n)_n$ converges (up to extraction) to a function φ_∞ in $W^{1,p/2;2,p}_{loc}\big(\mathbb{R} \times \mathbb{R}^N\big)$. Hence,

$$\varepsilon_n \ln\big(\phi_{\varepsilon_n}(t_n, x_n)\varphi_n(t,x)\big) = \varepsilon_n \ln \phi_{\varepsilon_n}(t_n,x_n) + \varepsilon_n \ln \varphi_n(t,x) \to \underline{\ell} \quad \text{as } n \to +\infty, \quad \text{locally in } (t,x).$$

Moreover, as $a_{i,j}$, b_i and c are uniformly continuous in $\mathbb{R} \times \mathbb{R}^N$, one can assume that the sequences $\big(a_{i,j}(\cdot + t_n, \cdot + x_n)\big)_n$, $\big(q_i(\cdot + t_n, \cdot + x_n)\big)_n$ and $\big(c(\cdot + t_n, \cdot + x_n)\big)_n$ converge in $\mathcal{C}^0_{loc}$ as $n \to +\infty$. Let $a^*_{i,j}$, q^*_i and c^* be their respective limits and $\mathcal{L}^* := -\partial_t + a^*_{i,j}\partial_{ij} + q^*_i \partial_i + c^*$. One has

$$\mathcal{L}^* \varphi_\infty = \underline{\ell}\varphi_\infty \quad \text{in } \mathbb{R} \times \mathbb{R}^N.$$

On the other hand, one has $\varphi_n(0,0) = 1$, $\varphi_n \geq 0$ and

$$\varphi_n(t,x) \leq \frac{\sup_{\mathbb{R}\times\mathbb{R}^N} \phi_{\varepsilon_n}}{\phi_{\varepsilon_n}(t_n,x_n)} \leq \frac{\sup_{\mathbb{R}\times\mathbb{R}^N} \phi_{\varepsilon_n}}{(1-1/n)\sup_{\mathbb{R}\times\mathbb{R}^N} \phi_{\varepsilon_n}} = \frac{1}{1-1/n} \to 1 \quad \text{as } n \to +\infty.$$

Hence, the strong maximum principle gives

$$\varphi_\infty > 0 \quad \text{and} \quad \varphi_\infty \leq 1.$$

It follows from the definition of $\underline{\mu_1}(\mathcal{L}^*)$ that

$$\overline{\ell} \leq \underline{\mu_1}(\mathcal{L}^*). \tag{59}$$

3. Next, one has

$$\mathcal{L}\phi_{\varepsilon_n} = (\varepsilon_n \ln \phi_{\varepsilon_n})\phi_{\varepsilon_n} \leq (\varepsilon_n \sup_{\mathbb{R}\times\mathbb{R}^N} \ln \phi_{\varepsilon_n})\phi_{\varepsilon_n}.$$

As ϕ_ε is bounded and uniformly positive, one can use ϕ_{ε_n} as a test-function in the definition of $\overline{\lambda_1}(\mathcal{L})$, implying $\varepsilon_n \sup_{\mathbb{R}\times\mathbb{R}^N} \ln \phi_{\varepsilon_n} \geq \overline{\lambda_1}(\mathcal{L})$. Letting $n \to +\infty$, we get

$$\overline{\ell} \geq \overline{\lambda_1}(\mathcal{L}). \tag{60}$$

4. Next, we will prove later in Proposition 4.2 that for all limit operator $\mathcal{L}^*$ of $\mathcal{L}$, one has

$$\underline{\lambda_1}(\mathcal{L}^*, \mathbb{R}\times\mathbb{R}^N) \geq \underline{\lambda_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N) \quad \text{and} \quad \overline{\lambda_1}(\mathcal{L}^*, \mathbb{R}\times\mathbb{R}^N) \leq \overline{\lambda_1}(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N). \tag{61}$$

Gathering inequalities (59), (60), (61) and Lemma 4.1, one gets

$$\overline{\lambda_1}(\mathcal{L}) \geq \overline{\lambda_1}(\mathcal{L}^*) \geq \overline{\mu_1}(\mathcal{L}^*) \geq \underline{\mu_1}(\mathcal{L}^*) \geq \overline{\ell} \geq \overline{\lambda_1}(\mathcal{L}).$$

Hence, all these inequalities are equalities. In particular, $\overline{\mu_1}(\mathcal{L}^*) = \underline{\mu_1}(\mathcal{L}^*)$. Furthermore, if $\mathcal{L}^{**}$ is an arbitrary limit operator of $\mathcal{L}$, then (61) and the obvious inequality $\overline{\mu_1} \leq \overline{\lambda_1}$ give

$$\overline{\lambda_1}(\mathcal{L}) \geq \overline{\lambda_1}(\mathcal{L}^{**}) \geq \overline{\mu_1}(\mathcal{L}^{**})$$

from which the conclusion follows.

The proof for $\underline{\lambda_1}(\mathcal{L})$ follows the same steps.

□

The connection with Floquet exponents

In random stationary ergodic media, the proofs of spreading properties rely, in general, on large deviation principles. Without going into too many details, the idea is to consider the solution v of the linear parabolic equation $\mathcal{L}v = 0$ in $(0,\infty)\times\mathbb{R}^N$, $v(0,x) = u_0(x)$, and prove that $\lambda := \lim_{t\to+\infty}\frac{1}{t}\ln v(t,x)$ exists, in some sense. This quantity is called the Floquet exponent and coincides with the principal eigenvalue in periodic media.

The generalized principal eigenvalues will certainly provide bounds on $\frac{1}{t}\ln v(t,x)$ at large times. The authors are investigating this question in joint work with Rossi in [20]. Indeed, depending on the conditions we require on the test-functions involved in the definitions of generalized principal eigenvalues, we find estimates related to different notions of limits for $\frac{1}{t}\ln v(t,x)$. In random stationary ergodic media, all these notions coincide since the limit of $\frac{1}{t}\ln v(t,x)$ is well-defined as $t\to+\infty$ almost everywhere. However, it is not the case for more general classes of coefficients. As we want to handle general heterogeneous coefficients in the current memoir, we do not discuss this approach here and refer to [20] and possible future works.

5 Proof of the spreading property

5.1 The connection between asymptotic spreading and homogenization

It has long been known that there is a strong link between homogenization problems and spreading properties, that is, the investigation of sets $\overline{\mathcal{S}}$ and $\underline{\mathcal{S}}$ satisfying (2). However, to our knowledge, this link has never been fully established in a general framework. Xin in [104] provides mostly heuristic computations describing this link in the periodic setting. Actually, one of our aims in the present manuscript is to establish this link rigorously and in a general framework. Indeed, along the way in our proofs, we realized that heuristic arguments and homogenization methods need to be supplemented in order to derive the actual spreading properties for reaction-diffusion equations.

Before starting the proof of our main result, let us first describe this more precisely. Consider a solution u of the nonlinear reaction-diffusion equation (1). Assume that $A = I_N$ in order to simplify the presentation. In order to locate its level sets, following the homogenization approach, one lets $Z_\varepsilon(t,x) := \varepsilon\ln v_\varepsilon$, with $v_\varepsilon(t,x) := u(t/\varepsilon, x/\varepsilon)$. The aim is then to compute the limit of $(Z_\varepsilon)_\varepsilon$ when it exists. This function satisfies

$$\begin{cases} \partial_t Z_\varepsilon - \varepsilon\Delta Z_\varepsilon - H(t/\varepsilon, x/\varepsilon, \nabla Z_\varepsilon) = \frac{1}{v_\varepsilon} f(t/\varepsilon, x/\varepsilon, v_\varepsilon) - f'_u(t/\varepsilon, x/\varepsilon, 0) \text{ in } (0,\infty)\times\mathbb{R}^N, \\ \\ Z_\varepsilon(0,x) = \begin{cases} \varepsilon\ln u_0(x/\varepsilon) & \text{if } u_0(x/\varepsilon)\neq 0, \\ -\infty & \text{otherwise,} \end{cases} \end{cases}$$

with

$$H(s,y,p) := |p|^2 + q(s,y)\cdot p + f'_u(s,y,0).$$

If one replaces the initial datum by a function which does not depend on ε and if the right-hand side cancels, that is, if $f = f(t,x,u)$ is linear with respect to u, then this equation reduces to the following typical equation considered in the homogenization literature:

$$\begin{cases} \partial_t Z_\varepsilon - \kappa\varepsilon\Delta Z_\varepsilon - H(t/\varepsilon, x/\varepsilon, \nabla Z_\varepsilon) = 0 \text{ in } (0,\infty)\times\mathbb{R}^N, \\ Z_\varepsilon(0,x) = Z_0(x) \text{ otherwise}, \end{cases} \tag{62}$$

with $\kappa = 1$ here. Such problems are usually investigated in the framework where $Z_0 \in \mathcal{C}_b(\mathbb{R}^N)$, $\kappa \geq 0$ and H is continuous in (t,x,p), convex in p and $H(t,x,p)/|p| \to +\infty$ as $|p| \to +\infty$ uniformly in $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ (see for instance [67]).

Consider first the case when H is periodic in x and does not depend on t. The heuristics that give the characterization of the effective Hamiltonian H^{hom} are the following (we refer to [67] for a complete review on this topic). First, one looks for an approximation of the form

$$Z_\varepsilon(t,x) \simeq Z(t,x) + \varepsilon Y(t,x,x/\varepsilon),$$

where Y is periodic in x/ε. Then, in order to separate the two scales x and x/ε, a straightforward computation shows that Y has to satisfy an equation of the form

$$-\kappa\Delta_y Y - H(y, \nabla_x Z + \nabla_y Y) = H^{hom}(\nabla_x Z)$$

for some function H^{hom}. In other words, choosing (t,x) and letting $p = \nabla_x Z(t,x)$ and $v_p(y) = Y(t,x,y)$, one needs to find for all $p \in \mathbb{R}^N$ a solution $\big(v_p, H^{hom}(p)\big)$, with v_p periodic, of

$$-\kappa\Delta_y v_p - H(y, p + \nabla_y v_p) = H^{hom}(p) \quad \text{in } \mathbb{R}^N. \tag{63}$$

This equation is called the *cell problem* associated with (62) and v_p is called an *exact corrector* associated with this cell problem. If $H(y,p) = |p|^2 + c(y)$ and $\kappa = 1$, which is the Hamiltonian that comes from a linear elliptic equation, using the WKB change of variable $\phi_p = e^{v_p}$, we see that the existence of an exact corrector is equivalent to the existence of a periodic solution $(\phi_p, H^{hom}(p))$ of

$$\Delta_y\phi_p + 2p\cdot\nabla\phi_p + (|p|^2 + c(y))\phi_p = H^{hom}(p)\phi_p \quad \text{in } \mathbb{R}^N. \tag{64}$$

In other words, as $\phi_p > 0$, in this case $H^{hom}(p)$ is the *periodic principal eigenvalue* associated with the operator $L_p = \Delta + 2p\cdot\nabla + (|p|^2 + c(y))$. Indeed, it is always possible to find a solution $(v_p, H^{hom}(p))$ of the more general cell problem (63) when the Hamiltonian $H(y,p)$ is periodic in y. Then, a classical machinery yields that $\lim_{\varepsilon\to 0} Z_\varepsilon(t,x) = Z(t,x)$ locally in (t,x), where Z is the unique solution of the homogenized equation

$$\begin{cases} \partial_t Z - H^{hom}(\nabla Z) = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ Z(0,x) = Z_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{65}$$

When H is almost periodic, it is not always true that there exists a principal eigenvalue, and thus an exact corrector, associated with L_p. This problem was solved by Ishii [56] when $\kappa = 0$ and by Lions and Souganidis [66] for fully nonlinear almost periodic equations. They introduced the notion of *approximate correctors*. Namely, they proved the existence of a constant $H^{hom}(p)$ such that for all $\delta > 0$, there exist two bounded functions v_p^δ and $v_{p,\delta}$ that satisfy in $\mathbb{R}^N$:

$$-\kappa\Delta_y v_{p,\delta} + H(y, p + \nabla_y v_{p,\delta}) \leq H^{hom}(p) + \delta \quad \text{and} \quad -\kappa\Delta_y v_p^\delta + H(y, p + \nabla_y v_p^\delta) \geq H^{hom}(p) - \delta. \tag{66}$$

The existence of approximate correctors is sufficient in order to homogenize equation (62), as proved in [56, 66]. Now, if $H(y,p) = |p|^2 + c(y)$ and $\kappa = 1$, letting $\phi_{p,\delta} = \exp(-v_{p,\delta})$ and $\phi_p^\delta = \exp(-v_p^\delta)$, the existence of approximate correctors is equivalent to the existence of $\phi_{p,\delta}$ and ϕ_p^δ such that

$$L_p\phi_{p,\delta} \geq (H^{hom}(p) - \delta)\phi_{p,\delta} \quad \text{and} \quad L_p\phi_p^\delta \leq (H^{hom}(p) + \delta)\phi_p^\delta \quad \text{in } \mathbb{R}^N,$$

where $\phi_{p,\delta}$ and ϕ_p^δ are bounded and have a positive infimum. In other words, in terms of the generalized principal eigenvalues we have defined here, there exist approximate correctors if and only if

$$\underline{\lambda_1}(L_p, \mathbb{R}\times\mathbb{R}^N) = \overline{\lambda_1}(L_p, \mathbb{R}\times\mathbb{R}^N).$$

Ishii [56] and Lions and Souganidis [66] obtained such approximate correctors in the space almost periodic framework using Evan's perturbed test function method, that was first introduced in a periodic framework [37]. We also made use of this method to prove the equality of the two generalized principal eigenvalues in space-time almost periodic media in [19].

When H is random stationary ergodic with respect to x, it has been proved independently by Lions and Souganidis [67] and by Kosygina, Rezakhanlou and Varadhan [59] that it is possible to homogenize (62), that is, $Z_\varepsilon(t,x) \to Z(t,x)$ as $\varepsilon \to 0$ locally uniformly in (t,x) almost surely and the limit Z satisfies a deterministic equation of the form (65). This result has been extended to space-time random stationary ergodic equations by Kosygina and Varadhan [60] (see also [88] when $\kappa = 0$).

It is not always true that there exist approximate correctors in random stationary ergodic media. Lions and Souganidis [67] proved that there exists a global subsolution v of $-\kappa\Delta v + H(x, p + \nabla v) \leq H^{hom}(p)$ in $\mathbb{R}^N$ almost surely, where ∇v is a random stationary ergodic function with mean 0. It is well-known that such a function needs not necessarily be bounded nor stationary anymore but that it is sub-linear at infinity: $v(x)/|x| \to 0$ as $|x| \to +\infty$ almost surely. Hence, one needs to extend the notion of approximate correctors to sublinear functions at infinity. Moreover, even with this extended notion, it is not always true that there exists an upper approximate corrector. Indeed, Lions and Souganidis provided a counter-example in [65]. This is why they proposed a new notion of correctors (see Proposition 7.3 in [67]), which is tailored for homogenization problems of random stationary ergodic equations.

However, in dimension 1, for second order linear elliptic equations, we have proved in our earlier paper [19] that there exists an approximate corrector almost surely (see also [35] for a similar result concerning 1D first order nonlinear Hamilton-Jacobi equations). We thus derived the equality of the two generalized principal eigenvalues, from which the existence of an exact spreading speed followed for Fisher-KPP equations with random stationary ergodic diffusion and reaction terms. This result was obtained using different definitions for the generalized principal eigenvalues than in the present paper. Namely, in [19] we only asked the test-functions defining the generalized principal eigenvalues in Definition 2.1 to satisfy a sub-exponential growth at infinity $\lim_{|x|\to+\infty} \frac{1}{|x|}\ln\phi(x) = 0$, which is of course less restrictive than asking $\phi \in L^\infty$ and $\inf\phi > 0$. Unfortunately, in the present paper we were not able to construct exact eigenfunctions with sub-exponential growth at infinity in dimension N, since the method we used in [19] relied on one-dimensional arguments.

The introduction of a "metric problem" formulation by Armstrong and co-authors [2, 3] allowed for a new approach in homogenization theory. This "metric problem" provides an exact corrector in $\mathbb{R}^N \setminus B_1$. Our point of view bear some similarities with this approach in that our approximate correctors are only required to satisfy the equation in truncated cones $C_{R,\alpha}(e)$. The methods developed in [2, 3] might provide a path towards the construction of exact correctors. We leave these possible extensions as open problems.

As far as we know, homogenization results for (62) have never been investigated when the dependence of H with respect to x is general. Indeed, it is not possible to prove that the family $(Z_\varepsilon)_{\varepsilon>0}$ converges in general (see Proposition 13.1 above for example). The recent papers [59, 60, 67, 88] addressing this question focused on random stationary ergodic Hamiltonians H, but not all deterministic equations could be transformed into a relevant random stationary ergodic one, as already described in Section 1.1.

Thus, it is only possible to obtain bounds on the spreading speeds $w_*(e)$ and $w^*(e)$ for a general heterogeneous equation. Of course, we aim at constructing bounds as precisely as possible. In particular we identify some classes of equations where our bounds give $w_*(e) = w^*(e)$. Indeed, we show that this identity holds when the coefficients are periodic, almost periodic, asymptotically almost periodic and radially periodic. In these cases, the notions of generalized principal eigenvalues and approximate correctors are exactly the same since then we show that $\underline{\lambda}_1(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N) = \overline{\lambda}_1(\mathcal{L}, \mathbb{R}\times\mathbb{R}^N)$. But for other types of media, the two notions may differ.

Second, trying to find optimal bounds on the spreading speeds, we prove in the present paper that only what happens in the truncated cones $C_{R,\alpha}(e)$ enters into account in the computations of the propagation sets $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$ which give our bounds on the spreading speeds. These types of properties cannot be obtained using former homogenization techniques since the approximate correctors are global over $\mathbb{R}\times\mathbb{R}^N$ and do not take into account the direction of propagation. This enables us to handle the case of directionally homogeneous coefficients. Indeed, this very simple example lead us to a striking phenomenon: the expansion set we construct is not obtained through a Wulff-type construction like (8). Indeed, it is even possible to construct non-convex expansion sets as we have observed above (see the discussion following Proposition 3.15).

5.2 The WKB change of variables

We will now reformulate our problem by using the link between asymptotic spreading and homogenization described above. Define $v_\varepsilon(t,x) := u(t/\varepsilon, x/\varepsilon)$. In order to investigate the behavior of this function as $\varepsilon \to 0$, let introduce the WKB change of variables

$$Z_\varepsilon = \varepsilon \ln v_\varepsilon. \tag{67}$$

The first step of our proof relies on the classical half-limits method, developed in [6, 7, 55, 69]. Define

$$Z_*(t,x) := \liminf_{(s,y)\to(t,x),\varepsilon\to 0} Z_\varepsilon(s,y) \quad \text{and} \quad Z^*(t,x) := \limsup_{(s,y)\to(t,x),\varepsilon\to 0} Z_\varepsilon(s,y) \tag{68}$$

and let show that these functions are respectively super and subsolutions of some Hamilton-Jacobi equations.

Of course the general heterogeneity of the coefficients generates many new difficulties. As Z_ε satisfies an equation with oscillating coefficients depending on $(t/\varepsilon, x/\varepsilon)$, we need to identify approximate correctors, which will indeed be constructed through general principal eigenvalues. We refer to our previous one-dimensional work [19] (Section B) for a review on these difficulties and on the ways to overcome them. Here, in addition to these difficulties, we have to deal with dimension N in the present paper, unlike in [19]. The main change it induces is that we cannot always explicitly solve the upcoming Hamilton-Jacobi equations satisfied by Z_* and Z^*. This is why integral minimization problems will come up in the definitions of the expansion sets. This is not only a technical difficulty: this reflects, somehow, new multi-dimensional strategies of propagation for the population u, as observed in Propositions 3.13, 3.14 and 3.15.

Lemma 5.1 *The family $(Z_\varepsilon)_{\varepsilon>0}$ satisfies the following properties:*

1. *For all compact set $Q \subset (0,\infty)\times\mathbb{R}^N$, there exists a constant $C = C(Q)$ and $\varepsilon_0 = \varepsilon_0(Q)$ such that $|Z_\varepsilon(t,x)| \leq C$ for all $0 < \varepsilon < \varepsilon_0$ and $(t,x) \in Q$.*
2. *For all $t > 0$, one has $Z^*(t,0) = Z_*(t,0) = 0$.*
3. *Z_* is lower semicontinous and Z^* is upper semicontinuous.*

Note that assertion 1. yields that Z^* and Z_* are well-defined on $(0,\infty)\times\mathbb{R}^N$.

Proof. This Lemma is proved exactly as Lemma 4.1 of [19].

1. Take $s \geq 0$ such that for all $(t,x) \in Q$, one has $s \leq t$. We can assume that $(s,0) \in Q$. As $Z_\varepsilon(t,x) = \varepsilon \ln u(t/\varepsilon, x/\varepsilon)$, the Krylov-Safonov-Harnack inequality gives the existence of a constant $C > 0$ such that for all $(t,x,s,y) \in Q\times Q$ with $s < t$ and $\varepsilon > 0$, one has

$$|Z_\varepsilon(t,x) - Z_\varepsilon(s,y)| \leq C\Big(\frac{|x-y|^2}{t-s} + t - s + \varepsilon\Big). \tag{69}$$

Then for all $(t,x) \in Q$, (69) gives

$$|Z_\varepsilon(t,x)| \leq |Z_\varepsilon(s,0)| + C\Big(\frac{|x|^2}{t-s} + t - s + \varepsilon\Big). \tag{70}$$

As $Z^*(t,0) = Z_*(t,0) = 0$ by step 2. below, and Q is compact, the right hand-side of this inequality is bounded when ε is small enough.

2. We know from [13] that, by (15), there exists $w > 0$ such that

$$\lim_{t\to+\infty} \inf_{|x|\leq wt} u(t,x) = 1.$$

Take t_0 such that $\inf_{|x|\leq wt} u(t,x) \geq 1/2$ for all $t \geq t_0$. Consider now $t > 0$ and a sequence $(s_n, y_n) \in \mathbb{R}^+ \times \mathbb{R}^N$ such that $s_n \to t$ and $y_n \to 0$ as $n \to +\infty$. Thus $|y_n|/s_n \leq w$ and $s_n/\varepsilon \geq t_0$ when n is large and ε is small. This yields

$$0 \geq Z^\varepsilon(s_n, y_n) = \varepsilon \ln u(\frac{s_n}{\varepsilon}, \frac{y_n}{\varepsilon}) \geq \varepsilon \ln \inf_{|x|\leq ws_n/\varepsilon} u(\frac{s_n}{\varepsilon}, x) \geq -\varepsilon \ln 2 \to 0 \text{ as } \varepsilon \to 0.$$

Thus $Z^*(t,0) = Z_*(t,0) = 0$.

3. This immediately follows from the definition of Z_* and Z^*. □

Similarly, the extension to dimension N of the following lemma, which gives the link between the sign of Z_*, Z^* and the convergence of v_ε as $\varepsilon \to 0$, is straightforward.

Lemma 5.2 *The following convergence holds as $\varepsilon \to 0$:*

$$v_\varepsilon(t,x) \to \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right. \textit{locally uniformly in} \left\{ \begin{array}{c} \mathrm{int}\{Z_* = 0\}, \\ \{Z^* < 0\}. \end{array} \right. \tag{71}$$

Proof. We use the same arguments as in the proof of Lemma 4.2 in [19].

1. First, as $v_\varepsilon(t,x) = e^{Z_\varepsilon(t,x)/\varepsilon}$, one has $v_\varepsilon(t,x) \to 0$ as $\varepsilon \to 0$ locally uniformly with respect to (t,x) such that $Z^*(t,x) < 0$.

2. Take $(t_0,x_0) \in \mathrm{int}\{Z_* = 0\}$. As $u(t,x) \to 1$ as $t \to +\infty$ locally in x, one has $v_\varepsilon(t,0) \to 1$ as $\varepsilon \to 0$ for all $t > 0$. We thus exclude the case $x_0 = 0$. One has $Z^\varepsilon(t,x) \to 0$ as $\varepsilon \to 0$ uniformly in the neighborhood of (t_0,x_0). Define

$$\phi(t,x) = -|x - x_0|^2 - |t - t_0|^2.$$

As $Z_* = 0$ in the neighborhood of (t_0,x_0) and ϕ is nonpositive, the function $Z^\varepsilon - \phi$ reaches a minimum at a point $(t_\varepsilon, x_\varepsilon)$, with $(t_\varepsilon, x_\varepsilon) \to (t_0,x_0)$ as $\varepsilon \to 0$. Thus, the equation on Z_ε gives

$$\partial_t \phi - \varepsilon tr(A\nabla^2\phi) - \nabla\phi A \nabla\phi - q\cdot\nabla\phi - (v_\varepsilon)^{-1} f(t_\varepsilon/\varepsilon, x_\varepsilon/\varepsilon, v_\varepsilon) \geq 0,$$

where the derivatives of ϕ and v_ε are evaluated at $(t_\varepsilon, x_\varepsilon)$ and A and q are evaluated at $(t_\varepsilon/\varepsilon, x_\varepsilon/\varepsilon)$. An explicit computation of the left hand-side gives

$$(v_\varepsilon)^{-1} f(t_\varepsilon/\varepsilon, x_\varepsilon/\varepsilon, v_\varepsilon(t_\varepsilon,x_\varepsilon)) \leq o(1) \text{ at } (t_\varepsilon,x_\varepsilon) \text{ as } \varepsilon \to 0.$$

As f is of class $\mathcal{C}^{1+\gamma}$ with respect to s uniformly in (t,x), there exists $C > 0$ such that for all $(t,x,u) \in \mathbb{R} \times \mathbb{R}^N \times [0,1]$,

$$f(t,x,u) \geq f_s'(t,x,0)u - Cu^{1+\gamma}.$$

This gives

$$f_s'(t_\varepsilon/\varepsilon, x_\varepsilon/\varepsilon, 0) \leq Cv_\varepsilon(t_\varepsilon,x_\varepsilon)^\gamma + o(1) \text{ as } \varepsilon \to 0.$$

Lastly, hypothesis (16) together with $t_0 \neq 0$ and $x_0 \neq 0$ give

$$\liminf_{\varepsilon\to 0} f_s'(t_\varepsilon/\varepsilon, x_\varepsilon/\varepsilon, 0) > 0.$$

Thus $\liminf_{\varepsilon\to 0} v_\varepsilon(t_\varepsilon,x_\varepsilon) > 0$. Since all the above are clearly uniform for any compact subset of $\mathrm{int}\{Z_* = 0\}$, we have actually established

$$\inf_{(t,x)\in K} \liminf_{\varepsilon\to 0} v_\varepsilon(t,x) \geq \alpha > 0,$$

uniformly on any compact subset $K \subset \text{int}\{Z_* = 0\}$, for some $\alpha = \alpha(K)$.

3. Recall now that $v_\varepsilon(t,x) = u(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$, where u solves

$$\begin{cases} \partial_t u - tr(A(t,x)\nabla^2 u) - q(t,x)\cdot\nabla u = f(t,x,u) \text{ in } (0,\infty)\times\mathbb{R}^N, \\ u(0,x) = u_0(x) \text{ in } \mathbb{R}^N. \end{cases} \tag{72}$$

Take some sequences $(\varepsilon_n)_n$ and $(s_n, y_n) \in K$ such that $\varepsilon_n \to 0$ and

$$|u(\frac{s_n}{\varepsilon_n}, \frac{y_n}{\varepsilon_n}) - 1| \to \limsup_{\varepsilon\to 0} \sup_{(t,x)\in K} |u(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}) - 1|.$$

Set $t_n = \frac{s_n}{\varepsilon_n}$, $x_n = \frac{y_n}{\varepsilon_n}$ and $u_n(t,x) = u(t+t_n, x+x_n)$. This function satisfies:

$$\begin{aligned} &\partial_t u_n - tr(A(t+t_n, x+x_n)\nabla^2 u_n) - q(t+t_n, x+x_n)\cdot\nabla u_n \\ &= f(t+t_n, x+x_n, u_n) \text{ in } (-t_n, +\infty)\times\mathbb{R}^N. \end{aligned} \tag{73}$$

As K is a compact set, one may assume that $y_n \to y_\infty$ and $s_n \to s_\infty$, with $(s_\infty, y_\infty) \in K$. As $(0,x) \notin \{Z_* = 0\}$ for all $x \neq 0$, we know that $(0,x) \notin \text{int}\{Z_* = 0\}$ for all x and thus $s_\infty \neq 0$. Finally, $t_n \to +\infty$.

Up to some extraction, one may assume, as the coefficients are uniformly continuous over $\mathbb{R}\times\mathbb{R}^N$, that there exists some function (B, r, g) such that $A(t+t_n, x+x_n) \to B(t,x)$ and $q(t+t_n, x+x_n) \to r(t,x)$ locally uniformly in $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ and $f(t+t_n, x+x_n, s) \to g(t,x,s)$ locally uniformly in $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ and uniformly in $s \in [0,1]$.

Next, the parabolic regularity estimates yield that the sequence $(u_n)_n$ converges, up to some extraction, to some function u_∞ in $\mathcal{C}(\mathbb{R}\times\mathbb{R}^N)$ locally uniformly in $\mathbb{R}\times\mathbb{R}^N$. This function is a viscosity solution of

$$\partial_t u_\infty - tr(B(t,x)\nabla^2 u_\infty) - r(t,x)\cdot\nabla u_\infty = g(t,x,u_\infty) \text{ in } \mathbb{R}\times\mathbb{R}^N. \tag{74}$$

Consider a compact subset $K \subset \text{int}\{Z_* = 0\}$. Consider some $\delta > 0$ such that $K_\delta := K + \overline{B}_\delta(0) \subset \text{int}\{Z_* = 0\}$, where $\overline{B}_\delta(0)$ is the closed ball of radius δ and center 0 in $\mathbb{R}\times\mathbb{R}^N$. Consider some $\alpha > 0$ such that

$$\inf_{(t,x)\in K_\delta} \liminf_{\varepsilon\to 0} v_\varepsilon(t,x) \geq \alpha.$$

Take any $(t,x) \in (0,\infty)\times\mathbb{R}^N$ and n large enough so that $(\varepsilon_n t, \varepsilon_n x) \in \overline{B}_\delta(0)$. Then $(\varepsilon_n t + s_n, \varepsilon_n x + y_n) \in K_\delta$ and thus

$$u_n(t,x) = u(t+t_n, x+x_n) = v_{\varepsilon_n}(\varepsilon_n t + s_n, \varepsilon_n x + y_n) \geq \alpha,$$

when n is large. Thus $u_\infty(t,x) \geq \alpha$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$. Moreover, one has $u_\infty \leq 1$. Assume that $m = \inf_{\mathbb{R}\times\mathbb{R}^N} u_\infty < 1$. If this infimum is reached, consider (t_0, x_0) such that $u_\infty(t_0, x_0) = m$. Then as u_∞ is a viscosity solution of (74), one has $g(t_0, x_0, m) \leq 0$. But (12) gives $\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^N} f(t,x,m) > 0$ since $m \in (0,1)$. Hence, $g(t_0, x_0, m) > 0$, which is a contradiction.

Otherwise, consider a sequence $(t_k, x_k)_k$ such that $u_\infty(t_k, x_k) \to m$. As A, q and f are uniformly continuous in (t, x), B, r and g are also uniformly continuous and thus one can assume that $(B(\cdot + t_k, \cdot + x_k))_k$, $(r(\cdot + t_k, \cdot + x_k))_k$ and $(g(\cdot + t_k, \cdot + x_k, s))_k$ converge as $k \to +\infty$ locally uniformly in $(t, x, s) \in \mathbb{R} \times \mathbb{R}^N \times [0, 1]$. Thus $(u_\infty(\cdot + t_k, \cdot + x_k)_k$ converges to some solution of a parabolic equation that reaches its minimum m in $(0, 0)$. The same arguments as above lead to the contradiction.

This proves that

$$\limsup_{\varepsilon\to 0} \sup_{(t,x)\in K} |v_\varepsilon(t, x) - 1| = \lim_{n\to+\infty} |u(t_n, x_n) - 1| = 0.$$

□

5.3 The equations on Z_* and Z^*

We will now pass to the limit $\varepsilon \to 0$ in the equation satisfied by Z_ε:

$$\begin{cases} \partial_t Z_\varepsilon - \varepsilon tr(A(t/\varepsilon, x/\varepsilon)\nabla^2 Z_\varepsilon) - \nabla Z_\varepsilon A(t/\varepsilon, x/\varepsilon)\nabla Z_\varepsilon - q(t/\varepsilon, x/\varepsilon) \cdot \nabla Z_\varepsilon \\ = \frac{1}{v_\varepsilon} f(t/\varepsilon, x/\varepsilon, v_\varepsilon) \quad \text{in } (0, \infty) \times \mathbb{R}^N, \\ Z_\varepsilon(0, x) = \varepsilon \ln u_0(x/\varepsilon) \quad \text{if } x \in \varepsilon \text{ int}(\text{Supp} u_0), \\ \lim_{t\to 0^+} Z_\varepsilon(t, x) = -\infty \quad \text{if } x \notin \varepsilon \text{ int}(\text{Supp} u_0). \end{cases} \tag{75}$$

Proposition 5.3 *The functions Z^* and Z_* are discontinuous viscosity solutions of*

$$\begin{cases} \max\{\partial_t Z_* - \underline{H}(\frac{x}{|x|}, \nabla Z_*), Z_*\} \geq 0 \textit{ in } (0, \infty) \times \mathbb{R}^N \backslash \{0\}, \\ \max\{\partial_t Z^* - \overline{H}(\frac{x}{|x|}, \nabla Z^*), Z^*\} \leq 0 \textit{ in } (0, \infty) \times \mathbb{R}^N \backslash \{0\}, \\ Z^*(t, 0) = Z_*(t, 0) = 0 \textit{ for all } t > 0, \\ \lim_{t\to 0^+} Z_*(t, x) = \lim_{t\to 0^+} Z^*(t, x) = 0 \textit{ if } x = 0, \ -\infty \textit{ if } x \neq 0, \textit{ unif. with respect to } |x|. \end{cases} \tag{76}$$

The initial condition at $t = 0$ means that for all $r > 0$, one has

$$\lim_{t\to 0^+} \sup_{|x|=r} Z_*(t, x) = \lim_{t\to 0^+} \sup_{|x|=r} Z^*(t, x) = -\infty.$$

The proof will follow the same lines as that of Proposition 4.3 in [19] (which was itself inspired by [38, 69]). We underline that in [19], we were only dealing with Z_*, since $\overline{w}$ was constructed through direct arguments (see Section IV.A in [19]). Here we expect a more involved characterization of $\overline{\mathcal{S}}$ (28) and thus a direct proof as in [19] is unlikely. We thus have to work on Z^*. Indeed, the derivation of the equations on Z^* and Z_* are not similar, due in particular to the singular initial datum, and we thus need to provide some extra-arguments with respect to [19]. Moreover, we need to check that only what happens in the truncated cones $C_{R,\alpha}(e)$ needs to be taken into account, which is a new difficulty compared with our previous one-dimensional paper [19].

Proof.

1. We already know that $Z^*(t,x) \le 0$ for all (t,x). Take $T>0$ and a smooth test function χ and assume that $Z^*-\chi$ admits a strict maximum at some point $(t_0,x_0) \in (0,T]\times(\mathbb{R}^N\backslash\{0\})$ over the ball $\overline{B_r} := \{(t,x) \in (0,T] \times (\mathbb{R}^N\backslash\{0\}), |t-t_0| + |x-x_0| \le r\}$. Define $e = x_0/|x_0|$ and $p = \nabla\chi(t_0,x_0)$.

Take $R>0$ and $\alpha \in (0,1)$. Consider a function $\psi \in \mathcal{C}^{1,2}\big(C_{R,\alpha}(e)\big) \cap W^{1,\infty}\big(C_{R,\alpha}(e)\big)$ such that $\inf_{C_{R,\alpha}(e)} \psi > 0$ and $\Big(L_p - \overline{\lambda_1}\big(L_p, C_{R,\alpha}(e)\big)\Big)\psi \le \mu\psi$. Let $w = \ln\psi$, this function satisfies over $C_{R,\alpha}(e)$:

$$\partial_t w - a_{i,j}\big(\partial_{ij} w + (\partial_i w + p_i)(\partial_j w + p_j)\big) - q_i(\partial_i w + p_i) \ge f'_u(t,x,0) - \overline{\lambda_1}(L_p, C_{R,\alpha}(e)) - \mu. \quad (77)$$

Moreover, one has

$$\varepsilon w(t/\varepsilon, x/\varepsilon) \to 0 \text{ as } \varepsilon \to 0 \text{ locally in } (t,x) \in C_{R,\alpha}(e) \text{ since} w \text{ is bounded.} \quad (78)$$

Take a sequence $(\varepsilon_n)_n$ such that $\lim_{n\to+\infty} \varepsilon_n = 0$. Using the same arguments as in [19], one can prove that the definition of Z^* yields the existence of two sequences $(t_n)_n$ and $(x_n)_n$ such that

$$\begin{array}{l} Z^{\varepsilon_n}(t_n,x_n) \to Z^*(t_0,x_0), \\ (t_n,x_n) \to (t_0,x_0) \text{ as } n \to +\infty, \\ Z^{\varepsilon_n} - \chi - \varepsilon_n w(\cdot/\varepsilon_n, \cdot/\varepsilon_n) \text{ reaches a local maximum at } (t_n,x_n). \end{array} \quad (79)$$

As $t_0 \ne 0$ and $x_0 \ne 0$, one has $t_n/\varepsilon_n \to +\infty$ and $|x_n|/\varepsilon_n \to +\infty$. Moreover, $\frac{x_n}{|x_n|} - e \to 0$ as $n \to +\infty$.

2. Take n large enough so that $(t_n/\varepsilon_n, x_n/\varepsilon_n) \in C_{R,\alpha}(e)$. As $Z^{\varepsilon_n} - \big(\chi + \varepsilon_n w(\frac{\cdot}{\varepsilon_n}, \frac{\cdot}{\varepsilon_n})\big)$ reaches a local maximum in (t_n,x_n), we get:

$$\begin{array}{l} \partial_t\chi + \partial_t w - \partial_t Z_{\varepsilon_n} - \varepsilon_n tr(A(\nabla^2\chi + \varepsilon_n^{-1}\nabla^2 w - \nabla^2 Z_{\varepsilon_n})) \\ -(\nabla\chi + \nabla w - \nabla Z_{\varepsilon_n})A(\nabla\chi + \nabla w + \nabla Z_{\varepsilon_n}) - q\cdot(\nabla\chi + \nabla w - \nabla Z_{\varepsilon_n}) \le 0, \end{array} \quad (80)$$

where the derivatives of χ and Z_{ε_n} are evaluated at (t_n,x_n), A, q and the derivatives of w are evaluated at $(t_n/\varepsilon_n, x_n/\varepsilon_n)$. Using our KPP hypothesis (13) and the equation (75) satisfied by Z_ε, we get

$$\begin{array}{l} \partial_t\chi + \partial_t w - tr(A(\varepsilon_n\nabla^2\chi + \nabla^2 w)) - (\nabla\chi + \nabla w)A(\nabla\chi + \nabla w) - q\cdot(\nabla\chi + \nabla w) \\ \le f'_u(t_n/\varepsilon_n, x_n/\varepsilon_n, 0), \end{array}$$

where the derivatives of χ are evaluated at (t_n,x_n) and A, q and the derivatives of w are evaluated at $(t_n/\varepsilon_n, x_n/\varepsilon_n)$. Using (77) and the ellipticity property (14), this gives

$$\begin{array}{l} \partial_t\chi - \overline{\lambda_1}(L_p, C_{R,\alpha}(e)) \\ \le \mu + \varepsilon_n tr(A\nabla^2\chi) + q\cdot(\nabla\chi - p) + \Gamma|\nabla\chi - p|^2 + 2\Gamma|\nabla\chi - p||\nabla w + p|, \end{array}$$

where we remind to the reader that $p = \nabla\chi(t_0,x_0)$. Letting $n \to +\infty$ and $\mu \to 0$, this leads to $\partial_t\chi(t_0,x_0) - \overline{\lambda_1}(L_p, C_{R,\alpha}(e)) \le 0$.

Finally, letting $R \to +\infty$ and $\alpha \to 0$, the stability theorem for Hamilton-Jacobi equations (see for example Remark 6.2 in [33]) yields that:

$$\max\{\partial_t Z^* - \overline{H}(e, \nabla Z^*), Z^*\} \leq 0 \text{ in } (0,\infty) \times (\mathbb{R}^N \backslash \{0\}) \tag{81}$$

in the sense of viscosity solutions.

3. We next verify that the initial condition is satisfied. We first claim that if $\rho \in \mathcal{C}^\infty(\mathbb{R}^N)$ is such that $\rho(x) = 0$ if $x = 0$ and $\rho(x) > 0$ if $x \neq 0$, then

$$\min\Big\{\partial_t Z^* - \overline{H}\big(\frac{x}{|x|}, \nabla Z^*\big), Z^* + \rho\Big\} \leq 0 \text{ in } \{0\} \times (\mathbb{R}^N \backslash \{0\}). \tag{82}$$

In order to prove this variational inequality, consider some smooth test function χ such that $Z^* - \chi$ admits a strict local maximum at some point $(0, x_0)$. If $x_0 = 0$, then $\lim_{t\to 0^+} Z^*(t, x_0) + \rho(x_0) = 0$ is clearly true by Lemma 5.1.

Assume that $x_0 \neq 0$ and that $\lim_{t\to 0^+} Z^*(t, x_0) > -\rho(x_0)$. We need to prove that

$$\partial_t \chi(0, x_0) - \overline{H}\big(\frac{x_0}{|x_0|}, \nabla\chi(0, x_0)\big) \leq 0.$$

This can be done as previously by noting that since $Z^{\varepsilon_n}(0, x) = -\infty$ for all x near x_0 when ε_n is small enough, the points (t_n, x_n) above lie in $(0,\infty) \times \mathbb{R}^N$. Then the maximum principle argument leading to (80) is valid and (82) follows.

4. Clearly $Z^\varepsilon(0, 0) = \varepsilon \ln u_0(0)$ converges to 0 as ε goes to 0 and thus $\lim_{t\to 0^+} Z^*(t, 0) = 0$. Assume now that there exists $r > 0$ such that $\limsup_{t\to 0^+} \sup_{|x|=r} Z^*(t, x) > -\infty$. Take $\delta > 0$ and define

$$\chi^\delta(t, x) = \delta^{-1}(|x| - r)^2 + \lambda t,$$

where λ will be fixed later. As Z^* is upper semicontinuous and bounded from above, we know that $Z^* - \chi^\delta$ admits a maximum at a point $(t_\delta, x_\delta) \in [0,\infty) \times \mathbb{R}^N$ and that $x_\delta \neq 0$ when δ is sufficiently small.

Assume that $t_\delta > 0$. Then we know from (81) that

$$\partial_t \chi^\delta(t_\delta, x_\delta) - \overline{H}\Big(\frac{x_\delta}{|x_\delta|}, \nabla\chi^\delta(t_\delta, x_\delta)\Big) = \lambda - \overline{H}\Big(\frac{x_\delta}{|x_\delta|}, 2\delta^{-1}(|x_\delta| - r)\frac{x_\delta}{|x_\delta|}\Big) \leq 0.$$

On the other hand, one has for all x so that $|x| = r$,

$$\limsup_{t\to 0^+} Z^*(t, x) = \limsup_{t\to 0^+} \big(Z^*(t, x) - \chi^\delta(t, x)\big) \leq (Z^* - \chi^\delta)(t_\delta, x_\delta) \leq -\delta^{-1}(|x_\delta| - r)^2. \tag{83}$$

Thus we get from Proposition 2.2 that

$$\lambda \leq \overline{H}\Big(\frac{x_\delta}{|x_\delta|}, 2\delta^{-1}(|x_\delta| - r)\frac{x_\delta}{|x_\delta|}\Big) \leq C(1 + 4\delta^{-2}(|x_\delta| - r)^2) \leq C(1 - 4\delta^{-1} \limsup_{t\to 0^+} Z^*(t, x)). \tag{84}$$

This contradicts $\limsup_{t\to 0^+} \sup_{|x|=r} Z^*(t, x) > -\infty$ by taking $\lambda > 0$ large enough. Thus $t_\delta = 0$.

Consider a smooth radial function $\rho = \rho(|x|)$ so that $\rho(0) = 0$ and $\rho(r) > 0$ if $r > 0$. If $\lim_{t\to 0^+} \sup_{|x|=r} Z^*(t,x) > -\rho(r)$, then we know from (83) that one can find δ small enough so that $Z^*(0, x_\delta) > -\rho(x_\delta)$. But then (82) would lead to (84) and give a contradiction.

Thus $\lim_{t\to 0^+} \sup_{|x|=r} Z^*(t,x) \le -\rho(r)$. But as ρ is arbitrary in $r > 0$, this gives a contradiction.

5. The equation on Z_* could be derived from the same arguments as in the proof of Proposition 4.3 in [19], the arguments above ensuring that only what happens in $C_{R,\alpha}(e)$ is involved and thus that the corrector $\underline{H}(e,p)$ naturally emerges in the inequation on Z_*. $\square$

5.4 Estimates on Z_* and Z^* through some integral minimization problem

We first obtain comparisons with the solutions of Hamilton-Jacobi equations with continuous Hamiltonians H.

Proposition 5.4 *Assume that $H = H(x,p)$ is a Lipschitz-continuous function over $\mathbb{R}^N \times \mathbb{R}^N$, convex in p, such that $\overline{H}\big(\frac{x}{|x|}, p\big) \le H(x,p) \le C(1+|p|^2)$ for all $(x,p) \in (\mathbb{R}^N \backslash \{0\}) \times \mathbb{R}^N$ and for some given $C > 0$. Then*

$$-Z^*(t,x) \ge \inf \max_{a\in[0,t]} \Big\{ \int_0^a H^\star\big(\gamma(s), \gamma'(s)\big) ds, \quad \gamma(0) = x,\ \gamma(t) = 0 \Big\} \tag{85}$$

where $H^\star(e,q) := \sup_{p\in\mathbb{R}^N} \big(p\cdot q - H(e,p)\big)$.

The two difficulties here are the unboundedness of the domain $\mathbb{R}^N$ and the singular initial datum. For all $t > 0$, the functions $Z^*(t,\cdot)$ and $Z_*(t,\cdot)$ stay unbounded and thus one cannot directly apply classical doubling of variables method. We will thus compare the solutions with solutions of problems in bounded domains with smooth initial data, for which comparison results have been proved by Evans and Souganidis in [38].

Proof. We use the same approach as in Lemma 3.1 of [38] to prove this result. Hence we will just sketch the proof and focus on the differences with [38].

Consider a smooth function η such that $\eta(0) = 0$ and $0 > \eta(x) \ge -1$ for all $x \ne 0$. Let Z_k the solution of

$$\begin{cases} \max\{\partial_t Z^k - H(x, \nabla Z^k), Z^k\} = 0 \text{ in } (0,\infty)\times\mathbb{R}^N, \\ Z^k(0,x) = k\eta(x) \text{ for all } x \in \mathbb{R}^N, \end{cases} \tag{86}$$

which is a bounded and uniformly continuous function. Clearly, Z^* is a subsolution of equation (86).

Let u_k^ε the solution of the Cauchy problem (1) with initial datum $u_k^\varepsilon(0,x) = u_0(x) + e^{k\eta(\varepsilon x)/\varepsilon}$. The parabolic maximum principle yields $u(t,x) \le u_k^\varepsilon(t,x)$ for all $(t,x) \in (0,\infty)\times\mathbb{R}^N$ and thus $Z_\varepsilon(t,x) \le \varepsilon \ln u_k^\varepsilon(t/\varepsilon, x/\varepsilon)$. We could thus pass to the upper half-limit in this inequality: $Z^*(t,x) \le Y_k^*(t,x)$, where

$$Y_k^*(t,x) := \limsup_{(s,y)\to(t,x),\varepsilon\to 0} \varepsilon \ln u_k^\varepsilon(t/\varepsilon, x/\varepsilon). \tag{87}$$

The same arguments as in the proof of Proposition 5.3 yield that Y_k^* satisfies

$$\begin{cases} \max\{\partial_t Y_k^* - \overline{H}(x/|x|, \nabla Y_k^*), Y_k^*\} \le 0 \text{ in } (0,\infty)\times(\mathbb{R}^N\backslash\{0\}, \\ Y_k^*(t,0) = 0 \quad \text{ for all } t>0, \\ Y_k^*(0,x) = k\eta(x) \quad \text{ for all } x \in \mathbb{R}^N. \end{cases} \tag{88}$$

As $\overline{H} \le H$, Y_k^* is a subsolution of (86). Moreover, as $\eta \ge -1$, one has $u_k^\varepsilon(0,x) \ge e^{-k/\varepsilon}$ and thus $u_k^\varepsilon(t,x) \ge e^{-k/\varepsilon}$ for all $(t,x)\in[0,\infty)\times\mathbb{R}^N$ for $\varepsilon>0$ small enough since the positivity of f (12) implies that constants are subsolutions of (1). This eventually implies $Y_k^*(t,x) \ge -k$ for all $(t,x)\in[0,\infty)\times\mathbb{R}^N$. Hence, as Y_k^* and Z^k are bounded, we can adapt the doubling of variables argument of Theorem B.1 of [38] in order to obtain the comparison $Y_k^* \le Z^k$. We have thus proved $Z^*(t,x) \le Z^k(t,x)$ for all $(t,x)\in[0,\infty)\times\mathbb{R}^N$. The representation formula proved in Theorem D.1 of [38] yields

$$-Z^k(t,x) = \sup_{\theta\in\Theta}\inf\Big\{\int_0^{t\wedge\theta[\gamma(\cdot)]} H^\star\big(\gamma(s),\gamma'(s)\big)ds - 1_{\theta[\gamma(\cdot)]\ge t}k\eta\big(\gamma(t)\big), \quad \gamma(0)=x\Big\},$$

where Θ is the set of all stopping times (see [38]) and $\gamma\in H^1(0,t)$. In fact, the arguments of Lemma 2.4 in [43] yield that one can replace this expression by

$$-Z^k(t,x) = \inf\max_{a\in[0,t]}\Big\{\int_0^a H^\star\big(\gamma(s),\gamma'(s)\big)ds - 1_{a=t}k\eta\big(\gamma(t)\big), \quad \gamma(0)=x\Big\}. \tag{89}$$

We now pass to the limit $k\to+\infty$. The right hand-side in (89) is clearly nondecreasing since $\eta\le 0$. As $Z^*\le Z^k$, one can take a sequence $(\gamma_k)_k$ in $H^1(0,t)$ such that $\gamma_k(0)=x$ for all k and

$$-Z^*(t,x) \ge \max_{a\in[0,t]}\Big\{\int_0^a H^\star\big(\gamma_k(s),\gamma_k'(s)\big)ds - 1_{a=t}k\eta\big(\gamma_k(t)\big)\Big\} - 1/k.$$

As $H(x,p)\le C(1+|p|^2)$ for all $x,p\in\mathbb{R}^N$, one has $H^\star(x,q)\ge\frac{|q|^2}{4C}-C$, and we get $\forall a<t$, $\int_0^a|\gamma_k'(s)|^2ds \le 4C\big(Ct - Z^*(t,x)+1/k\big)$. Hence, as $\gamma_k(0)=x$ for all k, $(\gamma_k)_k$ is bounded in $H^1(0,t)$ and we can assume that this sequence converges weakly to a function γ such that $\gamma(0)=x$. It follows from the estimates above that $k\eta\big(\gamma_k(t)\big)$ is bounded from below by a constant independent of k, which implies that $\gamma(t)=0$. We could thus pass to the limit in (89) and obtain (85). □

Proposition 5.5 *Assume that $H=H(x,p)$ is a Lipschitz-continuous function over $\mathbb{R}^N\times\mathbb{R}^N$ such that $\underline{H}\big(\frac{x}{|x|},p\big)\ge H(x,p)\ge c(1+|p|^2)$ for all $(x,p)\in(\mathbb{R}^N\backslash\{0\})\times\mathbb{R}^N$ and for some given $c>0$. Then*

$$-Z_*(t,x) \le \inf\max_{a\in[0,t]}\Big\{\int_a^t H^\star\big(\gamma(s),\gamma'(s)\big)ds, \quad \gamma(0)=x,\ \gamma(t)=0\Big\}. \tag{90}$$

Proof. Take $T>0$ and j large enough so that $1/j<T$. The same arguments as in the second part of the proof of Lemma 2.1 in [38] yield that Z_* is Lipschitz-continuous over $(1/j,T)\times B_j$, where B_j is the open ball of center 0, since the estimates in [38] only depend

on L^∞ and ellipticity bounds on the coefficients. Let $m_j := \min_{(t,x)\in(1/j,T)\times B_j}$ and M_j the Lipschitz constant of Z_* on $(1/j,T)\times B_j$.

Consider the equation:

$$\begin{cases} \max\{\partial_t Z - H(x,\nabla Z), Z\} = 0 \text{ in } (1/j,T)\times B_j, \\ Z(t,x) = \min\{m_j, -M_j j\} \quad \text{for all } t\in(1/j,T), x\in\partial B_j, \\ Z(1/j,x) = -M_j|x| \quad \text{for all } x\in B_j. \end{cases} \tag{91}$$

We know (see [34]) that this equation admits a unique bounded Lipschitz-continuous solution Z_j. Moreover, as H is above its convex envelope, Z_j is a supersolution of the equation associated with the convex envelope of H instead of H. Hence, Theorem D.2 of [38] applies:

$$Z_j(t,x) \geq -\sup_{\theta\in\Theta}\inf\Big\{ \int_0^{(t-1/j)\wedge\theta[\gamma(\cdot)]\wedge t_\gamma} H^\star\big(\gamma(s),\gamma'(s)\big)ds \\ -1_{(t-1/j)\wedge\theta[\gamma(\cdot)]\geq t_\gamma} Z_*(t-t_\gamma,\gamma(t_\gamma)) \\ -1_{t_\gamma\wedge\theta[\gamma(\cdot)]\geq t-1/j} Z_*(1/j,\gamma(t-1/j)), \quad \gamma(0)=x\Big\},$$

where $t_\gamma := \inf\{s\geq 0, \gamma(s)\in\partial B_j\}$ is the exit time from B_j. Moreover, as $\underline{H}\big(\frac{x}{|x|},p\big)\geq H(x,p)$ for all (x,p) and due to our choice of m_j and M_j, Z_* is a supersolution of (91) and thus $Z_*\geq Z_j$.

Considering only paths γ such that $\gamma(t-1/j)=0$ and $|\gamma(s)|<j$ for all $s\in(0,t-1/j)$, with j large enough so that $|x|<j$, as $Z_*(t,0)=0$ for all $t>0$, we get

$$Z_j(t,x) \geq -\sup_{\theta\in\Theta}\inf\Big\{\int_0^{(t-1/j)\wedge\theta[\gamma(\cdot)]} H^\star\big(\gamma(s),\gamma'(s)\big)ds, \quad \gamma(0)=x,\ \gamma(t-1/j)=0,\ |\gamma|<j\Big\}.$$

The alternative formulation derived from [43] reads

$$Z_*(t,x)\geq Z_j(t,x)\geq -\inf\max_{a\in[0,t-1/j]}\Big\{\int_0^a H^\star\big(\gamma(s),\gamma'(s)\big)ds, \quad \gamma(0)=x,\ \gamma(t-1/j)=0,\ |\gamma|<j\Big\}.$$

For a given path $\gamma\in H^1(0,t)$ such that $\gamma(0)=x$ and $\gamma(t)=0$, taking j large enough so that $\|\gamma\|_\infty<j$ and defining $\gamma_j(s) := \gamma\big(\frac{st}{t-1/j}\big)$, we get

$$Z_*(t,x)\geq Z_j(t,x)\geq -\max_{a\in[0,t-1/j]}\int_0^a H^\star\big(\gamma_j(s),\gamma_j'(s)\big)ds = -\frac{t-1/j}{t}\max_{a\in[0,t]}\int_0^a H^\star\Big(\gamma(s),\frac{t}{t-1/j}\gamma'(s)\Big)ds.$$

We conclude by letting $j\to+\infty$ and taking the sup over all possible paths γ. □

Proposition 5.6 *For all $x\neq 0$, one has*

$$\begin{aligned} Z^*(1,x) &\leq -\inf\Big\{\max_{t\in[0,1]}\int_t^1 \overline{H}^\star\big(\gamma(s),-\gamma'(s)\big),\ \gamma\in H^1([0,1]),\ \gamma(0)=0,\ \gamma(1)=x\Big\} = -\overline{U}(x), \\ Z_*(1,x) &\geq -\inf\Big\{\max_{t\in[0,1]}\int_t^1 \underline{H}^\star\big(\gamma(s),-\gamma'(s)\big),\ \gamma\in H^1([0,1]),\ \gamma(0)=0,\ \gamma(1)=x\Big\} = -\underline{U}(x), \end{aligned} \tag{92}$$

where we recall to the reader that $\underline{U}$ and $\overline{U}$ were introduced in (27).

Proof. We will extend in the sequel the Hamiltonians $\overline{H}$ and $\underline{H}$ by $1-$homogeneity: for all $x \neq 0$ and $p \in \mathbb{R}^N$, $\overline{H}(x,p) := \overline{H}(x/|x|,p)$ and $\underline{H}(x,p) := \underline{H}(x/|x|,p)$. We also define $\overline{H}(0,p) := 2C(1+|p|^2)$ and $\underline{H}(0,p) := c/2(1+|p|^2)$, where c and C are given by Proposition 2.2, so that $\overline{H}$ (resp. $\underline{H}$) is upper (resp. lower) semicontinuous over $\mathbb{R}^N$.

1. For all n, consider the sup-convolution of $\overline{H}$:

$$\overline{H}_n(x,p) := \sup_{x' \in \mathbb{R}^N} \left\{ \overline{H}(x',p) - n|x'-x|^2 \right\}.$$

The semicontinuity of $\overline{H}$ in x, its continuity and convexity in p, and its coercivity yields that $\overline{H}_n$ is well-defined, convex in p and locally Lipschitz-continuous in (x,p). Hence, Proposition 5.4 applies and gives (up to the change of variables $\widetilde{s} = 1-s$ and $\widetilde{\gamma}(s) = \gamma(1-s)$):

$$Z^*(1,x) \leq -\overline{U}_n(x) := \sup \min_{t\in[0,1]} \left\{ \int_t^1 -\overline{H}_n^\star\Big(\gamma(s), -\gamma'(s)\Big) ds,\ \gamma(0)=0,\ \gamma(1)=x \right\} \qquad (93)$$

where $\overline{H}_n^\star$ is the convex conjugate of $\overline{H}_n$ and $\gamma \in H^1(0,1)$.

2. We now take $x \in \mathbb{R}^N$ and let $n \to +\infty$. For all n, let γ_n an admissible test-function such that

$$-\overline{U}_n(x) \leq \min_{t\in[0,1]} \int_t^1 -\overline{H}_n^\star\Big(\gamma_n(s), -\gamma_n'(s))\Big) ds + \frac{1}{n}. \qquad (94)$$

We know from Proposition 2.2 that

$$\forall (x,p) \in \mathbb{R}^N \times \mathbb{R}^N, \quad c(|p|^2-1) \leq \overline{H}(x,p) \leq C(1+|p|^2),$$

from which we easily derive the same estimate for $\overline{H}_n$, and thus

$$\frac{|q|^2}{4C} - C \leq \overline{H}_n^\star(x,q) \leq \frac{|q|^2}{4c} + c$$

for all $(x,q) \in \mathbb{R}^N \times \mathbb{R}^N$. Together with (94), this leads to

$$\min_{t\in[0,1]} \int_t^1 \Big(C - \frac{|\gamma_n'(s)|^2}{4C} \Big) ds \geq -\overline{U}_n(x) - \frac{1}{n} \geq -\frac{|x|^2}{4c} - c - \frac{1}{n}.$$

In particular, taking $t=0$, (γ_n') is bounded in $L^2([0,1])$. As $\gamma_n(0) = 0$ and $\gamma_n(1) = x$ for all n, we get that $(\gamma_n)_n$ is bounded in $H^1(0,1)$ and thus one can assume that it converges weakly in $H^1([0,1])$ and locally uniformly to a function γ. It is a well-known property of sup-convolutions that, as $\lim_{n\to+\infty} \gamma_n(s) = \gamma(s)$, one has for all $p \in \mathbb{R}^N$ and $s \in [0,1]$:

$$\limsup_{n\to+\infty} \overline{H}_n\big(\gamma_n(s), p\big) \leq \overline{H}\big(\gamma(s), p\big).$$

On the other hand, for all $s \in [0,1]$, take $p(s) \in \mathbb{R}^N$ such that

$$-\overline{H}^\star\big(\gamma(s), -\gamma'(s)\big) = \inf_{p\in\mathbb{R}^N} \Big(\overline{H}\big(\gamma(s), p\big) + p \cdot \gamma'(s) \Big) = \overline{H}\big(\gamma(s), p(s)\big) + p(s) \cdot \gamma'(s).$$

It follows from Proposition 2.2 that

$$-\overline{H}^\star\big(\gamma(s),-\gamma'(s)\big) \geq p(s)\cdot\gamma'(s)+c|p(s)|^2-c \geq -\frac{c}{2}|p(s)|^2-\frac{|\gamma'(s)|^2}{2c}+c|p(s)|^2-c$$

and thus as $\gamma'\in L^2(0,1)$ this implies that $p\in L^2(0,1)$. We thus get

$$\lim_{n\to+\infty}\int_t^1 \gamma_n'(s)\cdot p(s)ds=\int_t^1 \gamma(s)\cdot p(s)ds$$

for all $t\in[0,1]$. Hence, one has

$$\begin{aligned}
&\textstyle\int_t^1\Big(\overline{H}\big(\gamma(s),p(s)\big)+p(s)\cdot\gamma'(s)\Big)ds\\
\geq\ &\textstyle\int_t^1 \limsup_{n\to+\infty}\overline{H}_n\big(\gamma_n(s),p(s)\big)ds+\lim_{n\to+\infty}\int_t^1 p(s)\cdot\gamma_n'(s)ds\\
\geq\ &\textstyle\limsup_{n\to+\infty}\int_t^1\Big(\overline{H}_n\big(\gamma_n(s),p(s)\big)+p(s)\cdot\gamma_n'(s)\Big)ds && \text{by Fatou's lemma}\\
\geq\ &\textstyle\limsup_{n\to+\infty}\int_t^1 -\overline{H}_n^\star\big(\gamma_n(s),-\gamma_n'(s)\big)ds && \text{by definition of } \overline{H}_n^\star\\
\geq\ &\textstyle\limsup_{n\to+\infty}\min_{t\in[0,1]}\int_t^1 -\overline{H}_n^\star\big(\gamma_n(s),-\gamma_n'(s)\big)ds\\
\geq\ &\textstyle\limsup_{n\to+\infty}-\overline{U}_n(x)\geq Z^*(1,x) && \text{by (93) and (94).}
\end{aligned}$$

As $t\in[0,1]$ is arbitrary and γ is admissible, one gets

$$\begin{aligned}
&Z^*(1,x)\leq -\liminf_{n\to+\infty}\overline{U}_n(x)\\
&\leq -\inf\max_{t\in[0,1]}\Big\{\textstyle\int_t^1\overline{H}^\star\big(\gamma(s),-\gamma'(s)\big)ds,\quad \gamma(0)=0,\ \gamma(1)=x,\ \gamma\in H^1(0,1)\Big\}=-\overline{U}(x).
\end{aligned}$$

3. It is left to prove that one can assume that the test-functions satisfy $\gamma(s)\neq 0$ for all $s\in(0,1]$. Consider a test-function $\gamma\in H^1(0,1)$ such that $\gamma(0)=0$ and $\gamma(1)=x$. Assume that there exists $s_0\in(0,1)$ such that $\gamma(s_0)=0$. We can assume that $\gamma(s)\neq 0$ in $(s_0,1]$. Let

$$\widetilde{\gamma}(s):=\gamma\big(s_0+(1-s_0)s\big).$$

This function is an admissible path from 0 to x, such that $\widetilde{\gamma}(s)\neq 0$ for all $s\in(0,1)$. For all $t\in[0,1]$, one has

$$\int_t^1 -\overline{H}^\star\big(\widetilde{\gamma}(s),-\widetilde{\gamma}'(s)\big)ds=\int_{s_0+(1-s_0)t}^1 -\overline{H}^\star\big(\gamma(\tau),-(1-s_0)\gamma'(\tau)\big)\frac{d\tau}{1-s_0}$$

On the other hand, as $\overline{H}^\star$ is convex, one has for all $\tau\in(0,1)$ and $s_0\in(0,1)$:

$$\frac{-\overline{H}^\star\big(\gamma(\tau),-(1-s_0)\gamma'(\tau)\big)+\overline{H}^\star\big(\gamma(\tau),0\big)}{1-s_0}\geq -\overline{H}^\star\big(\gamma(\tau),-\gamma'(\tau)\big)+\overline{H}^\star\big(\gamma(\tau),0\big).$$

It follows that:

$$\int_t^1 -\overline{H}^\star\big(\widetilde{\gamma}(s),-\widetilde{\gamma}'(s)\big)ds \geq \int_{s_0+(1-s_0)t}^1 -\overline{H}^\star\big(\gamma(\tau),-\gamma'(\tau)\big)d\tau + \frac{s_0}{1-s_0}\int_{s_0+(1-s_0)t}^1 -\overline{H}^\star\big(\gamma(\tau),0\big)d\tau.$$

But Proposition 2.2 yields

$$-\overline{H}^\star\big(\gamma(\tau),0\big) = \inf_{p\in\mathbb{R}^N} \overline{H}\big(\gamma(\tau),p\big) \geq 0,$$

which leads to

$$\int_t^1 -\overline{H}^\star\big(\widetilde{\gamma}(s),-\widetilde{\gamma}'(s)\big)ds > \int_{s_0+(1-s_0)t}^1 -\overline{H}^\star\big(\gamma(\tau),-\gamma'(\tau)\big)d\tau$$

for all $t\in[0,1]$. Hence,

$$\min_{t\in[0,1]}\int_t^1 -\overline{H}^\star\big(\widetilde{\gamma}(s),-\widetilde{\gamma}'(s)\big)ds > \min_{t'\in[0,1]}\int_{t'}^1 -\overline{H}^\star\big(\gamma(\tau),-\gamma'(\tau)\big)d\tau.$$

Thus in order to maximize this quantity, replacing γ by $\widetilde{\gamma}$, one can always assume that $\gamma(s)\neq 0$ for all $s\in(0,s_0)$. The proof for the test-functions associated with $\underline{H}$ is similar.

4. Next, consider the inf-convolution of $\underline{H}$:

$$\underline{H}_n(x,p) := \inf_{x'\in\mathbb{R}^N}\Big(\underline{H}(x',p)+n|x'-x|^2\Big)\Big\}.$$

This function is well-defined since $\underline{H}(x,p)\geq c(1+|p|^2)$ for all $(x,p)\in\mathbb{R}^N\times\mathbb{R}^N$ and thus the set over which we take the supremum is non-empty. Moreover, for all $x\in\mathbb{R}^N$, if $p_n\to p$ as $n\to+\infty$, one has $\liminf_{n\to+\infty}\underline{H}_n(x,p_n)\geq(\underline{H}^\star)^\star(x,p)$ since the double convex-conjugate of $\underline{H}$ is the largest convex function below $\underline{H}$.

As $\underline{H}_n$ is Lipschitz-continuous and $\underline{H}_n\leq\underline{H}$, Proposition 5.5 yields

$$Z_*(1,x)\geq -\underline{U}_n(x) := \sup\min_{t\in[0,1]}\Big\{\int_t^1 -\underline{H}_n^\star\Big(\gamma(s),-\gamma'(s)\Big)ds,\ \gamma\in H^1([0,1]),\ \gamma(0)=0,\ \gamma(1)=x\Big\}. \tag{95}$$

Let γ an arbitrary admissible test-function and $t_n\in[0,1]$ such that

$$\min_{t\in[0,1]}\int_t^1 -\underline{H}_n^\star\Big(\gamma(s),-\gamma'(s)\Big)ds = \int_{t_n}^1 -\underline{H}_n^\star\Big(\gamma(s),-\gamma'(s)\Big)ds.$$

We can assume, up to extraction, that $(t_n)_n$ converges to $t_\infty\in[0,1]$.

For all n and for all $s\in[0,1]$, let $p_n(s)\in\mathbb{R}^N$ such that

$$-\underline{H}_n^\star\big(\gamma(s),-\gamma'(s)\big) = \inf_{p\in\mathbb{R}^N}\Big(p\cdot\gamma'(s)+\underline{H}_n\big(\gamma(s),p\big)\Big) = p_n(s)\cdot\gamma'(s)+\underline{H}_n\big(\gamma(s),p_n(s)\big).$$

With the same arguments as above, we could prove that $(p_n)_n$ is bounded uniformly in $L^2([0,1])$, we can thus assume that it converges to a limit $p_\infty\in L^2([0,1])$ for the weak topology. Mazur's theorem yields that there exists a family $(\widetilde{p}_n)_n$ of convex combination of the $(p_n)_n$, that we write

$$\widetilde{p}_n = \sum_{i=1}^{N_n} \lambda_i^n p_{k_i^n}, \quad \forall i \in [1, N_n], \quad k_i^n \geq n, \quad \lambda_i^n \geq 0, \quad \sum_{i=1}^{N_n} \lambda_i^n = 1,$$

and which converges to p_∞ almost everywhere and strongly in $L^2([0,1])$. One has

$$\begin{aligned}
\textstyle\int_{t_\infty}^1 (\underline{H}^\star)^\star\big(\gamma(s), p_\infty(s)\big)ds \;&\leq\; \textstyle\int_{t_\infty}^1 \liminf_{n\to+\infty} \underline{H}_n\big(\gamma(s), \widetilde{p}_n(s)\big)ds \\
&\leq\; \textstyle\liminf_{n\to+\infty} \int_{t_n}^1 \underline{H}_n\big(\gamma(s), \widetilde{p}_n(s)\big)ds && \text{by Fatou's lemma} \\
&\leq\; \textstyle\liminf_{n\to+\infty} \int_{t_n}^1 \sum_{i=1}^{N_n} \lambda_i^n \underline{H}_n\big(\gamma(s), p_{k_i^n}(s)\big)ds && \text{by convexity of } \underline{H}_n \\
&\leq\; \textstyle\liminf_{n\to+\infty} \sum_{i=1}^{N_n} \lambda_i^n \int_{t_{k_n^i}}^1 \underline{H}_{k_i^n}\big(\gamma(s), p_{k_i^n}(s)\big)ds && \text{as } k_n^i \geq n \text{ and } \underline{H}_n \nearrow .
\end{aligned}$$

Gathering all the previous inequalities, we eventually get

$$\begin{aligned}
Z_*(1,x) \;&\geq\; \textstyle\limsup_{n\to+\infty} -\sum_{i=1}^{N_n} \lambda_i^n \underline{U}_{k_i^n}(x) \geq \liminf_{n\to+\infty} -\sum_{i=1}^{N_n} \lambda_i^n \underline{U}_{k_i^n}(x) \\
&\geq\; \textstyle\liminf_{n\to+\infty} \sum_{i=1}^{N_n} \lambda_i^n \min_{t\in[0,1]} \int_t^1 -\underline{H}^\star_{k_i^n}\Big(\gamma(s), -\gamma'(s)\Big)ds \qquad \text{for any path } \gamma \\
&\geq\; \textstyle\liminf_{n\to+\infty} \sum_{i=1}^{N_n} \lambda_i^n \int_{t_{k_i^n}}^1 \big(p_{k_i^n}(s)\cdot\gamma'(s) + \underline{H}_{k_i^n}(\gamma(s), p_{k_i^n}(s))\big)ds \\
&\geq\; \textstyle\int_{t_\infty}^1 \big(p_\infty \cdot \gamma' + (\underline{H}^\star)^\star(\gamma, p_\infty)\big) \\
&\geq\; \textstyle\int_{t_\infty}^1 -\underline{H}^\star\big(\gamma, -\gamma'\big) \geq \min_{t\in[0,1]} \int_t^1 -\underline{H}^\star\big(\gamma, -\gamma'\big)
\end{aligned}$$

We have thus proved that

$$\begin{aligned}
&Z_*(1,x) \geq \textstyle\limsup_{n\to+\infty} -\sum_{i=1}^{N_n} \lambda_i^n \underline{U}_{k_i^n}(x) \geq \liminf_{n\to+\infty} -\sum_{i=1}^{N_n} \lambda_i^n \underline{U}_{k_i^n}(x) \\
&\geq -\inf \max_{t\in[0,1]} \Big\{ \textstyle\int_t^1 \underline{H}^\star\big(\gamma(s), -\gamma'(s)\big),\ \gamma \in H^1([0,1]),\ \gamma(0)=0,\ \gamma(1)=x \Big\} = -\underline{U}(x),
\end{aligned}$$

and we show that one can assume $\gamma(s) \neq 0$ for all $s \in (0,1)$ as above. □

It is easy to check that similar arguments as in the previous proof yield that $\overline{U}$ is indeed a minimum. That is, considering a minimizing sequence of admissible paths $(\gamma_n)_n$, one can extract a converging subsequence which minimizes the associated maximum of integrals over $t \in [0,1]$. We thus leave the complete proof of this result to the reader.

Lemma 5.7 *For all $x \neq 0$, the infimum defining $\overline{U}$ is indeed a minimum:*

$$\overline{U}(x) = \min\Big\{ \max_{t\in[0,1]} \int_t^1 \overline{H}^\star\big(\gamma(s), -\gamma'(s)\big),\ \gamma \in H^1([0,1]),\ \gamma(0)=0,\ \gamma(1)=x \Big\}.$$

As $\underline{H}$ is not upper semicontinuous in general, we do not expect such a result to hold for $\underline{U}$.

5.5 Conclusion of the proof of Theorem 2

Proof of Theorem 2. Gathering Lemma 5.2, Proposition 5.6 and the definition of v_ε, we immediately get that

$$u(1/\varepsilon, x/\varepsilon) \to \begin{cases} 0 & \text{loc. unif. in} \quad \{\overline{U} > 0\} \\ 1 & \text{loc. unif. in} \quad \text{int}\{\underline{U} = 0\} \end{cases} \qquad \text{as } \varepsilon \to 0.$$

Consider u, K and F as in the statement of the Theorem. As $K \subset \text{int}\underline{\mathcal{S}} = \text{int}\{\underline{U} = 0\}$, the previous convergence immediately implies:

$$\sup_{x \in tK} |u(t,x) - 1| = \sup_{x \in K} |v_{1/t}(1,x) - 1| = 1 - \inf_{x \in K} u(1/t, x) \to 0 \text{ as } t \to +\infty.$$

Similarly, if F is a compact set, then the local convergence above and the fact that $F \subset \mathbb{R}^N \backslash \{\overline{U} = 0\} = \{\overline{U} > 0\}$ yields

$$\sup_{x \in tF} |u(t,x)| = \sup_{x \in F} |u(1/t, x)| \to 0 \text{ as } t \to +\infty.$$

Consider a closed set $F \subset \mathbb{R}^N \backslash \overline{\mathcal{S}}$. We have proved in [13], together with Hamel, that there exists a speed $w^* > 0$ such that

$$\max_{|x| \geq w^* t} u(t,x) \to 0 \text{ as } t \to +\infty.$$

Define $F_1 = F \cap \{|x| \leq w^*\}$ and $F_2 = F \cap \{|x| \geq w^*\}$. We know that $\lim_{t \to +\infty} \max_{x \in F_2} u(t,x) = 0$. On the other hand, as F is closed, F_1 is compact and thus $\lim_{t \to +\infty} \max_{x \in tF_1} u(t,x) = 0$. Thus

$$\lim_{t \to +\infty} \max_{x \in tF} u(t,x) = 0.$$

□

5.6 The recurrent case

We now check that the two definitions (33) and (28) of the expansion sets $\underline{\mathcal{S}}$ and $\overline{\mathcal{S}}$ are equivalent when the coefficients are recurrent.

Proof of Proposition 3.3. Let $\alpha > 0$, $R > 0$, $p \in \mathbb{R}^N$ and $e \in \mathbb{S}^{N-1}$. Take $\phi \in W^{1,\infty}(C_{R,\alpha}(e))$ and λ' such that $\inf_{C_{R,\alpha}(e)} \phi > 0$ and $L_p \phi \geq \lambda \phi$ in $C_{R,\alpha}(e)$. Define $\phi_n(t,x) = \phi(t+n, x+ne)$ for all n. The sequence $(\phi_n)_{n > R}$ is equicontinuous and uniformly bounded since $\phi \in W^{1,\infty}(C_{R,\alpha}(e))$. We can assume that this sequence converges locally uniformly as $n \to +\infty$ to a function $\phi_\infty \in W^{1,\infty}(\mathbb{R} \times \mathbb{R}^N)$ such that $\inf_{\mathbb{R} \times \mathbb{R}^N} \phi_\infty > 0$. Similarly, one can assume, up to extraction, that there exist A_∞, q_∞ and c_∞ such that $A(t+n, x+ne) \to A_\infty(t,x)$, $q(t+n, x+ne) \to q_\infty(t,x)$ and $f'_u(t+n, x+ne, 0) \to c_\infty(t,x)$ as $n \to +\infty$ locally uniformly in $\mathbb{R} \times \mathbb{R}^N$. Define

$$L_p^* = -\partial_t + tr(A_\infty \nabla^2) + (2pA_\infty + q_\infty) \cdot \nabla + (pA_\infty p + q_\infty \cdot p + c_\infty).$$

Then $L_p^* \phi_\infty \geq \lambda \phi_\infty$ in $\mathbb{R} \times \mathbb{R}^N$, which give $\lambda \leq \underline{\lambda_1}(L_p^*, \mathbb{R} \times \mathbb{R}^N)$, and thus letting $\lambda \to \underline{\lambda_1}(L_p, C_{R,\alpha}(e))$, one gets

$$\underline{\lambda_1}(L_p, C_{R,\alpha}(e)) \leq \underline{\lambda_1}(L_p^*, \mathbb{R} \times \mathbb{R}^N).$$

Next, as A, q and $f_u'(\cdot,\cdot,0)$ are recurrent with respect to (t,x), there exists a sequence (s_n, y_n) such that $A_\infty(t-s_n, x-y_n) \to A(t,x)$, $q_\infty(t-s_n, x-y_n) \to q(t,x)$ and $c_\infty(t-s_n, x-y_n) \to f_u'(t,x,0)$ as $n \to +\infty$ locally uniformly in $\mathbb{R} \times \mathbb{R}^N$. Hence, the same arguments as above give

$$\underline{\lambda_1}(L_p^*, \mathbb{R} \times \mathbb{R}^N) \leq \underline{\lambda_1}(L_p, \mathbb{R} \times \mathbb{R}^N).$$

As $\underline{\lambda_1}(L_p, C_{R,\alpha}(e)) \geq \underline{\lambda_1}(L_p, \mathbb{R}\times\mathbb{R}^N)$ by (18), one eventually gets $\underline{\lambda_1}(L_p, C_{R,\alpha}(e)) = \underline{\lambda_1}(L_p, \mathbb{R}\times\mathbb{R}^N)$ for all $R > 0$, $\alpha > 0$ and $e \in \mathbb{S}^{N-1}$. This leads to

$$\underline{H}(e,p) = \underline{\lambda_1}(L_p, \mathbb{R} \times \mathbb{R}^N).$$

Similarly, one can prove that $\overline{H}(e,p) = \overline{\lambda_1}(L_p, \mathbb{R} \times \mathbb{R}^N)$. In other words, $\underline{H} = \underline{H}(p)$ and $\overline{H} = \overline{H}(p)$ do not depend on e.

It follows from the Jensen inequality that for all $\gamma \in H^1([0,1])$, with $\gamma(0) = 0$ and $\gamma(1) = x$:

$$\int_0^1 \underline{H}^\star\big(\gamma(s), -\gamma'(s)\big)ds = \int_0^1 \underline{H}^\star\big(-\gamma'(s)\big)ds \geq \underline{H}^\star\Big(-\int_0^1 \gamma'(s)ds\Big) = \underline{H}^\star(-x).$$

Hence, on one hand, taking $t = 0$ and $t = 1$ leads to:

$$\begin{aligned}&\inf \max\nolimits_{t\in[0,1]} \Big\{ \textstyle\int_t^1 \underline{H}^\star\big(-\gamma'(s)\big),\ \gamma \in H^1([0,1]),\ \gamma(0) = 0,\ \gamma(1) = x\Big\}\\ &\geq \max\Big\{0, \inf\Big\{ \textstyle\int_0^1 \underline{H}^\star\big(-\gamma'(s)\big),\ \gamma \in H^1([0,1]),\ \gamma(0) = 0,\ \gamma(1) = x\Big\}\Big\} \geq \max\{0, \underline{H}^\star(-x)\}.\end{aligned}$$

On the other hand, taking $\gamma(s) = sx$, one gets:

$$\inf \max_{t\in[0,1]} \Big\{ \int_t^1 \underline{H}^\star\big(-\gamma'(s)\big),\ \gamma \in H^1([0,1]),\ \gamma(0) = 0,\ \gamma(1) = x\Big\} \leq \max_{t\in[0,1]} \int_t^1 \underline{H}^\star(-x) = \max\{0, \underline{H}^\star(-x)\}.$$

We thus conclude that

$$\underline{\mathcal{S}} = \{x \in \mathbb{R}^N,\ \underline{H}^\star(-x) \geq 0\} = \{x \in \mathbb{R}^N,\ \exists p \in \mathbb{R}^N \mid -p\cdot x + \underline{\lambda_1}(L_{-p}, \mathbb{R} \times \mathbb{R}^N) \leq 0\}$$

from which the conclusion immediately follows. The identification of $\overline{\mathcal{S}}$ is similar. □

5.7 Geometry of the expansion sets

Proposition 5.8 *Under the assumptions and notations as in the proof of Proposition 5.5, assuming in addition that $x \mapsto H(x,p)$ is quasiconcave for all $p \in \mathbb{R}^N$, then the function Z_j is concave with respect to $(t,x) \in (1/j, \infty) \times B_k$.*

Proof. Take an arbitrary $T > 1/j$. We use the same approach as in [1], but we need to check that the quasiconcavity of the Hamiltonian is sufficient in order to get the concavity of the function. Let $\widetilde{Z}_j$ the concave envelope of Z_j, that is, the smallest concave function w.r.t (t, x) above Z_j in $(1/j, T) \times B_j$. We need to prove that $\widetilde{Z}_j \leq Z_j$ in order to conclude. We will prove that $\widetilde{Z}_j$ is a subsolution of (91), which is enough in order to derive the conclusion since (91) admits a comparison principle (see [34]). First note that $\widetilde{Z}_j \leq 0$ is obvious since $Z_j \leq 0$.

Let $(t, x) \in (1/j, T) \times B_j$ and consider a smooth function χ such that $\widetilde{Z}_j - \chi$ admits a strict local maximum (t, x). As in [1], we know that there exist $l \leq N+2$, $t_1, ..., t_l$ in $(1/j, T)$, $x_1, ..., x_l$ in B_j and $\lambda_1, ..., \lambda_l$ in $[0, 1]$ such that

$$t = \sum_{i=1}^{l} \lambda_i t_i, \quad x = \sum_{i=1}^{l} \lambda_i x_i, \quad \sum_{i=1}^{l} \lambda_i = 1 \quad \text{and} \quad \widetilde{Z}_j(t, x) = \sum_{1 \leq i \leq l} \lambda_i Z_j(t_i, x_i).$$

It is then standard that for all $i = 1, ..., l$,

$$(s_i, y_i) \mapsto \lambda_i Z_j(s_i, y_i) - \chi\big(\sum_{j \neq i} \lambda_j t_j + \lambda_i s_i, \sum_{j \neq i} \lambda_j x_j + \lambda_i y_i\big)$$

reaches a local maximum at (t_i, x_i). It follows from (91) that for all $i = 1, ..., l$:

$$\partial_t \chi(t, x) - H\big(x_i, \nabla \chi(t, x)\big) \leq 0.$$

We now check that the quasiconcavity is sufficient in order to conclude:

$$\begin{aligned} \partial_t \chi(t, x) - H\big(x, \nabla \chi(t, x)\big) &= \partial_t \chi(t, x) - H\big(\textstyle\sum_{1 \leq i \leq l} \lambda_i x_i, \nabla \chi(t, x)\big) \\ &\leq \partial_t \chi(t, x) - \inf_{1 \leq i \leq l} H\big(x_i, \nabla \chi(t, x)\big) \qquad \text{(by quasiconcavity)} \\ &\leq 0. \end{aligned}$$

Next, if $t = 1/j$, then necessarily $t_1 = ... = t_l = 1/j$. As $Z_j(1/j, x) = -M_j|x|$ is concave over B_j, one gets:

$$Z_j(1/j, x) \leq \widetilde{Z}_j(1/j, x) = \sum_{1 \leq i \leq l} \lambda_i Z_j(1/j, x_i) \leq Z_j(1/j, x).$$

Similarly, if $|x| = j$, then $x_1 = ... = x_l$ by strict convexity of the ball B_j and thus $\widetilde{Z}_j(t, x) = \min\{m_j, -M_j|j|\}$, which is concave, from which we get $\widetilde{Z}_j = Z_j$ in $(1/j, T) \times \partial B_j$.

We have thus proved that $\widetilde{Z}_j$ is a subsolution of (91) and thus $\widetilde{Z}_j \leq Z_j$, leading to $\widetilde{Z}_j \equiv Z_j$. Hence Z_j is concave with respect to (t, x). □

Proof of Proposition 2.3. The inf-convolution of $\underline{H}$:

$$\underline{H}_n(x, p) := \inf_{x' \in \mathbb{R}^N} \big(\underline{H}(x', p) + n|x - x'|^2\big) = \inf_{X \in \mathbb{R}^N} \big(\underline{H}(x + X, p) + n|X|^2\big).$$

is clearly quasiconcave in x as the infimum of a family of quasiconcave functions is quasiconcave.

For all n and j, we let $Z_{j,n}$ the function constructed in Proposition 5.5 with Hamiltonian $H = \underline{H}_n$, which is concave over $(1/j, \infty) \times B_j$ by Proposition 5.8. We also define

$$\underline{U}_n(x) := \inf \max_{t\in[0,1]} \left\{ \int_t^1 -\underline{H}_n^\star\Big(\gamma(s), -\gamma'(s)\Big) ds,\ \gamma \in H^1([0,1]),\ \gamma(0)=0,\ \gamma(1)=x \right\},$$

so that, we know from the proofs of Propositions 5.5 and 5.6 that for all $x \in \mathbb{R}^N$:

$$Z_{k,n}(1,x) \leq Z_*(1,x) \leq Z^*(1,x) \leq -\overline{U}(x),$$

$$Z_{j,n}(1,x) \geq -\underline{U}_n(x) \text{ for all } x \text{ when } j = j(x,n) \text{ is large enough,} \tag{96}$$

$$Z_*(1,x) \geq \limsup_{n\to+\infty} -\sum_{i=1}^{N_n} \lambda_i^n \underline{U}_{k_i^n}(x) \geq \liminf_{n\to+\infty} -\sum_{i=1}^{N_n} \lambda_i^n \underline{U}_{k_i^n}(x) \geq -\underline{U}(x) \tag{97}$$

for some families (k_n^i) and (λ_i^n) depending on x and satisfying $k_i^n \geq n$, $\lambda_i^n \geq 0$, $\sum_{i=1}^{N_n} \lambda_i^n = 1$.

Take now x_0, x_1 such that $\underline{U}(x_0) = \underline{U}(x_1) = 0$ and $\tau \in [0,1]$. We could consider common families (k_n^i) and (λ_i^n) such that (97) is satisfied.

One has:

$$\begin{aligned}
-\underline{U}((1-\tau)x_0+\tau x_1) &\geq Z_*(1,(1-\tau)x_0+\tau x_1) \\
&\geq \textstyle\sum_{i=1}^{N_n} \lambda_i^n \limsup_{j\to+\infty} Z_{j,k_n^i}(1,(1-\tau)x_0+\tau x_1) \\
&\geq \textstyle\sum_{i=1}^{N_n} \lambda_i^n \limsup_{j\to+\infty} \big((1-\tau)Z_{j,k_n^i}(1,x_0)+\tau Z_{j,k_n^i}(1,x_1)\big) \quad \text{by concavity} \\
&\geq -\textstyle\sum_{i=1}^{N_n} \lambda_i^n \big((1-\tau)\underline{U}_{k_n^i}(x_0)+\tau \underline{U}_{k_n^i}(x_1)\big) \quad \text{by (96).}
\end{aligned}$$

Taking the lim inf of the right-hand side, one gets

$$-\underline{U}((1-\tau)x_0+\tau x_1) \geq -(1-\tau)\underline{U}(x_0) + \tau\underline{U}(x_1) = 0.$$

As $\overline{U} \geq 0$, this implies

$$(1-\tau)x_0 + \tau x_1 \in \overline{\mathcal{S}} = \underline{\mathcal{S}} = cl\{\underline{U} = 0\}.$$

Hence, this set is convex. □

Proof of Proposition 2.4. Let $\sigma \in [0,1]$, $x \in \overline{\mathcal{S}}$, that is, $\overline{U}(x) = 0$, and take $\gamma \in H^1(0,1)$ such that $\gamma(0) = 0$, $\gamma(1) = x$ and $\gamma(s) \neq 0$ for all $s \in (0,1)$. We recall that $\overline{H}^\star(e,0) = -\inf_{p\in\mathbb{R}^N} \overline{H}(e,p) \leq -c < 0$ for all $e \in \mathbb{S}^{N-1}$. Consider the path

$$\gamma_\sigma(s) := \begin{cases} \sigma\gamma(s/\sigma) & \text{if} \quad s \in [0,\sigma], \\ \sigma x & \text{if} \quad s \in [\sigma,1]. \end{cases}$$

As it connects 0 to σx, we could use it as a test-function in the definition of $\overline{U}$:

$$\begin{aligned}
&\max_{t\in[0,1]}\int_t^1 \overline{H}^\star\Big(\tfrac{\gamma_\sigma(s)}{|\gamma_\sigma(s)|},-\gamma_\sigma'(s)\Big)ds\\
&=\Big(\max_{t\in[0,\sigma]}\int_t^1 \overline{H}^\star\Big(\tfrac{\gamma_\sigma(s)}{|\gamma_\sigma(s)|},-\gamma_\sigma'(s)\Big)ds\Big)_+ &&\text{since } \overline{H}^\star(x/|x|,0)<0\\
&=\Big(\max_{t\in[0,\sigma]}\int_t^\sigma \overline{H}^\star\Big(\tfrac{\gamma(s/\sigma)}{|\gamma(s/\sigma)|},-\gamma'(s/\sigma)\Big)ds+(1-\sigma)\overline{H}^\star(x/|x|,0)\Big)_+ &&\text{by definition of } \gamma_\sigma\\
&=\Big(\sigma\max_{t\in[0,1]}\int_t^1 \overline{H}^\star\Big(\tfrac{\gamma(\tau)}{|\gamma(\tau)|},-\gamma'(\tau)\Big)d\tau+(1-\sigma)\overline{H}^\star(x/|x|,0)\Big)_+ &&\text{letting } \tau:=s/\sigma\\
&=\Big((1-\sigma)\overline{H}^\star(x/|x|,0)\Big)_+=0 &&\text{by definition of } \gamma.
\end{aligned}$$

Hence, $\overline{U}(\sigma x)=0$, that is, $\overline{\mathcal{S}}$ is star-shaped. The star-shapedness of $\underline{\mathcal{S}}$ is proved similarly.

Next, take x and σ as above but consider the path

$$\widetilde{\gamma}_\sigma(s):=\begin{cases}\sigma\gamma(s/\sigma) & \text{if}\quad s\in[0,\sigma],\\ \sigma x+(s-\sigma)ce & \text{if}\quad s\in[\sigma,1],\end{cases}$$

where c appears in $c(1+|p|^2)\le \underline{H}(e,p)$ and $e\in\mathbb{S}^{N-1}$ is arbitrary. One easily computes $\overline{H}^\star(x/|x|,ce)\le 0$. One has:

$$\begin{aligned}
&\max_{t\in[0,1]}\int_t^1 \overline{H}^\star\Big(\tfrac{\widetilde{\gamma}_\sigma(s)}{|\widetilde{\gamma}_\sigma(s)|},-\widetilde{\gamma}_\sigma'(s)\Big)ds\\
&=\Big(\max_{t\in[0,\sigma]}\int_t^\sigma \overline{H}^\star\Big(\tfrac{\widetilde{\gamma}_\sigma(s/\sigma)}{|\widetilde{\gamma}_\sigma(s/\sigma)|},-\widetilde{\gamma}_\sigma'(s/\sigma)\Big)ds+(1-\sigma)\overline{H}^\star(x/|x|,ce)\Big)_+ &&\text{by definition of } \widetilde{\gamma}_\sigma\\
&=\Big(\sigma\max_{t\in[0,1]}\int_t^1 \overline{H}^\star\Big(\tfrac{\widetilde{\gamma}_\sigma(\tau)}{|\widetilde{\gamma}_\sigma(\tau)|},-\widetilde{\gamma}_\sigma'(\tau)\Big)d\tau+(1-\sigma)\overline{H}^\star(x/|x|,ce)\Big)_+=0.
\end{aligned}$$

Hence,

$$\overline{U}\big(\widetilde{\gamma}_\sigma(1)\big)=\overline{U}\big(\sigma x+(1-\sigma)ce\big)=0.$$

As $e\in\mathbb{S}^{N-1}$ is arbitrary, this means that for all $\sigma\in(0,1)$, there is a ball of radius $(1-\sigma)c$ around σx. In other words, there is a cone of angle β at x, such that $\sin\beta=|x|/c$, pointing into $\overline{\mathcal{S}}$. This exactly means that $\overline{\mathcal{S}}$ has a Lipschitz-continuous boundary. The smoothness of $\underline{\mathcal{S}}$ is proved similarly.

Next, as $c(1+|p|^2)\le \underline{H}(e,p)\le \overline{H}(e,p)\le C(1+|p|^2)$ for all $(e,p)\in\mathbb{S}^{N-1}\times\mathbb{R}^N$ by Proposition 2.2, one has $-\overline{H}^\star(e,q)\le C-|q|^2/4C$ for all (e,q) and thus, as in the proof of Proposition 3.3, Jensen inequality yields $\overline{U}(x)\le C-|x|^2/4C$. Hence, $\overline{\mathcal{S}}\subset\{|x|\le 2C\}$. Similarly, $\underline{U}(x)\ge c-|x|^2/4c$ and $\{|x|\le 2c\}\subset\underline{\mathcal{S}}$. □

6 The homogeneous, periodic and compactly supported cases

We have already described in details how to handle these cases in dimension 1 (see Sections II.D.1 and 3 in [19]). We provide here the proofs for sake of completeness.

Proof of Proposition 3.1.

1. Assume first that the coefficients are homogeneous, that is, A and $f'_u(\cdot, cdot, 0)$ do not depend on (t,x) and $q \equiv 0$. In this case $\mathcal{L} = -\partial_t + a_{i,j}\partial_{i,j} + f'_u(0)$, and

$$L_p = -\partial_t + a_{i,j}\partial_{i,j} + 2pA\nabla + (pAp + f'_u(0)).$$

It follows from Proposition 4.4 that

$$\underline{\lambda_1}(L_p, C_{R,\alpha}(e)) = \overline{\lambda_1}(L_p, C_{R,\alpha}(e)) = pAp + f'_u(0)$$

for all $\alpha, R > 0$. Hence,

$$\overline{H}(e,p) = \underline{H}(e,p) = pAp + f'_u(0)$$

for all $(e,p) \in \mathbb{S}^{N-1} \times \mathbb{R}^N$ and $\overline{\mathcal{S}} = \underline{\mathcal{S}}$. It is easy to compute that

$$\underline{w}(e) = \overline{w}(e) = 2\sqrt{eAef'(0)}.$$

This is consistent with the results of Kolmogorov, Petrovsky and Piskunov [57] when $N = 1$ and Aronson and Weinberger [4] for general N.

2. Assume now that the coefficients are periodic. We know that the operator L_p admits a unique periodic principal eigenvalue $k_p^{per}(\mathcal{L})$, defined by the existence of a solution ϕ_p of

$$\begin{cases} L_p\phi_p & = k_p^{per}(\mathcal{L})\phi_p, \\ \phi_p & > 0, \\ \phi_p & \text{is periodic.} \end{cases} \qquad (98)$$

Proposition 4.4 yields $\underline{\lambda_1}(L_p, \mathbb{R}\times\mathbb{R}^N) = \overline{\lambda_1}(L_p, \mathbb{R}\times\mathbb{R}^N) = k_p^{per}(\mathcal{L})$ and thus $\overline{H}(e,p) = \underline{H}(e,p) = k_p^{per}(\mathcal{L})$ for all $(e,p) \in \mathbb{S}^{N-1} \times \mathbb{R}^N$. Then, Proposition 3.3 gives

$$\overline{\mathcal{S}} = \underline{\mathcal{S}} = \{x \in \mathbb{R}^N, \ \exists p \in \mathbb{R}^N, k_p^{per}(\mathcal{L}) + x \cdot p < 0\},$$

and $\underline{w}(e) = \overline{w}(e) = \min_{p\cdot e>0} \dfrac{k_{-p}^{per}(\mathcal{L})}{p \cdot e}$, which is consistent with [13, 41, 42, 69, 81, 102].

3. Assume now that the coefficients satisfy (31). Taking constant test-functions in the definitions of the generalized principal eigenvalues, we immediately derive from this property that

$$\lim_{R\to+\infty} \underline{\lambda_1}(L_p, C_{R,\alpha}(e)) = \lim_{R\to+\infty} \overline{\lambda_1}(L_p, C_{R,\alpha}(e)) = pA^*p + q^* \cdot p + c^*$$

and thus $\overline{H}(e,p) = \underline{H}(e,p) = pA^*p + q^* \cdot p + c^*$ for all $(e,p) \in \mathbb{S}^{N-1} \times \mathbb{R}^N$. Easy computations then provide the conclusion, which is consistent with [13]

□

7 The almost periodic case

This case could be derived from Theorem 6 since almost periodic functions belong to the wider class of uniquely ergodic functions. However, we provide here a direct proof, inspired by the arguments of Lions and Souganidis [66], who proved the existence of approximate correctors in the framework of homogenization of Hamilton-Jacobi equations with almost periodic coefficients. Indeed, we only need to check that this proof still holds when there is an almost periodic time-dependence of the coefficients. We give the full proof here by sake of completeness and to illustrate the link between generalized principal eigenvalues and approximate correctors described in Section 5.1.

Proof of Theorem 4. 1. This proof is based on Evan's perturbed test-functions method [37]. Thus, we investigate the sequence of equations

$$-\partial_t u_\varepsilon + tr(A(t,x)\nabla^2 u_\varepsilon) + \nabla u_\varepsilon A(t,x)\nabla u_\varepsilon + q(t,x)\cdot\nabla u_\varepsilon + c(t,x) = \varepsilon u_\varepsilon \text{ in } \mathbb{R}\times\mathbb{R}^N, \quad (99)$$

and we will prove that the family $(\varepsilon u_\varepsilon)_{\varepsilon>0}$ converges as $\varepsilon\to 0$ uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}^N$.

First, as c is uniformly bounded, there exists some large M such that $-M$ is a subsolution and M is a supersolution of (99). As (99) admits a comparison principle, the Perron's method [54] gives the existence of a unique solution $u_\varepsilon\in\mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ of equation (99) such that $-M\le u_\varepsilon\le M$ (of course the bound M depends on ε). Moreover, there exists a constant $C>1$ such that

$$\|\nabla^2 u_\varepsilon\|_\infty + \|\partial_t u_\varepsilon\|_\infty + \|\nabla u_\varepsilon\|_\infty + \|\varepsilon u_\varepsilon\|_\infty \le C \text{ for all } \varepsilon>0.$$

2. Consider a sequence $\varepsilon_j\to 0$ such that $(\varepsilon_j u_{\varepsilon_j}(0,0))_j$ converges and define $-\lambda$ its limit. Let $v_j := u_{\varepsilon_j} - u_{\varepsilon_j}(0,0)$. We need to prove that $(\varepsilon_j v_j)_j$ converges to 0 uniformly over $\mathbb{R}\times\mathbb{R}^N$. Assume that this is not true. Then there exist $\kappa>0$ and a sequence $(t_j,x_j)_j$ such that

$$|\varepsilon_j v_j(t_j,x_j)|\ge\kappa \text{ for all } n.$$

As A, q and c are almost periodic in (t,x), one can assume, up to extraction, that

$$\begin{array}{l}\|A(\cdot+t_j,\cdot+x_j)-A(\cdot+t_k,\cdot+x_k)\|_\infty+\|q(\cdot+t_j,\cdot+x_j)-q(\cdot+t_k,\cdot+x_k)\|_\infty\\ +\|c(\cdot+t_j,\cdot+x_j)-c(\cdot+t_k,\cdot+x_k)\|_\infty\le\frac{\kappa}{8C^2} \text{ for all } j,k \text{ large enough.}\end{array}$$

Take k and let $w_j(t,x) := v_j(t+t_j-t_k, x+x_j-x_k)$. A straightforward computation shows that w_j satisfies

$$-\partial_t w_j + tr(A(t,x)\nabla^2 w_j) + \nabla w_j A(t,x)\nabla w_j + q(t,x)\cdot\nabla w_j + c(t,x) \ge \varepsilon_j w_j - \frac{\kappa}{2} \text{ in } \mathbb{R}\times\mathbb{R}^N. \quad (100)$$

As $v_j+\frac{\kappa}{2\varepsilon_j}$ is a super solution of (100), the comparison principle gives

$$\varepsilon_j v_j(t+t_j-t_k, x+x_j-x_k) \le \varepsilon_j v_j(t,x) + \frac{\kappa}{2} \text{ for all } (t,x)\in\mathbb{R}\times\mathbb{R}^N.$$

Hence, considering this inequality in (t_k,x_k), we get

$$\varepsilon_j v_j(t_k,x_k) \ge \varepsilon_j v_j(t_j,x_j) - \frac{\kappa}{2} \ge \frac{\kappa}{2}$$

for all j,k. The uniform Lipschitz bound on $(v_j)_j$ and the fact that $v_j(0,0)=0$ finally give $\varepsilon_j C(|t_k|+|x_k|)\ge\frac{\kappa}{2}$ which is a contradiction when $j\to+\infty$ since $\varepsilon_j\to 0$.

Hence, $(\varepsilon_j v_j)_j$ converges to 0 uniformly over $\mathbb{R}\times\mathbb{R}^N$ and thus $(\varepsilon u_\varepsilon)_{\varepsilon>0}$ converges to $-\lambda$ uniformly in $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ as $\varepsilon\to 0$.

3. We will conclude by proving that $\underline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N)=\overline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N)=\lambda$. Take $\delta>0$ and $\varepsilon>0$ small enough so that $\varepsilon u_\varepsilon(t,x)\geq\lambda-\delta$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$. Define $\phi:=e^{u_\varepsilon}$. One has $\phi\in W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N)\cap\mathcal{C}^2(\mathbb{R}\times\mathbb{R}^N)$ and $\inf_{\mathbb{R}\times\mathbb{R}^N}\phi>0$. Moreover, ϕ satisfies

$$\mathcal{L}\phi=\varepsilon u_\varepsilon\phi\geq(\lambda-\delta)\phi \text{ in } \mathbb{R}\times\mathbb{R}^N.$$

Hence, one has $\underline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N)\geq\lambda-\delta$ for all δ and thus $\underline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N)\geq\lambda$. Similarly, one can prove that $\overline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N)\leq\lambda$. As $\overline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N)\geq\underline{\lambda_1}(\mathcal{L},\mathbb{R}\times\mathbb{R}^N)$, this gives the conclusion. □

Proof of Theorem 3. Theorem 4 and (53) give $\overline{\lambda_1}(L_p,\mathbb{R}\times\mathbb{R}^N)=\underline{\lambda_1}(L_p,\mathbb{R}\times\mathbb{R}^N)$. Thus, using similar arguments as for homogeneous coefficients, one has

$$\underline{H}(e,p)=\overline{H}(e,p)=\overline{\lambda_1}(L_p,\mathbb{R}\times\mathbb{R}^N).$$

This concludes the proof. □

8 The uniquely ergodic case

Proof of Theorem 6. As in the proof of Theorem 4, we let u_ε the unique bounded solution of

$$a_{i,j}(x)\partial_{i,j}u_\varepsilon+a_{i,j}(x)\partial_i u_\varepsilon\partial_j u_\varepsilon+q_i(x)\partial_i u_\varepsilon+c(x)=\varepsilon u_\varepsilon \quad \text{in } \mathbb{R}^N \tag{101}$$

and the conclusion follows as in the almost periodic framework if we manage to prove that $(\varepsilon u_\varepsilon)_{\varepsilon>0}$ converges uniformly over $\mathbb{R}^N$ to a constant $\lambda\in\mathbb{R}$.

First, let $\Omega:=\mathcal{H}_{(A,q,c)}$ and, for all $\omega=(B,r,d)\in\Omega=\mathcal{H}_{(A,q,c)}$, $\widetilde{A}(x,\omega):=B(x)$, $\widetilde{q}(x,\omega):=r(x)$, and $\widetilde{c}(x,\omega):=d(x)$. This turns our problem into a random stationary ergodic one. Indeed, the stationarity immediately follows from the invariance of the measure $\mathbb{P}$ with respect to translations. If M is a measurable subset of Ω such that $\tau_x M=M$ for all $x\in\mathbb{R}^N$, then $\widetilde{\mathbb{P}}(A):=\mathbb{P}(A\cap M)/\mathbb{P}(M)$ would provide another invariant probability measure on Ω, unless $\mathbb{P}(M)=0$ or $\mathbb{P}(M)=1$. Hence, $\mathbb{P}$ is ergodic with respect to the translations $(\tau_x)_{x\in\mathbb{R}^N}$.

Under these hypotheses, Lions and Souganidis proved in [68] that there exists a constant $\lambda\in\mathbb{R}$ such that

$$\lim_{\varepsilon\to 0}\mathbb{P}\big(\{\omega\in\Omega,\ |\varepsilon u_\varepsilon(0,\omega)-\lambda|>\delta\}\big)=0 \quad \text{for all } \delta>0.$$

Let $A_\delta:=\{\omega\in\Omega,\ |\varepsilon u_\varepsilon(0,\omega)-\lambda|\leq\delta\}$ and $\varepsilon_\delta>0$ such that $\mathbb{P}(A_\delta)\geq 1-\delta$ for all $\varepsilon\in(0,\varepsilon_\delta)$.

Let $\delta\in(0,1/3)$ and $\varepsilon\in(0,\varepsilon_\delta)$. As $\Omega=\mathcal{H}_{(A,q,c)}$ is compact, there exists a continuous function $\Psi:\Omega=\mathcal{H}_{(A,q,c)}\to\mathbb{R}$ such that $\|\Psi-\mathbf{1}_{A_\delta}\|_{L^\infty(\Omega)}<\delta$. Proposition 3.7 yields that the following limit exists for all $\omega\in\Omega$:

$$\lim_{R\to+\infty}\frac{1}{|B_R(a)|}\int_{B_R(a)}\Psi(\tau_y\omega)dy=\mathbb{P}(\Psi) \quad \text{uniformly with respect to } a\in\mathbb{R}^N.$$

Hence:

$$\lim_{R\to+\infty}\frac{1}{|B_R(a)|}\int_{B_R(a)}\mathbf{1}_{A_\delta}(\tau_y\omega)dy > \mathbb{P}(\Psi)-\delta \geq \mathbb{P}(A_\delta)-2\delta \geq 1-3\delta > 0 \quad \text{uniformly w.r.t } a\in\mathbb{R}^N.$$

This implies in particular that, for all $\omega\in\Omega$, there exists $R>0$ such that, for all $a\in\mathbb{R}^N$, there exists $y\in B_R(a)$ such that $\tau_y\omega\in A_\delta$. Applying this property to $\omega=(A,q,c)$, we obtain in particular that for all $x\in\mathbb{R}^N$, there exists $y\in B_R(x)$ such that $|\varepsilon u_\varepsilon(y)-\lambda|\leq\delta$. But we also know that there exists a constant C, independent of ε, such that $|\nabla u_\varepsilon(z)|\leq C$ for all $z\in\mathbb{R}^N$. Hence:

$$|\varepsilon u_\varepsilon(x)-\lambda|\leq|\varepsilon u_\varepsilon(y)-\lambda|+\varepsilon C|x-y|\leq\delta+\varepsilon CR.$$

This implies that for all $\varepsilon>0$ small enough, one has $|\varepsilon u_\varepsilon(x)-\lambda|\leq 2\delta$ for all $x\in\mathbb{R}^N$, from which the conclusion follows.

□

9 The radially periodic case

The proof of Proposition 3.8 of course relies on the radial change of variables. This gives rise to some extra-terms which are indeed neglectible asymptotically, precisely because our construction only takes into account the values of the coefficients in the truncated cones $C_{R,\alpha}(e)$. We can thus construct approximated eigenvalues. This gives one more example where considering the generalized principal eigenvalues over the full space $\mathbb{R}^N$ would have given sub-optimal expansion sets.

Proof of Proposition 3.8. We will use the larger family of periodic operators for all $\widetilde{p}\in\mathbb{R}^N$ and $e\in\mathbb{S}^{N-1}$:

$$\widetilde{L}^{per}_{e,\widetilde{p}}\varphi := a_{per}(r)\varphi''+2\widetilde{p}\cdot e\ a_{per}(r)\varphi'+\big(|\widetilde{p}|^2a_{per}(r)+c_{per}(r)\big)\varphi.$$

Let φ the periodic principal eigenfunction associated with $\widetilde{L}^{per}_{e,-\widetilde{p}}$ and $\lambda_1^{per}(\widetilde{L}_{e,-\widetilde{p}})$ the associated eigenvalue: $\varphi=\varphi(r)$ is positive, $L-$periodic and one has $\widetilde{L}^{per}_{e,-\widetilde{p}}\varphi=\lambda_1^{per}(\widetilde{L}_{e,-\widetilde{p}})\varphi$. Take $e\in\mathbb{S}^{N-1}$, $\alpha>0$, $R>0$ and define $\phi(x)=\varphi(|x|)$. Then $\phi\in\mathcal{C}^2\big(C_{R,\alpha}(e)\big)$ and for all $\widetilde{p}\in\mathbb{R}^N$, coming back to our original operator $L_{-\widetilde{p}}$ defined by (25), one has over $C_{R,\alpha}(e)$:

$$\begin{aligned}
L_{-\widetilde{p}}\phi &= a_{per}(|x|)\Delta\phi-2a_{per}(|x|)\widetilde{p}\cdot\nabla\phi+\big(|\widetilde{p}|^2a_{per}(|x|)+c_{per}(|x|)\big)\phi\\
&= a_{per}(r)\varphi''+a_{per}(r)\tfrac{N-1}{r}\varphi'-2a_{per}(r)\widetilde{p}\cdot e_r\varphi'+\big(|\widetilde{p}|^2a_{per}(r)+c_{per}(r)\big)\varphi\\
&= \widetilde{L}^{per}_{e,-\widetilde{p}}\varphi+a_{per}(r)\tfrac{N-1}{r}\varphi'+2a_{per}(r)\widetilde{p}\cdot(e-e_r)\varphi'\\
&= \lambda_1^{per}(\widetilde{L}^{per}_{e,-\widetilde{p}})\varphi+a_{per}(r)\tfrac{\varphi'}{\varphi}\Big(\tfrac{(N-1)}{r}+2\widetilde{p}\cdot(e-e_r)\Big)\varphi\\
&= \Big(\lambda_1^{per}(\widetilde{L}^{per}_{e,-\widetilde{p}})+o(1/R)+o(\alpha)\Big)\varphi
\end{aligned}$$

since $r=|x|>R$, $|e-e_r|=|e-\frac{x}{|x|}|<\alpha$ and φ'/φ is bounded independently of R and α. Hence, taking φ as a test-function in the definition of $\underline{\lambda_1}$ and $\overline{\lambda_1}$ and letting $R\to+\infty$, $\alpha\to 0$, one gets $\overline{H}(e,\widetilde{p})=\underline{H}(e,\widetilde{p})=\lambda_1^{per}(\widetilde{L}^{per}_{e,-\widetilde{p}})$ for all $\widetilde{p}\in\mathbb{R}^N$ and $e\in\mathbb{S}^{N-1}$.

Next, noticing that $\widetilde{L}^{per}_{e,\widetilde{p}}\phi\leq \widetilde{L}^{per}_{e,(\widetilde{p}\cdot e)e}\phi+\max_{\mathbb{R}} a_{per}\big(|\widetilde{p}|^2-(\widetilde{p}\cdot e)^2\big)\phi$ for all ϕ, one gets

$$\overline{H}(e,\widetilde{p})=\underline{H}(e,\widetilde{p})=\lambda_1^{per}(\widetilde{L}^{per}_{e,-\widetilde{p}})\leq\lambda_1^{per}(\widetilde{L}^{per}_{e,-(\widetilde{p}\cdot e)e})+\max_{\mathbb{R}} a_{per}\big(|\widetilde{p}|^2-(\widetilde{p}\cdot e)^2\big).$$

An easy computation yields

$$\overline{H}^\star(e,\widetilde{q})=\underline{H}^\star(e,\widetilde{q})\geq k^\star(\widetilde{q}\cdot e)+\frac{|\widetilde{q}-(\widetilde{q}\cdot e)e|^2}{4\max_{\mathbb{R}} a_{per}}\geq k^\star(\widetilde{q}\cdot e)$$

where $p\in\mathbb{R}\mapsto k(p)$ is the convex function $k(p):=\lambda_1^{per}(\widetilde{L}^{per}_p)$ (as defined in the statement of the Proposition). Moreover, one can easily check that

$$\overline{H}^\star\big(e,(\widetilde{q}\cdot e)e\big)=\underline{H}^\star\big(e,(\widetilde{q}\cdot e)e\big)=k^\star(\widetilde{q}\cdot e).$$

It follows that for any admissible path γ connecting 0 to a given $x\in\mathbb{R}^N$, one has

$$\begin{aligned}
\max_{t\in[0,1]}\int_t^1\underline{H}^\star\big(\gamma(s),-\gamma'(s)\big)ds &\geq \max_{t\in[0,1]}\int_t^1 k^\star\Big(-\frac{\gamma(s)\cdot\gamma'(s)}{|\gamma(s)|}\Big)ds\\
&\geq \max_{t\in[0,1]}(1-t)k^\star\Big(-\int_t^1\frac{\gamma(s)\cdot\gamma'(s)}{|\gamma(s)|}\Big) \quad \text{(by Holder inequality)}\\
&\geq \max\big\{0,k^\star(-|x|)\big\} \quad \text{(taking } t=0 \text{ or } 1\text{).}
\end{aligned}$$

Hence,

$$\inf_\gamma\max_{t\in[0,1]}\int_t^1\underline{H}^\star\big(\gamma(s),-\gamma'(s)\big)\geq\big(k^\star(-|x|)\big)_+.$$

The reverse inequality is obtained with $\gamma(s)=sx$. The conclusion follows from classical arguments. □

10 The space-independent case

10.1 Computation of the generalized principal eigenvalues in the space-independent case

We first compute the two generalized principal eigenvalues when the coefficients do not depend on x.

Proposition 10.1 *Consider an operator* $\mathcal{L}\phi=-\partial_t\phi+tr(A(t)\nabla^2\phi)+q(t)\cdot\nabla\phi+c(t)\phi$, *where* A *and* q *are functions of* t *that satisfy the hypotheses of Section 2.1 and* $c\in\mathcal{C}^{\delta/2}_{loc}(\mathbb{R})$

is uniformly continuous and bounded. Consider $\omega \subset \mathbb{R}^N$ an open set that contains balls of arbitrary radii and $R \in \mathbb{R}$. Then

$$\underline{\lambda_1}\big(\mathcal{L},(R,\infty)\times\omega\big) = \liminf_{t\to+\infty}\inf_{s>R}\frac{1}{t}\int_s^{s+t} c \text{ and } \overline{\lambda_1}\big(\mathcal{L},(R,\infty)\times\omega\big) = \limsup_{t\to+\infty}\sup_{s>R}\frac{1}{t}\int_s^{s+t} c.$$

In order to prove this Proposition, we first prove that we can restrict ourselves to test-functions that only depend on t in the definition of $\underline{\lambda_1}$ and $\overline{\lambda_1}$:

Lemma 10.2 *Under the same hypotheses as in Proposition 10.1, one has*

$$\begin{aligned}&\underline{\lambda_1}\big(\mathcal{L},(R,\infty)\times\omega\big)\\ &= \sup\{\lambda\in\mathbb{R}, \exists\phi\in W^{1,\infty}(R,\infty)\cap\mathcal{C}^1(R,\infty), \inf_{(R,\infty)}\phi>0, -\phi'+c(t)\phi\geq\lambda\phi \text{ in } (R,\infty)\},\\ &\overline{\lambda_1}\big(\mathcal{L},(R,\infty)\times\omega\big)\\ &= \inf\{\lambda\in\mathbb{R}, \exists\phi\in W^{1,\infty}(R,\infty)\cap\mathcal{C}^1(R,\infty), \inf_{(R,\infty)}\phi>0, -\phi'+c(t)\phi\leq\lambda\phi \text{ in } (R,\infty)\}.\end{aligned} \tag{102}$$

Proof. Define

$$\underline{\mu_1} = \sup\{\lambda\in\mathbb{R}, \exists\phi\in W^{1,\infty}(R,\infty)\cap\mathcal{C}^1(R,\infty), \inf_{(R,\infty)}\phi>0, -\phi'+c(t)\phi\geq\lambda\phi \text{ for all } t>R\}. \tag{103}$$

Clearly, $\underline{\mu_1}\leq\underline{\lambda_1}$. Consider $\lambda\in\mathbb{R}$ such that there exists $\phi\in\mathcal{C}^{1,2}\big((R,\infty)\times\omega\big)$ with $\inf\phi>0$, $\phi\in W^{1,\infty}\big((R,\infty)\times\omega\big)$ and $\mathcal{L}\phi\geq\lambda\phi$. For all $n\in\mathbb{N}$, we know that there exists a ball of radius n in ω. Let x_n its center. We define

$$\phi_n(t) = \frac{1}{|B(x_n,n)|}\int_{B(x_n,n)}\phi(t,x)dx.$$

Clearly, $\inf_{(R,\infty)}\phi_n \geq \inf_{(R,\infty)\times\omega}\phi > 0$ for all n and $\|\phi_n\|_{W^{1,\infty}(R,\infty)} \leq \|\phi\|_{W^{1,\infty}\left((R,\infty)\times\omega\right)}$. The Ascoli theorem yields that we can assume, up to extraction, the existence of a continuous function ϕ_∞ such that $\phi_n\to\phi_\infty$ locally uniformly in (R,∞) as $n\to+\infty$. One has $\inf_{(R,\infty)}\phi_\infty \geq \inf_{(R,\infty)\times\omega}\phi>0$ and $\|\phi_\infty\|_{W^{1,\infty}(R,\infty)}\leq\|\phi\|_{W^{1,\infty}\left((R,\infty)\times\omega\right)}$.

On the other hand, integrating $\mathcal{L}\phi\geq\lambda\phi$ over $B(x_n,n)\subset\omega$, one gets

$$-\phi_n'(t)+\frac{1}{|B(x_n,n)|}\int_{\partial B(x_n,n)}\nu\cdot(A(t)\nabla\phi)d\sigma+\frac{1}{|B(x_n,n)|}\int_{\partial B(x_n,n)}q(t)\cdot\nu\phi d\sigma+c(t)\phi_n\geq\lambda\phi_n,$$

for all $t>R$, where ν is the outward unit normal to $B(x_n,n)$. Letting $n\to+\infty$, we obtain

$$-\phi_\infty'(t)+c(t)\phi_\infty\geq\lambda\phi_\infty \text{ almost everywhere in } (R,\infty)$$

since $\phi\in W^{1,\infty}\big((R,\infty)\times\omega\big)$.

We just need to check that we can assume the test-function to be smooth in order to conclude. Consider a convolution kernel K, that is, a smooth nonnegative function such that $\int_{\mathbb{R}}K=1$. Set $K_\sigma(t)=\frac{1}{\sigma}K(t/\sigma)$. Take $\varepsilon>0$ and let σ small enough so that $\|K_\sigma\star c-c\|_\infty\leq\varepsilon$.

Define $\ln\psi := K_\sigma \star \ln\phi_\infty$. Then $\psi \in W^{1,\infty}(R,\infty)\cap\mathcal{C}^1(R,\infty)$, $\inf_{(R,\infty)}\psi > 0$ and for all $t > R$:

$$-\frac{\psi'(t)}{\psi(t)} = K_\sigma \star \frac{-\phi'_\infty}{\phi_\infty} \geq \lambda - K_\sigma \star c(t) \geq \lambda - \varepsilon - c(t).$$

Thus, $\underline{\mu_1} \geq \lambda - \varepsilon$. As this is true for all $\varepsilon > 0$ and $\lambda < \underline{\lambda_1}$, one finally gets $\underline{\mu_1} \geq \underline{\lambda_1}$ and thus $\underline{\mu_1} = \underline{\lambda_1}$. The other equality is obtained similarly. □

Proof of Proposition 10.1.

1. Consider first some λ such that there exists $\phi \in W^{1,\infty}(R,\infty)\cap\mathcal{C}^1(R,\infty)$ with $\inf_{(R,\infty)}\phi > 0$ and $-\phi' + c(t)\phi \geq \lambda\phi$ for all $t > R$. Dividing by ϕ and integrating between s and $s+t$ for $s > R$ and $t > 0$, one gets

$$\ln\phi(s+t) - \ln\phi(s) \leq \int_s^{s+t} c - \lambda t.$$

Hence

$$\lambda + \frac{1}{t}\big(\ln \inf_{(R,\infty)} \phi - \ln \sup_{(R,\infty)} \phi\big) \leq \inf_{s>R} \frac{1}{t}\int_s^{s+t} c.$$

Taking the liminf when $t \to +\infty$, one gets

$$\lambda \leq \liminf_{t\to+\infty} \inf_{s>R} \frac{1}{t}\int_s^{s+t} c.$$

Thus $\underline{\lambda_1}\big(\mathcal{L}, (R,\infty)\times\omega\big) \leq \liminf_{t\to+\infty}\inf_{s>R}\frac{1}{t}\int_s^{s+t} c$ using Lemma 10.2.

2. Next, consider any small $\varepsilon > 0$ and let $\lambda := \liminf_{t\to+\infty}\inf_{s>R}\frac{1}{t}\int_s^{s+t} c - 2\varepsilon < \sup_{(R,\infty)} c$. In order to prove that $\underline{\lambda_1} \geq \lambda$, we need to construct an appropriate test-function ϕ. Up to some decreasing of ε, we can define ϕ the solution of the Cauchy problem

$$\begin{cases} \phi' = (c(t)-\lambda)\phi - \phi^2 \text{ in } (R,\infty), \\ \phi(R) = \phi_0, \end{cases} \tag{104}$$

with ϕ_0 an arbitrary initial datum in $\big(\varepsilon, \sup_{(R,\infty)} c - \lambda\big)$. Clearly, $-\phi' + c(t)\phi \geq \lambda\phi$ for all $t > R$ and as

$$\phi' \leq (\sup_{(R,\infty)} c - \lambda)\phi - \phi^2,$$

one has $0 \leq \phi \leq \sup_{(R,\infty)} c - \lambda$. Hence, $\phi \in W^{1,\infty}\big((R,\infty)\big)$. It is left to prove that $\inf_{(R,\infty)}\phi > 0$ in order to conclude that $\underline{\lambda_1} \geq \lambda$.

3. The definition of λ yields that

$$\text{there exists } T > 0 \text{ such that for all } t > T \text{ and } s > R, \text{ one has } \frac{1}{t}\int_s^{s+t} c \geq \lambda + \varepsilon. \tag{105}$$

Moreover, it clearly follows from (104) that ϕ'/ϕ is bounded over (R,∞) by some constant $M > 0$ (which depends on c and λ), which means that $\ln\phi$ is Lipschitz-continuous.

We will now prove that $\phi(s) \geq \phi(R)e^{-MT}$ for all $s > R$ and some $M > 0$. Assume that there exists $s > R$ such that $\phi(s) < \varepsilon$ and let

$$s_\varepsilon := \sup\{t < s, \phi(t) \geq \varepsilon\} \quad \text{and} \quad T_\varepsilon := \sup\{t > s_\varepsilon, \phi(t) \leq \varepsilon\} \in (s, \infty].$$

As $\phi(R) = \phi_0 > \varepsilon$, one has $s_\varepsilon > R$. Then $\phi(t) \leq \varepsilon$ for all $t \in (s_\varepsilon, T_\varepsilon)$ and thus $\phi'(t) \geq (c(t) - \lambda - \varepsilon)\phi(t)$ for all $t \in (s_\varepsilon, T_\varepsilon)$. Moreover, $\phi(s_\varepsilon) = \varepsilon$, which gives for all $t \in (0, T_\varepsilon - s_\varepsilon)$:

$$\phi(s_\varepsilon + t) \geq \varepsilon \exp\Big(\int_{s_\varepsilon}^{s_\varepsilon + t} c(s')ds' - (\lambda + \varepsilon)t \Big). \tag{106}$$

If $t > T$, then (105) gives $\phi(s_\varepsilon + t) \geq \varepsilon$. Thus, $T_\varepsilon \leq T + s_\varepsilon$. On the other hand, as $\ln \phi$ is Lipschitz-continuous for some constant M, one gets

$$\phi(s_\varepsilon + t) \geq \phi(s_\varepsilon)e^{-Mt} \geq \varepsilon e^{-MT} \quad \text{for all } t \in (0, T_\varepsilon - s_\varepsilon).$$

Finally, this gives $\phi(s) \geq \varepsilon e^{-MT}$ for all $s > R$.

4. Taking ϕ as a test-function in the definition of $\underline{\lambda_1}$, we obtain

$$\underline{\lambda_1} \geq \lambda = \liminf_{t \to +\infty} \inf_{s > R} \frac{1}{t} \int_s^{s+t} c - 2\varepsilon.$$

As this is true for all $\varepsilon > 0$, we conclude that $\underline{\lambda_1} \geq \liminf_{t\to+\infty} \inf_{s>R} \frac{1}{t}\int_s^{s+t} c$. Step 1. gives the reverse inequality. The proof for $\overline{\lambda_1}$ is similar. □

Let us mention that, as soon as Lemma 10.2 is known, one could prove Proposition 10.1 in a different way by using Lemma 3.2 in [75].

10.2 Computation of the speeds in the space-independent case

Proof of Proposition 3.9. Using the same notations as in the Proposition, we notice that Proposition 10.1 implies

$$\underline{H}(e, p) = \lim_{R\to+\infty, \alpha\to 0} \underline{\lambda_1}(L_p, C_{R,\alpha}(e)) = \lim_{R\to+\infty} \liminf_{t\to+\infty} \inf_{s>R} \frac{1}{t} \int_s^{s+t} (|p|^2 + f'_u(s', 0))ds'.$$

Let $\lfloor f \rfloor = \lim_{R\to+\infty} \liminf_{t\to+\infty} \inf_{s>R} \frac{1}{t} \int_s^{s+t} f'_u(s', 0)ds'$. Then,

$$\underline{w}(e) = \min_{p\cdot e > 0} \frac{\underline{H}(e, -p)}{p \cdot e} = \min_{p\cdot e>0} \frac{p^2 + \lfloor f \rfloor}{p \cdot e} = 2\sqrt{\lfloor f \rfloor}.$$

The computation of $\overline{w}(e)$ is similar. □

Proof of Proposition 3.10. We immediately get from Proposition 10.1 that

$$\overline{H}(e, p) = \underline{H}(e, p) = p\langle A\rangle p - \langle q\rangle p + \langle c\rangle.$$

The conclusion follows. □

11 The directionally homogeneous case

We will start this section by addressing the issue of existence of exact asymptotic spreading speeds for directionally homogeneous coefficients in $\mathbb{R}^2$. That is, when the coefficients are close to constants in radial sectors of $\mathbb{R}^2$ for sufficiently large $|x|$, we want to derive conditions ensuring that $\overline{\mathcal{S}} = \underline{\mathcal{S}}$. Indeed, when there only exists a finite number of such segments, such an equality holds.

It is well-known that discontinuous coefficients in Hamilton-Jacobi equations could cause a lack of uniqueness for the solutions. Indeed, comparison principles may fail (see [97] for such a counter-example). It is thus natural to try to identify conditions on the Hamiltonians ensuring uniqueness, but there are not many works on this topic (see [7, 97, 100] and the references therein). Another type of problems is to introduce additional properties on the solutions ensuring uniqueness (see for example [5, 45]), which is not relevant in the present framework since Z^* and Z_* are obtained as limits for which we do not have such properties. None of these references was directly well fitted to our present framework since we treat here a highly nonlinear equation involving convex conjugates. We thus needed to adapt the method developed in [97].

Proposition 11.1 *Assume that $N = 2$ and let identify $\mathbb{S}^1$ and $\mathbb{R}/\mathbb{Z}$. Assume that there exist $0 = e_0 < e_1 < \ldots < e_r < 1$, and a family of functions $H_1, \ldots, H_r$, such that for all $p \in \mathbb{R}^N$, for all $i \in [0, r-1]$:*

$$\forall e \in (e_i, e_{i+1}), \quad \underline{H}(e,p) = \overline{H}(e,p) = H_i(p).$$

Assume furthermore that for all $i \in [0, r]$, one has either $H_i(p) \geq H_{i+1}(p)$ for all $p \in \mathbb{R}^N$ or $H_i(p) \leq H_{i+1}(p)$ for all $p \in \mathbb{R}^N$, where $H_{r+1} := H_0$ by convention. Then $\overline{\mathcal{S}} = \underline{\mathcal{S}}$.

Proof. Consider an admissible path γ, that is, a function of $H^1([0,1], \mathbb{R}^2)$ such that $\gamma(0) = 0$, $\gamma(1) = x$ and $\gamma(s) \neq 0$ for all $s \in (0,1)$. We can construct a finite sequence of closed, nonempty, consecutive intervals $(I_k)_{k \in [1,K]}$ of $[0,1]$, which possibly intersect only at their extrema, whose union is $[0,1]$ and such that for all k:

- either there exists $j \in [1,n]$ such that $e_j < \gamma(s)/|\gamma(s)| < e_{j+1}$ for all s in the interior of I_k,
- or there exists $j \in [1,n]$ such that $\gamma(s)/|\gamma(s)| = e_j$ for all $s \in I_k$.

We do not modify the path γ in the intervals belonging to the first class. Consider an interval $I_k = [t_k, t_{k+1}]$ such that $\phi(s)/|\phi(s)| = e_j$ for some j in I_k.

By hypothesis, one has

$$\underline{H}(e,p) = \overline{H}(e,p) = \begin{cases} H_{j-1}(p) & \text{if} \quad e \in (e_{j-1}, e_j), \\ H_j(p) & \text{if} \quad e \in (e_j, e_{j+1}), \end{cases}$$

where we let $e_{-1} := e_r$ if needed, remembering that we have identified $\mathbb{S}^1$ and $\mathbb{R}/\mathbb{Z}$. Our hypotheses yield that one can assume $H_{j-1}(p) \leq H_j(p)$ for all p, which implies $-H_{j-1}^\star(q) \leq -H_j^\star(q)$ for all q.

As $\overline{H}(e,p)$ is upper semicontinuous with respect to e, one gets $\overline{H}(e_j,p) = H_j(p)$ for all $p \in \mathbb{R}^N$ and thus, as $\gamma/|\gamma| = e_j$ over I_k,

$$\int_t^{t_{k+1}} \overline{H}^\star\Big(\frac{\gamma(s)}{|\gamma(s)|}, -\gamma'(s)\Big)ds = \int_t^{t_{k+1}} H_j^\star\big(-\gamma'(s)\big)ds \tag{107}$$

for all $t \in I_k$.

Let ξ the orthonormal vector to e_j pointing in the radial segment where $\underline{H} = H_j$ (see Figure 11). Take $\delta > 0$ small and define the modified path in $I_k = [t_k, t_{k+1}]$:

$$\gamma_\delta(s) := \begin{cases} \gamma(t_k) + (s - t_k)\xi & \text{if} \quad t_k \le s \le t_k + \delta, \\ \delta\xi + \gamma(s-\delta) & \text{if} \quad t_k + \delta \le s \le t_{k+1} - \delta, \\ \frac{1}{\delta}\Big((t_{k+1} - s)\big(\delta\xi + \gamma(t_{k+1} - 2\delta)\big) + (s - t_{k+1} + \delta)\gamma(t_{k+1})\Big) & \text{if} \quad t_{k+1} - \delta \le s \le t_{k+1}. \end{cases}$$

The construction of γ_δ is illustrated in Figure 11.

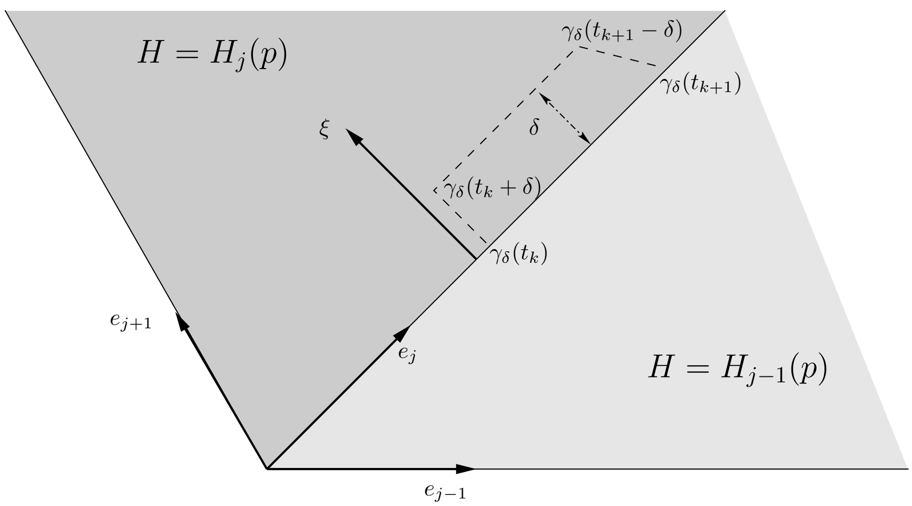

Figure 5: Construction of the modified path γ_δ.

Taking δ small enough, it is clear that $e_j < \frac{\gamma_\delta(s)}{|\gamma_\delta(s)|} < e_{j+1}$ for all $s \in (t_k, t_{k+1})$ (where we have identified $\mathbb{S}^1$ and $\mathbb{R}/\mathbb{Z}$) and thus $\underline{H}^\star\big(\frac{\gamma_\delta(s)}{|\gamma_\delta(s)|}, -\gamma_\delta'(s)\big) = H_j^\star\big(-\gamma_\delta'(s)\big)$. Moreover, as H_j is Locally Lipschitz-continuous by Proposition 2.2, one can easily show that there exists a constant $C > 0$ such that:

$$\Big|\int_t^{t_{k+1}} H_j^\star\big(-\gamma_\delta'(s)\big)ds - \int_t^{t_{k+1}} H_j^\star\big(-\gamma'(s)\big)ds\Big| \le C\delta \tag{108}$$

for all $t \in I_k$. Combining (107) and (108), we get

$$\int_t^{t_{k+1}} \overline{H}^\star\Big(\frac{\gamma(s)}{|\gamma(s)|}, -\gamma'(s)\Big)ds \ge \int_t^{t_{k+1}} \underline{H}^\star\Big(\frac{\gamma_\delta(s)}{|\gamma_\delta(s)|}, -\gamma_\delta'(s)\Big)ds - C\delta.$$

Repeating this construction on each such set I_k, we eventually obtain an admissible path γ_δ for each $\delta > 0$ small enough and a constant $C > 0$ such that for all $t \in [0,1]$:

$$\max_{t\in[0,1]} \int_t^1 \overline{H}^\star\Big(\frac{\gamma(s)}{|\gamma(s)|}, -\gamma'(s)\Big)ds \geq \max_{t\in[0,1]} \int_t^1 \underline{H}^\star\Big(\frac{\gamma_\delta(s)}{|\gamma_\delta(s)|}, -\gamma_\delta'(s)\Big)ds - C\delta.$$

The definition of $\underline{U}$ and $\overline{U}$ thus implies:

$$\overline{U}(x) \geq \underline{U}(x) - C\delta$$

and thus $\underline{U} \leq \overline{U}$. On the other hand, $\overline{H} \geq \underline{H}$ gives $\overline{U} \leq \underline{U}$. Hence $\underline{U} \equiv \overline{U}$ and thus $\overline{\mathcal{S}} = \underline{\mathcal{S}}$.
□

We are now in position to prove the results of Section 3.9.

Proof of Proposition 3.13. It is easy to see that

$$\underline{H}(e,p) = \overline{H}(e,p) = \begin{cases} a_+|p|^2 + f'(0) & \text{if} \quad e_1 > 0 \\ a_-|p|^2 + f'(0) & \text{if} \quad e_1 < 0 \end{cases}$$

since the coefficients converge uniformly in the truncated cones $C_{R,\alpha}(e)$ when $e_1 \neq 0$ and α is small enough. The semicontinuity yields $\underline{H}(\pm e_2, p) = a_-|p|^2 + f'(0)$ and $\overline{H}(\pm e_2, p) = a_+|p|^2 + f'(0)$.

Proposition 11.1 yields that we only need to compute

$$\begin{aligned} U(x) &= \inf\Big\{ \max_{t\in[0,1]} \textstyle\int_t^1 \overline{H}^\star\big(\frac{\gamma(s)}{|\gamma(s)|}, -\gamma'(s)\big)ds, \quad \gamma(0)=0,\ \gamma(1)=x,\ \gamma(s)\neq 0 \text{ for all } s\in(0,1)\Big\} \\ &= \inf\Big\{ \max_{t\in[0,1]} \textstyle\int_t^1 \frac{|\gamma'(s)|^2}{4a\big(\gamma(s)\big)}ds - f'(0)(1-t), \quad \gamma(0)=0,\ \gamma(1)=x,\ \gamma(s)\neq 0 \text{ for all } s\in(0,1)\Big\}. \end{aligned} \tag{109}$$

Such minimization problems are very close to other problems arising in geometric optics. The function

$$N(x) := \begin{cases} 1/4a_+ & \text{if} \quad x_1 \geq 0, \\ 1/4a_- & \text{if} \quad x_1 < 0, \end{cases}$$

can be viewed as a refraction index and the geodesics are the ray paths.

First notice that if $x \in \mathbb{R}^2$ satisfies $x_1 \geq 0$, then as $a_+ > a_-$, the function $\gamma(s) = sx$ minimizes (109) and thus

$$U(x) = \Big(|x|^2/4a_+ - f'(0)\Big)_+ \qquad \text{if } x_1 \geq 0.$$

More generally, as $a_- < a_+$, one always has $U(x) \geq |x|^2/4a_+ - f'(0)$ and thus $|x| > 2\sqrt{a_+ f'(0)}$ implies $U(x) > 0$. Consider now $x \in \mathbb{R}^2$ such that $x_1 < 0$ and $|x| \leq 2\sqrt{a_+ f'(0)}$.

3. Next, Lemma 5.7 yields that $U(x)$ is indeed a minimum. Take γ an admissible path. As γ is a minimizer, we can extract some properties of γ from the Euler-Lagrange equation associated with the minimization problem. Let

$$\tau = \max\{s \in [0,1),\ \gamma_1(s) \geq 0\},$$

where $\gamma_1(s)$ is the first coordinate of $\gamma(s)$. As γ is continuous, $\gamma(0) = 0$ and $\gamma_1(1) = x_1 < 0$, this maximum is well-defined. One has $\gamma_1(\tau) = 0$ and $\gamma_1(s) < 0$ for all $s \in (\tau, 1]$.

Next, assume that $\tau > 0$ and define

$$\widetilde{\gamma}(s) = \begin{cases} \frac{s}{\tau}\gamma(\tau) & \text{if} \quad s \in [0,\tau], \\ \frac{s-\tau}{1-\tau}x + \frac{s-\tau}{1-\tau}\gamma(\tau) & \text{if} \quad s \in (\tau, 1]. \end{cases}$$

One can take $\widetilde{\gamma}$ as a test-function in (109), which gives

$$\begin{aligned} U(x) &\leq \max_{t\in[0,1]} \Big(\textstyle\int_t^1 N(\widetilde{\gamma}(s))|\widetilde{\gamma}'(s)|^2 ds - f'(0)(1-t)\Big) \\ &= \max\Big\{0, \quad \frac{|x-\gamma(\tau)|^2}{4a_-(1-\tau)} - f'(0)(1-\tau), \quad \frac{|\gamma(\tau)|^2}{4a_+\tau} + \frac{|x-\gamma(\tau)|^2}{4a_-(1-\tau)} - f'(0)\Big\}. \end{aligned} \tag{110}$$

On the other hand, the Cauchy-Schwarz inequality yields

$$|\gamma(\tau)|^2 = \Big|\int_0^\tau \gamma'(s)ds\Big|^2 \leq \tau \int_0^\tau |\gamma'(s)|^2 ds \quad \text{and}$$
$$\frac{|x-\gamma(\tau)|^2}{4a_-(1-\tau)} = \frac{1}{4a_-(1-\tau)}\Big|\int_\tau^1 \gamma'\Big|^2 \leq \frac{1}{4a_-}\int_\tau^1 |\gamma'|^2$$

and these inequalities are equalities if and only if γ' is constant in $(0,\tau)$ and $(\tau, 1)$. Hence, the definition of $U(x)$ yields that (110) is smaller than $U(x)$ and thus γ' is constant in $(0,\tau)$ and in $(\tau, 1)$.

If $\tau = 0$ then $\gamma(s) = sx$ and thus $U(x) = \frac{|x|^2}{4a_-} - f'(0)$ in this case.

4. Assume that $\tau > 0$ and let $y = \gamma(\tau)$. We know that $y_1 = \gamma_1(\tau) = 0$. We assume that $x_2 \geq 0$, the case $x_2 < 0$ can be treated similarly. It is then easy to check that $y_2 \geq 0$, otherwise $\varphi(s) = sx$ is a better minimizer of (109), which is impossible. Similarly, one can prove that $\tau > 0$ implies $x_2 \neq 0$ and $y_2 \neq 0$.

For all $\sigma \in (0,1)$ and $z \in \mathbb{R}$, we define

$$\varphi_{\sigma,z}(s) = \begin{cases} \dfrac{sze_2}{\sigma} & \text{if} \quad s \in [0,\sigma], \\ \dfrac{(s-\sigma)x}{(1-\sigma)} + \dfrac{(1-s)ze_2}{(1-\sigma)} & \text{if} \quad s \in [\sigma, 1], \end{cases} \tag{111}$$

where e_2 is the unit vector associated with the second coordinate axis. We have proved in the previous step that $\gamma = \varphi_{\tau,y_2}$. But as any function of the form (111) is an appropriate test-function for the minimization problem (109), we get

$$U(x) = \min\Big\{\max\Big\{0, \quad \frac{|x-ze_2|^2}{4a_-(1-\sigma)} - f'(0)(1-\sigma), \quad \frac{z^2}{4a_+\sigma} + \frac{|x-ze_2|^2}{4a_-(1-\sigma)} - f'(0)\Big\}, \ \sigma \in (0,1), \ z \in \mathbb{R}\Big\} \tag{112}$$

and this minimum is reached when $\sigma = \tau$ and $z = y_2$.

Take $x \in \mathbb{R}^2$ such that $U(x) > 0$. Assume first that $|y| < 2\sqrt{f'(0)a_+}\tau$. Then

$$U(x) = \frac{|x-y|^2}{4a_-(1-\tau)} - f'(0)(1-\tau)$$

and $z = y_2$, $\sigma = \tau$ is a local minimizer of

$$(z, \tau) \mapsto \frac{|x - ze_2|^2}{4a_-(1-\sigma)} - f'(0)(1-\sigma),$$

which is a contradiction since this function is increasing with respect to σ and $\tau > 0$. Hence $|y| \geq 2\sqrt{f'(0)a_+}\tau$.

Next, assume that $|y| = 2\sqrt{f'(0)a_+}\tau$. Then τ is a minimizer of

$$\sigma \in (0,1) \mapsto \frac{|x - 2\sqrt{a_+ f'(0)}\sigma e_2|^2}{4a_-(1-\sigma)} - f'(0)(1-\sigma).$$

Derivating this function and computing, one obtains:

$$|x - 2\sqrt{a_+ f'(0)}e_2|^2 = 4f'(0)(1-\tau)^2(a_+ - a_-),$$

which gives, after some more computations:

$$U(x) = \frac{\sqrt{a_+ f'(0)}\big(x_2 - 2\sqrt{a_+ f'(0)}\big)}{4a_-}.$$

This yields a contradiction since $x_2 < |x| \leq 2\sqrt{a_+ f'(0)}$ and thus $U(x) = 0$.

Lastly, if $|y| > 2\sqrt{f'(0)a_+}\tau$, then as (τ, y_2) is a critical point for the right-hand side, one has

$$\begin{cases} \dfrac{y_2^2}{a_+\tau^2} & = & \dfrac{|x-y|^2}{a_-(1-\tau)^2}, \\ \\ \dfrac{y_2}{a_+\tau} & = & \dfrac{x_2 - y_2}{a_-(1-\tau)}. \end{cases} \tag{113}$$

Taking the square of the second line of (113) and multiplying by a_+, one gets

$$a_+(x_2 - y_2)^2 = a_-|x-y|^2. \tag{114}$$

In other words, $x_2 - y_2 = r|x_1|$, where

$$r := \sqrt{\frac{a_-}{a_+ - a_-}}$$

and, as $y_2 > 0$, one gets $x_2 > r|x_1|$. Using the second line of (113) to compute τ, one gets

$$\tau = \left(1 + \frac{a_+}{a_-} \times \frac{|x_1|r}{x_2 - r|x_1|}\right)^{-1}. \tag{115}$$

Eventually, a straightforward computation gives

$$U(x) = \frac{y_2^2}{4a_+\tau^2} - f'(0) = \frac{1}{4a_+}\left(x_2 + \frac{|x_1|}{r}\right)^2 - f'(0).$$

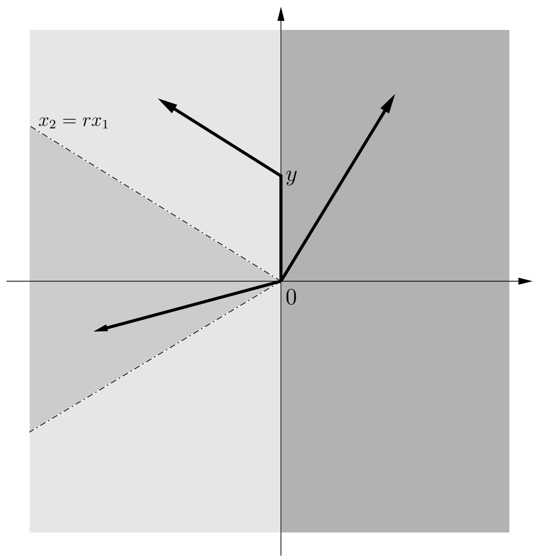

Figure 6: This figure represents the geodesics of the minimization problem (109). The darker area corresponds to the case $x_1 > 0$ and the lighter one to the case $x_1 < 0$ and $|x_1| \geq r|x_2|$. The large arrows represent the ray paths in each of these areas.

Similarly, one can prove that if $x_2 < 0$, then $-x_2 > r|x_1|$ and

$$U(x) = \frac{y_2^2}{4a_+\tau} - f'(0) = \frac{1}{4a_+}\Big(- x_2 + \frac{|x_1|}{r}\Big)^2 - f'(0).$$

5. There only remains to identify the condition $\tau > 0$ in order to conclude. We have already checked that $\tau > 0$ implies $|x_2| > r|x_1|$. On the other hand, if $|x_2| > r|x_1|$, then letting

$$\sigma = \Big(1 + \frac{a_+}{a_-} \times \frac{|x_1|r}{|x_2| - r|x_1|}\Big)^{-1} \text{ and } z = \begin{cases} x_2 - r|x_1| & \text{if} \quad x_2 > 0 \\ x_2 + r|x_1| & \text{if} \quad x_2 < 0 \end{cases},$$

the same computations as above gives

$$\int_0^1 N(\varphi_{\sigma,z}(s))|\varphi'_{\sigma,y_2}(s)|^2 ds = \frac{1}{4a_+}\big(|x_2| + \frac{|x_1|}{r}\big)^2.$$

On the other hand, we know that if $\tau = 0$, then $\gamma(s) = sx$ and $U(x) = \frac{|x|^2}{4a_-} - f'(0)$. But the condition $x_2 > rx_1$ then yields

$$U(x)+f'(0) = \frac{|x_1|^2 + |x_2|^2}{4a_-} = \frac{1}{4a_+}\Big(\frac{1}{r^2}+1\Big)|x|^2 = \frac{1}{4a_+}\Big(\frac{x_2}{r}-x_1\Big)^2+\frac{1}{a_+}\Big(x_2+\frac{x_1}{r}\Big)^2 > \frac{1}{4a_+}\big(x_2+\frac{x_1}{r}\big)^2.$$

Hence, γ is not a minimizer of (109), which is a contradiction. We derive a similar contradiction if $-x_2 > r|x_1|$. We conclude that $\tau > 0$ if and only if $|x_2| > r|x_1|$.

Gathering all these facts, we have proved that

$$U(x) + f'(0) = \begin{cases} |x|^2/4a_- & \text{if} \quad x_1 < 0 \text{ and } |x_1| \leq r|x_2|, \\ |x|^2/4a_+ & \text{if} \quad x_1 \geq 0, \\ \frac{1}{4a_+}\big(|x_2| + \frac{|x_1|}{r}\big)^2 & \text{if} \quad x_1 < 0 \text{ and } |x_1| > r|x_2|. \end{cases}$$

Eventually, $U(x) = 0$ is the equation of two circles of radii $2\sqrt{a_+ f'(0)}$ for $x_1 \geq 0$ and $2\sqrt{a_- f'(0)}$ for $x_1 < 0$ and $|x_1| \leq r|x_2|$. For $x_1 < 0$ and $x_2 > r|x_1|$ or $x_2 < -r|x_1|$, it is the equation of a line, which is the frontier of the convex hull of the two half-circles. This ends the proof. $\square$

Remark: Note that the population leaves the set $\{x_1 \geq 0\}$ with an angle $\pi/2$ and enters $\{x_1 < 0\}$ with an angle θ given by $\tan\theta = \dfrac{x_2 - y_2}{|x_1|} = r = \sqrt{\dfrac{a_-}{a_+ - a_-}}$, which also reads

$$\sin\theta = \cos\theta \times \tan\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \sqrt{\frac{a_-}{a_+}}.$$

Hence, θ is characterized by $\frac{1}{\sqrt{a_-}}\sin\theta = \frac{1}{\sqrt{a_+}}\sin\pi/2$, which is the classical Snell-Descartes law for geometric optics, with refraction indexes $\frac{1}{\sqrt{a_\pm}}$, which is consistent with the local speeds $2\sqrt{f'(0)a_\pm}$ in each half-space. It is the first time, as as we know, that such a characterization is identified in a reaction-diffusion setting.

Proof of Proposition 3.14. As $\overline{H}(e,p) = \underline{H}(e,p)$ for all $e \neq e_2$ and $p \in \mathbb{R}^N$, by Proposition 11.1 we only need to characterize the set $\{U > 0\}$, where

$$U(x) := \min \max_{t\in[0,1]} \Big\{ \int_t^1 \Big(\tfrac{1}{4}|\gamma'(s)|^2 - \mu\big(\gamma(s)\big)\Big)ds, \quad \gamma \in H^1([0,1]),\ \gamma(0) = 0,\ \gamma(1) = x, \\ \gamma(s) \neq 0 \text{ for all } s \in (0,1)\Big\}, \tag{116}$$

where $\mu(x) = \mu_+$ if $x_1 \geq 0$, $\mu(x) = \mu_-$ if $x_1 < 0$. If $x_1 \geq 0$, then $\gamma(s) = sx$ minimizes (116) and $U(x) = \big(\frac{|x|^2}{4} - \mu_+\big)_+$. Otherwise, the same arguments as above yield that there exists a minimizer $\gamma = \varphi_{\tau,y_2}$ of (116) defined by (111), with $\tau \in [0,1)$and $y = \gamma(\tau)$, $y_1 = 0$, and the maximum with respect to $t \in [0,1]$ is reached when $t = 0, \tau$ or 1.

If $\tau = 0$, then $U(x) = \big(\frac{|x|^2}{4} - \mu_-\big)_+$. We will now compute $U(x)$ when $\tau > 0$ and characterize this situation. Assume that $x_2 \geq 0$, the case $x_2 < 0$ being treated similarly. If $x_2 = 0$, then it is easy to check that $\gamma(s) = sx$ minimizes (116), which contradicts $\tau > 0$. Putting $\gamma = \gamma_{\sigma,z}$ and $t = 0, \sigma$ or 1 in (116) gives

$$U(x) = \min_{\sigma\in(0,1), z\in\mathbb{R}} \max\Big\{0, \frac{|x - ze_2|^2}{4(1-\sigma)} - \mu_-(1-\sigma), \frac{|z|^2}{4\sigma} - \mu_+\sigma + \frac{|x - ze_2|^2}{4(1-\sigma)} - \mu_-(1-\sigma)\Big\}$$

where $\sigma = \tau$ and $z = y_2$ minimizes this quantity.

Let $x \in \mathbb{R}^2$ such that $U(x) > 0$ and $|x| \leq 2\sqrt{\mu_+}$. Assume first that $|y| > 2\sqrt{\mu_+}\tau$. Then

$$U(x) = \min_{\sigma\in(0,1), z\in\mathbb{R}} \Big(\frac{|z|^2}{4\sigma} - \mu_+\sigma + \frac{|x - ze_2|^2}{4(1-\sigma)} - \mu_-(1-\sigma)\Big)$$

and this minimum is reached when $\sigma = \tau$ and $z = y_2$. As y is a critical point of this function to minimize, one has:

$$\frac{y_2}{\tau} = \frac{x_2 - y_2}{1-\tau}, \quad \text{leading to} \quad y_2 = \tau x_2.$$

But as $|y| > 2\sqrt{\mu_+}\tau$, this implies $|x| \geq |x_2| > 2\sqrt{\mu_+}$, a contradiction.

Hence, $|y| \leq 2\sqrt{\mu_+}\tau$ and

$$U(x) = \min_{\sigma\in(0,1),z\in\mathbb{R}} \Big(\frac{|x - ze_2|^2}{4(1-\sigma)} - \mu_-(1-\sigma)\Big).$$

As the right hand-side is increasing with respect to σ, we necessarily have $\tau = \frac{|y|}{2\sqrt{\mu_+}}$. Thus, in this case:

$$U(x) = \min_{\sigma\in(0,1)} \Big(\frac{|x - 2\sqrt{\mu_+}\sigma e_2|^2}{4(1-\sigma)} - \mu_-(1-\sigma)\Big).$$

Then τ is a critical point for the right-hand side and

$$\frac{|x - 2\sqrt{\mu_+}\tau e_2|^2}{4(1-\tau)^2} + \mu_- - \frac{x_2 - y_2}{2(1-\tau)} 2\sqrt{\mu_+} = 0.$$

Developing this expression, we find

$$\frac{|x_2 - 2\sqrt{\mu_+}|^2 + x_1^2}{4(1-\tau)^2} - \mu_+ + \mu_- = 0.$$

Putting back this expression in the computation of $U(x)$, we find that

$$\begin{aligned} U(x) &= (1-\tau)\big\{\tfrac{x_2-y_2}{2(1-\tau)} 2\sqrt{\mu_+} - 2\mu_-\big\} \\ &= (x_2 - y_2)\sqrt{\mu_+} - 2\mu_-(1-\tau) \\ &= x_2\sqrt{\mu_+} - 2\mu_- - 2(\mu_+ - \mu_-)\tau \\ &= x_2\sqrt{\mu_+} - 2\mu_+ + \sqrt{\mu_+ - \mu_-}|x - 2\sqrt{\mu_+}e_2|. \end{aligned}$$

Hence, $U(x) > 0$ and $|x| \leq 2\sqrt{\mu_+}$ implies

$$\mu_+(x_2 - 2\sqrt{\mu_+})^2 < (\mu_+ - \mu_-)\big((x_2 - 2\sqrt{\mu_+})^2 + x_1^2\big),$$

which eventually yields

$$2\sqrt{\mu_+} - x_2 < \sqrt{\frac{\mu_+}{\mu_-} - 1}|x_1|.$$

It is easy to check that $U(x) > 0$ when $|x| > 2\sqrt{\mu_+}$.

Reciprocally, one can check that if $2\sqrt{\mu_+} - x_2 \geq \sqrt{\frac{\mu_+}{\mu_-} - 1}|x_1|$, then $U(x) = 0$. The case $x_2 < 0$ is treated similarly.

These computations also yield that $\tau > 0$ implies $|x - 2\sqrt{\mu_+}e_2| < 2\sqrt{\mu_+ - \mu_-}$, which reads on the frontier of the set $\{U = 0\}$:

$$|x_1| = -x_1 < 2\sqrt{\mu_+ - \mu_-}\sqrt{\frac{\mu_-}{\mu_+}}.$$

The same comparison argument as in the proof of Proposition 3.13 yields that the reciprocal is true. We have thus proved that

$$U(x) = \begin{cases} |x|^2/4 - \mu_- & \text{if} \quad x_1 < 0 \text{ and } |x_1| \geq 2\sqrt{\mu_+ - \mu_-}\sqrt{\frac{\mu_-}{\mu_+}}, \\ |x|^2/4 - \mu_+ & \text{if} \quad x_1 \geq 0, \\ x_2\sqrt{\mu_+} - 2\mu_+ + \sqrt{\mu_+ - \mu_-}|x - 2\sqrt{\mu_+}e_2| & \text{if} \quad x_1 < 0 \text{ and } |x_1| < 2\sqrt{\mu_+ - \mu_-}\sqrt{\frac{\mu_-}{\mu_+}}. \end{cases}$$

The fact that $\{U = 0\}$ is the convex envelope of the half-disk of radius $2\sqrt{\mu_-}$ in the half-plane $\{x_1 < 0\}$ and $2\sqrt{\mu_+}$ in the half-plane $\{x_1 > 0\}$ easily follows, by noting that $x_1 = -2\sqrt{\mu_+ - \mu_-}\sqrt{\frac{\mu_-}{\mu_+}}$ is the abscissa of the point of the circle of radius $2\sqrt{\mu_-}$ from which the tangent hits the point $(0, 2\sqrt{\mu_+})$.

$\square$

Proof of Proposition 3.15. We will only sketch this proof since it is very similar to that of Proposition 3.13. First, one has

$$\underline{H}(e,p) = \overline{H}(e,p) = \begin{cases} a_+p^2 + f'(0) & \text{if } |e_2| > r_0 e_1, \\ a_-p^2 + f'(0) & \text{if } |e_2| < r_0 e_1. \end{cases}$$

Hence, $\overline{\mathcal{S}} = \underline{\mathcal{S}} = \{x \in \mathbb{R}^2, U(x) = 0\}$, where $U(x)$ is defined by the same minimization problem as (109) except that now $N(x) = 1/4a_+$ if $|x_2| \geq r_0 x_1$ and $1/4a_-$ if $|x_2| < r_0 x_1$. Clearly, $U(x) = |x|^2/4a_+ - f'(0)$ if $|x_2| \geq r_0 x_1$. If $0 < x_2 < r_0 x_1$ (the case $0 > x_2 > -r_0 x_1$ being treated similarly), the minimizer γ associated with U can be written

$$\gamma(s) = \begin{cases} \frac{s}{\tau} y & \text{if} \quad s \in [0,\tau], \\ y + \frac{s-\tau}{1-\tau}(x-y) & \text{if} \quad s \in [\tau, 1], \end{cases}$$

where $\tau \in [0,1)$ is the time when the geodesic leaves the set $x_2 \geq r_0|x_1|$ and $y = \gamma(\tau)$, which imposes $y_2 = r_0 y_1$.

Let X_2 is the projection of x on the axis $x_2 = r_0 x_1$ and X_1 is the projection of x on the orthogonal axis. Let $\theta_0 := \arctan r_0$ and $\theta := \arctan r$, where we remind to the reader that r is defined by

$$r = \sqrt{\frac{a_-}{a_+ - a_-}}.$$

The inequality $rr_0 < 1$ reads $\theta < \pi/2 - \theta_0$. It is easy to check from this inequality that if (x_1, x_2) belongs to the line $X_2 = rX_1$, with $x_1 > 0$, then one has $x_2 < 0$. Thus, as we are currently considering the case $0 < x_2 < r_0 x_1$, we have proved that $rr_0 < 1$ ensures that $X_2 > rX_1$. This implies in particular that $\tau > 0$ is always satisfied in this area, as observed in the proof of Proposition 3.13, from which it follows that

$$U(x) + f'(0) = \frac{1}{4a_+}(X_2 + X_1/r)^2.$$

Hence, $U(x) = 0$ is the equation of a line when $0 < x_2 < r_0 x_1$. Similarly, one can prove that $U(x) = 0$ is the equation of another line when $0 > x_2 > -r_0 x_1$, and we have already shown that it is the equation of a circle when $|x_2| \geq r_0 x_1$. It only remains to compute this intersection point of the two lines.

If $x_2 = 0$, one has $X_1 = x_1 \sin\theta_0$ and $X_2 = x_1 \cos\theta_0$. Hence,

$$U(x) + f'(0) = \frac{1}{4a_+}\Big(x_1 \cos\theta_0 + \frac{x_1 \sin\theta_0}{r}\Big)^2 = \frac{x_1^2 \cos^2\theta_0}{4a_+}\Big(1 + \frac{r_0}{r}\Big)^2 = \frac{x_1^2}{4a_+(1+r_0^2)}\Big(1 + \frac{r_0}{r}\Big)^2.$$

Finally, the intersection point is $\Big(2\sqrt{f'(0)a_+(1+r_0^2)}/\big(1+\frac{r_0}{r}\big), 0\Big)$. The equation of the two lines in (x_1, x_2) can then easily be computed, leading to the conclusion. □

12 Proof of the spreading property with the alternative definition of the expansion sets and applications

The proof of Lemma 3.16 will rely on the following non-existence result. A similar result was proved in the one-dimensional setting [19]. Here, the new difficulty is to take into account what happens on the boundary of the truncated cylinder $C_{R,\alpha}(e)$.

Lemma 12.1 *Assume that $z \in \mathcal{C}^2(C_{R,\alpha}(e))$ is positive and satisfies*

$$-\Delta z + M|\nabla z| \geq \delta z \quad \text{in } C_{R,\alpha}(e) \tag{117}$$

for some $M > 0$ and $\delta > 0$. Then one cannot have $\lim_{|x|\to+\infty} \frac{\ln z(x)}{|x|} = 0$.

Proof of Lemma 12.1. We could assume that $e = e_1$. Even if it means decreasing δ, we could assume that $\delta < M^2$. Define

$$P(k) := -k^2 + Mk - \delta$$

and denote $\kappa_\pm := \frac{1}{2}\big(M \pm \sqrt{M^2 - \delta}\big) > 0$ its two roots. Assume first that $\lim_{|x|\to+\infty} \dfrac{\ln z(x)}{|x|} = \kappa > 0$, with $\kappa/\cos\alpha < \kappa_-$.

Take $\kappa' > 0$ such that $\kappa' < \kappa < \frac{\kappa'}{\cos\alpha} < \kappa_-$. By monotonicity of the generalized principal eigenvalues with respect to R, we can assume that R is large enough so that

$$z(x) \geq e^{\kappa'|x|} \quad \text{if } |x| \geq R.$$

Similarly, we can take α small enough so that $\kappa_+ \cos^2\alpha > \kappa_-$.

Define

$$\underline{z}^B(x) := Ae^{\frac{\kappa'}{\cos\alpha}x_1} - Be^{\kappa_+ x_1} \quad \text{for all } x \leq \frac{1}{\kappa_+ - \frac{\kappa'}{\cos\alpha}} \ln\Big(\frac{A\kappa'}{B\kappa_+\cos\alpha}\Big) =: X_B$$

$$\text{with} \quad A = e^{-\frac{\kappa'}{\cos\alpha}R + \kappa' R} < 1$$

$$\text{and} \quad \underline{z}^B(x+X_B) := \underline{z}^B(X_B - x) \quad \text{for all } x \geq 0.$$

As $\underline{z}^B(X_B) = \max_R \underline{z}^B$, one has $(\underline{z}^B)'(X_B) = 0$ and the function $\underline{z}^B$ is $\mathcal{C}^1$ over $\mathbb{R}$. Also, note that $\underline{z}^B(x_1) \leq Ae^{\frac{\kappa'}{\cos\alpha}x_1}$ for all $x_1 \in \mathbb{R}$.

Let us check that $z \geq \underline{z}^B$ on $\partial\big(C_{R,\alpha}(e_1) \cap B_{R+h}(0)\big)$ for h large enough. When $|x| = R$, one has

$$\underline{z}^B(x) \leq Ae^{\frac{\kappa'}{\cos\alpha}x_1} \leq Ae^{\frac{\kappa'}{\cos\alpha}R} = e^{\kappa' R} \leq z(x).$$

When $x_1 = |x|\cos\alpha$, we compute

$$\underline{z}^B(x) \leq Ae^{\frac{\kappa'}{\cos\alpha}x_1} \leq e^{\kappa'|x|} \leq z(x).$$

As $\underline{z}^B$ is nondecreasing on $(-\infty, X_B)$, one has for all $x \leq X_B$:

$$-\Delta\underline{z}^B + M|\nabla\underline{z}^B| - \delta\underline{z}^B = -\partial_{x_1x_1}\underline{z}^B + M\partial_{x_1}\underline{z}^B - \delta\underline{z}^B = AP\big(\kappa'/\cos\alpha\big)e^{\frac{\kappa'}{\cos\alpha}x_1} \leq 0.$$

since $\kappa'/\cos\alpha < \kappa_-$. When $x_1 > X_B$, as $\underline{z}^B(x+X_B) := \underline{z}^B(X_B - x)$, one gets

$$-\Delta\underline{z}^B + M|\nabla\underline{z}^B| - \delta\underline{z}^B = -\partial_{x_1x_1}\underline{z}^B - M\partial_{x_1}\underline{z}^B - \delta\underline{z}^B = AP\big(\kappa'/\cos\alpha\big)e^{\frac{\kappa'}{\cos\alpha}x_1} \leq 0.$$

It remains to prove that $\underline{z}^B \leq z$ when $|x| = R + h$. Define:

$$h_B := -R + \frac{1}{\cos\alpha}X_B > 0 \quad \text{when } B \text{ is small enough.}$$

For all x such that $|x| = R + h$, with $h > h_B$, one has $R + h \geq x_1 \geq (R+h)\cos\alpha > X_B$. We thus write $x_1 = X_B + y_1$, with $y_1 > 0$, and we get:

$$\begin{aligned}
\underline{z}^B(y_1 + X_B) = \underline{z}^B(X_B - y_1) &\leq Ae^{\frac{\kappa'}{\cos\alpha}(X_B - y_1)} \\
&\leq Ae^{\frac{2\kappa'}{\cos\alpha}X_B - \kappa'(R+h)} = e^{\frac{\kappa'}{\cos\alpha}(2X_B - R) - \kappa' h} \\
&\leq e^{\kappa'(R+h)} \quad \text{as } h \geq h_B \\
&\leq e^{\kappa'|x|} = z(x).
\end{aligned}$$

We have used here the obvious inequality $2h_B \geq \frac{2X_B - R}{\cos\alpha} - R$. Hence, $\underline{z}^B$ is a subsolution of (117) on $C_{R,\alpha}(e_1)$.

The sub and super solution method provides a solution Z of (117), with $Z = z$ on $\partial\big(C_{R,\alpha}(e_1) \cap B_{R+h}(0)\big)$, such that $0 \leq Z \leq z$, and Z is above all the nonnegative sub solutions. In particular, $z \geq \underline{z}^B$ on $C_{R,\alpha}(e_1) \cap B_{R+h}(0)$ for all $h \geq h_B$. Letting $B \to 0$, as $\lim_{h\to+\infty} B = 0$, as $h_B \to +\infty$, one gets

$$z(x) \geq Ae^{\frac{\kappa'}{\cos\alpha}x_1} \quad \text{in } C_{R,\alpha}(e_1).$$

Hence,

$$\lim_{x_1\to+\infty} \frac{1}{x_1}\ln z(x_1, 0) = \kappa \geq \frac{\kappa'}{\cos\alpha},$$

a contradiction with our choice of κ'.

Now, if $\lim_{|x|\to+\infty}\frac{\ln z(x)}{|x|}=0$, then $\widetilde{z}(x):=z(x)e^{\kappa|x|}$ satisfies the hypotheses of the previous step if κ is sufficiently small, and thus a contradiction follows. $\square$

Proof of Lemma 3.16. Assume that $\overline{\eta}(C_{R,\alpha}(e))<\underline{\eta}(C_{R,\alpha}(e))$ and let η',η'' such that

$$\underline{\eta}(C_{R,\alpha}(e))>\eta'>\eta''>\overline{\eta}(C_{R,\alpha}(e)).$$

There exist $\phi,\psi\in\mathcal{B}$ such that $\mathcal{L}\phi\geq\eta'\phi$ and $\mathcal{L}\psi\leq\eta''\psi$ in $C_{R,\alpha}(e)$. Let $z:=\psi/\phi$. The function z is nonnegative and $\lim_{|x|\to+\infty}\frac{\ln z(x)}{|x|}=0$ and satisfies

$$-a_{i,j}\partial_{i,j}z-\big(q_i+2a_{i,j}\frac{\partial_j\phi}{\phi}\big)\partial_i z\geq(\eta'-\eta'')z \text{ in } C_{R,\alpha}(e).$$

The contradiction then follows from Lemma 12.1. $\square$

Proof of Theorem 7. The proof is the same as that of Theorem 2. Indeed, one only needs to check that Proposition 5.3 holds with $\overline{H}(p)=\lim_{R\to+\infty}\overline{\eta_1}(L_{-p},C_{R,\alpha}(e))$ and $\underline{H}(p)=\lim_{R\to+\infty}\underline{\eta_1}(L_{-p},C_{R,\alpha}(e))$. The reader could check that the only place where the fact that the test-functions in the definition of the generalized principal eigenvalue are bounded and has positive infimum is equation (78). Indeed, with our alternative definitions based on $\underline{\eta_1}$ and $\overline{\eta_1}$, if ϕ is a test-function in the definition of $\underline{\eta_1}$ or $\overline{\eta_1}$, then $w(x):=\ln\phi(x)$ still satisfies

$$\varepsilon w(x/\varepsilon)\to 0 \text{ as } \varepsilon\to 0 \text{ locally in } x\in C_{R,\alpha}(e) \text{ since } \lim_{|x|\to+\infty} w(x)/|x|=0. \quad (118)$$

Then one could continue the proof as in that of Proposition 5.3 and, with the comparison $\overline{\eta_1}(C_{R,\alpha}(e))\geq\underline{\eta_1}(C_{R,\alpha}(e))$ in hand, conclude as in the proof of Theorem 2. Moreover, as the new Hamiltonians $\overline{H}(p)=\lim_{R\to+\infty}\overline{\eta_1}(L_{-p},C_{R,\alpha}(e))$ and $\underline{H}(p)=\lim_{R\to+\infty}\underline{\eta_1}(L_{-p},C_{R,\alpha}(e))$ do not depend on $e=x/|x|$, the expansion sets $\overline{T}$ and $\underline{T}$ could be written in a Wulff-type form with the same arguments as in Proposition 3.3. $\square$

Proof of Proposition 3.18. We know from [64] that v_p is semiconcave: there exists a constant C such that $\Delta v_p(x)\leq C$ for a.e. x. Hence, we compute

$$
\begin{aligned}
\nabla\varphi_p(x) &= \left(\left(1-\frac{L'(|x|)|x|}{L(|x|)}\right)\nabla v_p\big(x/L(|x|)\big)+\frac{L'(|x|)x}{|x|}v_p\big(x/L(|x|)\big)\right)\varphi_p(x),\\
\frac{\Delta\varphi_p(x)}{\varphi_p(x)} &= \frac{1}{L(|x|)}\left(1-\frac{L'(|x|)|x|}{L(|x|)}\right)^2\Delta v_p\big(x/L(|x|)\big)\\
&\quad -\left(\frac{L''(|x|)}{L(|x|)}-\frac{L'(|x|)^2}{L(|x|)^2}+\frac{L'(|x|)}{L(|x|)}\right)x\cdot\nabla v_p\big(x/L(|x|)\big)\\
&\quad +\left|\left(1-\frac{L'(|x|)|x|}{L(|x|)}\right)\nabla v_p\big(x/L(|x|)\big)+\frac{L'(|x|)x}{|x|}v_p\big(x/L(|x|)\big)\right|^2\\
&\quad +\left(\left(1-\frac{L'(|x|)|x|}{L(|x|)}\right)\frac{L'(|x|)x}{L(|x|)|x|}\nabla v_p\big(x/L(|x|)\big)\right)+L''(|x|)v_p\big(x/L(|x|)\big)\\
&\le \left|\nabla v_p\big(x/L(|x|)\big)\right|^2+o_{|x|\to+\infty}(1).
\end{aligned}
$$

This gives

$$\limsup_{|x|\to+\infty}\frac{L_p\varphi_p(x)-H(p)\varphi_p(x)}{\varphi_p(x)}\le 0.$$

Moreover, one has

$$\frac{\ln\varphi_p(x)}{|x|}=\frac{L(|x|)v_p\big(x/L(|x|)\big)}{|x|}\to 0 \quad \text{as } |x|\to+\infty$$

since v_p is periodic and $L(|x|)/|x|\to 0$.

Hence, taking φ_p as a test-function in the definitions of $\overline{\eta_1}(L_p,\mathbb{R}^N\backslash B_R)$, one gets

$$\overline{\eta_1}(L_p,\mathbb{R}^N\backslash B_R)\le H(p)+2\delta.$$

We conclude that

$$\lim_{R\to+\infty}\overline{\eta_1}(L_p,\mathbb{R}^N\backslash B_R)\le H(p).$$

One proves similarly that $\lim_{R\to+\infty}\underline{\eta_1}(L_p,\mathbb{R}^N\backslash B_R)\ge H(p)$, by using $-v_p$ instead of v_p, which is the unique viscosity solution of

$$-\left|\nabla\widetilde{v}_p(y)+p\right|^2-c_0(y)=-H(p)\quad \text{over } \mathbb{R}, \tag{119}$$

and is semiconvexe.

Hence,

$$\lim_{R\to+\infty}\underline{\eta_1}(L_p,\mathbb{R}^N\backslash B_R)=\lim_{R\to+\infty}\overline{\eta_1}(L_p,\mathbb{R}^N\backslash B_R)=H(p).$$

The conclusion then immediately follows from Theorem 7.

□

13 Further examples and other open problems

In order to conclude the statement of the results, we discuss their optimality analyzing in detail various examples.

13.1 An example of recurrent media which does not admit an exact spreading speed

We have described in Section 3.2 how the results simplify when the coefficients are recurrent. Then we applied these results to various classes of recurrent media, such as homogeneous, periodic and almost periodic ones, for which we have proved that $\underline{w}(e) = \overline{w}(e)$, showing that there exists an exact asymptotic spreading speed in every directions. It could thus be tempting to conjecture that any equation with recurrent coefficients admits an exact asymptotic spreading speed in every directions. We will indeed construct a counter-example to this conjecture.

The next Proposition gives a generic way to construct examples for which $w_*(e) < w^*(e)$. We recall here that another such example was provided by the second author, together with Garnier and Giletti [44], for an equation with a non-recurrent reaction term depending on x (but not on t).

Proposition 13.1 *Consider a uniformly continuous and bounded function $\omega \in \mathcal{C}^{\delta}_{loc}(\mathbb{R})$ and let*

$$\overline{\omega} = \limsup_{T\to+\infty} \frac{1}{T}\int_0^T \omega(t)dt \quad \text{and} \quad \underline{\omega} = \liminf_{T\to+\infty} \frac{1}{T}\int_0^T \omega(t)dt.$$

Let $e \in \mathbb{S}^{N-1}$, consider a bounded, nonnegative, mesurable and compactly supported function $u_0 \not\equiv 0$ and let u the solution of the Cauchy problem

$$\begin{cases} \partial_t u - \Delta u - \omega(t)e\cdot\nabla u = u(1-u) \text{ in } (0,\infty)\times\mathbb{R}^N, \\ u(0,x) = u_0(x) \text{ in } \mathbb{R}^N. \end{cases} \tag{120}$$

Then if $\overline{\omega} - \underline{\omega} < 4$, one has

$$w_*(e) = 2 + \underline{\omega} \quad \text{and} \quad w^*(e) = 2 + \overline{\omega}.$$

Moreover, if $w \in (w_*(e), w^*(e))$, then for all $s \in [0,1]$, there exists a sequence $t_n \to +\infty$ such that $u(t_n, wt_ne) \to s$ as $n \to +\infty$.

Let us postpone the proof of this result for a moment and display some of its applications.

Example 1. Let first construct an explicit example of non-recurrent coefficients for which $w_*(e) < w^*(e)$. Consider the same equation as in Proposition 13.1 with

$$\omega(t) = \begin{cases} \omega_2 & \text{if} \quad t \in [s_n+1, t_n], \\ \omega_1 & \text{if} \quad t \in [t_n+1, s_{n+1}], \end{cases}$$

where $(s_n)_{n\geq1}$ and $(t_n)_{n\geq1}$ are two sequences of $\mathbb{R}^+$ such that $t_n - s_n = n$ and $s_{n+1} - t_n = n$, $0 < \omega_1 < \omega_2 < 4 + \omega_1$, ω is smooth and $\omega(t) \in [\omega_1, \omega_2]$ for all $t \in \mathbb{R}$. Then it follows from

Proposition 13.1 that $w_*(e) = 2 + \omega_1$ and $w^*(e) = 2 + \omega_2$. Moreover, one easily computes using the Remark below Proposition 3.9 that $\underline{w}(e) = 2 + \omega_1$ and $\overline{w}(e) = 2 + \omega_2$. Thus, in this case, $w_*(e) < w^*(e)$ but our result is optimal since $\underline{w}(e) = w_*(e)$ and $\overline{w}(e) = w^*(e)$.

Example 2. We now construct a similar example but with recurrent coefficients. It has long been known that recurrent functions do not necessarily admit a mean value, but there does not exist many explicit examples in the literature. One was exhibited by Lewin and Lewitan in 1939 [63]. Let ω such a function: ω is uniformly continuous, bounded and depends recurrently on t, and one has

$$\liminf_{T\to+\infty} \frac{1}{T}\int_0^T \omega(t)dt < \limsup_{T\to+\infty} \frac{1}{T}\int_0^T \omega(t)dt.$$

Under the same hypotheses as in Proposition 13.1, one then immediatley gets $w_*(e) < w^*(e)$, that is, equation (120) does not admit an exact spreading speed in direction e, despite it has recurrent coefficients.

In these Examples, as in [44], the spreading is not linear: the level lines of $u(t,\cdot)$ do not move with a given speed but oscillate between two speeds. Hence, instead of considering the limit of $t \mapsto u(t, wte)$ with $w \in \mathbb{R}^+$, one should try to localize the level sets of $u(t,\cdot)$ by computing the limit of $t \mapsto u\big(t, e\int_0^t w(s)ds\big)$, with $w \in \mathcal{C}^0(\mathbb{R}^+,\mathbb{R}^+)$. We introduced with Hamel some notions that are useful when one tries to identify such "nonlinear" spreading properties in [13]. The method we present in this manuscript only fits to the investigation of "linear" spreading properties.

We leave as an open problem the existence of spreading surfaces, in the sense of [13], involving generalized principal eigenvalues.

Proof of Proposition 13.1. The proof relies on the change of variable

$$v(t,x) = u(t, x + e\int_0^t \omega(s)ds).$$

This function satisfies

$$\begin{cases} \partial_t v - \Delta v = v(1-v) \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ v(0,x) = u_0(x) \text{ in } \mathbb{R}^N. \end{cases} \tag{121}$$

Thus $\min_{|x|\leq wt} v(t,x) \to 1$ if $0 < w < 2$ and $\max_{|x|\geq wt} v(t,x) \to 0$ if $w > 2$, leading to

$$w_*(e) \geq \underline{\omega} + 2 \quad \text{ and } \quad w^*(e) \leq \overline{\omega} + 2.$$

Now if $\underline{\omega} + 2 > \overline{\omega}$ and $w \in \big(2+\underline{\omega}, 2+\overline{\omega}\big)$, there exist two sequences $(t_n)_n$ and $(t'_n)_n$ such that

$$\overline{\omega} = \lim_{n\to+\infty} \frac{1}{t_n}\int_0^{t_n} \omega(t)dt \text{ and } \underline{\omega} = \lim_{n\to+\infty} \frac{1}{t'_n}\int_0^{t'_n} \omega(t)dt.$$

One also has $u(t_n, wt_ne) = v\big(t_n, t_ne(w - \frac{1}{t_n}\int_0^{t_n}\omega(s)ds)\big)$. But as $-2 < w - \overline{\omega}$ (since $4 \geq \overline{\omega} - \underline{\omega}$) and $2 > w - \overline{\omega}$, there exists some small positive ε such that

$$-2 + \varepsilon < w - \frac{1}{t_n}\int_0^{t_n} \omega(s)ds < 2 - \varepsilon$$

for n sufficiently large. Hence, one gets

$$u(t_n, wt_n e) \geq \min_{|x|\leq(2-\varepsilon)t_n} u(t_n, x) \to 1 \text{ as } n \to +\infty.$$

Similarly, one can prove that $u(t'_n, wt'_n e) \to 0$ as $n \to +\infty$.

Define the ω-limit set as $t \to +\infty$ of the function $t \mapsto u(t, wte)$:

$$\Omega = \{s \in [0,1], \exists (t_n)_n,\ t_n \to +\infty,\ u(t_n, wt_n e) \to s\}.$$

As the function $t \mapsto u(t, cte)$ is continuous, this set is connected. Moreover, 0 and 1 both belong to Ω. Hence $\Omega = [0,1]$, which concludes the proof. □

13.2 A time-heterogeneous example where our construction is not optimal

In the next example, Proposition 13.1 shows that $w^*(e) = w_*(e)$, that is, there exists an exact spreading speed, but the speeds we construct through Theorem 2 are not equal: $\underline{w}(e) < \overline{w}(e)$. Thus, Theorem 2 does not give optimal bounds on the level sets of $u(t, \cdot)$ in this case.

Example 3. Consider the same ω as in Example 1 but with $s_{n+1} - t_n = n^2$. Then on one hand, Proposition 13.1 gives

$$w_*(e) = w^*(e) = 2 + \omega_1 \quad \text{since} \quad \frac{1}{t}\int_0^t \omega(s)ds \to \omega_1 \text{ as } t \to +\infty.$$

On the other hand, one can easily prove that

$$\limsup_{t\to+\infty} \sup_{s>0} \frac{1}{t}\int_s^{s+t} \omega = \omega_2 \quad \text{and} \quad \liminf_{t\to+\infty} \inf_{s>0} \frac{1}{t}\int_s^{s+t} \omega = \omega_1.$$

The Remark below Proposition 3.9 gives

$$\underline{w}(e) = 2 + \omega_1 = w_*(e) \quad \text{and} \quad \overline{w}(e) = 2 + \omega_2 > w^*(e).$$

13.3 A multi-dimensional example where our construction is not optimal

We conclude with an example showing that our construction of $\underline{w}(e)$ might not be optimal in dimension N. In this example a direct approach, through sub and supersolutions, gives more accurate results.

Proposition 13.2 *Assume that u satisfies*

$$\partial_t u - a(x)\Delta u = u(1-u), \quad \text{in } (0,\infty) \times \mathbb{R}^2,$$

where $u_0 \not\equiv 0$ is compactly supported, nonnegative and continuous, a is smooth and

$$a(x) = \begin{cases} a_1 & \text{if} \quad x_1 \geq x_2^2 + 1, \\ a_2 & \text{if} \quad x_1 \leq x_2^2, \end{cases}$$

with $a_1 > a_2 > 0$.

Then, $\underline{\mathcal{S}} = \{x \in \mathbb{R}^N,\ |x| \le 2\sqrt{a_1}\}$ *and* $\overline{\mathcal{S}}$ *is the closed convex envelope of* $B(0, 2\sqrt{a_1}) \cup \{(2\sqrt{a_2}, 0)\}$.

However, for all compact subset $K \subset \mathrm{int}\overline{\mathcal{S}}$*, one has*

$$\lim_{t\to+\infty} \sup_{x\in tK} |u(t,x) - 1| = 0.$$

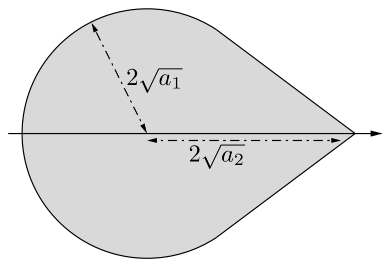

Figure 7: The set $\overline{\mathcal{S}}$ of Proposition 13.2.

This example indicates that considering what happens in the full truncated cones $C_{R,\alpha}(e)$ in the computations of the Hamiltonians might not be optimal in some situations. In the present case, we believe that the equation obtained in Proposition 5.3 on Z_* is not optimal[4]. However, in order to get a more precise equation on Z_*, one would have to control the location of (t_n, x_n) with respect to (t_0, x_0), which is technically challenging and requires new ingredients in the method. We thus leave such a generalization as an open problem.

In order to construct a more optimal Hamiltonian, as already observed in our previous paper with Hamel [13], only the value of the coefficients at finite distance from the propagation paths should matter. The present Hamilton-Jacobi approach requires us to consider what happens in the truncated cones $C_{R,\alpha}(e)$, which is sub-optimal.

We leave as an open question a refinement of the method described in this paper providing a unified approach giving optimal results in this framework.

Proof of Proposition 13.2. One easily computes $\overline{H}(e,p) = \underline{H}(e,p) = a_2|p|^2+1$ if $e \neq e_1$, since a is close to a_2 in the cones $C_{R,\alpha}(e)$ if $e \neq e_1$, R is large and α is small. Similarly, using appropriate balls with increasing radii, one gets $\overline{H}(e_1,p) = a_1|p|^2+1$ and $\underline{H}(e_1,p) = a_2|p|^2+1$. Hence, $\underline{\mathcal{S}} = \{w \in \mathbb{R}^2,\ |x| \le 2\sqrt{a_2}\}$ and the same arguments as in the proof of Proposition 3.13 yield that $\overline{\mathcal{S}}$ is the closed convex envelope of $B(0, 2\sqrt{a_2})$ and $(2\sqrt{a_1}, 0)$.

Next, let $0 \le w_1 < 2\sqrt{a_1}$ and $0 \le w_2 < 2\sqrt{a_2}$. For $i = 1, 2$, let (λ_i, ϕ_i) the principal eigenelements associated with the operator $-a_i\Delta - 1 + w_i^2/4a_i$ in the ball of radius R_i, with Dirichlet boundary conditions. As $0 \le w_i < 2\sqrt{a_i}$, there exist $\delta > 0$ and $R_1 > R_2$ large enough such that $\lambda_i < -\delta$ for $i = 1, 2$. Up to multiplication by a positive constant, we can assume that $\|\phi_i\|_\infty < \delta e^{w_i R/(2a_i)}$ and that

$$\phi_1(x)e^{-w_1 x\cdot\xi_1/2a_1} \ge \phi_2(x)e^{-w_2 x\cdot\xi_2/2a_2} \quad \text{in } B(0, R_2). \tag{122}$$

[4]We thank an anonymous referee for raising this point.

Define

$$u_i(t,x) := \phi_i(x - w_i t\xi_i)e^{\frac{w_i}{2a_i}(w_i t - x\cdot\xi_i)},$$

where $\xi_1 = e_1$ and $\xi_2 \neq e_1$ is a unit vector. These functions satisfy:

$$\partial_t u_i - a_i \Delta u_i = u_i + \lambda_i u_i < u_i(1-u_i) \quad \text{ in } B(w_i t\xi_i, R_i)$$

since $u_i < \delta$, and vanish on the boundary of these balls. Moreover, this inequation stays true if we multiply u_i by any positive constant $\kappa \in (0,1)$.

Let $T_1 > 0$ large enough such that $a(x) = a_1$ in $B(w_1 t e_1, R_1)$ for all $t \geq T_1$. Let $\kappa_1 > 0$ such that

$$u(T_1, x + w_1 T_1 e_1) \geq \kappa_1 \phi_1(x) e^{w_1 R_1/2a_1} \geq \kappa_1 u_1(T_1, x + w_1 T_1 e_1) \quad \text{ in } B(0, R_1).$$

It follows from the parabolic maximum principle that for all $t \geq 0$ and $x \in \mathbb{R}^2$,

$$u\big(t+T_1, x + w_1(t+T_1)e_1\big) \geq \kappa_1 u_1\big(t+T_1, x + w_1(t+T_1)e_1\big) = \kappa_1 \phi_1(x) e^{-w_1 x\cdot e_1/2a_1}. \quad (123)$$

Let T_2 large enough such that $a(x + w_1 T_1 e_1 + w_2 T_2 \xi_2) = a_2$ for all $x \in B(0, R_2)$. It follows from the definition of a that $a\big(x + w_1(t+T_1)e_1 + w_2(t+T_2)\xi_2\big) = a_2$ in $B(0,R_2)$ for all $t \geq 0$. Moreover, the parabolic Harnack inequality yields that there exists $\kappa_2 > 0$, independent of t, such that:

$$u\big(t+T_1+T_2, x + w_1(t+T_1)e_1 + w_2 T_2 \xi_2\big) \geq \kappa_2 u\big(t+T_1, x + w_1(t+T_1)e_1\big) \quad \text{ in } B(0,R_1).$$

This implies

$$u\big(t+T_1+T_2, x + w_1(t+T_1)e_1 + w_2 T_2\xi_2\big) \geq \kappa_1\kappa_2 \phi_1(x) e^{-w_1 x\cdot e_1/2a_1} \geq \kappa_1\kappa_2 u_2\big(T_2, x + w_2 T_2 \xi_2\big),$$

by (122). The parabolic maximum principle gives, for all $s \geq 0$, $t \geq 0$:

$$\begin{array}{rcl} u\big(s+t+T_1+T_2, x + w_1(t+T_1)e_1 + w_2(s+T_2)\xi_2\big) & \geq & \kappa_1\kappa_2 u_2\big(s+T_2, x + w_2(s+T_2)\xi_2\big) \\ & = & \kappa_1\kappa_2\phi_2(x) e^{-\frac{w_i}{2a_i}x\cdot\xi_i}. \end{array} \quad (124)$$

Consider now a given w in the interior of the closed convex envelope of $B(0, 2\sqrt{a_2})$ and $\{(2\sqrt{a_1}, 0)\}$. We could write $w = (1-\tau)w_1 e_1 + \tau w_2 \xi_2$, where $\tau \in (0,1)$, $w_1 \in [0, 2\sqrt{a_1})$ and $w_2\xi_2 \in B(0, 2\sqrt{a_2})$, that is, $0 \leq w_2 < 2\sqrt{a_2}$ and $|\xi_2| = 1$.

We now apply the above results. First, if $\xi_2 = e_1$, then inequality (123) immediately implies

$$\liminf_{t\to+\infty} u\big(t, (1-\tau)w_1 e_1 t + \tau w_2 e_1 t\big) = \liminf_{t\to+\infty} u(t, tw) \geq \kappa_1\phi_1(0) > 0.$$

Next, if $\xi_2 \neq e_1$, replacing $t + T_1$ by $(1-\tau)t$ and $s + T_2$ by τt in (124), which is possible if t is large enough since $\tau \in (0,1)$, one gets

$$u\big(t, (1-\tau)w_1 e_1 t + \tau w_2 \xi_2 t\big) = u(t, tw) \geq \kappa_1\kappa_2\phi_2(0).$$

Moreover, the reader could check that these estimates hold locally uniformly with respect to τ, ξ_2, w_1, w_2, that is, locally uniformly with respect to w. It follows that

$$\liminf_{t\to+\infty} u(t, tw) > 0,$$

and thus our hypotheses on f (12) and classical arguments (see for example Theorem 1.6 and Proposition 1.8 of [13]) yield

$$\liminf_{t\to+\infty} u\big(t,tw\big) = 1.$$

Moreover, as this convergence is locally uniform around any w in the interior of the closed convex envelope of $B(0,2\sqrt{a_2})$ and $\{(2\sqrt{a_1},0)\}$, it is also uniform in any of its compact subset, which concludes the proof. $\square$

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