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# Gevrey order and summability of formal series solutions of some classes of inhomogeneous linear partial differential equations with variable coefficients

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## Abstract

We investigate Gevrey order and summability properties of formal power series solutions of some classes of inhomogeneous linear partial differential equations with variable coefficients and analytic initial conditions. In particular, we give necessary and sufficient conditions under which these solutions are convergent or are  $k$ -summable, for a convenient  $k$ , in a given direction.

**Keywords.** Linear partial differential equation, divergent power series, Gevrey order, summability

**AMS subject classification.** 35C10, 35C20, 40B05

## 1 Introduction

In recent years, various works have been done towards the summability of divergent solutions of partial differential equations with constant coefficients (see [1, 3, 5, 6, 8, 13] etc.) or variable coefficients (see [4, 9–11, 15, 16] etc.) in two variables.

In the present article, we are interested in some classes of inhomogeneous linear partial differential equation with variable coefficients and analytic initial conditions. More precisely, we consider Cauchy problems of the form

$$(1.1) \quad \begin{cases} \partial_t^\kappa u - a(x, t) \partial_x^p u = \tilde{q}(x, t) \\ \partial_t^j u(x, t)|_{t=0} = \varphi_j(x) \quad , j = 0, \dots, \kappa - 1 \end{cases}$$

where

- $\kappa$  and  $p$  are two positive integers,
- $\varphi_j(x) \in \mathcal{O}(D_{\rho_1})$  is holomorphic for all  $j = 0, \dots, \kappa - 1$  in a disc  $D_{\rho_1}$  with center  $0 \in \mathbb{C}$  and radius  $\rho_1 > 0$ ,
- $a(x, t) \in \mathcal{O}(D_{\rho_1} \times D_{\rho_2})$  is holomorphic in the two variables  $x$  and  $t$  in a polydisc  $D_{\rho_1} \times D_{\rho_2}$  centered at  $(0, 0) \in \mathbb{C}^2$  and satisfies  $a(0, 0) \neq 0$ ,
- $\tilde{q}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]$ <sup>1</sup> may be smooth or not.

Note that Cauchy problems of type (1.1) play an important role in physics since many classical problems, such as the heat initial conditions problem, the wave initial conditions problem, the beams initial conditions problem, etc. are of this form.

A first study of problem (1.1) has been done by D. A. Lutz, M. Miyake and R. Schäfke in 1999 in the special case where  $a \equiv 1$  and  $\tilde{q} \equiv 0$  [8, 13]. In particular, they proved that this problem has a unique formal series solution  $\tilde{u}(x, t)$  in  $\mathcal{O}(D_{\rho_1})[[t]]$  which converges for  $1 \leq p \leq \kappa$  and diverges (in general) in the opposite case  $1 \leq \kappa < p$ ; in this latter case, they more precisely showed that  $\tilde{u}(x, t)$  is a  $s$ -Gevrey series (see definition 3.1 below for the exact definition of a  $s$ -Gevrey series) with  $s = p/\kappa - 1$  and they gave necessary and sufficient conditions under which  $\tilde{u}(x, t)$  is  $k$ -summable, with  $k = 1/s$ , in a given direction  $\arg(t) = \theta$ . More recently, in a 2009 article [4], W. Balsler and M. Loday-Richaud investigated problem (1.1) in the case  $(\kappa, p) = (1, 2)$  and  $a(x, t) = \alpha(x)$  analytic at  $x = 0$ . Again, they proved that this problem has a unique formal series solution and they gave necessary and sufficient conditions under which it is 1-summable.

The aim of this article is to extend the results above to the very general problem (1.1), where no generic assumption on  $a$  and  $\tilde{q}$  is made. For notational convenience, we rewrite from now problem (1.1) in the form

$$(1.2) \quad \boxed{(1 - \partial_t^{-\kappa} (a(x, t) \partial_x^p)) u = \tilde{f}(x, t)}$$

where  $\partial_t^{-1}u$  stands for the anti-derivative  $\int_0^t u(x, s)ds$  of  $u$  with respect to  $t$  which vanishes at  $t = 0$  and where  $\tilde{f}(x, t) := \partial_t^{-\kappa} \tilde{q}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]$  satisfies  $\partial_t^j \tilde{f}(x, t)|_{t=0} = \varphi_j(x)$  for all  $j = 0, \dots, \kappa - 1$ .

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<sup>1</sup>We denote  $\tilde{q}$  with a tilde to emphasize the possible divergence of the series  $\tilde{q}$ .

The organization of the article is as follows. In section 2, we prove that problem (1.2) has a unique formal series solution  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]$  and we give a characterization of its coefficients. In section 3, we show that  $\tilde{u}(x, t)$  and the coefficient  $\tilde{f}(x, t)$  are together convergent when  $1 \leq p \leq \kappa$  and  $s$ -Gevrey with  $s = p/\kappa - 1$  when  $1 \leq \kappa < p$ . In section 4, we restrict ourselves to this latter case and we investigate the summability of  $\tilde{u}(x, t)$ . In particular, we give necessary and sufficient conditions under which  $\tilde{u}(x, t)$  is  $k$ -summable with  $k = 1/s$  in a given direction  $\arg(t) = \theta$  (theorem 4.3), conditions which coincide with those given in [4, 8, 13]. We provide thus a new proof of the results of [4, 8, 13].

From now on, we denote by  $D_{\kappa, p}$  the operator  $D_{\kappa, p} := 1 - \partial_t^{-\kappa} (a(x, t) \partial_x^p)$  and, for any series  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]$ , we denote

$$\tilde{u}(x, t) = \sum_{j \geq 0} u_{j,*}(x) \frac{t^j}{j!} = \sum_{n \geq 0} \tilde{u}_{*,n}(t) \frac{x^n}{n!} = \sum_{j,n \geq 0} u_{j,n} \frac{t^j x^n}{j! n!}.$$

## 2 Existence and uniqueness of formal series solutions

Let us first observe that  $D_{\kappa, p}$  is a linear operator acting inside  $\mathcal{O}(D_{\rho_1})[[t]]$ . Indeed,  $(\mathcal{O}(D_{\rho_1})[[t]], \partial_x, \partial_t)$  is a  $\mathbb{C}$ -differential algebra and  $a(x, t) \in \mathcal{O}(D_{\rho_1} \times D_{\rho_2}) \subset \mathcal{O}(D_{\rho_1})[[t]]$ . More precisely, we have the following.

**Theorem 2.1** *Let  $\kappa, p \geq 1$ . The map  $D_{\kappa, p} : \mathcal{O}(D_{\rho_1})[[t]] \longrightarrow \mathcal{O}(D_{\rho_1})[[t]]$  is a linear isomorphism.*

**Proof.** Let  $\tilde{f}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]$ . A series  $\tilde{u}(x, t) = \sum_{j \geq 0} u_{j,*}(x) \frac{t^j}{j!}$  is solution

of  $D_{\kappa, p} \tilde{u} = \tilde{f}$  is and only if its coefficients  $u_{j,*}(x)$  satisfy, for all  $j \geq 0$ , the identities

$$(2.1) \quad u_{j,*}(x) = f_{j,*}(x) + \sum_{m=0}^{j-\kappa} \binom{j-\kappa}{m} a_{m,*}(x) \partial_x^p u_{j-\kappa-m,*}(x)$$

with the classical convention that the sum is 0 if  $j < \kappa$ . Thereby, equation  $D_{\kappa, p} \tilde{u} = \tilde{f}$  admits a unique solution  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]$ ; hence, the bijectivity of  $D_{\kappa, p}$ . The remark just above achieves the proof. ■

As a direct consequence of theorem 2.1, we deduce the following result on the existence and the uniqueness of formal series solutions of problems (1.2).

**Corollary 2.2** *Problem (1.2) admits, for any  $\kappa, p \geq 1$ , a unique formal series solution  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]$ . Moreover, its coefficients  $u_{j,*}(x)$  are recursively determined for all  $j \geq 0$  by identities (2.1).*

Recall that the solution  $\tilde{u}(x, t)$  may be divergent or not (see for example the case  $a \equiv 1$  and  $\tilde{q} \equiv 0$  treated in [8, 13]). In section 3 below, we shall investigate in great details Gevrey properties of  $\tilde{u}(x, t)$ . In particular, we shall show that  $\tilde{u}(x, t)$  and the inhomogeneity  $\tilde{f}(x, t)$  have the same Gevrey order.

### 3 Gevrey properties

Before starting the study of Gevrey properties of formal solutions  $\tilde{u}(x, t)$ , let us recall the definition and some results about the  $s$ -Gevrey formal series.

#### 3.1 $s$ -Gevrey formal series

In this article, we consider  $t$  as the variable and  $x$  as a parameter. The classical notion of  $s$ -Gevrey formal series is then extended to  $x$ -families as follows.

**Definition 3.1** Let  $s \geq 0$ . A series  $\tilde{u}(x, t) = \sum_{j \geq 0} u_{j,*}(x) \frac{t^j}{j!} \in \mathcal{O}(D_{\rho_1})[[t]]$  is said to be *Gevrey of order  $s$*  (in short,  *$s$ -Gevrey*) if there exist  $0 < r_1 \leq \rho_1$ ,  $C > 0$  and  $K > 0$  such that inequalities

$$|u_{j,*}(x)| \leq CK^j \Gamma(1 + (s + 1)j)$$

hold for all  $j \geq 0$  and  $x \in D_{r_1}$ .

Observe that definition 3.1 means that  $\tilde{u}(x, t)$  is  $s$ -Gevrey in  $t$  uniformly in  $x$  on a neighborhood of  $x = 0$ .

We denote by  $\mathcal{O}(D_{\rho_1})[[t]]_s$  the set of all the formal series in  $\mathcal{O}(D_{\rho_1})[[t]]$  which are  $s$ -Gevrey. Note that the set  $\mathcal{O}(D_{\rho_1})[[t]]_0$  coincides with the set  $\mathbb{C}\{x, t\}$  of germs of analytic functions at the origin  $(0, 0) \in \mathbb{C}^2$ .

**Proposition 3.2** *Let  $s \geq 0$ . Then,  $(\mathcal{O}(D_{\rho_1})[[t]]_s, \partial_x, \partial_t)$  is a  $\mathbb{C}$ -differential algebra stable under anti-derivations  $\partial_x^{-1}$  and  $\partial_t^{-1}$ .*

**Proof.** Since proposition 3.2 is true for  $\mathcal{O}(D_{\rho_1})[[t]]$  instead of  $\mathcal{O}(D_{\rho_1})[[t]]_s$ , it is sufficient to prove that  $\mathcal{O}(D_{\rho_1})[[t]]_s$  is stable under multiplication, derivations and anti-derivations.

◁ *Multiplication.* Let  $\tilde{u}(x, t), \tilde{v}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]_s$ . We can always assume that  $\tilde{u}(x, t)$  and  $\tilde{v}(x, t)$  satisfy conditions of definition 3.1 with the same constants  $r_1$ ,  $C$  and  $K$ . Denote by  $\tilde{w}(x, t)$  their product. Since the coefficients  $w_{j,*}(x)$  of  $\tilde{w}(x, t)$  are given by

$$w_{j,*}(x) = \sum_{k=0}^j \binom{j}{k} u_{k,*}(x) v_{j-k,*}(x)$$

we have, for all  $j \geq 0$ ,

$$\sup_{x \in D_{r_1}} |w_{j,*}(x)| \leq C^2 K^j \sum_{k=0}^j \binom{j}{k} \underbrace{\Gamma(1 + (s+1)k) \Gamma(1 + (s+1)(j-k))}_{a_{j,k}}.$$

where, according to relations between the Gamma and Beta functions,

$$a_{j,k} = \Gamma(2 + (s+1)j) \int_0^1 t^{(s+1)k} (1-t)^{(s+1)(j-k)} dt \leq \Gamma(2 + (s+1)j).$$

Thereby,

$$\begin{aligned} \sup_{x \in D_{r_1}} |w_{j,*}(x)| &\leq C^2 K^j \Gamma(2 + (s+1)j) \sum_{k=0}^j \binom{j}{k} \\ &= C^2 (2K)^j (1 + (s+1)j) \Gamma(1 + (s+1)j) \end{aligned}$$

and, consequently, there exist  $C', K' > 0$  such that

$$\sup_{x \in D_{r_1}} |w_{j,*}(x)| \leq C' K'^j \Gamma(1 + (s+1)j) \quad \text{for all } j \geq 0.$$

◁ *Derivation*  $\partial_x$ . Let  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]_s$  and  $\tilde{w}(x, t) = \partial_x \tilde{u}(x, t)$ . For a given  $r'_1 < r_1$ , Cauchy integral formula gives us

$$w_{j,*}(x) = \partial_x u_{j,*}(x) = \frac{1}{2i\pi} \int_{|x'-x|=r_1-r'_1} \frac{u_{j,*}(x')}{(x'-x)^2} dx'$$

for all  $j \geq 0$  and  $x \in D_{r'_1}$ . Hence, the inequalities

$$\sup_{x \in D_{r'_1}} |w_{j,*}(x)| \leq C' K^j \Gamma(1 + (s+1)j) \quad \text{with } C' = \frac{C}{r_1 - r'_1} \text{ for all } j \geq 0.$$

◁ *Derivation*  $\partial_t$ . Let  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]_s$  and  $\tilde{w}(x, t) = \partial_t \tilde{u}(x, t)$ . From relations  $w_{j,*}(x) = u_{j+1,*}(x)$ , we deduce

$$\sup_{x \in D_{r_1}} |w_{j,*}(x)| \leq CK^{j+1} \Gamma(1 + (s+1)(j+1)) \quad \text{for all } j \geq 0.$$

Let us now choose an integer  $S \geq s+1$ . Inequalities

$$2 \leq 1 + (s+1)(j+1) \leq 1 + (s+1)j + S$$

and the increase of the Gamma function on  $[2, +\infty[$  then imply

$$\Gamma(1 + (s+1)(j+1)) \leq \Gamma(1 + (s+1)j + S) = \Gamma(1 + (s+1)j) \prod_{\ell=1}^S ((s+1)j + \ell).$$

Hence, there exist  $C', K' > 0$  such that

$$\sup_{x \in D_{r_1}} |w_{j,*}(x)| \leq C' K'^j \Gamma(1 + (s+1)j) \quad \text{for all } j \geq 0.$$

◁ *Anti-derivation*  $\partial_x^{-1}$ . Let  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]_s$  and  $\tilde{w}(x, t) = \partial_x^{-1} \tilde{u}(x, t)$ . Since  $w_{j,*}(x) = \partial_x^{-1} u_{j,*}(x)$ , we clearly have

$$\sup_{x \in D_{r_1}} |w_{j,*}(x)| \leq C' K^j \Gamma(1 + (s+1)j) \quad \text{with } C' = Cr_1 \text{ for all } j \geq 0.$$

◁ *Anti-derivation*  $\partial_t^{-1}$ . Let  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]_s$  and  $\tilde{w}(x, t) = \partial_t^{-1} \tilde{u}(x, t)$ . We have  $w_{0,*} \equiv 0$  and  $w_{j,*}(x) = u_{j-1,*}(x)$  for all  $j \geq 1$ ; hence, the inequalities

$$\sup_{x \in D_{r_1}} |w_{j,*}(x)| \leq CK^{j-1} \Gamma(1 + (s+1)(j-1)) \quad \text{for all } j \geq 1.$$

From the increase of the Gamma function on  $[2, +\infty[$ , we get

$$\Gamma(1 + (s+1)(j-1)) \leq \Gamma(1 + (s+1)j)$$

for all  $j \geq 2$  and

$$\Gamma(1 + (s+1)(j-1)) = \Gamma(1) = \Gamma(2) \leq \Gamma(1 + (s+1)j)$$

for  $j = 1$ . Consequently,

$$\sup_{x \in D_{r_1}} |w_{j,*}(x)| \leq C' K^j \Gamma(1 + (s+1)j) \quad \text{with } C' = \frac{C}{K} \text{ for all } j \geq 0.$$

The proof is complete. ■

Note that the stability under  $\partial_x$  is guaranteed by the condition “*there exists*  $r_1 \leq \rho_1 \dots$ ” in definition 3.1. Note also that proposition 3.2 implies that the linear operators  $D_{\kappa,p}$  act inside  $\mathcal{O}(D_{\rho_1})[[t]]_s$  for any  $\kappa, p \geq 1$  and  $s \geq 0$ . Theorem 3.3 below shall precise this result.

### 3.2 Gevrey order of formal series solutions

We are now able to state the main result of this section.

**Theorem 3.3** *Let  $\kappa, p \geq 1$ . Let  $s \geq 0$  be defined by*

$$s = \begin{cases} 0 & \text{if } 1 \leq p \leq \kappa \\ p/\kappa - 1 & \text{if } 1 \leq \kappa < p \end{cases} .$$

*Then, the map  $D_{\kappa,p} : \mathcal{O}(D_{\rho_1})[[t]]_s \longrightarrow \mathcal{O}(D_{\rho_1})[[t]]_s$  is a linear isomorphism.*

Following corollary 3.4 is straightforward from theorem 3.3 and gives us some properties about the Gevrey orders of formal series solutions of problems (1.2).

**Corollary 3.4** *Let  $\kappa, p$  and  $s$  as in theorem 3.3.*

*Let  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]$  be the unique formal series solution of problem (1.2). Then,  $\tilde{u}(x, t)$  is a  $s$ -Gevrey series if and only if  $\tilde{f}(x, t)$  is a  $s$ -Gevrey series.*

In particular, in the case  $1 \leq p \leq \kappa$ , this provides us a necessary and sufficient condition under which the formal solution  $\tilde{u}(x, t)$  is convergent.

**Corollary 3.5** *Let  $1 \leq p \leq \kappa$  and  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]$  be the unique formal series solution of problem (1.2). Then,  $\tilde{u}(x, t)$  is convergent if and only if  $\tilde{f}(x, t)$  is convergent.*

The proof of theorem 3.3 is developed in next section 3.3. Before starting it, let us first recall the definition and some main properties of Nagumo norms on which we are going to be based. For more details, we refer for instance to [14] or [7].

**Definition 3.6 (Nagumo norms)** *Let  $f \in \mathcal{O}(\tilde{D}_\rho)$ ,  $q \geq 0$  and  $0 < r \leq \rho$ . Let  $d_r(x) = r - |x|$  denote the Euclidian distance of  $x \in D_r$  to the boundary of the disc  $D_r$ . Then, the Nagumo norm  $\|f\|_{q,r}$  of  $f$  is defined by*

$$\|f\|_{q,r} := \sup_{x \in D_r} |f(x) d_r(x)^q| .$$

**Proposition 3.7 (Properties of Nagumo norms)** *Let  $f, g \in \mathcal{O}(D_\rho)$ . Let  $q, q' \geq 0$  and  $0 < r \leq \rho$ . One has the following properties:*

1.  $\|\cdot\|_{q,r}$  is a norm on  $\mathcal{O}(D_\rho)$ .
2. For all  $x \in D_r$ ,  $|f(x)| \leq \|f\|_{q,r} d_r(x)^{-q}$ .



3.  $\|f\|_{0,r} = \sup_{x \in D_r} |f(x)|$  is the usual sup-norm on  $D_r$ .

4.  $\|fg\|_{q+q',r} \leq \|f\|_{q,r} \|g\|_{q',r}$ .

5.  $\|\partial_x f\|_{q+1,r} \leq e(q+1) \|f\|_{q,r}$ .

Note that the same index  $r$  occurs on both sides of inequalities 4 and 5. In particular, we get estimates for the product  $fg$  in terms of  $f$  and  $g$  and for the derivative  $\partial_x f$  in terms of  $f$  without having to shrink the disc  $D_r$ .

Let us now turn to the proof of theorem 3.3.

### 3.3 Proof of theorem 3.3

Calculations below are based on similar arguments to those detailed in [4] in the case  $(\kappa, p) = (1, 2)$ . Nevertheless, they are much more complicated because  $s$  may not be an integer.

Let us begin by observing that proposition 3.2 implies  $D_{\kappa,p}(\mathcal{O}(D_{\rho_1})[[t]]_s) \subset \mathcal{O}(D_{\rho_1})[[t]]_s$  and that theorem 2.1 implies the linearity and the injectivity of  $D_{\kappa,p}$ . Thereby, we are left to prove that  $D_{\kappa,p}$  is surjective. To do that, let us fix  $\tilde{f}(x, t) = \sum_{j \geq 0} f_{j,*}(x) \frac{t^j}{j!} \in \mathcal{O}(D_{\rho_1})[[t]]_s$  and let  $\tilde{u}(x, t) = \sum_{j \geq 0} u_{j,*}(x) \frac{t^j}{j!} \in \mathcal{O}(D_{\rho_1})[[t]]$  denote the unique formal series solution of  $D_{\kappa,p} \tilde{u} = \tilde{f}(x, t)$  (see theorem 2.1). The coefficients  $f_{j,*}(x)$  satisfy conditions

- $f_{j,*}(x) \in \mathcal{O}(D_{\rho_1})$  for all  $j \geq 0$ ,
- there exist  $0 < r_1 \leq \rho_1$ ,  $C > 0$  and  $K > 0$  such that  $|f_{j,*}(x)| \leq CK^j \Gamma(1 + (s+1)j)$  for all  $j \geq 0$  and  $x \in D_{r_1}$

and we must prove that the coefficients  $u_{j,*}(x)$  satisfy similar conditions.

◁ It results from identities (2.1) that relations

$$\frac{u_{j,*}(x)}{\Gamma(1 + (s+1)j)} = \frac{f_{j,*}(x)}{\Gamma(1 + (s+1)j)} + \sum_{m=0}^{j-\kappa} \binom{j-\kappa}{m} a_{m,*}(x) \frac{\partial_x^p u_{j-\kappa-m,*}(x)}{\Gamma(1 + (s+1)j)}$$

hold for all  $j \geq 0$  (as before, we use the classical convention that the sum is 0 when  $j < \kappa$ ). Applying then the Nagumo norms of indices  $(pj, r_1)$ , we

deduce from property 4 of proposition 3.7 that

$$\frac{\|u_{j,*}(x)\|_{p_j,r_1}}{\Gamma(1+(s+1)j)} \leq \frac{\|f_{j,*}(x)\|_{p_j,r_1}}{\Gamma(1+(s+1)j)} + \sum_{m=0}^{j-\kappa} \binom{j-\kappa}{m} \|a_{m,*}(x)\|_{p(\kappa+m-1),r_1} \frac{\|\partial_x^p u_{j-\kappa-m,*}(x)\|_{p(j-\kappa-m+1),r_1}}{\Gamma(1+(s+1)j)}$$

and from property 5 of proposition 3.7 that

$$\frac{\|u_{j,*}(x)\|_{p_j,r_1}}{\Gamma(1+(s+1)j)} \leq \frac{\|f_{j,*}(x)\|_{p_j,r_1}}{\Gamma(1+(s+1)j)} + \sum_{m=0}^{j-\kappa} e^p A_{\kappa,p,m} \frac{\|a_{m,*}(x)\|_{p(\kappa+m-1),r_1}}{m!} \|u_{j-\kappa-m,*}(x)\|_{p(j-\kappa-m),r_1}$$

where

$$A_{\kappa,p,m} = \frac{\left( \prod_{\ell=0}^{m-1} (j-\kappa-\ell) \right) \left( \prod_{\ell'=0}^{p-1} (p(j-\kappa-m+1)-\ell') \right)}{\Gamma(1+(s+1)j)}$$

with the convention that the first product is 1 when  $m = 0$ . The following two lemmas allow to bound  $A_{\kappa,p,m}$ .

**Lemma 3.8** *Let  $j \geq \kappa$  and  $m \in \{0, \dots, j - \kappa\}$ . Then,*

$$\frac{\prod_{\ell=0}^{m-1} (j-\kappa-\ell)}{\Gamma(1+(s+1)j)} \leq \frac{1}{\Gamma(1+(s+1)(j-m))}.$$

**Proof.** Since the inequality is clear when  $m = 0$ , we assume below  $m \geq 1$  (hence,  $j > \kappa$ ). From relation

$$\Gamma(1+(s+1)j) = \Gamma(1+(s+1)j-m) \prod_{\ell=0}^{m-1} ((s+1)j-\ell),$$

we first deduce that

$$\frac{\prod_{\ell=0}^{m-1} (j-\kappa-\ell)}{\Gamma(1+(s+1)j)} = \frac{\prod_{\ell=0}^{m-1} \frac{j-\kappa-\ell}{(s+1)j-\ell}}{\Gamma(1+(s+1)j-m)} \leq \frac{1}{\Gamma(1+(s+1)j-m)}.$$

Lemma 3.8 follows then from inequalities

$$1 + (s + 1)j - m \geq 1 + (s + 1)(j - m) \geq 1 + (s + 1)\kappa \geq 2$$

and from the increase of the Gamma function on  $[2, +\infty[$ . ■

**Lemma 3.9** *Let  $j \geq \kappa$  and  $m \in \{0, \dots, j - \kappa\}$ . Then,*

$$\frac{\prod_{\ell'=0}^{p-1} (p(j - \kappa - m + 1) - \ell')}{\Gamma(1 + (s + 1)(j - m))} \leq \frac{\kappa^p}{\Gamma(1 + (s + 1)(j - \kappa - m))}.$$

**Proof.** • When  $1 \leq p \leq \kappa$  (hence,  $s = 0$ ), lemma 3.9 stems from relations

$$\begin{aligned} \prod_{\ell'=0}^{p-1} (p(j - \kappa - m + 1) - \ell') &= \kappa^p \prod_{\ell'=0}^{p-1} \left( \frac{p}{\kappa}(j - m + 1) - p - \frac{\ell'}{\kappa} \right) \\ &\leq \kappa^p \prod_{\ell'=0}^{p-1} \left( j - m + 1 - p - \frac{\ell'}{\kappa} \right) \end{aligned}$$

and

$$\Gamma(1 + j - m) = \Gamma(1 + j - \kappa - m) \prod_{\ell'=0}^{\kappa-1} (j - m - \ell').$$

Indeed, we clearly have

$$\begin{aligned} \frac{\prod_{\ell'=0}^{p-1} (p(j - \kappa - m + 1) - \ell')}{\Gamma(1 + j - m)} &\leq \frac{\kappa^p \prod_{\ell'=0}^{p-1} \frac{j - m + 1 - p - \frac{\ell'}{\kappa}}{j - m - \ell'}}{\Gamma(1 + j - \kappa - m)} \\ &\leq \frac{\kappa^p}{\Gamma(1 + j - \kappa - m)}. \end{aligned}$$

• In the opposite case  $1 \leq \kappa \leq p$  (hence,  $s = p/\kappa - 1$ ), lemma 3.9 is proved in a similar way by using relations

$$\prod_{\ell'=0}^{p-1} (p(j - \kappa - m + 1) - \ell') = \kappa^p \prod_{\ell'=0}^{p-1} \left( (s + 1)(j - m + 1) - p - \frac{\ell'}{\kappa} \right)$$

and

$$\Gamma(1 + (s + 1)(j - m)) = \Gamma(1 + (s + 1)(j - m) - p) \prod_{\ell'=0}^{p-1} ((s + 1)(j - m) - \ell').$$

Indeed, we get

$$\frac{\prod_{\ell'=0}^{p-1} (p(j - \kappa - m + 1) - \ell')}{\Gamma(1 + (s + 1)(j - m))} = \frac{\kappa^p \prod_{\ell'=0}^{p-1} \frac{(s + 1)(j - m + 1) - p - \frac{\ell'}{\kappa}}{(s + 1)(j - m) - \ell'}}{\Gamma(1 + (s + 1)(j - m) - p)}$$

$$\leq \frac{\kappa^p}{\Gamma(1 + (s + 1)(j - m) - p)}$$

and we conclude by observing that  $\Gamma(1 + (s + 1)(j - m) - p) = \Gamma(1 + (s + 1)(j - \kappa - m))$ . ■

Hence, the following inequalities

$$\boxed{\frac{\|u_{j,*}(x)\|_{p,j,r_1}}{\Gamma(1 + (s + 1)j)} \leq g_j + \sum_{m=0}^{j-\kappa} \alpha_m \frac{\|u_{j-\kappa-m,*}(x)\|_{p(j-\kappa-m),r_1}}{\Gamma(1 + (s + 1)(j - \kappa - m))}}$$

hold for all  $j \geq 0$  with

$$\boxed{g_j = \frac{\|f_{j,*}(x)\|_{p,j,r_1}}{\Gamma(1 + (s + 1)j)} \text{ and } \alpha_m = \frac{(e\kappa)^p \|a_{m,*}(x)\|_{p(\kappa+m-1),r_1}}{m!}.$$

◁ Let us now bound the  $\|u_{j,*}(x)\|_{p,j,r_1}$ 's. To do that, we shall use a technique of majorant series. Let us consider the numerical sequence  $(v_j)$  defined for all  $j \geq 0$  by the recursive relations

$$v_j = g_j + \sum_{m=0}^{j-\kappa} \alpha_m v_{j-\kappa-m}$$

(with the same classical convention as above on the sum). By construction, we have

$$0 \leq \frac{\|u_{j,*}(x)\|_{p,j,r_1}}{\Gamma(1 + (s + 1)j)} \leq v_j \quad \text{for all } j \geq 0$$

and the  $v_j$ 's can be bounded as follows. By assumption, we have

$$0 \leq g_j \leq \frac{CK^j \Gamma(1 + (s + 1)j)}{\Gamma(1 + (s + 1)j)} r_1^{pj} = C(Kr_1^p)^j$$

for all  $j \geq 0$  and the series  $g(X) = \sum_{j \geq 0} g_j X^j$  is convergent. On the other

hand, since  $a(x, t) \in \mathcal{O}(D_{\rho_1})\{t\}$ , there exist  $C', K' > 0$  such that  $|a_{m,*}(x)| \leq C' K'^m m!$  for all  $m \geq 0$  and  $x \in D_{r_1}$ . Then,

$$0 \leq \alpha_m \leq \frac{(e\kappa)^p C' K'^m m!}{m!} r_1^{p(\kappa+m-1)} = \frac{(e\kappa r_1^\kappa)^p C'}{r_1^p} (K' r_1^p)^m$$

for all  $m \geq 0$  and, consequently, the series  $A(X) = \sum_{j \geq 0} \alpha_j X^j$  is convergent too. From this and from the recurrence relations defining the  $v_j$ 's, we then deduce that the series  $v(X) = \sum_{j \geq 0} v_j X^j$  is also convergent. Indeed, it satisfies identity  $(1 - X^\kappa A(X))v(X) = g(X)$ . Therefore, there exist  $C''', K'' > 0$  such that  $v_j \leq C''' K''^j$  for all  $j \geq 0$ .

This leads then us to the following inequalities:

$$\|u_{j,*}(x)\|_{pj,r_1} \leq C''' K''^j \Gamma(1 + (s+1)j) \quad \text{for all } j \geq 0$$

and we are left to prove similar estimates on the sup-norm of the  $u_{j,*}(x)$ 's. To this end, we proceed by shrinking the domain  $D_{r_1}$ . Let  $0 < r'_1 < r_1$ . Then, for all  $j \geq 0$  and  $x \in D_{r'_1}$ , we have

$$|u_{j,*}(x)| = \left| u_{j,*}(x) d_{r_1}(x)^{pj} \frac{1}{d_{r_1}(x)^{pj}} \right| \leq \frac{1}{(r_1 - r'_1)^{pj}} |u_{j,*}(x) d_{r_1}(x)^{pj}|$$

and, thereby,

$$\sup_{x \in D_{r'_1}} |u_{j,*}(x)| \leq \frac{\|u_{j,*}(x)\|_{pj,r_1}}{(r_1 - r'_1)^{pj}} \leq C''' \left( \frac{K''}{(r_1 - r'_1)^p} \right)^j \Gamma(1 + (s+1)j).$$

This achieves the proof of theorem 3.3.

## 4 Summability

As we saw in corollary 3.4, the unique formal series solution  $\tilde{u}(x, t)$  and the inhomogeneity  $\tilde{f}(x, t)$  of problem (1.2) are together  $s$ -Gevrey. In particular, this provides us in the case  $1 \leq p \leq \kappa$  a necessary and sufficient condition under which  $\tilde{u}(x, t)$  is convergent (see corollary 3.5). In this section, we consider the opposite case  $1 \leq \kappa < p$  and we are interested in the summability of  $\tilde{u}(x, t)$ . More precisely, our aim is to display necessary and sufficient conditions under which  $\tilde{u}(x, t)$  is  $k$ -summable for  $k = 1/s$  in a given direction  $\arg(t) = \theta$ .

Before starting the calculations, let us recall the definition and some properties of the  $k$ -summability.

### 4.1 $k$ -summable formal series

As in section 3, we consider  $t$  as the variable and  $x$  as a parameter. Doing that,  $\tilde{u}(x, t)$  can be seen as a formal power series in  $t$  with coefficients in the

Banach space  $\mathcal{O}(D_{\rho_1})$ . To define the  $k$ -summability of such formal series, one then extends the classical notion of  $k$ -summability to families parametrized by  $x$  in requiring similar conditions, the estimates being however uniform with respect to  $x$ . For a general treatment of this theory, we refer for instance to [2]. Here below, we choose, among the many equivalent definitions of  $k$ -summability in a given direction  $\arg(t) = \theta$  at  $t = 0$ , a generalization of Ramis' definition which states that a formal series  $\tilde{g}$  is  $k$ -summable in direction  $\theta$  if there exists a holomorphic function  $g$  which is  $1/k$ -Gevrey asymptotic to  $\tilde{g}$  in an open sector  $\Sigma_{\theta, > \pi/k}$  bisected by  $\theta$  and with opening larger than  $\pi/k$  [17, Def. 3.1]. To express the  $1/k$ -Gevrey asymptotic, there also exist various equivalent ways. We choose here the one which sets conditions on the successive derivatives of  $g$  (see [12, p. 171] or [17, Thm. 2.4] for instance).

**Definition 4.1 ( $k$ -summability)** Let  $k > 0$  and  $s = 1/k$ . A formal series  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]$  is said to be  $k$ -summable in the direction  $\arg(t) = \theta$  if there exist a sector  $\Sigma_{\theta, > \pi s}$ , a radius  $0 < r_1 \leq \rho_1$  and a function  $u(x, t)$  called  $k$ -sum of  $\tilde{u}(x, t)$  in direction  $\theta$  such that

1.  $u$  is defined and holomorphic on  $D_{r_1} \times \Sigma_{\theta, > \pi s}$ ;
2. For any  $x \in D_{r_1}$ , the map  $t \mapsto u(x, t)$  has  $\tilde{u}(x, t) = \sum_{j \geq 0} u_{j,*}(x) \frac{t^j}{j!}$  as Taylor series at 0 on  $\Sigma_{\theta, > \pi s}$ ;
3. For any proper<sup>2</sup> subsector  $\Sigma \Subset \Sigma_{\theta, > \pi s}$ , there exist constants  $C > 0$  and  $K > 0$  such that, for all  $\ell \geq 0$ , all  $t \in \Sigma$  and all  $x \in D_{r_1}$ ,

$$|\partial_t^\ell u(x, t)| \leq CK^\ell \Gamma(1 + (s + 1)\ell).$$

We denote by  $\mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}$  the subset of  $\mathcal{O}(D_{\rho_1})[[t]]$  made of all the  $k$ -summable formal series in the direction  $\arg(t) = \theta$ . Obviously, we have  $\mathcal{O}(D_{\rho_1})\{t\}_{k;\theta} \subset \mathcal{O}(D_{\rho_1})[[t]]_s$ .

Note that, for any fixed  $x \in D_{r_1}$ , the  $k$ -summability of  $\tilde{u}(x, t)$  coincides with the classical  $k$ -summability. Consequently, Watson's lemma implies the unicity of its  $k$ -sum, if any exists.

Note also that the  $k$ -sum of a  $k$ -summable formal series  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}$  may be analytic with respect to  $x$  on a disc  $D_{r_1}$  smaller than the common

---

<sup>2</sup>A subsector  $\Sigma$  of a sector  $\Sigma'$  is said to be a *proper subsector* and one denotes  $\Sigma \Subset \Sigma'$  if its closure in  $\mathbb{C}$  is contained in  $\Sigma' \cup \{0\}$ .

disc  $D_{\rho_1}$  of analyticity of the coefficients  $u_{j,*}(x)$  of  $\tilde{u}(x, t)$ .

Obviously, the set  $\mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}$  is a subspace of  $\mathcal{O}(D_{\rho_1})[[t]]$ . Proposition 4.2 below precises its algebraic structure.

**Proposition 4.2** *Let  $k > 0$  and  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ . Then,  $(\mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}, \partial_x, \partial_t)$  is a  $\mathbb{C}$ -differential algebra stable under anti-derivatives  $\partial_x^{-1}$  and  $\partial_t^{-1}$ .*

**Proof.** The stability of  $\mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}$  under  $\partial_x^{-1}$ ,  $\partial_t$  and  $\partial_t^{-1}$  is straightforward. As for the stability under  $\partial_x$ , it is obtained in the same way of the stability of  $s$ -Gevrey formal series by using Cauchy integral formula on a disc  $D_{r'_1}$  with  $0 < r'_1 < r_1$ .

We are left to prove that  $\mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}$  is stable under multiplication. Let  $\tilde{u}(x, t)$  and  $\tilde{v}(x, t)$  be two  $k$ -summable formal series in direction  $\theta$ . We can always assume that  $\tilde{u}(x, t)$  and  $\tilde{v}(x, t)$  satisfy conditions of definition 4.1 with the same constants  $r_1$ ,  $C$  and  $K$  and the same sector  $\Sigma_{\theta, >\pi s}$ . Denote by  $\tilde{w}(x, t)$  their product. It obvious satisfies conditions 1 and 2 of definition 4.1. Moreover, given a proper subsector  $\Sigma \Subset \Sigma_{\theta, >\pi s}$  and using Leibniz formula, we get, for all  $\ell \geq 0$ ,  $x \in D_{r_1}$  and  $t \in \Sigma$ ,

$$\begin{aligned} |\partial_t^\ell \tilde{w}(x, t)| &\leq \sum_{j=0}^{\ell} \binom{\ell}{j} |\partial_t^j \tilde{u}(x, t)| |\partial_t^{\ell-j} \tilde{v}(x, t)| \\ &\leq C^2 K \sum_{j=0}^{\ell} \binom{\ell}{j} \Gamma(1 + (s+1)j) \Gamma(1 + (s+1)(\ell-j)). \end{aligned}$$

Similar calculations to those detailed in the proof of proposition 3.2 (see page 5) lead then us to an inequality of the form

$$|\partial_t^\ell \tilde{w}(x, t)| \leq C' K'^\ell \Gamma(1 + (s+1)\ell)$$

with convenient constants  $C', K' > 0$ ; this proves condition 3 of definition 3.1. Hence the stability of  $\mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}$  under multiplication. ■

## 4.2 Main result

The main result of this section is the following.

**Theorem 4.3** *Let  $1 \leq \kappa < p$ ,  $s = p/\kappa - 1$  and  $k = 1/s$ . Let  $\arg(t) = \theta \in \mathbb{R}/2\pi\mathbb{Z}$  be a direction issuing from 0. Then,*

1. The unique formal series solution  $\tilde{u}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]$  of problem (1.2) is  $k$ -summable in the direction  $\theta$  if and only if the inhomogeneity  $\tilde{f}(x, t)$  and the coefficients  $\tilde{u}_{*,n}(t) \in \mathbb{C}[[t]]$  for  $n = 0, \dots, p-1$  are  $k$ -summable in the direction  $\theta$ .
2. Moreover, the  $k$ -sum  $u(x, t)$ , if any exists, satisfies problem (1.2) in which  $\tilde{f}(x, t)$  is replaced by its  $k$ -sum  $f(x, t)$  in direction  $\theta$ .

Note that theorem 3.4 coincides with the result stated by W. Balser and M. Loday-Richaud in [4] in the case  $(\kappa, p) = (1, 2)$  and  $a(x, t) = \alpha(x)$  independent of  $t$ .

The proof of theorem 4.3 is the subject of next section 4.3. Before starting it, let us first show how theorem 4.3 allows to find the result formulated by M. Miyake in [13].

The formal series  $\tilde{u}_{*,n}(t) \in \mathbb{C}[[t]]$  for  $n = 0, \dots, p-1$  can be computed (at least theoretically) in terms of  $\tilde{f}(x, t)$  from the formula

$$\tilde{u}(x, t) = \sum_{m \geq 0} (\partial_t^{-\kappa} (a(x, t) \partial_x^p))^m \tilde{f}(x, t).$$

Let us assume from now that  $a(x, t) = a \in \mathbb{C}^*$  is a nonzero constant and that  $\tilde{f}(x, t) = f(x) = \sum_{n \geq 0} f_{0,n} \frac{x^n}{n!}$  is independent of  $t$ . Note that this case is equivalent to problem (1.1) with  $\tilde{q} \equiv 0$  and with initial conditions  $\tilde{u}(x, 0) = f(x)$  and  $\partial_t^j \tilde{u}(x, t)|_{t=0} \equiv 0$  for all  $j = 1, \dots, \kappa - 1$ .

Since operators  $a$ ,  $\partial_x$  and  $\partial_t$  commute, we get

$$\tilde{u}(x, t) = \sum_{m \geq 0} a^m \partial_t^{-\kappa m} \partial_x^{pm} f(x)$$

and thereby

$$\tilde{u}_{*,n}(t) = \sum_{m \geq 0} \frac{(at^\kappa)^m}{(\kappa m)!} f_{0, pm+n} \quad \text{for all } n = 0, \dots, p-1.$$

Let us now denote by  $\tilde{F}$  the function defined by  $\tilde{F}(x) = \sum_{n \geq 0} f_{0,n} \frac{x^n}{[\kappa n/p]!}$ , where  $[\kappa n/p]$  stands for the integer part of  $\kappa n/p$ . Then,

$$\tilde{F}((at^\kappa)^{1/p}) = \sum_{n=0}^{p-1} (at^\kappa)^{n/p} \tilde{u}_{*,n}(t)$$

and the following result may be proved.



**Proposition 4.4** *Let  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ . Then, the following three assertions are equivalent.*

1. *The  $\tilde{u}_{*,n}(t)$ 's are  $k$ -summable for all  $n \in \{0, \dots, p-1\}$  in direction  $\theta$ .*
2.  *$\tilde{F}(x)$  is  $(k+1)$ -summable in all the directions  $(\kappa\theta + \arg(a))/p \pmod{2\pi/p}$ .*
3.  *$f(x)$  is analytic near 0 and can be analytically continued to sectors neighbouring the directions  $(\kappa\theta + \arg(a))/p \pmod{2\pi/p}$  with exponential growth of order  $k+1$  at infinity.*

Observe that assertion 3 with  $a = 1$  (hence,  $\arg(a) = 0$ ) is the necessary and sufficient condition stated by M. Miyake in [13] for the  $k$ -summability of  $\tilde{u}(x, t)$  and proved via direct  $k$ -Borel-Laplace transformations. In particular, our method provides a new proof of this result.

Observe also that proposition 4.4 coincides with the result proved by W. Balsler and M. Loday-Richaud in [4] in the case  $(\kappa, p) = (1, 2)$ .

When  $\tilde{f}(x, t) \in \mathcal{O}(D_{\rho_1})\{t\}_{k;\theta}$  is a more general  $k$ -summable series in a given direction  $\theta$ , a result of the same type can be written. Nevertheless, calculations are much more complicated and require in general to use Borel and Laplace transforms of  $\tilde{f}(x, t)$  in both variables. For an exemple in the case  $(\kappa, p) = (1, 2)$ , we refer for instance to [4].

### 4.3 Proof of theorem 4.3

Let us start this proof with a preliminary remark on  $\tilde{u}(x, t)$ . Writing as before  $a(x, t) = \sum_{n \geq 0} a_{*,n}(t) \frac{x^n}{n!}$  with  $a_{*,n}(t) \in \mathcal{O}(D_{\rho_2})$ , an identification of the powers in  $x$  in equation

$$(1 - \partial_t^{-\kappa}(a(x, t)\partial_x^p)) \sum_{n \geq 0} \tilde{u}_{*,n}(t) \frac{x^n}{n!} = \sum_{n \geq 0} \tilde{f}_{*,n}(t) \frac{x^n}{n!}$$

brings us to the recurrence relations

$$a_{*,0}(t)\tilde{u}_{*,n+p} = \partial_t^\kappa(\tilde{u}_{*,n} - \tilde{f}_{*,n}) - \sum_{m=1}^n \binom{n}{m} a_{*,m}(t)\tilde{u}_{n+p-m}$$

with  $n \geq 0$  and the classical convention that the sum is 0 if  $n = 0$ . By assumption, we have  $a(0, 0) \neq 0$  (see page 2); hence,  $1/a_{*,0}(t)$  is well-defined in  $\mathbb{C}[[t]]$  and, consequently, each  $\tilde{u}_{*,\ell}(t)$  is uniquely determined from  $\tilde{f}(x, t)$  and from the  $\tilde{u}_{*,n}(t)$ 's with  $n = 0, \dots, p-1$ . In particular, the same applies to  $\tilde{u}(x, t)$ .

Let us now turn to the proof of theorem 4.3.

**Point 1. Necessary condition.** This is straightforward from proposition 4.2. Indeed, since  $\tilde{f}(x, t) = D_{\kappa, p}\tilde{u}(x, t)$  and since  $\tilde{u}_{*,n}(t) = \partial_x^n \tilde{u}(x, t)|_{x=0}$ , the  $k$ -summability of  $\tilde{u}(x, t)$  implies the  $k$ -summability of  $\tilde{f}(x, t)$  and the  $\tilde{u}_{*,n}(t)$ 's.

**Point 1. Sufficient condition.** Prove now that the condition is sufficient. To do that, we proceed in a similar way as the proof of [4, Thm. 3.4].

By assumption, we have  $a(0, 0) \neq 0$ . Then,  $b(x, t) := 1/a(x, t)$  is well-defined and holomorphic on a domain  $D_{\rho'_1} \times D_{\rho'_2}$  with convenient  $\rho'_1, \rho'_2 > 0$ .

Let us now write

$$\tilde{u}(x, t) = \sum_{n=0}^{p-1} \tilde{u}_{*,n}(t) \frac{x^n}{n!} + \partial_x^{-p} \tilde{v}(x, t)$$

with  $\tilde{v}(x, t) \in \mathcal{O}(D_{\rho_1})[[t]]$  and let us set  $\tilde{w} := \partial_t^{-\kappa}(a(x, t)\tilde{v})$ . Then, problem (1.2) can be rewritten on the form

$$(4.1) \quad (1 - \partial_x^{-p}(b(x, t)\partial_t^\kappa))\tilde{w} = \tilde{g}(x, t) \text{ with } \tilde{g}(x, t) = \sum_{n=0}^{p-1} \tilde{u}_{*,n}(t) \frac{x^n}{n!} - \tilde{f}(x, t).$$

Consequently, assuming  $\tilde{f}(x, t)$  and the  $\tilde{u}_{*,n}(t)$ 's  $k$ -summable in a given direction  $\theta$ , it suffices to prove that so is  $\tilde{w}(x, t)$ .

To this end, we proceed through a fixed point method as follows. Let us set  $\tilde{w}(x, t) = \sum_{m \geq 0} \tilde{w}_m(x, t)$  and let us consider the solution of equation (4.1),

where the  $\tilde{w}_m(x, t)$ 's belong to  $\mathcal{O}(D_\rho)[[t]]$  for a suitable common  $\rho > 0$  and are recursively defined by the relations

$$(4.2) \quad \begin{cases} \tilde{w}_0 = \tilde{g}, \\ \tilde{w}_m = \partial_x^{-p}(b(x, t)\partial_t^\kappa \tilde{w}_{m-1}) \text{ for } m \geq 1. \end{cases}$$

Note that, for all  $m \geq 0$ , the formal series  $\tilde{w}_m(x, t)$  are of order  $O(x^{pm})$  in  $x$  and, consequently, the series  $\tilde{w}(x, t)$  itself makes sense as a formal series in  $t$  and  $x$ .

Let  $w_0(x, t)$  denote the  $k$ -sum of  $\tilde{w}_0 = \tilde{g}$  in direction  $\theta$  and, for all  $m \geq 0$ , let  $w_m(x, t)$  be determined as the solution of system (4.2) in which all the  $\tilde{w}_m$  are replaced by  $w_m$ . By construction, all the  $w_m(x, t)$ 's are defined and holomorphic on a common domain  $D_{\rho''_1} \times \Sigma_{\theta, > \pi_s}$ , where the radius  $\rho''_1$  of  $D_{\rho''_1}$  and the radius  $\rho''_2$  of  $\Sigma_{\theta, > \pi_s}$  can always be chosen so that  $0 < \rho''_1 < \min(\rho_1, \rho'_1)$  and  $0 < \rho''_2 < \rho'_2$ . To end the proof, we shall now show that the series  $\sum_{m \geq 0} w_m(x, t)$  is convergent and that its sum  $w(x, t)$  is the  $k$ -sum of  $\tilde{w}(x, t)$  in direction  $\theta$ .

According to definition 3.1, the  $k$ -summability of  $\tilde{w}_0$  implies that there exists  $0 < r_1 < \rho_1''$  such that, for any proper subsector  $\Sigma \Subset \Sigma_{\theta, > \pi s}$ , there exist constants  $C, K > 0$  such that, for all  $\ell \geq 0$  and  $(x, t) \in D_{r_1} \times \Sigma$ , the function  $w_0$  satisfies the inequalities

$$(4.3) \quad \left| \partial_t^\ell w_0(x, t) \right| \leq CK^\ell \Gamma(1 + (s + 1)\ell).$$

Let us now fix a proper subsector  $\Sigma \Subset \Sigma_{\theta, > \pi s}$  and let us denote by  $r_2$  its radius. Note that inequalities (4.3) still hold with the same constants  $C$  and  $K$  for any  $0 < r_1' < r_1$ .

Let  $B := \max_{(x, t) \in \overline{D}_{\rho_1''} \times \overline{D}_{\rho_2''}} |b(x, t)|$ , where  $\overline{D}_\rho$  denotes the closed disc with center 0 and radius  $\rho$ . Note that  $B$  is well-defined since  $b(x, t)$  is holomorphic on  $D_{\rho_1'} \times D_{\rho_2'}$  and  $\rho_j'' < \rho_j'$  for  $j = 1, 2$ . Note also that Cauchy integral formula gives us

$$\partial_t^\ell b(x, t) = \frac{\ell!}{(2i\pi)^2} \int_{\substack{|x'-x|=\rho_1''-r_1 \\ |t'-t|=\rho_2''-r_2}} \frac{b(x', t')}{(x' - x)(t' - t)^{\ell+1}} dx' dt';$$

hence, inequalities

$$\left| \partial_t^\ell b(x, t) \right| \leq \ell! B \left( \frac{1}{\rho_2'' - r_2} \right)^\ell$$

for all  $\ell \geq 0$  and  $(x, t) \in D_{r_1} \times \Sigma$ . In particular, these estimates only depend on the radius  $r_2$  of sector  $\Sigma$  and not on  $r_1$ . Thereby, the constant  $K$  being chosen  $\geq 1/(\rho_2'' - r_2)$ , we get

$$(4.4) \quad \left| \partial_t^\ell b(x, t) \right| \leq \ell! BK^\ell \quad \text{for all } \ell \geq 0 \text{ and } (x, t) \in D_{r_1} \times \Sigma.$$

Proposition 4.5 below provides us some estimates on the derivatives  $\partial_t^\ell w_m$ .

**Proposition 4.5** *Let  $B' := (\kappa + 1)B$ . Then, the following inequalities*

$$\left| \partial_t^\ell w_m(x, t) \right| \leq CB'^m K^{\kappa m + \ell} \Gamma(1 + (s + 1)(\kappa m + \ell)) \frac{|x|^{pm}}{(pm)!}$$

hold for all  $m, \ell \geq 0$  and all  $(x, t) \in D_{r_1} \times \Sigma$ .

**Proof.** Proposition 4.5 is clear for  $m = 0$ . Prove it for  $m = 1$ . From relation  $w_1 = \partial_x^{-p}(b(x, t)\partial_t^\kappa w_0)$ , we deduce that, for all  $(x, t) \in D_{r_1} \times \Sigma$ ,

$$(4.5) \quad \left| \partial_t^\ell w_1(x, t) \right| \leq \frac{|x|^p}{p!} \sup_{(x, t) \in D_{r_1} \times \Sigma} \left| \partial_t^\ell (b(x, t)\partial_t^\kappa w_0)(x, t) \right|.$$

On the other hand, Leibniz formula implies

$$|\partial_t^\ell(b(x, t)\partial_t^\kappa w_0)(x, t)| \leq \sum_{j=0}^{\ell} \frac{\ell!}{j!} \left| \frac{\partial_t^{\ell-j} b}{(\ell-j)!}(x, t) \right| |\partial_t^{\kappa+j} w_0(x, t)|;$$

hence, using inequalities (4.3) and (4.4),

$$|\partial_t^\ell(b(x, t)\partial_t^\kappa w_0)(x, t)| \leq CBK^{\kappa+\ell} \sum_{j=0}^{\ell} \frac{\ell!}{j!} \Gamma(1 + (s+1)(\kappa+j))$$

for all  $(x, t) \in D_{r_1} \times \Sigma$ . Then, applying successively technical lemmas 4.6 and 4.7 below, we get

$$\begin{aligned} |\partial_t^\ell(b(x, t)\partial_t^\kappa w_0)(x, t)| &\leq CBK^{\kappa+\ell} \sum_{j=0}^{\ell} \Gamma(1 + (s+1)(\kappa+j) + \ell - j) \\ &= CBK^{\kappa+\ell} \Gamma(1 + (s+1)(\kappa+\ell)) \times \\ &\quad \sum_{j=0}^{\ell} \frac{\Gamma(1 + (s+1)(\kappa+j) + \ell - j)}{\Gamma(1 + (s+1)(\kappa+\ell))} \\ &\leq CBK^{\kappa+\ell} \Gamma(1 + (s+1)(\kappa+\ell)) \times (\kappa+1) \\ &= CB'K^{\kappa+\ell} \Gamma(1 + (s+1)(\kappa+\ell)). \end{aligned}$$

Inequality (4.5) ends the proof for  $m = 1$ . For  $m \geq 2$ , we proceed by recursion on  $m$  by using relations  $w_{m+1} = \partial_x^{-p}(b(x, t)\partial_t^\kappa w_m)$  and same arguments as above. This achieves the proof. ■

**Lemma 4.6** For all  $\ell \geq 0$ ,  $j \in \{0, \dots, \ell\}$  and  $m \geq 1$ ,

$$(4.6) \quad \frac{\ell!}{j!} \Gamma(1 + (s+1)(\kappa m + j)) \leq \Gamma(1 + (s+1)(\kappa m + j) + \ell - j).$$

**Proof.** Lemma 4.6 is clear for  $j = \ell$ . Assume now  $j < \ell$  and write  $\ell!/j!$  on the form

$$\frac{\ell!}{j!} = \prod_{n=j+1}^{\ell} n = \prod_{n=j+1}^{\ell} (j+n-j).$$

We get

$$\frac{\ell!}{j!} \leq \prod_{n=j+1}^{\ell} ((s+1)(\kappa m + j) + n - j) = \prod_{n=1}^{\ell-j} ((s+1)(\kappa m + j) + n)$$

and relation

$$\Gamma(1+(s+1)(\kappa m+j)+\ell-j) = \Gamma(1+(s+1)(\kappa m+j)) \prod_{n=1}^{\ell-j} ((s+1)(\kappa m+j)+n)$$

proves then inequality (4.6), which ends the proof. ■

**Lemma 4.7** For all  $\ell \geq 0$  and  $m \geq 1$ ,

$$(4.7) \quad \sum_{j=0}^{\ell} \frac{\Gamma(1+(s+1)(\kappa m+j)+\ell-j)}{\Gamma(1+(s+1)(\kappa m+\ell))} \leq \kappa+1.$$

**Proof.** ◁ Let us first suppose  $\ell \leq \kappa$ . For all  $j \in \{0, \dots, \ell\}$ , we have

$$1+(s+1)(\kappa m+j)+\ell-j \leq 1+(s+1)\kappa m+\ell+s j \leq 1+(s+1)(\kappa m+\ell)$$

and

$$1+(s+1)(\kappa m+j)+\ell-j \geq 1+\kappa m(s+1) = 1+pm \geq 3.$$

Hence, using the increasing of the Gamma function on  $[2, +\infty[$ ,

$$\sum_{j=0}^{\ell} \frac{\Gamma(1+(s+1)(\kappa m+j)+\ell-j)}{\Gamma(1+(s+1)(\kappa m+\ell))} \leq \sum_{j=0}^{\ell} 1 = \ell+1 \leq \kappa+1$$

and so inequality (4.7).

◁ Let us now suppose  $\ell > \kappa$  and let us write the sum of (4.7) on the form

$$(4.8) \quad \sum_{j=0}^{\ell} \dots = \sum_{j=0}^{\ell-\kappa} \dots + \sum_{j=\ell-\kappa+1}^{\ell} \dots$$

The second sum of the right-hand side of (4.8) is treated as in the first case and we get

$$\sum_{j=\ell-\kappa+1}^{\ell} \frac{\Gamma(1+(s+1)(\kappa m+j)+\ell-j)}{\Gamma(1+(s+1)(\kappa m+\ell))} \leq \sum_{j=\ell-\kappa+1}^{\ell} 1 = \kappa.$$

On the other hand, for  $j \in \{0, \dots, \ell-\kappa\}$ , similar calculations as above lead us to the following inequalities

$$3 \leq 1+(s+1)(\kappa m+j)+\ell-j \leq 1+(s+1)(\kappa m+\ell) - \kappa s.$$

Thereby, the first sum of the right-hand side of (4.8) gives us

$$\begin{aligned} \sum_{j=0}^{\ell-\kappa} \frac{\Gamma(1+(s+1)(\kappa m+j)+\ell-j)}{\Gamma(1+(s+1)(\kappa m+\ell))} &\leq (\ell-\kappa+1) \frac{\Gamma(1+(s+1)(\kappa m+\ell)-\kappa s)}{\Gamma(1+(s+1)(\kappa m+\ell))} \\ &= \frac{\ell-\kappa+1}{(s+1)(\kappa m+\ell)} \frac{\Gamma(1+(s+1)(\kappa m+\ell)-\kappa s)}{\Gamma(1+(s+1)(\kappa m+\ell)-1)} \\ &\leq \frac{\ell-\kappa+1}{(s+1)(\ell+\kappa m)}. \end{aligned}$$

Indeed, we have  $\ell > \kappa$  and  $\kappa s = p - \kappa \geq 1$ ; hence,

$$3 \leq 1 + pm \leq 1 + (s+1)(\kappa m + \ell) - \kappa s \leq 1 + (s+1)(\kappa m + \ell) - 1$$

and, consequently,

$$\frac{\Gamma(1+(s+1)(\kappa m+\ell)-\kappa s)}{\Gamma(1+(s+1)(\kappa m+\ell)-1)} \leq 1.$$

We then conclude by observing that

$$\frac{\ell-\kappa+1}{(s+1)(\ell+\kappa m)} \leq \frac{1}{s+1} \leq 1$$

for all  $\ell \geq 0$ . This ends the proof of lemma 4.7. ■

From proposition 4.5, we deduce that, for all  $\ell \geq 0$  and  $(x, t) \in D_{r_1} \times \Sigma$ ,

$$\sum_{m \geq 0} |\partial_t^\ell w_m(x, t)| \leq CK^\ell \Gamma(1+(s+1)\ell) \sum_{m \geq 0} A_{m,\ell}(x)$$

with

$$A_{m,\ell}(x) = \frac{\Gamma(1+(s+1)(\kappa m+\ell))}{\Gamma(1+(s+1)\ell)} \frac{(B'K^\kappa |x|^p)^m}{(pm)!}.$$

Let us now observe that inequality  $s+1 \leq p$  implies

$$\begin{aligned} \Gamma(1+(s+1)(\kappa m+\ell)) &= \Gamma(1+(s+1)\ell+pm) \\ &= \Gamma(1+(s+1)\ell) \prod_{j=1}^{pm} ((s+1)\ell+j) \\ &\leq \Gamma(1+(s+1)\ell) \prod_{j=1}^{pm} (p\ell+j) \end{aligned}$$

and, thereby,

$$\frac{\Gamma(1 + (s+1)(\kappa m + \ell))}{(pm)! \Gamma(1 + (s+1)\ell)} \leq \binom{p\ell + pm}{pm} \leq \sum_{j=0}^{p\ell + pm} \binom{p\ell + pm}{j} = 2^{p\ell + pm}.$$

Consequently,

$$\sum_{m \geq 0} |\partial_t^\ell w_m(x, t)| \leq C(2^p K)^\ell \Gamma(1 + (s+1)\ell) \sum_{m \geq 0} (2^p B' K^\kappa |x|^p)^m$$

for all  $\ell \geq 0$  and  $(x, t) \in D_{r_1} \times \Sigma$ . Let  $L = 2^p B' K^\kappa r^p$  and choose  $0 < r < r_1$  small enough so that  $L < 1$ . Denote  $C' = C \sum_{m \geq 0} L^m \in \mathbb{R}^+$  and  $K' = 2^p K$ .

Then, for all  $\ell \geq 0$  and  $(x, t) \in D_r \times \Sigma$ ,

$$(4.9) \quad \sum_{m \geq 0} |\partial_t^\ell w_m(x, t)| \leq C' K'^\ell \Gamma(1 + (s+1)\ell).$$

In particular, for  $\ell = 0$ , the series  $\sum_{m \geq 0} w_m(x, t)$  is normally convergent on  $D_r \times \Sigma$ . Therefore, its sum  $w(x, t)$  is well-defined and holomorphic on  $D_r \times \Sigma$ . This proves condition 1 of definition 3.1 if we choose for  $\Sigma$  a sector bisected by  $\theta$  and opening larger than  $\pi s = \pi/k$ . Note that such a choice is already possible due to the definition of proper subsector (see note 2).

For all  $\ell \geq 1$ , the series  $\sum_{m \geq 0} \partial_t^\ell w_m(x, t)$  is also normally convergent on  $D_r \times \Sigma$ . Thereby, the series  $\sum_{m \geq 0} w_m(x, t)$  can be derivated termwise infinitely many times with respect to  $t$  and inequalities (4.9) imply

$$|\partial_t^\ell w_m(x, t)| \leq C' K'^\ell \Gamma(1 + (s+1)\ell)$$

for all  $\ell \geq 0$  and  $(x, t) \in D_r \times \Sigma$ . This proves condition 3 of definition 3.1.

Note that the fact that all derivatives  $\partial_t^\ell w(x, t)$  of  $w(x, t)$  are bounded on  $\Sigma$  implies the existence of  $\lim_{\substack{t \rightarrow 0 \\ t \in \Sigma}} \partial_t^\ell w(x, t)$  for all  $x \in D_r$ ; hence, the existence of

the Taylor series of  $w$  at 0 on  $\Sigma$  for all  $x \in D_r$ . On the other hand, considering recurrence relations (4.2) with the  $k$ -sums  $w_m$  and  $g$  instead of  $\tilde{w}_m$  and  $\tilde{g}$ , it is clear that  $w(x, t)$  satisfies equation (4.1) with right-hand side  $g(x, t)$  in place of  $\tilde{g}(x, t)$ . Consequently, the Taylor series of  $w(x, t)$  also satisfies this equation. Then, since equation (4.1) admits a unique formal series solution  $\tilde{w}(x, t)$  (see theorem 2.1 by exchanging the roles of  $x$  and  $t$ ), it results that the Taylor expansion of  $w(x, t)$  is  $\tilde{w}(x, t)$ ; hence, condition 2 of definition 3.1.

This shows the  $k$ -summability of  $\tilde{w}(x, t)$  in direction  $\theta$  and proves thereby that the condition is sufficient.

**Point 2.** As for point 2 of theorem 4.3, let us observe that the fact that the  $k$ -sum  $u(x, t)$  of  $\tilde{u}(x, t)$  in direction  $\theta$  satisfies problem (1.2) in which  $\tilde{f}(x, t)$  is replaced by its  $k$ -sum  $f(x, t)$  in direction  $\theta$  is equivalent to the fact that  $w(x, t)$  satisfies equation (4.1) with right-hand side  $g(x, t)$  instead of  $\tilde{g}(x, t)$ , which we proved just above. Hence, point 2.

This achieves the proof of theorem 4.3.

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