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Chapter 1

Coalescing complex planar stationary points

Loïc Teyssier

Among all bifurcation behaviors of parametric families of real planar vector fields $Z_\bullet = (Z_\lambda)_{\lambda \in \Lambda}$, those that stand out most prominently are confluences of distinct stationary points. The qualitative change is so drastic that in some classes of families (*e.g.* fold-like bifurcations) the stationary points generically annihilate each others in the process (Sotomayor's theorem).

The simplest example of such a behavior, an instance of saddle-node bifurcation, is the polynomial family $(X_\lambda^\infty)_{\lambda \in \mathbb{R}}$ given in the canonical basis of \mathbb{R}^2 by

$$X_\lambda^\infty(x, y) := \begin{bmatrix} x^2 + \lambda \\ y \end{bmatrix} . \quad (1.1)$$

The bifurcation value occurs at $\lambda = 0$: for negative λ the system has two stationary points located at $(\pm\sqrt{-\lambda}, 0)$ which collide as λ reaches 0, while none remain for $\lambda > 0$. The stationary points have left the real plane, true enough, but only to slip into the complex domain. Let us elaborate a bit on this observation in order to motivate the need for complexifying the whole setting, even in the context of real dynamics.

The trajectories $t \mapsto (x(t), y(t))$ of X_λ^∞ appear naturally as solutions of the autonomous flow-system of X_λ^∞ :

$$\begin{cases} \dot{x}(t) &= x(t)^2 + \lambda \\ \dot{y}(t) &= y(t) \end{cases}$$

and can be implicitly expressed by solving the associated non-autonomous differential equation. This equation is obtained by eliminating the time in the flow-system using the rule $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}$:

$$(x^2 + \lambda) y'(x) = y(x) .$$

Separation of variables yields multivalued complex solutions

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$$y_\lambda : z \mapsto c \left(\frac{z - i\sqrt{\lambda}}{z + i\sqrt{\lambda}} \right)^{1/2i\sqrt{\lambda}}, \quad c \in \mathbb{C}. \quad (1.2)$$

On the one hand, if $\lambda < 0$ real solutions are given on appropriate intervals by

$$y_\lambda : x \mapsto c \left| \frac{x - \sqrt{-\lambda}}{x + \sqrt{-\lambda}} \right|^{1/2\sqrt{-\lambda}}, \quad c \in \mathbb{R}, \quad (1.3)$$

from which we deduce that $(-\sqrt{-\lambda}, 0)$ is a saddle-point and $(\sqrt{-\lambda}, 0)$ a node-point. On the other hand for $\lambda > 0$ we have

$$y_\lambda : x \mapsto c \exp\left(\frac{1}{\sqrt{\lambda}} \arctan \frac{x}{\sqrt{\lambda}}\right), \quad c \in \mathbb{R}. \quad (1.4)$$

The latter is a perfectly honest real-analytic function on \mathbb{R} . One might wonder why, despite the fact of being so regular a function, its Taylor expansion at 0 does not have infinite convergence radius instead of $\sqrt{\lambda}$. One can *explain* the discrepancy by, say, direct use of Cauchy-Hadamard formula, although one cannot *understand* its source without noticing the imaginary singularities $\pm i\sqrt{\lambda}$ quietly sitting on the boundary of the disk of convergence. Also it is hard to understand why, when playing the movie backwards starting from positive values of λ and reaching negative ones, a stationary point somehow pops out of nowhere. One can see the singularity coming only when looking along the imaginary axis.

At a less commonplace level, when $\lambda < 0$ both stationary points organize the dynamics of X_λ^∞ and there is no reason why they should stop to when $\lambda > 0$, or even when λ is not real, and we will present how.

We begin this text by performing the detailed study of affine families perturbing X_\bullet^∞ in Section (1.1)

$$X_\bullet : X_\lambda(x, y) := \begin{bmatrix} x^2 + \lambda \\ y - (x^2 + \lambda) a_\lambda(x) \end{bmatrix}, \quad (1.5)$$

where $(\lambda, x) \mapsto a_\lambda(x)$ is a given analytic function near the origin of \mathbb{C}^2 . Let us denote Affine(1) the collection of all such vector fields. Although for elements of Affine(1) all computations can be performed explicitly (variation of constant), some natural questions and non-trivial answers arise already in this case-study. Generalizing the constructions and objects introduced in that simple situation to arbitrary bifurcation-preserving analytic perturbations of the model family X_\bullet^∞ is the main concern of the rest of the chapter.

Reducing the setting to analytic parametric families may seem rather restrictive. Yet the geometric approach we present here could be inherited by less regular situations, or could give insights as to where sources of peculiar behaviors may lie. On the other end of the argument, the obvious added benefit stemming from this restriction is the rigidity of holomorphic functions and diffeomorphisms of complex (compact) manifolds. Also the analytic class encompasses polynomial vector fields, of special interest for planar vector fields *e.g.* regarding Hilbert's 16th problem on the number / position of limit cycles, or Poincaré's problem on the existence of rational first integrals.

In the sequel we investigate the links between local dynamics on the one hand, local analytic classification on the other hand, while at the same time hinting at how they can help measuring divergence of some class of «summable» power series. We particularly explain the role of complex geometry and analysis in understanding saddle-node bifurcations. We wish to underline that the two objects Z_0 and $(Z_\lambda)_{\lambda \neq 0}$ are intertwined, as dynamical properties for one can be deduced from studying objects attached to the other and *vice versa*.

We do not wish to emphasize too much the link between orbital classification of Z_\bullet and classification of its strong holonomy \mathfrak{h}_\bullet , the family of holomorphic first-return map of Z_λ on a fixed horizontal disc which crosses $\{x^2 + \lambda = 0\}$ for $|\lambda|$ small enough. The connection is very clearly explained by C. Rousseau for instance in [15] for saddle-node bifurcations, or again in [16] for deformations of a resonant saddle stationary point. Although both objects encode the same dynamics, and are classified by the same invariant under local analytic equivalence and change of parameters, we take advantage of the extra dimension the complex plane \mathbb{C}^2 offers to deploy more geometrical constructions *à la* Martinet-Ramis [12, 22]. Instead of simply deducing the classification of vector fields from that of holonomies, which would frankly spoil the fun, the present text is focused on building objects specifically from the continuous nature of the dynamics of Z_λ . Although both moduli spaces end up with the same presentation, some formulations for vector fields yield different characterizations of *e.g.* the orbital «compatibility condition» as compared to holonomies [18]. A by-product of this approach is an explicit family of normal forms for bifurcations Z_\bullet having persisting heteroclinic connections, generalizing [20] to the case $\lambda \neq 0$. There is as yet no such known explicit universal family for holonomies \mathfrak{h}_\bullet (not even for \mathfrak{h}_0).

The choice has been made to focus mainly on precise constructions and sketches of proof whenever doing so helps the exposition, while detailing the proofs of original material. The missing technical details are to be found mostly in [19].

All notations and basic notions pertaining to the theory of holomorphic vector fields and foliations shall be recalled in Section 1.2.

Section 1.3 is devoted to an introductory text, giving a brief historical overview and explaining the main relationships between the former themes.

In that respect we present a more detailed account of the formal and local classifications in Section 1.4, where the main theorems are stated, and subsequent sections. We particularly address in Section 1.9 the dynamical interpretation of the modulus of classification.

1.1 Affine saddle-node bifurcations

The study of affine saddle-node vector fields was initiated in the second half of the nineteenth century by C. Bouquet and C. Briot [2] as a family of examples of invariant manifolds existing at a formal level but not at an analytic one, generalizing the famous Euler's differential equation $x^2 y' = y - x$. Their first significant result is the existence of a formal **weak separatrix** for X_0 , that is an invariant formal curve $\{y = \hat{\mathfrak{s}}(x)\}$ with

$\hat{\mathfrak{s}} \in \mathbb{C}[[x]]$. They obtained an explicit criterion for convergence of $\hat{\mathfrak{s}}$ in terms of the Taylor coefficients of a_0 , which we recover in Proposition 3 after a study aimed at understanding how the trajectories of X_λ , $\lambda \neq 0$, evolve into those of X_0 .

We begin by explaining the easiest instance $a_\bullet = 0$. Standard results describing the regularity of parametric solutions state that solutions y_λ in (1.3) and (1.4) converge to corresponding solutions for $\lambda = 0$

$$x^2 y'(x) = y(x) \tag{1.6}$$

$$y_0 : x \mapsto c \exp\left(-\frac{1}{x}\right), \quad c \in \mathbb{C}$$

uniformly on compact subsets of $\mathbb{R}_{\neq 0}$ as $\lambda \rightarrow 0$ (it suffices to wait until the stationary points have left the compact set). Now, can we say something about the convergence near 0? Obviously the question only makes sense for families of solutions bounded near 0 as $\lambda \rightarrow 0$. For $\lambda = 0$ the limiting objects are center manifolds of the saddle-node stationary point $(0, 0)$ of X_0^∞ . As a real vector field X_0^∞ has infinitely many center manifolds passing through $(0, 0)$, each one given by the graph $\{y = s(x)\}$ of the smooth (meaning C^∞) function

$$\begin{aligned} s : \mathbb{R} &\longrightarrow \mathbb{R} \\ x \leq 0 &\longmapsto 0 \\ x > 0 &\longmapsto c \exp\left(-\frac{1}{x}\right) \end{aligned}$$

for arbitrary $c \in \mathbb{R}$. Those are the only bounded solutions of (1.6) at 0. Only one of them is analytic there, namely $\mathcal{S}_0 := \{y = 0\}$, all others being non-zero flat functions. This property identifies uniquely a distinguished center manifold, called the **weak separatrix** of X_0^∞ , with the most regular dynamics. The weak separatrix is the limiting curve of the family $(\mathcal{S}_\lambda)_\lambda$ collecting the only smooth integral curve connecting both stationary points $(\pm\sqrt{-\lambda}, 0)$ for $\lambda < 0$. In this simple situation the only such **heteroclinic** integral curve is $\mathcal{S}_\lambda = \{y = 0\}$, since y_λ in 1.3 is not of class C^{r+1} at the node, $r := \left\lceil \frac{1}{2\sqrt{-\lambda}} \right\rceil$, save for $c = 0$.

Consider next a quadratic perturbation of X_\bullet^∞ , the Euler family

$$E_\lambda(x, y) := \begin{bmatrix} x^2 + \lambda \\ y - (x^2 + \lambda) \end{bmatrix} \tag{1.7}$$

whose stationary points are again located at $(\pm s, 0)$ where, for the sake of simplicity, we set:

$$s := \sqrt{-\lambda}.$$

This is a special member of Affine (1) (see (1.5)) obtained by setting $a_\bullet := 1$, yet we are to prove that together with X_\bullet^∞ they somehow span all possible behaviors for members of Affine (1).

For $\lambda = 0$ infinitely many smooth center manifolds persist through $(0, 0)$, given by the graphs of

$$\begin{aligned} \mathfrak{s}_0 : \mathbb{R} &\longrightarrow \mathbb{R} \\ x < 0 &\longmapsto \exp\left(-\frac{1}{x}\right) \int_x^0 \exp\left(\frac{1}{u}\right) du \\ 0 &\longmapsto 0 \\ x > 0 &\longmapsto \exp\left(-\frac{1}{x}\right) \left(c + \int_x^1 \exp\left(\frac{1}{u}\right) du\right) . \end{aligned} \tag{1.8}$$

A standard calculus exercise consists in checking for the smoothness of \mathfrak{s}_0 . Yet none of these functions can be analytic, as if one were it would possess a convergent Taylor series $\hat{\mathfrak{s}}$ at 0 solving

$$x^2 \hat{\mathfrak{s}}'(x) = \hat{\mathfrak{s}}(x) - x^2 . \tag{1.9}$$

A straightforward computation yields the unique formal power series

$$\hat{\mathfrak{s}}(x) = x \sum_{n \geq 0} n! x^{n+1} ,$$

which has null radius of convergence. We say in that case that we encounter a **divergent weak separatrix**. It is worth mentioning that the Taylor expansion of each \mathfrak{s}_0 at 0 is $\hat{\mathfrak{s}}$.

Here we cannot distinguish a preferred center manifold in the class of analytic objects at $(0, 0)$. Although the divergence of the weak separatrix can be explained computationally for the Euler family, the generic perturbation $X_\bullet \in \text{Affine}(1)$ is impossible to deal with this way since no reasonable closed-form formulas for the coefficients of $\hat{\mathfrak{s}}$ exist in general. Even so the basic formal approach, computing coefficients of $\hat{\mathfrak{s}}$ one after the other, cannot prove nor disprove the power series convergence in finite time. We propose a dynamical approach instead to trace back the source of the divergence (Theorem 1), which leads to the semi-decidability of the convergence of $\hat{\mathfrak{s}}$: there exists an algorithm taking a «computable» a_\bullet as input and stopping in finite time if, and only if, $\hat{\mathfrak{s}}$ diverges. The key is to check whether the complex contour integral

$$\varphi_0^n := \frac{1}{2i\pi} \oint_{r\mathbb{S}^1} a_0(z) \left(\frac{z+s}{z-s}\right)^{1/2s} dz \in \mathbb{C}$$

vanishes (meaning convergence), for $r > 0$ small enough. This viewpoint also allows us to find a complete collection of normal forms (Theorem 2).

When $\lambda < 0$ write \mathcal{S}_λ^- the (analytic) stable manifold of the saddle-point located at $(-s, 0)$ and, when it exists, \mathcal{S}_λ^+ the (analytic) unstable manifold of the node-point at $(s, 0)$. What happens in the Euler family is that no heteroclinic connection between stationary

points takes place: \mathcal{S}_λ^- does not coincide with \mathcal{S}_λ^+ . We aim to establish that this property has a predominant bearing on the convergence of the weak separatrix.

Theorem 1. *Consider a family $X_\bullet \in \text{Affine}(1)$ as in (1.5). The implications (1) \Rightarrow (2) \Rightarrow (3) hold, and if moreover $\frac{\partial a_\lambda}{\partial \lambda} = 0$ then (3) \Rightarrow (1).*

1. The vector field X_λ has a heteroclinic connection for all $\lambda < 0$ sufficiently close to 0.
2. The vector field X_λ has a heteroclinic connection for values of $\lambda < 0$ accumulating on 0.
3. The vector field X_0 admits a convergent weak separatrix (that is, an analytic center manifold).

Remark 1.

1. In each item of the theorem the corresponding property is equivalent to the existence of an open interval $I \ni 0$ such that the differential equation

$$(x^2 + \lambda) y' (x) = y(x) - (x^2 + \lambda) a_\lambda(x) \quad (1.10)$$

admits a solution analytic on I , for every corresponding values of λ . The solution is necessarily unique.

2. The practical usefulness of the theorem is by contraposition: if we happen to know that X_0 has a divergent weak separatrix then *any* unfolding $X_\bullet \in \text{Affine}(1)$ of X_0 eventually sheds all heteroclinic connections as λ goes to 0.

We prove this theorem for the Euler family E_\bullet in the next Section 1.1.1 for (2) \Rightarrow (3) and Section 1.1.2 for (3) \Rightarrow (1). After that step there are two ways to process the general case. On the one hand the proof performed in Euler's case could be adapted straightforwardly to fit the more general setting. On the other hand we can provide a collection of normal forms X_\bullet^κ for $\text{Affine}(1)$ on which the validity of the equivalences are easily read. This approach brings also the benefit of characterizing completely situations for which (3) \Rightarrow (1) holds.

Theorem 2. *Consider a family $X_\bullet \in \text{Affine}(1)$ as in (1.5).*

1. There exists a unique

$$\kappa \in \overline{\mathbb{N}} := \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

such that X_\bullet is conjugate to one of the models X_\bullet^κ

$$X_\lambda^\kappa(x, y) := \begin{bmatrix} x^2 + \lambda \\ y - \lambda^\kappa (x^2 + \lambda) \end{bmatrix}$$

where we conventionally identify λ^∞ and 0. This conjugacy can be chosen fibred in the variables x and λ . Moreover families X_\bullet^κ are mutually orbitally non-equivalent for different values of κ .

2. The implication (3) \Rightarrow (1) in Theorem 1 holds if, and only if, $\kappa \in \{0, \infty\}$. Notice that the condition $\frac{\partial a_\lambda}{\partial \lambda} = 0$ implies $\kappa \in \{0, \infty\}$.

This theorem, proved in Section 1.1.3 below, discriminates all three possible qualitative dynamical behaviors occurring in $\text{Affine}(1)$.

- $\kappa = 0$ **Pure divergence.** For every $\lambda \neq 0$ sufficiently close to 0 the vector field X_λ has no heteroclinic connection while X_0 has a divergent weak separatrix.
- $\kappa \in \mathbb{N}_{>0}$ **Sly convergence.** For every $\lambda \neq 0$ sufficiently close to 0 the vector field X_λ has no heteroclinic connection although X_0 has a convergent weak separatrix.
- $\kappa = \infty$ **Pure convergence.** For every $\lambda \neq 0$ sufficiently close to 0 the vector field X_λ has a heteroclinic connection so that X_0 has a convergent weak separatrix.

Here the modulus space $\overline{\mathbb{N}}$ for analytical orbital classification is discrete. The property no longer persists for families unfolding a more degenerate saddle-node, *i.e.* the coalescence of $k + 1$ stationary points with $k > 1$. We refer to [19] for this more involved situation.

1.1.1 From heteroclinic connections to convergence

We must exclude values of the parameter λ for which there are no analytic unstable manifold through the node of the Euler vector field E_λ . Although this phenomenon is not generic, it still turns up for an infinite discrete set of parameters. The stable manifold is always unique, given by the graph of

$$\begin{aligned} \mathfrak{s}_\lambda^- :] - s, s[&\longrightarrow \mathbb{R} \\ x &\longmapsto \left(\frac{s-x}{s+x} \right)^{1/2s} \int_x^{-s} \left(\frac{s+u}{s-u} \right)^{1/2s} du . \end{aligned} \quad (1.11)$$

Proposition 1.

1. E_λ admits a (unique) analytic unstable manifold \mathcal{S}_λ^+ if, and only if,

$$\lambda \in \hat{\Lambda} := \mathbb{R}_{<0} \setminus \frac{-1}{4\mathbb{N}^2} .$$

2. There exists a unique function $\mathfrak{c} : \sqrt{-\hat{\Lambda}} \rightarrow \mathbb{R}$ such that for all $\lambda \in \hat{\Lambda}$ the manifold \mathcal{S}_λ^+ coincides with the graph of

$$\begin{aligned} \mathfrak{s}_\lambda^+ :] - s, s[&\longrightarrow \mathbb{R} \\ x &\longmapsto \left(\frac{s-x}{s+x} \right)^{1/2s} \left(\mathfrak{c}(s) + \int_x^0 \left(\frac{s+u}{s-u} \right)^{1/2s} du \right) . \end{aligned}$$

3. \mathfrak{c} is analytic.

Proof. First notice that whatever the value of $\mathfrak{c}(s)$ may be, the graph of \mathfrak{s}_λ^+ is an integral curve of E_λ even when $\lambda \notin \hat{\Lambda}$. Swapping the order of summation and integration operations in the expansion

$$(u+s)^{1/2s} =: \sum_{n=0}^{\infty} \alpha_n^+(s) (u-s)^n ,$$

which converges uniformly on compact subsets of $] -s, 3s[$, we isolate the candidate singular term of \mathfrak{s}_λ^+ at s :

$$\mathfrak{s}_\lambda^+(x) = \left(\frac{s-x}{s+x} \right)^{1/2s} \left(\mathfrak{c}(s) - \sum_{n+1 \neq \frac{1}{2s}} \alpha_n^+(s) (-1)^n \frac{s^{n+1-1/2s}}{n+1-\frac{1}{2s}} - \alpha_*(s) \ln \frac{s-x}{s} \right) + (\text{analytic at } s)$$

where

$$\alpha_*(s) := \begin{cases} 0 & \text{if } \frac{1}{2s} \notin \mathbb{N} \\ 1 & \text{otherwise} \end{cases} .$$

If $\frac{1}{2s} \in \mathbb{N}$ no choice of $\mathfrak{c}(s) \in \mathbb{C}$ may yield an analytic \mathfrak{s}_λ^+ . On the contrary for $\lambda \in \hat{\Lambda}$ we can only have

$$\mathfrak{c}(s) = \sum_{n=0}^{\infty} \alpha_n^+(s) (-1)^n \frac{s^{n+1-1/2s}}{n+1-\frac{1}{2s}} ,$$

which is an analytic function of s . □

A consequence of the proposition is the following: if $\frac{1}{2s} \in \mathbb{N}$ there are no heteroclinic connection, while a heteroclinic connection for $\lambda \in \hat{\Lambda}$ occurs exactly if

$$\mathfrak{s}_\lambda^-(0) = \mathfrak{s}_\lambda^+(0) ,$$

that is if

$$\varphi(s) := \mathfrak{c}(s) + \int_{-s}^0 \left(\frac{s+u}{s-u} \right)^{1/2s} du$$

vanishes.

Corollary 1. *If φ vanishes on a set accumulating on 0 then $\hat{\mathfrak{s}}$ converges.*

The proof requires our switching to complex analysis in order to use compactness of normal families of holomorphic functions. The main ingredient is therefore to show that $(\mathfrak{s}_\lambda^-)_{-1 < \lambda < 0}$ extends to a uniformly bounded family of analytic functions on the slit unit disc

$$\{z \in \mathbb{C} \setminus [s, \infty[: |z| < 1\} .$$

We need to slit the disc because the complex (multivalued) extension of \mathfrak{s}_λ^- is given by taking path integrals in the variation of constant method

$$\mathfrak{s}_\lambda^-(z) = \left(\frac{s-z}{s+z} \right)^{1/2s} \int_{\gamma(z)} \left(\frac{s+u}{s-u} \right)^{1/2s} du \quad (1.12)$$

where γ is a piecewise smooth path linking $z \neq s$ to $-s$. We choose the determination of the logarithm in such a way that $\mathfrak{s}_\lambda^-(z)$ coincides with (1.11) on $] -s, s[$.

Remark 2. We will discuss the relevance of the multivaluedness of \mathfrak{s}_λ^- regarding the question of convergence of $\hat{\mathfrak{s}}$ in the next section, when proving the converse of the corollary.

Lemma 1. *There exists $C > 0$ such that for every $(\lambda, z) \in]-1, 0[\times (\mathbb{S}^1 \setminus \{1\})$ we have*

$$|\mathfrak{s}_\lambda^-(z)| \leq C .$$

Proof. Let us build an adequate integration path $\gamma(z)$ for which bounds are easily obtained.

- ⊗ When $\Im(z) < 0$ we first follow the shortest anticlockwise arc $\gamma^-(z)$ of \mathbb{S}^1 joining z to -1 , then the interval

$$I_\lambda := [-1, -s] .$$

- ⊗ Otherwise we follow the shortest clockwise arc $\gamma^+(z)$ of \mathbb{S}^1 joining z to -1 before I_λ .

For $u \in I_\lambda$ we have $0 < \frac{u+s}{u-s} < 1$ so that

$$\int_{-1}^{-s} \left(\frac{u+s}{u-s} \right)^{1/2s} du \leq (1-s) .$$

Moreover there exists $C_1 \geq 0$ for which

$$|\mathfrak{s}_\lambda^-(z)| \leq \left| \frac{s-z}{s+z} \right|^{1/2s} \left((1-s) + \int_{\gamma^\pm(z)} \left| \frac{s+u}{s-u} \right|^{1/2s} du \right) \leq C_1 (1-s) + \pi$$

because on the one hand $\left| \frac{s+u}{s-u} \right| \leq \left| \frac{s+z}{s-z} \right|$ when $z \in \mathbb{S}^1$ and $u \in \gamma^\pm(z)$, while on the other hand $\left| \frac{s+z}{s-z} \right| \leq \frac{1+s}{1-s}$ and $\lim_{s \rightarrow 0} \left(\frac{1+s}{1-s} \right)^{1/2s} = e$. □

We get on now with proving Corollary 1.

Proof. Each function \mathfrak{s}_λ^- , holomorphic on the slit unit disc, can be analytically extended to the whole \mathbb{D} precisely when it is analytic near s or, in other words, $\varphi(s) = 0$. Let $\Omega \subset \varphi^{-1}(0)$ be a set accumulating on 0. Because of Lemma (1) and of the maximum principle we know that \mathfrak{s}_λ^- is bounded on \mathbb{D} uniformly in $-1 < \lambda < 0$, *i.e.* the family $(\mathfrak{s}_\lambda^-)_{\lambda \in \Omega}$ is normal. Thus by Montel's theorem we can consider an adherence value (for uniform convergence on compact sets of \mathbb{D}) which must be a solution to Euler's equation (1.9) with analytic Taylor expansion at 0. But there is only one such formal power series solving Euler's equation, namely $\hat{\mathfrak{s}}$. □

Remark 3. We can give a series representation for φ using the expansions

$$(s \pm u)^{\pm 1/2s} =: \sum_{n=0}^{\infty} \alpha_n^\pm(s) (u \mp s)^n$$

where the determination of the logarithm on the left hand side is chosen in such a way that the function is real on $] -s, s[$. In particular

$$\alpha_n^+(-s) = (-1)^{1/2s} \alpha_n^-(s) = \alpha_n^-(s) \exp \frac{i\pi}{2s} .$$

In that setting

$$\mathbf{c}(s) := \sum_{n=0}^{\infty} \frac{\alpha_n^+(s)}{n+1-1/2s} (-1)^n s^{n+1-1/2s}$$

and it is easy to compute

$$\int_{-s}^0 \left(\frac{s+u}{s-u} \right)^{1/2s} du = \sum_{n=0}^{\infty} \frac{\alpha_n^-(s)}{n+1+1/2s} (-1)^n s^{n+1+1/2s} = -\mathbf{c}(-s) ,$$

therefore

$$\varphi(s) = \mathbf{c}(s) - \mathbf{c}(-s) . \quad (1.13)$$

1.1.2 From convergence to heteroclinic connections

We just observed that if \mathfrak{s}_λ^- is uniform (that is, not multivalued) then $\hat{\mathfrak{s}}$ converges. We want to establish the converse statement in the following way. When $\hat{\mathfrak{s}}$ converges it defines a real, entire holomorphic function, in particular for given $x_* > s$

$$(x_*, y_*) := (x_*, \hat{\mathfrak{s}}(x_*))$$

is a well-defined point in \mathbb{R}^2 . Consider the solution y_λ of (1.10) with initial value (x_*, y_*) and its local analytic continuation over

$$\overline{\mathbb{D}}_\lambda := \overline{\mathbb{D}} \setminus \{\pm s\} .$$

We are more particularly interested in the analytic continuation of y_λ along the unit circle, which can be performed in the universal cover of $\overline{\mathbb{D}} \setminus [-s, s]$ as we explain below. We identify the action of the deck transform of this covering with the symbolic multiplication of z by $\exp 2i\pi$, so that the analytic continuation of y_λ along \mathbb{S}^1 can be conveniently written $y_\lambda(x_* \exp 2i\pi)$. Because $(y_\lambda)_\lambda$ converges uniformly on compact subsets of $\rho^{-1}(\mathbb{S}^1)$ as $\lambda \xrightarrow{<} 0$, if $\hat{\mathfrak{s}}$ converges then we must have

$$\lim_{\lambda \xrightarrow{<} 0} y_\lambda(x_* \exp 2i\pi) = y_* .$$

Let us see how this observation relates to the presence of heteroclinic connections.

Proposition 2. *For every $\lambda \in \hat{\Lambda}$ and $(x_*, y) \in \mathbb{R}_{>s} \times \mathbb{R}$ write $z \mapsto y_\lambda(z, y)$ the solution of (1.10) with initial value (x_*, y) . The local analytic continuation of $y_\lambda(\bullet, y)$ over \mathbb{S}^1 follows the rule*

$$y_\lambda(x_* \exp 2i\pi, y) = y - 2i \left(\frac{x_* - s}{x_* + s} \right)^{1/2s} \varphi(s) \sin \frac{\pi}{2s} \varphi.$$

In particular

$$\lim_{\lambda \underset{<}{\rightarrow} 0} y_\lambda(x_* \exp 2i\pi) = y_*$$

if, and only if,

$$\lim_{s \rightarrow 0} \varphi(s) \sin \frac{\pi}{2s} = 0.$$

Remark 4. Notice that when $y \in \mathbb{R}$ and $\lambda \in \hat{\Lambda}$ the continued value $y_\lambda(x_* \exp 2i\pi, y)$ is never real since $\left(\frac{x_* - s}{x_* + s} \right)^{1/2s} \varphi(s) \sin \frac{\pi}{2s} \in \mathbb{R}$.

In order to establish the proposition we need to understand the monodromy of

$$\hat{\mathbf{g}}_s : z \neq \pm s \mapsto \left(\frac{s+z}{s-z} \right)^{1/2s}.$$

We fix a determination $\hat{\mathbf{g}}_s^*$ of $\hat{\mathbf{g}}_s$ on $(\overline{\mathbb{D}} \setminus \mathbb{R}) \cup]-s, s[$ in such a way that $\hat{\mathbf{g}}_s^*|_{]-s, s[}$ coincides with the canonical real determination used previously. For any path γ , starting from 0 with image included in $\overline{\mathbb{D}}_\lambda$, we define

$$\hat{\mathbf{g}}_s(\gamma) := \hat{\mathbf{g}}_s^*(0) \exp \int_\gamma \frac{du}{u^2 + \lambda}.$$

Fix a system γ^\pm of generators of $\pi_1(\overline{\mathbb{D}}_\lambda, 0)$ whose index around $\pm s$ is 1 and 0 around the other point. The monodromy of $\hat{\mathbf{g}}_s$ is multiplicative and given by

$$\hat{\mathbf{g}}_s(\gamma^\pm \wedge \gamma) = \hat{\mathbf{g}}_s(\gamma) \exp \oint_{\gamma^\pm} \frac{du}{u^2 + \lambda} = \hat{\mathbf{g}}_s(\gamma) \exp \frac{\pm i\pi}{s},$$

according to the residue formula and the identity

$$\frac{-1}{z^2 + \lambda} = \frac{1}{2s} \left(\frac{1}{s+z} + \frac{1}{s-z} \right).$$

In particular

$$\hat{\mathbf{g}}_s(\gamma^- \wedge \gamma^+ \wedge \gamma) = \mathbf{g}_s(\gamma)$$

so that $\hat{\mathbf{g}}_s^*$ is actually holomorphically extendable to $\overline{\mathbb{D}} \setminus [-s, s]$, as claimed. We prove now the proposition.

Proof. The difference

$$y_\lambda(x_* \exp 2i\pi, y) - y = -\frac{1}{\hat{\mathbf{g}}_s^*(x_*)} \oint_{\mathbb{S}^1} \hat{\mathbf{g}}_s^*(u) du$$

can be computed by deforming \mathbb{S}^1 into the concatenation $\gamma^- \wedge \gamma^+$ of generators of $\pi_1(\overline{\mathbb{D}}_\lambda, 0)$ given by

$$\begin{aligned}\gamma^- &: t \in [0, 1] \mapsto -s + s \exp(2i\pi t) \\ \gamma^+ &: t \in [0, 1] \mapsto s - s \exp(2i\pi(t-1))\end{aligned}$$

using the relation

$$\oint_{\mathbb{S}^1} \hat{\mathbf{g}}_s^*(u) du = \int_{\gamma^-} \hat{\mathbf{g}}_s(u) du + \int_{\gamma^+} \hat{\mathbf{g}}_s(u) du$$

(notice that we do not use the symbol \oint for the paths γ^\pm because this integration really happens in the universal cover of $\overline{\mathbb{D}}_\lambda$ and the lift of γ^\pm is not a loop). Using the notations and formulas presented in Remark 3 we compute

$$\begin{aligned}\int_{\gamma^-} \hat{\mathbf{g}}_s(u) du &= \sum_{n=0}^{\infty} \frac{\alpha_n^-(s)}{n+1+1/2s} \left[z^{n+1+1/2s} \right]_s^{s \exp 2i\pi} \\ &= \left(1 - \exp \frac{i\pi}{s} \right) \mathbf{c}(-s)\end{aligned}$$

with $z := u + s$, then

$$\begin{aligned}\int_{\gamma^+} \hat{\mathbf{g}}_s(u) du &= \sum_{n=0}^{\infty} \frac{\alpha_n^+(s) (-1)^{n+1}}{n+1-1/2s} \left[z^{n+1-1/2s} \right]_s^{s \exp(-2i\pi)} \\ &= \left(\exp \frac{i\pi}{s} - 1 \right) \mathbf{c}(s)\end{aligned}$$

with $z := s - u$. The conclusion follows from

$$\frac{1}{\hat{\mathbf{g}}_s^*(x_*)} = \left(\frac{x_* - s}{x_* + s} \right)^{1/2s} \exp \frac{-i\pi}{2s}$$

and from (1.13). \square

We end the story by an explicit computation which settles the question of the divergence of $\hat{\mathbf{s}}$.

Lemma 2. *For every $\lambda \in \hat{\Lambda}$ we have*

$$\varphi(s) \sin \frac{\pi}{2s} = \pi .$$

Proof. We just proved

$$\begin{aligned}\varphi(s) \sin \frac{\pi}{2s} &= \pi \times \exp \frac{-i\pi}{2s} \times \frac{1}{2i\pi} \oint_{\mathbb{S}^1} \hat{\mathbf{g}}_s^*(u) du \\ &= \pi \times \frac{1}{2i\pi} \oint_{\mathbb{S}^1} \left(\frac{u+s}{u-s} \right)^{1/2s} du .\end{aligned}$$

The latter integral can be evaluated using the residue formula at ∞ since $z \mapsto \left(\frac{z+s}{z-s}\right)^{1/2s}$ is holomorphic at this point. Setting $w := \frac{1}{u}$ we compute

$$\left(\frac{z+s}{z-s}\right)^{1/2s} = \left(\frac{1+ws}{1-ws}\right)^{1/2s} = \exp(w + o(w))$$

and

$$\frac{1}{2i\pi} \oint_{\mathbb{S}^1} \left(\frac{u+s}{u-s}\right)^{1/2s} du = \frac{1}{2i\pi} \oint_{\mathbb{S}^1} \exp(w + o(w)) \frac{dw}{w^2} = 1 .$$

1.1.3 Normal forms

We just established the equivalence in the Euler family between

- ⊗ divergence of $\hat{\mathfrak{s}}$,
- ⊗ absence of heteroclinic connections (non-vanishing of φ),
- ⊗ non-vanishing of the integral

$$\varphi_s^n := \frac{1}{2i\pi} \oint_{\mathbb{S}^1} \left(\frac{u+s}{u-s}\right)^{1/2s} du = \frac{1}{\pi} \varphi(s) \sin \frac{\pi}{2s} = 1 .$$

In order to establish the classification Theorem 2 we need to find an (almost) invariant quantity under changes of coordinates. This invariant turns out to be φ_\bullet^n . One can argue that it suffices to consider φ instead, which is somehow nicer because of its dynamical flavor. Yet φ is afflicted of serious drawbacks:

- ⊗ φ presents an accumulation of poles as $s \xrightarrow{>} 0$, and there is no hope of extending it analytically at 0,
- ⊗ φ is odd, and there is no hope of extending it holomorphically on an annulus surrounding 0 as a function of λ .

None of these shortcomings hinder φ_\bullet^n , even in the more general setting of affine unfoldings.

Proposition 3. For $X_\bullet \in \text{Affine}(1)$ as in (1.5) we may find $\rho > 0$ such that $(\lambda, x) \mapsto a_\lambda(x)$ is holomorphic on $\rho^2\mathbb{D} \times \rho\overline{\mathbb{D}}$. For $s \in \rho\mathbb{D} \setminus \{0\}$ define

$$\varphi_s^n := \frac{1}{2i\pi} \oint_{\rho\mathbb{S}^1} a_{-s^2}(u) \left(\frac{u+s}{u-s}\right)^{1/2s} du . \quad (1.14)$$

1. The holomorphic mapping $s \mapsto \varphi_s^n$ can be continued to an even germ of a holomorphic function at 0 satisfying

$$\varphi_0^n = \frac{1}{2i\pi} \oint_{\mathbb{S}^1} a_0(u) \exp \frac{1}{u} du .$$

2. Write $a_\lambda(x) = \sum_{n=0}^{\infty} \phi_n(\lambda) x^n$. Then for $s \in \rho\mathbb{D} \setminus \{0\}$

$$\varphi_s^n = \sum_{n=0}^{\infty} \frac{\phi_n(-s^2)}{(n+1)!} \times \frac{1}{2^n} \sum_{p+q=n} \binom{n}{p} \prod_{j=1}^p (1+2sj) \prod_{j=1}^q (1-2sj) ,$$

with limit

$$\varphi_0^n = \sum_{n=0}^{\infty} \frac{\phi_n(0)}{(n+1)!} .$$

3. The formal solution \hat{s} with $\hat{s}(0) = 0$ of

$$x^2 y'(x) = y(x) - x^2 a_0(x)$$

converges if, and only if, $\varphi_0^n = 0$.

Remark 5. We deduce the determination of $\mathfrak{g}_s := \left(\begin{smallmatrix} \bullet+s \\ \bullet-s \end{smallmatrix} \right)^{1/2s}$ from that of the function $\hat{\mathfrak{g}}_s$ built in Section 1.1.2 by setting

$$\mathfrak{g}_s := \hat{\mathfrak{g}}_s \exp \frac{-i\pi}{2s} .$$

The multiplicative monodromy of \mathfrak{g}_s is the same as that of $\hat{\mathfrak{g}}_s$.

Proof.

1. Although it is a consequence of 2. we can prove directly the property. First notice that $\varphi_{-s}^n = \varphi_s^n$. Also $z \mapsto a_{-s^2}(z) \mathfrak{g}_s(z)$ converges uniformly to $z \mapsto a_0(z) \exp \frac{1}{z}$ on $\rho\mathbb{S}^1$ as $s \rightarrow 0$, so that $s \mapsto \varphi_s^n$ is bounded on a pointed neighborhood of 0. Riemann's removable singularity theorem yields the conclusion. This is a trick used broadly in this text.
2. For $n \in \mathbb{Z}_{\geq 0}$ let us evaluate

$$t_s(n) := \frac{1}{2i\pi} \oint_{\mathbb{S}^1} u^n \mathfrak{g}_s(u) du .$$

The residue formula used in Lemma 2 to compute $t_s(0)$ sure works here, yet one would have to formally derive a closed-form for the Taylor coefficients of $z \mapsto \mathfrak{g}_s(z)$ at ∞ , which is no trivial task. We relate instead the computation at hands to the Beta function, more precisely its integral representation along a Pochhammer contour around 0 and 1. Introduce first the contour around $-s$ and s

$$\mathcal{P} := \gamma^+ \wedge \gamma^- \wedge (-\gamma^+) \wedge (-\gamma^-) \tag{1.15}$$

where γ^\pm are generators of $\pi_1(\overline{\mathbb{D}} \setminus \{\pm s\}, 0)$ as described in Section 1.1.2. The identity

$$\oint_{\mathcal{P}} u^n \mathfrak{g}_s(u) du = \left(\exp \frac{-i\pi}{s} - 1 \right) \oint_{\mathbb{S}^1} u^n \mathfrak{g}_s(u) du$$

holds because the value of \mathfrak{g}_s above $\gamma^+ \wedge \gamma^-$ is multiplied by $\exp \frac{-i\pi}{s}$ as compared to that above $(-\gamma^+) \wedge (-\gamma^-)$.

We invoke now the standard formula

$$(1 - \exp 2ia\pi)(1 - \exp 2ib\pi) B(a, b) = \oint_{\hat{\mathcal{P}}} z^{a-1} (1-z)^{b-1} dz, \quad (1.16)$$

where $\hat{\mathcal{P}}$ is a Pochhammer contour around 0 and 1. We can take for $\hat{\mathcal{P}}$ the image of \mathcal{P} under the change of variable

$$z := \frac{1}{2s} (s - u)$$

which transforms $u - s$ into $-2sz$ (maps s on 0) and $u + s$ into $2s(1 - z)$ (maps $-s$ on 1). It is therefore relevant to work with the expansion

$$u^n = \frac{1}{2^n} \sum_{p+q=n} \binom{n}{p} (u+s)^p (u-s)^q.$$

From (1.16) we compute, for $p + q = n$ non-negative integers,

$$\begin{aligned} t_{p,q} &:= \oint_{\mathbb{S}^1} (u+s)^{p+1/2s} (u-s)^{q-1/2s} du \\ &= \frac{1}{\exp \frac{-i\pi}{s} - 1} \oint_{\mathcal{P}} (u+s)^{p+1/2s} (u-s)^{q-1/2s} du \\ &= \frac{(2s)^{n+1}}{1 - \exp \frac{-i\pi}{s}} \exp \frac{-i\pi}{2s} \oint_{\hat{\mathcal{P}}} (1-z)^{p+1/2s} z^{q-1/2s} dz \\ &= \frac{(2s)^{n+1}}{1 - \exp \frac{-i\pi}{s}} \exp \frac{-i\pi}{2s} \left(1 - \exp \frac{-i\pi}{s}\right) \left(1 - \exp \frac{i\pi}{s}\right) B\left(1+q - \frac{1}{2s}, 1+p + \frac{1}{2s}\right) \\ &= 2i (2s)^{n+1} \sin \frac{\pi}{2s} B\left(1+q - \frac{1}{2s}, 1+p + \frac{1}{2s}\right) \\ &= \frac{2i}{(n+1)!} (2s)^{n+1} \sin \frac{\pi}{2s} \Gamma\left(1+q - \frac{1}{2s}\right) \Gamma\left(1+p + \frac{1}{2s}\right). \end{aligned}$$

Since $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$ we deduce finally

$$t_{p,q} = \frac{2i\pi}{(n+1)!} \prod_{j=1}^p (1+2sj) \prod_{j=1}^q (1-2sj)$$

and

$$t_s(n) = \frac{1}{2^n (n+1)!} \sum_{p+q=n} \binom{n}{p} \prod_{j=1}^p (1+2sj) \prod_{j=1}^q (1-2sj).$$

Because φ_s^n is obtained by integrating a holomorphic 1-form on a compact loop we can swap the order of summation operators:

$$\oint_{\mathbb{S}^1} a_{-s^2}(u) \mathbf{g}_s(u) du = \sum_{n=0}^{\infty} \phi_n(-s^2) t_s(n) .$$

3. After applying a convenient linear scaling of the x -coordinate we can assume that a_0 is holomorphic on $\overline{\mathbb{D}}$. For $z \in \overline{\mathbb{D}} \setminus [0, 1]$ consider a path $\gamma(z)$ joining 0 directly to -1 , then reaching z within the domain. The function

$$\mathfrak{s}_0^- : z \in \overline{\mathbb{D}} \setminus [0, 1] \mapsto \exp \frac{-1}{z} \int_{\gamma(z)} a_0(u) \exp \frac{1}{u} du$$

is well-defined and holomorphic on $\overline{\mathbb{D}} \setminus [0, 1]$. It is the only solution to the equation which tends to 0 at 0 over $\mathbb{R}_{<0}$. It must therefore coincide with $\hat{\mathfrak{s}}$ when one of the two objects represents a holomorphic function on $\overline{\mathbb{D}}$. The conclusion follows from the fact that φ_0^n embodies the monodromy of the multivalued continuation of \mathfrak{s}_0^- on $\overline{\mathbb{D}} \setminus \{0\}$. \square

Let us present now the classification theorem.

Theorem 3. *Take two families X_\bullet and \tilde{X}_\bullet of Affine (1). The following properties are equivalent.*

1. *There exists a germ of a holomorphic function $\lambda \mapsto c(\lambda)$ with $c(0) \neq 0$ such that for all s sufficiently close to 0*

$$\tilde{\varphi}_s^n = c(-s^2) \varphi_s^n .$$

2. *X_\bullet and \tilde{X}_\bullet are conjugate.*

Any conjugacy between the two families must fix λ , and in that case a change of coordinates Ψ_\bullet such that $\Psi_\bullet^ X_\bullet = \tilde{X}_\bullet$ exists in the form*

$$(\lambda, x, y) \mapsto (\lambda, x, yc(\lambda) + \phi_\lambda(x)) .$$

Proof.

1. \Rightarrow 2. We find a germ of a holomorphic function $(s, x) \mapsto \psi_s(x)$ such that

$$\Psi_{-s^2}(x, y) := (x, yc(-s^2) + \psi_s(x))$$

satisfies $\Psi_{-s^2}^* X_{-s^2} = \tilde{X}_{-s^2}$. We prove next that $s \mapsto \psi_s$ is even, so that there exists a holomorphic function $(\lambda, x) \mapsto \phi_\lambda(x)$ with $\phi_{-s^2} = \psi_s$. By definition we need to solve the equation

$$D\Psi_\lambda(X_\lambda) = \tilde{X}_\lambda \circ \Psi_\lambda$$

where $\lambda := -s^2$, that is

$$z^2 \psi'_s(z) = \psi_s(z) - \delta_\lambda(z) (z^2 + \lambda)$$

where

$$\delta_\lambda(z) := \tilde{a}_\lambda(z) - c(\lambda) a_\lambda(z) .$$

Without loss of generality we can assume that c is holomorphic on \mathbb{D} . Suppose first that $0 < s < 1$. The method of variation of the constant yields

$$\psi_s(z) = \frac{1}{\mathfrak{g}_s(z)} \int_z^{-s} \delta_\lambda(u) \mathfrak{g}_s(u) du ,$$

which is holomorphic on $\overline{\mathbb{D}} \setminus [s, 1]$. Because $\tilde{\varphi}_s^n = c(-s^2) \varphi_s^n$ the function ψ_s extends to a uniform (holomorphic) function on $\overline{\mathbb{D}} \setminus \{s\}$. As in Lemma 1 it is easy to prove that ψ_s is bounded on \mathbb{S}^1 (uniformly in s). Using the maximum modulus principle and Riemann's removable singularity theorem we deduce that ψ_s extends holomorphically to $\overline{\mathbb{D}}$. Montel's theorem ensures that $(\psi_s)_s$ converges uniformly on $\overline{\mathbb{D}}$ to some function ψ_0 for which $\Psi_0^* X_0 = \tilde{X}_\lambda$.

The above construction can be holomorphically continued for all $s \in \mathbb{D} \setminus \mathbb{R}$, in that case the graph of ψ_s coincides with the invariant manifold of the collection $\Delta_\bullet \in \text{Affine}(1)$,

$$\Delta_\lambda(x, y) := \left[\begin{array}{c} x^2 + \lambda \\ y - \delta_\lambda(x)(x^2 + \lambda) \end{array} \right] ,$$

passing through the hyperbolic point $(-s, 0)$ and transverse to the line $\{z = -s\}$. This manifold is unique, as other non-vertical trajectories of Δ_λ are multivalued. This property guarantees that a heteroclinic connection occurs in Δ_λ , otherwise ψ_s would not be uniform near $(s, 0)$. Therefore the local graph of ψ_s near $(s, 0) = (-(-s), 0)$ coincides with that of ψ_{-s} . From the analytic continuation principle we derive $\psi_{-s} = \psi_s$ for $s \in \mathbb{D} \setminus \mathbb{R}$, which allows to extend holomorphically $(s, x) \mapsto \psi_s(x)$ to $\mathbb{D} \times \mathbb{D}$ to an even function of s , as expected.

2. \Rightarrow 1. Take an orbital equivalence

$$\Psi : (\lambda, x, y) \mapsto (\phi(\lambda), \Psi_\lambda(x, y)) \in \text{Diff}(\mathbb{C}^3, 0)$$

between X_\bullet and \tilde{X}_\bullet . Assuming that Ψ is holomorphic on $\mathbb{D} \times \overline{\mathbb{D}} \times \mathbb{D}$ does not lessen the generality of our argument. We prove that $\phi = \text{Id}$. The key ingredient is the following classical fact.

Lemma 3. *Take $p \in \mathbb{C}^2$ a stationary point of a holomorphic vector field X , and consider the linear part of X at p , i.e. the linear mapping $DX(p)$. Let $L_p(X)$ denote the equivalence class of its spectrum under the equivalence*

$$\{\lambda_1, \lambda_2\} \rightsquigarrow \{\tilde{\lambda}_1, \tilde{\lambda}_2\} \iff (\exists c \in \mathbb{C}_{\neq 0}) : \{\lambda_1, \lambda_2\} = c \{\tilde{\lambda}_1, \tilde{\lambda}_2\} .$$

Then $L_p(X)$ is invariant under orbital equivalence.

In our situation for given λ the diffeomorphism Ψ_λ maps $p_\pm := (\pm\sqrt{-\lambda}, 0)$ to $\tilde{p}_\pm = (\pm(-1)^\ell \sqrt{-\phi(\lambda)}, 0)$ for some integer ℓ . Because the spectrum of the linearization of X_λ at p_\pm is $\{\pm 2\sqrt{-\lambda}, 1\}$ we must have

$$\begin{cases} \sqrt{-\lambda} &= \sqrt{-\phi(\lambda)}(-1)^\ell \\ \text{or} \\ 1 &= \sqrt{-\lambda}\sqrt{-\phi(\lambda)}(-1)^\ell \end{cases}.$$

The former identity yields $\lambda = \phi(\lambda)$ while the latter $\lambda\phi(\lambda) = 1$ cannot hold on a neighborhood of 0. Also Ψ_λ must fix each stationary point $(\pm s, 0)$. We will not prove that Ψ_λ can be taken to act identically on the x -variable, although it is the case (see *e.g.* [19, 20]). The other claims can be recovered by a formal computation. \square

Corollary 2. *For $X_\bullet \in \text{Affine}(1)$ there exists a unique*

$$\kappa \in \overline{\mathbb{N}} := \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

such that X_\bullet is conjugate to one of the models X_\bullet^κ

$$X_\lambda^\kappa(x, y) := \begin{bmatrix} x^2 + \lambda \\ y - \lambda^\kappa(x^2 + \lambda) \end{bmatrix}$$

where we conventionally identify λ^∞ and 0. Moreover families X_\bullet^κ are mutually orbitally non-equivalent for different values of κ .

Proof. There exists a unique κ such that

$$\varphi_s^n = \frac{s^{2\kappa}}{c(-s^2)}$$

for a germ c of a holomorphic function at 0 satisfying $c(0) \neq 0$. Observe that the invariant $\tilde{\varphi}_s^n$ associated to X_λ^κ equals $s^{2\kappa}$. Using Theorem 3 we obtain the first claim. The theorem also implies that if X_\bullet^κ is orbitally equivalent to $X_\bullet^{\tilde{\kappa}}$ then $\lambda^{\tilde{\kappa}} = \lambda^\kappa c(\lambda)$ for some holomorphic function c with $c(0) \neq 0$ and every λ close enough to 0. Therefore $\kappa = \tilde{\kappa}$. \square

1.2 Basic objects and notations

We chose to frame our study in a geometric setting, with its own standard terminology. We present below basic objects attached to singular holomorphic vector fields: directional derivative, flow, change of coordinates, and most importantly singular foliations, first integrals and normal forms. We recall related basic results of differential geometry. Readers familiar with these concepts should skim briefly through this section mainly to fix notations. The core of the exposition starts in Section 1.3.

1.2.1 Standard notations

In this paragraph n is a positive integer. All rings are commutative and unital.

- ⊗ We use the convention $\mathbb{N} = \{1, 2, \dots\}$. By putting expressions as index of \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} or \mathbb{C} we build subsets of the space satisfying said expressions, *e.g.* $\mathbb{R}_{<-1} =]\infty, -1[$ or $\mathbb{Z}_{\geq 0} = \{0\} \cup \mathbb{N}$.
- ⊗ The open unit disc of \mathbb{C} is written

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$$

and we denote by $\overline{\mathbb{D}} := \text{adh}(\mathbb{D})$ the closed unit disc. Also

$$\mathbb{S}^1 := \overline{\mathbb{D}} \setminus \mathbb{D} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$$

stands for the unit circle of the complex line.

- ⊗ A complex number $z \in \mathbb{C}$ has real part $\Re(z)$ and imaginary part $\Im(z)$.
- ⊗ The multiplicative group of invertible elements of a ring \mathcal{R} is written \mathcal{R}^\times .
- ⊗ The ring of polynomials in the variables $(z_j)_{1 \leq j \leq n}$ over a ring \mathcal{R} is written

$$\mathcal{R}[z_1, \dots, z_n],$$

while for $\star \in \{<, \leq, =, \geq, >\}$ and $d \in \mathbb{N}$ the notation $\mathcal{R}[z_1, \dots, z_n]_{\star d}$ stands for the set of such polynomials of homogeneous degree d satisfying $\delta \star d$.

- ⊗ The ring of all formal power series in the variables $(z_j)_{1 \leq j \leq n}$ over \mathcal{R} is written

$$\mathcal{R}[[z_1, \dots, z_n]].$$

- ⊗ For $p \in \mathbb{C}^n$ the notation

$$(\mathbb{C}^n, p)$$

should stand for the set of domains of \mathbb{C}^n containing p , but by a standard and convenient abuse of notations we actually write (\mathbb{C}^n, p) to mean some small enough such domain, much like the usage for Landau's $o(\bullet)$ and $O(\bullet)$ notations.

- ⊗ The algebra of holomorphic functions on an open set $\mathcal{U} \subset \mathbb{C}^n$ is written $\text{Holo}(\mathcal{U})$. We say that a function is holomorphic on $A \subset \mathbb{C}^n$ if it belongs to some $\text{Holo}(\mathcal{U})$ for $A \subset \mathcal{U}$. The algebra of all such germs of a function is denoted by $\text{Holo}(A)$.
- ⊗ In the special case $A = \{p\}$ we more conventionally refer to $\text{Holo}(\{p\})$ as

$$\text{Holo}(\mathbb{C}^n, p),$$

the algebra of germs at p of a holomorphic functions. The group $\text{Holo}(\mathbb{C}^n, p)^\times$ consists of all germs $U \in \text{Holo}(\mathbb{C}^n, p)$ such that $U(p) \neq 0$.

- ⊗ If moreover $p = 0$ we identify $\text{Holo}(\mathbb{C}^n, 0)$ with the sub-algebra

$$\mathbb{C}\{z_1, \dots, z_n\}$$

of $\mathbb{C}[[z_1, \dots, z_n]]$ consisting in formal power series which are absolutely convergent on a neighborhood of 0.

- ⊗ The $\text{Holo}(\mathcal{U})$ -module of all holomorphic vector fields on \mathcal{U} is written $\mathfrak{X}(\mathcal{U})$. The $\text{Holo}(\mathbb{C}^n, p)$ -module of all germs at p of a holomorphic vector field is written

$$\mathfrak{X}(\mathbb{C}^n, p) .$$

- ⊗ The set of biholomorphic mappings $\mathcal{U} \rightarrow \tilde{\mathcal{U}}$ from an open set $\mathcal{U} \subset \mathbb{C}^n$ onto another one $\tilde{\mathcal{U}}$ is written $\text{Diff}(\mathcal{U} \rightarrow \tilde{\mathcal{U}})$. As before this construction can be germified near any $A, \tilde{A} \subset \mathbb{C}^n$, yielding the set $\text{Diff}(A \rightarrow \tilde{A})$ whose elements Ψ belong to some $\text{Diff}(\mathcal{U} \rightarrow \tilde{\mathcal{U}})$ with $A \subset \mathcal{U}, \tilde{A} \subset \tilde{\mathcal{U}}$ and $\Psi(A) = \tilde{A}$.
- ⊗ In the special case $A = \{p\}$ and $\tilde{A} = \{\tilde{p}\}$ we conventionally write $\text{Diff}((\mathbb{C}^n, p) \rightarrow (\mathbb{C}^n, \tilde{p}))$ instead. If moreover $p = \tilde{p}$ we name

$$\text{Diff}(\mathbb{C}^n, p)$$

the (pseudo)group of germs of a diffeomorphism fixing p .

- ⊗ A tuple of power series $\Psi = (\Psi_j)_{1 \leq j \leq n} \in \mathbb{C}[[z_1, \dots, z_n]]^n$ is a **formal diffeomorphism** when $\Psi(0) = 0$ and Ψ is invertible for the composition of formal power series (that is, $D\Psi(0) \in \text{GL}_n(\mathbb{C})$). The group of all such formal diffeomorphisms is written

$$\widehat{\text{Diff}}(\mathbb{C}^n, 0) .$$

1.2.2 Lie derivative

Till the end of Section 1.2 we are given a vector field $Z \neq 0$ holomorphic on a domain $\mathcal{U} \subset \mathbb{C}^2$, which we understand as a section

$$\begin{aligned} \mathcal{U} &\longrightarrow \text{T}\mathcal{U} = \mathcal{U} \times \mathbb{C}^2 \\ p &\longmapsto (p, Z(p)) \end{aligned}$$

of the tangent bundle of \mathcal{U} . We write vector fields as derivations expressed in the canonical basis $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, say

$$Z = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y}$$

for two unique functions $A, B \in \text{Holo}(\mathcal{U})$ not both identically zero.

Example 1. The simplest saddle-node encountered in (1.1) can be written

$$X_0^\infty(x, y) = x^2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} .$$

The associated **Lie (directional) derivative** on functions f (or formal power series) is defined by

$$Z \cdot f := A \frac{\partial f}{\partial x} + B \frac{\partial f}{\partial y} .$$

Considering $Z(p)$ as an element of the tangent space of \mathcal{U} at p we have

$$Z \cdot f = Df(Z) .$$

The **Lie bracket** of two vector fields X and Y is the vector field whose action by derivation is

$$[X, Y] \cdot f := X \cdot Y \cdot f - Y \cdot X \cdot f .$$

We write for short $[X, Y] = X \cdot Y - Y \cdot X$, which makes sense component-wise and endows the space of vector fields with a Lie algebra structure. When $[X, Y] = 0$ we say that X and Y **commute**.

We define inductively for $m \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned} Z \cdot^0 f &:= f \\ Z \cdot^{m+1} &:= Z \cdot (Z \cdot^m f) . \end{aligned}$$

The action is extended component-wise to vectors or matrices of functions.

Any holomorphic function $H \in \text{Holo}(\mathcal{U})$ such that

$$Z \cdot H = 0$$

is called a **first integral** of Z .

Example 2. The function $H : (x, y) \mapsto y \exp \frac{1}{x}$ is a first integral of the saddle-node X_0^∞ on $\mathbb{C}^\times \times \mathbb{C}$.

1.2.3 Flow, integral curves and singularities

The **local flow** of Z at $p \in \mathcal{U}$ is the germ of a mapping

$$\begin{aligned} \Phi^\bullet : (\mathbb{C}^2, p) \times (\mathbb{C}, 0) &\longrightarrow \mathbb{C}^2 \\ (x, y, t) &\longmapsto \Phi(x, y) \end{aligned}$$

defined as the unique local solution to the flow-system of Z

$$\begin{aligned} \frac{d\Phi_Z^t(x, y)}{dt} &= Z \circ \Phi_Z^t(x, y) \\ \Phi_Z^0(x, y) &= (x, y) . \end{aligned}$$

The Lie formula gives a series expansion, normally convergent near $p \times \{0\}$, in the form

$$\Phi_Z^t = \sum_{m=0}^{\infty} \frac{t^m}{m!} Z \cdot^m \text{Id} , \tag{1.17}$$

where $\text{Id} : (x, y) \mapsto (x, y)$ is the identity of the complex plane. More generally for any $G \in \text{Holo}(\mathcal{U})$ we have (locally for all $t \in (\mathbb{C}, 0)$)

$$G \circ \Phi_Z^t = \sum_{m=0}^{\infty} \frac{t^m}{m!} Z \cdot^m G . \quad (1.18)$$

(In particular G is a first integral of Z if, and only if, G is constant along every integral curves of Z .) If $0 \in \mathcal{U}$ the formula also holds for any formal power series $G \in \mathbb{C}[[x, y]]$, the right-hand side belonging to $\mathbb{C}[[x, y, t]]$.

Example 3. We compute easily

$$\Phi_{X_0^\infty}^t(x, y) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (n!x^{n+1}, y) = \left(\frac{x}{1-tx}, y \exp t \right)$$

For fixed p we perform the maximal analytic continuation of $t \mapsto \Phi_Z^t(p)$ by patching in the appropriate fashion well-chosen solutions to nearby flow systems. The result is a curve parameterization $\Phi_Z^*(p) : S_p \rightarrow \mathcal{U}$ from a connected Riemann surface S_p onto the **integral curve** of Z passing through p . One encounters also the terminology «orbit of p under (the flow of) Z », which is not use as such here but helps explaining some terminology we employ below for changes of coordinates. The parameterization itself may be referred to as the **trajectory** of Z passing through p . It is the natural parameterization of the integral curve by the time of Z . Two vector fields Z and X on \mathcal{U} have same integral curves if, and only if,

$$Z = UX$$

for some $U \in \text{Holo}(\mathcal{U})^\times$.

Notice that according to (1.18) the following identity holds (locally for all $t \in (\mathbb{C}, 0)$)

$$Z \cdot \Phi_Z^t = Z \circ \Phi_Z^t .$$

A **singularity** (or stationary point) of Z is a point $p \in \mathcal{U}$ such that $Z(p) = 0$. The set of singular points of Z is written $\text{Sing}(Z)$. Outside $\text{Sing}(Z)$ we say that Z is **regular**. Singularities of Z are the only constant trajectories.

Example 4. The only singularity of X_0^∞ is located at $(0, 0)$. It is therefore isolated. Any other integral curve, distinct from $\{x = 0, y \neq 0\}$, coincides with a level curve of the first integral $H(x, y) = y \exp \frac{1}{x}$. For arbitrary $p = (x, y) \in \mathbb{C}^2$, the trajectory of the integral curve passing through p is defined on $S_p = \mathbb{C} \setminus \{\frac{1}{x}\}$.

1.2.4 Holomorphic foliations

We wish to describe the holomorphic **singular foliation** $\mathcal{F} = \mathcal{F}_Z$ associated to Z on \mathcal{U} . Roughly speaking it is the partition of \mathcal{U} into singular points and leaves, the latter corresponding to non-constant integral curves (without referring to a particular parameterization). There is a small catch, though, when Z is singular at p but the singularity is not isolated. In that case we can factor out a greatest common divisor in the components of Z , yielding a (local) decomposition $Z = UX$, where $U \in \text{Holo}(\mathbb{C}^2, p)$ vanishes at p

and $X \in \mathfrak{X}(\mathbb{C}^2, p)$ either is regular or has an isolated singularity at p . All such eventually isolated singularities $p \in \text{Sing}(X)$ form the **singular set** $\text{Sing}(\mathcal{F})$ of \mathcal{F} . By each point $p \notin \text{Sing}(\mathcal{F})$ passes a unique **leaf** \mathcal{L}_p of the foliation, which is the maximal connected smooth complex curve tangent to Z and containing p . It is obtained by gluing integral curves of corresponding local vector fields X .

Two foliations \mathcal{F}_Z and \mathcal{F}_X are identical if, and only if, there exists $V, W \in \text{Holo}(\mathcal{U}) \setminus \{0\}$ such that $VZ = WX$. If Z has only isolated singularities in \mathcal{U} then the conditions boils down to $Z = UX$ for some $U \in \text{Holo}(\mathcal{U})^\times$.

The **restriction** of \mathcal{F} to a subdomain $\mathcal{V} \subset \mathcal{U}$ is the foliation

$$\mathcal{F}|_{\mathcal{V}}$$

of \mathcal{V} , with singularities located at points of $\mathcal{V} \cap \text{Sing}(\mathcal{F})$ and whose leaves are the connected components of $\mathcal{V} \cap \mathcal{L}_p$ for each $p \in \mathcal{V}$.

Example 5. Take $\mathcal{U} := \mathbb{C}^2$ and $Z : (x, y) \mapsto yX_0^\infty(x, y)$. the vector field Z has the line $\{y = 0\}$ for singular set. Yet $\mathcal{F}_Z = \mathcal{F}_{X_0^\infty}$ has only one singularity at $(0, 0)$, all other leaf is either of the form $\{y = c \exp \frac{-1}{x}, x \neq 0\}$ for some $c \in \mathbb{C}$, or coincides with $\{x = 0, y \neq 0\}$.

1.2.5 Changes of coordinates

We define the action of $\text{Diff}(\tilde{\mathcal{U}} \rightarrow \mathcal{U})$ by change of coordinates on vector fields. On the source space \mathcal{U} of the vector field, $\Psi \in \text{Diff}(\tilde{\mathcal{U}} \rightarrow \mathcal{U})$ acts as a usual mapping by composition. The action on the range space $\text{T}\tilde{\mathcal{U}}$ is induced by the direct product $\Psi \oplus D\Psi$, sending $(p, \mathbf{v}) \in \tilde{\mathcal{U}} \times \mathbb{C}^2$ to $(\Psi(p), D\Psi(p)(\mathbf{v}))$. We write Ψ^*Z the element of $\mathfrak{X}(\tilde{\mathcal{U}})$ defined in such a way that the following diagram commute

$$\begin{array}{ccc} \tilde{\mathcal{U}} & \xrightarrow{\Psi} & \mathcal{U} \\ \downarrow \Psi^*Z & & \downarrow Z \\ \text{T}\tilde{\mathcal{U}} & \xrightarrow{\Psi \oplus D\Psi} & \text{T}\mathcal{U} \end{array}$$

that is

$$\Psi^*Z = (D\Psi)^{-1}(Z \circ \Psi) . \quad (1.19)$$

The vector field Ψ^*Z is called the **pullback** of Z by Ψ . In that situation trajectories of Ψ^*Z are mapped to trajectories of Z , leaving the natural time unchanged (locally for all $t \in (\mathbb{C}, 0)$):

$$\Psi \circ \Phi_{\Psi^*Z}^t = \Phi_Z^t \circ \Psi . \quad (1.20)$$

We say that $Z \in \mathfrak{X}(\mathcal{U})$ and $\tilde{Z} \in \mathfrak{X}(\tilde{\mathcal{U}})$ are **analytically conjugate** if there exists $\Psi \in \text{Diff}(\tilde{\mathcal{U}} \rightarrow \mathcal{U})$ such that $\tilde{Z} = \Psi^*Z$. This is equivalent to the **conjugacy equation**

$$\tilde{Z} \cdot \Psi = Z \circ \Psi \quad (1.21)$$

being satisfied.

We say that Z and \tilde{Z} are **analytically orbitally equivalent** when there exists $\tilde{U} \in \text{Holo}(\tilde{\mathcal{U}})^\times$ such that Z is conjugate to $\tilde{U}\tilde{Z}$. This means that Z is conjugate to a vector field with same integral curves as \tilde{Z} , in other words that integral curves of \tilde{Z} are mapped under Ψ onto integral curves of Z , yet the natural time changes in general.

Naturally all these notions can be germified. We then speak of **local conjugacy** and **local orbital equivalence**. If (1.21) holds at a formal level for some $\Psi \in \widehat{\text{Diff}}(\mathbb{C}^2, 0)$ then Z is **formally conjugate** to \tilde{Z} . If Z is formally conjugate to some $\tilde{U}\tilde{Z}$ with $\tilde{U} \in \mathbb{C}[[x, y]]^\times$ then Z is **formally orbitally equivalent** to \tilde{Z} .

In case Z is (analytically, locally) orbitally equivalent to \tilde{Z} , a bijection Ψ realizing the equivalence maps $\text{Sing}(\mathcal{F}_{\tilde{Z}})$ onto $\text{Sing}(\mathcal{F}_Z)$ and sends each leaf of $\mathcal{F}_{\tilde{Z}}$ onto a leaf of \mathcal{F}_Z . We say the foliations \mathcal{F}_Z and $\mathcal{F}_{\tilde{Z}}$ are **(analytically, locally) conjugate** and define

$$\Psi^*\mathcal{F}_Z := \mathcal{F}_{\Psi^*Z} .$$

We extend the terminology in the obvious way for formal diffeomorphisms, speaking of **formal conjugacy** between foliations.

Example 6. If one lets $\widehat{\Psi}$ be $(x, y) \mapsto (x, y - \widehat{\mathfrak{s}}(x))$, where $\widehat{\mathfrak{s}}$ is the formal solution to (1.9), then $E_0 = \widehat{\Psi}^*X_0^\infty$ since

$$\begin{aligned} X_0^\infty \circ \widehat{\Psi}(x, y) &= x^2 \frac{\partial}{\partial x} + (y - \widehat{\mathfrak{s}}(x)) \frac{\partial}{\partial y} \\ E_0 \cdot \widehat{\Psi}(x, y) &= x^2 \frac{\partial}{\partial x} + (y - x^2 \widehat{\mathfrak{s}}'(x) - x^2) \frac{\partial}{\partial y} . \end{aligned}$$

1.2.6 Flow-boxes and first integrals

Around a regular point $p \notin \text{Sing}(\mathcal{F})$ we can apply the rectification theorem to some regular $X \in \mathfrak{X}(\mathbb{C}^2, p)$ defining the foliation: there exists a local diffeomorphism $\Psi : (\mathbb{C}^2, p) \rightarrow (\mathbb{C}^2, p)$ such that $\Psi^*X = \frac{\partial}{\partial x}$. Hence the leaves of $\mathcal{F}|_{(\mathbb{C}^2, p)}$ are images of small «horizontal» discs included in $\{y = \text{cst}\}$. A pair (\mathcal{D}, Ψ) of a domain $\mathcal{D} = (\mathbb{C}^2, p)$ and a map $\Psi \in \text{Diff}(\mathcal{D} \rightarrow (\mathbb{C}^2, p))$ sending X to $\frac{\partial}{\partial x}$ is called a **rectifying chart** (or **flow-box**) for \mathcal{F} .

Level sets of

$$\begin{aligned} H &: \Psi(\mathcal{D}) \longrightarrow \mathbb{C} \\ \Psi(x, y) &\longmapsto y \end{aligned}$$

coincide with leaves of $\mathcal{F}|_{\Psi(\mathcal{D})}$. Because H is constant on integral curves of X it is a first integral of X . Analogically any holomorphic function $H \in \text{Holo}(\mathcal{V})$ on a subdomain $\mathcal{V} \subset \mathcal{U}$, which is a first integral of $Z|_{\mathcal{V}}$, is called a **first integral** of \mathcal{F} on \mathcal{V} . Level sets of H are saturated by $\mathcal{F}|_{\mathcal{V}}$. When a connected level set of a non-constant first integral H does not contain a singularity of \mathcal{F} then it coincides with a single leaf of $\mathcal{F}|_{\mathcal{V}}$. We say that H has **connected fibers** when it is non-constant and every level set is connected. When there exists a first integral with connected fibers H then the algebra of all first integrals $\text{FirstIntegral}(\mathcal{V})$ on \mathcal{V} is functionally generated by H :

$$\begin{aligned} \text{Holo}(H(\mathcal{V})) &\longrightarrow \text{FirstIntegral}(\mathcal{V}) \\ f &\longmapsto f \circ H \end{aligned}$$

is bijective: any first integral factors uniquely holomorphically through H .

Example 7. The function $H(x, y) := xy$ is a first integral of $X(x, y) := x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$. Its fibers are the connected Riemann surfaces $\{xy = c\}$. Notice that the two branches of $\{xy = 0\}$ are disconnected when the singularity $(0, 0)$ is removed from them.

The equation $X \cdot F = 0$ has formal solutions $F(x, y) = \sum_{n, m > 0} f_{n, m} x^n y^m$ satisfying $f_{n, m} = 0$ if $n \neq m$, while each $f_{n, n}$ is free to choose in \mathbb{C} . Therefore $F(x, y) = \sum_{n \geq 0} h_{n, n} (xy)^n = f(xy)$ where $f(t) := \sum_{n \geq 0} f_{n, n} t^n$.

1.2.7 Moduli spaces, normal forms

The local rectification theorem says that there is a single equivalence class for local conjugacy near a regular point. One important goal in the theory of vector fields is therefore to understand qualitative behaviors near singular points up to diverse conjugacy notions (and their orbital counterparts for foliations). This means to describe the quotients, called **moduli spaces**, of $\mathfrak{X}(\mathbb{C}^2, 0)$ under the action by conjugacy of $\text{Diff}(\mathbb{C}^2, 0)$ or $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ respectively, *i.e.* to perform the (local, formal) classification by identifying a complete set of objects invariant under conjugacy. We call such objects **(local, formal) invariants**.

An important invariant is the following. Take $p \in \text{Sing}(Z)$ and consider the linear part of Z at p , *i.e.* the linear mapping $DZ(p)$. Then its spectrum, written $\text{Spec}(Z, p)$ for the sake of simplicity, is invariant under formal conjugacy:

$$\forall \Psi \in \text{Diff}((\mathbb{C}^n, \tilde{p}) \rightarrow (\mathbb{C}^n, p)) \quad \text{Spec}(\Psi^* Z, \tilde{p}) = \text{Spec}(Z, p) .$$

Besides, let $\Lambda(Z, p)$ be the equivalence class of $\text{Spec}(Z, p)$ under

$$\{l_1, l_2\} \rightsquigarrow \{\tilde{l}_1, \tilde{l}_2\} \iff (\exists c \in \mathbb{C}^\times) : \{l_1, l_2\} = c \{\tilde{l}_1, \tilde{l}_2\} .$$

Then $\Lambda(Z, p)$ is invariant under formal orbital equivalences. The quotient space is naturally isomorphic to the double cover of $\mathbb{P}_1(\mathbb{C})$ ramified over $[1 : 1]$ and $[-1 : 1]$, obtained under the obvious $\mathbb{Z}/2\mathbb{Z}$ -action $[l_1 : l_2] \mapsto [l_2 : l_1]$ in homogeneous coordinates. The quotient is a smooth, compact Riemann surface of genus 0 parameterized by $[l_1 : l_2] \mapsto [l_1^2 + l_2^2 : l_1 l_2]$, therefore itself a conformal projective line $\mathbb{P}_1(\mathbb{C})$.

Generically this is the only invariant, as if $l_1 l_2 \neq 0$ and $l_1/l_2 \notin \mathbb{R}$ then the vector field is **hyperbolic** and Poincaré's theorem guarantees that Z is locally **linearizable**: there exists $\Psi \in \text{Diff}(\mathbb{C}^2, 0)$ such that $\Psi^*Z = l_1 x \frac{\partial}{\partial x} + l_2 y \frac{\partial}{\partial y}$.

To be altogether correct, we need to mention that the group $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ does not really act on $\mathfrak{X}(\mathbb{C}^2, 0)$. If $\Psi \in \widehat{\text{Diff}}(\mathbb{C}^2, 0)$ there is no reason why Ψ^*Z should be a holomorphic vector field, even though (1.19) defines a perfectly valid vector field with formal power series components. Yet being formally conjugate defines an equivalence relation, and we write resulting quotients as if they were quotients of a group action, for convenience sake.

The complete invariants we seek should differ in nature from simply stating «the equivalence class in the quotient». We particularly wish to build non-trivial bijective mappings between the various flavors of moduli spaces and some functional spaces. Classifying vector fields is out of reach in such a general form, although it can be carried out for smaller classes of vector fields. We take $\mathfrak{F} \subset \mathfrak{X}(\mathbb{C}^2, 0)$ and write respectively

$$\begin{aligned} \text{Mod}_{\text{loc}}(\mathfrak{F}) &:= \mathfrak{F}/\text{Diff}(\mathbb{C}^2, 0) \\ \text{Mod}_{\text{for}}(\mathfrak{F}) &:= \mathfrak{F}/\widehat{\text{Diff}}(\mathbb{C}^2, 0) \end{aligned}$$

the corresponding moduli spaces. Since formal conjugacy is weaker than local conjugacy there is a canonical map $\text{Mod}_{\text{loc}}(\mathfrak{F}) \rightarrow \text{Mod}_{\text{for}}(\mathfrak{F})$, and this is why in practice we fix a formal equivalence class and perform the local classification within. The notation

$$[Z]_{\star} \in \text{Mod}_{\star}(\mathfrak{F}) \quad , \quad \star \in \{\text{for}, \text{loc}\}$$

stands for the equivalence class of $Z \in \mathfrak{F}$ with respect to corresponding class of conjugacy.

- Let $\text{Mod}_{\star}(\mathfrak{F}) = \mathfrak{F}/G$ stand for one of the above quotients and let Ω be a set. We call
- ⊗ an injective mapping $C : \text{Mod}_{\star}(\mathfrak{F}) \rightarrow \Omega$ a **classification** of $\text{Mod}_{\star}(\mathfrak{F})$ (it is **complete** when surjective),
 - ⊗ a surjective mapping $R : \Omega \rightarrow \text{Mod}_{\star}(\mathfrak{F})$ a **realization** of $\text{Mod}_{\star}(\mathfrak{F})$.

The best way to realize a moduli space $\text{Mod}_{\star}(\mathfrak{F})$ in a concrete form is to find a collection of **(local, formal) normal forms** $\text{NF}_{\star}(\mathfrak{F}) \subset \mathfrak{F}$ satisfying the first two following properties:

- Versality** The natural map $\text{NF}_{\star}(\mathfrak{F})/G \rightarrow \mathfrak{F}/G$ is bijective
- Uniqueness** There exists $\nu \in \mathbb{N}$ and a smooth \mathbb{C}^{ν} -action on $\text{NF}_{\star}(\mathfrak{F})$ such that for any $Z \in \text{NF}_{\star}(\mathfrak{F})$ the whole equivalence class $[Z] \in \text{Mod}_{\star}(\mathfrak{F})$ is included in a single orbit.
- Simplicity** Although being primarily opinion-based, it is generally expected that elements of $\text{NF}_{\star}(\mathfrak{F})$ have «simple» expressions in some «natural» basis of the tangent bundle.

Remark 6. The clause of uniqueness states that a normal form $Z \in \text{NF}_{\star}(\mathfrak{F})$ is unique «up to a finite-dimensional space». A notion of smoothness on spaces of germs (endowed with a convenient locally convex topology) adapted to this context can be found for instance in [21]. Once these normal forms are given it is in general straightforward to refine the study and pinpoint unique representatives for a given equivalence class. This work can

be messy, though in practice seldom reaching further than linear algebra. In this text we stick to finite-dimensional uniqueness.

Example 8. Let $\mathfrak{H} := \{Z \in \mathfrak{X}(\mathbb{C}^2, 0) : Z \text{ is hyperbolic at } (0, 0)\}$. Then

$$\begin{aligned} C_{\text{loc}} : \text{Mod}_{\text{loc}}(\mathfrak{H}) &\longrightarrow (\mathbb{C}^\times)^2 / \mathbb{Z}/2\mathbb{Z} \\ [Z]_{\text{loc}} &\longmapsto \text{Spec}(Z, 0) \end{aligned}$$

is a classification for local conjugacy, which is not complete. Injective realizations are given by

$$\begin{aligned} R_{\text{loc}} : (\mathbb{C} \setminus \mathbb{R}) \times \mathbb{C}^\times &\longrightarrow \text{Mod}_{\text{loc}}(\mathfrak{H}) \\ (\rho, l_2) &\longmapsto \left[l_2 \rho x \frac{\partial}{\partial x} + l_2 y \frac{\partial}{\partial y} \right]_{\text{loc}} . \end{aligned}$$

Normal forms are given by

$$\text{NF}_{\text{loc}}(\mathfrak{H}) := \left\{ l_1 x \frac{\partial}{\partial x} + l_2 y \frac{\partial}{\partial y} : (l_1, l_2) \in (\mathbb{C}^\times)^2, l_1/l_2 \notin \mathbb{R} \right\} .$$

Example 9. Theorem 2 asserts

$$\text{Mod}_{\text{loc}}(\text{Affine}(1)) \simeq \overline{\mathbb{N}}$$

with normal forms

$$\text{NF}_{\text{loc}}(\text{Affine}(1)) := \{X_\bullet^\kappa : \kappa \in \overline{\mathbb{N}}\} .$$

1.3 General saddle-node bifurcations

From now on we deal with the general case of a holomorphic germ of a planar saddle-node bifurcation. For the bifurcation value of the parameter $\lambda \in A$, which we conveniently locate at the origin of a complex affine space of which A is a (sufficiently small) domain, the vector field Z_0 is of **saddle-node** type near, say, the origin of \mathbb{C}^2 , that is:

- ⊗ 0 is an isolated singularity of Z_0 ,
- ⊗ the differential at 0 of the vector field has exactly one non-zero eigenvalue or, with notations introduced earlier, $\text{Spec}(Z, 0) = \{0, l_2\}$ for some $l_2 \in \mathbb{C}^\times$ (the singularity is elementary degenerate).

To stick to general terminology, a (holomorphic germ of a) parametric family of (germs at $0 \in \mathbb{C}^2$ of) vector fields $Z_\bullet = (Z_\lambda)_{\lambda \in A}$ is called a holomorphic germ of an **unfolding** of Z_0 . We study in details only «generic» unfoldings, those which possess the «right number» of parameters to encode the bifurcation structure. Let us be more specific.

Definition 1. Let $k \in \mathbb{N}$ be given. A **generic unfolding of multiplicity k** is a germ of an unfolding Z_\bullet for which the following conditions hold.

- ⊛ There exists $\Lambda = (\mathbb{C}^k, 0)$ and $\mathcal{U} = (\mathbb{C}^2, 0)$ such that $(\lambda, x, y) \mapsto Z_\lambda(x, y)$ is holomorphic on $\Lambda \times \mathcal{U}$, and the vector field Z_0 has only one singularity in \mathcal{U} .
- ⊛ For a dense open set $\hat{\Lambda} \subset \Lambda$ and all $\lambda \in \hat{\Lambda}$ the vector field Z_λ has exactly $k+1$ (distinct) singularities in \mathcal{U} , which are all hyperbolic and merge to 0 as $\lambda \rightarrow 0$.
- ⊛ $\lambda \in \hat{\Lambda} \mapsto \text{Sing}(Z_\lambda)$ is injective.

The limiting saddle-node Z_0 has multiplicity (codimension) k .

Families with $\tilde{k} > k$ parameters can be dealt with by changing the parameters (say, using the implicit function theorem, after desingularization if need be) in such a way that the first k components govern the location of the singularities, while the rest (seen as extra parameters) do not move them around. To be more specific, we mean that each fiber of $\lambda \in \hat{\Lambda} \mapsto \text{Sing}(Z_\lambda)$ is included in a single fiber of the natural projection on the first k components. All results presented here hold also for these extra-parametric generic unfoldings, as it will appear clearly that the constructions depend holomorphically on extra parameters.

Families $(\tilde{Z}_{\tilde{\lambda}})_{\tilde{\lambda} \in \tilde{\Lambda}}$ having singularities either generically elementary degenerate (e.g. coalescing saddle-nodes) or non-degenerate but reached multiple times, can be studied through (extra-parametric) generic unfoldings $(Z_\lambda)_{\lambda \in \Lambda}$ by specializing values $\lambda = \phi(\tilde{\lambda})$.

We postpone a formal definition of conjugacy / orbital equivalence between unfoldings till the end of this section. Just keep in mind that we do not allow parameter changes involving the spatial coordinates (x, y) .

Affine unfoldings, as detailed in Section 1.1, suggest that dynamical questions regarding saddle-node bifurcations can be understood from the local classification of generic unfoldings. This point is made in Section 1.9. More generally the classification of unfoldings contains the classification of Z_0 by specialization. For this reason we deliberately elude presenting this degenerate situation in details. Yet as the strategies adopted to address saddle-nodes will serve us well, we present them briefly as a stepping stone to the unfolded case. We refer the reader to the works cited below for a comprehensive study of the subject.

The analytic unstable manifold of Z_0 , tangent at 0 to the eigenspace associated to l_2 , is called the **strong separatrix**. The other eigenspace corresponds to a «formal separatrix» called the **weak separatrix** (generically divergent [14], always summable in the sens of Borel [9]). We say that a saddle-node is convergent or divergent according to the nature of its weak separatrix.

The formal orbital classification was performed by H. Poincaré & H. Dulac [5, 6], yielding polynomial normal forms. It was known from the very beginning that the formal conjugacy cannot always converge, and as a matter of fact divergence is the rule. After some inspiring works by D. Birkhoff [1] on local classification of resonant diffeomorphisms, a complete local orbital classification was achieved in the early 1980's by J. Martinet & J.-P. Ramis [12]. At about the same time A. Bruno [3] presented formal normal forms for saddle-node vector fields. These works were complemented with a complete local classification in the early 2000's simultaneously by Y. Meshcheryakova & S. Voronin [25] ($k = 1$) and by L. Teyssier [22] for the general case. These studies make apparent that classifying vector fields can be dissociated into two independent process:

1. classify the orbital part (the foliation),

2. classify the «time» (the vector field for fixed foliation).

Item 2. is a linear problem, simpler to deal with. Hence in the current introduction we only present how orbital classification is achieved.

The foundational viewpoint introduced in [12] bridges the gaps between classification on the one hand, dynamical and analytical properties on the other hand. It consists in interpreting the orbital invariants as transition maps defining an analytic atlas of (roughly speaking) the space of leaves Ω_0 of \mathcal{F}_{Z_0} . The conformal class of Ω_0 is reasonably a complete orbital invariant of Z_0 . As illustrated in Figure ##, this space is obtained by

- ⊗ cutting $(\mathbb{C}^2, 0)$ up into $2k$ fibred sectors,
- ⊗ taking $2k$ copies of the Riemann sphere $\overline{\mathbb{C}}$ (sectorial spaces of leaves), sectorial weak separatrices of Z_0 attached to points 0 and the strong separatrix to ∞ ,
- ⊗ gluing them sequentially near 0 (*resp.* ∞) by germs of a diffeomorphism (*resp.* translations) like beads on a necklace (Martinet-Ramis's *chapelet de sphères*), the identification coming from the inclusions of the sectors in $(\mathbb{C}^2, 0)$.

The key is the conformal rigidity of $\overline{\mathbb{C}}$: its automorphism group is $\mathbb{PGL}_2(\mathbb{C})$ (acting by homography). With the choice of leaves for 0 and ∞ , and yet another canonical normalization, the only remaining degree of freedom for analytic atlases of Ω_0 is the linear action of \mathbb{C}^\times (simultaneously on all spheres). Notice that Ω_0 is not exactly the space of leaves $\widehat{\Omega}_0$ of \mathcal{F}_{Z_0} . Firstly the strong separatrix should be in the adherence of each point of $\widehat{\Omega}_0$, since there exist $2k$ directions over which each sectorial leaf of \mathcal{F}_{Z_0} accumulates on the strong separatrix. The general attitude toward this is to identify only 0 and ∞ , since these are the only two points in Ω_0 that can be reached over the intersection of two neighboring sectors while tending to the strong separatrix (see Figure ##). Secondly $\widehat{\Omega}$ is obtained from Ω_0 by modding out the action of the global monodromy of the necklace. This fact is explained in details later on for unfoldings (Section 1.9).

The first technique based on a deformation of a saddle-node vector field in order to recover Martinet-Ramis invariants was presented in the early 2000's, after an analogous work by J. Martinet [11] for unfoldings of parabolic diffeomorphisms. A. Glutsyuk [8] embedded Z_0 in a generic unfolding of multiplicity k . Restricting λ to $\widehat{\Lambda}$, he let the $k + 1$ hyperbolic singularities merge as $\lambda \rightarrow 0$. He proved that the domains $(D_{\lambda,j})_j$ of linearization of Z_λ (near each singular point) overlap and their union D_λ contains a domain $(\mathbb{C}^2, 0)$ independent on λ . The space of leaves Ω_λ of $\mathcal{F}_{Z_\lambda}|_{D_\lambda}$ is therefore built by gluing the $2k$ spaces of leaves of $\mathcal{F}_{Z_\lambda}|_{D_{\lambda,j}}$, which are (rigid) conformal tori $\mathbb{C}^\times/\mathbb{Z}$ (plus one point, corresponding to local separatrices, in the adherence of every other point). The gluing mappings come from the inclusions $D_{\lambda,j} \hookrightarrow D_\lambda$. As $\lambda \rightarrow 0$ each torus gets pinched more and more sharply along a meridian, so that the limiting surface is a sphere $\overline{\mathbb{C}}$ with the points 0 and ∞ identified. We refer to Figure ##.

The natural continuation of A. Glutsyuk's use of the saddle-node bifurcation would be to classify all generic unfoldings. Yet this approach is doomed to fail, because $\widehat{\Lambda}$ is not connected. Glutsyuk's construction critically depends on the local behavior of hyperbolic singularities, and as such cannot be extended to $\Lambda \setminus \widehat{\Lambda}$. Indeed most (resonant) saddle points are not linearizable and, even if they were, their local leaves space would be a conformal disc, which is not rigid: the automorphism group of germs at 0 of conformal discs coincides with the infinite dimensional group $\text{Diff}(\mathbb{C}, 0)$. Roughly speaking, one

cannot turn a saddle or node singularity into a hyperbolic singularity. But one can the converse, as done by C. Rousseau and L. Teyssier in [19]. By cutting «sectors» attached to hyperbolic points with a special, asymptotically spiraling shape, the restricted foliation can be forced to behave locally very much like a node or a saddle. The process yields a necklace Ω_λ obtained exactly in the same way as for $\lambda = 0$, with an additional linear identification. This approach allows to cover the whole parameter space. The conformal structure of the necklace Ω_λ depends locally analytically on λ and is continuous as $\lambda \rightarrow 0$ (in particular sectors for Z_λ converge toward standard sectors for Z_0 in the Hausdorff distance). Therefore unfoldings of foliations are locally classified by families of gluing mappings unfolding the local orbital invariants of Z_0 .

The problem of giving a complete local classification of generic unfoldings (identifying the total image of the classification) is still open, except for the case $k = 1$ if one compiles the results of [15] and [18]. The realization problem is double:

1. realize, for fixed λ , a given necklace as the leaves space of some Z_λ ,
2. glue all Z_λ for $\lambda \in A$ to form an unfolding of Z_0 .

Item 1. poses no specific problem and can be dealt with in the usual manner using tools borrowed from complex geometry, by building an abstract almost-complex realization, then invoking Newlander-Nirenberg theorem to incarnate it as a germ of a foliated analytic manifold. On the contrary Item 2. is linked with the combinatorial structure of the covering of the parameter space by (contractible) open cells on which the invariants $\lambda \mapsto C_\lambda$ of the unfoldings are holomorphic. This decomposition is not trivial: it is indeed impossible to perform the previous sectorial decomposition uniformly for all values of λ . The reason is the following: the transfiguration of a saddle point into a sectorial saddle-like hyperbolic point cannot be pursued after a certain point, corresponding to parameter values for which a saddle-like singularity tends to a genuine node. A node cannot be tricked into believing it behaves like a saddle. One must therefore deal with finite families $((C_\lambda)_{\lambda \in A_\ell})_\ell$ of unfoldings of invariants of Z_0 , where $A = \bigcup_\ell \text{adh}(A^\ell)$. On neighboring intersections $A^\ell \cap A^{\tilde{\ell}}$ all singularities are hyperbolic and the configuration is that of a Glutsyuk deformation. The invariants C_λ^ℓ and $C_\lambda^{\tilde{\ell}}$ must relate to Glutsyuk tori decomposition, since all three objects encode the same leaves space Ω_λ and the same underlying dynamics. Expressing this identity yields necessary **compatibility conditions** (Section 1.9.3) that C_λ^ℓ and $C_\lambda^{\tilde{\ell}}$ must obey, as explained clearly in [17]. Save for the case $k = 1$, where $A = A^+ \cup A^- \cup \{0\}$ and the compatibility conditions guarantee that Newlander-Nirenberg theorem applies in parameter space, those conditions have not been written down and proved sufficient (there is little doubt, though, that they are). We do not present the details of this approach. Another way of achieving a complete classification would be to describe a collection of normal forms for generic unfoldings. Normal forms have been devised recently by R. Schäfke & L. Teyssier [20] for convergent saddle-nodes. These families can be unfolded to families of normal forms, in the case of **pure convergence**: every member Z_λ of the unfolding has as a heteroclinic connection or, equivalently, sectorial weak separatrices patch continuously for all $\lambda \in (\mathbb{C}, 0)$. We refer to Section 1.10.2. This method leaves open the generic case where there are no homoclinic connection in the unfolding, although general normal forms are expected to be worked out soon.

To conclude this introduction we give a precise definition of what changes of variables we allow between unfoldings. Section 1.2 recalled the diverse notions of (formal, local) conjugacy and orbital equivalence between vector fields in $\mathfrak{X}(\mathbb{C}^2, 0)$. We need to precise the corresponding notions for unfoldings in order to perform their classification.

Definition 2. We say that two unfoldings $(Z_\lambda)_{\lambda \in \Lambda}$ and $(\tilde{Z}_{\tilde{\lambda}})_{\tilde{\lambda} \in \tilde{\Lambda}}$ are (formally, locally) **conjugate** (*resp.* **orbitally equivalent**) if there exists an association

$$\Psi : (\lambda, x, y) \mapsto (\phi(\lambda), \Psi_\lambda(x, y))$$

in the corresponding regularity class, such that:

1. $\lambda \in (\mathbb{C}^k, 0) \mapsto \tilde{\lambda} = \phi(\lambda)$ has invertible derivative at 0,
2. for each $\lambda \in (\mathbb{C}^k, 0)$ the component Ψ_λ is a conjugacy (*resp.* orbital equivalence) between Z_λ and $\tilde{Z}_{\phi(\lambda)}$.

If the above conditions are fulfilled we write

$$\Psi^*(Z_\lambda)_\lambda = (\tilde{Z}_{\tilde{\lambda}})_{\tilde{\lambda}} .$$

We wish to describe the (formal, local) classification in the set

$$\text{SNU}(k) := \{(Z_\lambda)_\lambda : (Z_\lambda)_\lambda \text{ generic unfolding of multiplicity } k\} .$$

Definition 3. We use the notations $\text{Mod}_{\text{for}}(k)$, $\text{Mod}_{\text{loc}}(k)$ and $\text{Mod}_{\text{top}}(k)$ to stand for the **moduli spaces** of $\text{SNU}(k)$ under corresponding conjugacy. The **orbital moduli spaces** under orbital equivalence is written $\text{Mod}_{\text{for}}^{\text{orb}}(k)$ and $\text{Mod}_{\text{loc}}^{\text{orb}}(k)$ respectively. The same notational convention is used for the class of an unfolding:

$$[(Z_\lambda)_\lambda]_\star^\sharp \in \text{Mod}_\star^\sharp(k) .$$

We need to slacken a little the close of uniqueness for **normal forms**: we require that there exist $\nu \in \mathbb{N}$ such that, for every fixed value of the parameter λ , the equivalence class of $[(Z_\lambda)_\lambda]_\star^\sharp$ is contained in the orbit of a \mathbb{C}^ν -action.

1.4 Every step of the way

For the sake of clarity we present only the case $k = 1$. Unlike saddle-node singularities, where a general complete classification is not harder to obtain than for $k = 1$, unfoldings are more difficult to deal with this way, due to the need of splitting the parameter space into many cells. The specific problems and corresponding results are detailed in [19].

Let us summarize the different steps leading to the classification of generic saddle-node unfoldings Z_\bullet of multiplicity 1. The section ends by the statement of the main theorems. Most important items in the list below are developed later on in the course of the chapter. There we will outline the precise setting of and problems occurring in the upcoming constructions, while referring to [19] for all details concerning actual proofs. The exceptions to this regard the complete local temporal classification (Section 1.10.1)

and a collection of normal form for purely convergent unfoldings (Section 1.10.2), new results to which we give detailed proofs.

Preparation of the family (Section 1.5.1)

We can choose local conformal coordinates (λ, x, y) in which Z_\bullet can be put under **prepared form**

$$Z_\lambda = U_\lambda \left(X_\lambda^\infty + (\dots) \frac{\partial}{\partial y} \right)$$

where X_\bullet^∞ , called the **orbital model**, is formally orbitally equivalent to Z_\bullet , the notation $(\dots) \frac{\partial}{\partial y}$ denotes a transverse holomorphic perturbation and $U_\bullet \in \mathbb{C} \{ \lambda, x, y \}^\times$. Moreover the singularities of Z_λ are located at points in $\{y = 0\}$ corresponding to the roots of the polynomial

$$P_\lambda(x) := x^2 + \lambda. \tag{1.22}$$

The parameter λ is then a formal invariant. In the sequel we only consider prepared unfoldings, which allows us to work for fixed λ . To lighten notations we omit to mention the subscript « λ » in all the following items.

Sectorial decomposition (Section 1.7)

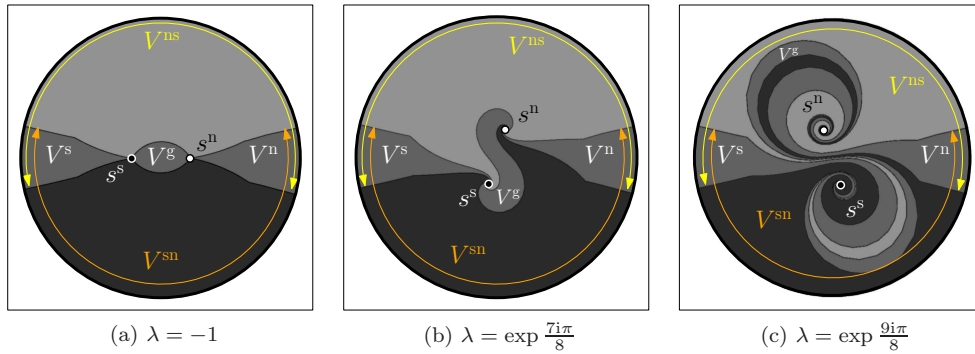
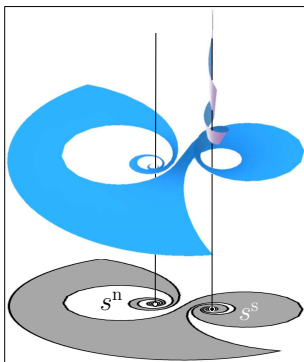


Fig. 1.1: A sectorial decomposition of the x -variable when $k = 1$ for three values of the parameter

The local invariants of Z_\bullet are built by comparing transition maps between two neighboring normalizing charts. By this we mean to cut $(\mathbb{C}^2, 0) \setminus P^{-1}(0)$ up into two overlapping open **canonical sectors** \mathcal{V}^{ns} and \mathcal{V}^{sn} on every



one of which Z is orbitally equivalent to the model X^∞ . The canonical sectors are fibred over **squid sectors** V^\sharp in the x -variable, displayed in Figure 1.1. The intersection $V^{ns} \cap V^{sn}$ has three components:

- ⊗ a **saddle-part** V^s having only s^s in its adherence,
- ⊗ a **node-part** V^n having only s^n in its adherence,
- ⊗ a **gate-part** V^g having both points in its adherence.

We write

$$\mathcal{V}^\sharp := V^\sharp \times (\mathbb{C}, 0) \quad , \quad \sharp \in \{n, ns, s, sn, g\}$$

the corresponding fibred sectors. There are two transitions to consider for orbital equivalence (happening in the saddle- and node-part), and one more to account for conjugacy (over the saddle-part).

We denote by \mathcal{F} the (germ of a) singular foliation induced by Z . Each canonical sector is attached to both singular points s^n and s^s . The boundary of a sector is carved in such a way that the leaves of $\mathcal{F}|_{\mathcal{V}^\sharp}$ near s^n behave like those of a node: every leaf accumulates on the singular point. Near s^s the foliation is similar to a saddle: every leaf but one (the local invariant manifold) stays far away from the singular point (Figure 1.2). This topological configuration allows us to mimic constructions performed in Section 1.1 for affine unfoldings. In particular the sectorial space of leaves is a conformal line

$$\Omega^\sharp = H^\sharp(\mathcal{V}^\sharp) = \mathbb{C} \ ,$$

where $H^\sharp \in \text{Holo}(\mathcal{V}^\sharp)$ is the **canonical first integral** of $\mathcal{F}|_{\mathcal{V}^\sharp}$, having connected fibers.

Straightening of the weak separatrices

If a singularity p of X is not a node, there exists only one integral curve with smooth analytic closure passing through p (a separatrix, or invariant manifold) and transverse to the vertical lines $\{x = \text{cst}\}$. It is given by the graph of a holomorphic function $x \mapsto \mathfrak{s}(x)$, with holomorphic continuation over every canonical sector. We call such an analytic continuation a **sectorial weak separatrix**. It corresponds to the level 0 of the canonical first integral H^\sharp . In particular both sectorial weak separatrices coincides in the saddle- and gate-part.

Applying the change of coordinates $\psi : (x, y) \mapsto (x, y + \mathfrak{s}(x))$ to Z straightens the sectorial weak separatrix into $\{y = 0\}$. Notice that \mathfrak{s} cannot (in general) be analytically continued on a whole neighborhood of the other singularity, as a given sectorial weak separatrix may not coincide with the other one when continued (the typical \mathfrak{s} is multivalued). When \mathfrak{s} does extend holomorphically on a neighborhood of all singular points we say that a **heteroclinic connection** occurs between s^s and s^n .

Normalization strategy (Section 1.5.2)

After straightening, we can write:

$$\begin{aligned} Z &= UX \\ X &= X^\infty + yR \frac{\partial}{\partial y} \end{aligned}$$

for some (sectorial) holomorphic functions U and R . It turns out the conjugacy equation $\mathcal{O}^*X^\infty = X$ is equivalent to the following **orbital cohomological equation**

$$X \cdot O = -R \tag{1.23}$$

if one seeks a conjugacy in the form

$$\mathcal{O}(x, y) = (x, y \exp O(x, y)) .$$

Also the conjugacy equation $\mathcal{T}^*X = UX$ takes the form of a **temporal cohomological equation**

$$X \cdot T = \frac{1}{U} - 1 \tag{1.24}$$

for changes of coordinates

$$\mathcal{T}(x, y) = \Phi_X^{T(x, y)}(x, y)$$

obtained by taking a dependant time in the flow $\Phi_X^t(x, y)$ of X . Therefore the normalization process has been reduced to solving two cohomological equations.

Formal classification (Section 1.5.3)

The straightening step can be realized at a formal level. The orbital cohomological equation (1.23) can be solved formally, more or less by construction of the model X^∞ and of the prepared form. The temporal cohomological equation (1.24) needs some adjustment since $\frac{1}{U} - 1$ may not belong to the image of the Lie derivative associated to X . We can find an affine function $u \in \mathbb{C}[x]_{\leq 1}$, relatively prime with P , such that $\frac{1}{U} - \frac{1}{u}$ belongs to that image. Hence uX^∞ is formally conjugate to Z , finally yielding a polynomial **formal model**

$$Z^u := uX^\infty .$$

This complete formal classification of generic unfoldings ranges in the space of germs $\mathbb{C}\{\lambda\}^3$.

Local classification (Section 1.8)

The normalizing cohomological equations (1.23) and (1.24) admit a bounded solution on each canonical sector. They are obtained by integrating the right-hand side against $\frac{dx}{P}$ along **asymptotic paths** tangent to \mathcal{F} , ending at the point (x, y) and accumulating on s^n in backward time. The node-like nature of s^n guarantees that every point in the sector can be reached by asymptotic paths, and that both solutions coincide in the node- and gate-part.

Therefore Z is conjugate to the formal model Z^u on each canonical sectors. The local class of Z_\bullet is thus completely determined by the following data.

- ⊗ Its orbital class $\varphi := (\varphi^n, \varphi^s)$, obtained by comparing:
 - ⊙ the sectorial weak separatrices in the node-part, measuring how far the vector field is from having a heteroclinic connection and encoded in a translation $h \mapsto h + \varphi^n$,
 - ⊙ the sectorial solutions to (1.23) in the saddle-part, measuring how far the continued sectorial solution of the orbital cohomological equation is from uniformity (that is, continuity).
- ⊗ Its temporal class f , encoded by comparing the sectorial solutions to (1.24) in the saddle-part.

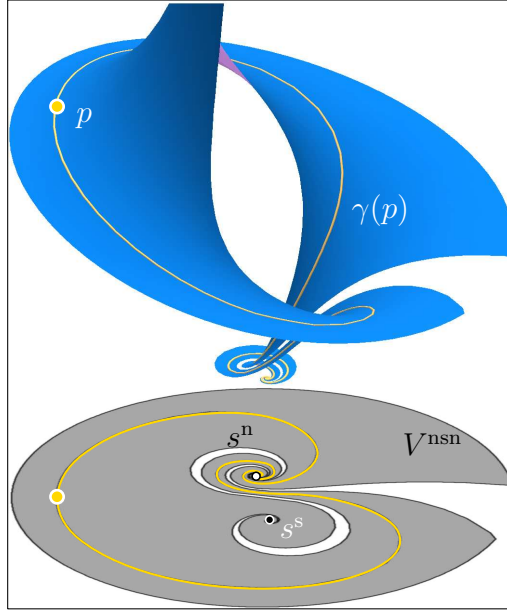


Fig. 1.3: An asymptotic cycle and its projection on the x -variable. (Modulus of the y -coordinate of the leaf as a height-map)

In both cases where cohomological equations are involved, comparing sectorial solutions means to compute the **period**

$$g(p) := \frac{1}{2i\pi} \int_{\gamma(p)} G \frac{dx}{P}$$

of the right-hand side G along an **asymptotic cycle** $\gamma(p)$ circling around s^s and tending asymptotically to s^n both in forward and backward time, while passing through $p \in \mathcal{V}^s$, as displayed in Figure 1.3. This gives an integral representation of the invariants over the saddle-part (Section 1.8.5.2). The value of the integral does not depend on the choice of p in a fixed leaf of the restriction of \mathcal{F} to the sector

$$\mathcal{V}^{\text{nsn}} := (\mathbb{C}^2, 0) \setminus (\mathcal{V}^n \cup \mathcal{V}^g)$$

because the leaf is simply connected. Hence g is a first integral of X and for that matter factors as a (germ of a) holomorphic function of the canonical first integral H^{ns} , holomorphic on \mathcal{V}^{ns} :

$$g = \mathfrak{T}_{\bullet}^X(G) \circ H^{\text{ns}} \quad , \quad \mathfrak{T}_{\bullet}^X(G) \in h\mathbb{C}\{h\} \quad .$$

Remark 7. For the period $\mathfrak{T}_{\bullet}^X(G)$ to be well-defined, we need to mod it out by the degree of freedom in the choice of the sectorial first integrals H^{ns} . This freedom results from a faithful action of \mathbb{C}^\times by linear changes of variables $h \mapsto ch$. We leave such essentially irrelevant technicalities out till subsequent sections.

For a fixed formal model we finally obtain a local classification ranging in a functional space

$$[(Z_\lambda)_\lambda]_{\text{loc}} \mapsto (\lambda \mapsto C_\lambda) \quad , \quad C_\lambda = (\varphi_\lambda^n, \varphi_\lambda^s, f_\lambda) \in \mathbb{C} \times h\mathbb{C}\{h\} \times h\mathbb{C}\{h\} \quad .$$

Notice how cowardly we shy away from discussing the dependence in the parameter λ . The result to come, proved in Sections 1.8 and 1.10.1, gives a precise formulation.

Theorem 4. *There exists a cover of $(\mathbb{C}, 0)$ by the adherence of two germs of sectors Λ^+ , Λ^- attached to 0 (called cells), for which we can find a complete local classification of generic unfoldings of multiplicity 1 with fixed formal class, ranging in the space of those collections $(C_\bullet^-, C_\bullet^+)$ satisfying:*

- ⊗ $(\lambda, h) \in \Lambda^\pm \times (\mathbb{C}, 0) \mapsto C_\lambda^\pm(h)$ is holomorphic with continuous extension to $\text{adh}(\Lambda^\pm) \times (\mathbb{C}, 0)$, and C_λ^\pm is holomorphic for every $\lambda \in \partial\Lambda^\pm$,
- ⊗ $C_\lambda^\pm \in \mathbb{C} \times h\mathbb{C}\{h\} \times h\mathbb{C}\{h\}$,
- ⊗ the **compatibility condition**, given explicitly in Definitions 16 and 17.

By construction the model $\mathbf{u}_\bullet X_\bullet^\infty$ has local class $C_\bullet^\pm = 0$.

Normal forms for pure convergence (Section 1.10.2)

When $\varphi_\bullet^n = 0$ (we say the unfolding is **purely convergent**) we provide normal forms for Z_\bullet . Remark both sectorial separatrices glue to form a holomorphic weak separatrix (heteroclinic connection) and therefore purely convergent unfoldings are locally conjugate to prepared unfoldings for which $\{y = 0\}$ is a leaf, and *vice versa*.

Theorem 5. *Let Convergent (1) be the space of all purely convergent, generic unfoldings of multiplicity 1. Define*

$$\tau := \begin{cases} 0 & \text{if } \mu_0 \notin \mathbb{R}_{\leq 0} \\ 1 + \lfloor -\mu_0 \rfloor & \text{otherwise} \end{cases}$$

$$\text{Section (1)} := x^{\tau+1}y\mathbb{C}\{\lambda, x^\tau y\} .$$

Then the collection

$$\left\{ \frac{u_\bullet}{1 + u_\bullet G_\bullet} \left(X_\bullet^\infty + yR_\bullet \frac{\partial}{\partial y} \right) : G_\bullet, R_\bullet \in \text{Section (1)} \right\}$$

is a family of normal forms for Convergent (1).

1.5 Preparation and formal classification

1.5.1 Preparation

Theorem 6. *There exists local conformal coordinates (λ, x, y) in which Z_\bullet has the following prepared form*

$$Z_\lambda = U_\lambda X_\lambda$$

$$X_\lambda(x, y) = P_\lambda(x) \frac{\partial}{\partial x} + (y(1 + \mu_\lambda x) + P_\lambda(x) R_\lambda(x, y)) \frac{\partial}{\partial y} \quad (1.25)$$

where

$$\begin{cases} \lambda & \in (\mathbb{C}, 0) \\ \mu_\bullet & \in \mathbb{C}\{\lambda\} \\ R_\bullet & \in \mathbb{C}\{\lambda, x, y\} \\ U_\bullet & \in \mathbb{C}\{\lambda, x, y\}^\times \end{cases}$$

are arbitrary.

Notice that every prepared form is a generic unfolding of multiplicity 1, for $\text{Sing}(X_\lambda) = \{y = 0\} \cap P_\lambda^{-1}(0)$. A formal changes of coordinates fixing the general form of the family (1.25) must leave the parameter λ invariant. It suffices to work with fixed λ in order to perform the local classification.

1.5.2 From normalization to cohomological equations

The whole procedure relies on writing the conjugacy equation $\Psi^*Z = \tilde{Z}$ as **cohomological equations**

$$Z \cdot F = G \quad (1.26)$$

for well-chosen right-hand sides. The key computation is the following proposition.

Proposition 4. *Let X and Y be two germs of a holomorphic vector field on a domain \mathcal{U} such that $[X, Y] = 0$. If f is holomorphic on \mathcal{U} (resp. a formal power series at some point $p \in \mathcal{U}$) then*

$$\Psi(x, y) := \Phi_Y^{f(x, y)}(x, y)$$

has same regularity as f , and satisfies

$$\Psi^* X = X - \frac{X \cdot f}{1 + Y \cdot f} Y .$$

In particular, the following properties hold.

1. **(Temporal conjugacy)** UZ is conjugate to VZ by Φ_{UZ}^T if, and only if,

$$Z \cdot T = \frac{1}{U} - \frac{1}{V} .$$

2. **(Orbital conjugacy)** Assume $X \pitchfork Y$. Then X is conjugate to $X + RY$ by Φ_Y^O if, and only if,

$$X \cdot O = R .$$

Proof. It is sufficient to perform computations at a formal level. We use Lie formula (1.18) so that, because $X \cdot Y = Y \cdot X$,

$$\begin{aligned} X \circ \Psi &= \sum_{n \geq 0} \frac{f^n}{n!} Y \cdot^n (X \cdot \text{Id}) \\ &= \sum_{n \geq 0} \frac{f^n}{n!} X \cdot (Y \cdot^n \text{Id}) . \end{aligned}$$

Besides

$$\begin{aligned} D\Psi(X - RY) &= (X - RY) \cdot \Psi \\ &= \sum_{n \geq 0} (X - RY) \cdot \left(\frac{f^n}{n!} Y \cdot^n \text{Id} \right) \\ &= (X - RY) \cdot f \times \sum_{n \geq 0} \frac{f^n}{n!} Y \cdot^{n+1} \text{Id} + \sum_{n \geq 0} \frac{f^n}{n!} (X - RY) \cdot Y \cdot^n \text{Id} \\ &= X \circ \Psi + (X \cdot f - R(1 + Y \cdot f)) \times Y \circ \Psi . \end{aligned}$$

Therefore

$$R = \frac{X \cdot f}{1 + Y \cdot f} .$$

To prove 1. it suffices to take $X := UZ$ and $Y := UZ$. \square

By taking $(Z_s)_s$ in prepared form (1.25) we can write

$$\begin{aligned} Z_\lambda &= U_\lambda X_\lambda \\ X_\lambda &= \left(X_\lambda^\infty + P_\lambda R_\lambda \frac{\partial}{\partial y} \right) \\ X_\lambda^\infty &:= P_\lambda \frac{\partial}{\partial x} + y(1 + \mu_\lambda x) \frac{\partial}{\partial y}. \end{aligned}$$

Notice that

$$\left[X_\lambda^\infty, y \frac{\partial}{\partial y} \right] = 0$$

so we can apply Proposition 4 Item 2. as soon as R_s can be factored by y , that is once the weak separatrix is straightened into $\{y = 0\}$.

1.5.3 Formal classification

The strategy is the following.

- ⊗ There exists a unique weak separatrix family, that is a formal family of curves $\{y - \mathfrak{s}_\lambda(x) = 0\}$ such that

$$P_\lambda(x) \widehat{\mathfrak{s}}_\lambda'(x) = \widehat{\mathfrak{s}}_\lambda(x) (1 + \mu_\lambda x) + P_\lambda(x) R_\lambda(x, \widehat{\mathfrak{s}}_\lambda(x)) \quad , \quad \widehat{\mathfrak{s}}_\bullet \in \mathbb{C}[[\lambda, x, y]] \quad .$$

- ⊗ After applying the change of coordinates $(x, y) \mapsto (x, y - \widehat{\mathfrak{s}}_\lambda(x))$ to X_λ we obtain the formal vector field

$$\widehat{X}_\lambda = X_\lambda^\infty + P_\lambda \widehat{R}_\lambda y \frac{\partial}{\partial y}$$

where

$$\widehat{R}_\lambda(x, y) := \frac{R_\lambda(x, y + \widehat{\mathfrak{s}}_\lambda(x)) - R_\lambda(x, \widehat{\mathfrak{s}}_\lambda(x))}{y} \quad . \quad (1.27)$$

- ⊗ The cohomological equation

$$\widehat{X}_\lambda \cdot \widehat{O}_\lambda = -P_\lambda \widehat{R}_\lambda$$

admits a unique formal family of solutions $\widehat{O}_\bullet \in \mathbb{C}[[\lambda, x, y]]$ such that $\widehat{O}_\lambda(0) = 0$. Therefore X_\bullet^∞ is orbitally formally conjugate to \widehat{X}_\bullet by

$$\widehat{O}_\bullet := \Phi_{y \frac{\partial}{\partial y}}^{\widehat{O}_\bullet} \quad ,$$

thus formally conjugate to X_\bullet by a (λ, x) -fibred formal conjugacy. Remark that the previous cohomological equation is equivalent to one involving X_λ in the original coordinates:

$$X_\lambda \cdot O_\lambda = -P_\lambda \frac{R_\lambda(x, y) - R_\lambda(x, \widehat{\mathfrak{s}}_\lambda(x))}{y - \widehat{\mathfrak{s}}_\lambda(x)}. \quad (1.28)$$

⊗ The cohomological equation with $u_\bullet \in \mathbb{C}[[\lambda, x]]^\times$

$$X_\lambda \cdot \widehat{T}_\lambda = \frac{1}{U_\lambda} - \frac{1}{u_\lambda} \quad (1.29)$$

admits a formal solution if, and only if,

$$U_\lambda(x, y) = u_\lambda(x) + \mathcal{O}(x^2) + \mathcal{O}(y)$$

where u_λ is relatively prime with P_λ in the factorial ring $\mathbb{C}[[\lambda, x, y]]$. The holomorphic germ u_\bullet can therefore be taken as the remainder of the Euclidean division of $U_\bullet(\star, 0)$ by P_\bullet . In particular for each value of the parameter u_λ is affine. Because $\widehat{\mathcal{O}}_\lambda$ is x -fibred, the vector field $u_\lambda X_\lambda^\infty$ is formally conjugate to $u_\lambda X_\lambda$, thus to Z_λ .

The last two claims derive from the following easy computational lemma.

Lemma 4. *Let $\widehat{G}_\bullet \in \mathbb{C}[[\lambda, x, y]]$ be given. The cohomological equation*

$$X_\lambda \cdot \widehat{F}_\lambda = \widehat{G}_\lambda$$

admits a formal solution $\widehat{F}_\bullet \in \mathbb{C}[[\lambda, x, y]]$ if, and only if, $\widehat{G}_\lambda = \mathcal{O}(P_\lambda(x)) + \mathcal{O}(y)$. This family of solution is unique up to the addition with an arbitrary formal power series belonging to $\mathbb{C}[[\lambda]]$, corresponding to the choice of the value $\widehat{F}_\lambda(0)$.

Discounting additional straightforward computations, we just established a formal classification with normal forms.

Theorem 7. *We have complete classifications*

$$\text{Mod}_{\text{for}}(1) \simeq \mathbb{C}\{\lambda\}[x]_{\leq 1}^\times \times \text{Mod}_{\text{for}}^{\text{orb}}(1)$$

$$\text{Mod}_{\text{for}}^{\text{orb}}(1) \simeq \mathbb{C}\{\lambda\}$$

with normal forms

$$\begin{aligned} \text{NF}_{\text{for}}(1) &:= \mathbb{C}\{\lambda\}[x]_{\leq 1}^\times \text{NF}_{\text{for}}^{\text{orb}}(1) \\ \text{NF}_{\text{for}}^{\text{orb}}(1) &:= \left\{ P_\bullet(x) \frac{\partial}{\partial x} + y(1 + \mu_\bullet x) \frac{\partial}{\partial y} : \mu_\bullet \in \mathbb{C}\{\lambda\} \right\}. \end{aligned}$$

Definition 4. An unfolding \widehat{Z}_\bullet in $\text{NF}_{\text{for}}^{\text{orb}}(1)$ is now fixed, corresponding to the choice of the holomorphic germs $\mu_\bullet : \lambda \mapsto \mu_\lambda$ and $u_\bullet : (\lambda, x) \mapsto u_\lambda(x)$. It is referred to as the **model**.

The local classification is obtained by repeating the construction on sectors over which \hat{O}_λ and \hat{T}_λ have holomorphic «sums». The local invariant measures how far from converge these power series are.

1.6 Parameter space

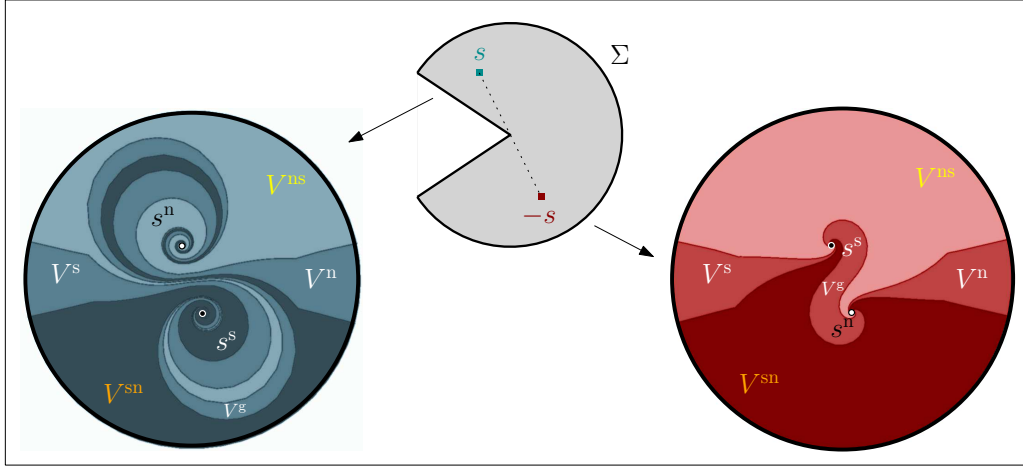


Fig. 1.4: Two non-equivalent partitions by squid sectors with the same value of the parameter $\lambda = -s^2$. Notice that the nature of the singular points (node- or saddle-like) swaps from one configuration to the other.

To carry out the construction of the sectorial decomposition we need to follow singularities as λ varies. Although the *set* $\text{Sing}(Z_\lambda)$ depends continuously on λ , it is not possible to *mark* and follow continuously singularities, as these get exchanged by turning around the bifurcation value of λ (corresponding to $P_0(x) = x^2$). This phenomenon has a prominent bearing on the construction of sectors, since for the same value of λ one obtains dynamically non-equivalent coverings, as illustrated in Figure 1.4. We resolve the ambiguity in the labeling of the singularities by using the two-fold branched covering

$$\hat{P}_\bullet : s \in \mathbb{C} \mapsto (x - s)(x + s)$$

and take s , which determines completely the position of the roots $\pm s$, as new parameter for the constructions.

Definition 5. Let us call Param the complex line \mathbb{C} viewed as the s -space. The branched covering

$$\begin{aligned} \lambda : \text{Param} &\longrightarrow \mathbb{C} \\ s &\longmapsto \lambda(s) := -s^2, \end{aligned}$$

satisfying the identity

$$P_{\lambda(s)} = \widehat{P}_s ,$$

is called the **canonical re-parameterization**. We will actually use only strict subsectors of Param, given for $\rho > 0$ by

$$\Sigma := \left\{ s : 0 < |s| < \rho , |\arg s| < \frac{2\pi}{3} \right\}$$

(where the principal determination of the argument on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is used). For each $s \in \text{Param}$ let us define

$$\begin{aligned} s^n &:= (s, 0) \\ s^s &:= (-s, 0) , \end{aligned}$$

which are the singularities of X_λ .

Remark 8. Notice that $\lambda(\Sigma \cup \{0\}) = (\mathbb{C}, 0)$ so every original parameter is covered this way. The explanation as to why we cannot take $\Sigma \cup \{0\} = (\mathbb{C}, 0)$ will be given in the course of the upcoming sections (especially Section 1.7.3).

The next properties will be used without explicitly referencing the trivial lemma beneath.

Lemma 5.

1. The automorphism group of the covering λ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.
2. The critical set Δ of λ is the origin.
3. A (germ of a) holomorphic function $\widehat{f} : \text{Param} \rightarrow \mathbb{C}$ factors as $f \circ \lambda$ with f holomorphic if, and only if, \widehat{f} is even (i.e. $\mathbb{Z}/2\mathbb{Z}$ -invariant).

Definition 6. In all the remaining text, we make the following notational conventions:

- ⊗ when an object Ω is subscripted with « s » we imply $s \mapsto \Omega_s$ depends on s in a (holomorphic, continuous) way on Σ ,
- ⊗ when an object $\widetilde{\Omega}$ is subscripted with « λ » we imply $\lambda \mapsto \widetilde{\Omega}_\lambda$ depends (formally, holomorphically, continuously) on $\lambda \in (\mathbb{C}, 0)$.

1.7 Canonical sectors

1.7.1 Splitting vector fields

The boundary of a squid sector will be defined by real trajectories of vector fields

$$\Xi_s(x) := \vartheta P_\lambda(x) \frac{\partial}{\partial x}$$

for some suitable choice of a direction $\vartheta = \vartheta(s) \in \mathbb{S}^1$. In order to ensure the upcoming construction matches our needs, we must ensure that the vector field is sufficiently generic. Let us explain what we mean by describing some dynamical data attached to the planar real-analytic foliation \mathcal{F} induced by Ξ_s on $\overline{\mathbb{C}}$.

- ⊗ $\text{Sing}(\mathcal{F}) = P_\lambda^{-1}(0)$, in particular \mathcal{F} is regular near ∞ . We call **local separatrix** of Ξ_s any one of the two leaves of $\mathcal{F}|_{(\overline{\mathbb{C}}, \infty) \setminus \{\infty\}}$ accumulating on ∞ . We call separatrix the corresponding leaf in $\mathcal{F}|_{\mathbb{C}}$.
- ⊗ Because Ξ_s is holomorphic, \mathcal{F} is free from limit (poly)cycles. Therefore the fate of a non-singular trajectory Γ of Ξ_s can only be one of the following (Bendixon-Poincaré theorem).
 - ⊙ Γ is a separatrix and its adherence links ∞ to either a singular point $\pm s$ of Ξ_s , or ∞ . In the former case we say that Γ **lands** at $\pm s$. In the latter case we say Γ is a **homoclinic connection** (happening exactly when both separatrices meet *en route*).
 - ⊙ Γ connects (asymptotically) both singular points s and $-s$.
 - ⊙ Γ is a non-isolated simple loop. In that case both $\pm s$ are center points.

Definition 7. Take $s \in \text{Param}$. We say that Ξ_s is **splitting** if it admits no homoclinic connection.

Lemma 6. *The following conditions are equivalent.*

1. Ξ_s is splitting.
2. There exists a leaf connecting s and $-s$.
3. $\vartheta P'_\lambda(\pm s) \notin i\mathbb{R}$.

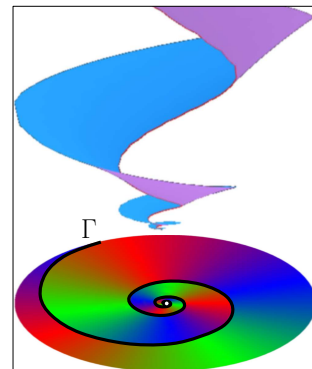
Let us briefly explain why this lemma holds. When $s \neq 0$ the vector field Ξ_s is locally linearizable around each root $\pm s$ of P_λ , and its linear part is given by $\vartheta P'_\lambda(\pm s)(x \mp s) \frac{\partial}{\partial x}$. Therefore $\vartheta P'_\lambda(\pm s)$ is purely imaginary (non-zero) if, and only if, $\pm s$ is a center point of Ξ_s .

- ⊗ When $\pm s$ is a center, it lies within an open basin B^\pm of periodic trajectories. In particular no leaf can connect s and $-s$. Moreover ∂B^\pm must be a homoclinic connection, and Ξ_s is not splitting.
- ⊗ When $\pm s$ is not a center, it is either a focus or a sink. In any case it lies within an open basin of attraction B^\pm (respectively in forward or backward time) and any integral curve of Ξ_s crossing the basin must accumulate on $\pm s$ one way or the other. There are no trajectory accumulating on $\pm s$ in both directions.

1.7.2 Transvestite hyperbolic points

For $l \in \mathbb{C}^\times$ we consider the linear vector field

$$W(x, y) := lx \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.$$



Save for the separatrix $\{x = 0\}$, the leaves of \mathcal{F}_W are included in level sets of the (in general) multivalued first integral

$$H(x, y) := yx^{-1/l} .$$

The space of leaves of \mathcal{F}_W is the quotient

$$\Omega_W := \mathbb{C}^\times / \mathbb{Z} \cup \{0\} ,$$

where 0 represents the separatrices $\{xy = 0\}$. The quotient corresponds to the multiplicative action of \mathbb{Z} on the space of initial values \mathbb{C}^\times

$$y \longmapsto y \exp^{-2i\pi n/l} \quad , \quad n \in \mathbb{Z} ,$$

encoding the monodromy of H .

Lemma 7.

1. If $l \notin \mathbb{R}$ then $\Omega_W \setminus \{0\}$ is a torus. The only neighborhood of 0 in Ω_W is Ω_W .
2. If $l \in \mathbb{Q}$ then Ω_W is a complex line.
3. If $l \in \mathbb{R} \setminus \mathbb{Q}$ then Ω_W is the quotient $\mathbb{S}^1 / \mathbb{Z}$ by the irrational rotation $y \mapsto y \exp^{-2i\pi/l}$ (not Hausdorff).

Choose $\vartheta \in \mathbb{S}^1$ and pick a real-time trajectory of $\vartheta l x \frac{\partial}{\partial x}$, given by

$$t \in \mathbb{R} \longmapsto x(t) := x_* \exp(\vartheta l t) \quad , \quad x_* \in \mathbb{C}^\times .$$

We can lift this path into \mathcal{F}_W through the projection

$$\Pi : (x, y) \longmapsto x ,$$

starting from some $(x_*, y_*) \in \mathbb{C}^\times \times \mathbb{C}$. We obtain the path tangent to W

$$t \in \mathbb{R} \longmapsto (x(t), y(t)) \quad , \quad y(t) = y_* \exp(\vartheta t)$$

satisfying the identity $H(x(\bullet), y(\bullet)) = \text{cst}$.

Notice that

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} x(t) = 0 &\iff \pm \Re(\vartheta l) < 0 \\ \lim_{t \rightarrow \pm\infty} y(t) = 0 &\iff \pm \Re(\vartheta) < 0 \text{ or } y_* = 0 . \end{aligned}$$

Definition 8. For given $l \in \mathbb{C}^\times$ we say that $\vartheta \in \mathbb{S}^1$ is a **saddle-direction** (*resp.* **node-direction**) for l if:

- ⊗ $\Re(\vartheta) > 0$,
- ⊗ $\Re(\vartheta l) < 0$ (*resp.* $\Re(\vartheta l) > 0$).

Remark 9. Just pointing out the obvious.

1. l admits a saddle-direction if, and only if, $l \notin \mathbb{R}_{>0}$ (i.e. W is not a node).
2. l admits a node-direction if, and only if, $l \notin \mathbb{R}_{<0}$ (i.e. W is not a saddle).

For such a choice of ϑ , the curve $t \mapsto x(t)$ is a spiral (specializing to a straight line when $\vartheta l \in \mathbb{R}$). Consider the domain

$$V := \mathbb{C} \setminus \exp(\vartheta l \overline{\mathbb{R}})$$

obtained by slitting the complex line along the adherence Γ of a real integral curve of $\vartheta l x \frac{\partial}{\partial x}$, and build the fibred domain of the complex plane

$$\mathcal{V} := V \times \mathbb{C} .$$

Then $\mathcal{F}_W|_{\mathcal{V}}$ is **saddle-like** (*resp.* **node-like**) in the sense that only one (*resp.* every) leaf accumulates on 0. Notice that a saddle-like (*resp.* node-like) singularity is reached in positive (*resp.* negative) time. Also, (any determination of) the first integral H on \mathcal{V} is holomorphic. The following properties are immediate to establish.

Lemma 8.

1. $\mathcal{F}_W|_{\mathcal{V}}$ is saddle-like if, and only if, for every $\mathcal{U} = (\mathbb{C}^2, 0)$ we have $H(\mathcal{U} \cap \mathcal{V}) = (\mathbb{C}, 0)$ and its diameter goes to 0 as that of \mathcal{U} does.
2. $\mathcal{F}_W|_{\mathcal{V}}$ is node-like if, and only if, for every $\mathcal{U} = (\mathbb{C}^2, 0)$ we have $H(\mathcal{U} \cap \mathcal{V}) = \mathbb{C}$.

1.7.3 Sectorial decomposition

We work here within a fixed formal class μ_{\bullet} . We find a covering of $(\mathbb{C}, 0) \setminus \{\pm s\}$ by two squid sectors V_s^{ns} and V_s^{sn} attached to $\pm s$, such that near $s^{\text{n}} = (s, 0)$ (*resp.* $s^{\text{s}} = (-s, 0)$) the linear part of X_{λ} defines a node-like (*resp.* saddle-like) foliation over both sectors, except for forbidden values of s we shall describe afterward.

The linear part of X_{λ} at the singularity $(\pm s, 0)$ is

$$\pm 2s(x \mp s) \frac{\partial}{\partial x} + y(1 \pm \mu_{\lambda} s) \frac{\partial}{\partial y} ,$$

while its local analytic invariant is given by

$$l_s^{\pm} := \frac{\pm 2s}{1 \pm \mu_{\lambda} s} . \tag{1.30}$$

Both functions $s \mapsto l_s^{\pm}$ are holomorphic on $(\mathbb{C}, 0)$. Notice that

$$l_s^+ = l_{-s}^-$$

and, when $s \neq 0$,

$$\frac{1}{l_s^+} + \frac{1}{l_s^-} = \mu_{\lambda} .$$

Definition 9. The curves

$$\{s \neq 0 : \pm l_s^\pm < 0\}$$

are called the **forbidden curves** associated to μ_\bullet . Values of the parameter for «+» (*resp.* «-») correspond to configurations where X_λ is a saddle at s^n (*resp.* a node at s^s).

Till the end of the section we consider the principal determination of the argument on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. For each $s \in \Sigma$ set

$$\vartheta(s) := \exp \frac{-i \arg s}{2}. \quad (1.31)$$

Since

$$l_s^\pm \sim_0 \pm 2s$$

the following lemma holds.

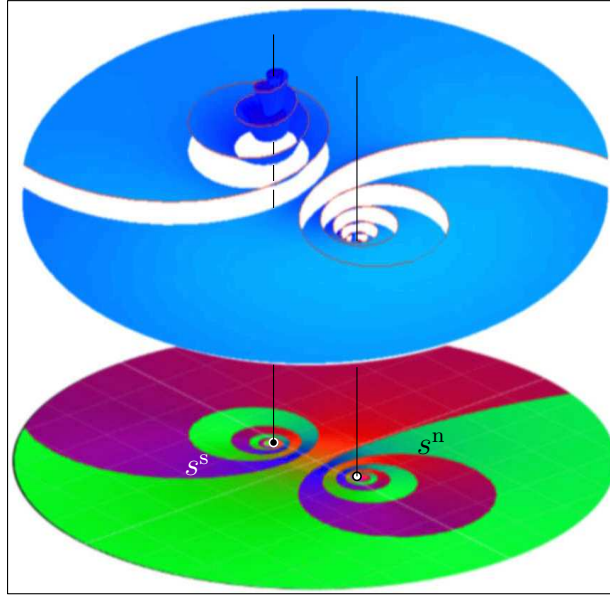
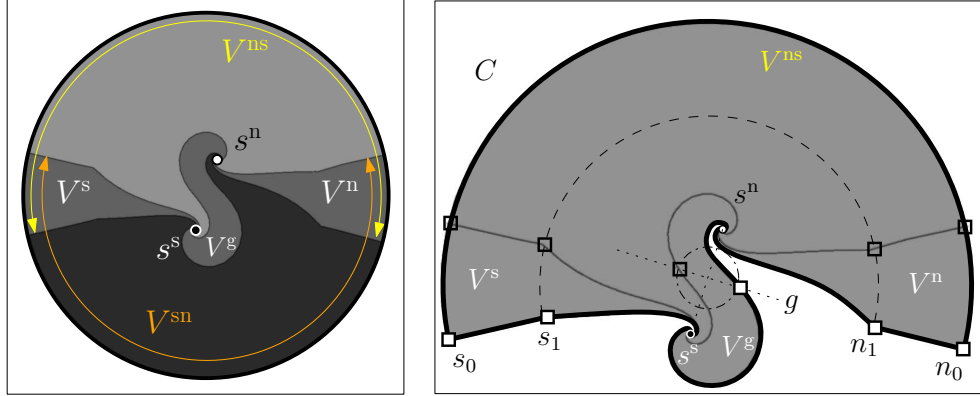


Fig. 1.6: The typical leaf above a slit disc. (Modulus represented as a height-map, argument as colors in the base)

Lemma 9. *There exists $\rho > 0$ so that for all $s \in \Sigma$*

$$\begin{aligned} |\arg(\vartheta(s) l_s^+)| &< \frac{3\pi}{8} \\ |\arg(\vartheta(s) l_s^-)| &> \frac{5\pi}{8}. \end{aligned} \quad (1.32)$$

In particular $\vartheta(s)$ is a node-direction (resp. a saddle-direction) for l_s^+ (resp. l_s^-), and Σ meets no forbidden curve.



(a) A sectorial covering by squid sectors

(b) Structure of a single squid sector. White squares represent construction points of V_s^{ns} , while unfilled squares do those of V_s^{sn}

Fig. 1.7

Definition 10. We refer to Figure 1.7 (b). Let $r > 2\rho > 0$ and $s \in \Sigma \cup \{0\}$. We recall the vector field

$$\Xi_s = \vartheta(s) P_\lambda \frac{\partial}{\partial x}$$

where ϑ is given by (1.31). The **squid sector** V_s^{ns} of radius r is the domain bounded by

- ⊗ the forward trajectory of $P_0 \frac{\partial}{\partial x}$ starting from $n_0 := r \exp \frac{-i\pi}{8}$ till it reaches $2|s|$ at a point n_1 , then the complete forward trajectory of Ξ_s ,
- ⊗ the backward trajectory of $P_0 \frac{\partial}{\partial x}$ starting from $s_0 := r \exp \frac{9i\pi}{8}$ till it reaches $2|s|$ at a point s_1 , then the complete backward trajectory of Ξ_s ,
- ⊗ the integral curve of Ξ_s passing through the «outward» point of intersection g between the perpendicular bisector of $[-s, s]$ and a circle of radius small enough not to meet the already built paths (of the order of $|s|(1 + \cos \arg s)$),
- ⊗ the circular arc $C := r \exp i \left[-\frac{i\pi}{8}, \frac{9i\pi}{8} \right]$.

The squid sector V_s^{sn} of radius r is build in much the same way, replacing the circular arc C by $r \exp i \left[-\frac{9i\pi}{8}, \frac{i\pi}{8} \right]$.

We mention without proof the next descriptive lemma.

Lemma 10. (See [19])

1. The intersection $V_s^{ns} \cap V_s^{sn}$ has three components if $s \in \Sigma$:

- ⊗ a **saddle-part** V_s^s having only s^s in its adherence,
- ⊗ a **node-part** V_s^n having only s^n in its adherence,
- ⊗ and if $s \neq 0$, a **gate-part** V_s^g having both points in its adherence.

When $s = 0$ we define the saddle- or node-part as the components crossing $\mathbb{R}_{<0}$ or $\mathbb{R}_{>0}$ respectively.

2. As $s \rightarrow 0$ in Σ , the squid sectors $V_s^\#$ tends (for the Hausdorff distance) to the sector $V_0^\#$ associated to the saddle-node Z_0 .
3. The length of $\partial V_s^\#$ is uniformly bounded for $s \in \Sigma$.

Remark 10. Item 3. above is really important in order to get uniform bounds in $s \in \Sigma$ for functions obtained by integrating over spirals included in $V_s^\#$, which includes almost all the upcoming material.

1.8 Local classification

We work here within a fixed formal class μ_\bullet . We take $r, r' > 0$ and $\rho > 0$ sufficiently small so that

$$\Sigma \times \mathcal{U} := \left\{ s : 0 < |s| < \rho, |\arg s| < \frac{2\pi}{3} \right\} \times (r\mathbb{D} \times r'\mathbb{D})$$

is a domain on which:

- ⊗ every data appearing in the preparation Theorem 6 is holomorphic and bounded,
- ⊗ $(s, x) \mapsto \frac{P_\lambda(x)}{1+\mu_\lambda x}$ is holomorphic and bounded,
- ⊗ Lemma 9 holds.

The actual values of ρ, r, r' may be implicitly decreased finitely many times in the course of the construction. For any $s \in \Sigma$ we denote $V_s^\#$ the squid sector of radius r associated to s built in Definition 10. We follow now the strategy introduced for the formal classification, only on canonical sectors

$$\mathcal{V}_s^\# \subset V_s^\# \times r'\mathbb{D} \quad , \quad \# \in \{n, ns, s, sn, g\} .$$

Before building the sectors, let us first introduce the space

$$\text{Holo}_c(\mathcal{D}) := \left\{ f_\bullet \in C^0(\text{adh}(\mathcal{D})) : f_\bullet \in \text{Holo}(\mathcal{D}), (\forall s \in \text{adh}(\Sigma)) f_s \in \text{Holo}(\mathcal{V}_s) \right\} \quad (1.33)$$

where \mathcal{D} is a subdomain $\Sigma \times \mathbb{C}^n$ of the form

$$\mathcal{D} = \bigcup_{s \in \Sigma} \{s\} \times \mathcal{V}_s .$$

The key point is to provide weak sectorial separatrices $\mathfrak{s}_\bullet^\#$ and solutions $F_\bullet^\#$ to cohomological equations (1.26) which belong to $\text{Holo}_c(D^\#)$ and $\text{Holo}_c(\mathcal{D}^\#)$ respectively, with

$$\begin{aligned} \mathcal{D}^\sharp &:= \bigcup_{s \in \Sigma} \{s\} \times \mathcal{V}_s^\sharp, \\ D^\sharp &:= \bigcup_{s \in \Sigma} \{s\} \times V_s^\sharp. \end{aligned} \tag{1.34}$$

1.8.1 Sectorial weak separatrices

Theorem 8. (See [19]) *Up to decrease ρ , r , r' there exists two unique families of functions $\mathfrak{s}_s^\sharp \in \text{Holoc}(D^\sharp)$, called in the following **sectorial (weak) separatrices**, such that for any $s \in \Sigma \cup \{0\}$:*

1. $\{y = \mathfrak{s}_s^\sharp(x)\} \subset V_s^\sharp \times r'\mathbb{D}$ is an integral curve of X_λ ,
2. $\lim_{x \rightarrow \pm s} \mathfrak{s}_s^\sharp(x) = 0$,
3. $\mathfrak{s}_s^{\text{ns}}(x) = \mathfrak{s}_s^{\text{sn}}(x)$ for all $x \in V_s^{\text{g}} \cup V_s^{\text{s}}$.

1.8.2 Asymptotic paths and canonical sectors

Definition 11. Let $s \in \Sigma \cup \{0\}$. An **asymptotic path** over V_s^\sharp ending at $p \in \mathcal{U} \cap (V_s^\sharp \times \mathbb{C})$ is a regular, smooth curve $\gamma : \mathbb{R}_{\leq 0} \rightarrow V_s^\sharp \times \mathbb{C}$ meeting the next requirements.

- ⊗ $\gamma(0) = p$.
- ⊗ $\dot{\gamma} = cX_\lambda \circ \gamma$ for some smooth function c . In other words, γ is tangent to X_λ or, equivalently, its image is contained in a single leaf of \mathcal{F}_λ .
- ⊗ $\lim_{t \rightarrow -\infty} \gamma(t) = s^n$.

We abusively write $\gamma : (s^n \rightarrow p)$ or $(s^n \rightarrow p)$ to stand for such an asymptotic path ending at p .

Theorem 9. (See [19]) *Up to decrease ρ , r , r' the following properties hold for $s \in \Sigma \cup \{0\}$.*

1. The sets (called **canonical sectors**)

$$\mathcal{V}_s^\sharp := \{p \in \mathcal{U} \cap (V_s^\sharp \times \mathbb{C}) : \exists (s^n \rightarrow p)\} \quad , \quad \sharp \in \{\text{ns}, \text{sn}\}$$

are domains containing smaller sectors $V_s^\sharp \times \tilde{r}\mathbb{D}$ with $\tilde{r} > 0$ independent on s . The union

$$\mathcal{V}_s := \mathcal{V}_s^{\text{ns}} \cup \mathcal{V}_s^{\text{sn}}$$

is a pointed neighborhood of $\{x = \pm s\} \cap \mathcal{U}$.

2. Each leaf of $\mathcal{F}_\lambda|_{\mathcal{V}_s^\sharp}$ is simply connected, and if γ and $\tilde{\gamma}$ are two asymptotic paths ending at $p \in \mathcal{V}_s^\sharp$ then the **asymptotic cycle**

$$-\gamma \wedge \tilde{\gamma} := \begin{cases} t \leq 0 & \mapsto \tilde{\gamma}(t) \\ t \geq 0 & \mapsto \gamma(-t) \end{cases}$$

is trivial, in the sense that there exists an **asymptotic tangential homotopy** $h : [-\infty, 0] \times \mathbb{R} \rightarrow \mathcal{V}_s^\sharp$ between $-\gamma \wedge \tilde{\gamma}$ and s^n , a mapping such that:

- ⊗ h is uniformly continuous,
- ⊗ $h(0, \bullet) = -\gamma \wedge \tilde{\gamma}$,
- ⊗ $h(-\infty, \bullet) = s^n$,
- ⊗ for every t real $h(\bullet, t) : s^n \rightarrow h(0, t)$,
- ⊗ for every τ negative real $h(\tau, \bullet) : s^n \rightarrow p$.

3. Take $p \in \mathcal{V}_s^{\text{ns}} \cap \mathcal{V}_s^{\text{sn}}$ and consider an asymptotic cycle $\gamma(p)$ obtained by concatenating two asymptotic paths ending at p , one of which is in $\mathcal{V}_s^{\text{ns}}$ and the other one in $\mathcal{V}_s^{\text{sn}}$.

- a. If $p \in \mathcal{V}_s^{\text{g}} \cup \mathcal{V}_s^{\text{n}}$ then $\gamma(p)$ is trivial.
- b. If $p \in \mathcal{V}_s^{\text{s}}$ then $\gamma(p)$ is trivial if, and only if, p belongs to the sectorial weak separatrix. Two such asymptotic cycles $\gamma(p)$ and $\tilde{\gamma}(\tilde{p})$ are tangentially homotopic.

1.8.3 Sectorial solutions to cohomological equations

Theorem 10. (See [19]) For $m \in \Sigma$ define

$$\begin{aligned} \mathcal{V}_s^{\text{nsn}} &:= \mathcal{V}_s \setminus (\mathcal{V}_s^{\text{g}} \cup \mathcal{V}_s^{\text{n}}) \\ \mathcal{D}^{\text{nsn}} &:= \bigcup_{s \in \Sigma} \{s\} \times \mathcal{V}_s^{\text{nsn}} \end{aligned}$$

and take $G_\bullet \in \text{Holo}_c(\mathcal{D}^{\text{nsn}})$ such that $(s, x) \mapsto \frac{G_s(x, 0)}{P_\lambda(x)}$ is bounded. For $s \in \Sigma$ and $p \in \mathcal{V}_s^\sharp$ define

$$F_s^\sharp(p) := \int_{(s^n \rightarrow p)} G_s \frac{dx}{P_\lambda}$$

where $(s^n \rightarrow p)$ is an asymptotic path of X_\bullet ending at p . The following properties hold.

1. $F_s^\sharp(p)$ is an absolutely convergent integral.
2. F_\bullet^\sharp is the unique family of solutions of the cohomological equation $X_\bullet \cdot F_\bullet = G_\bullet$ which belongs to $\text{Holo}_c(\mathcal{D}^\sharp)$ and vanishes at each s^n . Another such solution differs from F_\bullet^\sharp by the addition of a function $f_\bullet \in \text{Holo}_c(\Sigma)$.
- 3.

$$F_s^{\text{sn}}(p) - F_s^{\text{ns}}(p) = \begin{cases} 0 & \text{if } p \in \mathcal{V}_s^{\text{n}} \cup \mathcal{V}_s^{\text{g}} \\ \int_{\gamma(p)} G_s \frac{dx}{P_\lambda} & \text{if } p \in \mathcal{V}_s^{\text{s}} \end{cases}$$

where $\gamma(p)$ is an asymptotic cycle passing through p .

4. The following properties are equivalent.

- a. There exists $F_\bullet \in \text{Holo}_c(\mathcal{D}_s)$ such that $X_\bullet \cdot F_\bullet = G_\bullet$.
- b. For all $s \in \Sigma$ and $p \in \mathcal{V}_s^{\text{s}}$

$$\int_{\gamma(p)} G_s \frac{dx}{P_\lambda} = 0 .$$

The fact that F_s^\sharp is a solution to the cohomological equation can be understood by covering $(s^n \rightarrow p)$ with flow-boxes. In such a local rectifying chart $\Psi : \mathcal{D} \rightarrow (\mathbb{C}^2, \gamma(t))$ one has to solve

$$\frac{\partial F_s^\sharp \circ \Psi}{\partial x} = G_\lambda \circ \Psi$$

so that for $p_*, q_* \in \mathcal{D}$ in the same leaf of $\mathcal{F}_{\frac{\partial}{\partial x}}|_{\mathcal{D}}$ we have

$$F_s^\sharp \circ \Psi(q_*) - F_s^\sharp \circ \Psi(p_*) = \int_{p_* \rightarrow q_*} G_\lambda \circ \Psi dx$$

and *vice versa*. As a consequence $\int_{\gamma(p)} G_s \frac{dx}{P_\lambda}$ encodes the additive monodromy of the analytic continuation of F_s^\sharp . Riemann's removable singularity theorem yields 4.

1.8.4 Sectorial normalization and space of leaves

From Theorem 8 and Proposition 4 we obtain a vector field X_s and a right-hand side $G_s := -P_\lambda R_s^\sharp$ with

$$R_s^\sharp : (x, y) \in \mathcal{V}_s^\sharp \mapsto \frac{R_\lambda(x, y) - R_\lambda(x, \mathfrak{s}_s^\sharp(x))}{y - \mathfrak{s}_s^\sharp(x)} \quad (1.35)$$

satisfying the hypothesis of Theorem 10. The orbital normalization equation (1.28) therefore admits solutions $O_\bullet^\sharp \in \text{Holo}_c(\mathcal{D}^\sharp)$. The situation is the same for the temporal normalization with $\widehat{X}_s := X_\lambda$ and $G_\lambda := \frac{1}{U_\lambda} - \frac{1}{u_\lambda}$, and (1.29) admits solutions $T_\bullet^\sharp \in \text{Holo}_c(\mathcal{D}^\sharp)$.

Corollary 3. Z_\bullet is conjugate to its formal model $\mathfrak{u}_\bullet X_\bullet^\infty$ on \mathcal{D}^\sharp . It is orbitally conjugate to X_\bullet^∞ by (λ, x) -fibred transformations.

The model X_λ^∞ has a first integral $H_\lambda^{\infty, \sharp} \in \text{Holo}(\mathcal{V}^\sharp)$ with connected fibers

$$H_\lambda^{\infty, \sharp}(x, y) := y P_\lambda^{-\mu\lambda/2}(x) \left(\frac{x+s}{x-s} \right)^{1/2s}, \quad s \neq 0$$

$$H_0^{\infty, \sharp}(x, y) := y x^{-\mu} \exp \frac{1}{x},$$

with $\lim_{\lambda \rightarrow 0} H_\lambda^{\infty, \sharp} = H_0^{\infty, \sharp}$ uniformly on compact subsets; for the remaining of the section we only deal with $s \neq 0$ to lighten notations, although everything is valid for $s = 0$ by continuity. Recalling notations and conventions of Section 1.1.3, we choose determinations of the multivalued functions involved above:

- ⊗ $\left(\frac{\bullet+s}{\bullet-s}\right)^{1/2s} = \mathfrak{g}_\lambda$ coincides with the principal determination on V_s^n (the one given in Remark 5 for real s); note that $\mathfrak{g}_\lambda \in \text{Holo}(\mathbb{C} \setminus V_s^{\mathfrak{g}})$,
- ⊗ $P_\lambda^{-\mu\lambda/2}$ agrees with the principal determination on V_s^n .

Invoking Corollary 3 above and the formulas given in Section 1.5.3, we obtain **canonical sectorial first integrals** with connected fibers

$$\begin{aligned} H_s^\sharp(x, y) &:= H_\lambda^{\infty, \sharp}(x, (y - \mathfrak{s}_s^\sharp(x)) \exp O_s^\sharp(x, y)) \\ &= (y - \mathfrak{s}_s^\sharp(x)) P_\lambda^{-\mu\lambda/2}(x) \mathfrak{g}_\lambda(x) \exp O_s^\sharp(x, y) . \end{aligned} \quad (1.36)$$

The next result can be proved by studying the linearized system on squid sectors as in Lemma 8.

Theorem 11. (See [19]) *For $s \in \Sigma$ define the **sectorial spaces of leaves** as*

$$\begin{aligned} \Omega_s^\sharp &:= H_s^\sharp(\mathcal{V}_s^\sharp) \quad , \quad \sharp \in \{\text{ns} , \text{sn}\} \\ \Omega_s^{\flat} &:= H_s^{\text{ns}}(\mathcal{V}_s^{\flat}) \quad , \quad \flat \in \{\text{n} , \text{g} , \text{s}\} . \end{aligned}$$

1. $\Omega_s^{\text{n}} = \Omega_s^{\text{ns}} = \Omega_s^{\text{sn}} = \Omega_s^{\mathfrak{g}} = \mathbb{C}$, the sectorial weak separatrices corresponding to 0.
2. $\Omega_s^{\text{s}} = (\mathbb{C}, 0)$, the sectorial weak separatrices corresponding to 0. The size of Ω_s^{s} goes to 0 as r or r' does.

1.8.5 Classification

1.8.5.1 Orbital necklace

Take $s \in \Sigma \cup \{0\}$. A given point $p \in \mathcal{V}_s^{\text{ns}} \cap \mathcal{V}_s^{\text{sn}}$ corresponds to a point $h^{\text{ns}} \in \Omega_s^{\text{ns}}$ and a point $h^{\text{sn}} \in \Omega_s^{\text{sn}}$. These points must be identified in order to encode the local orbital class of X_λ . Because each H_s^\sharp has connected fibers, the (holomorphic) identifications must be injective.

- ⊗ If $s \neq 0$ and $p \in \mathcal{V}_s^{\mathfrak{g}}$ then every function involved in (1.36) for $\sharp = \text{ns}$ and $\sharp = \text{sn}$ coincide at p except $\mathfrak{g}_\lambda P_\lambda^{-\mu\lambda/2}$. The monodromy of $\mathfrak{g}_\lambda P_\lambda^{-\mu\lambda/2}$ around s^n acts as

$$h^{\text{sn}} = h^{\text{ns}} \exp i\pi \left(\frac{1}{s} + \mu_\lambda \right) =: \psi_s^{\mathfrak{g}}(h^{\text{ns}}) .$$

- ⊗ If $p \in \mathcal{V}_s^{\text{n}}$ then every function involved in (1.36) for $\sharp = \text{ns}$ and $\sharp = \text{sn}$ coincide at p except \mathfrak{s}_s^\sharp . Because $\Omega_s^{\text{n}} = \mathbb{C}$ the mapping ψ_s^{n} must be affine, and it is not hard to check it is a translation:

$$h^{\text{sn}} = h^{\text{ns}} + \varphi_s^{\text{n}} =: \psi_s^{\text{n}}(h^{\text{ns}}) . \quad (1.37)$$

- ⊗ If $p \in \mathcal{V}_s^{\text{s}}$ then every function involved in (1.36) for $\sharp = \text{ns}$ and $\sharp = \text{sn}$ coincide at p except $P_\lambda^{-\mu\lambda/2} \exp O_s^\sharp$. Because 0 corresponds to both sectorial weak separatrices, and because those two functions agree on $V_s^{\mathfrak{s}}$ the mapping $\psi_s^{\mathfrak{s}}$ must fix the point 0:

$$h^{\text{sn}} = h^{\text{ns}} \exp 2i\pi\mu_\lambda + o(h^{\text{ns}}) =: \psi_s^{\text{s}}(h^{\text{ns}}) \in \text{Diff}(\mathbb{C}, 0) . \quad (1.38)$$

Definition 12. Let $\mu_\bullet \in \mathbb{C} \setminus \{\lambda\}$ be given.

1. Take

$$\varphi_\bullet := (\varphi_\bullet^{\text{n}}, \varphi_\bullet^{\text{s}}) \in \text{Holo}_c(\Sigma) \times \text{Holo}_c(\Sigma \times (\mathbb{C}, 0))$$

with $\varphi_\bullet^{\text{s}}(0) = 0$. We call **orbital necklace** associated to φ_\bullet (and μ_\bullet implicitly) the complex manifold $\Omega(\varphi_\bullet)$ obtained by the analytic atlas consisting in two copies Ω^{ns} and Ω^{sn} of $(\Sigma \cup \{0\}) \times \mathbb{C}$, with transitions maps

$$\begin{aligned} \psi_\bullet^{\text{n}} : \Omega^{\text{ns}} &\longrightarrow \Omega^{\text{sn}} \\ (s, h) &\longmapsto (s, h + \varphi_s^{\text{n}}) \end{aligned}$$

$$\begin{aligned} \psi_\bullet^{\text{s}} : (\Omega^{\text{ns}}, 0) &\longrightarrow (\Omega^{\text{sn}}, 0) \\ (s, h) &\longmapsto (s, h \exp(2i\pi\mu_\lambda + \varphi_s^{\text{s}}(h))) . \end{aligned}$$

2. A **diffeomorphism** between two necklaces $\Omega(\varphi_\bullet)$ and $\Omega(\tilde{\varphi}_\bullet)$ is the data $(\Psi_\bullet^{\text{ns}}, \Psi_\bullet^{\text{sn}})$ of s -fibred injective, holomorphic mappings inducing a conjugacy between atlases:

$$\Psi_\bullet^{\text{ns}} \circ \psi_\bullet^{\text{b}} = \tilde{\psi}_\bullet^{\text{b}} \circ \Psi_\bullet^{\text{sn}} \quad , \quad \text{b} \in \{\text{n}, \text{s}\}$$

and such that for $s \neq 0$

$$\Psi_\bullet^{\text{ns}} \circ \psi_\bullet^{\text{g}} = \psi_\bullet^{\text{g}} \circ \Psi_\bullet^{\text{sn}} \quad (1.39)$$

where

$$\psi_s^{\text{g}}(h) := h \exp i\pi \left(\frac{1}{s} + \mu_\lambda \right) .$$

We say that the necklaces are **analytically conjugate**.

3. Let Z_\bullet be a generic unfolding of formal orbital class μ_\bullet as in Theorem 7. We call **orbital class** of Z_\bullet the necklace $\text{Orb}(Z_\bullet) := \Omega(\varphi_\bullet)$ where φ_\bullet is built from the mappings (1.37) and (1.38).

Remark 11. We do not take the gate-part identification in account to build the necklace because the construction does not make it at the limit. It is not needed anyway to perform the local classification, because both sectorial normalizing maps always glue on \mathcal{V}_s^{g} . However the actual space of leaves Ω_s on \mathcal{V}_s is the quotient of the corresponding s -fiber of $\text{Orb}(Z_\bullet)$ by ψ_s^{g} for $s \neq 0$. This is why we must include the condition (1.39) to capture all information relating to orbital conjugacy.

Because holomorphic automorphisms of the complex line are rigid there are not many necklaces diffeomorphisms. Hence the orbital necklace of an unfolding is nearly a local invariant, and we prove in the section that it suffices to characterize its local class.

Lemma 11. $(\Psi_\bullet^{\text{ns}}, \Psi_\bullet^{\text{sn}})$ is a diffeomorphism of necklaces if, and only if, there exists $c_\bullet \in \text{Holo}_c(\Sigma)^\times$ such that $\Psi_s^\#(h) = c_s h$.

Proof. One direction is trivial. Assume then that $(\Psi_{\bullet}^{\text{ns}}, \Psi_{\bullet}^{\text{sn}})$ is a diffeomorphism between necklaces. For each $s \in \Sigma \cup \{0\}$ the mapping Ψ_s^{\sharp} is biholomorphic, thus an invertible linear map $h \mapsto c_s^{\sharp} h$. The conjugacy equation $\Psi_s^{\text{ns}} \circ \psi_s^{\text{g}} = \psi_s^{\text{g}} \circ \Psi_s^{\text{sn}}$ implies $c_s^{\text{ns}} = c_s^{\text{sn}}$ for $s \neq 0$, thus also for $s = 0$ by continuity. \square

1.8.5.2 Integral representation of the saddle orbital invariant

Thanks to Theorem 8 we know the sectorial separatrices \mathfrak{s}_s^{\sharp} glue to a holomorphic function $\mathfrak{s}_s^{\text{nsn}}$ on V_s^{nsn} , therefore so does R_s^{\sharp} appearing in (1.35), yielding a function $R_{\bullet}^{\text{nsn}} \in \text{Hol}_c(\mathcal{D}^{\text{nsn}})$.

Proposition 5. *Let $G_{\bullet} \in \text{Hol}_c(\mathcal{D}^{\text{nsn}})$ such that $(s, x) \mapsto \frac{G_s(x, 0)}{P_{\lambda}(x)}$ is bounded, and let $\text{Orb}(X_{\bullet})$ be the orbital necklace of X_{\bullet} . The mapping (given in the chart Ω^{ns})*

$$\begin{aligned} \mathfrak{T}_{\bullet}(G_{\bullet}) : (\text{Orb}(X_{\bullet}), 0) &\longrightarrow \mathbb{C} \\ (s, h) &\longmapsto \frac{1}{2i\pi} \int_{\gamma(p)} G_s \frac{dx}{P_{\lambda}} , \end{aligned}$$

where $\gamma(p)$ is an asymptotic cycle defined in Theorem 9 Item 2. such that $H_s^{\text{ns}}(p) = h$, is well-defined and vanishes along $\{h = 0\}$. This mapping defines the (linear) **period operator**

$$\mathfrak{T}_{\bullet} : G_{\bullet} \longmapsto \mathfrak{T}_{\bullet}(G_{\bullet}) \in \text{Hol}_c(\text{Orb}(X_{\bullet}), 0) .$$

Proof. Because $X_{\lambda} \cdot F_s^{\text{ns}} = X_{\lambda} \cdot F_s^{\text{sn}} = G_{\lambda}$ the difference $F_s^{\text{sn}} - F_s^{\text{ns}}$ is a first integral of X_{λ} , and therefore factors as a map τ_s defined on the sectorial space of leaves Ω_s^{s} . This can also be seen from Theorem 9. Indeed the value of the integral depends only on the asymptotic tangential homotopy class of $\gamma(p)$, as an asymptotic tangential homotopy is uniformly continuous. Hence only asymptotic cycles with $p \in \mathcal{V}_s^{\text{s}}$ contribute to the period, and the value of the integral only depends on the sectorial leaf $\mathcal{F}_{\lambda}|_{\mathcal{V}_s^{\text{s}}}$, since there is at most one non-trivial homotopy class of asymptotic cycles per leaf. The same argument shows that $\tau_s(0) = 0$, for any asymptotic cycle within the sectorial separatrix is trivial. \square

The transition map $\psi_{\bullet}^{\text{s}}$ of the orbital necklace of X_{\bullet} obey the identity

$$H_s^{\text{sn}}(p) = \psi_s^{\text{s}}(H_s^{\text{ns}}(p))$$

while at the same time

$$H_s^{\sharp}(p) = H_s^{\infty, \sharp}(p) \exp O_s^{\sharp}(p) ,$$

following (1.36). For $p \in \mathcal{V}_s^{\text{s}}$ we have $H_s^{\infty, \text{sn}}(p) = H_s^{\infty, \text{ns}}(p) \exp 2i\pi\mu_{\lambda}$ so that

$$H_s^{\text{sn}}(p) = H_s^{\text{ns}}(p) \exp(2i\pi\mu_{\lambda} + O_s^{\text{sn}}(p) - O_s^{\text{ns}}(p)) .$$

Recalling how we just defined the period operator, we obtain an integral representation for the saddle-part of the orbital invariant.

Corollary 4. *Let $\Omega(\varphi_\bullet^n, \varphi_\bullet^s)$ be the orbital necklace of X_\bullet . Then for all $s \in \Sigma \times \{0\}$*

$$\varphi_s^s = -2i\pi \mathfrak{T}_s(P_\bullet R_\bullet^{\text{nsn}}) .$$

1.8.5.3 Temporal invariant

Definition 13. Let $\mu_\bullet \in \mathbb{C}\{\lambda\}$ and $\mathbf{u}_\bullet \in \mathbb{C}\{\lambda\}[x]_{\leq 1}^\times$ be given.

1. The data of an orbital necklace $\Omega(\varphi_\bullet)$ and

$$f_\bullet \in \text{Holo}_c(\Omega(\varphi_\bullet), 0)$$

with $f_\bullet(0) = 0$ is called **temporal necklace**.

2. A **diffeomorphism** between two temporal necklaces $(\varphi_\bullet, f_\bullet)$ and $(\tilde{\varphi}_\bullet, \tilde{f}_\bullet)$ is the data $(\Psi_\bullet^{\text{ns}}, \Psi_\bullet^{\text{sn}})$ of a diffeomorphism between corresponding orbital necklaces and satisfying:

$$f_\bullet = \tilde{f}_\bullet \circ \Psi_\bullet^{\text{sn}} .$$

We say that the necklaces are **analytically conjugate**.

3. Let $Z_\bullet = U_\bullet X_\bullet$ be a generic unfolding under prepared form (1.25) with formal invariants $(\mu_\bullet, \mathbf{u}_\bullet)$ as in Theorem 7. We call **local class** of Z_\bullet the temporal necklace

$$\text{Class}(Z_\bullet) := \left(\text{Orb}(X_\bullet), \mathfrak{T}_\bullet \left(\frac{1}{U_\bullet} - \frac{1}{\mathbf{u}_\bullet} \right) \right)$$

where the period operator \mathfrak{T}_\bullet is defined in Proposition 5.

1.8.5.4 Classification theorem

Theorem 12. (See [19]) *Two generic unfoldings of multiplicity 1 in the same formal class $(\mu_\bullet, \mathbf{u}_\bullet) \in \mathbb{C}\{\lambda\} \times \mathbb{C}\{\lambda\}[x]_{\leq 1}^\times$ are locally (resp. orbitally) equivalent if, and only if, their temporal (resp. orbital) necklaces are analytically conjugate. In other words we have local classifications*

$$\begin{aligned} \text{Orb} &: \text{Mod}_{\text{loc}}^{\text{orb}}(1) \longrightarrow \text{Holo}_c(\Sigma) \times \text{Holo}_c(\Sigma \times (\mathbb{C}, 0)) / \text{Holo}_c(\Sigma, 0)^\times \\ \text{Class} &: \text{Mod}_{\text{loc}}(1) \longrightarrow \text{Holo}_c(\Sigma) \times \text{Holo}_c(\Sigma \times (\mathbb{C}, 0)) \times \text{Holo}_c(\Sigma \times (\mathbb{C}, 0)) / \text{Holo}_c(\Sigma, 0)^\times \end{aligned}$$

where the action of $\text{Holo}_c(\Sigma, 0)^\times$ is described in Lemma 11.

Proof. Let us present only the orbital part of the proof, the temporal part being easier according to Theorem 10 Item 4. and Proposition 4 Item 1. One way is clear: if Z_\bullet and \tilde{Z}_\bullet are locally orbitally conjugate by Ψ then the conjugacy factors as a diffeomorphism $(\Psi_\bullet^{\text{ns}}, \Psi_\bullet^{\text{sn}})$ between the orbital necklaces $\text{Orb}(Z_\bullet)$ and $\text{Orb}(\tilde{Z}_\bullet)$. Conversely, assume the existence of a diffeomorphism $(\tilde{\Psi}_\bullet^{\text{ns}}, \tilde{\Psi}_\bullet^{\text{sn}})$ conjugating the necklaces $\text{Orb}(Z_\bullet)$ and $\text{Orb}(\tilde{Z}_\bullet)$. Invoking Lemma 11 we can apply the rescaling $(s, x, y) \mapsto (s, x, c_s y)$ to \tilde{Z} . Without changing

notations, in the new coordinates we must have $\Psi_s^\sharp = \text{Id}$ for each $s \in \Sigma \cup \{0\}$. This particularly implies the identities $\varphi_s^b = \tilde{\varphi}_s^b$ for $b \in \{\mathfrak{n}, \mathfrak{s}\}$. But these quantities measure the obstruction to glue over $\mathcal{V}_s^{\text{ns}} \cap \mathcal{V}_s^{\text{sn}}$ the transitions between corresponding sectorial normalization mappings $\mathcal{Y}_s^\sharp := \tilde{\mathcal{O}}_s^\sharp \circ (\mathcal{O}_s^\sharp)^{-1}$. Hence $\Psi_s|_{\mathcal{V}_s^{\text{ns}}} := \mathcal{Y}_s^{\text{ns}}$ and $\Psi_s|_{\mathcal{V}_s^{\text{sn}}} := \mathcal{Y}_s^{\text{sn}}$ defines a holomorphic conjugacy between X_λ and \tilde{X}_λ on $\mathcal{V}_s = \mathcal{V}_s^{\text{ns}} \cup \mathcal{V}_s^{\text{sn}}$. This mapping is bounded, thus extends biholomorphically to $\text{adh}(\mathcal{V}_s)$ by Riemann's removable singularity theorem.

To conclude the proof we only need to check that $\Psi_s = \Psi_{-s}$. Define $\mathcal{S}_s := \Psi_s^{\circ-1} \circ \Psi_{-s}$ which, by construction, is a symmetry of X_λ , that is $\mathcal{S}_s^* X_\lambda = X_\lambda$. A direct computation at a formal level on $\mathcal{S}_s(x, y) = (x, y + \sum_{n+m>1} S_{n,m}(s) x^n y^m)$ establishes $\mathcal{S}_s = \text{Id}$. \square

The classification presented above cannot be complete, as we explain in the upcoming Section 1.9.4.

1.9 Dynamical interpretation of the orbital necklace

We describe the relationship between the actual dynamics of X_λ (the holonomy of the underlying foliation \mathcal{F}_λ) and what could be called the *necklace dynamics*. We define in the first place what we mean by «the dynamics of X_λ » in Section 1.9.1. It encodes the «monodromy» of the canonical first integrals H_s^\sharp , which really is what the orbital necklace is all about. As X_λ does not depend on the choice of s or $-s$, the splitting of \mathcal{U} into squid sectors is artificially superimposed on the dynamics. By rewording this acknowledgment as a relationship between orbital necklaces $\Omega(\varphi_s)$ and $\Omega(\varphi_{-s})$, we derive the orbital compatibility condition in Section 1.9.3. The temporal compatibility condition will be derived while performing the temporal realization in Section 1.10.1.

We finally use the compatibility condition to characterize even necklaces, corresponding to invariants $\varphi_\bullet = (\varphi_\bullet^{\mathfrak{n}}, \varphi_\bullet^{\mathfrak{s}})$ holomorphic as a function of the initial parameter λ . We show this configuration to be very rare, underlying the lack of completeness of the classification provided by Theorem (12).

1.9.1 Weak holonomy

Fix once and for all $x_* \in V_s^{\mathfrak{n}} \setminus \rho\overline{\mathbb{D}}$ (by construction of the squid sector in Definition 10 this domain is independent on $s \in \Sigma$) and a transverse disc

$$T := \{x = x_*\} \cap \bigcap_{s \in \Sigma} \mathcal{V}_s^{\mathfrak{s}} = \{x_*\} \times (\mathbb{C}, 0) .$$

Because X_λ is in the prepared form (1.25) its integral curves are everywhere transverse to the fibers of the natural projection

$$H : (x, y) \longmapsto x$$

outside $P_\lambda^{-1}(0)$. As a consequence we can lift (smooth) paths in the punctured base $\gamma : [0, 1] \rightarrow \mathbb{D}_\lambda$,

$$\mathbb{D}_\lambda := r\mathbb{D} \setminus P_\lambda^{-1}(0) ,$$

through Π into leaves of \mathcal{F}_λ and starting from points in T . More precisely, being given $p_* := (x_*, y_*) \in T$ there exists a unique (germ at 0 of a) solution

$$t \mapsto \gamma_{p_*}(t) = (\gamma(t), y(t))$$

to the constrained flow-system

$$\dot{\gamma}_{p_*} = \frac{\dot{\gamma}}{P_\lambda} X_\lambda \circ \gamma_{p_*} \quad , \quad \gamma_{p_*}(0) = p_* .$$

Notice that the image of γ_{p_*} is included in a single leaf of \mathcal{F}_λ . Of course if γ is «too long» then γ_{p_*} may eventually escape from \mathcal{U} . On the contrary if γ_{p_*} is defined on the whole $[0, 1]$ we call

$$\mathfrak{h}_\lambda^\gamma(p_*) := \gamma_{p_*}(1)$$

the image of p_* by the **holonomy** \mathfrak{h}^γ of \mathcal{F}_λ along the path γ . The holonomy $\mathfrak{h}_\lambda^\gamma$ is holomorphic and locally invertible. When γ is a loop the holonomy defines a germ of a biholomorphic self-map $(T, p_*) \rightarrow (T, \mathfrak{h}_\lambda^\gamma(p_*))$ of the transversal.

We are particularly interested in the case where γ is a generator γ_s or Γ of the fundamental group $\pi_1(\mathbb{D}_\lambda, x_*)$ when $s \neq 0$.

Definition 14. Let $s \in \Sigma$. Consider a system $\{\Gamma, \gamma_s\}$ of generators of $\pi_1(\mathbb{D}_\lambda, x_*)$ such that $\Gamma = |x_*| \mathbb{S}^1$ and γ_s winds directly once around s and do not around $-s$. The holonomy

$$\mathfrak{h}_s^n := \mathfrak{h}_\lambda^{\gamma_s^n}$$

is called the **(weak) nodal holonomy**. Similarly we name **(weak) holonomy** the mapping $\mathfrak{h}_\lambda^\Gamma$.

We state a consequence of Theorem 8.

Lemma 12. *The nodal holonomy \mathfrak{h}_s^n is an injective holomorphic map on subdomain T_s of T containing both points of intersection of T with the sectorial weak separatrices $\{y = \mathfrak{s}_s^{ns}(x)\} \cup \{y = \mathfrak{s}_s^{sn}(x)\}$.*

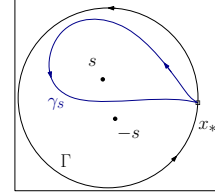


Fig. 1.8: Generators of $\pi_1(\mathbb{D}_\lambda, x_*)$ for $s \in \Sigma$

1.9.2 Necklace holonomy

While walking along a loop $\gamma : [0, 1] \rightarrow \mathbb{D}_\lambda$ and lifting it in the foliation \mathcal{F}_λ to build the holonomy $\mathfrak{h}_\lambda^\gamma$, one may follow what happens in the orbital necklace. More precisely, since the image of γ in the (global) leaves space corresponds with just a point, one may wish to understand the «trajectory» induced by γ in the orbital necklace $\Omega(\varphi_\bullet)$. We recall that

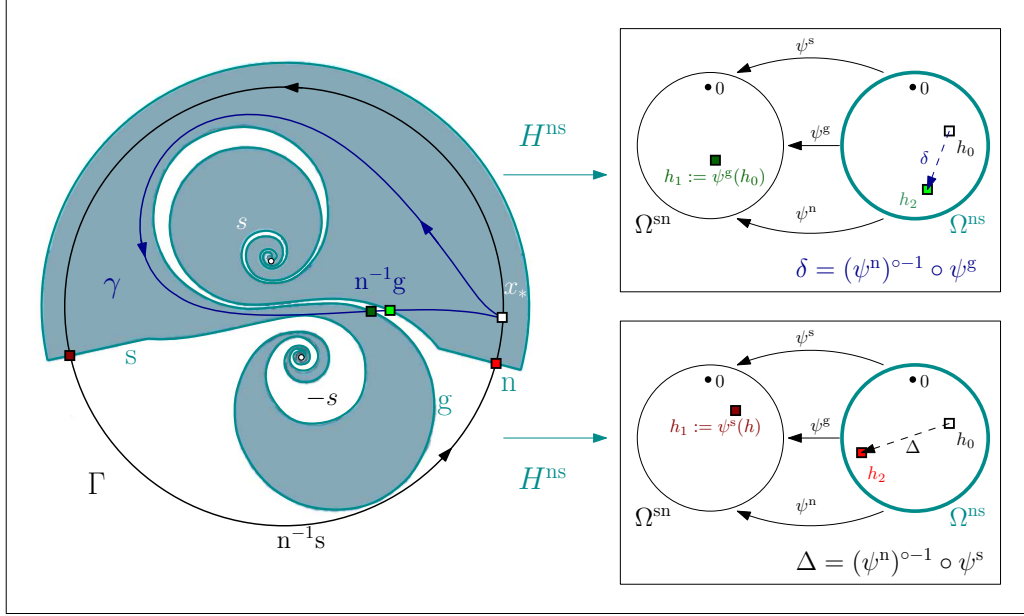


Fig. 1.9: The dynamics induced by the weak holonomies in the orbital necklace

a basepoint $x_* = \gamma(0)$ is fixed once and for all now. The restriction of the canonical first integral to the transverse disc T induces an invertible mapping

$$H_s : (x_*, y) \in T_s \mapsto H_s^{\text{ns}}(x_*, y) \in (\Omega^{\text{ns}}, 0)$$

whose image contains $\{0, \varphi_s^n\}$ (Lemma 12). Starting from $h_0 := H_s(p_*)$ we build a sequence of points $(h_n)_{0 \leq n \leq 2d}$ such that $h_{2m} \in \Omega_s^{\text{ns}}$ and $h_{2m+1} \in \Omega_s^{\text{sn}}$, the connection between h_{n-1} and h_n being given by the action of $(\psi^{b_n})^{\circ \pm 1}$ for $b_n \in \{n, g, s\}$ corresponding to the connected component V_s^b being crossed by γ , the sign being determined by whether the path leaves V_s^{ns} ($\ll + \gg$) or enters it ($\ll - \gg$), as long as the partial lift of γ in \mathcal{F}_λ is defined. We name Δ_s^γ the **necklace holonomy**

$$\Delta_s^\gamma : h_0 \mapsto h_{2d}.$$

The maps Δ_s^γ and \mathfrak{h}^γ represent the same dynamics since they are conjugate:

$$H_s^* \Delta_s^\gamma = \mathfrak{h}_\lambda^\gamma. \quad (1.40)$$

We refer to Figure 1.9 for a depiction of this construction in the case of the weak holonomies \mathfrak{h}_s^n and $\mathfrak{h}_\lambda^\Gamma$.

Let us generalize the construction to abstract orbital necklaces $\Omega(\varphi_\bullet)$. Let \mathfrak{W} be the free group on the letters $\{n, g, s\}$. For every $s \in \Sigma$ there exists a group morphism

$$\begin{aligned} \mathfrak{w}_s &: \pi_1(\mathbb{D}_\lambda, x_*) \longrightarrow \mathfrak{W} \\ \gamma &\longmapsto b_1^{\epsilon_1} \circ \dots \circ b_{2d}^{\epsilon_{2d}} \quad , \quad b_j \in \{n, g, s\} \quad , \quad \epsilon_j \in \{\pm 1\} \end{aligned}$$

defined in such a way that $(b_j, \epsilon_j)_{j \leq 2d}$ is the sequence obtained as before: b_j corresponds to the connected component $V_s^{b_j}$ currently crossed by γ , the sign being determined by whether the path leaves V_s^{ns} or enters it. For instance

$$\begin{aligned} \mathfrak{w}_s(\gamma_s) &= n^{-1}g =: \widehat{\gamma} \\ \mathfrak{w}_s(\Gamma) &= n^{-1}s =: \widehat{\Gamma} . \end{aligned}$$

We omit the proof of the following lemma.

Lemma 13.

1. For every $\gamma \in \pi_1(\mathbb{D}_\lambda, x_*)$ the word $\mathfrak{w}_s(\gamma)$ has the form

$$\mathfrak{w}_s(\gamma) = \prod_{j=1}^d b_{j,1}^{-1} b_{j,0} .$$

2. The image \mathfrak{W} of \mathfrak{w}_s is generated by the words $n^{-1}g$ and $n^{-1}s$:

$$\mathfrak{W} = \langle \widehat{\gamma} , \widehat{\Gamma} \rangle .$$

3. The mapping $\mathfrak{w}_s : \pi_1(\mathbb{D}_\lambda, x_*) \rightarrow \mathfrak{W}$ is bijective. For any $s \in \Sigma$ we write

$$\begin{aligned} \mathfrak{p}_s &: \mathfrak{W} \longrightarrow \pi_1(\mathbb{D}_\lambda, x_*) \\ \widehat{\gamma} &\longmapsto \gamma_s \\ \widehat{\Gamma} &\longmapsto \Gamma \end{aligned}$$

its inverse.

Definition 15. Being given an orbital necklace $\Omega(\varphi_\bullet)$ we build a dynamics in the following manner. Take $s \in \Sigma$ and $w \in \mathfrak{W}$, then denote $\gamma := \mathfrak{p}_s(w)$. We define the **necklace holonomy** associated to γ (or w) as the following symbolic expression

$$\Delta_s^\gamma := \bigcirc_{1 \leq j \leq 2d} \left(\psi_s^{b_j} \right)^{\circ \epsilon_j} .$$

Depending on γ the expression Δ_s^γ may not represent an actual germ of a diffeomorphism, because $\mathfrak{h}_\lambda^\gamma$ may not be geometrically defined. For that reason the map Δ_s^\bullet ranges in the pseudogroup of local diffeomorphisms of $\mathcal{O}_s^{\text{ns}}$. The **necklace holonomy representation** is the collection $\Delta_\bullet = (\Delta_s)_{s \in \Sigma}$ of (pseudo-)group morphisms

$$\Delta_s : w \in \mathfrak{W} \longmapsto \Delta_s(w) := \Delta_s^{\mathfrak{p}_s(w)} .$$

1.9.3 Orbital compatibility condition

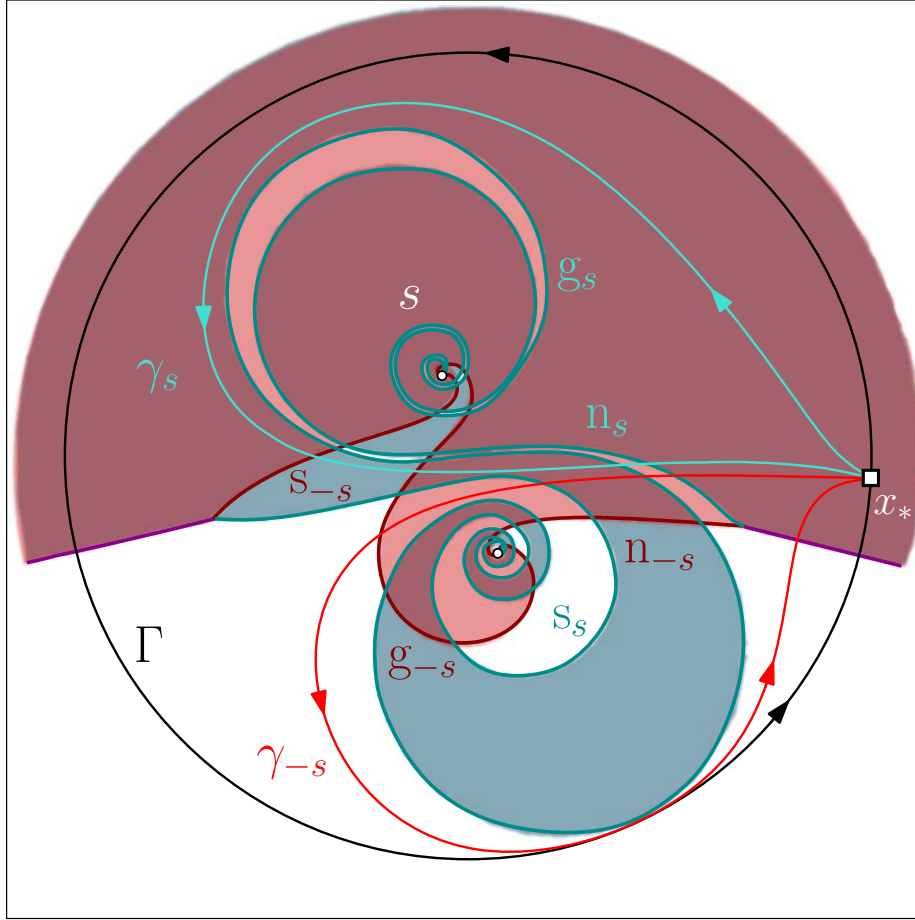


Fig. 1.10: Comparing the necklace dynamics

Define

$$\begin{aligned}
 \Sigma^\cap &:= \Sigma \cap (-\Sigma) \\
 &= \{s \in \Sigma : -s \in \Sigma\} \\
 &= \Sigma^+ \cup \Sigma^- \\
 \Sigma^\pm &:= \Sigma^\cap \cap \{\pm \mathfrak{S}(s) > 0\}
 \end{aligned}$$

the union of two domains on which we can compare the necklace dynamics for s and $-s$. According to (1.40) and Figure 1.10 we have for $s \in \Sigma^\cap$ the identities between actual diffeomorphisms

$$\begin{cases} H_s^* \Delta_s (n^{-1}g) = H_{-s}^* \Delta_{-s} (g^{-1}s) & \text{if } s \in \Sigma^+ \\ H_s^* \Delta_s (n^{-1}g) = H_{-s}^* \Delta_{-s} (n^{-1}sg^{-1}n) & \text{if } s \in \Sigma^- \\ H_s^* \Delta_s (n^{-1}s) = H_{-s}^* \Delta_{-s} (n^{-1}s) \end{cases} . \quad (1.41)$$

In order to motivate the definition of compatibility condition, we must explicit the bridge between relations on Σ^+ and on Σ^- . We obtain this connection by rewording algebraically the topological fact that $\Gamma = \gamma_{-s} \wedge \gamma_s$ if $s \in \Sigma^+$ while $\Gamma = \gamma_s \wedge \gamma_{-s}$ if $s \in \Sigma^-$. The monomorphism

$$\begin{aligned} \sigma : \mathfrak{W} &\longrightarrow \mathfrak{W} \\ \hat{\gamma} &\longmapsto \hat{\gamma}^{-1} \hat{F} \\ \hat{F} &\longmapsto \hat{F} \end{aligned}$$

with inverse

$$\begin{aligned} \sigma^{\circ-1} : \mathfrak{W} &\longrightarrow \mathfrak{W} \\ \hat{\gamma} &\longmapsto \hat{F} \hat{\gamma}^{-1} \\ \hat{F} &\longmapsto \hat{F} \end{aligned}$$

satisfies

$$\mathfrak{p}_s = \mathfrak{p}_{-s} \circ \sigma^{\circ\pm 1} \quad \text{for } s \in \Sigma^\pm .$$

The system (1.41) expresses that the necklace holonomy associated to an actual unfolding is compatible with the latter identity. Yet the system explicitly involves the sectorial first integrals H_\bullet^{ns} and is therefore not intrinsic to the orbital necklace. The key to resolve this issue is to observe that for $s \in \Sigma^+$ the mapping

$$\Delta_{-s} (g^{-1}s) \in \text{Diff}(\mathbb{C}, 0)$$

is hyperbolic and therefore locally analytically linearizable near the fixed-point 0. There exist only one such analytic linearization with prescribed linear part. Because $\Delta_s (n^{-1}g)$ is an affine map the invertible function

$$\eta_s := H_s \circ H_{-s}^{\circ-1} \quad (1.42)$$

is a holomorphic linearization of $\Delta_{-s} (g^{-1}s)$. Hence η_s can be recovered uniquely, up to its linear part, from the knowledge of the orbital necklace.

Definition 16. For an orbital necklace $\Omega(\varphi_\bullet)$ recall the symbolic holonomy representation Δ_\bullet . We say that the orbital necklace $\Omega(\varphi_\bullet)$ is a **compatible orbital necklace** when there exists $\eta_\bullet \text{Holo}_c(\Sigma \times (\mathbb{C}, 0))$ such that for every $s \in \Sigma^+$ (*resp.* $s \in \Sigma^-$) the mapping η_s is a local linearization of $\Delta_{-s} (g^{-1}s)$ (*resp.* $\Delta_{-s} (n^{-1}sg^{-1}n)$) satisfying the next properties.

⊗ For every $s \in \Sigma^\cap$

$$\eta_s \circ \eta_{-s} = \text{Id} .$$

⊗ For every $s \in \Sigma^\cap$

$$\eta_s^* \Delta_s (n^{-1}s) = \Delta_{-s} (n^{-1}s) ,$$

which is equivalent to the conjugacy of the whole dynamics: $\eta_s^* \Delta_s = \Delta_{-s} \circ \sigma^{\circ \pm 1}$ for all $s \in \Sigma^\pm$.

1.9.4 Characterization of even purely convergent unfoldings

For the sake of concision we only deal with the case $\mu_\bullet = 0$.

Proposition 6. *Take a purely convergent generic unfolding of multiplicity 1 with $\mu_\bullet = 0$, i.e. its orbital necklace φ_\bullet satisfies $\varphi_\bullet^n = 0$. There exists $p \in \mathbb{N}$ and $\alpha_\bullet \in \mathbb{C} \setminus \{\lambda\}$ such that*

$$\varphi_\bullet^s(h) = -\frac{1}{p} \log(1 + \alpha_\bullet h^p)$$

if, and only if, $\varphi_\bullet^s = \varphi_{-s}^s$.

Because the local classification for saddle-node vector fields is complete, and since φ_\bullet extends continuously at $s = 0$, the configuration presented in the proposition is rather rare. The classification presented in Theorem 12 is not complete.

We also mention that such unfoldings are locally orbitally conjugate to a normal form (Theorem 5)

$$X_\bullet^\infty + \lambda^\kappa xy^{p+1} \frac{\partial}{\partial y} , \quad \kappa \in \overline{\mathbb{N}} .$$

As in [24] it is indeed possible to show that for the above normal form one has

$$\varphi_\bullet^s(h) = -\frac{1}{p} \log(1 - 2i\pi\lambda^\kappa \mathfrak{T}_\bullet(xy^p)(h)) ,$$

where the period operator is the one associated to the model X_\bullet^∞ . The explicit computations done in Corollary 6 therefore proves our claim, as well as one direction of the proposition since the period $\mathfrak{T}_\bullet(xy^p)$ is actually even.

Let us prove the other direction. If φ_\bullet^s is even then the orbital compatibility condition writes

$$\eta_s^* \psi_s^s = \psi_s^s .$$

Therefore $\langle \eta_s, \psi_s^s \rangle < \text{Diff}(\mathbb{C}, 0)$ is Abelian. Since ψ_s^s is tangent-to-identity the subgroup is rigid, and is conjugate [10, 4] by $\phi_s \in \text{Diff}(\mathbb{C}, 0)$ tangent-to-identity to

$$\begin{aligned} \widehat{G} &:= \left\langle \frac{\beta h}{(1 + \xi_\bullet h^p)^{1/p}} , \frac{h}{(1 + \alpha_\bullet h^p)^{1/p}} \right\rangle \\ &= \left\langle \widehat{\eta}_\bullet , \widehat{\psi}_\bullet^s \right\rangle \end{aligned}$$

for some $p \in \mathbb{N}$, $\alpha_\bullet, \xi_\bullet \in \text{Holo}_c(\Sigma)$ and $\beta^p = 1$. Taking into account the fact that η_s linearizes $\Delta_{-s}(g^{-1}s) = \ell_s \psi_s^s$ for $\ell_s := \exp \frac{i\pi}{s}$, we obtain the identity

$$L_s^* \widehat{\eta}_s = \left(\widehat{\psi}_s^s \right)^{\circ -1} \circ \widehat{\eta}_s = \frac{\beta h}{(1 + (\xi_\bullet - \alpha_\bullet) h^p)^{1/p}}$$

where

$$L_s := \phi_s^{\circ -1} \circ (\ell_s \phi_s) .$$

Hence L_s must be of the form

$$L_s(h) = \frac{\ell_\bullet h}{(1 + \delta_s h^p)^{1/p}} ,$$

and we deduce

$$\xi_\bullet = \frac{\alpha_\bullet}{1 - \ell_\bullet^p} . \quad (1.43)$$

Similarly ϕ_s linearizes L_s , and therefore

$$\phi_s(h) = \frac{h}{(1 + \chi_s h^p)^{1/p}}$$

for some $\chi_s \in \mathbb{C}$. As a conclusion $\phi_\bullet^* \psi_\bullet^s = \widehat{\psi}_\bullet^s$ and ψ_\bullet^s has the expected form.

Remark 12. In this specific configuration we observe explicitly that the transition mapping η_s cannot be defined for all values of $s \in \Sigma$, because of (1.43).

1.10 Instances of complete classifications

1.10.1 Complete temporal classification

We first describe the range of the period operator (Proposition 5). The next theorem is showed in Section 1.10.1.1.

Theorem 13. *For a generic unfolding X_\bullet of multiplicity 1 we recall its orbital necklace $\text{Orb}(X_\bullet) = \Omega(\varphi_\bullet)$ and associated transition map η_\bullet as in (1.42).*

1. *Let $f_\bullet \in \text{Holo}_c(\text{Orb}(X_\bullet), 0)$ such that $f_\bullet(0) = 0$. There exists $G_\bullet \in \text{Holo}_c(\Sigma \times (\mathbb{C}^2, 0))$ with $G_\bullet = \mathcal{O}(P_\bullet) + \mathcal{O}(y)$ such that $\mathfrak{T}_\bullet(G_\bullet) = f_\bullet$.*
2. *We can find such a function G_\bullet satisfying $G_\bullet = G_{-\bullet}$ on Σ^\cap if, and only if*

$$(\forall s \in \Sigma^+) \quad \sum_{n=0}^{\infty} f_s \circ \Delta_s \left((g^{-1}s)^n \right) \circ \eta_s = \sum_{n=0}^{\infty} f_{-s} \circ \Delta_{-s} \left((s^{-1}g)^n \right) . \quad (1.44)$$

Notice that because $\Delta_s(g^{-1}s)$ and $\Delta_{-s}(s^{-1}g)$ are tangent to $\exp i\pi(-\frac{1}{s} + \mu_\lambda)\text{Id}$ and $\exp i\pi(-\frac{1}{s} - \mu_\lambda)\text{Id}$ respectively, both sums converge geometrically.

Definition 17. Let $(\varphi_\bullet, f_\bullet)$ be a temporal necklace (Definition 13). We say that it is a **compatible temporal necklace** if $\Omega(\varphi_\bullet)$ is a compatible orbital necklace and if f_\bullet fulfills the condition (1.44). This is equivalent to the following statement: for $s \in \Sigma^+$ let $\phi_{\pm s}$ denote the unique solution vanishing at 0 of the discrete cohomological equations

$$\begin{cases} \phi_s - \phi_s \circ \Delta_s(g^{-1}s) & = f_s \\ \phi_{-s} - \phi_{-s} \circ \Delta_{-s}(s^{-1}g) & = f_{-s} \end{cases},$$

then

$$\eta_s^* \phi_s = \phi_{-s}.$$

Remark 13. It is sufficient to ensure the relation holds on Σ^+ because the orbital necklace is compatible. We refer to Remark 14 for more details.

Corollary 5. *Being given X_\bullet define*

$$\text{Fol}(X_\bullet) := \left\{ U_\bullet X_\bullet : U_\bullet \in \text{Holo}(\mathbb{C}^3, 0)^\times \right\}$$

and

$$\text{Compat}(X_\bullet) := \{ f_\bullet \in \text{Holo}_c(\Sigma \times (\mathbb{C}, 0)) : f_\bullet \text{ satisfies (1.44)} \}.$$

1. *We have a complete classification*

$$\text{Mod}_{\text{loc}}(\text{Fol}(X_\bullet)) \simeq \text{Compat}(X_\bullet) / \text{Holo}_c(\Sigma, 0)^\times.$$

2. *If $X_\bullet = X_\bullet^\infty$ we have normal forms*

$$\text{NF}_{\text{loc}}(\text{Fol}(X_\bullet^\infty)) := \left\{ \frac{\mathbf{u}_\bullet}{1 + \mathbf{u}_\bullet G_\bullet} : \mathbf{u}_\bullet \in \mathbb{C}\{\lambda\}[x]_{\leq 1}^\times, G_\bullet \in \text{Section}(1) \right\} X_\bullet^\infty$$

where

$$\tau := \begin{cases} 0 & \text{if } \mu_0 \notin \mathbb{R}_{\leq 0} \\ 1 + \lfloor -\mu_0 \rfloor & \text{otherwise} \end{cases}$$

$$\text{Section}(1) := x^{\tau+1}y\mathbb{C}\{\lambda, x^\tau y\}.$$

Item 1. is a direct restatement of Theorem 13, Theorem 12 and Definition 13 Item 3. We give the proof of Item 2. in Section 1.10.1.2, based on the explicit computation of the period operator for the formal model in terms of the Gamma function. The dominant terms are $\Gamma(1 + m(\tau + \mu_\bullet))$ for $m \in \mathbb{Z}_{\geq 0}$: the presence of the monomial x^τ helps keeping far away from poles for small $|s|$.

1.10.1.1 Range of the period operator (proof of Theorem 13)

The proof relies on solving two Cousin problems, one in (x, y) -space and the other one in s -space.

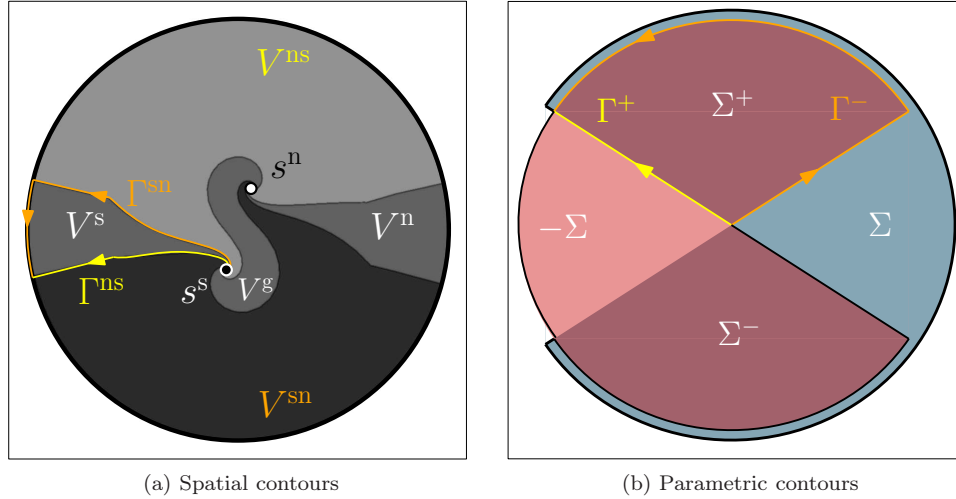


Fig. 1.11: The contour used for the Cauchy-Heine transforms

Lemma 14. Recall the total sectorial spaces \mathcal{D}^\sharp as in (1.34). Given $\delta_\bullet \in \text{Holo}_c(\text{Orb}(X_\bullet), 0)$ such that $s \mapsto \frac{\delta_s}{s}$ is bounded, there exist two functions $F_\bullet^\sharp \in \text{Holo}_c(\mathcal{D}^\sharp)$ such that

$$F_s^{\text{sn}} - F_s^{\text{ns}} = \begin{cases} \delta_s \circ H_s^{\text{ns}} & \text{on } \mathcal{V}_s^{\text{s}} \\ 0 & \text{on } \mathcal{V}_s^{\text{g}} \cup \mathcal{V}_s^{\text{n}} \end{cases}.$$

Proof. This is a slight variation on the classical Cauchy-Heine transform. See Figure 1.11 (a). For $(x, y) \in \mathcal{V}_s^\sharp$ we set

$$F_s^\sharp(x, y) := \frac{1}{2i\pi} \int_{\Gamma_s^\sharp} \frac{\delta_s \circ H_s^{\text{ns}}(z, y)}{z - x} dz,$$

which by hypothesis defines an element of $\text{Holo}_c(\mathcal{D}^\sharp)$ if Γ_s^\sharp is deformed slightly to lie outside $\text{adh}(\mathcal{V}_s^\sharp)$. The rest follows from Cauchy's formula. \square

The same construction applies for the parametric Cousin problem, using the corresponding contours given in Figure 1.11 (b).

Lemma 15. Given $d_\bullet \in \text{Holo}_c(\Sigma^+ \times \mathcal{U})$ such that $s \mapsto \frac{d_s}{s}$ is bounded, there exist a function $D_\bullet \in \text{Holo}_c(\Sigma \times \mathcal{U})$ such that for all $s \in \Sigma^+$ we have $D_s - D_{-s} = d_s$.

Proof. For $s \in \Sigma$ simply set

$$D_s := \frac{1}{2i\pi} \int_{\Gamma^+} \frac{dz}{z-s} + \frac{1}{2i\pi} \int_{\Gamma^-} \frac{dz}{z+s} .$$

□

Let us proceed now with the proof of Theorem 13. Lemma 14, applied to the temporal invariant $\delta_s := f_s$, yields two sectorial functions F_s^\sharp with prescribed difference over the saddle sector. Because $F_s^{\text{sn}} - F_s^{\text{ns}}$ is a first integral of X_λ the function defined by

$$G_s := \begin{cases} X_\lambda \cdot F_s^{\text{ns}} & \text{on } \mathcal{V}_s^{\text{ns}} \\ X_\lambda \cdot F_s^{\text{sn}} & \text{on } \mathcal{V}_s^{\text{sn}} \end{cases}$$

is holomorphic and bounded on \mathcal{V}_s , therefore extends holomorphically to \mathcal{U} . By construction its period coincides with f_s . Because F_s^\sharp has bounded derivatives [19], $G_\bullet \in \text{Hol}_c(\Sigma \times \mathcal{U})$ is of the desired form:

$$G_s(x, y) = P_s(x) \left(\frac{\partial F_s^\sharp}{\partial x} + R_\lambda(x, y) \frac{\partial F_s^\sharp}{\partial y} \right) + y(1 + \mu_\lambda x) \frac{\partial F_s^\sharp}{\partial y} .$$

This proves Item 1.

In general there is no reason for $G_s = G_{-s}$ to hold when $s \in \Sigma^\cap$. We must therefore modify G_\bullet by adding to it a function of the form $X_\bullet \cdot D_\bullet$ with $D_\bullet \in \text{Hol}_c(\Sigma \times \mathcal{U})$, in such a way that $\tilde{G}_\bullet := G_\bullet + X_\bullet \cdot D_\bullet$ be even. The equation $\tilde{G}_{-s} = \tilde{G}_s$ reads

$$G_s - G_{-s} = X_\lambda \cdot (D_{-s} - D_s) .$$

According to Theorem 10 Item 4. we need the identity

$$f_s = \mathfrak{T}_s(G_s) = \mathfrak{T}_s(G_{-s})$$

to hold for $s \in \Sigma^\cap$. We prove below the hypothesis of Theorem 13 Item 2. guarantees that very property. Taking this fact for granted, we deduce the existence of d_\bullet such that $G_s - G_{-s} = X_\lambda \cdot d_s$ for all $s \in \Sigma^+$ using Theorem 10 Item 4. again. This function can be so chosen that $\lim_{s \rightarrow 0} d_s = 0$, in which case $s \mapsto \frac{d_s}{s}$ is bounded. Then Lemma 15 yields the expected D_\bullet , completing the proof of Item 2.

Remark 14. From Theorem 10 Item 4. we know that $\mathfrak{T}_s(G_s - G_{-s}) = 0$ for all $s \in \Sigma^+$ if, and only if, $\mathfrak{T}_{-s}(G_s - G_{-s}) = 0$ for all $s \in \Sigma^+$. Therefore $f_s = \mathfrak{T}_s(G_{-s})$ for all $s \in \Sigma^\cap$ if, and only if, the equality holds merely on Σ^+ .

Proposition 7.

1. For all $s \in \Sigma^+$ we have

$$\eta_s^* \sum_{n=0}^{\infty} \mathfrak{T}_s(G_{-s}) \circ \Delta_s \left((g^{-1}s)^n \right) = \sum_{n=0}^{\infty} f_{-s} \circ \Delta_{-s} \left((s^{-1}g)^n \right) .$$

2. $f_\bullet = \mathfrak{T}_\bullet(G_{-\bullet})$ if, and only if, condition (1.44) holds.

Proof. Set $\phi_s := \sum_{n=0}^{\infty} f_s \circ \Delta_s \left((g^{-1}s)^n \right)$, $\phi_{-s} := \sum_{n=0}^{\infty} f_{-s} \circ \Delta_{-s} \left((s^{-1}g)^n \right)$ and $\tilde{\phi}_s := \sum_{n=0}^{\infty} \mathfrak{T}_s(G_{-s}) \circ \Delta_s \left((g^{-1}s)^n \right)$.

1. Take $p_* \in T$. Any asymptotic cycle $\gamma(p_*)$, used to compute the period in Proposition 5, is tangentially asymptotically homotopic to the lift in \mathcal{F}_λ of the limit of nested cycles $\lim_{m \rightarrow \infty} \gamma_m$

$$\begin{aligned} \gamma_m &:= \mathfrak{p}_{-s} \left((s^{-1}g)^{-m} (n^{-1}s) (s^{-1}g)^m \right) : [-m, m] \longrightarrow \mathbb{D}_\lambda \\ \gamma_{m+1}|_{[-m, m]} &= \gamma_m . \end{aligned} \quad (1.45)$$

We let $\tilde{\gamma}_m$ be the lift of γ_m in \mathcal{F}_λ with $\tilde{\gamma}_m(0) = p_*$. The quantity $F_{-s}^{\text{ns}}(\tilde{\gamma}_m(-m)) + \int_{\tilde{\gamma}_m} G_{-s} \frac{dx}{P_\lambda}$ represent the analytic continuation of F_{-s}^{ns} along $\tilde{\gamma}_m$. By construction the additive monodromy of the continuation of F_{-s}^{ns} is given by the period f_{-s} when turning around the saddle-like singularity of $\mathcal{V}_{-s}^{\text{ns}}$. Hence

$$\begin{aligned} &\int_{\tilde{\gamma}_m} G_{-s} \frac{dx}{P_\lambda} + F_{-s}^{\text{ns}}(\tilde{\gamma}_m(-m)) - F_{-s}^{\text{ns}}(\tilde{\gamma}_m(m)) = \\ &\left(\sum_{n=0}^m f_{-s} \circ \Delta_{-s} \left((s^{-1}g)^{-n} \right) - \sum_{n=1}^m f_{-s} \circ \Delta_{-s} \left((s^{-1}g)^{-n} n^{-1}s \right) \right) \circ H_{-s}^{\text{ns}}(p_*) . \end{aligned}$$

Because F_{-s}^{ns} extends continuously to $\{x = s\}$ and $\lim_{n \rightarrow \infty} \tilde{\gamma}_n(\pm n) = (s, 0)$, taking the limit we obtain in the chart Ω_{-s}^{ns}

$$\mathfrak{I}_s(G_{-s}) \circ \eta_s = \phi_{-s} - \phi_{-s} \circ \Delta_{-s}(s^{-1}gn^{-1}s) .$$

According to Definition 16, for $s \in \Sigma^+$ the identity $\eta_s^* \Delta_s(g^{-1}s) = \Delta_{-s}(s^{-1}gn^{-1}s)$ holds, so that summing over all terms $\eta_s^*(\mathfrak{I}_s(G_{-s}) \circ \Delta_s((g^{-1}s)^n))$ for $n \in \mathbb{Z}_{\geq 0}$ yields the expected result:

$$\eta_s \tilde{\phi}_s = \phi_{-s} .$$

2. The direct implication is trivial. Assume conversely that $\eta_s^* \phi_s = \sum_{n=0}^{\infty} f_{-s} \circ \Delta_{-s}((s^{-1}g)^n)$, i.e. $\phi_s = \tilde{\phi}_s$. Because $\phi_s - \phi_s \circ \Delta_s(g^{-1}s) = f_s$ and $\tilde{\phi}_s - \tilde{\phi}_s \circ \Delta_s(g^{-1}s) = \mathfrak{I}_s(G_{-s})$ we recover $f_s = \mathfrak{I}_s(G_{-s})$. □

1.10.1.2 Computation of the period of the model $\mathbf{X}_\bullet^\infty$

Proposition 8. *Let $G_{n,m}(x, y) := x^n y^m$ and $X_\bullet := X_\bullet^\infty$. Then for all $s \in \Sigma$ we have (in the chart Ω_s^{ns})*

$$\begin{aligned} \mathfrak{I}_s(G_{n,m})(h) &= h^m \times \frac{(-m)^{n+m\mu_\lambda}}{\Gamma(n+m\mu_\lambda)} \times \left(-\frac{2s}{m} \right)^{m\mu_\lambda} \frac{\Gamma\left(-\frac{m}{2s} + \frac{m\mu_\lambda}{2}\right)}{\Gamma\left(-\frac{m}{2s} - \frac{m\mu_\lambda}{2}\right)} \times t_{s,n,m} \\ t_{s,n,m} &:= \frac{1}{2^n} \sum_{p+q=n} \binom{n}{p} \prod_{j=0}^{p-1} \left(1 - s \left(\mu_\lambda + \frac{2j}{m} \right) \right) \prod_{j=0}^{q-1} \left(1 + s \left(\mu_\lambda + \frac{2j}{m} \right) \right) . \end{aligned}$$

Remark 15. Notice that taking the limit $s \rightarrow 0$ in Σ leads to

$$\lim_{s \rightarrow 0} \left(-\frac{2s}{m} \right)^{m\mu_\lambda} \frac{\Gamma\left(-\frac{m}{2s} + \frac{m\mu_\lambda}{2}\right)}{\Gamma\left(-\frac{m}{2s} - \frac{m\mu_\lambda}{2}\right)} = \lim_{s \rightarrow 0} t_{s,n,m} = 1$$

recovering classical computations [7, 23, 24] performed for $\lambda = 0$. Also, if $\mu_\bullet = 0$ then $\mathfrak{T}_\bullet(G_{n,m})$ is a holomorphic function of λ .

Proof. We perform the computation over $V_s^{\text{nsn}} = \mathbb{D}_\lambda \setminus (V_s^g \cup V_s^n)$. Because

$$H_s(x, y) := y(x-s)^{-\frac{1}{2s} - \frac{\mu_l}{2}} (x+s)^{\frac{1}{2s} - \frac{\mu_l}{2}}$$

is constant on the leaves of \mathcal{F}_λ we can parameterize an asymptotic path as

$$x \in \gamma_\infty \mapsto \left(x, h(x-s)^{\frac{1}{2s} + \frac{\mu_l}{2}} (x+s)^{-\frac{1}{2s} + \frac{\mu_l}{2}} \right)$$

where $h = H_s^{\text{ns}}(p_*)$ and $\gamma_\infty = \lim_{p \rightarrow \infty} \gamma_p$ (as in (1.45)) is the projection on $\{y = 0\}$ of the asymptotic cycle $\gamma(p_*)$. This projection does not depend on the choice of $p_* \in T$.

Remembering the computations performed in Proposition 3, we introduce the Pochhammer contour $\mathcal{P}_s \in \pi_1(\mathbb{D}_\lambda, x_*)$ whose encoding in the dynamics necklace is given by

$$\mathfrak{w}_s(\mathcal{P}_s) = \mathfrak{n}^{-1} \mathfrak{g} \mathfrak{s}^{-1} \mathfrak{n} \mathfrak{g}^{-1} \mathfrak{s} .$$

For $h \in (\Omega_s^{\text{ns}}, 0)$ let $\tilde{\mathcal{P}}_s(h)$ be the lift in \mathcal{F}_λ of \mathcal{P}_s starting from $p_* \in T$ with $h = H_s(p_*)$. Both necklace holonomies $\Delta_s(\tilde{\gamma})$ and $\Delta_s(\tilde{\Gamma})$ are linear in the same coordinate $H_s|_T$, and therefore commute. Because the Pochhammer contour is a commutator we have

$$\Delta_s(\mathfrak{n}^{-1} \mathfrak{g} \mathfrak{s}^{-1} \mathfrak{n} \mathfrak{g}^{-1} \mathfrak{s}) = \text{Id} .$$

Hence $\tilde{\mathcal{P}}_s(h)$ is a (non-trivial) element of the fundamental group of the corresponding leaf of \mathcal{F}_λ . As a matter of consequence

$$\oint_{\tilde{\mathcal{P}}_s(h)} G_{n,m} \frac{dx}{P_\lambda} = \mathfrak{T}_s(G_{n,m}) - \mathfrak{T}_s(G_{n,m}) \circ \Delta_s(\mathfrak{g}^{-1} \mathfrak{s}) .$$

Summing over the forward orbit of $\Delta_s(\mathfrak{g}^{-1} \mathfrak{s})$ we obtain

$$\begin{aligned} \mathfrak{T}_s(G_{n,m})(h) &= \sum_{\ell=0}^{\infty} \oint_{\tilde{\mathcal{P}}_s \circ \Delta_s((\mathfrak{g}^{-1} \mathfrak{s})^\ell)(h)} G_{n,m} \frac{dx}{P_\lambda} \\ &= \sum_{\ell=0}^{\infty} \left(\frac{\Delta_s((\mathfrak{g}^{-1} \mathfrak{s})^\ell)(h)}{h} \right)^m \oint_{\tilde{\mathcal{P}}_s(h)} G_{n,m} \frac{dx}{P_\lambda} \\ &= \frac{1}{1 - \exp mi\pi \left(-\frac{1}{s} + \mu_\lambda\right)} \oint_{\tilde{\mathcal{P}}_s(h)} G_{n,m} \frac{dx}{P_\lambda} \end{aligned}$$

because

$$\oint_{\tilde{\mathcal{P}}_s(h)} G_{n,m} \frac{dx}{P_\lambda} = h^m \oint_{\mathcal{P}_s} x^n (x-s)^{\frac{m}{2s} + \frac{m\mu_\lambda}{2} - 1} (x+s)^{-\frac{m}{2s} + \frac{m\mu_\lambda}{2} - 1} dx .$$

We recognize the integral representation of the Beta function. Up to the presence of μ_λ and m , the remaining computations are done identically to those in Proposition 3. \square

Remark 16. The fact that the complete invariant φ_\bullet^n introduced in Section 1.1.3 for affine unfoldings is obtained from the above proposition for $\mu_\bullet := 0$ and $m := -1$ is not fortuitous and can be explained very much like in [24]. The best heuristics is the relation

$$X_\lambda = X_\lambda^\infty - a_\lambda P_\lambda \frac{\partial}{\partial y} = X_\lambda^\infty - \frac{a_\lambda P_\lambda}{y} \times y \frac{\partial}{\partial y}$$

so that locally conjugating X_\bullet^∞ to X_\bullet is somehow equivalent to solving analytically

$$X_\lambda^\infty \cdot F_\lambda = \frac{a_\lambda P_\lambda}{y} .$$

A more precise approach would require to study the generic saddle-node unfolding near $(0, 0, \infty)$ whose node- and saddle-parts corresponds to saddle- and node-parts near $(0, 0, 0)$.

We deduce from this proposition the following result, concluding the proof of Corollary 5.

Corollary 6. *Recall the notations of Corollary 5. The operator*

$$\begin{aligned} \text{Section (1)} &\longrightarrow \{f_\bullet \in \text{Holo}_c(\Sigma \times \mathbb{C}, 0) : f_\bullet(0) = 0\} \\ G_\bullet &\longmapsto \mathfrak{F}_\bullet(G_\bullet) \end{aligned}$$

is bijective.

In the next paragraph we generalize this result for all purely convergent unfoldings.

Proof. This is a consequence of the following two facts:

- ⊗ the period operator sends $y^m x^n$ to some monomial $h^m T_{s,n,m}$,
- ⊗ $T_{s,n,m} = 0$ for $s \in (\Sigma, 0)$ if, and only if, $T_{0,n,m} = 0$, that is $n + m\mu_\lambda \in \mathbb{Z}_{\leq 0}$.

The choice of τ prevents $T_{s,m\tau,m}$ to vanish so that the operator is formally invertible. The fact that the inverse operator maps convergent power series f_\bullet to convergent power series is a consequence of $\lim_{s \rightarrow 0} T_{s,1+\tau m,m} = T_{0,1+\lambda m,m} \neq 0$ and of estimates established in [24] for $T_{0,1+m\lambda,m}$. \square

1.10.2 Normal forms for pure convergence

Here we assume that Z_\bullet is a generic unfolding of multiplicity 1 whose orbital invariant has null node-part:

$$\varphi_\bullet^n = 0 .$$

We say that the unfolding is **purely convergent**, and write $\text{Convergent}(1)$ the set of all such unfoldings.

Theorem 14. *Recall the definition of Section (1) given in Corollary 5 Item 2. The collection*

$$\text{NF}_{\text{loc}}(\text{Convergent}(1)) := \left\{ \frac{u_{\bullet}}{1 + u_{\bullet}G_{\bullet}} \left(X_{\bullet}^{\infty} + yR_{\bullet} \frac{\partial}{\partial y} \right) : G_{\bullet}, R_{\bullet} \in \text{Section}(1) \right\}$$

is a family of normal forms for Convergent(1). Two vector fields in normal forms are locally analytically conjugate if, and only if, there exists $c_{\bullet} \in \text{Holo}_c(\Sigma)^{\times}$ such that

$$\begin{cases} \tilde{R}_{\lambda}(x, y) &= R_{\lambda}(x, c_{\lambda}y) \\ \tilde{G}_{\lambda}(x, y) &= G_{\lambda}(x, c_{\lambda}y) \end{cases} .$$

Remark 17. Notice that the normal forms are *not* in prepared form (1.25).

We show this result in three steps, following the strategy presented in [20] for $\lambda = 0$. For the sake of clarity we only deal here with the case $\Re(\mu_0) > 0$, particularly implying $\tau = 0$.

The initial data is a formal class $(\mu_{\bullet}, u_{\bullet})$ and a compatible temporal necklace $(\varphi_{\bullet}, f_{\bullet})$, as in Definition 17, with $\varphi_{\bullet}^n = 0$.

Proposition 9. *One can find $r' > 0$ and a covering of $(\mathbb{C} \setminus \{\pm s\}) \times r'\mathbb{D}$ into two modified, infinite canonical sectors \mathcal{V}_s^{\sharp} , $\sharp \in \{\text{ns}, \text{sn}\}$, such that the following properties hold.*

1. *There exists a collection of holomorphic vector fields $X_{\bullet} \in \mathfrak{X}(\Sigma \times \mathbb{C} \times r'\mathbb{D})$ of the form*

$$X_s = X_{\lambda}^{\infty} + yxR_s \frac{\partial}{\partial y} \quad , \quad R_{\bullet} \in \text{Holo}_c(\Sigma \times r'\mathbb{D}) \quad , \quad (1.46)$$

and such that the associated canonical first integrals $H_{\bullet}^{\sharp} \in \text{Holo}(\mathcal{D}^{\sharp})$ have connected fibers and satisfy $H_s^{\text{sn}} = \varphi_s^{\flat} \circ H_s^{\text{ns}}$ for $\flat \in \{\text{n}, \text{g}, \text{s}\}$ and $s \in \Sigma$.

2. *The action of holomorphic automorphisms of the orbital necklace $\Omega(\varphi_{\bullet})$ by $h \mapsto c_{\bullet}h$ induces an action*

$$c_{\bullet}^* : R_{\bullet} \mapsto R_{\bullet} \circ (c_{\bullet}\text{Id}) \quad .$$

In other words, two vector fields as above are locally orbitally conjugate for all $s \in \Sigma$ if, and only if, $\tilde{R}_{\bullet}(y) = R_{\bullet}(c_{\bullet}y)$ for some $c_{\bullet} \in \text{Holo}_c(\Sigma)^{\times}$.

It may happen that $R_{\bullet} \neq R_{-\bullet}$ in Item 1., preventing X_{\bullet} to be a generic unfolding, so we seek a function $c_{\bullet} \in \text{Holo}_c(\Sigma)^{\times}$ such that

$$c_{\bullet}^* R_{\bullet} = c_{-\bullet}^* R_{-\bullet} \quad .$$

If this is possible at all then $X_{\bullet}^{\infty} + yxc_{\bullet}^* R_{\bullet}$ is the expected normal form. The following lemma hence completes orbital realization with normal forms.

Lemma 16. *Let X_{\bullet} be a collection of vector fields in the form (1.46), with associated orbital necklace $\Omega(\varphi_{\bullet})$. There exists a function $c_{\bullet} \in \text{Holo}_c(\Sigma)^{\times}$ such that $c_{\bullet}^* R_{\bullet} = c_{-\bullet}^* R_{-\bullet}$ if, and only if, $\Omega(\varphi_{\bullet})$ is compatible.*

The temporal realization is a straightforward consequence of the next concluding result, as was done to prove Corollary 5. This is a generalization of Corollary 6 to generic unfoldings under normal forms.

Proposition 10. *Let X_\bullet be a generic unfolding of multiplicity 1 in normal form (1.46). The operator*

$$\begin{aligned} \text{Section (1)} &\longrightarrow \{f_\bullet \in \text{Holo}_c(\Sigma \times \mathbb{C}, 0) : f_\bullet(0) = 0\} \\ G_\bullet &\longmapsto \mathfrak{T}_\bullet(G_\bullet) \end{aligned}$$

is bijective.

1.10.2.1 Orbital realization on Σ (proof of Proposition 9)

The proof is achieved by iterating a Cauchy-Heine integral transformation solving a certain sectorial Cousin problem, like in Lemma 14, to obtain H_\bullet^\sharp . In fact we seek two functions $\mathcal{O}_\bullet^\sharp$ such that

$$H_\bullet^\sharp := H_\bullet^{\infty, \sharp} \exp(\mathcal{O}_\bullet^\sharp)$$

as in (1.36).

Definition 18. Take $\rho, r > 0$ such that $r < \frac{1}{|\mu_\lambda|}$ for every $|s| < \rho$. We refer to Figure ##.

1. The **modified squid sector** V_s^{ns} is obtained from the union of a squid sector of radius r as in Definition 10 and the half-rays $\exp\frac{-i\pi}{8}\mathbb{R}_{\geq 0}$ and $\exp\frac{9i\pi}{8}\mathbb{R}_{\geq 0}$. The construction is analogous for V_s^{sn} , as well as their saddle-, gate- and node-parts.
2. Let $r' > 0$ be given. We call **modified canonical sector** \mathcal{V}_s^\sharp the product $V_s^\sharp \times r'\mathbb{D}$.
3. We say that a triple $(r', \mathcal{O}_\bullet^{\text{sn}}, \mathcal{O}_\bullet^{\text{ns}})$, with $\mathcal{O}_\bullet^\sharp \in \text{Holo}_c(\mathcal{D}^\sharp)$, is **adapted** to a domain $\Omega = (\mathbb{C}, 0)$ if $H_s^{\text{ns}}(\mathcal{V}_s^\sharp) \subset \Omega$ for all $s \in \Sigma$.

The next result is the basis of the construction. We omit the proof, which is a straightforward generalization of its counterpart in [20] for $s = 0$. Instead we focus on the constructions involved, stressing the few steps where the case $s = 0$ does not extend straightforwardly.

Theorem 15. *Consider some $(r', \mathcal{O}_\bullet^{\text{sn}}, \mathcal{O}_\bullet^{\text{ns}})$ adapted to Ω . Take any $f_\bullet \in \text{Holo}_c(\Sigma \times \Omega)$ vanishing along $\{h = 0\}$ and with bounded derivative f'_\bullet on $\Sigma \cup \{0\}$, then define F_\bullet^\sharp by*

$$F_s^\sharp(x, y) := \frac{x}{2i\pi} \int_{\Gamma_s^\sharp} \frac{f_s(H_s^\sharp(z, y))}{z - x} \times \frac{dz}{z}, \quad s \in \Sigma, \quad (x, y) \in \mathcal{V}_s^\sharp, \quad (1.47)$$

where the paths were described in Definition 18. The following properties hold.

1. $F_\bullet^\sharp \in \text{Holo}_c(\mathcal{D}^\sharp)$.
2. $F_s^{\text{sn}} - F_s^{\text{ns}} = f_s \circ H_s^{\text{ns}}$ on \mathcal{V}_s^\sharp and vanishes elsewhere.
3. $F_\bullet^\sharp(\star, 0) = 0$ and $F_\bullet^\sharp(0, \star) = 0$.
4. For $|s| < \rho$ one has estimates

a.

$$\sup_{\mathcal{V}_s^\sharp} |F_s^\sharp| \leq r' K \sup_{\Omega} |f'_s| \exp \sup_{\mathcal{V}_s^\sharp} |\mathcal{O}_s^\sharp|$$

b.

$$\sup_{\mathcal{V}_s^\sharp} \left| y \frac{\partial F_s^\sharp}{\partial y} \right| \leq r' K \sup_{\Omega} |f'_s| \exp \sup_{\mathcal{V}_s^\sharp} |\mathcal{O}_s^\sharp| \left(1 + \sup_{\mathcal{V}_s^\sharp} \left| y \frac{\partial \mathcal{O}_s^\sharp}{\partial y} \right| \right)$$

c.

$$\sup_{\mathcal{V}_s^\sharp} \left| x \frac{\partial F_s^\sharp}{\partial x} \right| \leq r' K \sup_{\Omega} |f'_s| \exp \sup_{\mathcal{V}_s^\sharp} |\mathcal{O}_s^\sharp| \left(1 + \sup_{\mathcal{V}_s^\sharp} \left| x \frac{\partial \mathcal{O}_s^\sharp}{\partial x} \right| \right)$$

with some constant $K > 0$ depending only on μ_0 , ρ and r .

Remark 18. We mention the fact that the assumption $\Re(\mu_\lambda) > 0$ guarantees the convergence of the integrals near ∞ , since $f_s(H_s^\sharp(z, y)) \sim_\infty C z^{-\mu_\lambda}$ for fixed $y \in r'\mathbb{D}$.

For $r' > 0$ and $\mathcal{O} := (\mathcal{O}_\bullet^{\text{sn}}, \mathcal{O}_\bullet^{\text{ns}})$ adapted to Ω , we write $\mathcal{E}(\mathcal{O})$ the pair $(F_\bullet^{\text{ns}}, F_\bullet^{\text{sn}})$ built in the previous theorem for $f_\bullet := \varphi_\bullet^s$. Define the recursive sequence $(\mathcal{O}_n)_{n \in \mathbb{Z}_{\geq 0}}$ starting with $\mathcal{O}_0 := (0, 0)$ and

$$\mathcal{O}_{n+1} := \mathcal{E}(\mathcal{O}_n), \quad n \geq 0. \quad (1.48)$$

Show it converges in the Banach space $\mathcal{H} := \text{Holo}_c(\mathcal{D}^{\text{ns}}) \times \text{Holo}_c(\mathcal{D}^{\text{sn}})$ (equipped with the sup-norm). For the sake of clarity we write H_n the canonical first integral H_s^{ns} built from $\mathcal{O}_s^{\text{ns}}$. We can assume that all φ_s^s are holomorphic and have bounded derivatives on some disc $\eta\mathbb{D} = (\mathbb{C}, 0)$. Then we choose

$$\rho \leq \frac{\eta}{M} \exp \left(-\frac{\eta}{M} K \sup_{\eta\mathbb{D}} \left| \frac{d\varphi_s^s}{dh} \right| \right)$$

where

$$M = M(\mu_\bullet) := \sup_{|s| < \rho, z \in V_s^s} \left| P_\lambda^{-\mu_\lambda/2} \mathbf{g}_s \right| \exp 2\pi |\mu_\lambda|,$$

and K is the constant appearing in Theorem 15. We wish to ensure that

$$(\forall s \in \Sigma) (\forall n \in \mathbb{N}) (\forall y \in r'\mathbb{D}) (\forall z \in V_s^s) \quad |H_n(z, y)| \leq \eta. \quad (1.49)$$

By construction of H_n we have for $(z, y) \in \mathcal{V}_s^{\text{ns}}$

$$|H_n(z, y)| \leq r' M \exp \sup_{\mathcal{V}_s^{\text{ns}}} |\mathcal{O}_s^{\text{ns}}|.$$

Therefore if for some $n \in \mathbb{N}$ we have $\sup_{\mathcal{V}_s^{\text{ns}}} |\mathcal{O}_n^{\text{ns}}| \leq \frac{2\pi}{M} \eta K \sup_{\eta\mathbb{D}} \left| \frac{d\varphi_s^s}{dh} \right|$ then we first find that

$$|H_n(z, y)| < r' M \exp\left(\frac{\eta}{M} K \sup_{\eta\mathbb{D}} \left|\frac{d\varphi_s^s}{dh}\right|\right) = \eta,$$

i.e. (r', \mathcal{O}_n) is adapted to $\eta\mathbb{D}$ and then, using Theorem 15 4a., we obtain

$$\begin{aligned} \sup_{\mathcal{V}_s^{\text{ns}}} |\mathcal{O}_{n+1}^{\text{ns}}| &\leq r' K \sup_{\eta\mathbb{D}} \left|\frac{d\varphi_s^s}{dh}\right| \exp \sup_{\mathcal{V}_s^{\text{ns}}} |\mathcal{O}_n^{\text{ns}}| \\ &\leq \frac{\eta}{M} K \sup_{\eta\mathbb{D}} \left|\frac{d\varphi_s^s}{dh}\right|. \end{aligned}$$

These estimates show by induction on n that, with the above choice of r' , the relation (1.48) defines a bounded sequence $(\mathcal{O}_n)_n \subset \mathcal{H}$. It so happens that the components of the sequence $(\mathcal{O}_n)_n$ converge for the Krull topology in $\text{Holo}_c(D^\sharp)[[y]]$, almost by construction, hence we can ensure it converges in the Banach space $\text{Holo}_c(\mathcal{D}^\sharp)$ by using the same argument as [20]. The cited reference likewise provides the remaining claims of the upcoming proposition.

Proposition 11. *Let $\Omega = (\mathbb{C}, 0)$ be a domain and a collection $\varphi_\bullet^s \in \text{Holo}_c(\Sigma \times \Omega)$ be given. Then there exists $(r', \mathcal{O}_\bullet^{\text{sn}}, \mathcal{O}_\bullet^{\text{ns}}) \in \mathbb{R}_{>0} \times \mathcal{H}$ adapted to Ω such that*

1.

$$H_s^{\text{sn}} = H_s^{\text{ns}} \exp(2i\pi\mu_\lambda + \varphi_s^s(H_s^{\text{ns}})),$$

2.

$$\sup_{\mathcal{V}_s^\sharp} \left| y \frac{\partial \mathcal{O}_s^\sharp}{\partial y} \right| < 1 \quad , \quad \sup_{\mathcal{V}_s^\sharp} \left| x \frac{\partial \mathcal{O}_s^\sharp}{\partial x} \right| < 1.$$

We can now complete the proof of the first item of the proposition.

Corollary 7. *Let $r' > 0$ and $\mathcal{O}_\bullet^\sharp$ be given by Proposition 11.*

1. *For $s \in \Sigma$ the vector field*

$$X_s^\sharp := X_s^\infty - y \frac{X_s^\infty \cdot \mathcal{O}_s^\sharp}{1 + y \frac{\partial \mathcal{O}_s^\sharp}{\partial y}} \frac{\partial}{\partial y}, \quad \sharp \in \{\text{ns}, \text{sn}\}$$

is holomorphic on \mathcal{V}_s^\sharp and admits H_s^\sharp as first integral.

2. *The vector fields X_s^\sharp are restrictions to the sectors \mathcal{V}_s^\sharp of a vector field*

$$\begin{aligned} X_s(x, y) &= X_s^\infty(x, y) + x R_s(y) y \frac{\partial}{\partial y} \\ R_\bullet &\in \text{Holo}_c(\Sigma \times r'\mathbb{D}), \quad R_\bullet(0) = 0. \end{aligned}$$

Proof. Define

$$R_s^\sharp := -\frac{X_s^\infty \cdot \mathcal{O}_s^\sharp}{1 + y \frac{\partial \mathcal{O}_s^\sharp}{\partial y}}. \quad (1.50)$$

1. is a straightforward consequence of Proposition 11.
2. On the one hand we have

$$X_s^\# \cdot H_s^{\text{sn}} = X_s^\# \cdot (H_s^{\text{sn}} \exp(2i\pi\mu_\lambda + \varphi_s^s(H_s^{\text{sn}}))) = 0 .$$

On the other hand a short calculation shows that

$$X_s^\# \cdot H_s^{\text{sn}} = H_s^{\text{sn}} \left(X_s^\infty \cdot \mathcal{O}_s^{\text{sn}} + \left(1 + y \frac{\partial \mathcal{O}_s^{\text{sn}}}{\partial y} \right) R_s^{\text{ns}} \right) .$$

Therefore the functions $R_s^\#$ glue other the intersection of canonical sectors to a holomorphic function \widehat{R}_s on $(\mathbb{C} \setminus \{\pm s\}) \times r'\mathbb{D}$, bounded near $\{x = \pm s\}$. Hence \widehat{R}_s is holomorphic on $\mathbb{C} \times r'\mathbb{D}$ by Riemann's removable singularity theorem. From (1.50) follows, for $|x| > 1$,

$$\left| \widehat{R}_s \right| \leq \frac{\left| \frac{P_\lambda}{x} \times x \frac{\partial \mathcal{O}_s^\#}{\partial x} \right| + (1 + |\mu_\lambda x|) \left| y \frac{\partial \mathcal{O}_s^\#}{\partial y} \right|}{1 - \left| y \frac{\partial \mathcal{O}_s^\#}{\partial y} \right|} \leq C |x|$$

for some constant $C > 0$ whose existence is guaranteed by Proposition 11 Item 2. Therefore for any fixed $y \in r'\mathbb{D}$ the partial function $x \mapsto \widehat{R}_s(x, y)$ is affine. Taking Theorem 15 Item 3. into account we derive that $\widehat{R}_s = xR_s$, so that R_s can only depend on y , concluding the proof. \square

So far we have proven Proposition 9 Item 1. The second item can be shown in exactly the same way as in [20], so we shall skip additional details.

1.10.2.2 Gluing antipodal realizations (proof of Lemma 16)

Assume the existence of $c_\bullet \in \text{Hol}_c(\Sigma)^\times$ for which $\widehat{R}_\lambda := c_s^* R_s$ is holomorphic with respect to λ for $s \in \Sigma$. Let \widehat{X}_\bullet be the corresponding vector field in normal form. We put a hat over objects associated to this vector field, to avoid confusing the same objects attached to X_\bullet . The orbital necklace $\Omega(\widehat{\varphi}_\bullet)$ is compatible: $\widehat{\eta}_\bullet := \widehat{H}_\bullet \circ \widehat{H}_{-\bullet}$ conjugates the necklace dynamics $\widehat{\Delta}_\bullet$ and $\widehat{\Delta}_{-\bullet}$. Because \widehat{X}_\bullet and X_\bullet are conjugate by $(s, x, y) \mapsto (s, x, c_s y)$ their orbital necklaces are diffeomorphic: $\widehat{\Delta}_s = c_s^* \Delta_s$. As a conclusion, setting $\eta_s := \frac{1}{c_s} \widehat{\eta}_s \circ (c_{-s} \text{Id})$ we check easily that η_\bullet conjugates the necklace dynamics Δ_\bullet and $\Delta_{-\bullet}$, while $\eta_\bullet^{\circ-1} = \eta_{-\bullet}$.

Conversely, if $\Omega(\varphi_\bullet)$ is compatible then $\psi_\bullet := H_\bullet^{\circ-1} \circ \eta_\bullet \circ H_{-\bullet}$ is a self-map of T conjugating the holonomy representation of R_\bullet and $R_{-\bullet}$ on T . Because the union of leaves \mathcal{L}_p of \mathcal{F}_s for $p \in T$ contains a uniform, connected neighborhood \widehat{U} of $\{(\pm s, 0)\}$, there exists a family of paths $\gamma(x)$ linking x to x_* so that $\mathfrak{h}_s^{\gamma(x)}(x, y) \in T$ for every $(x, y) \in \widehat{U}$. The map built à la Mattei-Moussu [13]

$$\Psi_s : (x, y) \mapsto \left(x , \mathfrak{h}_s^{-\gamma(x)} \circ \psi_s \circ \mathfrak{h}_s^{\gamma(x)}(x, y) \right)$$

is therefore well-defined, biholomorphic and locally conjugates X_\bullet and $X_{-\bullet}$. The conclusion follows from Proposition 9 Item 2.

1.10.2.3 A section to the period operator (proof of Proposition 10)

This is really Theorem 15. Being given $X_\bullet \in \text{Holo}_c(\Sigma \times \mathbb{C} \times r'\mathbb{D})$ in normal form, the functions F_\bullet^\sharp of (1.47) induces a function $G_\bullet := X_\bullet \cdot F_\bullet^\sharp \in \text{Holo}_c(\Sigma \times \mathbb{C} \times r'\mathbb{D})$ with prescribed period as in Section 1.10.1.1. Applying again the arguments of Corollary 7 Item 2., we can give a polynomial bound on the growth of $x \mapsto G_s(x, y)$, so that it must be of the expected form. The claim follows from the same reasoning as in Theorem 13 to perturb G_\bullet so that the resulting function is even in s and has same period.

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