



HAL
open science

A POSSIBLE EXPLANATION FOR THE VALIDITY OF THE RIEMANN HYPOTHESIS

Kevin Duffy

► **To cite this version:**

Kevin Duffy. A POSSIBLE EXPLANATION FOR THE VALIDITY OF THE RIEMANN HYPOTHESIS. 2015. hal-01170637

HAL Id: hal-01170637

<https://hal.science/hal-01170637>

Preprint submitted on 2 Jul 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A POSSIBLE EXPLANATION FOR THE VALIDITY OF THE RIEMANN HYPOTHESIS

KEVIN JAN DUFFY

ABSTRACT. The renowned Riemann Hypothesis is considered from the perspective of symmetry in the functional equation for the Riemann zeta function developed by Bernhard Riemann. The validity of the Riemann Hypothesis is shown by considering the complex conjugations of the leading terms of this equation.

1. INTRODUCTION

$\zeta(s)$ is defined as the series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ where $s = \alpha + \beta i$ with α, β real and n a positive integer, and where $\zeta(s)$ is convergent for $\alpha > 1$. The Riemann zeta function is an analytical continuation of this series by Bernhard Riemann which is holomorphic for all s except at $s = 1$ [3]. Using a natural symmetry in the result Riemann [3, 2] obtained the functional equation:

$$(1) \quad \zeta(s) = \aleph(s)\zeta(1-s) = 2^s \pi^{s-1} \sin \frac{s\pi}{2} \Gamma(1-s)\zeta(1-s)$$

with $\Gamma(1-s) = \int_0^{\infty} e^{-z} z^{-s} dz$.

The renowned Riemann Hypothesis (RH) is that non-trivial zeroes of the Riemann zeta function $\zeta(s)$ all have real part on the 'critical line' $Re(s) = \alpha = \frac{1}{2}$ [1]-[4].

My explanation for the validity of RH will focus on the complex conjugations of this equation. Starting with $\zeta(s) = 0$ and using the fact that $\aleph(s) \neq 0$ on the critical strip $0 < \alpha < 1$ where all non-trivial zeroes are known to occur [1]-[4] it is shown that $\zeta(s) = 0 \Rightarrow \aleph(s) \overline{\aleph(s)} = 1$ and why this is only possible if $\alpha = 0.5$.

1991 *Mathematics Subject Classification.* AMS Subject Classification 11S40.

Key words and phrases. Riemann zeta-function; functional equation.

2. DETAILS

Here I will define the complex functions resulting from (1) as:

$$(2) \quad \zeta(s) = u + vi, \zeta(1-s) = w + ti, \aleph(s) = x + yi$$

where u, v, w, t, x, y are all real. Thus,

$$(3) \quad (u + vi) = (x + yi)(w + ti)$$

resulting in

$$(4) \quad x = \frac{uw + vt}{w^2 + t^2}, y = \frac{vw - ut}{w^2 + t^2}, x^2 + y^2 = \frac{u^2 + v^2}{w^2 + t^2} > 0$$

Note in (4) $x^2 + y^2 \neq 0$ and exists on the critical strip $0 < \alpha < 1$ where all non-trivial zeroes are known to occur [1]-[4] because: $2^s \neq 0$, $\pi^{s-1} \neq 0$, $\int_0^\infty e^{-z} z^{-s} dz \neq 0$, and $\sin \frac{s\pi}{2} \neq 0$ for $0 < \alpha < 1$, and all of these are holomorphic for the same range, $\Rightarrow \aleph(s) \neq 0$ and is holomorphic for $0 < \alpha < 1$. Thus, using (1), $\zeta(s) = 0 \Rightarrow \zeta(1-s) = 0 \Rightarrow \zeta(s) = \zeta(1-s) \Rightarrow u = w, v = t \Rightarrow x = 1, y = 0 \Rightarrow x^2 + y^2 = 1$ from (4).

Thus the RH is true if

$$(5) \quad \aleph(s) \cdot \overline{\aleph(s)} = x^2 + y^2 = 1 \iff \alpha = 0.5.$$

Theorem 1. $\aleph(s) \cdot \overline{\aleph(s)} = 1 \iff \alpha = 0.5$ for $0 < \alpha < 1$ and $\beta > 10$.

Proof. First, for $\aleph(0.5 + \beta i) \overline{\aleph(0.5 + \beta i)}$ using (1):

- (i) $2^s \pi^{s-1} \overline{2^s \pi^{s-1}} = 2^{0.5 + \beta i} \pi^{-0.5 + \beta i} \overline{2^{0.5 + \beta i} \pi^{-0.5 + \beta i}} = \frac{2}{\pi}$
- (ii) $\sin \frac{s\pi}{2} \overline{\sin \frac{s\pi}{2}} = (\sin \frac{\alpha\pi}{2} \cos \frac{\beta i\pi}{2} + \cos \frac{\alpha\pi}{2} \sin \frac{\beta i\pi}{2})(\sin \frac{\alpha\pi}{2} \cos \frac{\beta i\pi}{2} - \cos \frac{\alpha\pi}{2} \sin \frac{\beta i\pi}{2})$
 $= 0.5(\cos^2 \frac{\beta\pi i}{2} - \sin^2 \frac{\beta\pi i}{2})$ at $\alpha = 0.5$
 $= 0.5 \cos(\beta\pi i) = 0.5 \sin(\beta\pi i + \frac{\pi}{2})$
- (iii) $\overline{\Gamma(1-s)} = \Gamma(\overline{1-s}) = \Gamma(s)$ at $\alpha = 0.5$ and so $\overline{\Gamma(1-s)}\Gamma(1-s)$
 $= \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\beta\pi i + \frac{\pi}{2})}$ by Euler's reflection formula.

Thus, combining these equalities, $\aleph(0.5 + \beta i) \cdot \overline{\aleph(0.5 + \beta i)} = 1$.

Second, consider $\aleph(\alpha+\beta i).\overline{\aleph(\alpha+\beta i)}$ in general on the critical strip $0 < \alpha < 1$ where all non-trivial zeroes are known to occur. To simplify some of what follows define functions f,g,h such that:

- (i) $f = 2^s \pi^{s-1} \overline{2^s \pi^{s-1}} = 4^\alpha \pi^{2\alpha-2}$ and $0 < 4^\alpha \pi^{2\alpha-2} < 4$ for $0 < \alpha < 1$.
- (ii) $g = \sin \frac{s\pi}{2} \overline{\sin \frac{s\pi}{2}} = (\sin \frac{\alpha\pi}{2} \cos \frac{\beta i\pi}{2} + \cos \frac{\alpha\pi}{2} \sin \frac{\beta i\pi}{2})(\sin \frac{\alpha\pi}{2} \cos \frac{\beta i\pi}{2} - \cos \frac{\alpha\pi}{2} \sin \frac{\beta i\pi}{2})$
 $= (\sin^2 \frac{\alpha\pi}{2} \cos^2 \frac{\beta i\pi}{2} - \cos^2 \frac{\alpha\pi}{2} \sin^2 \frac{\beta i\pi}{2})$
 $\simeq \cos^2 \frac{\beta i\pi}{2} (\cos^2 \frac{\alpha\pi}{2} + \sin^2 \frac{\alpha\pi}{2})$ for $0 < \alpha < 1$ and as for β large enough
 $\cos^2 \frac{\beta i\pi}{2} \simeq -\sin^2 \frac{\beta i\pi}{2}$
 $= \cos^2 \frac{\beta i\pi}{2}$ which is constant on $0 < \alpha < 1$ for each β large enough.

Thus, $\sin \frac{s\pi}{2} \overline{\sin \frac{s\pi}{2}} \simeq \text{constant}$ for β large enough and $0 < \alpha < 1$.

Even $\beta > 10$ suffices which is less than the first known zero of the zeta function (for the same reason this criteria will be used in what follows).

Also, the value increasingly converges as β increases.

- (iii) $h = \Gamma(1-s) \overline{\Gamma(1-s)}$.

Now $\frac{\partial |\aleph(s)|^2}{\partial \alpha} = f'gh + fg'h + fgh'$ and using the partial derivatives:

- (i) $f' = \frac{\partial(4^\alpha \pi^{2\alpha-2})}{\partial \alpha} = 2^{(2\alpha+1)} \pi^{(2\alpha-2)} (\ln(\pi) + \ln(2))$
- (ii) $h' = \frac{\partial(\Gamma(1-s)\overline{\Gamma(1-s)})}{\partial \alpha} =$
 $-(\Psi(1-\alpha-i\beta) + \Psi(1-\alpha+i\beta))\Gamma(1-\alpha+i\beta)\Gamma(1-\alpha-i\beta),$
 where $\Psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$
- (iii) $g' = \frac{\partial \sin \frac{s\pi}{2} \overline{\sin \frac{s\pi}{2}}}{\partial \alpha} = (\frac{1}{2})\cos(\frac{\pi}{2}(\alpha+i\beta))\pi \sin((\frac{\pi}{2}(\alpha-i\beta))) + (\frac{1}{2})\sin(\frac{\pi}{2}(\alpha+i\beta))\cos(\frac{\pi}{2}(\alpha-i\beta))\pi$
 $= \pi \cos(\frac{\pi}{2}\alpha) \cosh(\frac{\pi}{2}\beta)^2 \sin(\frac{\pi}{2}\alpha) - \pi \sin(\frac{\pi}{2}\alpha) \sinh(\frac{\pi}{2}\beta)^2 \cos(\frac{\pi}{2}\alpha)$
 $= \cos(\frac{\pi}{2}\alpha) \sin(\frac{\pi}{2}\alpha) \pi$
 and $0 < \cos(\frac{\pi}{2}\alpha) \sin(\frac{\pi}{2}\alpha) \pi < 1.6$ for $0 < \alpha < 1$,

then for $0 < \alpha < 1$ and $\beta > 10$,

$$\frac{\partial |\aleph(s)|^2}{\partial \alpha} = fgh(\frac{h'}{h} + \frac{g'}{g} + \frac{f'}{f}) < fgh(\frac{h'}{h} + \frac{1.6}{g} + \frac{f'}{f}) \simeq fgh(\frac{h'}{h} + \frac{1.6}{\cos^2 \frac{\beta i\pi}{2}} + \frac{f'}{f})$$

$$\simeq fgh(\frac{h'}{h} + \frac{f'}{f}) \text{ because } \cos^2 \frac{\beta i\pi}{2} \text{ will be very large.}$$

So, taking the overall partial derivative and simplifying gives

$$(6) \quad \frac{\partial |\aleph(s)|^2}{\partial \alpha} < -4^\alpha \pi^{(2\alpha-2)} \cosh^2 \frac{\pi\beta}{2} \Gamma(1-\alpha+i\beta) \Gamma(1-\alpha-i\beta) \\ (\Psi(1-\alpha-i\beta) + \Psi(1-\alpha+i\beta) - 2\ln(\pi) - 2\ln(2))$$

The $\Re(\Psi(\alpha+i\beta)) > 0$ if $\beta > 5/3$ and $\alpha \geq 0$ (see [5] for details). Thus, $\Re(\Psi(1-\alpha-i\beta) + \Psi(1-\alpha+i\beta)) > 0$ on $0 < \alpha < 1$ and $\beta > 5/3$. Now $(\Psi(1-\alpha-i\beta) + \Psi(1-\alpha+i\beta) - 2\ln(\pi) - 2\ln(2)) > 0$ for $\beta > 10$ on $0 < \alpha < 1$ because the first two terms are then at least > 0.3676 , the approximate absolute value of the last two terms.

The term $4^\alpha \pi^{(2\alpha-2)} \cosh^2 \frac{\pi\beta}{2}$ is a very large positive real number for all $0 < \alpha < 1$ and $\beta > 10$.

The remaining term $\Gamma(1-\alpha+i\beta)\Gamma(1-\alpha-i\beta)$ is by definition a positive real number.

From these points, and considering (6), we have $\frac{\partial |\aleph(s)|^2}{\partial \alpha} < 0$ and so $|\aleph(s)|^2$ is strictly monotonic in terms of α on $0 < \alpha < 1$ for each $\beta > 10$.

Now, as $\aleph(\alpha+\beta i) \cdot \overline{\aleph(\alpha+\beta i)}$ is strictly monotonic, this function can equal 1 at only a single unique value of α for each $\beta > 10$ on the critical strip $0 < \alpha < 1$. It is already shown above that this occurs at $\alpha = 0.5$.

Thus, $\aleph(s) \cdot \overline{\aleph(s)} = 1 \implies \alpha = 0.5$ for $\beta > 10$. □

3. SUMMARY

In this simple way, as $\zeta(s) = 0 \implies \aleph(s) \cdot \overline{\aleph(s)} = 1$ and $\aleph(s) \cdot \overline{\aleph(s)} = 1 \iff \alpha = 0.5$, the Riemann Hypothesis is shown to be true. If these results are correct then Riemann had already laid the foundation for the proof of his hypothesis by establishing the functional equation in his original paper [3].

ACKNOWLEDGMENTS

Obiora Collins is thanked for reading the manuscript many times, very fruitful discussions and valuable suggestions. Kian Duffy is thanked for pointing out the requirement that $\aleph(s) \neq 0$.

REFERENCES

- [1] E. Bombieri, Problems of the Millenium the Riemann Hypothesis. Clay Mathematics Institute (2000).
http://www.claymath.org/sites/default/files/official_problem_description.pdf. Accessed January 2015.
- [2] H.M. Edwards, *Riemann's Zeta Function*, Dover Publications, New York (1974).
- [3] B. Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse (On the Number of Prime Numbers less than a Given Quantity). Translated by D. R. Wilkins, (1998).
- [4] P. Sarnak P, Problems of the Millenium: the Riemann Hypothesis. Clay Mathematics Institute (2004). http://www.claymath.org/sites/default/files/sarnak_rh_0.pdf. Accessed January 2015.
- [5] G.K. Srinivasan and P. Zvengrowski, On the horizontal monotonicity of $|\Gamma(s)|$, *Can. Math. Bull.*, **54** (2011), 538–543.

INSTITUTE OF SYSTEMS SCIENCE, DURBAN UNIVERSITY OF TECHNOLOGY, DURBAN 4000, SOUTH AFRICA AND SCHOOL OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE, UNIVERSITY OF KWAZULU-NATAL, DURBAN 4000, SOUTH AFRICA.

E-mail address: `kevind@dut.ac.za`