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Paul Dorbec, Eric Duchêne, André Fabbri, Julien Moncel, Aline Parreau, Eric
Sopena

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ICE SLIDING GAMES

PAUL DORBEC, ÉRIC DUCHÊNE, ANDRÉ FABBRI, JULIEN MONCEL,
ALINE PARREAU, AND ÉRIC SOPENA

ABSTRACT. This paper deals with sliding games, which are a variant of the
better known PUSHPUSH game. On a given structure (grid, torus...), a robot can
move in a specific set of directions, and stops when it hits a block or boundary
of the structure. The objective is to place the minimum number of blocks such
that the robot can visit all the possible positions of the structure. In particular,
we give the exact value of this number when playing on a rectangular grid and
a torus. Other variants of this game are also considered, by constraining the
robot to stop on each case, or by replacing blocks by walls.

Keywords: Combinatorial game theory; Graph theory; Sliding games

1. BACKGROUND AND DEFINITIONS

Sliding/pushing puzzles are classical problems used for entertainment. In a slid-
ing puzzle, entities (often described as robots) are moving around on a grid, and
trying to reach a final position. Everytime a robot starts a move in a direction, it
slides and cannot stop until it hits another element on the grid (a wall, a block or
another robot). In a pushing puzzle, the entities often may stop a move without
hitting a wall, but mostly they are also allowed to move some inert blocks by push-
ing them. Pushing games are known for example as Sokoban, but can also appear
as enigmas in video games such as the Zelda sequel. They have been thoroughly
studied before, and we refer the reader to the pleasant survey of Demaine [4] which
also provides a classification of these games. He in particular gives a very detailed
overview of known complexity results about such games. Without any surprise, a
large majority of these are proved to be NP- or PSPACE-complete. A difference
is also made between PUSH and PUSHPUSH-games: in the first game the blocks are
pushed one square at a time, while in the second they slide until they meet an
obstacle whenever they are pushed.

Sliding games are still less studied, though there are many commercial games
using this principle. RASENDE ROBOTER, LUNAR LOCKOUT (marketed in 1999 by
Binary Arts) or its predecessor UFO are such examples. Before describing these

games, we propose a general definition. An ICE SLIDE game is a puzzle where one or more robots are on a grid, trying to reach a flag (the name was initially given in [11]). Each move of a robot consists in sliding in one direction until it meets an obstacle, which stops him. Obstacles include static obstacles, such as walls or blocks, but also other robots. In such a setting, the natural question is whether there is a sequence of moves for one robot to reach the flag (see Figure 1 for an example of this game).
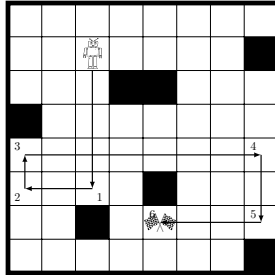

Figure 1 – *Robot sliding game: a solution with six moves*

This question, where obstacles are predefined, and a single robot is trying to reach a final position was posed recently by K. Burke, on his excellent blog devoted to combinatorial game theory [2]. K. Burke also asked for the minimum number of moves for the robot to reach the goal. Actually, the resolution of this problem is not very hard, as explained in [11]. Indeed, it reduces to the search for a shortest path between the starting and final squares in an underlying subgraph of the grid. By way of consequence, a simple application of Djikstra's algorithm yields to a polynomial computation of the minimum number of moves of a solution (provided the game representation is not sublinear in the number of squares).

Yet, another natural question arises when dealing with this problem: what if the player can choose the positions of the obstacles on the board? Of course, if the initial and final squares are fixed, at most one block is necessary, yielding to a solution with at most two moves. But if we ask the player to be able to move the robot to any position of the grid, the problem becomes considerably more challenging. More precisely, given an initial square P of the grid, we ask what is the minimum number of blocks needed (and also how they must be placed) to move the robot from P to any other square. (In that context, the question of the minimum number of moves becomes secondary.)

In correlation with this problem, we have identified two research parameters that slightly change the rules detailed above but raise new reflexion on the topic:

- Blocks can be substituted by walls in the problem. In other words, the robot bounces when hitting an edge of the grid (instead of a square).
- One can suppose the final square not to be a flag to go through, and that the robot must stop on it. Note that it forces any square of the grid to be adjacent to a block (or a wall).

In the following, we differentiate four situations of the game, depending on the precise question and the nature of the obstacles. We denote the game ICE SLIDE$_{B/W}^{P/S}$-K

using a superscript P or S depending whether we require that the robot simply **P**asses or need to **S**top on the flagged position, a subscript B or W depending whether the obstacles are **B**locks or **W**alls, and K for the number of robots that we are allowed to move.

As a first example, the marketed game RASENDE ROBOTER is related to the game ICE SLIDE$_W^S$-4 with a particular setting of the obstacles. In RASENDE ROBOTER, the objective is to find the minimum number of moves of a solution where the initial and final positions are fixed. Its originality is that three additional robots can also be moved and be used as blocks on which bouncing could help the player. RASENDE ROBOTER had a good marketing success and also raised the curiosity of researchers in algorithmic game theory. Because of the presence of other robots and the size of the grid (16×16), simple shortest path algorithms are no more efficient. To the best of our knowledge, only AI algorithms based on multi-agent systems have been proposed to solve the game [3].

Another popular game called LUNAR LOCKOUT (marketed in 1999 by Binary Arts) is very close to ICE SLIDE$_B^S$-K, with the of a board with no bounding walls. As in RASENDE ROBOTER, other robots can be used as movable blocks. Before its marketing, this puzzle was first introduced in 1998 by Yoshigahara and called UFO. It was studied in the literature under this original name. Hock proved in [8] that deciding the existence of a solution in UFO is NP-complete. In [7], a variant of UFO is proposed, called GLLV, where fixed robots are allowed and played on a general rectangular grid. This version clearly meets ICE SLIDE$_B^S$-K. Whereas the PSPACE-completeness of UFO is open, it is proved in [7] that the variant which includes fixed blocks is PSPACE-complete.

Note that the two games ICE SLIDE$_{B/W}^P$-2 played on a rectangular grid are trivial, in the sense that no block is needed. Engels and Kamphans [5] proved that the two games ICE SLIDE$_{B/W}^S$-3 are also trivial in any rectangular grid. Hence the most interesting instances, at least on rectangular grids, seem to be ICE SLIDE-1 and ICE SLIDE$_{B/W}^S$-2. In this paper, we will focus in this paper on the games ICE SLIDE-1, using a single robot, and denote it simply by ICE SLIDE. We use the following definition.

Definition 1. *Let $G_{n,m}$ be a rectangular grid with n rows and m columns. Each square of the grid is denoted by a pair (i,j) with $1 \leq i \leq n$ and $1 \leq j \leq m$. The parameter $\iota\sigma_{B/W}^{P/S}(G_{n,m})$ is the minimum number of **B**locks (resp. **W**alls) that needs to be placed on the grid so that, from the starting position $(1,1)$ and for every position (i,j) of $G_{n,m}$ which does not contain a block, there exists a sequence of moves making the robot **P**ass over (resp. **S**top on) (i,j).*

Remark 2. *In what follows, we shall see that starting from the position $(1,1)$ is not very restrictive. Indeed, our constructions often guarantee that from any starting position, it is possible to move the robot and stop on $(1,1)$.*

Remark 3. *In the definition, one can replace the rectangular grid by any kind of grids for which the moves of the robot is clear. We will for example consider in Section 3 king grids and tori.*

We proceed with the resolution of these problem for several instances, which should also stress how this game is related to some well-known problems in graph
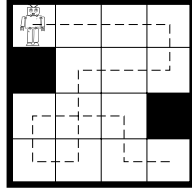
Figure 2 – $\iota\sigma_B^P(G_{4,4}) \leq 2$.
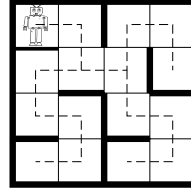
Figure 3 – $\iota\sigma_W^S(G_{4,4}) \leq 9$.

theory. We first give in Section 2 the definitions and results of related graph problems. In Section 3, we study $\iota\sigma_B^P$ on different kinds of grids, such as rectangular grids, king grids or tori. In Section 4, the three other variants of the game are explored.

2. GRAPH PARAMETERS RELATED TO ICE SLIDE

The current section presents two optimization problems in graph theory which are in direct correlation with our game.

DOMINATION IN GRAPHS

Given a graph G, a set $D \subseteq V(G)$ is said to be a dominating set (resp. a total dominating set) of G if every vertex of $V(G) \setminus D$ (resp. $V(G)$) is adjacent to a vertex of D.

Definition 4. *Let G be a graph. The value $\gamma(G)$ (resp. $\gamma_t(G)$), called domination number (resp. total domination number) of G, corresponds to the minimum cardinality of a dominating set (resp. total dominating set) in G.*

In the next section, we observe that the dominating set problem is directly related to ICE SLIDE$_B^S$. Indeed, the ability to stop the robot anywhere means that a block must be adjacent to any square of the grid. In other words, the set of blocks needs to be a dominating set of the grid. Total dominating sets appear in the study of the game ICE SLIDE$_B^P$, as explained further in Subsection 3.1.

Given any graph G, the computation of $\gamma(G)$ and $\gamma_t(G)$ are known to be NP-hard problems. However, their values are known for simple classes of graphs, such as grids or paths. As needed later, we give below the total domination number of paths.

Proposition 5 (Klobucar [9]). *Let P_n denote the path with n vertices. We have*

$$\gamma_t(P_n) = \begin{cases} 2\lfloor \frac{n}{4} \rfloor + 1 & if\ n \equiv 1 \pmod 4 \\ 2\lfloor \frac{n}{4} \rfloor & otherwise \end{cases}$$

EDGE COVER IN GRAPHS

Given a graph G, a set $S \subseteq E(G)$ is said to be an edge cover of G if every vertex of G is incident to at least one edge of S.

Definition 6. *Let G be a graph. The value $\rho(G)$, called the edge covering number of G, corresponds to the size of a minimum edge cover of G.*

Unlike the domination number, the value $\rho(G)$ can be computed in polynomial time for any graph G. In Subsections 4.2 and 4.3, we observe how edge covers interact with the game ICE SLIDE played with walls. For example, we use the value $\rho(P_n)$, which is straightforwardly equal to $\lceil n/2 \rceil$.

3. ICE SLIDE$_B^P$ PLAYED ON VARIOUS GRIDS

In all the paper, we consider grids with n rows and m columns. We denote by R_i (resp. C_j) the ith row (resp. j^{th} column) of G. Position (i, j) is at the intersection of R_i and C_j and in the figures, unless something else is mentionned, $(1, 1)$ is the top-left square.

3.1. **Rectangular grids.** In this part, we consider the game ICE SLIDE$_B^P$ on a rectangular grid $G = G_{n,m}$: we place *blocks* on G and the robot must be able to *pass* everywhere from the square $(1, 1)$.

Definition 7. *Let B be a set of blocks. The row R_i, $2 \le i \le n - 1$ (resp. column C_j, $2 \le j \le m - 1$) is said to be totally dominated by B if there is at least one block of B in row R_{i-1} or R_{i+1} (resp. in column C_{i-1} or C_{i+1}).*

If B is a set of blocks in $G_{n,m}$ which totally dominates every column C_j, $2 \le j \le m - 2$, then, by moving blocks from columns C_1 and C_m to columns C_3 and C_{m-2} respectively, every column is still totally dominated by B. Hence we get the following:

Observation 8. *If B is a set of blocks in $G_{n,m}$ which totally dominates every column C_j, $2 \le j \le m - 2$, then $|B| \ge \gamma_t(P_{m-2})$.*

Proposition 9. *For each $n \ge m$, we have $\iota\sigma_B^P(G_{n,m}) \ge \gamma_t(P_{m-2})$.*

Proof. Consider an optimal solution B of ICE SLIDE$_B^P$ for $G_{n,m}$. If every column C_j, $2 \le j \le m - 2$, is totally dominated by B then, by Observation 8, we get $\iota\sigma_B^P(G_{n,m}) \ge \gamma_t(P_{m-2})$.

Assume now that there exists a column C_j, $2 \le j \le m - 2$, which is not totally dominated by B. Let $I_j = \{i : (i, j) \in B\}$ be the line indices of the blocks of C_j. The robot can not initiate a vertical move in C_j since to stop in C_j it needs a block in column C_{j-1} or C_{j+1}. Hence it has to go horizontally trough every row R_i, $i \in [2, n-1] \setminus I_j$. Therefore, each such row must be totally dominated by $B' = B \setminus \{(i, j) : i \in I_j\}$. We define the set of blocks B^* as follows:

$$B^* = B' \cup \{(i - 1, j) : i \in I_j, i > 1\}.$$

We claim that each row R_i, $2 \le i \le n - 1$, is totally dominated by the set B^*. Indeed, if $i \notin I_j$, then R_i is totally dominated by B' as observed above. Otherwise there is a block in row R_{i-1}. We finally get

$$|B| \ge |B^*| \ge \gamma_t(P_{n-2}) \ge \gamma_t(P_{m-2})$$

by Observation 8 and the result follows. □

Lemma 10. *Assume that the pattern P_n of size $n \times 8$ depicted on Figure 4 (a) is in position $(1, j)$ ($(1, j)$ being the top-left square of the pattern). Then, if a robot is able to enter horizontally in each of the four positions $(1, j)$, (n, j), $(2, j + 7)$, and $(n - 1, j + 7)$, it can pass vertically through all the columns C_j to C_{j+7}. In addition, if a robot enters horizontally the pattern P_n in position $(1, j)$, it can leave*
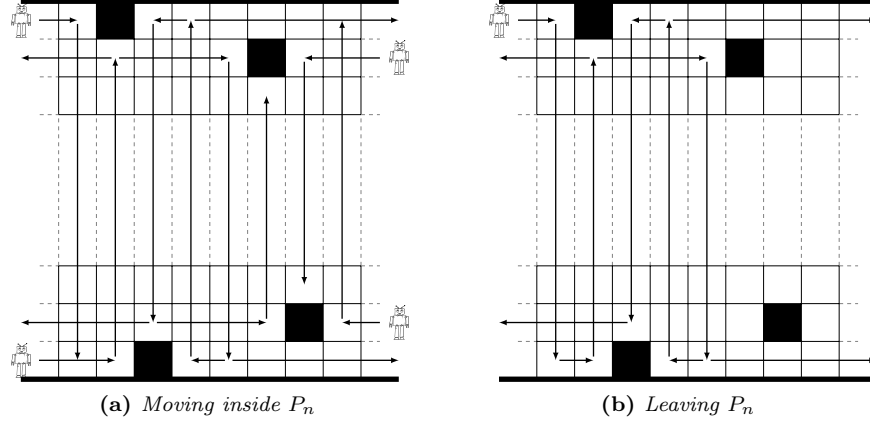
(a) *Moving inside P_n*   **(b)** *Leaving P_n*

Figure 4 – *Pattern of size $n \times 8$ for* ICE SLIDE$_B^P$ *on grid and its representation.*

it horizontally on rows R_2 and R_{n-1} by going left and on rows R_1 and R_n by going right (see Figure 4b).

Proof. The routes followed by the robot to satisfy the above lemma are shown on Figure 4. □

Theorem 11. *For any $n \geq m$, $m \neq 10$, we have $\iota\sigma_P^B(G_{n,m}) = \gamma_t(P_{m-2})$. For any $n \geq 10$, we have $\iota\sigma_P^B(G_{n,10}) = 5$.*

Proof. Let $n \geq m > 10$. We consider the following cases:

- $m-2 \equiv 3, 4, 5, 6$ or $7 \pmod 8$. The solution is build by gluing $\lfloor \frac{m-2}{8} \rfloor$ copies of P_n starting from column C_2 (leaving column C_1 empty), the remaining columns being filled by the patterns of Figure 5. From Lemma 10, all the columns of all the copies of P_n but the last one are visited. Figure 5 shows that all the remaining rightmost columns can be visited by entering from the left on rows R_1 and R_n. The robot is also able to leave these columns from rows R_2 and R_{n-2}, so that the columns of the last copy of P_n can be visited.
- $m-2 \equiv 1$ or $2 \pmod 8$. Using $\lfloor \frac{m-10}{8} \rfloor$ copies of P_n, the proof is similar to the previous case using the patterns depicted in Figure 6.
- $m-2 \equiv 0 \pmod 8$. The proof is again similar using $\lfloor \frac{m-18}{8} \rfloor$ copies of P_n (recall that $m \neq 10$) and the pattern depicted in Figure 7.

We now consider the case $m < 10$, $n \geq m$. If $m = 1, 2$, clearly no block is needed. For $m = 3$, put a block in position $(n,3)$. For $m = 4$, put two blocks in positions $(n,3)$ and $(1,4)$. For $5 \leq m \leq 9$, leave column C_1 empty glued to the pattern of width $(m-1)$ given in Figure 5.

In each case, the number of blocks used in our solutions is exactly $\gamma_t(P_{m-2})$.

In the case $m = 10$, one can easily show that five blocks are enough by using the pattern of Figure 6 (a) without the first column. It now remains to prove that $\gamma_t(P_8) = 4$ blocks are not sufficient. Suppose on the contrary that $\iota\sigma_B^P(G_{n,10}) = \gamma_t(P_8) = 4$ for $n \geq 10$, and consider an optimal solution B. According to the proof of Proposition 9, one can assume that every column C_j, $2 \leq j \leq m-2$ is

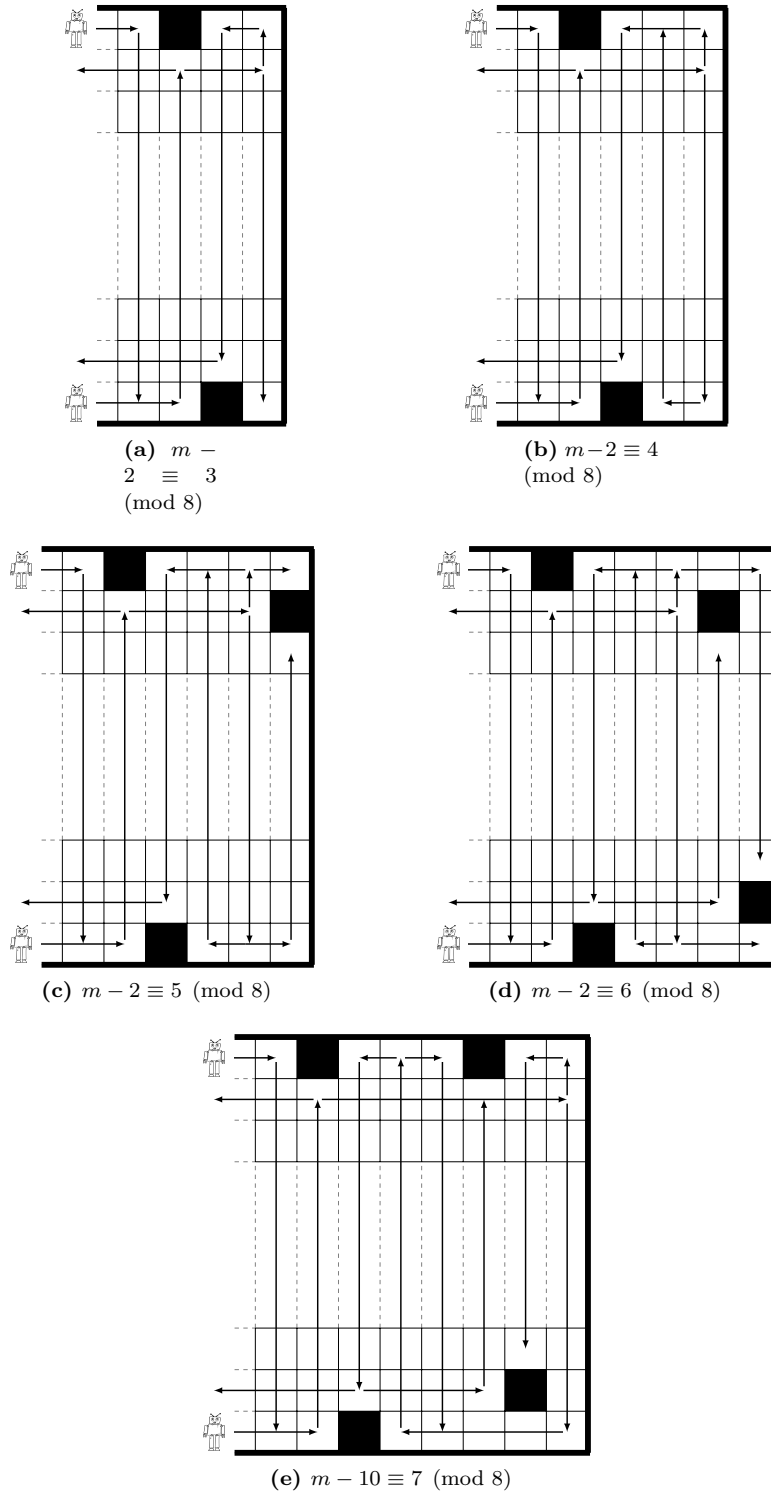**(a)** $m - 2 \equiv 3$ (mod 8)

(b) $m-2 \equiv 4$ (mod 8)

(c) $m - 2 \equiv 5$ (mod 8)

(d) $m - 2 \equiv 6$ (mod 8)

(e) $m - 10 \equiv 7$ (mod 8)

Figure 5 – *Visiting the remaining columns when $m - 2 \equiv 3, 4, 5, 6, 7$ (mod 8)*

(a) $m - 10 \equiv 1 \pmod{8}$
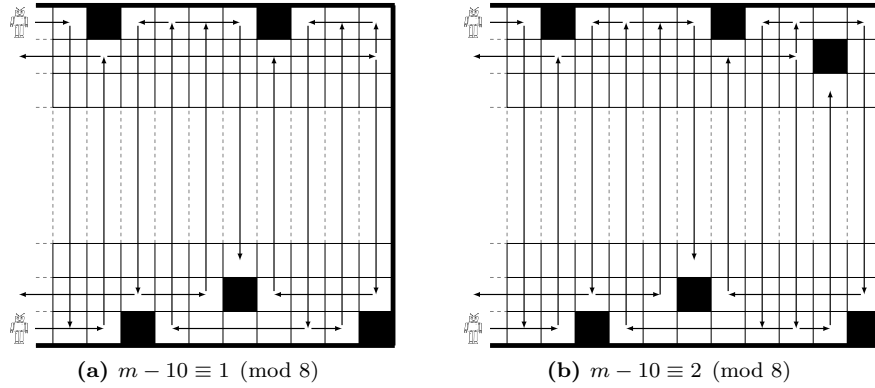
(b) $m - 10 \equiv 2 \pmod{8}$

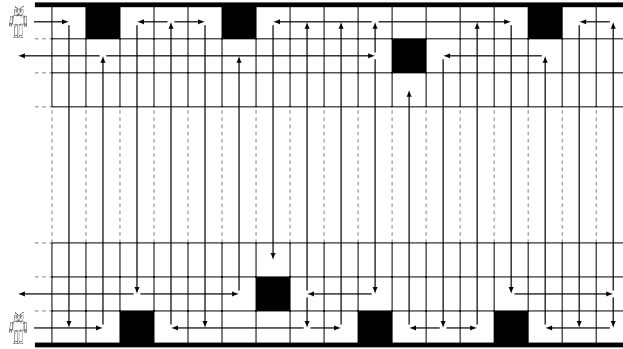**Figure 6** – *Visiting the remaining columns when $m - 10 \equiv 1, 2 \pmod{8}$*

Figure 7 – *Visiting the remaining columns when $m - 18 \equiv 0 \pmod{8}$*

totally dominated by B. There exists a unique minimum total dominating set in P_8, depicted in Figure 8. Consequently, the four blocks of B are located in columns C_3, C_4, C_7 and C_8.
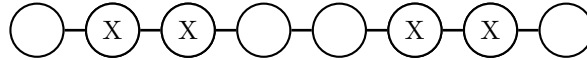

Figure 8 – *Minimum total dominating set in P_8*

Now, since column C_2 is totally dominated by column C_3, the block in C_3 is necessarily located on the first or last row. Otherwise, it would not be possible to make the robot stop in column C_2 since there is no block in C_1. By symmetry, the block in C_8 is also located on the first or last row. Now, with the same argument, since C_3 is totally dominated by C_4 and there are no blocks in C_1 and C_2, the only way to make the robot stop on column C_3 is to place the block of C_4 in the first or last row, and not adjacent to the block in C_3. Ditto for the block in C_7. There are thus only 4 possible sets for B, but for each of them, the position $(1, 5)$ is not reachable. □

Remark 12. *Note that our constructions preserves the accessibility of the robot to every square of the grid if the starting position is different from $(1,1)$. Indeed, one can verify that from any position in P_n or in the ending patterns, there exists a sequence of moves landing in $(1,1)$.*

3.2. **Tori.** We now consider the torus grid $T_{n,m}$ and compute the value $\iota\sigma_B^P(T_{n,m})$ for any values of n and m. Indices are taken modulo m for columns and modulo n for rows.

We first give a lower bound, similar to Proposition 9.

Proposition 13. *For any $n \geq m$, $\iota\sigma_B^P(T_{n,m}) \geq \frac{m-1}{2}$.*

Proof. Consider an optimal solution B of ICE SLIDE$_B^P$ for $T_{n,m}$. If for every column C_j which is not the starting column of the robot, there is a block in column C_{j-1} or C_{j+1}, then there are at least $\frac{m-1}{2}$ blocks in the solution.

Hence we assume that there exists a column C_j, not the starting column of the robot, such that there is no block in columns C_{j-1} and C_{j+1}. Let n_j be the number of blocks in column C_j. The robot can not go vertically in column C_j since to stop in column C_j it needs a block in column C_{j-1} or C_{j+1}. Hence it has to go horizontally trough the $n-n_j$ free squares of the column. But to go horizontally on row R_i, either the robot starts on this row, or there is a block in row R_{i-1} or R_{i+1}. Thus, to go through the $n-n_j$ free squares of C_j, the robot needs at least $\frac{n-n_j-1}{2}$ blocks. Note that these blocks are not on column C_j. At the end, the solution has at least $\frac{n-n_j-1}{2} + n_j \geq \frac{m-1}{2}$ blocks. $\square$

We will prove that this lower bound is reached for $m \geq 6$. For that, we will prove by induction the following stronger statement:

Proposition 14. *For any $m \geq 6$, there exists a solution of ICE SLIDE$_P^B$ on $T_{m,m}$ with $\frac{m-1}{2}$ blocks, such that:*
  (1) *if i_m is the maximum index of a row with a block, the robot can pass horizontally on the row R_{i_m+1},*
  (2) *the robot can pass vertically on all the columns.*

Proof. We will use for the induction the pattern of size 4×8 of Figure 9. The figure gives a proof of the following lemma:

Lemma 15. *Assume that the pattern of Figure 9 is in position (i,j) ((i,j) is the top-left square of the pattern) and that there is no block in any column and row intersecting the pattern (i.e. rows R_i to R_{i+3} and columns C_j to C_{j+7}). Then, if a robot enters in position (i,j) horizontally in the pattern of Figure 9, it can pass vertically on all the columns C_j to C_{j+7} and can go out of the pattern horizontally on rows R_{i-1} and R_{i+4}, in any direction.*

Using this pattern, if we know a solution for $T_{m,m}$ of size $\frac{m-1}{2}$ satisfying Conditions (1) and (2) of the proposition, we can get a solution for $T_{m+8,m+8}$ of size $\frac{m-1}{2} + 4$ still satifsfying the conditions. Indeed, one can copy the solution of $T_{m,m}$ and if i_m (resp. j_m) denotes the largest index of a row (resp. a column) containing a block, add the pattern in position $(i_m + 1, j_m + 1)$ (see Figure 10).
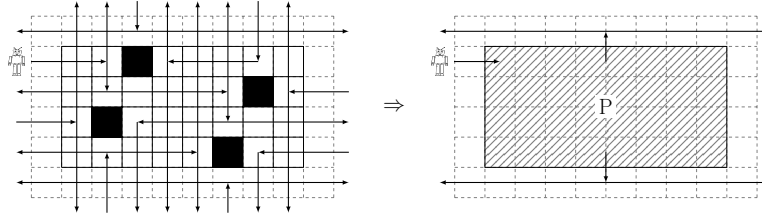
Figure 9 – *Pattern for* ICE SLIDE$_B^P$ *on torus and its representation.*
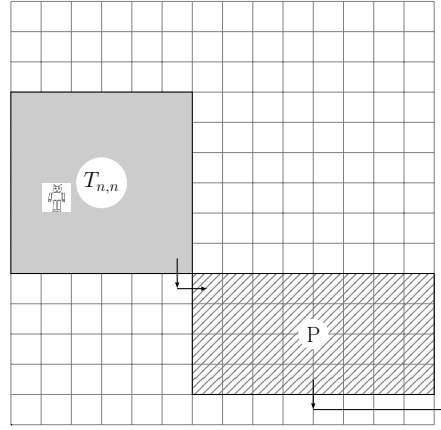

Figure 10 – *Construction of a solution for* $T_{m+8,m+8}$ *from a solution of* $T_{m,m}$. *The conditions 1 and 2 of Proposition 14 remain satisfied.*

Hence we just have to prove the proposition for $m = 6$ to $m = 13$. This is proved by Figure 11. On this figure, we show the solutions for odd m. We get the solutions for even m by removing the row and the column in gray.

$\square$

Since all the columns are passed vertically, we can add as many rows as we want to the torus and we get optimal solutions for $T_{n,m}$ with $n \geq m \geq 6$:

Corollary 16. *For* $n \geq m \geq 6$, $\iota\sigma_P^B(T_{n,m}) = \frac{m-1}{2}$

We now complete the study with the small values of m:

Proposition 17. *We have:*
- $\iota\sigma_P^B(T_{n,1}) = 0$
- $\iota\sigma_P^B(T_{n,2}) = n - 2$
- $\iota\sigma_P^B(T_{n,3}) = 3$
- $\iota\sigma_P^B(T_{n,4}) = 3$
- $\iota\sigma_P^B(T_{n,5}) = 3$

Proof. The case $m = 1$ is trivial.

If $m = 2$, assume that the robot starts at position $(1, 1)$. Then it can only go vertically on column 1, horizontally on row 1 and eventually horizontally on two more rows $i - 1$ and $i + 1$ if there is a block in position $(i, 1)$. Hence $n - 3$ squares are not reachable and we need to add blocks on these squares.
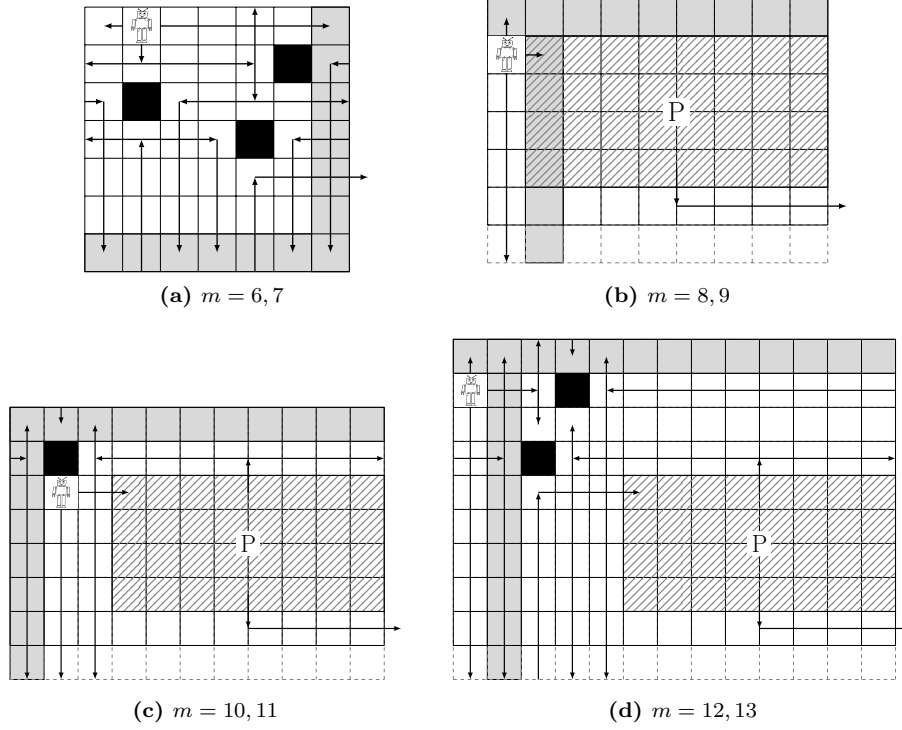
(a) $m = 6, 7$

(b) $m = 8, 9$

(c) $m = 10, 11$

(d) $m = 12, 13$

Figure 11 – *Small cases for induction for* ICE SLIDE$_P^B$ *on torus. Since all columns are passed vertically, we did not represent all the horizontal rows but just the significant ones. To get the even grids, remove the gray row and column.*

For $m \in \{3, 4, 5\}$, assume there is a solution with starting position $(1, 1)$ and two blocks in positions (i_1, j_1) and (i_2, j_2). We must have $1 \in \{i_1, j_1, i_2, j_2\}$. Assume that $i_1 = 1$ (other cases are similar). We have $j_1 \neq 1$, $j_1 \neq j_2$ and $i_2 \neq 1$ (otherwise the robot can not go everywhere). Then one can check that the robot can not go in position (i_2, i_1), which leads to a contradiction. Blocks in positions $\{(1, 3), (2, 4), (3, 2)\}$ is a solution with three blocks. □

Remark 18. *In the torus, all the positions are equivalent. Also, if we knows the starting position before placing the blocks, we can always use our constructions. However, if the solution must work for any starting position, then one need at least $\min(n, m)$ blocks. Indeed, if there exists a row R_i and a column C_j without any block, a robot starting in (i, j) will never be able to leave the squares of $R_i \cup C_j$.*

3.3. **King grids.** We now consider the game ICE SLIDE$_B^P$ played on the King grid $n \times m$, denoted $\mathcal{K}_{n,m}$, also known as the strong product of two paths $P_n \boxtimes P_m$. In the King grid, the robot is also allowed to initiate moves diagonally.

Theorem 19. *Let $m \geq n$ be positive integers.*
If $\gcd(n, m) \leq 3$, then $\iota\sigma_B^P(\mathcal{K}_{n+1, m+1}) = 0$.

(a) *On $\mathcal{K}_{n+1,m+1}$ when $\gcd(n, m) = 3$, the robot can stop on grey squares.*
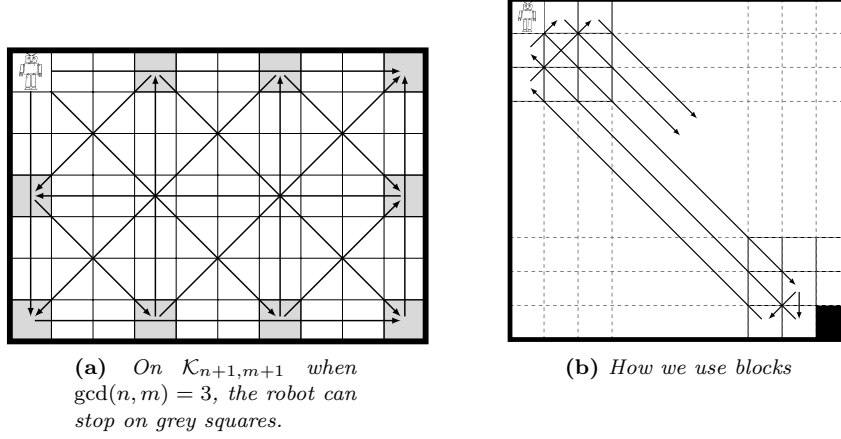
(b) *How we use blocks*

Figure 12 – *Block pass in the king grid.*

Proof. Let $m \geq n$ be positive integers such that $\gcd(n, m) \leq 3$, and consider the game on the King grid $\mathcal{K}_{n+1,m+1}$. We first observe that the robot can stop on all positions $(1, 1 + \alpha n - \beta m)$ where $0 \leq \alpha n - \beta m \leq m$ (see Figure 12a). From $(1, 1)$, the robot can go diagonally to $(1 + n, 1 + n)$ then up to $(1, n + 1)$. Iterating the process, it reaches $(1, 1 + \alpha n)$ for all $0 \leq \alpha \leq \frac{m}{n}$. Eventually, the robot reaches the position $(1, 1 + \lfloor \frac{m}{n} \rfloor n)$. Then, going diagonally, it reaches $(1 + m - \lfloor \frac{m}{n} \rfloor n, m + 1)$, then can slide to $(1 + m - \lfloor \frac{m}{n} \rfloor n, 1)$, and can continue diagonally its movement to $(1 + n, 1 + (\lfloor \frac{m}{n} \rfloor + 1)n - m)$ from which it can go up to $(1, 1 + (\lfloor \frac{m}{n} \rfloor + 1)n - m)$. Iterating the same process, we get that it reaches all positions $(1, 1 + \alpha n - \beta m)$ where $0 \leq \alpha n - \beta m \leq m$. Then, thanks to Bézout's identity, we deduce that the robot reaches all positions $(1, 1 + \alpha \gcd(m, n))$ where $0 \leq \alpha \leq \frac{m}{\gcd(m, n)}$.

Now, if $\gcd(m, n) = 1$, from these positions, the robot can pass vertically any position. If $\gcd(m, n) = 2$, then the robot passes vertically all positions $(x, 1 + 2y)$. From $(1, 1 + 2y)$, it can also reach by a diagonal $(1 + 2y, 1)$, and thus can pass horizontally all $(1 + 2x, y)$. Positions $(2x, 2y)$ can be reached from either $(1 + 2(x - y), 1)$ (if $x \geq y$) or from $(1, 1 + 2(y - x))$ (otherwise). So the robot can pass all positions. Finally, if $\gcd(m, n) = 3$, the robot can pass vertically all $(x, 1 + 3y)$ and horizontally all $(1 + 3x, y)$. The positions $(2 + 3x, 2 + 3y)$ and $(3x, 3y)$ can be reached diagonally from $(1 + 3(x - y), 1)$ or from $(1, 1 + 3(y - x))$. The positions $(3x, 2 + 3y)$ and $(3x - 1, 3y)$ can be reached diagonally from $(n + 1, 3(y + x) - n + 1)$ or from $(3(y + x) + 1, 1)$. □

Theorem 20. *For $m \geq n$ positive integers, we have*

$$\iota\sigma_B^P(\mathcal{K}_{n+1,m+1}) \leq \left\lceil \frac{\gcd(n, m) - 3}{8} \right\rceil.$$

Proof. Let $k = \left\lceil \frac{\gcd(n,m)-3}{8} \right\rceil$, we prove now that using k blocks, the robot may pass on every unoccupied position. For $0 \leq i < k$, we place a block B_i on position $(n + 1, n + 1 - 2i)$ when i is even, and $(1, n + 1 - 2i)$ when i is odd (see Figure 13).

We first prove by induction that the robot can reach position $(1, n + 1 - j)$ or position $(n + 1, n + 1 - j)$ for all $0 \le j \le 2k$, and thus, with two diagonal moves and possibly one vertical move, the robot also reaches positions $(1, 1 + j)$ and $(n + 1, 1 + j)$. Let j be a positive even integer in the range $0 \le j \le 2k - 2$, and assume the robot can reach the positions $(1, 1+j)$ and $(n+1, 1+j)$. The base case, when $j = 0$, is obvious. If j is even, from $(1, j + 1)$, the robot can slide to $(j + 1, 1)$ then to $(n, n - j)$ where it get stopped by block $B_{\frac{j}{2}}$ (see Figure 12b). Then it can go up to $(1, n - j)$ and diagonally to $(n + 1, n - j - 1)$, proving the property for $j + 1$ and $j + 2$.

From these positions, with similar arguments as in the previous lemma, we get that the robot can always reach positions of type $(1, 1 + \alpha \gcd(n, m) - j)$ or $(n + 1, 1 + \alpha \gcd(n, m) - j)$ as well as $(1 + \alpha \gcd(n, m) + j, 1)$ or $(1 + \alpha \gcd(n, m) + j, n + 1)$ with $0 \le j \le 2k$ (see Figure 13. Then, the robots can reach all positions whose line or column can be written in such a way. So consider now a position of type $(1 + \alpha \gcd(n, m) + x, 1 + \beta \gcd(n, m) + y)$ where x and y are larger than $2k$ but less than $\gcd(n, m) - 2k \le 6k + 3$. If $|x - y| \le 2k$, then the robot can reach that position by sliding diagonally from $(1 + (\alpha - \beta) \gcd(n, m) + (x - y), 1)$ or $(1, 1 + (\alpha - \beta) \gcd(n, m) + (x - y))$. If $|x - y| \ge 2k + 1$, we deduce that

$$x + y - gcd(n, m) \le 2(gcd(n, m) - 2k - 1) - (2k + 1) - gcd(n, m) \le 2k$$

$$x + y - gcd(n, m) \ge 2(2k + 1) + 2k + 1 - gcd(n, m) \ge -2k$$

Therefore, if $y' = 1 + \alpha \gcd(n, m) + x + n + 1 - (1 + \beta \gcd(n, m) + y)$ is non negative, then it can be writen in the form $1 + \gamma \gcd(n, m) + j$ or $1 + \gamma \gcd(n, m) - j$ where $0 \le j \le 2k$, and the position can be reached diagonally from $(n + 1, x')$, otherwise $x' = n + 1 - y'$ and then the position can be reached from $(x', 1)$.

$\square$

When $\gcd(m, n) > 3$, it is not difficult to verify that there is no solution without any block, by checking all the possible moves. We can observe that there is no possibility to reach the position $(2, 3)$ for example. However, we do not have any general proof that our bound is optimal, and we leave it as an open question.

4. Other instances of ice slide

4.1. ICE SLIDE$_B^S$. In what follows, we consider the game ICE SLIDE played with blocks on a grid where the robot has to stop on each square. An inner square of the grid is a square (i, j) such that $2 \le i \le n - 1$ and $2 \le j \le m - 1$. Other squares are called border squares.

We obtain the following lower bound for $\iota\sigma_B^S(G_{n,m})$:

Proposition 21. *For each $n \ge m \ge 3$, we have $\iota\sigma_B^S(G_{n,m}) \ge \gamma(G_{n-2,m-2})$.*

Proof. Consider an optimal solution B of ICE SLIDE$_B^S$ on $G_{n,m}$. Let (i, j) be an inner square of the grid. There must a block adjacent to (i, j) otherwise the robot cannot stop on it. Let G' be the grid induced by the inner squares. It is isomorphic to $G_{n-2,m-2}$. Let B_1 be the set of blocks of B that are located on inner squares. Let V_b be the set of squares of G' that are not adjacent to a block of B_1. Note that all these squares are on the border of G', and that, for each square (i, j) of V_b, there is a block on an adjacent square of (i, j) which is on the border of $G_{n,m}$. Thus we have $|B| \ge |B_1| + |V_b|$. Now let us consider B' defined as the disjoint union of
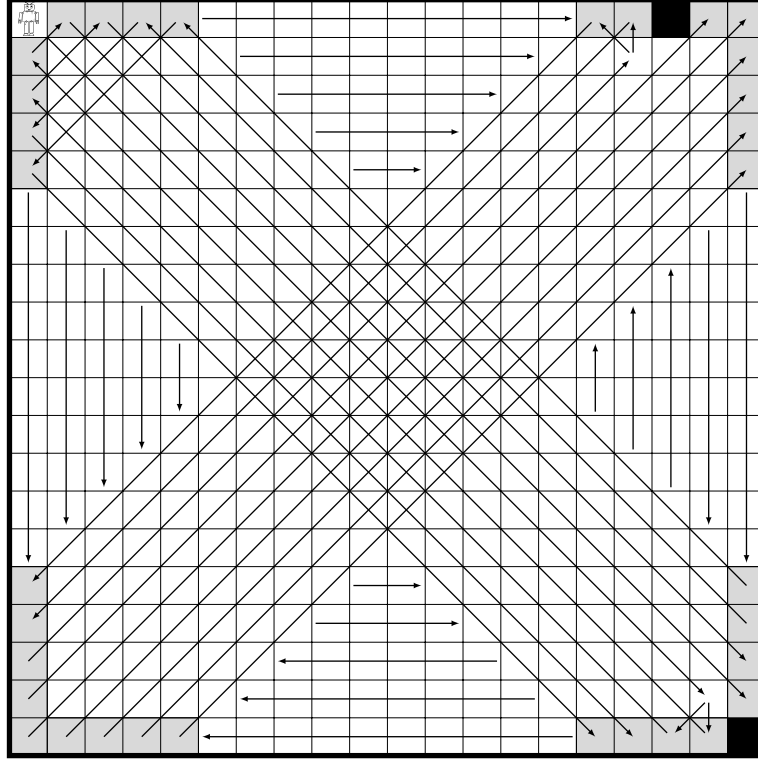
Figure 13 – *Two blocks for the King grid $\mathcal{K}_{20,20}$.*

B_1 and V_b. Clearly, B' is a dominating set of $G_{n-2,m-2}$, and we have $|B'| \leq |B|$. Hence we have:

$$\iota\sigma_B^S(G_{n,m}) = |B| \geq |B'| \geq \gamma(G_{n-2,m-2}).$$

$\square$

We now give a construction of a solution of ICE SLIDE_B^S on $G_{n,m}$.

Proposition 22. *For any $n \geq m \geq 3$, we have*

$$\iota\sigma_B^S(G_{n,m}) \leq \gamma(G_{n-2,m-2}) + \frac{18}{25}(m+n) + \frac{28}{5}.$$

Proof. Let $n \geq m \geq 2$.

We construct a solution B of ICE SLIDE_B^S in three steps. The construction is illustrated on Figures 14 and 15.

Let first B_1 be the set of blocks that are the translation of the block located on square $(1, 2)$ by linear combinations of the two vectors $\{(1, 2), (3, 1)\}$ (see Figure 14). Note that B_1 covers all the inner squares of $G_{n,m}$ and that each inner square is covered exactly once.

Let us call B_1' the subset of blocks which are located on the border. By construction, there are at most $2\left(\left\lceil \frac{n-2}{5} \right\rceil + \left\lceil \frac{m-2}{5} \right\rceil\right) \leq \frac{2}{5}(m+n) + 2$ such blocks. Clearly, the blocks of B_1' cover a subset S' of the inner squares such that $|B_1'| = |S'|$.

Let us denote S'' the set of inner squares which are not in S'. Since each inner square is covered exactly once, then the subset B_1'' of blocks of B_1 that are located on inner squares satisfies $5|B_1''| = |S''|$.

On the other hand, since each square of the grid has at most four neighbours, then we have $\gamma(G_{n-2,m-2}) \geq \frac{1}{5}(|S'|+|S''|)$. Hence we have $\gamma(G_{n-2,m-2}) \geq \frac{1}{5}|B_1'|+|B_1''|$, and since B_1 is the disjoint union of B_1' and B_1'' then we have $|B_1| \leq \gamma(G_{n-2,m-2})+ \frac{8}{25}(m+n) + \frac{8}{5}$.

The blocks of B_1 will be enough to ensure that the robot will be able to stop on almost each inner square as soon as it may reach some of them. Indeed, if the robot enters in a rectangle of size 4×4 that has four blocks on its border (see as an example the gray rectangle on Figure 14), then it will be able to stop in the four inner squares of this rectangle and thus to go out of the rectangle from these four places (see Figure 15). To ensure that the robot will be able to enter in those rectangles from the border of the grid and to stop on the border, we add some blocks that will form the set B_2.
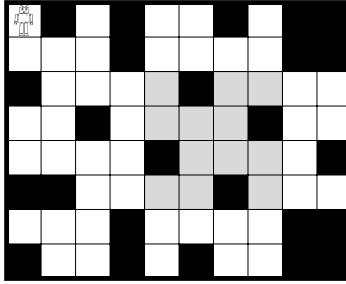

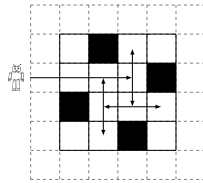**Figure 14** – *Solution of* ICE SLIDE$_B^S$ *on* $G_{8,10}$

Figure 15 – *Pattern of the solution of* ICE SLIDE$_B^S$

For each block of B_1 located on an inner square which is neighbour of a border square, we add a block in B_2 in the following way:

- for each block $(2, j) \in B_1$, we add the block $(1, j)$ to B_2;
- for each block $(n - 1, j) \in B_1$, we add the block $(n, j - 1)$ to B_2;
- for each block $(i, 2) \in B_1$, we add the block $(i, 1)$ to B_2;
- for each block $(i, m - 1) \in B_1$, we add the block (i, m) to B_2.

By construction, we have $|B_2| \leq 2 \left(\lceil \frac{n-2}{5} \rceil + \lceil \frac{m-2}{5} \rceil \right) \leq \frac{2}{5}(m+n) + 2$.

Finally, we add blocks in the corners of $G_{n,m}$ if these squares are not reachable by the robot:

- if there is a block on $(2, m-1)$, then we add the block $(1, m)$;
- if there is a block on $(n-1, 2)$, then we add the block $(n, 1)$;
- if there is a block on $(n-1, m-1)$, then we add the block (n, m).

These blocks form the set B_3. Note that at most two blocks are added in B_3, since $(1, m)$ is added if and only if $m \equiv 0 \mod 5$, and $(n, 1)$ is added if and only if $n \equiv 2 \mod 5$.

The set of blocks $B = B_1 \cup B_2 \cup B_3$ form a solution of ICE SLIDE$_B^S$, of cardinality at most $\gamma(G_{n-2,m-2}) + \frac{18}{25}(m+n) + \frac{28}{5}$.

$\square$

By combining the lower and uppers bounds and since $\gamma(G_{n,m}) = \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4$ [6], we obtain an asymptotic value for $\iota\sigma_B^S(G_{n,m})$:

Corollary 23. *If m and n are large enough, $\iota\sigma_B^S(G_{n,m}) = \frac{mn}{5} + O(m+n)$.*

Notice that the construction of Proposition 22 is optimal for $n = m = 2$, $n = m = 3$, and $m = n = 4$. We did not try to find the exact value of $\iota\sigma_B^S(G_{n,m})$ in the general case since we think that such a result does not deserve much interest compared to the tedious case study it would need to be proved.

Remark 24. *Remark that, in the torus, using an optimal dominating set we immediately get $\iota\sigma_B^S(T_{n,m}) = \frac{nm}{5}$ when $n \equiv 0 \mod 5$ and $m \equiv 0 \mod 5$. Similarly, in the general case, one could also show that $\iota\sigma_B^S(T_{n,m}) = \frac{nm}{5} + O(m+n)$.*

4.2. ICE SLIDE$_W^P$. In what follows, we consider the game ICE SLIDE played with walls on a grid and where the robot has to pass over each square. Unlike blocks, there will be two types of walls: horizontal and vertical ones (see Figure 3).

Definition 25. *Let W be a set of walls. The row R_i, $2 \leq i \leq n-1$ (resp. column C_j, $2 \leq j \leq m-1$) is said to be covered by W if there is at least one horizontal wall of W adjacent to row R_i (resp. one vertical wall adjacent to column C_i).*

By similar considerations to Subsection 3.1, we get the following observation:

Observation 26. *If W is a set of walls in $G_{n,m}$ which covers every column C_j, $2 \leq j \leq m-1$, then $|W| \geq \rho(P_{m-2})$.*

This result leads to the following lower bound for $\iota\sigma_W^P(G_{n,m})$:

Proposition 27. *For each $n \geq m$, we have $\iota\sigma_W^P(G_{n,m}) \geq \rho(P_{m-2})$.*

Proof. Similar to the one of Proposition 9. If every column is covered by a wall, then the proposition holds from Observation 26. Now consider an optimal solution W such that there exists a column C_j, $2 \leq j \leq m-1$ which is not covered by a wall. The robot cannot move vertically in C_j since there is no vertical wall adjacent to this column. Hence it has to go horizontally through every row R_i, $2 \leq i \leq n-1$, implying that each such row must be covered by a wall. Therefore we get the desired result:

$$|W| \geq \rho(P_{n-2}) \geq \rho(P_{m-2}).$$

$\square$

The following lemma is the key to reach the lower bound of the above proposition:

Lemma 28. *Assume that the pattern P_n of size $n \times 4$ depicted on Figure 16 is in position $(1, j)$ ($(1, j)$ being the top-left square of the pattern). Then, if a robot is able to enter horizontally either in $(1, j)$ or $(n, j + 3)$, it can pass vertically through all the columns C_j to C_{j+3}. In addition, if a robot enters horizontally the pattern P_n in position $(1, j)$, it can leave it horizontally on rows R_1 and R_n in either direction.*

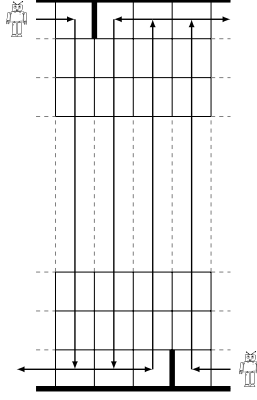A proof of this result is given by Figure 16.

Figure 16 – *Pattern of size $n \times 4$ for* ICE SLIDE$_W^P$ *on grid*

Theorem 29. *For any $n \geq m \geq 4$, we have $\iota\sigma_W^P(G_{n,m}) = \rho(P_{m-2})$.*
For $m = 3$ and $n \geq m$, we have $\iota\sigma_W^P(G_{n,m}) = 1$.
For $m \leq 2$ and $n \geq m$, we have $\iota\sigma_W^P(G_{n,m}) = 0$.

Proof. Let $n \geq m \geq 4$. A solution is build by gluing $\lceil \frac{m-2}{4} \rceil$ copies of P_n starting from column C_2 and ending in C_{m-1} (leaving columns C_1 and C_m empty). In this process, if $(m - 2) \not\equiv 0 \pmod 4$, then remove the $(m - 2) \pmod 4$ last columns of the rightmost copy of P_n. Figure 16 ensures that all the columns (except the last ones) of all the copies are visited by entering from row R_1. It also shows that all the rightmost columns of each pattern can be visited by entering from the right on row R_n. Note that it is also true for the rightmost copy of P_n since the robot can move vertically in column C_m.
In the case where $m = 3$, it is straightforward to see that from position $(1, 1)$, the robot cannot visit all the squares if there is no wall. A vertical wall adjacent to position $(1, 2)$ is however sufficient. $\square$

Remark 30. *Note that Theorem 29 remains valid for any starting position. It is not hard to see that the pattern of Figure 16 ensures the robot to reach $(1, 1)$ from any position.*

4.3. ICE SLIDE$_W^S$. In what follows, we consider the game ICE SLIDE played with walls on a grid where the robot has to stop on each square. Constructions and proofs are similar to the ones of Subsection 4.1. We obtain the following lower bound for $\iota\sigma_W^S(G_{n,m})$:

Proposition 31. *For each $n \geq m \geq 3$, we have $\iota\sigma_W^S(G_{n,m}) \geq \rho(G_{n-2,m-2})$.*

Proof. Consider an optimal solution W of ICE SLIDE$_W^S$ on $G_{n,m}$. Let (i,j) be an inner square of the grid, i.e $2 \leq i \leq n-1$ and $2 \leq j \leq m-1$. There must a wall adjacent to (i,j) otherwise the robot cannot stop on it. Let G' be the grid induced by the inner squares. It is isomorphic to $G_{n-2,m-2}$. Let W_1 be the set of walls that are adjacent to two inner squares of $G_{n,m}$. Let V_b be the set of squares of G' that are not adjacent to a wall of W_1. Note that all these squares are on the border of G' and for each square of V_b, there is a wall between this square and the adjacent square on the border of $G_{n,m}$. Hence one can add to W_1 one wall adjacent to each square of V_b and this new set W' will have size at most $|W|$. The set W' naturally corresponds to a set of edges of G' that cover the vertices of G'. Hence we have:

$$\iota\sigma_W^S(G_{n,m}) = |W| \geq |W'| \geq \rho(G_{n-2,m-2}).$$

$\square$

We now give a construction of a solution of ICE SLIDE$_W^S$ on $G_{n,m}$.

Proposition 32. *For any $n \geq m \geq 3$, we have*

$$\iota\sigma_W^S(G_{n,m}) \leq \rho(G_{n-2,m-2}) + \frac{3(m+n)}{4} + 3.$$

Proof. Let $n \geq m \geq 3$. We construct a solution W of ICE SLIDE$_W^S$ in three steps. The construction is illustrated on Figures 17 and 18.

We denote by $(i,j) - (i,j+1)$ the vertical wall between the square (i,j) and $(i,j+1)$ and use similar notations for horizontal walls. An *inner wall* is a wall that is not touching the border (if it is a vertical wall, it means that $2 \leq j \leq m-2$ and i is not constrained in the previous notation).

Let first W_1 be the set of inner walls that are the translation of the walls $(2,2) - (2,3)$ and $(3,2)-(4,2)$ by linear combinations of the three vectors $\{(1,1),(0,4),(4,0)\}$ (see the inner walls of Figure 17). Note that W_1 covers all the inner squares of $G_{n,m}$ and that each square is covered exactly once. Some walls of W_1 are located between an inner square and a border square. There are at most $2\left(\lceil \frac{n-2}{4} \rceil + \lceil \frac{m-2}{4} \rceil\right) \leq \frac{m+n}{2} + 2$ such walls. Hence $|W_1| \leq \rho(G_{n-2,m-2}) + \frac{m+n}{4} + 1$.

The walls of W_1 will be enough to ensure that the robot will stop on almost each inner square as soon as it will enter to some of them. Indeed, if the robot enters in a rectangle of size 3×2 that has four walls on its border (see as an example the gray rectangle on Figure 17), then it will be able to stop in the four corners of this rectangle and thus to go out of the rectangle from these four places (see Figure 18). To ensure that the robot will enter in the rectangles of the border of the grid and stop on the border, we add some walls that will form the set W_2. For each wall of W_1 located between an inner square and a border square, we add a wall in W_2 in the following way:

- for each $(1,j) - (2,j) \in W_1$, we add the wall $(1,j) - (1,j+1)$ to W_2;
- for each $(n-1,j) - (n,j) \in W_1$, we add the wall $(n,j-1) - (n,j)$ to W_2;
- for each $(i,1) - (i,2) \in W_1$, we add the wall $(i,1) - (i+1,1)$ to W_2;
- for each $(i,m-1) - (i,m) \in W_1$, we add the wall $(i-1,m) - (i,m)$ to W_2.

We furthermore add the wall $(2,1) - (3,1)$ to W_2. Finally, if $(n-2,m-1) - (n-1,m-1)$ (respectively $(n-1,m-2) - (n-1,m-1)$) is a wall, we add the wall $(n,m-2) - (n,m-1)$ (resp. $(n-2,m) - (n-1,m)$) to W_2. Note that the

first case holds if and only if $n - m \equiv 2 \mod 4$ and whereas the second case holds if and only if $n - m \equiv 3 \mod 4$.

The final set of walls $W_1 \cup W_2$ is a solution of ICE SLIDE$_W^S$ of size at most
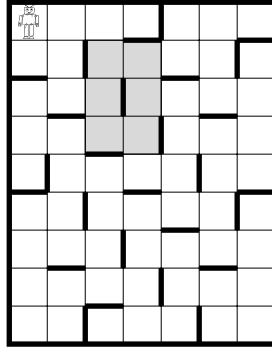
$$\rho(G_{n-2,m-2}) + \frac{3(m+n)}{4} + 3.$$

$\square$

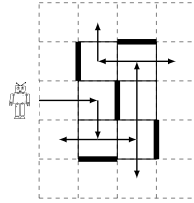**Figure 17** – *Solution of* ICE SLIDE$_W^S$ *on* $G_{9,7}$

Figure 18 – *Pattern of the solution of* ICE SLIDE$_W^S$

By combining the lower and uppers bounds and noticing that $\rho(G_{n,m}) = \left\lceil \frac{nm}{2} \right\rceil$, we obtain an asymptotic value for $\iota\sigma_W^S(G_{n,m})$:

Corollary 33. *If m and n are large enough, $\iota\sigma_W^S(G_{n,m}) = \frac{mn}{2} + O(m+n)$.*

As for Subsection 4.1, we did not try to find the exact value of $\iota\sigma_W^S(G_{n,m})$.

Remark 34. *The solution built in Proposition 32 remains valid for any starting position except the part of the border around the bottom-right corner when $n - m \equiv 0 \mod 4$ or $n - m \equiv 1 \mod 4$.*

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