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Small moving rigid body into a viscous incompressible fluid

Christophe Lacave & Takéo Takahashi

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Abstract

We consider rigid bodies moving under the influence of a viscous fluid and we study the asymptotic as the size of the solids tends to zero.

In a bounded domain, if the solids shrink to “massive” pointwise particles, we obtain a convergence to the solution of the Navier-Stokes equations independently to any possible collision of the bodies with the exterior boundary.

In the case of “massless” pointwise particles, we obtain a result for a single disk moving in the full plane. In this situation, the energy equality is not sufficient, and we obtain a uniform estimate for the solid velocity thanks to the optimal $L^p - L^q$ decay estimates of the Stokes semigroup.

1 Introduction

We study in this paper the asymptotic of a fluid-solids system as the solids shrink to points.

More precisely, we consider N rigid bodies $\mathcal{S}_i^\varepsilon(t)$ of shape $\mathcal{S}_{i,0}$ and of size ε immersed into a viscous incompressible fluid. The rigid bodies occupy the sets $\mathcal{S}_{i,0}^\varepsilon$ at the initial time:

$$\mathcal{S}_{i,0}^\varepsilon := h_{i,0} + \varepsilon \mathcal{S}_{i,0},$$

where we assume that $\mathcal{S}_{i,0}$ is a smooth simply-connected compact subset of \mathbb{R}^2 , with nonempty interior and where the center of mass of $\mathcal{S}_{i,0}$ is 0. At $t = 0$, the viscous fluid fills the domain

$$\mathcal{F}_0^\varepsilon := \Omega \setminus \left(\bigcup_{i=1}^N \mathcal{S}_{i,0}^\varepsilon \right)$$

where Ω is either a bounded domain of \mathbb{R}^2 or the full plane. The positions $h_{i,0} \in \Omega$ are distinct, hence for ε small enough we have

$$\mathcal{S}_{i,0}^\varepsilon \cap \mathcal{S}_{j,0}^\varepsilon = \emptyset \quad (i \neq j) \quad \text{and} \quad \mathcal{S}_{i,0}^\varepsilon \subset \Omega. \quad (1.1)$$

At time $t > 0$, the rigid bodies follow the trajectory

$$\mathcal{S}_i^\varepsilon(t) := h_i^\varepsilon(t) + R_{\theta_i^\varepsilon(t)}(\mathcal{S}_{i,0}^\varepsilon - h_{i,0}), \quad (1.2)$$

where for all t ,

$$h_i^\varepsilon(t) \in \mathbb{R}^2, \quad \theta_i^\varepsilon(t) \in \mathbb{R}, \quad R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

The domain of the fluid evolves through the formula

$$\mathcal{F}^\varepsilon(t) := \Omega \setminus \left(\bigcup_{i=1}^N \mathcal{S}_i^\varepsilon(t) \right). \quad (1.3)$$

We denote by $n := n^\varepsilon(t, x)$ the exterior unit normal of $\partial\mathcal{F}^\varepsilon(t)$. The equations for the fluid-solids system read

$$\frac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon \cdot \nabla)u^\varepsilon - \operatorname{div} \sigma(u^\varepsilon, p^\varepsilon) = 0 \quad t > 0, \quad x \in \mathcal{F}^\varepsilon(t), \quad (1.4)$$

$$\operatorname{div} u^\varepsilon = 0 \quad t > 0, \quad x \in \mathcal{F}^\varepsilon(t), \quad (1.5)$$

$$u^\varepsilon = 0 \quad t > 0, \quad x \in \partial\Omega, \quad (1.6)$$

$$u^\varepsilon = (h_i^\varepsilon)'(t) + (\theta_i^\varepsilon)'(t)(x - h_i^\varepsilon(t))^\perp \quad t > 0, \quad x \in \partial\mathcal{S}_i^\varepsilon(t), \quad i = 1, \dots, N, \quad (1.7)$$

$$m_i^\varepsilon (h_i^\varepsilon)''(t) = - \int_{\partial\mathcal{S}_i^\varepsilon(t)} \sigma(u^\varepsilon, p^\varepsilon) n \, d\gamma \quad t > 0, \quad i = 1, \dots, N, \quad (1.8)$$

$$J_i^\varepsilon (\theta_i^\varepsilon)''(t) = - \int_{\partial\mathcal{S}_i^\varepsilon(t)} (x - h_i^\varepsilon)^\perp \cdot \sigma(u^\varepsilon, p^\varepsilon) n \, d\gamma \quad t > 0, \quad i = 1, \dots, N, \quad (1.9)$$

$$u^\varepsilon(0, \cdot) = u_0^\varepsilon \quad \text{in } \mathcal{F}_0^\varepsilon, \quad (1.10)$$

$$h_i^\varepsilon(0) = h_{i,0}, \quad (h_i^\varepsilon)'(0) = \ell_{i,0}^\varepsilon, \quad \theta_i^\varepsilon(0) = 0, \quad (\theta_i^\varepsilon)'(0) = r_{i,0}^\varepsilon, \quad i = 1, \dots, N. \quad (1.11)$$

Here and in what follows

$$\sigma(u, p) = 2\nu D(u) - pI_2,$$

with $\nu > 0$ is the constant viscosity and

$$D(u) := \frac{1}{2} ((\nabla u) + (\nabla u)^*).$$

We write for any $x \in \mathbb{R}^2$,

$$x^\perp := \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = R_{\pi/2}x.$$

In the unbounded case $\Omega = \mathbb{R}^2$, (1.6) should be understood as

$$\lim_{|x| \rightarrow \infty} u^\varepsilon(t, x) = 0 \quad t > 0.$$

It is convenient to extend the velocity field u^ε inside the domains of the rigid bodies as follows:

$$u^\varepsilon(t, x) = (h_i^\varepsilon)'(t) + (\theta_i^\varepsilon)'(t)(x - h_i^\varepsilon(t))^\perp \quad t > 0, \quad x \in \mathcal{S}_i^\varepsilon(t), \quad i = 1, \dots, N.$$

For any smooth open set \mathcal{O} , we define

- $V(\mathcal{O}) := \{\varphi \in C_0^\infty(\mathcal{O}) \mid \operatorname{div} \varphi = 0 \text{ in } \mathcal{O}\}$;
- $\mathcal{H}(\mathcal{O})$ the closure of $V(\mathcal{O})$ in the norm L^2 :

$$\mathcal{H}(\mathcal{O}) = \{\varphi \in L^2(\mathcal{O}) \mid \operatorname{div} \varphi = 0 \text{ in } \mathcal{O}, \varphi \cdot n = 0 \text{ at } \partial\mathcal{O}\};$$

- $\mathcal{V}(\mathcal{O})$ the closure of $V(\mathcal{O})$ in the norm H^1 :

$$\mathcal{V}(\mathcal{O}) = \{\varphi \in H_0^1(\mathcal{O}) \mid \operatorname{div} \varphi = 0 \text{ in } \mathcal{O}\}$$

and its dual space by $\mathcal{V}'(\mathcal{O})$ with respect to $\mathcal{H}(\mathcal{O})$.

We also define $\mathcal{V}_R(\mathcal{F}^\varepsilon(t))$ the subspace of $\mathcal{V}(\Omega)$ of velocity fields that are rigid in the solids:

$$\mathcal{V}_R(\mathcal{F}^\varepsilon(t)) := \{\varphi \in H_0^1(\Omega) \mid D(\varphi) = 0 \text{ in } \mathcal{S}_i^\varepsilon(t) \forall i \in \{1, \dots, N\}, \operatorname{div} \varphi = 0\}. \quad (1.12)$$

In this paper, we will consider the asymptotic $\varepsilon \rightarrow 0$ in the two following situations:

- several bodies shrinking to “massive” pointwise particles in a bounded domain;
- a single disk shrinking to a “massless” pointwise particle in the full plane.

The precise definition of massive/massless is given below.

1.1 Massive pointwise particles in a bounded domain

In this situation, Ω is a bounded domain of \mathbb{R}^2 and we are considering the case where the solids tend to *massive pointwise particles*:

$$m_i^\varepsilon = m_i^1 > 0 \quad \text{and} \quad J_i^\varepsilon = \varepsilon^2 J_i^1 > 0. \quad (1.13)$$

For instance, this is the case when the density ρ_i^ε of the rigid bodies satisfies

$$\rho_i^\varepsilon = \frac{\rho_i}{\varepsilon^2}$$

with ρ_i a constant.

Let us define a global density in Ω by

$$\rho^\varepsilon(t, x) = \begin{cases} 1 & x \in \mathcal{F}^\varepsilon(t), \\ \rho_i^\varepsilon & x \in \mathcal{S}_i^\varepsilon(t). \end{cases} \quad t \geq 0.$$

Under the following hypotheses on the initial conditions

$$\begin{aligned} u_0^\varepsilon &\in L^2(\mathcal{F}_0^\varepsilon), \quad \operatorname{div} u_0^\varepsilon = 0, \quad u_0^\varepsilon \cdot n = 0 \text{ on } \partial\Omega, \\ u_0^\varepsilon \cdot n &= (\ell_{i,0}^\varepsilon + r_{i,0}^\varepsilon(x - h_{i,0}^\varepsilon)) \cdot n \text{ on } \partial\mathcal{S}_{i,0}^\varepsilon, \quad i \in \{1, \dots, N\}, \end{aligned} \quad (1.14)$$

(that is $u_0^\varepsilon \in \mathcal{V}_R(\mathcal{F}_0^\varepsilon)$ after the extension defined above) together with (1.1), there exists a global weak solution $(u^\varepsilon, h_i^\varepsilon, \theta_i^\varepsilon)$ see [31] (see also [12], [18]), in the sense of the definition below.

Definition 1.1. *We say that $(u^\varepsilon, h_i^\varepsilon, \theta_i^\varepsilon)$ is a global weak solution of (1.4)–(1.11) if, for any $T > 0$, we have*

$$u^\varepsilon \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad u^\varepsilon(t, \cdot) \in \mathcal{V}_R(\mathcal{F}^\varepsilon(t)),$$

if

$$\mathcal{S}_i^\varepsilon(t) \cap \mathcal{S}_j^\varepsilon(t) = \emptyset \quad \forall i \neq j, \quad t \in [0, T];$$

if it satisfies the weak formulation

$$- \int_0^T \int_\Omega \rho^\varepsilon u^\varepsilon \cdot \left(\frac{\partial \varphi^\varepsilon}{\partial t} + (u^\varepsilon \cdot \nabla) \varphi^\varepsilon \right) dx ds + 2\nu \int_0^T \int_\Omega D(u^\varepsilon) : D(\varphi^\varepsilon) dx ds = \int_\Omega \rho^\varepsilon u_0^\varepsilon(x) \cdot \varphi^\varepsilon(0, x) dx, \quad (1.15)$$

for any $\varphi^\varepsilon \in C_c^1([0, T]; H^1(\Omega))$ such that $\varphi^\varepsilon(t, \cdot) \in \mathcal{V}_R(\mathcal{F}^\varepsilon(t))$ and if it satisfies the energy inequality

$$\frac{1}{2} \int_\Omega \rho^\varepsilon(t, x) |u^\varepsilon(t, x)|^2 dx + 2\nu \int_0^t \int_\Omega |D(u^\varepsilon)|^2 dx ds \leq \frac{1}{2} \int_\Omega \rho_0^\varepsilon(x) |u_0^\varepsilon(x)|^2 dx \quad \text{a.e. } t \in (0, T). \quad (1.16)$$

Remark 1.2. For $\partial\Omega$ and $\mathcal{S}_{i,0}$ of class C^2 , San Martín, Starovoitov and Tucsnak established in [31] the existence of a weak solution $(u^\varepsilon, h_i^\varepsilon, \theta_i^\varepsilon)$ globally in time, even with possible contact between the rigid bodies or between a rigid body and the exterior boundary $\partial\Omega$.

When $\varepsilon \rightarrow 0$, we establish the convergence of u^ε to the unique solution of the Navier-Stokes equations in the full domain Ω .

Theorem 1.3. *Assume (1.13), (1.14) and that*

$$u_0^\varepsilon \rightharpoonup u_0 \quad \text{in} \quad L^2(\Omega), \quad (1.17)$$

$$|\ell_{i,0}^\varepsilon| \leq C, \quad \varepsilon |r_{i,0}^\varepsilon| \leq C, \quad \forall i \in \{1, \dots, N\}. \quad (1.18)$$

Then there exists $T > 0$ such that

$$u^\varepsilon \xrightarrow{*} u \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \quad (1.19)$$

where u is the weak solution of the Navier-Stokes equations associated to u_0 : for any $\varphi \in C_c^1([0, T]; \mathcal{V}(\Omega))$,

$$- \int_0^T \int_\Omega u \cdot \left(\frac{\partial \varphi}{\partial t} + (u \cdot \nabla) \varphi \right) dx ds + \nu \int_0^T \int_\Omega \nabla u : \nabla \varphi dx ds = \int_\Omega u_0(x) \cdot \varphi(0, x) dx. \quad (1.20)$$

Remark 1.4. To prove Theorem 1.3, we need the existence of a positive distance between the rigid bodies, independent of ε . Therefore, denoting by h_i the limit of h_i^ε , the time T in the above theorem corresponds to a time such that

$$|h_i(t) - h_j(t)| > 0 \quad (i \neq j, \quad t \in [0, T]).$$

In contrast, we do not need a positive distance between the bodies and the exterior boundary $\partial\Omega$. In particular, in the case of a single rigid body (i.e. $N = 1$), we can take T arbitrary large.

There are also several works showing that for some particular geometries, there are no contact in finite time (see [20], [19], [21]). In that case, we can also consider any $T \in \mathbb{R}^+$.

The asymptotic behavior of the fluid motion around shrinking obstacles is already considered in several recent papers. Iftimie, Lopes Filho and Nussenzveig Lopes [23] have studied the case of one small fixed obstacle in an incompressible viscous fluid in 2D. Iftimie and Kelliher [22] have treated the same situation in 3D. In [26, 27] Lacave has considered the case of one thin obstacle shrinking to a curve in 2D and 3D.

There is also a large literature about porous medium in the homogenization framework. Since the pioneer work of Cioranescu and Murat [6] for the Laplace problem, the Navier-Stokes system was studied, in particular, by Allaire [1, 2]. We also mention [7, 8, 29, 32, 33, 38] for the fluid motion through a perforated domain.

In all the above studies, the general strategy relies on energy estimate to get a uniform estimate in H^1 . It turns out that such an estimate is sufficient to pass to the limit in the weak formulation by a truncature procedure. Namely, for a test function $\varphi \in \mathcal{D}(\Omega)$ and for a cutoff function χ^ε , we note that $\chi^\varepsilon\varphi$ is an admissible test function for the Laplace problem in the perforated domain Ω^ε . If the inclusions are far enough, for standard cutoff function, $\|\chi^\varepsilon\|_{W^{1,p}}$ remains bounded only for $p \leq 2$ in dimension two, which allows to pass to the limit in terms such as $\int \nabla u^\varepsilon : \nabla(\chi^\varepsilon\varphi)$. For the Navier-Stokes equations, the cutoff procedure is more complicated and relies on Bogovskiĭ operators. Indeed, we need approximated test functions that are divergence free (see [27] and Section 3.2).

When the obstacles can move under the influence of the fluid, we also need to control uniformly the velocities of the solids $(h_i^\varepsilon)'(t)$. When the masses are independent of ε (1.13), this estimate follows easily from the energy estimate (1.16).

Remark 1.5. We do not use that the domain is bounded in Theorem 1.3, and a similar result holds in the case of $\Omega = \mathbb{R}^2$. The only difference is that the existence of global weak solutions is only stated in the literature in the case of a single rigid body (see [34]). The well-posedness in the case of several bodies in the full plane could be established with similar arguments as [31] (for instance). One could also use the result in [9] for strong solutions in the case of several rigid bodies in the plane.

Indeed, as the compactness argument holds up to collision between the bodies, it is easy to adapt our result with strong solutions, up to choosing more regular u_0^ε . But even in this situation, our main argument is related to a cutoff procedure and a passing to the limit in the weak formulation (1.15).

1.2 Massless pointwise particle in the full plane

In contrast to the massive case, when the masses tend to zero, it is no more possible from the energy estimate to deduce estimates of $(h_i^\varepsilon)'(t)$ independently of ε . One could try to get an estimate of $(h_i^\varepsilon)'(t)$ from the boundary condition. However, since the size of the solids tends to zero, this leads to look for a C^0 -estimate for the fluid velocity, and thus for H^s estimates with $s > 1$. It was the strategy followed in [11, 36] with a H^2 analysis. Unfortunately, these articles are based on uniform elliptic estimates in the exterior of a small obstacle which fail for $s > 1$; this can be noted by a scaling argument (see also a counter-example related to these estimates in [5]).

Our strategy is different here. Our basic remark is that the small obstacle limit is related to the long-time behavior though the scaling property of the Navier-Stokes equations $u^\varepsilon(t, x) = \varepsilon^{-1}u^1(\varepsilon^{-2}t, \varepsilon^{-1}x)$. For one disk moving in the plane, the long-time behavior was recently studied by Ervedoza, Hillairet and Lacave in [13]. In particular, the optimal decay estimates of the Stokes semigroup are the key to treat the massless pointwise particle.

More precisely, we consider one rigid disk moving in the plane:

$$N = 1, \quad \Omega = \mathbb{R}^2, \quad \text{and } \mathcal{S}_0 = B(0, 1). \quad (1.21)$$

Without loss of generality, we can assume that $h_0 = 0$ (we also suppress the subscript $i = 1$ to ease notation). To apply the result in [13], we also need to assume that the center of the mass corresponds with the center of the disk. For simplicity, let us assume that the density is constant $\rho_0 > 0$ in the disk.

In this setting and for initial data satisfying (1.14), the existence and uniqueness of a global weak solution $(u^\varepsilon, h^\varepsilon, \theta^\varepsilon)$ were established by Takahashi and Tucsnak [37].

The goal of the second theorem is to treat the case where the disk shrinks to a *massless pointwise particle*:

$$\rho^\varepsilon = \rho_0 \quad (1.22)$$

hence,

$$m^\varepsilon = \varepsilon^2 m^1 \quad \text{and} \quad J_i^\varepsilon = \varepsilon^4 J^1. \quad (1.23)$$

We consider the massless case for small data:

Theorem 1.6. *Assume (1.21) and (1.22). Then there exists λ_0 such that the following holds.*

Let (u_0^ε) be a family of vector fields verifying (1.14) and such that

$$\varepsilon |\ell_0^\varepsilon|, \quad \varepsilon^2 |r_0^\varepsilon|, \quad \|u_0^\varepsilon\|_{L^2(\mathbb{R}^2 \setminus B(0,1))} \leq \lambda_0 \quad (1.24)$$

and

$$u_0^\varepsilon \rightharpoonup u_0 \quad \text{in} \quad L^2(\mathbb{R}^2). \quad (1.25)$$

Then for any $T > 0$ we have

$$u^\varepsilon \xrightarrow{*} u \quad \text{in} \quad L^\infty(0, T; L^2(\mathbb{R}^2)) \cap L^2(0, T; H_0^1(\mathbb{R}^2)) \quad (1.26)$$

where u is the weak solution of the Navier-Stokes equations in \mathbb{R}^2 associated to u_0 in the sense of (1.20).

For a 2D ideal incompressible fluid governed by the Euler equations, the case of a massive pointwise particle in the full plane was treated in [15], a massless pointwise particle in the full plane in [16] and both case in a bounded domain in [17]. In these works, non-trivial limit was obtained (namely, Kutta-Joukowski lift force or vortex-wave system) when we consider non-zero initial circulations around the points.

The remainder of this work is organized in four sections. In the next section, we provide three examples where the initial convergence (1.17) or (1.25) holds.

Section 3 is dedicated to the proof of Theorem 1.3. We introduce the cutoff procedure which follows the trajectories of the solid and provide some uniform estimates. A crucial point for the proof of our main result is to construct a corrected test function φ^η which satisfies the divergence free condition. This will be obtained by the Bogovskiĭ operator [3, 4]. Then we follow the analysis developed in [27]. Roughly, we pass first to the limit $\varepsilon \rightarrow 0$ far away from the solids to get that u satisfies the Navier-Stokes equations in this region. Next, we pass to the limit $\eta \rightarrow 0$ in the cutoff function, to prove that the equations are also verified in the vicinity of the massive pointwise particles.

Section 4 concerns the proof of Theorem 1.6. The main point here is to get a uniform estimate of the disk velocity. This will be obtained thanks to the results of [13]. Then, the passing to the limit will be performed in the same way as in the massive case.

Remark 1.7. In dimension three, the case of massive pointwise particles could be treated by the same strategy. The case of a massless pointwise particle would require to adapt [13] in dimension three, which should be possible if there is only one rigid ball in \mathbb{R}^3 . The decay estimates for the Stokes solution for other geometry is a complicated open question.

2 Examples of initial conditions

In this paragraph, we develop three examples of family $(u_0^\varepsilon)_\varepsilon$ satisfying the compatibility condition (1.14) which converges in $L^2(\Omega)$.

Example 2.1. The first trivial example is the case where u_0^ε is independent of ε . Namely, let us consider ε_0 small enough such that the solids are disjoint (1.1), then we set

$$u_0^\varepsilon := u_0^{\varepsilon_0} \quad \forall \varepsilon \in (0, \varepsilon_0]$$

where $u_0^{\varepsilon_0}$ satisfies (1.14) in $\mathcal{F}_0^{\varepsilon_0}$. In this case, the vector field u_0^ε is solid in a neighborhood of the solid and obviously verifies (1.17) or (1.25).

Example 2.2. In domains depending on ε , a standard setting is to give an initial data in terms of an independent vorticity $\omega_0 = \text{curl } u_0^\varepsilon$ (see, e.g., [15, 16, 22, 23, 26, 27]).

More precisely, let us assume here that Ω is simply connected. We fix $\omega_0 \in L^q(\Omega)$, $q > 1$, $\ell_{i,0} \in \mathbb{R}^2$, $r_{i,0} \in \mathbb{R}$. Then we consider u_0^ε as the unique solution of

$$\text{div } u_0^\varepsilon = 0 \quad \text{in } \mathcal{F}_0^\varepsilon, \quad \text{curl } u_0^\varepsilon = \omega_0 \quad \text{in } \mathcal{F}_0^\varepsilon, \quad (2.1)$$

$$u_0^\varepsilon \cdot n = (\ell_{i,0} + r_{i,0}(x - h_{i,0})^\perp) \cdot n \quad \text{on } \partial \mathcal{S}_{i,0}^\varepsilon, \quad (i \in \{1, \dots, N\}), \quad (2.2)$$

$$u_0^\varepsilon \cdot n = 0 \quad \text{on } \partial \Omega, \quad (2.3)$$

$$\oint_{\partial \mathcal{S}_{i,0}^\varepsilon} u_0^\varepsilon \cdot \tau \, ds = 0 \quad (i \in \{1, \dots, N\}). \quad (2.4)$$

Lemma 2.3. *Let Ω be a bounded simply connected open subset of \mathbb{R}^2 . For $\omega_0 \in L^q(\Omega)$, $q > 1$, $\ell_{i,0} \in \mathbb{R}^2$, $r_{i,0} \in \mathbb{R}$ fixed, we have*

$$u_0^\varepsilon \rightarrow u_0 \quad \text{in } L^2(\Omega),$$

where u_0^ε is the unique solution of (2.1)-(2.4) and where u_0 is the unique vector field in $L^2(\Omega)$ such that

$$\text{div } u_0 = 0 \quad \text{in } \Omega, \quad \text{curl } u_0 = \omega_0 \quad \text{in } \Omega, \quad u_0 \cdot n = 0 \quad \text{on } \partial \Omega.$$

Proof. By standard results related to the Hodge–De Rham theorem (see e.g. [15, 17]), there is a unique solution of (2.1)-(2.4) and it can be decomposed as

$$u_0^\varepsilon = \nabla^\perp \psi_0^\varepsilon + \sum_{i=1}^N \left(\sum_{j=1}^2 (\ell_{i,0})_j \nabla \phi_{i,j}^\varepsilon + r_{i,0} \nabla \phi_{i,3}^\varepsilon \right)$$

with ψ_0^ε the unique solution of

$$\begin{cases} \Delta \psi_0^\varepsilon = \omega_0 & \text{in } \mathcal{F}_0^\varepsilon \\ \psi_0^\varepsilon = 0 & \text{on } \partial \Omega, \\ \partial_\tau \psi_0^\varepsilon = 0 & \text{on } \partial \mathcal{S}_{i,0}^\varepsilon \quad \forall i = 1, \dots, N, \\ \oint_{\partial \mathcal{S}_{i,0}^\varepsilon} \partial_n \psi_0^\varepsilon \, ds = 0 & \forall i = 1, \dots, N, \end{cases}$$

and $\phi_{i,j}^\varepsilon$ are the Kirchoff potentials:

$$\begin{cases} \Delta \phi_{i,j}^\varepsilon = 0 & \text{in } \mathcal{F}_0^\varepsilon \\ \partial_n \phi_{i,j}^\varepsilon = 0 & \text{on } \partial \Omega \cup \left(\bigcup_{k \neq i} \mathcal{S}_{k,0}^\varepsilon \right), \\ \partial_n \phi_{i,j}^\varepsilon = K_{i,j} & \text{on } \partial \mathcal{S}_{i,0}^\varepsilon, \end{cases}$$

where

$$K_{i,1} = n_1, \quad K_{i,2} = n_2, \quad K_{i,3} = n \cdot (x - h_{i,0})^\perp.$$

As the Kirchhoff potentials are defined up to a constant, we can assume that

$$\int_{B(h_{i,0}, \alpha \varepsilon) \setminus \mathcal{S}_{i,0}^\varepsilon} \phi_{i,j}^\varepsilon dx = 0 \quad \text{with } \alpha \text{ fixed such that } \mathcal{S}_{i,0} \Subset B(0, \alpha).$$

By an energy estimate together with a trace estimate we have

$$\begin{aligned} \|\nabla \phi_{i,j}^\varepsilon\|_{L^2(\mathcal{F}_0^\varepsilon)}^2 &= \int_{\partial \mathcal{S}_{i,0}^\varepsilon} \phi_{i,j}^\varepsilon K_{i,j} ds \leq \varepsilon^{1+\delta_{j,3}} \int_{\partial \mathcal{S}_{i,0}} |\phi_{i,j}^\varepsilon(\varepsilon s + h_{i,0})| ds \\ &\leq C \varepsilon^{1+\delta_{j,3}} \|\phi_{i,j}^\varepsilon(\varepsilon \cdot + h_{i,0})\|_{L^2(\partial \mathcal{S}_{i,0})} \leq C \varepsilon^{1+\delta_{j,3}} \|\varepsilon (\nabla \phi_{i,j}^\varepsilon)(\varepsilon \cdot + h_{i,0})\|_{L^2(B(0, \alpha) \setminus \mathcal{S}_{i,0})} \\ &\leq C \varepsilon^{1+\delta_{j,3}} \|\nabla \phi_{i,j}^\varepsilon\|_{L^2(\mathcal{F}_0^\varepsilon)} \end{aligned}$$

where we have used Poincaré–Wirtinger in $B(0, \alpha) \setminus \mathcal{S}_{i,0}$. Therefore, we deduce that

$$\sum_{i=1}^N \left(\sum_{j=1}^2 (\ell_{i,0})_j \nabla \phi_{i,j}^\varepsilon + r_{i,0} \nabla \phi_{i,3}^\varepsilon \right) \rightarrow 0 \text{ strongly in } L^2(\Omega)$$

where we have extended $\nabla \phi_{i,j}^\varepsilon$ (by e_j for $j = 1, 2$ and by $(x - h_{i,0})^\perp$ for $j = 3$) in $\mathcal{S}_{i,0}^\varepsilon$ and by 0 in $\mathcal{S}_{k,0}^\varepsilon$ for all $k \neq i$.

Concerning the last part, we introduce $q' \in [1, \infty)$ such that $1/q + 1/q' = 1$ and we use that ψ_0^ε is constant on each boundary to perform the energy estimate

$$\begin{aligned} \|\nabla \psi_0^\varepsilon\|_{L^2(\mathcal{F}_0^\varepsilon)}^2 &= \int_{\mathcal{F}_0^\varepsilon} \omega_0 \psi_0^\varepsilon + \sum_{i=1}^N \oint_{\partial \mathcal{S}_{i,0}^\varepsilon} \psi_0^\varepsilon \partial_n \psi_0^\varepsilon = \int_{\mathcal{F}_0^\varepsilon} \omega_0 \psi_0^\varepsilon \\ &\leq \|\omega_0\|_{L^q(\Omega)} \|\psi_0^\varepsilon\|_{L^{q'}(\Omega)} \leq C \|\omega_0\|_{L^q(\Omega)} \|\psi_0^\varepsilon\|_{H^1(\Omega)} \\ &\leq C \|\omega_0\|_{L^q(\Omega)} \|\nabla \psi_0^\varepsilon\|_{L^2(\Omega)} \end{aligned}$$

where we have extended ψ_0^ε inside the obstacles by the constant values at the boundary, and applied the Poincaré inequality in Ω . Therefore, ψ_0^ε is uniformly bounded in $H_0^1(\Omega)$, so for any sequence, we can extract a subsequence which converges weakly to ψ_0 in $H_0^1(\Omega)$ (Banach-Alaoglu theorem) and strongly in $L^2 \cap L^{q'}(\Omega)$ (Rellich theorem). Passing to the limit in the Laplace problem, we have

$$\Delta \psi_0 = \omega_0 \quad \text{in } \mathcal{D}'(\Omega \setminus \cup \{h_{i,0}\}).$$

For any $\varphi \in \mathcal{D}(\Omega)$, we introduce a cutoff function $\chi \equiv 1$ on $B(0, 1)^c$ and that $\chi \equiv 0$ on $B(0, 1/2)$, so that $\varphi \prod \chi(\frac{x-h_{i,0}}{\eta})$ belongs to $\mathcal{D}(\Omega \setminus \cup \{h_{i,0}\})$ and converges weakly in H_0^1 to φ as $\eta \rightarrow 0$. With this remark, we prove that

$$\Delta \psi_0 = \omega_0 \quad \text{in } \mathcal{D}'(\Omega).$$

As the solution of this laplace problem is unique in $H_0^1(\Omega)$ we infer that the limit holds for the full sequence (without extraction).

Moreover, by the strong convergence of ψ_0^ε to ψ_0 in $L^{q'}(\Omega)$, the weak convergence of $\omega_0 \mathbf{1}_{\mathcal{F}_0^\varepsilon}$ to ω_0 in $L^q(\Omega)$, and the equation verified by ψ_0^ε and ψ_0 we note:

$$\|\nabla \psi_0^\varepsilon\|_{L^2(\Omega)}^2 = \int_{\mathcal{F}_0^\varepsilon} |\nabla \psi_0^\varepsilon|^2 = \int_{\mathcal{F}_0^\varepsilon} \psi_0^\varepsilon \omega_0 = \int_{\Omega} \psi_0^\varepsilon \omega_0 \mathbf{1}_{\mathcal{F}_0^\varepsilon} \rightarrow \int_{\Omega} \psi_0 \omega_0 = \|\nabla \psi_0\|_{L^2(\Omega)}^2$$

which gives with the weak limit in $H_0^1(\Omega)$ that

$$\psi_0^\varepsilon \rightarrow \psi_0 \text{ strongly in } H_0^1(\Omega).$$

This ends the proof of $u_0^\varepsilon \rightarrow \nabla^\perp \psi_0 = u_0$ strongly in $L^2(\Omega)$. \square

Remark 2.4. If we consider non-zero initial circulations in (2.4), then some singular terms appear at the limit of the form $\gamma_i \frac{(x-h_{i,0})^\perp}{2\pi|x-h_{i,0}|^2}$, which does not belong to $L^2(\Omega)$. In this case, we can only show L^p estimate for $p \in [1, 2)$. For more details, we refer to [23, 15, 16, 17].

In exterior domains, we recall that

$$u_0^\varepsilon \in L^2(\mathcal{F}_0^\varepsilon) \iff \int_{\mathcal{F}_0^\varepsilon} \omega_0 = 0.$$

Adding the zero mean-value condition, we get a convergence result for unbounded domains $\mathcal{F}_0^\varepsilon = \mathbb{R}^2 \setminus \overline{B(0, \varepsilon)}$.

Lemma 2.5. *Let $\Omega = \mathbb{R}^2$. For $\omega_0 \in L^q_c(\mathbb{R}^2 \setminus \{0\})$, $q > 1$, $\ell_0 \in \mathbb{R}^2$ fixed such that $\int \omega_0 = 0$, we have, for ε small enough such that $\text{supp } \omega_0 \cap B(0, \varepsilon) = \emptyset$, a unique solution u_0^ε in $L^2(\mathcal{F}_0^\varepsilon)$ of*

$$\begin{aligned} \text{div } u_0^\varepsilon &= 0 \text{ in } \mathcal{F}_0^\varepsilon, \quad \text{curl } u_0^\varepsilon = \omega_0 \text{ in } \mathcal{F}_0^\varepsilon, \quad \lim_{|x| \rightarrow \infty} u_0^\varepsilon(x) = 0, \\ u_0^\varepsilon \cdot n &= \ell_0 \cdot n \text{ on } \partial B(0, \varepsilon), \quad \oint_{\partial B(0, \varepsilon)} u_0^\varepsilon \cdot \tau \, ds = 0. \end{aligned}$$

Moreover, extending u_0^ε by $\ell_0 + r_0 x^\perp$ in $B(0, \varepsilon)$ (with any $r_0 \in \mathbb{R}$) we have

$$u_0^\varepsilon \rightarrow u_0 \quad \text{in } L^2(\mathbb{R}^2),$$

where $u_0 = K_{\mathbb{R}^2}[\omega_0] = \frac{x^\perp}{2\pi|x|^2} * \omega_0$ is the unique vector field in $L^2(\Omega)$ such that

$$\text{div } u_0 = 0 \text{ in } \mathbb{R}^2, \quad \text{curl } u_0 = \omega_0 \text{ in } \mathbb{R}^2, \quad \lim_{|x| \rightarrow \infty} u_0(x) = 0.$$

Proof. The existence and uniqueness of u_0^ε is well-known (see e.g. [15, Section 2]), and we can find therein the following explicit formula:

$$u_0^\varepsilon(x) = \frac{1}{2\pi} \int_{B(0, \varepsilon)^c} \frac{(x-y)^\perp}{|x-y|^2} \omega_0(y) dy + \frac{1}{2\pi} \int_{B(0, \varepsilon)^c} \left(\frac{x}{|x|^2} - \frac{x - \varepsilon^2 y^*}{|x - \varepsilon^2 y^*|^2} \right)^\perp \omega_0(y) dy - \varepsilon^2 \sum_{j=1}^2 (\ell_0)_j \nabla \left(\frac{x_j}{|x|^2} \right)$$

with the notation $y^* = y/|y|^2$. By a standard computation, we note that

$$\left\| \varepsilon^2 \sum_{j=1}^2 (\ell_0)_j \nabla \left(\frac{x_j}{|x|^2} \right) \right\|_{L^2(\mathcal{F}_0^\varepsilon)} \leq C\varepsilon |\ell_0|.$$

It is also rather classical to prove that the second integral in the right hand side tends to zero as $\varepsilon \rightarrow 0$. For instance, we can do as follows:

- Squaring both side, we verify that $\left| \frac{a}{|a|^2} - \frac{b}{|b|^2} \right| = \frac{|a-b|}{|a||b|}$. Hence

$$\left| \int_{B(0, \varepsilon)^c} \left(\frac{x}{|x|^2} - \frac{x - \varepsilon^2 y^*}{|x - \varepsilon^2 y^*|^2} \right)^\perp \omega_0(y) dy \right| \leq \int_{B(0, \varepsilon)^c} \frac{\varepsilon^2 |y^*|}{|x||x - \varepsilon^2 y^*|} |\omega_0(y)| dy.$$

- Defining R_0 such that $\text{supp } \omega_0 \subset B(0, R_0) \setminus B(0, 1/R_0)$ we have for all $x \in B(0, 2\varepsilon)^c$:

$$\int_{B(0, \varepsilon)^c} \frac{\varepsilon^2 |y^*|}{|x||x - \varepsilon^2 y^*|} |\omega_0(y)| dy \leq \frac{\varepsilon^2}{(|x| - \varepsilon)^2} \int \frac{|\omega_0(y)|}{|y|} dy \leq \frac{\varepsilon^2 R_0 \|\omega_0\|_{L^1}}{(|x| - \varepsilon)^2}$$

so we compute

$$\left\| \int_{B(0, \varepsilon)^c} \left(\frac{x}{|x|^2} - \frac{x - \varepsilon^2 y^*}{|x - \varepsilon^2 y^*|^2} \right)^\perp \omega_0(y) dy \right\|_{L^2(B(0, 2\varepsilon)^c)} \leq CR_0 \varepsilon \|\omega_0\|_{L^1}.$$

- For $x \in B(0, 2\varepsilon) \setminus B(0, \varepsilon)$, $y \in \text{supp } \omega_0$ and $\varepsilon \leq 1/(2R_0)$, we verify that $|x - \varepsilon^2 y^*| \geq \varepsilon - \frac{\varepsilon^2}{|y|} \geq \varepsilon(1 - \varepsilon R_0) \geq \varepsilon/2$, hence

$$\left| \int_{B(0, \varepsilon)^c} \left(\frac{x}{|x|^2} - \frac{x - \varepsilon^2 y^*}{|x - \varepsilon^2 y^*|^2} \right)^\perp \omega_0(y) dy \right| \leq \int_{B(0, \varepsilon)^c} \frac{\varepsilon^2 |y^*|}{\varepsilon^2/2} |\omega_0(y)| dy \leq 2R_0 \|\omega_0\|_{L^1}$$

which implies

$$\left\| \int_{B(0, \varepsilon)^c} \left(\frac{x}{|x|^2} - \frac{x - \varepsilon^2 y^*}{|x - \varepsilon^2 y^*|^2} \right)^\perp \omega_0(y) dy \right\|_{L^2(B(0, 2\varepsilon) \setminus B(0, \varepsilon))} \leq 2R_0 \|\omega_0\|_{L^1} (\pi 3\varepsilon^2)^{1/2}.$$

This ends the proof of

$$\left\| \int_{B(0, \varepsilon)^c} \left(\frac{x}{|x|^2} - \frac{x - \varepsilon^2 y^*}{|x - \varepsilon^2 y^*|^2} \right)^\perp \omega_0(y) dy \right\|_{L^2(\mathcal{F}_\varepsilon^c)} \leq CR_0 \varepsilon \|\omega_0\|_{L^1}.$$

Now we note that $\frac{1}{2\pi} \int_{B(0, \varepsilon)^c} \frac{(x-y)^\perp}{|x-y|^2} \omega_0(y) dy = K_{\mathbb{R}^2}[\omega] = u_0$ for $\varepsilon \leq 1/R_0$, then for all $\varepsilon \leq 1/(2R_0)$ we have

$$\begin{aligned} \|u_0^\varepsilon - u_0\|_{L^2(\mathbb{R}^2)} &\leq \|\ell_0 + r_0 x^\perp - u_0\|_{L^2(B(0, \varepsilon))} + CR_0 \varepsilon \|\omega_0\|_{L^1} + C\varepsilon |\ell_0| \\ &\leq (|\ell_0| + |r_0| \varepsilon + \frac{1/(2\pi)}{1/R_0 - \varepsilon} \|\omega_0\|_{L^1}) (\pi \varepsilon^2)^{1/2} + CR_0 \varepsilon \|\omega_0\|_{L^1} + C\varepsilon |\ell_0| \\ &\leq C(|\ell_0| + |r_0| \varepsilon + \|\omega_0\|_{L^1}) \varepsilon \end{aligned}$$

which goes to zero as $\varepsilon \rightarrow 0$. □

Example 2.6. Another example of initial conditions satisfying (1.14) can be obtained by truncating a stream function associated to a vector field u_0 defined on Ω .

In a bounded domain Ω , let us consider $u_0 = \nabla^\perp \psi_0 \in L^2(\Omega)$ such that

$$\text{div } u_0 = 0 \quad \text{in } \Omega, \quad \text{curl } u_0 \in L^q(\Omega) \quad \text{with } q > 1, \quad u_0 \cdot n = 0 \quad \text{on } \partial\Omega.$$

We denote by χ a smooth cutoff function such that $\chi(x) \equiv 0$ in $B(0, \alpha)$ and $\chi(x) \equiv 1$ in $B(0, 2\alpha)^c$, where α is chosen large enough such that $\mathcal{S}_{i,0} \subset B(0, \alpha)$ for any i . We consider $(\ell_{i,0}, r_{i,0}) \in \mathbb{R}^3$ given and since the positions $h_{i,0} \in \Omega$ are distinct, we can consider ε small enough such that

$$B(h_{i,0}, 2\alpha\varepsilon) \cap B(h_{j,0}, 2\alpha\varepsilon) = \emptyset \quad (i \neq j) \quad \text{and } B(h_{i,0}, 2\alpha\varepsilon) \subset \Omega.$$

Then, we define

$$u_0^\varepsilon := \nabla^\perp \left(\psi_0(x) \prod_{i=1}^N \chi \left(\frac{x - h_{i,0}}{\varepsilon} \right) + \sum_{i=1}^N \left(1 - \chi \left(\frac{x - h_{i,0}}{\varepsilon} \right) \right) (\ell_{i,0} \cdot (x - h_{i,0})^\perp + r_{0,i} \frac{|x - h_{i,0}|^2}{2}) \right)$$

which is divergence free, tangent to $\partial\Omega$, equal to u_0 far away the solids, and equal to $\ell_{i,0} + r_{i,0}(x - h_{i,0})^\perp$ in the vicinity of $\mathcal{S}_{i,0}^\varepsilon$. Using that $\psi_0 \in W^{2,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ and that $\frac{1}{\varepsilon} \nabla \chi \left(\frac{\cdot - h_{i,0}}{\varepsilon} \right)$ converges weakly to 0 in L^2 , one can check that $u_0^\varepsilon \rightharpoonup u_0$ in $L^2(\Omega)$.

In the case of one solid in the full plane, we chose ψ_0 in the previous formula such that $\psi_0(h_{1,0}) = 0$ and in that case, we can prove that the convergence holds strongly in $L^2(\mathbb{R}^2)$.

3 Case of massive pointwise particles

The goal of this section is to prove Theorem 1.3, hence we are considering the following settings:

- Ω is a bounded domain of \mathbb{R}^2 and the fluid domain is the exterior of N rigid solids ($N \geq 1$) see (1.2)-(1.3);
- (u_0^ε) is a family of divergence vector fields verifying (1.14) such that $u_0^\varepsilon \rightharpoonup u_0$ in $L^2(\Omega)$ and $(\ell_{i,0}^\varepsilon, \varepsilon r_{i,0}^\varepsilon)_{i,\varepsilon}$ are bounded (for instance see Section 2);
- the solids tend to massive pointwise particles (1.13).

From the result in [31], we know that, for every $\varepsilon > 0$, there exists a global weak solution $(u^\varepsilon, h_i^\varepsilon, \theta_i^\varepsilon)$ of (1.4)–(1.11) in the sense of Definition 1.1.

3.1 First convergences

Thanks to the assumptions on the initial data, we deduce from the energy inequality (1.16) that

$$(u^\varepsilon)_\varepsilon \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad (3.1)$$

$$((h_i^\varepsilon)')_\varepsilon \text{ is bounded in } L^\infty(0, T), \quad (3.2)$$

$$(\varepsilon(\theta_i^\varepsilon)')_\varepsilon \text{ is bounded in } L^\infty(0, T), \quad (3.3)$$

Then up to a subsequence,

$$u^\varepsilon \xrightarrow{*} u \quad \text{in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \quad (3.4)$$

and

$$h_i^\varepsilon \rightarrow h_i \quad \text{uniformly in } [0, T] \quad (i = 1, \dots, N), \quad (3.5)$$

with

$$h_i \in W^{1,\infty}(0, T). \quad (3.6)$$

By abuse of notation, we continue to write u^ε and h_i^ε the subsequences.

As $(h_i)'$ is bounded, there exist $T > 0$ and $\eta_0 > 0$, such that for any $t \in [0, T]$,

$$|h_i(t) - h_j(t)| \geq 2\eta_0 \quad (i \neq j). \quad (3.7)$$

We will prove the convergence up to this time T .

Remark 3.1. We can note that this time does not take into account the distance between h_i and $\partial\Omega$. Therefore, in the case of one obstacle ($N = 1$), we can choose any $T > 0$.

3.2 Modified test functions

The key to treat shrinking obstacles problem is to approximate test functions in Ω by admissible test functions in the perforated domain.

Proposition 3.2. *Let $T > 0$, $\varphi \in C_c^\infty([0, T] \times \Omega)$ with $\operatorname{div} \varphi = 0$ and consider $\eta_1 > 0$ such that*

$$|h_i(t) - h_j(t)| \geq 2\eta_1 \quad \text{for all } t \in [0, T] \text{ and } i \neq j \quad (3.8)$$

and

$$\operatorname{dist}(\operatorname{supp} \varphi(t, \cdot), \partial\Omega) \geq 2\eta_1 \quad \text{for all } t \in [0, T]. \quad (3.9)$$

For any $\eta \leq \eta_1$ there exists $\varphi^\eta \in W_c^{1,\infty}([0, T]; H_0^1(\Omega))$ satisfying

$$\operatorname{div} \varphi^\eta = 0 \quad \text{in } [0, T] \times \Omega, \quad (3.10)$$

$$\varphi^\eta \equiv 0 \quad t \in (0, T), \quad x \in B\left(h_i(t), \frac{\eta}{2}\right), \quad (3.11)$$

$$\varphi^\eta \xrightarrow{*} \varphi \quad L^\infty(0, T; H^1(\Omega)), \quad (3.12)$$

$$\partial_t \varphi^\eta \xrightarrow{*} \partial_t \varphi \quad L^\infty(0, T; L^2(\Omega)). \quad (3.13)$$

Proof. We introduce a cutoff function $\chi \in C^\infty(\mathbb{R}^2, [0, 1])$ such that $\chi \equiv 1$ in $B(0, 1)^c$ and $\chi \equiv 0$ in $B(0, 1/2)$. Let us denote the annulus $B(0, 1) \setminus \overline{B(0, 1/2)}$ by A .

We remark that the function $\tilde{\varphi}_i^\eta : (t, y) \mapsto \varphi(t, \eta y + h_i(t)) \nabla \chi(y)$ belongs to $W^{1, \infty}(0, T; L^2(A))$ and verifies for any t

$$\begin{aligned} \int_A \tilde{\varphi}_i^\eta(t, y) \, dy &= \int_A \operatorname{div} \left(\varphi(t, \eta y + h_i(t)) \chi(y) \right) \, dy = \int_{\partial B(0, 1)} \varphi(t, \eta y + h_i(t)) \cdot n(y) \, ds \\ &= \int_{B(0, 1)} \operatorname{div} \left(\varphi(t, \eta y + h_i(t)) \right) \, dy = 0, \end{aligned}$$

where we have used twice that φ is divergence free, that $\chi \equiv 1$ on $\partial B(0, 1)$ and that $\chi \equiv 0$ on $\partial B(0, 1/2)$. With these properties, it is known by [14, Theorem III.3.1] (and Exercice III.3.6) that there exists C depending only on A such that the problem

$$\operatorname{div} \tilde{g}_i^\eta = \tilde{\varphi}_i^\eta, \quad \tilde{g}_i^\eta \in W^{1, \infty}(0, T; H_0^1(A))$$

has a solution such that

$$\begin{aligned} \|\tilde{g}_i^\eta\|_{L^\infty(0, T; H^1(A))} &\leq C \|\tilde{\varphi}_i^\eta\|_{L^\infty(0, T; L^2(A))}, \\ \|\partial_t \tilde{g}_i^\eta\|_{L^\infty(0, T; H^1(A))} &\leq C \|\partial_t \tilde{\varphi}_i^\eta\|_{L^\infty(0, T; L^2(A))}. \end{aligned}$$

Extending \tilde{g}_i^η by zero in the exterior of A , we define

$$\varphi^\eta(t, x) = \varphi(t, x) \prod_{i=1}^N \chi\left(\frac{x - h_i(t)}{\eta}\right) - \sum_{i=1}^N g_i^\eta(t, x)$$

where

$$g_i^\eta(t, x) := \tilde{g}_i^\eta\left(t, \frac{x - h_i(t)}{\eta}\right).$$

From the assumption (3.8), we deduce that for all i , $\prod_{j \neq i} \chi\left(\frac{x - h_j(t)}{\eta}\right) \equiv 1$ on the support of $\nabla \chi\left(\frac{\cdot - h_i(t)}{\eta}\right)$. So we easily verify the divergence free condition (3.10). Moreover, with a change of variables, we also note that

$$\frac{1}{\eta} \left\| \sum_{i=1}^N g_i^\eta \right\|_{L^\infty(0, T; L^2(\mathbb{R}^2))} + \left\| \nabla \sum_{i=1}^N g_i^\eta \right\|_{L^\infty(0, T; L^2(\mathbb{R}^2))} \leq C \sum_{i=1}^N \|\tilde{\varphi}_i^\eta\|_{L^\infty(0, T; L^2(A))} \leq C \|\varphi\|_{L^\infty((0, T) \times \Omega)} \quad (3.14)$$

so we check that

$$\frac{1}{\eta} \|\varphi^\eta - \varphi\|_{L^\infty(0, T; L^2(\mathbb{R}^2))} + \|\nabla \varphi^\eta\|_{L^\infty(0, T; L^2(\mathbb{R}^2))} \leq C \|\varphi\|_{W^{1, \infty}((0, T) \times \Omega)}$$

which gives directly that φ^η converges to φ strongly in $L^\infty(0, T; L^2(\Omega))$ and weak-* in $L^\infty(0, T; H^1(\Omega))$. By uniqueness of the limit, we do not need to extract a subsequence and we get the weak limit (3.12).

Now we compute

$$\begin{aligned} \partial_t \varphi^\eta - \partial_t \varphi &= \partial_t \varphi \left(\prod_{i=1}^N \chi\left(\frac{x - h_i(t)}{\eta}\right) - 1 \right) - \frac{\varphi}{\eta} \sum_{i=1}^N h_i'(t) \cdot (\nabla \chi)\left(\frac{x - h_i(t)}{\eta}\right) \\ &\quad - \sum_{i=1}^N \partial_t \tilde{g}_i^\eta\left(t, \frac{x - h_i(t)}{\eta}\right) + \sum_{i=1}^N h_i'(t) \cdot \nabla g_i^\eta(t, x). \end{aligned}$$

It is obvious that the first right hand side term converges to zero strongly in $L^\infty(0, T; L^2(\mathbb{R}^2))$. It is also an easy computation to check that the second term is bounded in $L^\infty(0, T; L^2(\mathbb{R}^2))$ and tends to zero strongly in $L^\infty(0, T; L^p(\mathbb{R}^2))$ for $p \in [1, 2)$. Hence, it converges weak-* to zero in $L^\infty(0, T; L^2(\mathbb{R}^2))$. From (3.14), we know that $\sum g_i^\eta$ converges weak-* to 0 in $L^\infty(0, T; H^1(\Omega))$, hence the last term converges also weak-* to zero in $L^\infty(0, T; L^2(\mathbb{R}^2))$. Finally, we note that

$$\frac{1}{\eta} \left\| \sum_{i=1}^N \partial_t \tilde{g}_i^\eta \left(t, \frac{x - h_i(t)}{\eta} \right) \right\|_{L^\infty(0, T; L^2(\mathbb{R}^2))} \leq C \|\partial_t \varphi\|_{L^\infty((0, T) \times \Omega)},$$

hence the third right hand side term tends to zero strongly in $L^\infty(0, T; L^2(\mathbb{R}^2))$. This gives (3.13).

Due to the support of χ and g_i^η , it is clear that $\varphi^\eta \equiv 0$ on $B(h_i(t), \frac{\eta}{2})$. It is also obvious that $\varphi \prod \chi(\frac{x - h_i(t)}{\eta})$ is compactly supported in $[0, T) \times \Omega$. To prove that φ^η vanishes at $\partial\Omega$ it is then enough to show that $g_i^\eta \equiv 0$ on $\partial\Omega$, for any t and i .

If $\text{dist}(h_i(t), \partial\Omega) > \eta$, then we use that $\text{supp } g_i^\eta \subset B(h_i(t), \eta) \setminus B(h_i(t), \eta/2) \Subset \Omega$.

If $\text{dist}(h_i(t), \partial\Omega) < \eta$, then $\text{dist}(\eta y + h_i(t), \partial\Omega) \leq 2\eta_1$ for $y \in B(0, 1)$. Therefore, we infer from (3.9) that $\tilde{\varphi}_i^\eta(y) = \varphi(t, \eta y + h_i(t)) \nabla \chi(y) = 0$ for $y \in B(0, 1)$, which implies that $g_i^\eta \equiv \tilde{g}_i^\eta \equiv 0$.

This ends the proof. \square

Remark 3.3. An important consequence is the approximation of any test function. Let T and η_0 verifying (3.7) and let $\varphi \in C_c^\infty([0, T) \times \Omega)$ with $\text{div } \varphi = 0$. Then, there exists $\eta_1 \leq \eta_0$ such that we have (3.9) and we can apply Proposition 3.2 to construct a family $(\varphi^\eta)_\eta$ of divergence free test functions which tends to φ in the sense of (3.12)-(3.13). Moreover, for any $\eta \in (0, \eta_1]$ fixed, we put together the strong convergence of h_i^ε (3.5) with the support of φ^η (3.11) to deduce the existence of $\varepsilon_\eta > 0$ such that

$$\varphi^\eta \equiv 0 \quad \text{for } t \in [0, T), \quad x \in \mathcal{S}_i^\varepsilon(h_i^\varepsilon(t)), \quad \varepsilon \leq \varepsilon_\eta.$$

This implies that φ^η is an admissible test function for the fluid-solid problem (see Definition 1.1).

In the proof of the above proposition, we note that H^1 is the critical space in dimension two: $\chi(\frac{\cdot}{\eta}) - 1$ tends to zero strongly in $W^{1,p}$ for any $p \in [1, 2)$, is bounded in H^1 (then tends weakly to zero), and goes to infinity in $W^{1,p}$ for $p > 2$. This explains why the standard framework for shrinking obstacles problems is H^1 (see, e.g., [1, 2, 6, 22, 23, 38]). Nevertheless, as we need an estimate of the solid velocities, it is natural to look for a C^0 estimate of the velocity, hence a H^s estimate for $s > 1$. Unfortunately, the H^2 analysis developed in [11, 36] fails (see [5]).

In the case of massive pointwise particles, the estimate of the solid velocities comes directly from the energy estimate, but for the case of a massless pointwise particle, we need a new argument (see Section 4)

In dimension three, the critical space for the cutoff argument is $W^{1,3}$ which is again not embedded in C^0 .

3.3 Proof of Theorem 1.3

The first step is to pass to the limit $\varepsilon \rightarrow 0$ for η fixed.

Theorem 3.4. *Let T and η_0 verifying (3.7) and let $\varphi \in C_c^\infty([0, T) \times \Omega)$ with $\text{div } \varphi = 0$. From Remark 3.3, we consider $\eta_1 \leq \eta_0$ and $(\varphi^\eta)_{\eta \leq \eta_1}$ which approximate φ .*

Then, for any $\eta \in (0, \eta_1]$, the limit u of u^ε (see (3.4)) verifies

$$-\int_0^T \int_\Omega u \cdot \left(\frac{\partial \varphi^\eta}{\partial t} + (u \cdot \nabla) \varphi^\eta \right) dx + 2\nu \int_0^T \int_\Omega D(u) : D(\varphi^\eta) dx = \int_\Omega u^0(x) \cdot \varphi^\eta(0, x) dx.$$

Proof. Let $\eta \in (0, \eta_1]$ be fixed. Since h_i is Lipschitz continuous, there exists a subdivision $t_0 = 0 < t_1 < \dots < t_{M+1} = T$ such that for all $j = 0, \dots, M$, if $t \in (t_j, t_{j+1})$, then

$$|h_i(t) - h_i(t_j)| \leq \frac{\eta}{6}.$$

From (3.11) in Proposition 3.2, we deduce

$$\varphi^\eta \equiv 0 \quad \text{in } (t_j, t_{j+1}) \times \bigcup_{i=1}^N B(h_i(t_j), \frac{\eta}{3}).$$

Putting together this relation with (3.5), there exist open relatively compact sets \mathcal{O}_j and $\tilde{\varepsilon}_\eta > 0$ such that for all $\varepsilon < \tilde{\varepsilon}_\eta$

$$\mathcal{S}_i^\varepsilon(t) \cap \mathcal{O}_j = \emptyset \quad \text{for all } t \in (t_j, t_{j+1}) \quad \text{and} \quad \text{supp } \varphi^\eta \subset \sum_{j=0}^M (t_j, t_{j+1}) \times \mathcal{O}_j. \quad (3.15)$$

For any $j = 0, \dots, M$, we write the Helmholtz-Weyl decomposition

$$u^\varepsilon = \mathbb{P}_{\mathcal{O}_j} u^\varepsilon + \nabla q^\varepsilon,$$

where $\mathbb{P}_{\mathcal{O}_j}$ is the Leray projection on $\mathcal{H}(\mathcal{O}_j)$ (see the introduction for the definition of $\mathcal{H}(\mathcal{O})$). This projection is orthogonal in L^2 and by a standard estimate on the Laplace problem with Neumann boundary condition, there exists a constant $C_{\mathcal{O}_j} > 0$ such that

$$\|\mathbb{P}_{\mathcal{O}_j} u^\varepsilon\|_{L^2(\mathcal{O}_j)} \leq \|u^\varepsilon\|_{L^2(\mathcal{O}_j)} \quad \text{and} \quad \|\mathbb{P}_{\mathcal{O}_j} u^\varepsilon\|_{H^1(\mathcal{O}_j)} \leq C_{\mathcal{O}_j} \|u^\varepsilon\|_{H^1(\mathcal{O}_j)}.$$

Thus, by (3.1),

$$(\mathbb{P}_{\mathcal{O}_j} u^\varepsilon, \nabla q^\varepsilon)_\varepsilon \quad \text{is bounded in } L^\infty(0, T; L^2(\mathcal{O}_j)) \cap L^2(0, T; H^1(\mathcal{O}_j)).$$

In particular,

$$\mathbb{P}_{\mathcal{O}_j} u^\varepsilon \xrightarrow{*} \mathbb{P}_{\mathcal{O}_j} u \quad \text{in } L^\infty(0, T; L^2(\mathcal{O}_j)) \cap L^2(0, T; H^1(\mathcal{O}_j)), \quad (3.16)$$

$$\nabla q^\varepsilon \xrightarrow{*} \nabla q = u - \mathbb{P}_{\mathcal{O}_j} u \quad \text{in } L^\infty(0, T; L^2(\mathcal{O}_j)) \cap L^2(0, T; H^1(\mathcal{O}_j)). \quad (3.17)$$

Now we derive a time estimate for $\mathbb{P}_{\mathcal{O}_j} u^\varepsilon$ in order to get a strong convergence. For any divergence free test function $\psi \in C_c^\infty((t_j, t_{j+1}) \times \mathcal{O}_j)$, we have by (3.15) that $\psi(t, \cdot) \in \mathcal{V}_R(\mathcal{F}^\varepsilon(t))$ (see (1.12)), hence (1.15) gives

$$\begin{aligned} \langle \partial_t \mathbb{P}_{\mathcal{O}_j} u^\varepsilon, \psi \rangle_{L^2((t_j, t_{j+1}); \mathcal{V}(\mathcal{O}_j)'), L^2((t_j, t_{j+1}); \mathcal{V}(\mathcal{O}_j))} &= - \int_{t_j}^{t_{j+1}} \int_{\mathcal{O}_j} \mathbb{P}_{\mathcal{O}_j} u^\varepsilon \cdot \partial_t \psi \, dx dt \\ &= - \int_{t_j}^{t_{j+1}} \int_{\mathcal{O}_j} u^\varepsilon \cdot \partial_t \psi \, dx dt \\ &= \int_{t_j}^{t_{j+1}} \int_{\mathcal{O}_j} u^\varepsilon \cdot (u^\varepsilon \cdot \nabla) \psi \, dx dt - 2\nu \int_{t_j}^{t_{j+1}} \int_{\mathcal{O}_j} D(u^\varepsilon) : D(\psi) \, dx dt. \end{aligned}$$

Thus, by using (3.1) and the interpolation inequality $\|f\|_{L^4(\Omega)} \leq \|f\|_{L^2(\Omega)}^{1/2} \|\nabla f\|_{L^2(\Omega)}^{1/2}$, we get

$$\begin{aligned} &\left| \langle \partial_t \mathbb{P}_{\mathcal{O}_j} u^\varepsilon, \psi \rangle_{L^2((t_j, t_{j+1}); \mathcal{V}(\mathcal{O}_j)'), L^2((t_j, t_{j+1}); \mathcal{V}(\mathcal{O}_j))} \right| \\ &\leq \|u^\varepsilon\|_{L^4((t_j, t_{j+1}); L^4(\mathcal{O}_j))}^2 \|\psi\|_{L^2((t_j, t_{j+1}); \mathcal{V}(\mathcal{O}_j))} + \|Du^\varepsilon\|_{L^2((t_j, t_{j+1}); L^2(\mathcal{O}_j))} \|\psi\|_{L^2((t_j, t_{j+1}); \mathcal{V}(\mathcal{O}_j))} \\ &\leq C \|\psi\|_{L^2((t_j, t_{j+1}); \mathcal{V}(\mathcal{O}_j))}. \end{aligned}$$

Consequently, $(\partial_t \mathbb{P}_{\mathcal{O}_j} u^\varepsilon)_\varepsilon$ is bounded in $L^2((t_j, t_{j+1}); \mathcal{V}(\mathcal{O}_j)'),$ and the Aubin-Lions lemma in $H^1 \cap \mathcal{H}(\mathcal{O}_j) \hookrightarrow L^4 \cap \mathcal{H}(\mathcal{O}_j) \hookrightarrow \mathcal{V}'(\mathcal{O}_j)$ allows us to extract a subsequence such that

$$\mathbb{P}_{\mathcal{O}_j} u^\varepsilon \rightarrow \mathbb{P}_{\mathcal{O}_j} u \quad \text{strongly in } L^2((t_j, t_{j+1}); L^4(\mathcal{O}_j)). \quad (3.18)$$

Actually, by the uniqueness of the limit, we do not need to extract a subsequence in (3.18).

These convergences are enough to pass to the limit in the Navier-Stokes equations. Indeed, for any $\varepsilon \in (0, \tilde{\varepsilon}_\eta]$, we know from (3.15) that φ^η is an admissible test function, and (1.15) reads

$$\begin{aligned} - \int_0^T \int_\Omega u^\varepsilon \cdot \frac{\partial \varphi^\eta}{\partial t} dx dt - \sum_{j=1}^M \int_{(t_j, t_{j+1}) \times \mathcal{O}_j} (u^\varepsilon \otimes u^\varepsilon) : \nabla \varphi^\eta dx dt + 2\nu \int_0^T \int_\Omega D(u^\varepsilon) : D(\varphi^\eta) dx dt \\ = \int_\Omega u_0^\varepsilon(x) \cdot \varphi^\eta(0, x) dx. \end{aligned}$$

Using the weak limits (1.17) and (3.4), we easily pass to the limit in the linear term

$$\begin{aligned} - \int_0^T \int_\Omega u^\varepsilon \cdot \frac{\partial \varphi^\eta}{\partial t} dx dt + 2\nu \int_0^T \int_\Omega D(u^\varepsilon) : D(\varphi^\eta) dx dt \\ \rightarrow - \int_0^T \int_\Omega u \cdot \frac{\partial \varphi^\eta}{\partial t} dx dt + 2\nu \int_0^T \int_\Omega D(u) : D(\varphi^\eta) dx dt \end{aligned}$$

and

$$\int_\Omega u_0^\varepsilon(x) \cdot \varphi^\eta(0, x) dx \rightarrow \int_\Omega u_0(x) \cdot \varphi^\eta(0, x) dx.$$

For the non-linear term, we decompose in $(t_j, t_{j+1}) \times \mathcal{O}_j$ as

$$u^\varepsilon \otimes u^\varepsilon = \mathbb{P}_{\mathcal{O}_j} u^\varepsilon \otimes u^\varepsilon + \nabla q^\varepsilon \otimes \mathbb{P}_{\mathcal{O}_j} u^\varepsilon + \nabla q^\varepsilon \otimes \nabla q^\varepsilon.$$

Let us note that for any harmonic \tilde{q} (i.e. $\Delta \tilde{q} = 0$), we have the following relation:

$$\int_{\mathcal{O}_j} (\nabla \tilde{q} \otimes \nabla \tilde{q}) : \nabla \varphi^\eta = - \int_{\mathcal{O}_j} \operatorname{div}(\nabla \tilde{q} \otimes \nabla \tilde{q}) \cdot \varphi^\eta = - \int_{\mathcal{O}_j} \left(\frac{1}{2} \nabla |\nabla \tilde{q}|^2 \cdot \varphi^\eta + \Delta \tilde{q} \nabla \tilde{q} \cdot \varphi^\eta \right) = 0, \quad (3.19)$$

because φ^η is divergence free and compactly supported in \mathcal{O}_j . From (3.18), (3.16) and (3.17), we have

$$\int_{(t_j, t_{j+1}) \times \mathcal{O}_j} (\mathbb{P}_{\mathcal{O}_j} u^\varepsilon \otimes u^\varepsilon) : \nabla \varphi^\eta dx dt \rightarrow \int_{(t_j, t_{j+1}) \times \mathcal{O}_j} (\mathbb{P}_{\mathcal{O}_j} u \otimes u) : \nabla \varphi^\eta dx dt,$$

and

$$\int_{(t_j, t_{j+1}) \times \mathcal{O}_j} (\nabla q^\varepsilon \otimes \mathbb{P}_{\mathcal{O}_j} u^\varepsilon) : \nabla \varphi^\eta dx dt \rightarrow \int_{(t_j, t_{j+1}) \times \mathcal{O}_j} (\nabla q \otimes \mathbb{P}_{\mathcal{O}_j} u) : \nabla \varphi^\eta dx dt.$$

Gathering the two above convergences and (3.19) applied to q^ε and to q , we conclude

$$\int_{(t_j, t_{j+1}) \times \mathcal{O}_j} (u^\varepsilon \otimes u^\varepsilon) : \nabla \varphi^\eta dx dt \rightarrow \int_{(t_j, t_{j+1}) \times \mathcal{O}_j} (u \otimes u) : \nabla \varphi^\eta dx dt.$$

This ends the proof of Theorem 3.4. □

To end the proof of Theorem 1.3, it is sufficient to pass to the limit $\eta \rightarrow 0$, thanks to Proposition 3.2.

Proof of Theorem 1.3. From Proposition 3.2, Remark 3.3 and Theorem 3.4, we consider T and η_0 verifying (3.7) and $\varphi \in C_c^\infty([0, T] \times \Omega)$ with $\operatorname{div} \varphi = 0$, then there exist $\eta_1 \leq \eta_0$ and $(\varphi^\eta)_{\eta \leq \eta_1}$ which approximate φ such that the limit u verifies for any $\eta \in (0, \eta_1]$

$$- \int_0^T \int_\Omega u \cdot \left(\frac{\partial \varphi^\eta}{\partial t} + (u \cdot \nabla) \varphi^\eta \right) dx + 2\nu \int_0^T \int_\Omega D(u) : D(\varphi^\eta) dx = \int_\Omega u_0(x) \cdot \varphi^\eta(0, x) dx.$$

As u belongs to $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, we deduce from the convergences (3.12)-(3.13) of φ^n to φ that

$$-\int_0^T \int_\Omega u \cdot \left(\frac{\partial \varphi}{\partial t} + (u \cdot \nabla) \varphi \right) dx + 2\nu \int_0^T \int_\Omega D(u) : D(\varphi) dx = \int_\Omega u_0(x) \cdot \varphi(0, x) dx.$$

By density, this equality is also true for any $\varphi \in C_c^1([0, T]; \mathcal{V}(\Omega))$. Noting that

$$\int D(u) : D(\varphi) = \frac{1}{2} \int \nabla u : \nabla \varphi + \frac{1}{2} \int \operatorname{div} u \operatorname{div} \varphi = \frac{1}{2} \int \nabla u : \nabla \varphi,$$

we end the proof of Theorem 1.3. \square

Remark 3.5. Since there exists a unique weak solution of the Navier–Stokes system associated to u_0 , the weak convergence (1.19) holds for all sequence (ε_n) converging to 0:

$$u^{\varepsilon_n} \rightharpoonup^* u \quad \text{in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

as $n \rightarrow \infty$, without extracting a subsequence.

4 Case of a massless pointwise particle

This section is dedicated to the proof of Theorem 1.6, hence we now consider the following situation:

- $\Omega = \mathbb{R}^2$ and the fluid domain is the exterior of one disk ($N = 1$);
- (u_0^ε) is a family of divergence vector fields verifying (1.14) such that $u_0^\varepsilon \rightharpoonup u_0$ in $L^2(\Omega)$ and $(\ell_0^\varepsilon, \varepsilon r_0^\varepsilon)_\varepsilon$ are bounded (for instance see Section 2);
- the solid tends to a massless pointwise particles: (1.22)-(1.23).

For one rigid ball in the full plane, [37] established that there exists a unique weak solution to problem (1.4)–(1.11). As there is only one solid, we omit the subscript 1 in the notation of this section:

$$\mathcal{S}_0^\varepsilon := \mathcal{S}_{0,1}^\varepsilon = B(0, \varepsilon), \quad \mathcal{S}^\varepsilon(t) := \mathcal{S}_1^\varepsilon(t) = B(h^\varepsilon(t), \varepsilon).$$

4.1 Uniform estimates

4.1.1 Energy estimate

As in the massive case, we first use the energy inequality (1.16) so that

$$(u^\varepsilon)_\varepsilon \text{ bounded in } L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L_{\text{loc}}^2(\mathbb{R}^+; H^1(\mathbb{R}^2)),$$

hence we can extract a subsequence such that

$$u^\varepsilon \rightharpoonup^* u \quad \text{in } L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L_{\text{loc}}^2(\mathbb{R}^+; H^1(\mathbb{R}^2)).$$

By abuse of notation, we continue to write u^ε the subsequence.

In the case of massless pointwise particle, ρ_0^ε is independent of ε , and the energy estimate only gives

$$\varepsilon(h^\varepsilon)'(t), \varepsilon^2(\theta^\varepsilon)'(t) \text{ bounded in } L^\infty(\mathbb{R}^+).$$

The goal of the sequel is to obtain an estimate of $(h^\varepsilon)'$ independently of ε .

4.1.2 Duhamel's formula

As in [13, 37], we make the change of variables

$$v^\varepsilon(t, x) = u^\varepsilon(t, x - h^\varepsilon(t)), \quad q^\varepsilon(t, x) = p^\varepsilon(t, x - h^\varepsilon(t)),$$

and we define

$$\ell^\varepsilon(t) = (h^\varepsilon)'(t), \quad r^\varepsilon(t) = (\theta^\varepsilon)'(t).$$

The vector fields v^ε is the weak solution of a system similar to (1.4)–(1.11):

$$\frac{\partial v^\varepsilon}{\partial t} + ([v^\varepsilon - \ell^\varepsilon] \cdot \nabla)v^\varepsilon - \operatorname{div} \sigma(v^\varepsilon, q^\varepsilon) = 0 \quad t > 0, \quad x \in \mathcal{F}_0^\varepsilon, \quad (4.1)$$

$$\operatorname{div} v^\varepsilon = 0 \quad t > 0, \quad x \in \mathcal{F}_0^\varepsilon, \quad (4.2)$$

$$\lim_{|x| \rightarrow \infty} v^\varepsilon(x) = 0 \quad t > 0, \quad (4.3)$$

$$v^\varepsilon(t, x) = \ell^\varepsilon(t) + r^\varepsilon(t)x^\perp \quad t > 0, \quad x \in \partial \mathcal{S}_0^\varepsilon, \quad (4.4)$$

$$m^\varepsilon(\ell^\varepsilon)'(t) = - \int_{\partial \mathcal{S}_0^\varepsilon} \sigma(v^\varepsilon, q^\varepsilon) n \, d\gamma \quad t > 0, \quad (4.5)$$

$$J^\varepsilon(r^\varepsilon)'(t) = - \int_{\partial \mathcal{S}_0^\varepsilon(t)} x^\perp \cdot \sigma(v^\varepsilon, q^\varepsilon) n \, d\gamma \quad t > 0, \quad (4.6)$$

$$v^\varepsilon(0, \cdot) = v_0^\varepsilon \quad \text{in } \mathcal{F}_0^\varepsilon, \quad (4.7)$$

$$\ell^\varepsilon(0) = \ell_0^\varepsilon, \quad r^\varepsilon(0) = r_0^\varepsilon. \quad (4.8)$$

We set the global density in \mathbb{R}^2 :

$$\rho^\varepsilon(x) = \begin{cases} 1 & x \in \mathcal{F}_0^\varepsilon, \\ \rho & x \in \mathcal{S}_0^\varepsilon. \end{cases}$$

We can define a weak solution

Definition 4.1. *We say that $(v^\varepsilon, \ell^\varepsilon, r^\varepsilon)$ is a global weak solution of (4.1)–(4.8) if, for any $T > 0$, we have*

$$v^\varepsilon \in L^\infty(0, T; L^2(\mathbb{R}^2)) \cap L^2(0, T; H^1(\mathbb{R}^2)), \quad v^\varepsilon(t, x) = \ell^\varepsilon(t) + r^\varepsilon(t)x^\perp \quad \text{in } \mathcal{S}_0^\varepsilon,$$

if it satisfies the weak formulation

$$- \int_0^T \int_{\mathbb{R}^2} \rho^\varepsilon v^\varepsilon \cdot \left(\frac{\partial \varphi^\varepsilon}{\partial t} + (v^\varepsilon \cdot \nabla)\varphi^\varepsilon \right) \, dx ds + 2\nu \int_0^T \int_{\mathbb{R}^2} D(v^\varepsilon) : D(\varphi^\varepsilon) \, dx ds = \int_{\mathbb{R}^2} \rho^\varepsilon v_0^\varepsilon(x) \cdot \varphi^\varepsilon(0, x) \, dx,$$

for any $\varphi^\varepsilon \in C_c^1([0, T]; H^1(\mathbb{R}^2))$ such that $\varphi^\varepsilon(t, \cdot) \in \mathcal{V}_R(\mathcal{F}_0^\varepsilon)$ and if it satisfies the energy inequality

$$\frac{1}{2} \int_{\mathbb{R}^2} \rho^\varepsilon(x) |v^\varepsilon(t, x)|^2 \, dx + 2\nu \int_0^t \int_{\mathbb{R}^2} |D(v^\varepsilon)|^2 \, dx ds \leq \frac{1}{2} \int_{\mathbb{R}^2} \rho^\varepsilon(x) |v_0^\varepsilon(x)|^2 \, dx \quad \text{a.e. } t \in (0, T).$$

One can check that u^ε is a weak solution in the sense of Definition 1.1 if and only if v^ε is a weak solution in the sense of the above definition.

We also define the following functional spaces for $p \in [1, \infty]$,

$$\mathcal{L}_\varepsilon^p = \{v \in L^p(\mathbb{R}^2) ; \operatorname{div} v = 0 \text{ in } \mathbb{R}^2, \quad D(v) = 0 \text{ in } \mathcal{S}_0^\varepsilon\},$$

with the norm (for $p \neq \infty$)

$$\|v\|_{\mathcal{L}_\varepsilon^p} = \left(\int_{\mathbb{R}^2} \rho^\varepsilon |v|^p \, dx \right)^{1/p}.$$

For $p = \infty$, the norm \mathcal{L}^∞ is the classical L^∞ norm. We recall that for any $v \in \mathcal{L}_\varepsilon^p$, there exists $(\ell_v, r_v) \in \mathbb{R}^3$ such that

$$v(y) = \ell_v + r_v y^\perp \quad (y \in \mathcal{S}_0^\varepsilon).$$

Moreover, one can deduce ℓ_v from v by

$$\ell_v := \frac{1}{|\mathcal{S}_0^\varepsilon|} \int_{\mathcal{S}_0^\varepsilon} v \, dy. \quad (4.9)$$

One can write the system (4.1)–(4.8) in the following abstract form:

$$\partial_t v^\varepsilon + A^\varepsilon v^\varepsilon = \mathbb{P}^\varepsilon \operatorname{div} F^\varepsilon(v^\varepsilon), \quad v^\varepsilon(0) = v_0^\varepsilon,$$

where

$$\begin{aligned} \mathcal{D}(\mathcal{A}^\varepsilon) &:= \{v \in H^2(\mathbb{R}^2) ; \operatorname{div} v = 0 \text{ in } \mathbb{R}^2, \quad D(v) = 0 \text{ in } \mathcal{S}_0^\varepsilon\}, \\ \mathcal{A}^\varepsilon v &:= \begin{cases} -\nu \Delta v & \text{in } \mathcal{F}_0^\varepsilon, \\ \frac{2\nu}{m^\varepsilon} \int_{\partial \mathcal{S}_0^\varepsilon} D(v)n \, ds + \frac{2\nu}{J^\varepsilon} \left(\int_{\partial \mathcal{S}_0^\varepsilon} y^\perp \cdot D(v)n \, dy \right) x^\perp & \text{in } \mathcal{S}_0^\varepsilon. \end{cases} \quad (v \in \mathcal{D}(\mathcal{A}^\varepsilon)), \\ A^\varepsilon &:= \mathbb{P}^\varepsilon \mathcal{A}^\varepsilon, \\ F^\varepsilon(v^\varepsilon) &= \begin{cases} v^\varepsilon \otimes (\ell_{v^\varepsilon} - v^\varepsilon) & \text{on } \mathcal{F}_0^\varepsilon \\ 0 & \text{on } \mathcal{S}_0^\varepsilon, \end{cases} \end{aligned} \quad (4.10)$$

and where \mathbb{P}^ε denotes the continuous projector from $L^p(\mathbb{R}^2)$ to $\mathcal{L}_\varepsilon^p$, and ℓ_{v^ε} is defined through (4.9). Note that in the definition of \mathcal{A}^ε , $D(v)n$ corresponds to the trace of the restriction of $D(v)$ to the fluid domain.

The operator $-A^\varepsilon$ is the infinitesimal generator of a semigroup of $(S^\varepsilon(t))_{t \geq 0}$ in $\mathcal{L}_\varepsilon^p$ for $p \in (1, \infty)$ (see [13]). Then, Duhamel's formula gives the following integral formulation of the above equations:

$$v^\varepsilon(t) = S^\varepsilon(t)v_0^\varepsilon + \int_0^t S^\varepsilon(t-s) \mathbb{P}^\varepsilon \operatorname{div} F^\varepsilon(v^\varepsilon(s)) \, ds. \quad (4.11)$$

4.1.3 Semigroup estimates

The key for the uniform estimate of ℓ^ε is the following theorem concerning the Stokes-rigid body semigroup.

Theorem 4.2. *For each $q \in (1, \infty)$, the semigroup $S^\varepsilon(t)$ on $\mathcal{L}_\varepsilon^q$ satisfies the following decay estimates:*

- For $p \in [q, \infty]$, there exists $K_1 = K_1(p, q) > 0$ such that for every $v_0^\varepsilon \in \mathcal{L}_\varepsilon^q$:

$$\|S^\varepsilon(t)v_0^\varepsilon\|_{\mathcal{L}_\varepsilon^p} \leq K_1 t^{\frac{1}{p} - \frac{1}{q}} \|v_0^\varepsilon\|_{\mathcal{L}_\varepsilon^q} \quad \text{for all } t > 0. \quad (4.12)$$

- For $2 \leq q \leq p < \infty$, there exists $K_2 = K_2(p, q) > 0$ such that for every $F^\varepsilon \in L^q(\mathbb{R}^2; M_{2 \times 2}(\mathbb{R}))$ satisfying $F^\varepsilon = 0$ in $\mathcal{S}_0^\varepsilon$:

$$\|S^\varepsilon(t) \mathbb{P}^\varepsilon \operatorname{div} F^\varepsilon\|_{\mathcal{L}_\varepsilon^p} \leq K_2 t^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} \|F^\varepsilon\|_{L^q(\mathbb{R}^2)} \quad \text{for all } t > 0. \quad (4.13)$$

- For $2 \leq q < \infty$, there exists $K_\ell = K_\ell(q) > 0$ such that for every $F^\varepsilon \in L^q(\mathbb{R}^2; M_{2 \times 2}(\mathbb{R}))$ satisfying $F^\varepsilon = 0$ on $\mathcal{S}_0^\varepsilon$:

$$|\ell_{S^\varepsilon(t) \mathbb{P}^\varepsilon \operatorname{div} F^\varepsilon}| \leq K_\ell t^{-(\frac{1}{2} + \frac{1}{q})} \|F^\varepsilon\|_{L^q(\mathbb{R}^2)} \quad \text{for all } t > 0. \quad (4.14)$$

For ε fixed, estimates like (4.12)–(4.13) were only established for the Stokes system with the Dirichlet boundary condition [10, 28]. For the fluid solid problem with one rigid disk in \mathbb{R}^2 , this result was recently obtained by Ervedoza, Hillairet and Lacave in [13]. The only point to check here is that the constants K_1, K_2, K_ℓ are independent of ε , which will be easily obtained by a scaling argument.

Proof. For $\varepsilon = 1$, the statements of the theorem were proved in [13], see therein Theorem 1.1, Corollaries 3.10 and 3.11.

We note that $v^\varepsilon(t) := S^\varepsilon(t)v_0^\varepsilon$ satisfies

$$\begin{aligned} \frac{\partial v^\varepsilon}{\partial t} - \operatorname{div} \sigma(v^\varepsilon, q^\varepsilon) &= 0 & t > 0, \quad x \in \mathcal{F}_0^\varepsilon, \\ \operatorname{div} v^\varepsilon &= 0 & t > 0, \quad x \in \mathcal{F}_0^\varepsilon, \\ \lim_{|x| \rightarrow \infty} v^\varepsilon(x) &= 0 & t > 0, \\ v^\varepsilon(t, x) &= \ell^\varepsilon(t) + r^\varepsilon(t)x^\perp & t > 0, \quad x \in \partial\mathcal{S}_0^\varepsilon, \\ m^\varepsilon(\ell^\varepsilon)'(t) &= - \int_{\partial\mathcal{S}_0^\varepsilon} \sigma(v^\varepsilon, q^\varepsilon)n \, d\gamma & t > 0, \\ J^\varepsilon(r^\varepsilon)'(t) &= - \int_{\partial\mathcal{S}_0^\varepsilon(t)} x^\perp \cdot \sigma(v^\varepsilon, q^\varepsilon)n \, d\gamma & t > 0, \\ v^\varepsilon(0, \cdot) &= v_0^\varepsilon & \text{in } \mathcal{F}_0^\varepsilon, \\ \ell^\varepsilon(0) &= \ell_0^\varepsilon, \quad r^\varepsilon(0) = r_0^\varepsilon. \end{aligned}$$

Setting

$$v(t, x) := v^\varepsilon(\varepsilon^2 t, \varepsilon x), \quad q(t, x) := \varepsilon q^\varepsilon(\varepsilon^2 t, \varepsilon x), \quad \ell(t) := \ell^\varepsilon(\varepsilon^2 t), \quad r(t) := \varepsilon r^\varepsilon(\varepsilon^2 t), \quad (4.15)$$

standard calculation gives that

$$\begin{aligned} \frac{\partial v}{\partial t} - \operatorname{div} \sigma(v, q) &= 0 & t > 0, \quad x \in \mathcal{F}_0^1, \\ \operatorname{div} v &= 0 & t > 0, \quad x \in \mathcal{F}_0^1, \\ \lim_{|x| \rightarrow \infty} v(x) &= 0 & t > 0, \\ v(t, x) &= \ell(t) + r(t)x^\perp & t > 0, \quad x \in \partial\mathcal{S}_0^1, \\ m^1 \ell'(t) &= - \int_{\partial\mathcal{S}_0^1} \sigma(v, q)n \, d\gamma & t > 0, \\ J^1 r'(t) &= - \int_{\partial\mathcal{S}_0^1(t)} x^\perp \cdot \sigma(v, q)n \, d\gamma & t > 0, \\ v(0, \cdot) &= v_0 & \text{in } \mathcal{F}_0^1, \\ \ell(0) &= \ell_0, \quad r(0) = r_0, \end{aligned}$$

where

$$v_0(x) := v_0^\varepsilon(\varepsilon x), \quad \ell_0 := \ell_0^\varepsilon, \quad r_0 := \varepsilon r_0^\varepsilon. \quad (4.16)$$

This means that $v(t) = v^1(t) = S^1(t)v_0$ and thus that

$$\|v(t)\|_{\mathcal{L}_1^p} \leq K_1(\nu t)^{\frac{1}{p} - \frac{1}{q}} \|v_0\|_{\mathcal{L}_1^q} \quad \text{for all } t > 0.$$

Using (4.15)-(4.16), this estimate is equivalent to

$$\|v^\varepsilon(t)\|_{\mathcal{L}_\varepsilon^p} \leq K_1(\nu t)^{\frac{1}{p} - \frac{1}{q}} \|v_0^\varepsilon\|_{\mathcal{L}_\varepsilon^q} \quad \text{for all } t > 0.$$

Relations (4.13) and (4.14) can be done similarly. In that case, we also set

$$F(x) := \frac{1}{\varepsilon} F^\varepsilon(\varepsilon x)$$

and we show that if $v^\varepsilon(t) = S^\varepsilon(t)\mathbb{P}^\varepsilon \operatorname{div} F^\varepsilon$, then v defined by (4.15) satisfies

$$v(t) = S^1(t)\mathbb{P}^1 \operatorname{div} F.$$

□

4.1.4 Uniform estimate on the solid velocity

The main goal of this paragraph is to get a uniform estimate on ℓ^ε . This will be obtained by assuming a smallness condition on the \mathcal{L}^2 norm of v_0^ε . We first show that there exists $\lambda_0 > 0$ such that if $\|v_0^\varepsilon\|_{\mathcal{L}_\varepsilon^2} \leq \lambda_0$, then there exists a unique

$$v^\varepsilon \in \mathcal{C}^0([0, T]; \mathcal{L}_\varepsilon^2) \cap \mathcal{C}_{3/8}^0([0, T]; \mathcal{L}_\varepsilon^8) \quad \text{with} \quad \ell^\varepsilon := \ell_{v^\varepsilon} \in \mathcal{C}_{1/2}^0([0, T]; \mathbb{R}^2) \quad (4.17)$$

satisfying (4.11) (that is a *mild* solution). Here we have denoted for any Banach space X by $\mathcal{C}_\alpha^0([0, T]; X)$ the Banach space of functions f such that $t \mapsto t^\alpha f(t)$ are continuous from $[0, T]$ in X . The norm associated is

$$\|f\|_{\mathcal{C}_\alpha^0([0, T]; X)} := \sup_{t \in [0, T]} t^\alpha \|f(t)\|_X.$$

Proposition 4.3. *There exist $\lambda_0, \mu_0 > 0$ independent of ε such that the following holds for any $T > 0$. If $v_0^\varepsilon \in \mathcal{L}_\varepsilon^2$ satisfies*

$$\|v_0^\varepsilon\|_{\mathcal{L}_\varepsilon^2} \leq \lambda_0 \quad (4.18)$$

then there exists a unique v^ε satisfying (4.11) and such that

$$\|v^\varepsilon\|_{\mathcal{C}^0([0, T]; \mathcal{L}_\varepsilon^2)}, \|v^\varepsilon\|_{\mathcal{C}_{3/8}^0([0, T]; \mathcal{L}_\varepsilon^8)}, \|\ell^\varepsilon\|_{\mathcal{C}_{1/2}^0([0, T]; \mathbb{R}^2)} \text{ are bounded by } \mu_0.$$

Moreover there exists a constant $C > 0$ independent of ε such that, if $v_{0a}^\varepsilon, v_{0b}^\varepsilon \in \mathcal{L}_\varepsilon^2$ are two initial conditions satisfying (4.18), then

$$\|v_a^\varepsilon - v_b^\varepsilon\|_{\mathcal{C}^0([0, T]; \mathcal{L}_\varepsilon^2)} \leq C \|v_{0a}^\varepsilon - v_{0b}^\varepsilon\|_{\mathcal{L}_\varepsilon^2}. \quad (4.19)$$

Proof. As we have proved in the previous theorem that the constant in the semigroup estimates are independent of ε , it is then enough to follow the fixed point argument in [13, pp. 364-371]. For completeness, let us write here the details.

Let us introduce the space

$$\mathcal{X}^\varepsilon := \left\{ v^\varepsilon \in \mathcal{C}^0([0, T]; \mathcal{L}_\varepsilon^2) \cap \mathcal{C}_{3/8}^0([0, T]; \mathcal{L}_\varepsilon^8) \quad \text{with} \quad \ell_{v^\varepsilon} \in \mathcal{C}_{1/2}^0([0, T]; \mathbb{R}^2) \right\}$$

endowed with the norm

$$\|v^\varepsilon\|_{\mathcal{X}^\varepsilon} := \|v^\varepsilon\|_{\mathcal{C}^0([0, T]; \mathcal{L}_\varepsilon^2)} + \|v^\varepsilon\|_{\mathcal{C}_{3/8}^0([0, T]; \mathcal{L}_\varepsilon^8)} + \|\ell_{v^\varepsilon}\|_{\mathcal{C}_{1/2}^0([0, T]; \mathbb{R}^2)}.$$

Let us also define the map

$$\mathcal{Z}^\varepsilon : \mathcal{X}^\varepsilon \rightarrow \mathcal{X}^\varepsilon,$$

defined by

$$\mathcal{Z}^\varepsilon(v^\varepsilon)(t) = S^\varepsilon(t)v_0^\varepsilon + \int_0^t S^\varepsilon(t-s)\mathbb{P}^\varepsilon \operatorname{div} F^\varepsilon(v^\varepsilon)(s) ds,$$

where F^ε is defined by (4.10). One can define

$$\Phi(v^\varepsilon, w^\varepsilon)(t) = \int_0^t S^\varepsilon(t-s)\mathbb{P}^\varepsilon \operatorname{div} G^\varepsilon(v^\varepsilon, w^\varepsilon)(s) ds,$$

where

$$G^\varepsilon(v^\varepsilon, w^\varepsilon) = \begin{cases} v^\varepsilon \otimes (\ell_{w^\varepsilon} - w^\varepsilon) & \text{on } \mathcal{F}_0^\varepsilon \\ 0 & \text{on } \mathcal{S}_0^\varepsilon. \end{cases}$$

We deduce from (4.13) that

$$t^{\frac{3}{8}} \|\Phi(v^\varepsilon, w^\varepsilon)(t)\|_{\mathcal{L}_\varepsilon^8} \leq t^{\frac{3}{8}} K_2(8, 4) \int_0^t (t-s)^{-5/8} (\|v^\varepsilon(s) \otimes w^\varepsilon(s)\|_{\mathcal{L}_\varepsilon^4} + |\ell_{w^\varepsilon}(s)| \|v^\varepsilon(s)\|_{\mathcal{L}_\varepsilon^4}) ds.$$

Using Hölder's inequalities, we obtain from the above inequality that

$$\begin{aligned} t^{\frac{3}{8}} \|\Phi(v^\varepsilon, w^\varepsilon)(t)\|_{\mathcal{L}_\varepsilon^8} &\leq t^{\frac{3}{8}} K_2(8, 4) \int_0^t (t-s)^{-5/8} \left(s^{-\frac{3}{8}} \|v^\varepsilon\|_{C_{3/8}^0 \mathcal{L}_\varepsilon^8} s^{-\frac{3}{8}} \|w^\varepsilon\|_{C_{3/8}^0 \mathcal{L}_\varepsilon^8} \right. \\ &\quad \left. + s^{-\frac{1}{2}} |\ell_{w^\varepsilon}|_{C_{1/2}^0} \|v^\varepsilon\|_{L^\infty \mathcal{L}_\varepsilon^2}^{1/3} (s^{-\frac{3}{8}})^{2/3} \|v^\varepsilon\|_{C_{3/8}^0 \mathcal{L}_\varepsilon^8}^{2/3} \right) ds \\ &\leq 2K_2(8, 4) B\left(\frac{5}{8}, \frac{3}{4}\right) \|v^\varepsilon\|_{\mathcal{X}^\varepsilon} \|w^\varepsilon\|_{\mathcal{X}^\varepsilon}, \end{aligned} \quad (4.20)$$

where $B(\cdot, \cdot)$ is the Beta function:

$$B(\alpha, \beta) := \int_0^1 (1-\tau)^{-\alpha} \tau^{-\beta} d\tau.$$

Similarly,

$$\|\Phi(v^\varepsilon, w^\varepsilon)(t)\|_{\mathcal{L}_\varepsilon^2} \leq 2K_2(2, 2) B\left(\frac{1}{2}, \frac{1}{2}\right) \|v^\varepsilon\|_{\mathcal{X}^\varepsilon} \|w^\varepsilon\|_{\mathcal{X}^\varepsilon}, \quad (4.21)$$

and

$$t^{\frac{1}{2}} |\ell_{\Phi(v^\varepsilon, w^\varepsilon)}(t)| \leq 2K_\ell(4) B\left(\frac{3}{4}, \frac{3}{4}\right) \|v^\varepsilon\|_{\mathcal{X}^\varepsilon} \|w^\varepsilon\|_{\mathcal{X}^\varepsilon}. \quad (4.22)$$

Gathering (4.20), (4.21) and (4.22) yields

$$\|\Phi(v^\varepsilon, w^\varepsilon)\|_{\mathcal{X}^\varepsilon} \leq C_0 \|v^\varepsilon\|_{\mathcal{X}^\varepsilon} \|w^\varepsilon\|_{\mathcal{X}^\varepsilon}, \quad (4.23)$$

where

$$C_0 = 2 \left(K_2(8, 4) B\left(\frac{5}{8}, \frac{3}{4}\right) + K_2(2, 2) B\left(\frac{1}{2}, \frac{1}{2}\right) + K_\ell(4) B\left(\frac{3}{4}, \frac{3}{4}\right) \right).$$

We assume (4.18) for λ_0 that we fix below and we apply (4.12) in order to obtain

$$\|S^\varepsilon v_0^\varepsilon\|_{\mathcal{X}^\varepsilon} \leq C_1 \lambda_0, \quad (4.24)$$

where

$$C_1 = K_1(8, 2) + K_1(2, 2) + K_1(\infty, 2).$$

Relations (4.23) and (4.24) imply that the mapping \mathcal{Z}^ε is well-defined and that

$$\|\mathcal{Z}^\varepsilon(v^\varepsilon)\|_{\mathcal{X}^\varepsilon} \leq C_1 \lambda_0 + C_0 \|v^\varepsilon\|_{\mathcal{X}^\varepsilon}^2.$$

Let us set

$$R := \frac{1}{4C_0} \quad \text{and} \quad \lambda_0 = \min\left(\frac{R}{2C_1}, R\right).$$

Then the closed ball $B_{\mathcal{X}^\varepsilon}(0, R)$ of \mathcal{X}^ε is invariant by \mathcal{Z}^ε and if $v^\varepsilon, w^\varepsilon \in B_{\mathcal{X}^\varepsilon}(0, R)$,

$$\|\mathcal{Z}^\varepsilon(v^\varepsilon) - \mathcal{Z}^\varepsilon(w^\varepsilon)\|_{\mathcal{X}^\varepsilon} = \|\Phi(v^\varepsilon, v^\varepsilon - w^\varepsilon) + \Phi(v^\varepsilon - w^\varepsilon, w^\varepsilon)\|_{\mathcal{X}^\varepsilon} \leq 2C_0 R \|v^\varepsilon - w^\varepsilon\|_{\mathcal{X}^\varepsilon} = \frac{1}{2} \|v^\varepsilon - w^\varepsilon\|_{\mathcal{X}^\varepsilon}.$$

Using the Banach fixed point, we deduce the existence and uniqueness results. Moreover, we have $\mu_0 = R = 1/(4C_0)$, which is independent of ε . This strategy comes from [24] and [25]. It is originally done through an iterative method, but it was adapted as a fixed point argument in [30] (see also [35]).

Sensitivity of v^ε to the initial data. Assume $v_{0a}^\varepsilon, v_{0b}^\varepsilon \in \mathcal{L}_\varepsilon^2$ satisfy (4.18). Then,

$$\begin{aligned} \|v_a^\varepsilon - v_b^\varepsilon\|_{\mathcal{X}^\varepsilon} &\leq \|S^\varepsilon(v_{0a}^\varepsilon - v_{0b}^\varepsilon)\|_{\mathcal{X}^\varepsilon} + \|\Phi(v_a^\varepsilon, v_a^\varepsilon - v_b^\varepsilon)\|_{\mathcal{X}^\varepsilon} + \|\Phi(v_a^\varepsilon - v_b^\varepsilon, v_b^\varepsilon)\|_{\mathcal{X}^\varepsilon} \\ &\leq C_1 \|v_{0a}^\varepsilon - v_{0b}^\varepsilon\|_{\mathcal{L}_\varepsilon^2} + C_0 \|v_a^\varepsilon - v_b^\varepsilon\|_{\mathcal{X}^\varepsilon} (\|v_a^\varepsilon\|_{\mathcal{X}^\varepsilon} + \|v_b^\varepsilon\|_{\mathcal{X}^\varepsilon}) \\ &\leq C_1 \|v_{0a}^\varepsilon - v_{0b}^\varepsilon\|_{\mathcal{L}_\varepsilon^2} + 2C_0 R \|v_a^\varepsilon - v_b^\varepsilon\|_{\mathcal{X}^\varepsilon}. \end{aligned}$$

We conclude that

$$\|v_a^\varepsilon - v_b^\varepsilon\|_{\mathcal{X}^\varepsilon} \leq 2C_1 \|v_{0a}^\varepsilon - v_{0b}^\varepsilon\|_{\mathcal{L}_\varepsilon^2}.$$

□

One can show that a mild solution in the above sense is also a weak solution (see [13]). For sake of completeness, we give the proof of this result here.

Lemma 4.4. *Assume v^ε satisfies (4.17) and (4.11). Then v^ε is the weak solution of (4.1)-(4.8) in the sense of Definition 4.1.*

Proof. Let us consider a sequence $(v_{0n}^\varepsilon)_n$ with values in $\mathcal{D}((A^\varepsilon)^{1/2})$ such that

$$v_{0n}^\varepsilon \rightarrow v_0^\varepsilon \quad \text{in } \mathcal{L}^2$$

and such that v_{0n}^ε satisfies (4.18). For all n , it is proved in [37] there exists a unique strong solution

$$v_n^\varepsilon \in H^1(0, T; \mathcal{L}_\varepsilon^2) \cap C([0, T]; \mathcal{D}((A^\varepsilon)^{1/2})) \cap L^2(0, T; \mathcal{D}(A^\varepsilon))$$

of (4.1)-(4.8) and it satisfies (4.11) since in this case

$$\operatorname{div} F^\varepsilon(v_n^\varepsilon) \in L^2(0, T; L^2(\mathbb{R}^2)).$$

It is also proved in [37] that (v_n^ε) converges towards the weak solution of (4.1)-(4.8) in the sense of Definition 4.1.

From (4.19), we also have that (v_n^ε) converges towards the mild solution of Proposition 4.3 in $L^\infty(0, T; \mathcal{L}_\varepsilon^2)$.

Consequently, the mild solution v^ε , that satisfies (4.17) and (4.11), is the weak solution associated to v_0^ε . \square

4.2 Proof of Theorem 1.6

Following the strategy of Section 3, we can now prove Theorem 1.6.

4.2.1 First convergences

We have already obtained in Section 4.1.1 the weak convergence of u^ε to u .

If the initial data satisfies the smallness condition (1.24), we deduce from the above section that

$$|(h^\varepsilon)'(t)| \leq \frac{\mu_0}{\sqrt{t}} \quad (t > 0),$$

where μ_0 is independent of ε . As a consequence,

$$|h^\varepsilon(t)| \leq 2\mu_0\sqrt{T} \quad (t \in [0, T]).$$

We fix $q \in (1, 2)$, thus (h^ε) is bounded in $W^{1,q}(0, T; \mathbb{R}^2)$ and we have, up to a subsequence,

$$h^\varepsilon \rightarrow h \quad \text{uniformly in } [0, T],$$

with

$$h \in W^{1,q}(0, T).$$

As in the massive case, let us note that this convergence is uniform (see (3.5)) but the estimate of h is slightly weaker than (3.6).

4.2.2 Modified test function

The differences comparing with Proposition 3.2 are the regularity and the convergence of $\partial_t \varphi^\eta$. Namely, for any $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^2)$ with $\operatorname{div} \varphi = 0$ and any $\eta > 0$, there exists $\varphi^\eta \in W_c^{1,q}([0, T]; H_0^1(\mathbb{R}^2))$ satisfying the divergence free condition (3.10), vanishing on $B(h(t), \frac{\eta}{2})$, which converges weak-* to φ in $L^\infty(0, T; H^1(\mathbb{R}^2))$ such that

$$\partial_t \varphi^\eta \rightharpoonup \partial_t \varphi \quad L^q(0, T; L^2(\mathbb{R}^2)). \quad (4.25)$$

4.2.3 Proof of Theorem 1.6

The proof of Theorem 3.4 can be done in a similar way. Indeed, we obtain the subdivision $t_0 = 0 < t_1 < \dots < t_{M+1} = T$ by fixing t_1 such that $2\mu_0\sqrt{t_1} = \eta/6$, and then we use that h is Lipschitz on $[t_1, T]$.

Finally, the passing to the limit $\eta \rightarrow 0$ holds in the same way, even with (4.25) instead to (3.13).

Remark 3.5 can be also applied to state that the convergence to the unique solution of the Navier-Stokes equations in \mathbb{R}^2 holds without extracting a subsequence. This ends the proof of Theorem 1.6.

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