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Eric Amar

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# On the $L^r$ Hodge theory in complete non compact riemannian manifolds.

Eric Amar

## Abstract

We study solutions for the Hodge laplace equation  $\Delta u = \omega$  on  $p$  forms with  $L^r$  estimates for  $r > 1$ . Our main hypothesis is that  $\Delta$  has a spectral gap in  $L^2$ . We use this to get *non classical*  $L^r$  Hodge decomposition theorems. An interesting feature is that to prove these decompositions we never use the boundedness of the Riesz transforms in  $L^s$ .

These results are based on a generalisation of the Raising Steps Method to complete non compact riemannian manifolds.

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# 1 Introduction.

In the sequel, a riemannian manifold  $(M, g)$  means a  $C^\infty$  smooth connected riemannian manifold of dimension  $\geq 3$ .

In this work we study the problem of  $L^r$  estimates of the Laplace equation  $\Delta u = \omega$  for the Hodge laplacian on  $p$ -forms and the Hodge decomposition theorems on complete non compact riemannian manifolds.

This problem was studied by a several authors, in particular N. Lohoué in [19] (see also the references therein). Also the problem of Hodge decompositions has a long history and for the recent developments one can see the papers by X.D. Li [15], [18], [17] and also related to several complex variables [16] (see also the references therein).

In all those works the boundedness of the Riesz transforms are explicitly used and in this work, where the Hodge decompositions are *not the classical ones*, we shall see that it is not the case.

Let me describe the method we shall use.

Suppose you are interested by solving an equation  $Du = \omega$ , in a manifold  $M$  with estimates of type Lebesgue  $L^r$  or Sobolev  $W^{d,r}$ ; you know how to solve it globally with "threshold" estimates  $L^s \rightarrow L^s$  and locally with estimates  $L^r \rightarrow L^t$  with a strict increase of the regularity, for instance  $\frac{1}{t} = \frac{1}{r} - \delta$ ,  $\delta > 0$  for any  $r \leq s$ , then the Raising Steps Method (RSM for short) gives a *global* solution  $v$  of  $Dv = \omega$  which is essentially in  $L^t(M)$  for  $\omega \in L^r(M)$ .

I introduced this method in [1] to get solutions for the  $\bar{\partial}$  equation with good estimates in relatively compact domains in Stein manifold. I extend it to linear partial differential operator  $D$  of any finite order  $m$  in [2] and I apply it to study the Poisson equation for the Hodge laplacian on forms in spaces  $L^r(M)$  where  $(M, g)$  is a compact riemannian manifold. This gave  $L^r$  Hodge decomposition theorems as was done by C. Scott [22], but by an entirely different approach.

The aim of this work is to extend it to the case of complete non compact riemannian manifold, and, as we shall see, at no point we shall use the boundedness of the Riesz transforms.

## 1.1 Solutions of the Poisson equation for the Hodge laplacian.

Let  $(M, g)$  be a  $C^\infty$  smooth connected riemannian manifold with metric tensor  $g$  and  $n = \dim M \geq 3$ ; let  $d$  be the exterior derivative,  $d^*$  its formal adjoint with respect to the Riemannian volume measure  $dv_g = \sqrt{\det g} dx$ , where  $dx$  is the Lebesgue measure in the chart  $x$ , and  $\Delta = \Delta_p := dd^* + d^*d$  the Hodge laplacian acting on  $p$  forms. Let  $L_p^r(M)$  be the space of  $p$  forms on  $M$  in the Lebesgue space  $L^r(M)$ .

We get the local solution of the Hodge Laplacian  $\Delta u = \omega$  in a ball  $B(x, R)$  in  $(M, g)$  with a radius  $R(x)$  small enough to make this ball "not too different" to a ball in the euclidean space  $\mathbb{R}^n$ ; this "admissible" radius is a special case of the "harmonic radius" of Hebey and Herzlich [13]. If  $\omega$  is a  $p$  form in  $L^r(B(x, R))$  then we get a  $p$  form solution  $u$  in the Sobolev space  $W^{2,r}(B(x, r))$  of the ball, hence in  $L^t(B(x, R))$  with  $\frac{1}{t} = \frac{1}{r} - \frac{2}{n}$  by the Sobolev embeddings. This is done classically by use of the Newtonian potential. So the first assumption for the RSM is true : we have locally a strict increase of the regularity.

In order to get global solutions we need to cover the manifold  $M$  with our "admissible balls" and for this we use a classical "Vitali type covering" with a uniformly finite overlap. We shall denote it

by  $\mathcal{C}$ .

When comparing non compact  $M$  to the compact case treated in [2], we have two important issues :

(i) the "admissible" radius may go to 0 at infinity, which is the case, for instance, if the canonical volume measure  $dv_g$  of  $(M, g)$  is finite and  $M$  is not compact ;

(ii) if  $dv_g$  is not finite, which is the case, for instance, if the "admissible" radius is bounded below, then  $p$  forms in  $L_p^t(M)$  are generally not in  $L_p^r(M)$  for  $r < t$ .

We address these problems by use of adapted weights on  $(M, g)$ . These weights are relative to the covering  $\mathcal{C}$  : they are positive functions which vary slowly on the balls of the covering  $\mathcal{C}$ .

To deal with the problem (i) we shall use a weight

$$w_0(x) = R(x)^{-2k} \quad (1.1)$$

for an adapted integer  $k$ , where  $R(x)$  is the admissible radius at the point  $x \in M$ .

To deal with the problem (ii) we shall use a weight  $\alpha(x)$  which is in  $L^\mu(M)$  with  $\mu := \frac{2t}{2-t}$ , for a  $t < 2$ , i.e.

$$\gamma(w, t) := \int_M w^{\frac{2t}{2-t}} dv_g < \infty. \quad (1.2)$$

This is done to get  $L_p^2(M) \subset L_p^t(M, \alpha)$ .

Our Hodge decompositions are *not the classical ones* because we do not use the laplacian adapted to those weights, but we always use the standard laplacian.

We define the Sobolev spaces  $W_p^{d,r}(M)$  of  $(M, g)$  following E. Hebey [12], and we set

**Definition 1.1.** We shall define the Sobolev exponents  $S_k(r)$  by  $\frac{1}{S_k(r)} := \frac{1}{r} - \frac{k}{n}$ .

Then our first result is a "twisted" Calderon Zygmund inequalities (CZI) with weight, different from results in [11] because we have weights and our forms are *not* asked to have compact support.

**Theorem 1.2.** Let  $(M, g)$  be a complete riemannian manifold. Let  $w$  be a weight relative to the  $\mathcal{C}_\epsilon$  associated covering  $\{B(x_j, 5r(x_j))\}_{j \in \mathbb{N}}$  and set  $w_0 := R(x)^{-2}$ . Let  $u \in L_p^r(M, ww_0^r)$  such that  $\Delta u \in L_p^r(M, w)$  ; then there are constants  $C_1, C_2$  depending only on  $n = \dim_{\mathbb{R}} M$ ,  $r$  and  $\epsilon$  such that:

$$\|u\|_{W^{2,r}(M,w)} \leq C_1 \|u\|_{L^r(M, ww_0^r)} + C_2 \|\Delta u\|_{L^r(M,w)}.$$

Moreover we have for  $t = S_2(r)$  that  $u \in L_p^t(M, w^t)$  with  $\|u\|_{L^t(M, w^t)} \leq c \|u\|_{W^{2,r}(M, w^r w_0^t)}$ .

We set, for a weight  $\alpha$ ,  $\mathcal{H}_p^r(M, \alpha) := L_p^r(M, \alpha) \cap \ker \Delta_p$ , the space of harmonic  $p$  forms in  $L^r(M, \alpha)$ .

This is our main hypothesis :

(HL2,p)  $\Delta = \Delta_p$  has a spectral gap in  $L_p^2(M)$ , i.e. there is no spectrum of  $\Delta_p$  in an open interval  $(0, \eta)$  with  $\eta > 0$ .

This assumption allows us to use  $L_p^2(M)$  as a threshold for the Raising Steps Method.

The (HL2,p) assumption is known to be true in the case of the hyperbolic manifold  $\mathbb{H}^{2n}$  of dimension  $2n$  for any value of  $p \in \{0, 2n\}$ . For  $p \neq n$  the space  $\mathcal{H}_p^2$  is reduced to 0. For  $\mathbb{H}^{2n+1}$  the

(HL2,p) is valid for  $p \neq n$  and  $p \neq n + 1$  and, out of these two cases, the space  $\mathcal{H}_p^2$  is reduced to 0 as was proved by Donnelly [6].

When  $\text{Ric}(M) \geq -c^2$  and  $M$  is open at infinity then  $0 \notin \text{Sp}\Delta_0$  by a result of Buser, see Lott [20], proposition 6, p. 353, hence (HL2,0) is true. If  $M$  is a normal covering of a compact manifold  $X$  with covering group  $\Gamma$ , then  $0 \notin \text{Sp}\Delta_0$  iff  $\Gamma$  is not amenable by a result of Brooks, see Lott [20], corollary 3, p. 354, for precise references. Hence (HL2,0) is true if  $\Gamma$  is not amenable.

For  $r = 2$ , there is the orthogonal projection  $H$  from  $L_p^2(M)$  on  $\mathcal{H}_p^2(M)$ ; we shall prove that this projection extends to  $L^r(M, w_0^r)$ , with  $w_0 := R(x)^{-2k}$  and  $R(x)$  the admissible radius at  $x \in M$ , as in (1.1), i.e.

$$\forall r \leq 2, H : L^r(M, w_0^r) \rightarrow \mathcal{H}_p^2(M) \quad (1.3)$$

boundedly and we get the following results on solutions of the Poisson equation.

**Theorem 1.3.** *Suppose that  $(M, g)$  is a complete riemannian manifold; let  $r < 2$  and choose a weight  $\alpha \in L^\infty(M)$  verifying  $\gamma(\alpha, r) < \infty$ . Set  $t := \min(2, S_2(r))$ . If  $t < 2$ , take the weight  $\alpha \in L^\infty(M)$  verifying also  $\gamma(\alpha, t) < \infty$ . Suppose we have conditions (HL2,p).*

*Take  $k$  big enough so that the threshold  $S_k(r) \geq 2$ , and set  $w_0(x) := R(x)^{-2k}$ , then for any  $\omega \in L_p^r(M, w_0^r)$  verifying  $H\omega = 0$ , for the orthogonal projection  $H$  defined in corollary 6.8, there is a  $u \in W_p^{2,r}(M, \alpha) \cap L_p^t(M, \alpha)$ , such that  $\Delta u = \omega$ .*

*Moreover the solution  $u$  is given linearly with respect to  $\omega$ .*

Here  $k$  was chosen such that  $S_k(r) \geq 2$  in order to use  $L_p^2(M)$  as a threshold for the Raising Steps Method.

Setting  $r'$  for the conjugate exponent for  $r$ ,  $\frac{1}{r'} + \frac{1}{r} = 1$ , by duality from theorem 1.3, we get

**Theorem 1.4.** *Suppose that  $(M, g)$  is a complete riemannian manifold; suppose we have  $r < 2$  and (HL2,p), then with  $k :: S_k(r) \geq 2$ , and  $w_0(x) := R(x)^{-k}$ , for any  $\varphi \in L_p^2(M) \cap L_p^{r'}(M)$ ,  $H\varphi = 0$ , there is a  $u \in L^{r'}(M, w_0^r)$  such that  $\Delta u = \varphi$ . This solution is linear with respect to  $\varphi$ .*

*If we add the hypothesis that the  $\epsilon_0$  admissible radius is bounded below, we get*

$$u := (T - C)^*\varphi, \quad u \in W_p^{2,r'}(M) \text{ and } u \text{ verifies } \Delta u = \varphi.$$

By theorem 1.3 in Hebey [12], we have that the harmonic radius  $r_H(1 + \epsilon, 2, 0)$  is bounded below if the Ricci curvature  $Rc$  verifies  $\|\nabla Rc\|_\infty < \infty$  and the injectivity radius is bounded below. This implies that the  $\epsilon$  admissible radius is also bounded below.

## 1.2 Hodge decomposition in $L^r$ spaces. Known results.

In 1949, Kodaira [14] proved that the  $L^2$ -space of  $p$ -forms on  $(M, g)$  has the orthogonal decomposition :

$$L_p^2(M) = \mathcal{H}_p^2 \oplus \overline{d\mathcal{D}_{p-1}(M)} \oplus \overline{d^*\mathcal{D}_{p+1}(M)},$$

and in 1991 Gromov [10] proved a strong  $L^2$  Hodge decomposition, under the hypothesis (HL2,p) :

$$L_p^2(M) = \mathcal{H}_p^2 \oplus dW_{p-1}^{1,2}(M) \oplus d^*W_{p+1}^{1,2}(M).$$

In 1995 Scott [22] proved a strong  $L^r$  Hodge decomposition but on *compact* riemannian manifold

$$\forall r > 1, L_p^r(M) = \mathcal{H}_p^r \oplus dW_{p-1}^{1,r}(M) \oplus d^*W_{p+1}^{1,r}(M).$$

Let  $d_\varphi^*$  be the formal adjoint of  $d$  relatively to the measure  $d\mu(x) = e^{-\varphi(x)} dv_g(x)$ , where  $\varphi \in \mathcal{C}^2(M)$ , and let  $\Delta_{\varphi,p} := dd_\varphi^* + d_\varphi^*d$  acting on  $p$  forms. Setting  $\Delta = \text{Tr}\nabla^2$  the covariant Laplace Beltrami operator acting on  $p$  forms and  $L = \Delta - \nabla\varphi \cdot \nabla$ , then, in 2009 X-D. Li [15] proved, among other nice results, a strong  $L^r$  Hodge decomposition on complete non compact riemannian manifold :

**Theorem 1.5.** (*X-D. Li*) Let  $r > 1$ ,  $r' = \frac{r}{r-1}$ . Let  $(M, g)$  be a complete riemannian manifold,  $\varphi \in \mathcal{C}^2(M)$ , and  $d\mu(x) = e^{-\varphi(x)} dv_g(x)$ . Suppose that the Riesz transforms  $d\Delta_{\varphi,p}^{-1/2}$  and  $d^*\Delta_{\varphi,p}^{-1/2}$  are bounded in  $L^r$  and  $L^{r'}$ , and the Riesz potential is bounded in  $L^r$ . Suppose also that  $(M, g)$  is  $L$  stochastically complete, then the strong  $L^r$  Hodge direct sum decomposition holds on  $p$  forms :

$$L_p^r(M, \mu) = \mathcal{H}_p^r(M, \mu) \oplus dW_{p-1}^{1,r}(M, \mu) \oplus d_\varphi^*W_{p+1}^{1,r}(M, \mu).$$

These results are valid for the family of weights  $\varphi \in \mathcal{C}^2(M)$  and for the Hodge laplacian associated to them, in the Witten sense [25]. Nevertheless it is worthwhile to notice that, even in the classical case  $\varphi \equiv 0$ , this result was new at the time it was proved, 2007, by X-D. Li.

### 1.3 Non classical Hodge decomposition in $L^r$ spaces. Main results.

The results of X-D. Li are based on the boundedness of the Riesz transforms in  $L^r$  and  $L^{r'}$  and the results we get use mainly the spectral gap hypothesis (HL2,p). X-D. Li was already concerned by the fact that the bottom of the spectrum of  $\Delta$  should be strictly positive ; the difference here is that we allow an eigenvalue 0 but a gap without spectrum after it, which gives the possible existence of non trivial harmonic functions in  $L^2$ . This is the meaning of (HL2,p).

In this way our results may appear to be the natural generalisation of Gromov results from  $L^2$  to  $L^r$ . On the other hand our results are proved only in the case  $\varphi = 0$ .

Our decompositions are *non classical* because we use weights to get estimates, but we use the usual laplacian, *not the Witten laplacian adapted to these weights*.

We shall need the following definition.

**Definition 1.6.** Let  $\alpha$  be a weight on  $M$ , we define the space  $\tilde{W}_p^{2,r}(M, \alpha)$  to be

$$\tilde{W}_p^{2,r}(M, \alpha) := \{u \in L_p^r(M, \alpha) :: \Delta u \in L_p^r(M, \alpha)\}$$

with the norm

$$\|u\|_{\tilde{W}_p^{2,r}(M, \alpha)} := \|u\|_{L_p^r(M, \alpha)} + \|\Delta u\|_{L_p^r(M, \alpha)}.$$

To get these decomposition theorems we shall apply our results on solutions of the Poisson equation.

**Theorem 1.7.** Let  $(M, g)$  be a complete riemannian manifold. Let  $r < 2$  and take a weight  $\alpha \in L^\infty(M)$  be such that  $\gamma(\alpha, r) < \infty$  ; with  $k :: S_k(r) \geq 2$ , set  $w_0 = R(x)^{-2k}$ , and suppose we have hypothesis (HL2,p). We have the direct decomposition given by linear operators :

$$L_p^r(M, w_0^r) = \mathcal{H}_p^2 \oplus \Delta(W_p^{2,r}(M, \alpha)).$$

With  $r' > 2$ , the conjugate exponent to  $r$ , we have the weaker decomposition, still given by linear operators :

$$L_p^{r'}(M) \cap L_p^2(M) = \mathcal{H}_p^2 \cap \mathcal{H}_p^{r'} + \Delta(\tilde{W}_p^{2,r'}(M)).$$

Because  $H : L^r(M, w_0^r) \rightarrow \mathcal{H}_p^2(M)$  boundedly by (1.3), where  $H$  is the orthogonal projection from  $L_p^2(M)$  on  $\mathcal{H}_p^2(M)$ , this explain the appearance of  $L_p^2(M)$  and  $\mathcal{H}_p^2(M)$  in the second part of the previous theorem.

To replace  $\tilde{W}_p^{2,r'}(M)$  by  $W_p^{2,r'}(M, \alpha)$  the price is the hypothesis that the  $\epsilon_0$  admissible radius is bounded below. So we get

**Corollary 1.8.** *Suppose the admissible radius is bounded below and suppose also hypothesis (HL2,p). Take  $r' > 2$ , then we have the direct decomposition given by linear operators*

$$L_p^{r'}(M) \cap L_p^2(M) = \mathcal{H}_p^2 \cap \mathcal{H}_p^{r'} \oplus \Delta(W_p^{2,r'}(M)).$$

As a corollary we get

**Corollary 1.9.** *Let  $r < 2$  and choose a weight  $\alpha \in L^\infty(M)$  such that  $\gamma(\alpha, r) < \infty$ ; with  $k :: S_k(r) \geq 2$ , set  $w_0 = R(x)^{-2k}$ , and suppose we have hypothesis (HL2,p). We have the direct decompositions given by linear operators*

$$L_p^r(M, w_0) = \mathcal{H}_p^2 \oplus d(W_p^{1,r}(M, \alpha)) \oplus d^*(W_p^{1,r}(M, \alpha)).$$

*With  $r' > 2$  the conjugate exponent of  $r$ , and adding the hypothesis that the  $\epsilon_0$  admissible radius is bounded below, we get*

$$L_p^{r'}(M) \cap L_p^2(M) = \mathcal{H}_p^2 \cap \mathcal{H}_p^{r'} \oplus d(W_p^{1,r'}(M)) \oplus d^*(W_p^{1,r'}(M)).$$

We also have weak  $L^r$  Hodge decompositions, where  $d^*$  is the adjoint of  $d$  with respect to the usual volume measure, not the weighted one, despite the weight appearing here.

We shall need another hypothesis :

(HWr) if the space  $\mathcal{D}_p(M)$  is dense in  $W_p^{2,r}(M)$ .

We already know that (HWr) is true if :

- either : the injectivity radius is strictly positive and the Ricci curvature is bounded [[12] theorem 2.8, p. 12].
- or :  $M$  is geodesically complete with a bounded curvature tensor [[11] theorem 1.1 p.3].

**Theorem 1.10.** *Suppose that  $(M, g)$  is a complete riemannian manifold, fix  $r < 2$  and choose a bounded weight  $\alpha$  with  $\gamma(\alpha, r) < \infty$ .*

*Take  $k$  with  $S_k(r) \geq 2$ , and set the weight  $w_0 := R(x)^{-2k}$ . Suppose we have (HL2,p) and (HW2). Then  $L_p^r(M, \alpha) = \mathcal{H}_p^r(M, \alpha) \oplus \overline{\Delta(\mathcal{D}_p(M))}$ , the closure being taken in  $L^r(M, \alpha)$ .*

We also have a weak  $L^r$  Hodge decomposition without hypothesis (HWr):

**Theorem 1.11.** *Suppose that  $(M, g)$  is a complete riemannian manifold and suppose we have (HL2,p). Fix  $r < 2$  and take a weight  $\alpha$  verifying  $\gamma(\alpha, r) < \infty$ . Then we have:*

$$L_p^r(M, \alpha) = \mathcal{H}_p^r(M, \alpha) \oplus \overline{d(\mathcal{D}_{p-1}(M))} \oplus \overline{d^*(\mathcal{D}_{p+1}(M))},$$

*the closures being taken in  $L^r(M, \alpha)$ .*

For the case  $r > 2$  we need a stronger hypothesis, namely that the  $\epsilon_0$  admissible radius is bounded below. Then we get a classical weak Hodge decompositions.

**Theorem 1.12.** *Suppose that  $(M, g)$  is a complete riemannian manifold and suppose the  $\epsilon_0$  admissible radius is bounded below and (HWr) and suppose also hypothesis (HL2,p). Fix  $r > 2$ , then we have*

$$L_p^r(M) = \mathcal{H}_p^r(M) \oplus \overline{\Delta(\mathcal{D}_p(M))}.$$

*Without (HWr) we still get*

$$L_p^r(M) = \mathcal{H}_p^r(M) \oplus \overline{d(\mathcal{D}_{p-1}(M))} \oplus \overline{d^*(\mathcal{D}_{p+1}(M))}.$$

*All the closures being taken in  $L^r(M)$ .*

**Remark 1.13.** *By theorem 1.3 in Hebey [12], we have that the harmonic radius  $r_H(1 + \epsilon, 2, 0)$  is bounded below if the Ricci curvature  $Rc$  verifies  $\|\nabla Rc\|_\infty < \infty$  and the injectivity radius is bounded below. This implies that the  $\epsilon$  admissible radius is also bounded below.*

*Moreover if we add the hypothesis that the Ricci curvature  $Rc$  is bounded below then by Proposition 2.10 in Hebey [12], we have hypothesis (HWr).*

These results are based on the raising steps method :

**Theorem 1.14.** *(Raising Steps Method) Let  $(M, g)$  be a riemannian manifold and take  $w$  a weight relative to the Vitali covering  $\{B(x_j, 5r(x_j))\}_{j \in \mathbb{N}}$ .*

*For any  $r \leq 2$ , any threshold  $s \geq r$ , take  $k \in \mathbb{N}$  such that  $t_k := S_k(r) \geq s$  then, with  $w_0(x) := w(x)R(x)^{-2k}$ ,*

*$\forall \omega \in L_p^r(M, w_0^r), \exists v \in L_p^r(M, w^r) \cap L_p^{s_1}(M, w^{s_1}) \cap W^{2,r}(M, w^r), \exists \tilde{\omega} \in L_p^s(M, w^s) :: \Delta v = \omega + \tilde{\omega}$*   
*with  $s_1 = S_2(r)$  and we have the control of the norms :*

$$\forall q \in [r, s_1], \|v\|_{L_p^q(M, w^q)} \leq C_q \|\omega\|_{L_p^r(M, w_0^r)} ; \|v\|_{W_p^{2,r}(M, w^r)} \leq C_r \|\omega\|_{L_p^r(M, w_0^r)} ;$$

$$\|\tilde{\omega}\|_{L_p^s(M, w^s)} \leq C_s \|\omega\|_{L_p^r(M, w_0^r)}.$$

*Moreover  $v$  and  $\tilde{\omega}$  are linear in  $\omega$ .*

*If  $M$  is complete and  $\omega$  is of compact support, so are  $v$  and  $\tilde{\omega}$ .*

I thank the referee for his pertinent questions and remarks making precise the meaning of these non classical Hodge decompositions.

This work will be presented in the following way.

In section 2 we define the admissible balls, the admissible radius and the basic facts relative to them.

In section 3 we use a Vitali type covering lemma with our admissible balls and we prove that its overlap is finite.

In section 4 we define the Sobolev spaces, following E. Hebey [12].

In section 5 we prove the local estimates for the Hodge Laplacian. This is essentially standard by use of classical results from Gilbarg and Trudinger [9].

In section 6 we develop the Raising Steps Method in the non compact case. The useful weights are defined here.

This is the basis of our results.

In section 7 we prove Calderon Zygmund inequalities with weights.

In section 8 we deduce the applications to the Poisson equation associated to the Hodge Laplacian.

In section 9 we use these solutions to get non classical strong  $L^r$  Hodge decomposition theorems. We also get non classical weak  $L^r$  Hodge decomposition theorems.

## 2 Basic facts.

**Definition 2.1.** *Let  $(M, g)$  be a riemannian manifold and  $x \in M$ . We shall say that the geodesic ball  $B(x, R)$  is  $\epsilon$  **admissible** if there is a chart  $\varphi : (x_1, \dots, x_n)$  defined on it with*

1)  $(1 - \epsilon)\delta_{ij} \leq g_{ij} \leq (1 + \epsilon)\delta_{ij}$  in  $B(x, R)$  as bilinear forms,

2)  $\sum_{|\beta|=1} \sup_{i,j=1,\dots,n, y \in B_x(R)} |\partial^\beta g_{ij}(y)| \leq \epsilon.$



**Definition 2.2.** Let  $x \in M$ , we set  $R'(x) = \sup \{R > 0 :: B(x, R) \text{ is } \epsilon \text{ admissible}\}$ . We shall say that  $R_\epsilon(x) := \min(1, R'(x))$  is the  $\epsilon$  **admissible radius** at  $x$ .

Our admissible radius is smaller than the harmonic radius  $r_H(1 + \epsilon, 1, 0)$  defined in Hebey [[12], p. 4].

By theorem 1.3 in Hebey [12], we have that the harmonic radius  $r_H(1 + \epsilon, 2, 0)$  is bounded below if the Ricci curvature  $Rc$  verifies  $\|\nabla Rc\|_\infty < \infty$  and the injectivity radius is bounded below. This implies easily that the  $\epsilon$  admissible radius is also bounded below.

**Remark 2.3.** By its very definition, we always have  $R_\epsilon(x) \leq 1$ .

Of course, without any extra hypotheses on the riemannian manifold  $M$ , we have  $\forall \epsilon > 0, \forall x \in M$ , taking  $g_{ij}(x) = \delta_{ij}$  in a chart on  $B(x, R)$  and the radius  $R$  small enough, the ball  $B(x, R)$  is  $\epsilon$  admissible.

We shall use the following lemma.

**Lemma 2.4.** Let  $(M, g)$  be a riemannian manifold then with  $R(x) = R_\epsilon(x) =$  the  $\epsilon$  admissible radius at  $x \in M$  and  $d(x, y)$  the riemannian distance on  $(M, g)$  we get :

$$d(x, y) \leq \frac{1}{4}(R(x) + R(y)) \Rightarrow R(x) \leq 4R(y).$$

Proof.

Let  $x, y \in M :: d(x, y) \leq \frac{1}{4}(R(x) + R(y))$  and suppose for instance that  $R(x) \geq R(y)$ . Then  $y \in B(x, R(x)/2)$  hence we have  $B(y, R(x)/4) \subset B(x, \frac{3}{4}R(x))$ . But by the definition of  $R(x)$ , the ball  $B(x, \frac{3}{4}R(x))$  is admissible and this implies that the ball  $B(y, R(x)/4)$  is also admissible for exactly the same constants and the same chart ; this implies that  $R(y) \geq R(x)/4$ . ■

### 3 Vitali covering.

**Lemma 3.1.** Let  $\mathcal{F}$  be a collection of balls  $\{B(x, r(x))\}$  in a metric space, with  $\forall B(x, r(x)) \in \mathcal{F}, 0 < r(x) \leq R$ . There exists a disjoint subcollection  $\mathcal{G}$  of  $\mathcal{F}$  with the following property :  
every ball  $B$  in  $\mathcal{F}$  intersects a ball  $C$  in  $\mathcal{G}$  and  $B \subset 5C$ .

This is a well known lemma, see for instance [7], section 1.5.1].

So fix  $\epsilon > 0$  and let  $\forall x \in M, r(x) := R_\epsilon(x)/120$ , where  $R_\epsilon(x)$  is the admissible radius at  $x$ , we built a Vitali covering with the collection  $\mathcal{F} := \{B(x, r(x))\}_{x \in M}$ . So lemma 3.1 gives a disjoint subcollection  $\mathcal{G}$  such that every ball  $B$  in  $\mathcal{F}$  intersects a ball  $C$  in  $\mathcal{G}$  and we have  $B \subset 5C$ . We set  $\mathcal{G}' := \{x_j \in M :: B(x_j, r(x_j)) \in \mathcal{G}\}$  and  $\mathcal{C}_\epsilon := \{B(x, 5r(x)), x \in \mathcal{G}'\}$  : we shall call  $\mathcal{C}_\epsilon$  the  $\epsilon$  **admissible covering** of  $(M, g)$ .

Then we have :

**Proposition 3.2.** *Let  $(M, g)$  be a riemannian manifold, then the overlap of the  $\epsilon$  admissible covering  $\mathcal{C}_\epsilon$  is less than  $T = \frac{(1 + \epsilon)^{n/2}}{(1 - \epsilon)^{n/2}}(120)^n$ , i.e.*

$$\forall x \in M, x \in B(y, 5r(y)) \text{ where } B(y, r(y)) \in \mathcal{G} \text{ for at most } T \text{ such balls.}$$

So we have

$$\forall f \in L^1(M), \sum_{j \in \mathbb{N}} \int_{B_j} |f(x)| dv_g(x) \leq T \|f\|_{L^1(M)}.$$

Proof.

Let  $B_j := B(x_j, r(x_j)) \in \mathcal{G}$  and suppose that  $x \in \bigcap_{j=1}^k B(x_j, 5r(x_j))$ . Then we have

$$\forall j = 1, \dots, k, d(x, x_j) \leq 5r(x_j)$$

hence

$$d(x_j, x_l) \leq d(x_j, x) + d(x, x_l) \leq 5(r(x_j) + r(x_l)) \leq \frac{1}{4}(R(x_j) + R(x_l)) \Rightarrow R(x_j) \leq 4R(x_l)$$

and by exchanging  $x_j$  and  $x_l$ ,  $R(x_l) \leq 4R(x_j)$ .

So we get

$$\forall j, l = 1, \dots, k, r(x_j) \leq 4r(x_l), r(x_l) \leq 4r(x_j).$$

Now the ball  $B(x_j, 5r(x_j) + 5r(x_l))$  contains  $x_l$  hence the ball  $B(x_j, 5r(x_j) + 6r(x_l))$  contains the ball  $B(x_l, r(x_l))$ . But, because  $r(x_l) \leq 4r(x_j)$ , we get

$$B(x_j, 5r(x_j) + 6 \times 4r(x_j)) = B(x_j, r(x_j)(5 + 24)) \supset B(x_l, r(x_l)).$$

The balls in  $\mathcal{G}$  being disjoint, we get, setting  $B_l := B(x_l, r(x_l))$ ,

$$\sum_{j=1}^k \text{Vol}(B_l) \leq \text{Vol}(B(x_j, 29r(x_j))).$$

The Lebesgue measure read in the chart  $\varphi$  and the canonical measure  $dv_g$  on  $B(x, R_\epsilon(x))$  are equivalent ; precisely because of condition 1) in the admissible ball definition, we get that :

$$(1 - \epsilon)^n \leq |\det g| \leq (1 + \epsilon)^n,$$

and the measure  $dv_g$  read in the chart  $\varphi$  is  $dv_g = \sqrt{|\det g_{ij}|} d\xi$ , where  $d\xi$  is the Lebesgue measure in  $\mathbb{R}^n$ . In particular :

$$\forall x \in M, \text{Vol}(B(x, R_\epsilon(x))) \leq (1 + \epsilon)^{n/2} \nu_n R^n,$$

where  $\nu_n$  is the euclidean volume of the unit ball in  $\mathbb{R}^n$ .

Now because  $R(x_j)$  is the admissible radius and  $4 \times 29r(x_j) < R(x_j)$ , we have

$$\text{Vol}(B(x_j, 29r(x_j))) \leq 29^n (1 + \epsilon)^{n/2} \nu_n r(x_j)^n.$$

On the other hand we have also

$$\text{Vol}(B_l) \geq \nu_n (1 - \epsilon)^{n/2} r(x_l)^n \geq \nu_n (1 - \epsilon)^{n/2} 4^{-n} r(x_j)^n,$$

hence

$$\sum_{j=1}^k (1 - \epsilon)^{n/2} 4^{-n} r(x_j)^n \leq 29^n (1 + \epsilon)^{n/2} r(x_j)^n,$$

so finally

$$k \leq (29 \times 4)^n \frac{(1 + \epsilon)^{n/2}}{(1 - \epsilon)^{n/2}},$$

which means that  $T \leq \frac{(1 + \epsilon)^{n/2}}{(1 - \epsilon)^{n/2}} (120)^n$ .

Saying that any  $x \in M$  belongs to at most  $T$  balls of the covering  $\{B_j\}$  means that  $\sum_{j \in \mathbb{N}} \mathbb{1}_{B_j}(x) \leq T$ , and this implies easily that :

$$\forall f \in L^1(M), \sum_{j \in \mathbb{N}} \int_{B_j} |f(x)| dv_g(x) \leq T \|f\|_{L^1(M)}. \blacksquare$$

**Lemma 3.3.** *Let  $(M, g)$  be a non compact connected complete riemannian manifold and  $\mathcal{C} := \{B_j\}_{j \in \mathbb{N}}$  a Vitali covering of  $M$  with balls of radius less than  $\delta > 0$ . For any compact set  $K$  in  $M$  covered by  $\mathcal{O} := \bigcup_{k \in F_K} B_k$ , with  $F_K$  finite, we can find a compact set  $K' \supset K$  such that  $\partial K'$  can be covered by elements of  $\mathcal{C}$  not intersecting  $\bar{\mathcal{O}}$ .*

Proof.

If this was not the case then there is a compact  $K$  covered by  $\mathcal{O} := \bigcup_{k \in F_K} B_k$  and such that for any compact  $K' \supset K$  and any covering of  $\partial K'$  by elements  $B_k$  of  $\mathcal{C}$ , then  $B_k \cap \bar{\mathcal{O}} \neq \emptyset$ . Because the balls have radius less than  $\delta$ , this means that  $\partial K'$  is at most at a distance  $2\delta$  of  $\mathcal{O}$  hence  $M$  is bounded, hence the completeness of  $M$  implies that  $M$  is compact.  $\blacksquare$

Clearly the assumption that the radii are uniformly bounded is necessary as the example of  $\mathbb{R}^n$  shows.

## 4 Sobolev spaces.

We have to define the Sobolev spaces in our setting, following E. Hebey [[12], p. 10].

First define the covariant derivatives by  $(\nabla u)_j := \partial_j u$  in local coordinates, while the components of  $\nabla^2 u$  are given by

$$(\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma_{ij}^k \partial_k u, \quad (4.4)$$

with the convention that we sum over repeated index. The Christoffel  $\Gamma_{ij}^k$  verify [3] :

$$\Gamma_{ij}^k = \frac{1}{2} g^{il} \left( \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right). \quad (4.5)$$

If  $k \in \mathbb{N}$  and  $r \geq 1$  are given, we denote by  $\mathcal{C}_k^r(M)$  the space of smooth functions  $u \in \mathcal{C}^\infty(M)$  such that  $|\nabla^j u| \in L^r(M)$  for  $j = 0, \dots, k$ . Hence

$$\mathcal{C}_k^r(M) := \{u \in \mathcal{C}^\infty(M), \forall j = 0, \dots, k, \int_M |\nabla^j u|^r dv_g < \infty\}$$

Now we have [12]

**Definition 4.1.** *The Sobolev space  $W^{k,r}(M)$  is the completion of  $\mathcal{C}_k^r(M)$  with respect to the norm :*

$$\|u\|_{W^{k,r}(M)} = \sum_{j=0}^k \left( \int_M |\nabla^j u|^r dv_g \right)^{1/r}.$$

We shall be interested only by  $k \leq 2$  and we extend in a natural way this definition to the case of  $p$  forms.

Let the Sobolev exponents  $S_k(r)$  as in the definition 1.1, then the  $k$  th Sobolev embedding is true if we have

$$\forall u \in W^{k,r}(M), u \in L^{S_k(r)}(M).$$

This is the case in  $\mathbb{R}^n$ , or if  $M$  is compact, or if  $M$  has a Ricci curvature bounded from below and  $\inf_{x \in M} v_g(B_x(1)) \geq \delta > 0$ , due to Varopoulos [24], see [12] theorem 3.14, p. 31].

**Lemma 4.2.** *We have the Sobolev comparison estimates where  $B(x, R)$  is a  $\epsilon$  admissible ball in  $M$  and  $\varphi : B(x, R) \rightarrow \mathbb{R}^n$  is the admissible chart relative to  $B(x, R)$ ,*

$$\forall u \in W^{2,r}(B(x, R)), \quad \|u\|_{W^{2,r}(B(x, R))} \leq (1 + \epsilon C) \|u \circ \varphi^{-1}\|_{W^{2,r}(\varphi(B(x, R)))},$$

and, with  $B_e(0, t)$  the euclidean ball in  $\mathbb{R}^n$  centered at 0 and of radius  $t$ ,

$$\|v\|_{W^{2,r}(B_e(0, (1-\epsilon)R))} \leq (1 + 2C\epsilon) \|u\|_{W^{2,r}(B(x, R))}.$$

Proof.

We have to compare the norms of  $u$ ,  $\nabla u$ ,  $\nabla^2 u$  with the corresponding ones for  $v := u \circ \varphi^{-1}$  in  $\mathbb{R}^n$ . First we have because  $(1 - \epsilon)\delta_{ij} \leq g_{ij} \leq (1 + \epsilon)\delta_{ij}$  in  $B(x, R)$  :

$$B_e(0, (1 - \epsilon)R) \subset \varphi(B(x, R)) \subset B_e(0, (1 + \epsilon)R).$$

Because  $\sum_{|\beta|=1} \sup_{i,j=1,\dots,n, y \in B(x, R)} |\partial^\beta g_{ij}(y)| \leq \epsilon$  in  $B(x, R)$ , we have the estimates, with  $\forall y \in B(x, R)$ ,  $z := \varphi(y)$ ,

$$\forall y \in B(x, R), \quad |u(y)| = |v(z)|, \quad |\nabla u(y)| \leq (1 + C\epsilon) |\partial v(z)|.$$

Because of (4.5) and (4.4) we get

$$\forall y \in B(x, R), \quad |\nabla^2 u(y)| \leq |\partial^2 v(z)| + \epsilon C |\partial v(z)|.$$

Integrating this we get

$$\|\nabla^2 u\|_{L^r(B(x, R))} \leq \| |\partial^2 v| + \epsilon C |\partial v| \|_{L^r(B_e(0, (1+\epsilon)R))} \leq \|\partial^2 v\|_{L^r(B_e(0, (1+\epsilon)R))} + C\epsilon \|\partial v\|_{L^r(B_e(0, (1+\epsilon)R))},$$

and

$$\|\nabla u\|_{L^r(B(x, R))} \leq (1 + C\epsilon) \|\partial v\|_{L^r(B_e(0, (1+\epsilon)R))}.$$

We also have the reverse estimates

$$\|\partial^2 v\|_{L^r(B_e(0, (1-\epsilon)R))} \leq \|\nabla^2 u\|_{L^r(B(x, R))} + C\epsilon \|\nabla u\|_{L^r(B(x, R))},$$

and

$$\|\partial v\|_{L^r(B_e(0, (1-\epsilon)R))} \leq (1 + C\epsilon) \|\nabla u\|_{L^r(B(x, R))}.$$

So, using that

$$\|u\|_{W^{2,r}(B(x, R))} = \|\nabla^2 u\|_{L^r(B(x, R))} + \|\nabla u\|_{L^r(B(x, R))} + \|u\|_{L^r(B(x, R))},$$

we get

$$\begin{aligned} \|u\|_{W^{2,r}(B(x, R))}^r &\leq \|\partial^2 v\|_{L^r(B_e(0, (1+\epsilon)R))}^r + C\epsilon \|\partial v\|_{L^r(B_e(0, (1+\epsilon)R))}^r + (1 + C\epsilon) \|\partial v\|_{L^r(B_e(0, (1+\epsilon)R))}^r + \\ &\quad + \|v\|_{L^r(B_e(0, (1+\epsilon)R))}^r \leq \\ &\leq (1 + 2\epsilon C) \|v\|_{W^{2,r}(B_e(0, (1+\epsilon)R))}^r. \end{aligned}$$

Of course all these estimates can be reversed so we also have

$$\|v\|_{W^{2,r}(B_e(0, (1-\epsilon)R))} \leq (1 + 2C\epsilon) \|u\|_{W^{2,r}(B(x, R))}.$$

This ends the proof of the lemma. ■

**Lemma 4.3.** *Let  $B := B(x, R)$  be a  $\epsilon$  admissible ball in  $M$ , we have the punctual estimates in  $B$*

(i)  $\exists C > 0$ ,  $\forall \chi \in \mathcal{C}^2(B)$ ,  $\forall u \in \mathcal{C}_p^2(B)$ ,  $|\nabla(\chi u)| \leq (1 + C\epsilon)(|\chi| |\nabla u| + |\nabla \chi| |u|)$ .

(ii)  $\exists C > 0$ ,  $\forall \chi \in \mathcal{C}^2(B)$ ,  $\forall u \in \mathcal{C}_p^2(B)$ ,  $|\nabla^2(\chi u)| \leq (1 + C\epsilon)(|\chi| |\nabla^2 u| + |\nabla^2 \chi| |u| + |\nabla \chi| |\nabla u|)$ .

Proof.

We have to compare the modulus of  $u$ ,  $\nabla u$ ,  $\nabla^2 u$  with the corresponding ones for  $v := u \circ \varphi^{-1}$  in  $\mathbb{R}^n$ .

First we have because  $(1 - \epsilon)\delta_{ij} \leq g_{ij} \leq (1 + \epsilon)\delta_{ij}$  in  $B(x, R)$  :

$$B_e(0, (1 - \epsilon)R) \subset \varphi(B(x, R)) \subset B_e(0, (1 + \epsilon)R).$$

Because  $\sum_{|\beta|=1} \sup_{i,j=1,\dots,n, y \in B_x(R)} |\partial^\beta g_{ij}(y)| \leq \epsilon$  in  $B(x, R)$ , we have the estimates, with  $\forall y \in$

$B(x, R)$ ,  $z := \varphi(y)$ ,

$$\forall y \in B(x, R), |u(y)| = |v(z)|, \quad |\nabla u(y)| \leq (1 + C\epsilon) |\partial v(z)|.$$

Now replacing  $u$  by  $\chi u$ , clearly we have for  $\chi \circ \varphi^{-1} v \circ \varphi^{-1}$  what we want, just using Leibnitz rule.

Then the computations above gives the existence of a new constant  $C$  such that

$$|\nabla(\chi u)| \leq (1 + C\epsilon)(|\chi| |\nabla u| + |\nabla \chi| |u|)$$

at all point of  $B$  which gives (i).

Because of (4.5) and (4.4) we get

$$\forall y \in B(x, R), |\nabla^2 u(y)| \leq |\partial^2 v(z)| + \epsilon C |\partial v(z)|.$$

Now replacing  $u$  by  $\chi u$ , clearly we have for  $\chi \circ \varphi^{-1} v \circ \varphi^{-1}$  what we want, again just using Leibnitz rule. Then the computations above gives the existence of a new constant  $C$  such that

$$|\nabla^2(\chi u)| \leq (1 + C\epsilon)(|\chi \nabla^2 u| + |\nabla^2 \chi u| + |\nabla \chi| |\nabla u|)$$

at all point of  $B$  which gives (ii) and ends the proof of this lemma. ■

We have to study the behavior of the Sobolev embeddings w.r.t. the radius. Set  $B_R := B_e(0, R)$ .

**Lemma 4.4.** *We have, with  $s = S_1(r)$ ,  $t = S_2(r)$ ,*

$$(i) \forall R, 0 < R \leq 1, \forall u \in W^{2,r}(B_R), \|u\|_{L^t(B_R)} \leq CR^{-2} \|u\|_{W^{2,r}(B_R)}$$

and

$$(ii) \forall R, 0 < R \leq 1, \forall u \in W^{2,r}(B_R), \|\partial u\|_{L^s(B_R)} \leq CR^{-1} \|u\|_{W^{2,r}(B_R)}$$

the constant  $C$  depending only on  $n, r$ .

Proof.

We start with  $R = 1$ , then we have by Sobolev embeddings with  $t = S_2(r)$ ,

$$\forall v \in W^{2,r}(B_1), \|v\|_{L^t(B_1)} \leq C \|v\|_{W^{2,r}(B_1)} \quad (4.6)$$

where  $C$  depends only on  $n$ . For  $u \in W^{2,r}(B_R)$  we set

$$\forall x \in B_1, y := Rx \in B_R, v(x) := u(y).$$

Then we have

$$\partial v(x) = \partial u(y) \times \frac{\partial y}{\partial x} = R \partial u(y); \quad \partial^2 v(x) = \partial^2 u(y) \times \left(\frac{\partial y}{\partial x}\right)^2 = R^2 \partial^2 u(y).$$

So we get, because the jacobian for this change of variables is  $R^{-n}$ ,

$$\|\partial v\|_{L^r(B_1)}^r = \int_{B_1} |\partial v(x)|^r dm(x) = \int_{B_R} |\partial u(y)|^r \frac{R^r}{R^n} dm(x) = R^{r-n} \|\partial u\|_{L^r(B_R)}^r.$$

So

$$\|\partial u\|_{L^r(B_R)} = R^{-1+n/r} \|\partial v\|_{L^r(B_1)}. \quad (4.7)$$

The same way we get

$$\|\partial^2 u\|_{L^r(B_R)} = R^{-2+n/r} \|\partial^2 v\|_{L^r(B_1)} \quad (4.8)$$

and of course

$$\|u\|_{L^r(B_R)} = R^{n/r} \|v\|_{L^r(B_1)}.$$

So with 4.6 we get

$$\|u\|_{L^t(B_R)} = R^{n/t} \|v\|_{L^t(B_1)} \leq CR^{n/t} \|v\|_{W^{k,r}(B_1)}. \quad (4.9)$$

But

$$\|u\|_{W^{2,r}(B_R)} := \|u\|_{L^r(B_R)} + \|\partial u\|_{L^r(B_R)} + \|\partial^2 u\|_{L^r(B_R)},$$

and

$$\|v\|_{W^{2,r}(B_1)} := \|v\|_{L^r(B_1)} + \|\partial v\|_{L^r(B_1)} + \|\partial^2 v\|_{L^r(B_1)},$$

so

$$\|v\|_{W^{2,r}(B_1)} := R^{-n/r} \|u\|_{L^r(B_R)} + R^{1-n/r} \|\partial u\|_{L^r(B_R)} + R^{2-n/r} \|\partial^2 u\|_{L^r(B_R)}.$$

Because we have  $R \leq 1$ , we get

$$\|v\|_{W^{2,r}(B_1)} \leq R^{-n/r} (\|u\|_{L^r(B_R)} + \|\partial u\|_{L^r(B_R)} + \|\partial^2 u\|_{L^r(B_R)}) = R^{-n/r} \|u\|_{W^{2,r}(B_R)}.$$

Putting it in (4.9) we get

$$\|u\|_{L^t(B_R)} \leq CR^{n/t} \|v\|_{W^{k,r}(B_1)} \leq CR^{-n(\frac{1}{r} - \frac{1}{t})} \|u\|_{W^{2,r}(B_R)}.$$

But, because  $t = S_2(r)$ , we get  $(\frac{1}{r} - \frac{1}{t}) = \frac{2}{n}$  and

$$\|u\|_{L^t(B_R)} \leq CR^{-2} \|u\|_{W^{2,r}(B_R)}.$$

To have the (ii) we proceed exactly the same way.

We start with  $R = 1$ , then we have by Sobolev embeddings with  $s = S_1(r)$ ,

$$\forall v \in W^{2,r}(B_1), \|\partial v\|_{L^s(B_1)} \leq C \|v\|_{W^{2,r}(B_1)}$$

and this leads as above to

$$\|\partial u\|_{L^t(B_R)} \leq CR^{-1} \|u\|_{W^{2,r}(B_R)}. \blacksquare$$

The constant  $C$  depends only on  $n, r$ .

**Lemma 4.5.** *Let  $x \in M$  and  $B(x, R)$  be a  $\epsilon$  admissible ball ; we have, with  $s = S_1(r)$ ,  $t = S_2(r)$ ,*

(i)  $\forall u \in W^{2,r}(B(x, R))$ ,  $\|u\|_{L^t(B(x, R))} \leq CR^{-2} \|u\|_{W^{2,r}(B(x, R))}$ ,

and

(ii)  $\forall u \in W^{2,r}(B(x, R))$ ,  $\|\nabla u\|_{L^s(B(x, R))} \leq CR^{-1} \|u\|_{W^{2,r}(B(x, R))}$ ,

the constant  $C$  depending only on  $n, r$  and  $\epsilon$ .

*Proof.*

This is true in  $\mathbb{R}^n$  by lemma 4.4 so we can apply the comparison lemma 4.2.  $\blacksquare$

**Lemma 4.6.** *Let  $B := B(0, R)$  be the ball in  $\mathbb{R}^n$  of center 0 and radius  $R \leq 1$  and  $B' = B(0, R/2)$ . Let  $u \in L^r(B)$  such that  $\Delta u \in L^r(B)$  then we have*

$$u \in W^{2,r}(B'), \quad \|u\|_{W^{2,r}(B')} \leq c_1 R^{-2} \|u\|_{L^r(B)} + c_2 \|\Delta u\|_{L^r(B)},$$

where the constants  $c_1, c_2$  depend only on  $n, r$ .

*Proof.*

We start with  $R = 1$ , then we have by the classical CZI for the usual laplacian  $\Delta_{\mathbb{R}}$  in  $\mathbb{R}^n$ , [9], Th. 9.11, p. 235]:

$$\forall v \in L^r(B), \Delta v \in L^r(B), \|v\|_{W^{2,r}(B')} \leq c_1 \|v\|_{L^r(B)} + c_2 \|\Delta v\|_{L^r(B)}, \quad (4.10)$$

the constants  $c_1, c_2$  depending only on  $n, r$ .

To go to any  $R$  we take  $u$  with the hypotheses of the lemma and we make the change of variables

$$y = Rx, \quad dm(y) = R^n dm(x), \quad v(x) := u(Rx).$$

We set  $B_R := B_e(0, R)$  then we get

$$\|v\|_{L^r(B_1)}^r := \int_{B_1} |v(x)|^r dm(x) = \int_{B_R} \left|v\left(\frac{y}{R}\right)\right|^r R^{-n} dm(y) = \int_{B_R} |u(y)|^r R^{-n} dm(y) = R^{-n} \|u\|_{L^r(B_R)}^r.$$

And

$$\partial_i v(x) = \partial_i u(Rx)R, \quad \partial_{ij} v(x) = \partial_{ij} u(Rx)R^2,$$

hence

$$\begin{aligned} \|\partial_i v\|_{L^r(B_1)}^r &:= \int_{B_1} |\partial_i v(x)|^r dm(x) = \\ &= \int_{B_R} |\partial_i v\left(\frac{y}{R}\right)|^r R^{-n} dm(y) = \int_{B_R} R^r |\partial_i u(y)|^r R^{-n} dm(y) = R^{r-n} \|\partial_i u\|_{L^r(B_R)}^r. \\ \|\partial_{ij} v\|_{L^r(B_1)}^r &:= \int_{B_1} |\partial_{ij} v(x)|^r dm(x) = \\ &= \int_{B_R} |\partial_{ij} v\left(\frac{y}{R}\right)|^r R^{-n} dm(y) = \int_{B_R} R^{2r} |\partial_{ij} u(y)|^r R^{-n} dm(y) = R^{2r-n} \|\partial_{ij} u\|_{L^r(B_R)}^r. \end{aligned}$$

So we get by (4.10)

$$(R/2)^{1-n/r} \|\partial_i u\|_{L^r(B_{R/2})} \leq c_1(r) R^{-n/r} \|u\|_{L^r(B_R)} + c_2(R) R^{2-n/r} \|\Delta u\|_{L^r(B_R)}$$

hence

$$\|\partial_i u\|_{L^r(B_{R/2})} \leq 2^{-1+n/r} (c_1(r) R^{-1} \|u\|_{L^r(B_R)} + c_2(r) R \|\Delta u\|_{L^r(B_R)}).$$

And the same way

$$\|\partial_{ij} u\|_{L^r(B_{R/2})} \leq 2^{-2+n/r} (c_1(r) R^{-2} \|u\|_{L^r(B_R)} + c_2(r) \|\Delta u\|_{L^r(B_R)}).$$

So we get finally

$$\|u\|_{W^{2,r}(B_{R/2})} \leq c_1(n, r) R^{-2} \|u\|_{L^r(B_R)} + c_2(n, r) \|\Delta u\|_{L^r(B_R)},$$

where the constants  $c_1, c_2$  depend only on  $n, r$ . ■

## 5 Local estimates for the laplacian.

All these local estimates are quite well known. I reprove them here to precise the notations and the dependences of the constants.

**Lemma 5.1.** *Let  $(M, g)$  be a riemannian manifold. For  $x \in M$ ,  $\epsilon > 0$ , we take a  $\epsilon$  admissible ball  $B_x(R)$ . Then there is a  $0 < \epsilon_0 \leq \epsilon$ , hence a  $R = R_{\epsilon_0}(x) > 0$ , and a constant  $C$  depending only on  $n = \dim_{\mathbb{R}} M$ ,  $r$  and  $\epsilon_0$  such that :*

$$\forall \omega \in L_p^r(B_x(R)), \exists u \in W_p^{2,r}(B_x(R)) :: \Delta u = \omega, \quad \|u\|_{W^{2,r}(B_x(R))} \leq C \|\omega\|_{L^r(B_x(R))}.$$

Moreover  $u$  is linear in  $\omega$ .

Proof.

For  $x \in M$  and  $\epsilon > 0$ , we take a  $\epsilon$  admissible ball  $B_x(R)$  and a chart  $\varphi : (y_1, \dots, y_n)$ , which means:

1)  $(1 - \epsilon)\delta_{ij} \leq g_{ij} \leq (1 + \epsilon)\delta_{ij}$  in  $B_x(R)$  as bilinear forms,

$$2) \sum_{|\beta|=1} \sup_{i,j=1,\dots,n, y \in B_x(R)} |\partial^\beta g_{ij}(y)| \leq \epsilon.$$

Of course the operator  $d$  on  $p$  forms is local and so is  $d^*$  as a first order differential operator.

So the Hodge laplacian  $\Delta_\varphi$  read by  $\varphi$  in  $U := \varphi(B_x(R))$  is still a second order partial differential system of operators and with  $\Delta_{\mathbb{R}}$  the usual laplacian in  $\mathbb{R}^n$  acting on forms in  $U$ , we set:  $A\omega_\varphi := \Delta_\varphi\omega_\varphi - \Delta_{\mathbb{R}}\omega_\varphi$ , where  $\omega_\varphi$  is the  $p$  form  $\omega$  read in the chart  $(B_x(R), \varphi)$  and  $A$  is a matrix valued second order operator with  $C^\infty$  smooth coefficients such that  $A := \Delta_\varphi - \Delta_{\mathbb{R}} : W^{2,r}(U) \rightarrow L^r(U)$ .

This difference  $A$  is controlled by the derivatives of the metric tensor up to order 1; for instance for a function  $f$  we have in the chart  $\varphi$

$$\Delta_\varphi f = \frac{1}{\sqrt{\det(g_{ij})}} \partial_i (g^{ij} \sqrt{\det(g_{ij})} \partial_j f) = g^{ij} \partial_{ij}^2 f + Y_0 f,$$

where  $Y_0$  is a first order differential operator depending on  $g$  and its first derivatives;

more generally for a  $p$  form  $u$ , still in the chart  $\varphi$ , the formula 21.23, p. 169 in [23] gives  $\Delta_\varphi u = g^{ij} \partial_{ij}^2 u + Y_p u$ , where  $Y_p$  is a first order differential operator.

So  $\Delta_\varphi$  depends on the first order derivatives of  $g$ , hence the difference  $A := \Delta_\varphi - \Delta_{\mathbb{R}}$ , where  $\Delta_{\mathbb{R}} u = \delta^{ij} \partial_{ij}^2 u$ , is controlled by the first order derivatives of  $g$ , which, by condition 2), can be made as small as we wish. So we have

$$\forall y \in U, |A(u)(y)| \leq |(g^{ij}(y) - \delta^{ij}) \partial_{ij}^2 u(y)| + |E(u)(y)|, \quad (5.11)$$

where  $E$  is a first order partial differential operator whose coefficients depend on the first order derivatives of  $g$ , and are 0 for  $y = x$ . So  $|E(u)(y)| \leq \eta |\nabla u(y)|$  for  $y \in U$ , where  $\eta$  is a continuous function of the metric  $g$  and  $\nabla g$  only; since  $|\nabla g| \leq \epsilon$ ,  $\eta$  may be chosen to depend on  $\epsilon > 0$  only and  $\eta(0) = 0$ .

Hence, integrating (5.11), we get:

$$\|Au\|_{L^r(U)} \leq \|\nabla g\|_{L^\infty(U)} \|u\|_{W^{2,r}(U)} + \eta(\epsilon) \|\nabla u\|_{L^r(U)}.$$

So, because  $\|\nabla u\|_{L^r(U)}$  is controlled by  $\|u\|_{W^{2,r}(U)}$ , there is a  $0 \leq c(\epsilon)$ ,  $c(0) = 0$  and  $c$  continuous at 0, such that  $\|Au\|_{L^r(U)} \leq c(\epsilon) \|u\|_{W^{2,r}(U)}$ .

Let  $\gamma$  be a  $p$  form in  $U \subset \mathbb{R}^n$ . We know that  $\Delta_{\mathbb{R}}$  operates component-wise on the  $p$  form  $\gamma$ , so we have:

$$\forall \gamma \in L_p^r(U), \exists v_0 \in W_p^{2,r}(U) :: \Delta_{\mathbb{R}} v_0 = \gamma, \|v_0\|_{W^{2,r}(U)} \leq C \|\gamma\|_{L^r(U)},$$

simply setting the component of  $v_0$  to be the Newtonian potential of the corresponding component of  $\gamma$  in  $U$ . These non trivial estimates are coming from Gilbarg and Trudinger [[9], Th 9.9, p. 230] and the constant  $C = C(n, r)$  depends only on  $n$  and  $r$ .

So we get  $\Delta_{\mathbb{R}} v_0 + Av_0 = \gamma + \gamma_1$ , with

$$\gamma_1 = Av_0 \Rightarrow \|\gamma_1\|_{L^r(U)} \leq c \|v_0\|_{W^{2,r}(U)} \leq cC \|\gamma\|_{L^r(U)}.$$

We solve again

$$\exists v_1 \in W_p^{2,r}(U) :: \Delta_{\mathbb{R}} v_1 = \gamma_1, \|v_1\|_{W^{2,r}(U)} \leq C \|\gamma_1\|_{L^r(U)} = C^2 c \|\gamma\|_{L^r(U)},$$

and we set

$$\gamma_2 := Av_1 \Rightarrow \|\gamma_2\|_{L^r(U)} \leq c \|v_1\|_{W^{2,r}(U)} \leq C \|\gamma_1\|_{L^r(U)} \leq C^2 c^2 \|\gamma\|_{L^r(U)}.$$

And by induction:

$$\forall k \in \mathbb{N}, \gamma_k := Av_{k-1} \Rightarrow \|\gamma_k\|_{L^r(U)} \leq c \|v_{k-1}\|_{W^{2,r}(U)} \leq C \|\gamma_{k-1}\|_{L^r(U)} \leq C^k c^k \|\gamma\|_{L^r(U)}$$

and

$$\exists v_k \in W_p^{2,r}(U) :: \Delta_{\mathbb{R}} v_k = \gamma_k, \|v_k\|_{W^{2,r}(U)} \leq C \|\gamma_k\|_{L^r(U)} \leq C^{k+1} c^k \|\gamma\|_{L^r(U)}.$$



Now we set  $v := \sum_{j \in \mathbb{N}} (-1)^j v_j$ . This series converges in norm  $W^{2,r}(U)$ , provided that we choose  $\epsilon_0 \leq \epsilon$  small enough to have  $c(\epsilon_0)C^2 < 1$ , and we get:

$$\Delta_\varphi v = \Delta_{\mathbb{R}} v + Av = \sum_{j \in \mathbb{N}} (-1)^j (\Delta_{\mathbb{R}} v_j + Av_j) = \gamma,$$

the last series converging in  $L^r_p(U)$ .

Going back to the manifold  $M$  with  $\gamma := \omega_\varphi$  and setting  $u_\varphi := v$ , we get the right estimates:

$$\exists u \in W^{2,r}(B_x(R)) :: \Delta u = \omega \text{ in } B_x(R), \quad \|u\|_{W^{2,r}(B_x(R))} \leq C \|\omega\|_{L^r(B_x(R))},$$

because, by use of the comparison lemma already seen, the Sobolev spaces for  $U$  go to the analogous Sobolev spaces for  $B_x(R)$  in  $M$ . Moreover  $C$  depends only on  $n, r, \epsilon_0$  and  $u$  is linear in  $\omega$ . ■

**Lemma 5.2.** *Let  $B_R := B(0, R)$  be the ball in  $\mathbb{R}^n$  of center 0 and radius  $R \leq 1$  and  $B'_R = B(0, 3R/4)$ . Suppose we have, with a constant  $C$  depending only on  $n, r$  and the  $\mathcal{C}^1$  bound of the coefficients of  $\Delta_\varphi$*

$$\forall v \in W^{2,r}(B_1), \quad \|v\|_{W^{2,r}(B'_1)} \leq C(\|v\|_{L^r(B_1)} + \|\Delta_\varphi v\|_{L^r(B_1)}).$$

*Let  $u \in L^r(B_R)$  such that  $\Delta_\varphi u \in L^r(B_R)$  and then we have:*

$$u \in W^{2,r}(B'_R), \quad \|u\|_{W^{2,r}(B'_R)} \leq c_1 R^{-2} \|u\|_{L^r(B_R)} + c_2 \|\Delta_\varphi u\|_{L^r(B_R)},$$

*where the constants  $c_1, c_2$  depend only on  $n, r$  and the  $\mathcal{C}^1$  bound of the coefficients of  $\Delta_\varphi$ .*

*Proof.*

We start with  $R = 1$ ,  $B := B(0, 1)$ , then we have by assumption:

$$\forall v \in W^{2,r}(B_1), \quad \|v\|_{W^{2,r}(B'_1)} \leq C(\|v\|_{L^r(B_1)} + \|\Delta_\varphi v\|_{L^r(B_1)}),$$

the constants  $C$  depending only on  $n, r$  and the  $\mathcal{C}^1$  bound of the coefficients of  $\Delta_\varphi$ . It remains to make the simple change of variables  $y = Rx$ ,  $dm(y) = R^n dm(x)$ ,  $v(x) := u(Rx)$  and to notice that  $\partial_j v(x) = R \partial_j(u)(Rx)$ ,  $\partial_{ij}^2 v(x) = R^2 \partial_{ij}^2(u)(Rx)$  in the integrals defining the  $L^r$  norm to get the result. ■

**Lemma 5.3.** *Let  $\Delta_\varphi$  be a second order elliptic matrix operator with  $\mathcal{C}^\infty$  coefficients operating on  $p$  forms  $v$  defined in  $U \subset \mathbb{R}^n$ . Let  $B := B(0, R)$  a ball in  $\mathbb{R}^n$ ,  $B' := B(0, 3R/4)$  and suppose that  $B \Subset U$ . Then we have an interior estimate: there are constants  $c_1, c_2$  depending only on  $n = \dim_{\mathbb{R}} M$ ,  $r$  and the  $\mathcal{C}^1$  norm of the coefficients of  $\Delta_\varphi$  in  $\bar{B}$  such that*

$$\forall v \in W_p^{2,r}(B), \quad \|v\|_{W^{2,r}(B')} \leq c_1 R^{-2} \|v\|_{L^r(B)} + c_2 \|\Delta_\varphi v\|_{L^r(B)}. \quad (5.12)$$

*Proof.*

For a 0 form, this lemma is exactly theorem 9.11, in [9] plus lemma 5.2 to get the dependence in  $R$ . For  $p$  forms we cannot avoid the use of deep results on elliptic systems of equations.

Let  $v$  be a  $p$  form in  $B \subset \mathbb{R}^n$ . We use the interior estimates in [[21], §6.2, Thm 6.2.6]. In our context, second order elliptic system, and with our notations, with  $r > 1$ , we get:

$$\exists C > 0, \quad \forall v \in W_p^{2,r}(B), \quad \|v\|_{W^{2,r}(B')} \leq c_1 R^{-2} \|v\|_{L^r(B)} + c_2 \|\Delta_\varphi v\|_{L^r(B)},$$

already including the dependence in  $R$ .

The constants  $c_1, c_2$  depend only on  $r, n := \dim M$  and the bounds and moduli of continuity of all the coefficients of the matrix  $\Delta_\varphi$ . (In Morrey's book, p. 213: the constant depends *only on  $E$  and on  $E'$* .)

In particular, if  $\Delta_\varphi$  has its coefficients near those of  $\Delta_{\mathbb{R}}$  in the  $\mathcal{C}^1$  norm, then the constants  $c_1, c_2$  are near the ones obtained for  $\Delta_{\mathbb{R}}$ . ■

**Lemma 5.4.** *Let  $(M, g)$  be a riemannian manifold. For  $x \in M$ ,  $\epsilon > 0$ , we take a  $\epsilon$  admissible ball  $B_x(R)$ . We have a local Calderon Zygmund inequality on the manifold  $M$ . There are constants  $c_1, c_2$  depending only on  $n = \dim_{\mathbb{R}} M$ ,  $r$  and  $\epsilon$  such that:*

$$\forall u \in W^{2,r}(B_x(R)), \quad \|u\|_{W^{2,r}(B_x(R/2))} \leq c_1 R^{-2} \|u\|_{L^r(B_x(R))} + c_2 \|\Delta u\|_{L^r(B_x(R))}.$$

*Proof.*

We transcribe the problem in  $\mathbb{R}^n$  by use of a coordinates path  $(V, \varphi)$ . The Hodge laplacian is the second order elliptic matrix operator  $\Delta_\varphi$  with  $C^\infty$  coefficients operating in  $\varphi(V) \subset \mathbb{R}^n$ . By the choice of a  $\epsilon$  admissible ball  $B_x(R)$ , and with  $R' := 3R/4$ , we have:

$$U' := \varphi(B_x(R')) \subset B_e(0, (1 + \epsilon)R'), \quad U := \varphi(B_x(R)) \subset B_e(0, (1 + \epsilon)R) \subset \varphi(V).$$

We apply lemma 5.3, to the euclidean balls  $B' := B_e(0, (1 + \epsilon)R')$ ,  $B := B_e(0, (1 + \epsilon)R)$  and we get, with  $u_\varphi$  the  $p$  form  $u$  read in the chart  $(V, \varphi)$

$$\|u_\varphi\|_{W^{2,r}(B')} \leq c_1 R^{-2} \|u_\varphi\|_{L^r(B)} + c_2 \|\Delta_\varphi u_\varphi\|_{L^r(B)}.$$

The fact that the coefficients of  $\Delta_\varphi$  are  $\epsilon$  near, in the  $C^1$  norm, of those of  $\Delta_{\mathbb{R}}$ , by condition 2) in the definition of the  $\epsilon$  admissible ball, implies that the constants  $c_1, c_2$  depend only on  $n, r$  and  $\epsilon$ .

The Lebesgue measure on  $U$  and the canonical measure  $dv_g$  on  $B_x(R)$  are equivalent; precisely because of condition 1) we get that  $(1 - \epsilon)^n \leq |\det g| \leq (1 + \epsilon)^n$ , and the measure  $dv_g$  read in the chart  $\varphi$  is  $dv_g = \sqrt{|\det g_{ij}|} d\xi$ , where  $d\xi$  is the Lebesgue measure in  $\mathbb{R}^n$ . So the Lebesgue estimates and the Sobolev estimates up to order 2 on  $U$  are valid in  $B_x(R)$  up to a constant depending only on  $n, r$  and  $\epsilon$  by lemma 4.2. In particular:

$$\forall x \in M, \quad \text{Vol}(B_x(R)) \leq (1 + \epsilon)^{n/2} \nu_n R^n, \quad (5.13)$$

where  $\nu_n$  is the euclidean volume of the unit ball in  $\mathbb{R}^n$ .

So passing back to  $M$ , we get, with  $A := B_x((1 + 2\epsilon)R) \supset \varphi^{-1}(B)$ ,  $A' := \varphi^{-1}(B')$

$$\|u\|_{W^{2,r}(A')} \leq c_1 R^{-2} \|u\|_{L^r(A)} + c_2 \|\Delta u\|_{L^r(A)}.$$

Now we notice that  $A' \supset B_x(R')$  so a fortiori:

$$\|u\|_{W^{2,r}(B_x(R'))} \leq c_1 R^{-2} \|u\|_{L^r(B_x((1+2\epsilon)R))} + c_2 \|\Delta u\|_{L^r(B_x((1+2\epsilon)R))}.$$

Finally, choosing  $\epsilon \leq 1/4$  we get:

$$\|u\|_{W^{2,r}(B_x(R'))} \leq c_1 R^{-2} \|u\|_{L^r(B_x(3R/2))} + c_2 \|\Delta u\|_{L^r(B_x(3R/2))},$$

i.e. the *CZI* local interior inequalities on  $B_x(3R/4) \subset B_x(3R/2) \subset M$ . So changing  $R$  for  $3R/2$  we proved the lemma. ■

## 6 The raising steps method.

Let  $(M, g)$  be a riemannian manifold. From now on we take  $\epsilon = \epsilon_0$  with  $\epsilon_0$  given by lemma 5.1 and we take the  $\epsilon_0$  admissible radius and the Vitali covering  $\{B(x_j, 5r(x_j))\}_{j \in \mathbb{N}}$  associated to it.

**Definition 6.1.** *Let  $(M, g)$  be a riemannian manifold. A weight relative to the covering  $\{B(x_j, 5r(x_j))\}_{j \in \mathbb{N}}$  is a function  $w(x) > 0$  on  $M$  such that :*

*there are two constants  $0 < c_{iw} \leq 1 \leq c_{sw}$  such that, setting*

$$\forall j \in \mathbb{N}, \quad B_j := B(x_j, 5r(x_j)), \quad w_j := \frac{1}{v_g(B_j)} \int_{B_j} w(x) dv_g(x),$$

*we have  $\forall j \in \mathbb{N}, \forall x \in B_j, c_{iw} w_j \leq w(x) \leq c_{sw} w_j$ . By smoothing  $w$  if necessary, we shall also suppose that  $w \in C^\infty(M)$ .*

As an example we have the constant weight,  $\forall x \in M$ ,  $w(x) = 1$ .

This means that  $w$  varies slowly on  $B_j$ .

So let  $w(x) > 0$  be any weight we say that  $\omega \in L_p^r(M, w)$ , if :

$$\|\omega\|_{L^r(M, w)}^r := \int_M |\omega(x)|^r w(x) dv_g(x) < \infty.$$

## 6.1 The raising steps method.

We shall use the following lemma.

**Lemma 6.2.** For  $\chi \in \mathcal{D}(M)$  and  $u \in W_p^{2, r}(M)$ , set  $B(\chi, u) := \Delta(\chi u) - \chi \Delta(u)$ . We have :

$$|B(\chi, u)| \leq |\Delta\chi| |u| + 2 |\nabla\chi| |\nabla u|.$$

Proof.

Exactly as for Proposition G.III.6 in [3] we have in an exponential chart at a point  $x \in M$ ,

$$u = \sum_{J, |J|=p} u_J dx^J, \quad g^{ij}(x) = \delta_{ij} \quad \text{and the basis } \left\{ \frac{\partial}{\partial x_j} \right\}_{j=1, \dots, n} \text{ is orthogonal.}$$

In this chart and *at the point*  $x$  we have that the laplacian is diagonal so

$$\Delta u(x) = \sum_{J, |J|=p} \frac{\partial^2 u_J}{\partial x_j^2}(x) dx^J$$

hence, for any  $x \in M$ ,

$$B(\chi, u)(x) = \Delta\chi(x)u(x) - 2 \sum_{J, |J|=p} \left( \sum_{j=1}^n \frac{\partial u_J}{\partial x_j} \frac{\partial \chi}{\partial x_j} \right) dx^J.$$

So we get

$$|B(\chi, u)| \leq |\Delta\chi| |u| + 2 |\nabla\chi| |\nabla u|. \quad \blacksquare$$

**Lemma 6.3.** Let  $w$  be a weight relative to the covering  $\mathcal{C}_\epsilon$  and set  $w_j$  as in definition 6.1. If  $v := \sum_{j \in \mathbb{N}} \chi_j u_j$  then we have

$$(i) \|v\|_{L^s(M, w^s)}^s \leq T^s c_{sw}^s \sum_{j \in \mathbb{N}} w_j^s \|u_j\|_{L^s(B_j)}^s.$$

$$(ii) \|\nabla v\|_{L^s(M, w^s)}^s \leq 2^{s/s'} (1 + C\epsilon) T^s c_{sw}^s \sum_{j \in \mathbb{N}} w_j^s (R_j^{-s} \|u_j\|_{L^s(B_j)}^s + \|\nabla u_j\|_{L^s(B_j)}^s).$$

$$(iii) \|\nabla^2 v\|_{L^s(M, w^s)}^s \leq 3^{s/s'} (1 + C\epsilon) T^s c_{sw}^s \sum_{j \in \mathbb{N}} w_j^s (R_j^{-2s} \|u_j\|_{L^s(B_j)}^s + \|\nabla^2 u_j\|_{L^s(B_j)}^s + R_j^{-s} \|\nabla u_j\|_{L^s(B_j)}^s).$$

Proof.

We have for (i)

$$\|v\|_{L^s(M, w^s)}^s = \int_M |v|^s w^s dv_g \leq \sum_{k \in \mathbb{N}} \int_{B_k} \left| \sum_{j \in \mathbb{N}} \chi_j u_j \right|^s w^s dv_g.$$

But the support of  $\chi_j$  is in  $B_j$  and the overlap of the covering is less than  $T$  so let

$$I(k) := \{B_j :: B_j \cap B_k \neq \emptyset\}$$

then  $\text{Card} I(k) \leq T$  and we have

$$\|v\|_{L^s(M, w^s)}^s \leq \sum_{k \in \mathbb{N}} \int_{B_k} \left| \sum_{j \in I(k)} \chi_j u_j \right|^s w^s dv_g.$$

We have, comparing the  $l^1$  and  $l^s$  norms by Hölder inequalities,

$$\left| \sum_{j \in I(k)} \chi_j u_j \right|^s \leq T^{s-1} \sum_{j \in I(k)} |\chi_j u_j|^s$$

so

$$\|v\|_{L^s(M, w^s)}^s \leq T^{s-1} \sum_{k \in \mathbb{N}} \sum_{j \in I(k)} \int_{B_k} |\chi_j u_j|^s w^s dv_g. \quad (6.14)$$

We still have, because  $\chi_j$  is supported by  $B_j$ ,

$$\sum_{j \in I(k)} \int_{B_k} |\chi_j u_j|^s w^s dv_g = \sum_{j \in \mathbb{N}} \int_{B_k} |\chi_j u_j|^s w^s dv_g$$

hence, exchanging the order of summation, all terms being positive,

$$\|v\|_{L^s(M, w^s)}^s \leq T^{s-1} \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} \int_{B_k} |\chi_j u_j|^s w^s dv_g.$$

The overlap being less than  $T$  we get

$$\sum_{k \in \mathbb{N}} \int_{B_k} |\chi_j u_j|^s w^s dv_g \leq T \int_M |\chi_j u_j|^s w^s dv_g$$

so

$$\|v\|_{L^s(M, w^s)}^s \leq T^{s-1} T \sum_{j \in \mathbb{N}} \int_M |\chi_j u_j|^s w^s dv_g = T^s \sum_{j \in \mathbb{N}} \int_{B_j} |\chi_j u_j|^s w^s dv_g.$$

With the constants  $c_{sw}$  defined in definition 6.1,

$$\|v\|_{L^s(M, w^s)}^s \leq T^s c_{sw}^s \sum_{j \in \mathbb{N}} w_j^s \|\chi_j u_j\|_{L^s(B_j)}^s \leq T^s c_{sw}^s \sum_{j \in \mathbb{N}} w_j^s \|u_j\|_{L^s(B_j)}^s$$

hence we get the (i) :

$$\|v\|_{L^s(M, w^s)}^s \leq T^s c_{sw}^s \sum_{j \in \mathbb{N}} w_j^s \|u_j\|_{L^s(B_j)}^s.$$

For (ii).

$$v := \sum_{j \in \mathbb{N}} \chi_j u_j \Rightarrow |\nabla v| \leq (1 + C\epsilon) \sum_{j \in \mathbb{N}} (|\chi_j| |\nabla u_j| + |\nabla \chi_j| |u_j|)$$

by lemma 4.3.

Because  $\{\chi_j\}_{j \in \mathbb{N}}$  is a partition of unity relative to the covering  $\{B_j\}_{j \in \mathbb{N}}$ , we have

$$|\nabla \chi_j| \leq \frac{1}{R_j} ; \quad |\nabla^2 \chi_j| \leq \frac{1}{R_j^2}.$$

Hence for the first term,  $A := \sum_{j \in \mathbb{N}} |\chi_j| |\nabla u_j|$  we get, again exactly as above

$$\|A\|_{L^s(M, w^s)}^s \leq T^s c_{sw}^s \sum_{j \in \mathbb{N}} w_j^s \|\nabla u_j\|_{L^s(B_j)}^s.$$

For the second one,  $B := \sum_{j \in \mathbb{N}} |\nabla \chi_j| |u_j|$  we get also as above, using the estimate  $|\nabla \chi_j| \leq \frac{1}{R_j}$ ,

$$\|B\|_{L^s(M, w^s)}^s \leq T^s c_{sw}^s \sum_{j \in \mathbb{N}} R_j^{-s} w_j^s \|u_j\|_{L^s(B_j)}^s.$$

Because  $(a + b)^s \leq 2^{s/s'} (a^s + b^s)$  we get

$$\|\nabla v\|_{L^s(M, w^s)}^s \leq 2^{s/s'} (1 + C\epsilon) T^s c_{sw}^s \sum_{j \in \mathbb{N}} w_j^s (R_j^{-s} \|u_j\|_{L^s(B_j)}^s + \|\nabla u_j\|_{L^s(B_j)}^s).$$

Finally for (iii). By lemma 4.3 (ii), we get

$$|\nabla^2(v)| \leq (1 + C\epsilon) \sum_{j \in \mathbb{N}} (|\chi_j| |\nabla^2 u_j| + |\nabla^2 \chi_j| |u_j| + |\nabla \chi_j| |\nabla u_j|)$$

So we get, for the two first terms, as above

$$C := (1 + C\epsilon) \sum_{j \in \mathbb{N}} (|\chi_j| |\nabla^2 u_j|) \Rightarrow \|D\|_{L^s(M, w^s)}^s \leq (1 + C\epsilon) T^s c_{sw}^s \sum_{j \in \mathbb{N}} w_j^s \|\nabla^2 u_j\|_{L^s(B_j)}^s.$$

And using the estimate  $|\nabla^2 \chi_j| \leq \frac{1}{R_j^2}$ ,

$$D := (1 + C\epsilon) \sum_{j \in \mathbb{N}} (|\nabla^2 \chi_j| |u_j|) \Rightarrow \|C\|_{L^s(M, w^s)}^s \leq (1 + C\epsilon) T^s c_{sw}^s \sum_{j \in \mathbb{N}} R_j^{-2s} w_j^s \|u_j\|_{L^s(B_j)}^s.$$

For the third one, we get using the estimate  $|\nabla \chi_j| \leq \frac{1}{R_j}$ ,

$$E := (1 + C\epsilon) \sum_{j \in \mathbb{N}} (|\nabla \chi_j| |\nabla u_j|) \Rightarrow \|D\|_{L^s(M, w^s)}^s \leq (1 + C\epsilon) T^s c_{sw}^s \sum_{j \in \mathbb{N}} w_j^s R_j^{-s} \|\nabla u_j\|_{L^s(B_j)}^s.$$

Adding this, we get

$$\|\nabla^2 v\|_{L^s(M, w^s)}^s \leq 3^{s/s'} (1 + C\epsilon) T^s c_{sw}^s \sum_{j \in \mathbb{N}} w_j^s (R_j^{-2s} \|u_j\|_{L^s(B_j)}^s + \|\nabla^2 u_j\|_{L^s(B_j)}^s + R_j^{-s} \|\nabla u_j\|_{L^s(B_j)}^s). \blacksquare$$

**Lemma 6.4.** *Let  $w$  be a weight relative to the covering  $\mathcal{C}_\epsilon$  and set  $w_j$  as in definition 6.1. Suppose that*

$$I^s = \sum_{j \in \mathbb{N}} w_j^s \|u_j\|_{L^s(B_j)}^s,$$

and, with  $s \geq r$ ,

$$w_j \|\chi_j u_j\|_{L^s(B_j)} \leq w_j R_j^{-\gamma} c \|\omega\|_{L^r(B_j)},$$

Then we have, with  $\forall x \in M$ ,  $\tilde{w}(x) := R(x)^{-\gamma} w(x)$ ,

$$I \leq c_w T^{s/r} \|\omega\|_{L^r(M, \tilde{w}^r)}.$$

Proof.

By

$$\sum_{j \in \mathbb{N}} a_j^s \leq \left( \sum_{j \in \mathbb{N}} a_j^r \right)^{s/r} \text{ because } s \geq r,$$

we get

$$I^s \leq \left( \sum_{j \in \mathbb{N}} w_j^r R_j^{-\gamma r} \|\omega\|_{L^r(B_j)}^r \right)^{s/r}.$$

By lemma 2.4 we have

$$\forall x \in B_j, d(x, x_j) < R_j = 5r(x_j) \leq \frac{1}{4} R(x_j) \leq \frac{1}{4} (R(x_j) + R(x)) \Rightarrow R(x) \leq 4R(x_j),$$

hence, because  $r(x_j) = \frac{R(x_j)}{120}$  and  $R_j = 5r(x_j) = \frac{R(x_j)}{24}$ ,

$$\forall x \in B_j, R(x) \leq 4R_j \Rightarrow R_j^{-\gamma r} \leq 96^{\gamma r} R(x)^{-\gamma r}. \quad (6.15)$$

But, by definition 6.1, we have

$$R_j^{-2r} w_j^r \|\omega\|_{L^r(B_j)}^r \leq c_{iw}^{-r} R_j^{-\gamma r} \int_{B_j} |\omega|^r w^r dv_g.$$

So

$$\sum_{j \in \mathbb{N}} R_j^{-\gamma r} w_j^r \|\omega\|_{L^r(B_j)}^r \leq c_{iw}^{-r} \sum_{j \in \mathbb{N}} R_j^{-\gamma r} \int_{B_j} |\omega|^r w^r dv_g$$

and, by (6.15), we get

$$\sum_{j \in \mathbb{N}} R_j^{-\gamma r} w_j^r \|\omega\|_{L^r(B_j)}^r \leq 96^{\gamma r} c_{iw}^{-r} \sum_{j \in \mathbb{N}} \int_{B_j} |\omega|^r R(x)^{-\gamma r} w^r dv_g.$$

Set  $\forall x \in M$ ,  $\tilde{w}(x) := R(x)^{-\gamma} w(x)$ .

Now, because the overlap is less than  $T$ , by proposition 3.2, we get

$$\sum_{j \in \mathbb{N}} \int_{B_j} |\omega|^r \tilde{w}^r dv_g \leq 96^{\gamma r} T \int_M |\omega(x)|^r \tilde{w}(x)^r dv_g(x) = T \|\omega\|_{L^r(M, \tilde{w}^r)}^r.$$

Putting this in  $v$ , we get

$$I^s \leq \left( \sum_{j \in \mathbb{N}} R_j^{-\gamma r} w_j^r \|\omega\|_{L^r(B_j)}^r \right)^{s/r} \leq 96^{\gamma s} c_{iw}^{-s} C^s (T \|\omega\|_{L^r(M, \tilde{w}^r)}^r)^{s/r},$$

so, setting  $c_w := 96^\gamma c_{iw}^{-1} C$  we get

$$I \leq c_w T^{s/r} \|\omega\|_{L^r(M, \tilde{w}^r)}. \blacksquare$$

With  $R(x)$  the  $\epsilon_0$  admissible radius at the point  $x \in M$ , and  $\mathcal{C}_{\epsilon_0}$  the  $\epsilon_0$  admissible covering of  $M$ , defined in section 3, we shall prove now :

**Theorem 6.5.** (*Raising Steps Method*) *Let  $(M, g)$  be a riemannian manifold and take  $w$  a weight relative to the Vitali covering  $\{B(x_j, 5r(x_j))\}_{j \in \mathbb{N}}$ .*

*For any  $r \leq 2$ , any threshold  $s \geq r$ , take  $k \in \mathbb{N}$  such that  $t_k := S_k(r) \geq s$  then, with  $w_0(x) := w(x)R(x)^{-2k}$ ,*

*$\forall \omega \in L_p^r(M, w_0^r)$ ,  $\exists v \in L_p^r(M, w^r) \cap L_p^{s_1}(M, w^{s_1}) \cap W^{2,r}(M, w^r)$ ,  $\exists \tilde{\omega} \in L_p^s(M, w^s) :: \Delta v = \omega + \tilde{\omega}$  with  $s_1 = S_2(r)$  and we have the control of the norms :*

$$\forall q \in [r, s_1], \|v\|_{L_p^q(M, w^q)} \leq C_q \|\omega\|_{L_p^r(M, w_0^r)} ; \|v\|_{W_p^{2,r}(M, w^r)} \leq C_r \|\omega\|_{L_p^r(M, w_0^r)} ;$$

$$\|\tilde{\omega}\|_{L_p^s(M, w^s)} \leq C_s \|\omega\|_{L_p^r(M, w_0^r)}.$$

*Moreover  $v$  and  $\tilde{\omega}$  are linear in  $\omega$ .*

*If  $M$  is complete and  $\omega$  is of compact support, so are  $v$  and  $\tilde{\omega}$ .*

Proof.

To simplify notations we do not put the  $p$  referring to the degree of the forms, i.e. we shall write  $L^r$  instead of  $L_p^r$ ,  $W^{2,r}$  instead of  $W_p^{2,r}$ , etc...

Set  $R_j := 5r(x_j)$ ,  $B_j := B(x_j, R_j)$  and apply lemma 5.1 to get, with  $c = c(n, r, \epsilon_0)$ ,

$$\exists u_j \in W^{2,r}(B_j) :: \Delta u_j = \omega, \|u_j\|_{W^{2,r}(B_j)} \leq c \|\omega\|_{L^r(B_j)}, \quad (6.16)$$

with  $u_j$  linear in  $\omega|_{B_j}$ .

So by lemma 4.5 we get, with  $t = S_1(r)$ ,  $s = S_2(r)$ ,

$$u_j \in L^s(B_j), \|u_j\|_{L^s(B_j)} \leq CR_j^{-2} \|u\|_{W^{2,r}(B(x,R))} \leq cCR_j^{-2} \|\omega\|_{L^r(B_j)}$$

and

$$\nabla u_j \in L^t(B_j), \|\nabla u_j\|_{L^t(B_j)} \leq CR_j^{-1} \|u\|_{W^{2,r}(B(x,R))} \leq CcR_j^{-1} \|\omega\|_{L^r(B_j)}.$$

Hence, because  $u_j \in L^r(B_j)$ , we have by interpolation [4], that  $\forall s' \in [r, s]$ ,  $u_j \in L^{s'}(B_j)$  with  $\|u_j\|_{L^{s'}(B_j)} \leq cCR_j^{-2}\|\omega\|_{L^r(B_j)}$ .

The same way, because  $\nabla u_j \in L^r(B_j)$ , by interpolation we get  $\forall t' \in [r, t]$ ,  $\nabla u_j \in L^{t'}(B_j)$  with

$$\|\nabla u_j\|_{L^{t'}(B_j)} \leq cCR_j^{-1}\|\omega\|_{L^r(B_j)}. \quad (6.17)$$

Let  $\{\chi_j\}_{j \in \mathbb{N}}$  be a partition of unity associated to the covering  $\{B(x_j, R_j)\}_{j \in \mathbb{N}}$  then we set

$$v_0 := \sum_{j \in \mathbb{N}} \chi_j u_j.$$

Because the  $u_j$  are linear in  $\omega|_{B_j}$ ,  $v_0$  is linear in  $\omega$ .

We have, because  $\|\chi_j\|_\infty = 1$ ,

$$\|\chi_j u_j\|_{L^s(B_j)} \leq cCR_j^{-2}\|\omega\|_{L^r(B_j)},$$

and multiplying by the  $w_j$ , given in definition 6.1,

$$w_j \|\chi_j u_j\|_{L^s(B_j)} \leq w_j R_j^{-2} c \|\omega\|_{L^r(B_j)}. \quad (6.18)$$

By lemma 6.3 (i), we have

$$\|v_0\|_{L^s(M, w^s)}^s \leq T^s c_{sw}^s \sum_{j \in \mathbb{N}} w_j^s \|u_j\|_{L^s(B_j)}^s.$$

Now, because of (6.18), we can apply lemma 6.4 with  $I = \|v_0\|_{L^s(M, w^s)}$  and  $\gamma = 2$ ; we get, with  $c_w := c_w := 96^2 c_{iw}^{-1} C$ , and  $\tilde{w}_2(x) := R(x)^{-2} w(x)$ .

$$\|v_0\|_{L^s(M, w^s)} \leq c_w T^{s/r} \|\omega\|_{L^r(M, \tilde{w}_2^r)}.$$

We also have  $v_0 \in L^r(M, w^r)$  because  $u_j \in W^{2,r}(B_j) \Rightarrow u_j \in L^r(B_j)$  as well, this means that  $v_0 w \in L^r(M) \cap L^s(M)$  hence by interpolation we have that

$$v_0 w \in L^{t'}(M) \Rightarrow v_0 \in L^{t'}(M, w^{t'}) \text{ for any } t' \in [r, s] \text{ with the same control of the norms.}$$

Because  $u_j \in W^{2,r}(B_j)$  we shall apply the same procedure to  $\nabla v_0$  by use of lemma 6.3 (ii), with  $s = r$ ,  $v = v_0$ , we get

$$\|\nabla v\|_{L^r(M, w^r)}^r \leq 2^{r/r'} (1 + C\epsilon) T^r c_{sw}^r \sum_{j \in \mathbb{N}} w_j^r (R_j^{-r} \|u_j\|_{L^r(B_j)}^r + \|\nabla u_j\|_{L^r(B_j)}^r). \quad (6.19)$$

But, by (6.16),

$$\|u_j\|_{W^{2,r}(B_j)} \leq c \|\omega\|_{L^r(B_j)} \Rightarrow \|\nabla u_j\|_{L^r(B_j)} \leq c \|\omega\|_{L^r(B_j)}.$$

To the first term of (6.19),  $A := \sum_{j \in \mathbb{N}} w_j^s R_j^{-s} \|u_j\|_{L^s(B_j)}^s$ , we can apply lemma 6.4 with  $s = r$ ,  $I \rightarrow A$

and  $C \rightarrow T c_{sw}$ ,  $\gamma = 1$ ,  $\tilde{w}_1(x) := R(x)^{-1} w(x)$ , to get

$$\|A\|_{L^r(M, w^r)} \leq c_w T \|\omega\|_{L^r(M, \tilde{w}_1^r)}.$$

To the second term of (6.19),  $B := \sum_{j \in \mathbb{N}} w_j^r \|\nabla u_j\|_{L^r(B_j)}^r$  we can apply lemma 6.4 with  $s = r$ ,  $I \rightarrow$

$B$ ,  $u_j \rightarrow \nabla u_j$  and  $C \rightarrow T c_{sw}$ ,  $\gamma = 0$ ,  $\tilde{w}(x) := w(x)$ , to get

$$\|v\|_{L^r(M, w^r)} \leq c_w T \|\omega\|_{L^r(M, w^r)}.$$

Adding these terms, we get

$$\|\nabla v\|_{L^r(M, w^r)} \leq 2^{1/r'}(1 + C\epsilon)^{1/r} T c_{sw} c_w T (\|\omega\|_{L^r(M, \tilde{w}_1^r)} + \|\omega\|_{L^r(M, w^r)}).$$

Again because  $u_j \in W^{2,r}(B_j)$  we shall apply the same procedure to  $\nabla^2 v_0$  by use of lemma 6.3 (iii), with  $s = r$ ,  $v = v_0$ ,  $\tilde{w}_2(x) := R^{-2}(x)w(x)$ , we get

$$\|\nabla^2 v_0\|_{L^r(M, w^r)}^r \leq 3^{r/r'}(1 + C\epsilon)^{r/r'} T^r c_{sw}^r \sum_{j \in \mathbb{N}} w_j^r (R_j^{-2r} \|u_j\|_{L^r(B_j)}^r + \|\nabla^2 u_j\|_{L^r(B_j)}^r + R_j^{-r} \|\nabla u_j\|_{L^r(B_j)}^r).$$

But, by (6.16),

$$\|u_j\|_{W^{2,r}(B_j)} \leq c\|\omega\|_{L^r(B_j)} \Rightarrow \|\nabla u_j\|_{L^r(B_j)} \leq c\|\omega\|_{L^r(B_j)}$$

and

$$\|\nabla^2 u_j\|_{L^r(B_j)} \leq c\|\omega\|_{L^r(B_j)}.$$

So playing the same game for each term, we get

$$\|\nabla^2 v_0\|_{L^r(M, w^r)}^r \leq 3^{1/r'}(1 + C\epsilon)^{1/r} T c_{sw} c_w T (\|\omega\|_{L^r(M, \tilde{w}^{2r})} + \|\omega\|_{L^r(M, \tilde{w}_1^r)} + \|\omega\|_{L^r(M, w^r)}).$$

Because we always have  $R(x) \leq 1$ , we get that  $\|\omega\|_{L^r(M, \tilde{w}_2^r)} \geq \|\omega\|_{L^r(M, \tilde{w}_1^r)} \geq \|\omega\|_{L^r(M, w^r)}$  so finally

$$\begin{aligned} \|v_0\|_{L^r(M, w^r)} &\leq C_0 \|\omega\|_{L^r(M, w^r)} ; \\ \|\nabla v_0\|_{L^r(M, w^r)} &\leq C_1 \|\omega\|_{L^r(M, \tilde{w}_1^r)} ; \\ \|\nabla^2 v_0\|_{L^r(M, w^r)}^r &\leq C_2 \|\omega\|_{L^r(M, \tilde{w}_2^r)}. \end{aligned}$$

Because

$$\|v_0\|_{W^{2,r}(M, w^r)} := \|v_0\|_{L^r(M, w^r)} + \|\nabla v_0\|_{L^r(M, w^r)} + \|\nabla^2 v_0\|_{L^r(M, w^r)}$$

we get

$$\|v_0\|_{W^{2,r}(M, w^r)} \leq C \|\omega\|_{L^r(M, \tilde{w}_2^r)},$$

where the constant  $C$  depends only on  $n$ ,  $\epsilon$ ,  $T$  and the constants of the weight  $w$  relative to the covering  $\mathcal{C}_\epsilon$ .

If  $\omega$  is of compact support and if  $M$  is complete, by lemma 3.3 we can cover  $\text{Supp } \omega$  by a finite set  $\{B_j\}_{j=1, \dots, N}$  and then add a layer  $\{B_j\}_{j=N_0+1, \dots, N_1}$  not intersecting  $\text{Supp } \omega$ , to cover  $\partial K'$  where  $K'$  is a compact containing  $K$ . This means that we can cover  $K'$  by a finite set  $\{B_j\}_{j=1, \dots, N_1}$ . By

linearity we get  $\forall j = N_0 + 1, \dots, N_1$ ,  $\omega_j = 0 \Rightarrow u_j = 0$  and setting now  $v_0 := \sum_{j=1}^{N_1} \chi_j u_j$  we can extend

$v_0$  as 0 outside  $\bigcup_{j=1}^{N_1} B_j$  hence we get that  $v_0$  is compactly supported.

We set, as in lemma 6.2,  $B(\chi_j, u_j) = \Delta(\chi_j u_j) - \chi_j \Delta u_j$ . Now consider  $\Delta v_0$ , we get

$$\Delta v_0 = \sum_{j \in \mathbb{N}} \Delta(\chi_j u_j) = \sum_{j \in \mathbb{N}} \chi_j \Delta u_j + \sum_{j \in \mathbb{N}} B(\chi_j, u_j) = \omega + \omega_1,$$

with  $\omega_1 := \sum_{j \in \mathbb{N}} B(\chi_j, u_j)$ .

Clearly  $\Delta v_0$  is linear in  $\omega$  so is  $\omega_1$ .

The  $\{\chi_j\}_{j \in \mathbb{N}}$  being a partition of unity relative to the covering  $\{B_j\}_{j \in \mathbb{N}}$ , we have  $|\nabla \chi_j| \leq \frac{1}{R_j}$  and

$|\Delta \chi_j| \leq \frac{1}{R_j^2}$ . We also have, because  $\|u_j\|_{W^{2,r}(B_j)} \leq c\|\omega\|_{L^r(B_j)}$ ,

$$\|\nabla u_j\|_{L^r(B_j)} \leq cR_j^{-1} \|\omega\|_{L^r(B_j)}$$

by lemma 4.4 (ii), and



$$\|u_j\|_{L^s(B_j)} \leq cR_j^{-2}\|\omega\|_{L^r(B_j)}, \quad (6.20)$$

with  $t = S_1(r)$ ,  $s = S_2(r)$  still by lemma 4.4 (i). Let  $q \in [r, t]$ .

By Young's inequality we get, because  $\frac{1}{t} = \frac{1}{s} + \frac{1}{n}$ ,

$$\|u_j\|_{L^t(B_j)} = \|\mathbb{1}_{B_j} u_j\|_{L^t(B_j)} \leq \|u_j\|_{L^s(B_j)} \|\mathbb{1}_{B_j}\|_{L^n(B_j)} = \|u_j\|_{L^s(B_j)} |B_j|^{1/n}.$$

Because  $|B(x, R)| := \text{Vol}(B(x, R)) \leq (1 + \epsilon)^{n/2} \nu_n R^n$  by equation (5.13), we get, with  $c_v = c \sqrt[n]{\nu_n(1 + \epsilon)^{n/2}}$ ,  $|B_j|^{1/n} \leq R_j$ .

Hence

$$\|u_j\|_{L^t(B_j)} \leq R_j \|u_j\|_{L^s(B_j)} \leq c_v R_j^{-1} \|\omega\|_{L^r(B_j)},$$

the last inequality given by (6.20).

Hence a fortiori  $\|u_j\|_{L^q(B_j)} \leq c_v R_j^{-1} \|\omega\|_{L^r(B_j)}$ .

By lemma 6.2 we have  $|B(\chi_j, u_j)| \leq |\Delta\chi_j| |u_j| + 2|\nabla\chi_j| |\nabla u_j|$ , so we get, because  $\nabla u_j \in L^q(B_j)$  by (6.17),

$$\|B(\chi_j, u_j)\|_{L^q(B_j)} \leq \|\nabla\chi_j\|_\infty \|\nabla u_j\|_{L^q(B_j)} + \|\Delta\chi_j\|_\infty \|u_j\|_{L^q(B_j)} \leq c_v R_j^{-2} \|\omega\|_{L^r(B_j)}.$$

Multiplying by  $w_j$  we get

$$w_j \|B(\chi_j, u_j)\|_{L^q(B_j)} \leq R_j^{-2} w_j c_v \|\omega\|_{L^r(B_j)}.$$

Set  $\omega_1 := \sum_{j \in \mathbb{N}} B(\chi_j, u_j)$ , then

$$\|\omega_1\|_{L^q(M, w^q)}^q \leq \sum_{j \in \mathbb{N}} \|B(\chi_j, u_j)\|_{L^q(B_j, w^q)}^q$$

Notice that  $\chi_j B(\chi_j, u_j) = B(\chi_j, u_j)$ , so again we apply lemma 6.4 with  $s = q$ ,  $I \rightarrow \|\omega_1\|_{L^q(M, w^q)}^q$ ,  $u_j \rightarrow B(\chi_j, u_j)$  and  $\gamma = 2$ ,  $\tilde{w}(x) := R(x)^{-2} w(x)$ , to get

$$\|\omega_1\|_{L^q(M, w^q)} \leq c_w T^{q/r} \|\omega\|_{L^r(M, \tilde{w}^r)}.$$

Set  $t_1 = t = S_1(r)$ , we have, with  $w_1(x) = w(x)$ ,  $w_0(x) = \tilde{w}(x) := w(x)R(x)^{-2}$ ,  $\forall q \in [r, t_1]$

$$\|\omega_1\|_{L^q(M, w_1^q)} \leq c_w T^{q/r} \|\omega\|_{L^r(M, w_0^r)}.$$

If  $\omega$  is of compact support and if  $M$  is complete, by lemma 3.3 we have seen that  $v_0$  is also of compact support hence so is  $\Delta v_0 = \omega + \omega_1$ . Which means that  $\omega_1$  is also of compact support.

Now we play the same game starting with  $\omega_1$  in place of  $\omega$  and we get, with

$$s_2 = S_2(t_1), t_2 = S_1(t_1) = S_2(r), w_2(x) = w(x), w_1(x) = w(x)R(x)^{-2}, w_0(x) = w(x)R(x)^{-4},$$

that

$$\forall q \in [r, t_1], \forall s \in [r, s_1], \exists v_1 \in L^s(M, w_2^s) \cap W^{2,q}(M, w_1) :: \Delta v_1 = \omega_1 + \omega_2$$

and

$$\forall t \in [r, t_2], \omega_2 \in L^t(M, w_2^t), \|\omega_2\|_{L^t(M, w_2^t)} \lesssim \|\omega_1\|_{L^{t_1}(M, w_1^{t_1})} \lesssim \|\omega\|_{L^r(M, w_0^r)}.$$

We keep the linearity of  $v_1$  w.r.t. to  $\omega_1$  hence to  $\omega$ . So  $\omega_2$  is still linear w.r.t.  $\omega$ .

So by induction we have, with

$$t_k = S_k(r), w_k(x) := w(x), w_{k-1}(x) = w(x)R(x)^{-2}, \dots, w_0(x) = w(x)R(x)^{-2k},$$

and, with  $s_{j+1} = S_2(t_j)$ ,

$$\forall s \in [r, s_{j+1}], \forall q \in [r, s_j] \forall j = 0, \dots, k-1, v_j \in L^q(M, w_{j+1}^q) \cap W^{2,q}(M, w_j),$$

and

$$\forall q \in [r, t_k], \omega_k \in L^q(M, w_k^q), \|\omega_k\|_{L^q(M, w_k^q)} \lesssim \cdots \lesssim \|\omega_1\|_{L^{t_1}(M, w_1^{t_1})} \lesssim \|\omega\|_{L^r(M, w_0^r)}. \quad (6.21)$$

Setting now  $v := \sum_{j=0}^{k-1} (-1)^j v_j$  and  $\tilde{\omega} := (-1)^k \omega_k$ , we have that  $\Delta v = \omega + \tilde{\omega}$  and

$$\forall q \in [r, s_1], v_j \in L^q(M, w_{j+1}^q) \cap W^{2,r}(M, w), \quad s_1 = S_2(t_1), \quad w_{j+1} = w(x)R(x)^{2(j+1-k)},$$

this implies, because  $w_k = w \leq w_{j+1}$ ,

$$\forall q \in [r, s_1], v_j \in L^q(M, w^q), \quad \|v_j\|_{L^q(M, w^q)} \leq c_l T^{1/r} \|\omega\|_{L^r(M, w_0^r)}.$$

So we have also for  $v := \sum_{j=0}^{k-1} (-1)^j v_j$  :

$$\forall q \in [r, s_1], v \in L^q(M, w^q), \quad \|v\|_{L^q(M, w^q)} \leq k c_l T^{1/r} \|\omega\|_{L^r(M, w_0^r)}. \quad (6.22)$$

We cannot go beyond  $s_1 := S_2(r)$  for  $v$  because of  $v_0$ . For the same reason, we cannot go beyond  $W^{2,r}(M, w)$ .

For the remaining term  $\tilde{\omega}$ , we get a better regularity, still because we set  $w_k = w$ ,  $\tilde{\omega} = (-1)^k \omega_k$ ,

$$\forall q \in [r, t_k], \tilde{\omega} \in L^q(M, w^q), \quad \|\tilde{\omega}\|_{L^q(M, w^q)} \lesssim \|\omega\|_{L^r(M, w_0^r)}. \quad (6.23)$$

Clearly the linearity is kept along the induction.

Now we choose  $k$  such that the threshold  $t_k := S_k(r) \geq s$ .

If  $\omega$  is of compact support and if  $M$  is complete, by lemma 3.3 we have seen that  $v_0$  and  $\omega_1$  also and by induction all the  $v_j$  and  $\omega_j$  are also of compact support. ■

We shall refer to this theorem as RSM for short. We notice that we have no completeness assumption on  $M$  to get the first part of the result.

**Lemma 6.6.** *Set, for  $k \in \mathbb{N}$ ,  $w_k^q = R(x)^{-qk}$ , we have  $L_p^q(M, w_k^q) \subset L_p^q(M)$  and*

$$\forall q > 1, \forall f \in L_p^q(M, w_k^q), \quad \|f\|_{L_p^q(M)} \leq \|f\|_{L_p^q(M, w_k^q)}.$$

*Proof.*

We have, because  $\forall x \in M, R(x) \leq 1 \Rightarrow w_k^q(x) \geq 1$ ,

$$\|f\|_{L_p^q(M)}^q = \int_M |f|^q dv_g \leq \int_M |f|^q w_k^q dv_g = \|f\|_{L_p^q(M, w_k^q)}^q. \quad \blacksquare$$

**Remark 6.7.** *We have, by inequalities (6.21), that  $\forall q \in [r, t_k], \tilde{\omega} \in L^q(M, w^q)$ . With the choice of  $w \equiv 1$  for the weight relative to the covering, with the notations of the RSM, we get  $\forall q \in [r, t_k], \tilde{\omega} \in L^q(M)$ .*

*We also have that  $w \equiv 1 \Rightarrow \forall q \in [r, s_1], v \in L^q(M)$ , with  $s_1 := S_2(r)$ .*

**Corollary 6.8.** *Let  $(M, g)$  be a complete riemannian manifold. For  $r \leq 2$ , take  $\omega \in L^r(M, w_0^r)$  with  $k \in \mathbb{N}$ ,  $w_0(x) := R(x)^{-2k}$  and  $s_1 := S_2(r)$ .*

*Choosing  $k$  big enough for the threshold  $t_k := S_k(r) \geq 2$  then the orthogonal projection  $H : L_p^2(M) \rightarrow \mathcal{H}_p^2(M)$  extends boundedly from  $L_p^r(M, w_0^r)$  to  $\mathcal{H}_p^2(M)$ . This implies  $H\omega = 0 \iff H\tilde{\omega} = 0$ .*

Proof.

For  $\omega \in L_p^r(M, w_0^r)$  and  $\forall s \in [r, s_1]$ ,  $\forall q \in [r, t_k]$ , the RSM, theorem 6.5, gives us two forms  $v \in L_p^s(M, w^s)$ ,  $\tilde{\omega} \in L_p^q(M, w^q)$ , such that

$$v = T\omega, \tilde{\omega} = A\omega, \Delta v = \omega + \tilde{\omega}. \quad (6.24)$$

where  $T$  and  $A$  are bounded linear operators :

$$T : L_p^r(M, w_0^r) \rightarrow L_p^s(M, w^s) ; A : L_p^r(M, w_0^r) \rightarrow L_p^q(M, w^q).$$

Now choose  $w \equiv 1 \Rightarrow w_0 = R(x)^{-2k}$ . Then we have  $\tilde{\omega} \in L_p^q(M)$ , and if  $k$  is such that the threshold  $t_k = S_k(r) \geq 2$ , we have  $\tilde{\omega} \in L_p^2(M)$ .

Hence the projection  $H$  is well defined on  $\tilde{\omega}$ . Suppose that  $H\Delta v = 0$  then we were done because, by (6.24), we would have  $0 = H\Delta v = H\omega + H\tilde{\omega} \Rightarrow H\omega = -H\tilde{\omega}$ .

We start by approximating  $\omega$  by a sequence  $\omega_l \in \mathcal{D}_p(M)$ ,  $\omega_l \rightarrow \omega$  in  $L_p^r(M, w_0^r)$ . Then apply the RSM to  $\omega_l$  ; we get  $v_l = T\omega_l$ ,  $\tilde{\omega}_l = A\omega_l$ ,  $\Delta v_l = \omega_l + \tilde{\omega}_l$ . We have that  $v_l$ ,  $\tilde{\omega}_l$  have compact support and by linearity with (6.22)

$$\forall s \in [r, s_1], (v - v_l) \in L^s(M), \|v - v_l\|_{L^s(M)} \leq kc_l T^{1/r} \|\omega - \omega_l\|_{L^r(M, w_0^r)}$$

so  $\|v - v_l\|_{L^s(M)} \rightarrow 0$  and the same way with (6.23) we get

$$\forall q \in [r, t_k], (\tilde{\omega} - \tilde{\omega}_l) \in L^q(M), \|\tilde{\omega} - \tilde{\omega}_l\|_{L^q(M)} \lesssim \|\omega - \omega_l\|_{L^r(M, w_0^r)}$$

hence  $\|\tilde{\omega} - \tilde{\omega}_l\|_{L^q(M)} \rightarrow 0$ .

Then  $H$  is well defined on  $v_l$ ,  $\Delta v_l$ ,  $\omega_l$  and  $\tilde{\omega}_l$ , because they are  $C^\infty$  and compactly supported hence in  $L_p^2(M)$ , and we have

$$\Delta v_l = \omega_l + \tilde{\omega}_l \Rightarrow H\Delta v_l = H\omega_l + H\tilde{\omega}_l.$$

Take  $t_k \geq 2$ ,  $h \in L_p^2(M)$  then  $\langle H\Delta v_l, h \rangle_{L^2(M)} = \langle \Delta v_l, Hh \rangle_{L^2(M)}$  because  $H$  is self adjoint. But because  $M$  is complete,  $\Delta$  is essentially self adjoint on  $L_p^2(M)$  by [8] and  $v_l$  has compact support, we have

$$\langle \Delta v_l, Hh \rangle_{L^2(M)} = \langle v_l, \Delta Hh \rangle_{L^2(M)} = 0,$$

because  $Hh \in \mathcal{H}_p^2(M)$ .

So we have  $\forall l \in \mathbb{N}$ ,  $H\Delta v_l = 0$  and this implies

$$\forall l \in \mathbb{N}, H\omega_l + H\tilde{\omega}_l = 0.$$

Now we have  $\tilde{\omega} \in L_p^2(M)$  and the convergence  $\|\tilde{\omega} - \tilde{\omega}_l\|_{L_p^2(M)} \rightarrow 0$  by lemma 6.6. So, because  $H$  is bounded on  $L_p^2(M)$ , we get  $H\tilde{\omega}_l \rightarrow H\tilde{\omega}$  in  $L_p^2(M)$ , and this means  $H\omega_l \rightarrow -H\tilde{\omega}$  also in  $L_p^2(M)$ . So we define, for any sequence  $\omega_l \in \mathcal{D}_p(M)$ ,  $\omega_l \rightarrow \omega$  in  $L_p^r(M, w_0)$ , by :

$$H\omega := \lim_{l \rightarrow \infty} H\omega_l = -\lim_{l \rightarrow \infty} H\tilde{\omega}_l = -H\tilde{\omega},$$

so we proved that  $H\omega_l$  converges in  $L_p^2(M)$  to  $-H\tilde{\omega}$ , with  $\tilde{\omega}$  given by the Raising Steps Method. This limit is independent of the sequence of approximations  $\omega_l$ , and it is clearly a extension of the projection  $H$  to  $L^r(M, w_0^r)$ .

This implies that  $H\omega = 0 \iff H\tilde{\omega} = 0$ . ■

**Corollary 6.9.** *Let  $(M, g)$  be a complete riemannian manifold. We get :  $\forall s \geq 2$ ,  $\mathcal{H}_p^2(M) \hookrightarrow \mathcal{H}_p^s(M)$  with*

$$\forall h \in \mathcal{H}_p^2(M), h \in \mathcal{H}_p^s(M) \text{ and } \|h\|_{L_p^s(M)} \leq C_s \|h\|_{L_p^2(M)}.$$

Proof.

Let  $\omega \in \mathcal{D}_p(M)$  and  $\varphi \in L_p^2(M)$ , then we have  $\langle H\omega, \varphi \rangle = \langle \omega, H^*\varphi \rangle$  by duality ; on the other hand, because  $\omega \in L_p^2(M)$ , we get

$$\langle H\omega, \varphi \rangle = \langle \omega, H\varphi \rangle ;$$

so, against  $\mathcal{D}_p(M)$ , we have  $H = H^*$ .

Now take  $r \leq 2$ , and  $\omega \in L^r(M, w_0^r)$  with  $k \in \mathbb{N}$ ,  $w_0(x) := R(x)^{-k}$  and  $s_1 := S_2(r)$ . Chosing  $k$  big enough for the threshold  $t_k := S_k(r) \geq 2$ , then the orthogonal projection  $H : L_p^2(M) \rightarrow L_p^2(M)$  extends boundedly from  $L_p^r(M, w_0^r)$  to  $L_p^2(M)$  by corollary 6.8 hence by duality  $H^* : L_p^2(M) \rightarrow L_p^{r'}(M, w_0^r)$ .

By density of  $\mathcal{D}_p(M)$  in  $L^r(M, w_0^r)$  we get that  $H = H^* : L_p^2(M) \rightarrow L_p^{r'}(M, w_0^r)$ .

We also have by lemma 6.6  $L_p^{r'}(M, w_0^r) \subset L_p^{r'}(M)$  with norm less than one, hence  $H : L_p^2(M) \rightarrow L_p^{r'}(M)$  boundedly and

$$h \in \mathcal{H}_p^2(M) \Rightarrow h = Hh \in L_p^{r'}(M) \Rightarrow h \in \mathcal{H}_p^{r'}(M).$$

Now we choose  $r = s'$  the conjugate exponent of  $s$  to end the proof of the corollary. ■

We already know that harmonic forms are smooth, see for instance [[5] corollary 5.4], so corollary 6.9 gives another kind of smoothness.

## 7 Weighted Calderon Zygmund inequalities.

In the same spirit of theorem 1.2 by Guneyusu and Pigola [11], we get the following "twisted" Calderon Zygmund inequality with weights and being valid directly for forms not a priori in  $\mathcal{D}_p(M)$ .

These CZI are twisted because there are 2 different weights in the inequality.

**Theorem 7.1.** *Let  $(M, g)$  be a complete riemannian manifold. Let  $w$  be a weight relative to the  $\mathcal{C}_\epsilon$  associated covering  $\{B(x_j, 5r(x_j))\}_{j \in \mathbb{N}}$  and set  $w_0 := R(x)^{-2}$ . Let  $u \in L_p^r(M, ww_0^r)$  such that  $\Delta u \in L_p^r(M, w)$  ; then there are constants  $C_1, C_2$  depending only on  $n = \dim_{\mathbb{R}} M$ ,  $r$  and  $\epsilon$  such that:*

$$\|u\|_{W^{2,r}(M,w)} \leq C_1 \|u\|_{L^r(M, ww_0^r)} + C_2 \|\Delta u\|_{L^r(M,w)}.$$

Moreover we have for  $t = S_2(r)$  that  $u \in L_p^t(M, w^t)$  with  $\|u\|_{L^t(M, w^t)} \leq c \|u\|_{W^{2,r}(M, w^r w_0^t)}$ .

Proof.

Let  $u \in L^r(M, ww_0^r)$ ,  $\Delta u \in L^r(M, w)$ . Set  $R_j := 5r(x_j)$ ,  $B_j := B(x_j, R_j)$ ,  $B'_j = B(x_j, 2R_j)$  and apply lemma 5.4 to get :

there are constants  $c_1, c_2$  depending only on  $n = \dim_{\mathbb{R}} M$ ,  $r$ ,  $\epsilon$  such that

$$\|u\|_{W^{2,r}(B_j)} \leq c_1 R_j^{-2} \|u\|_{L^r(B'_j)} + c_2 \|\Delta u\|_{L^r(B'_j)}. \quad (7.25)$$

Recall that

$$\|u\|_{W^{2,r}(M,w)} := \|\nabla^2 u\|_{L^r(M,w)} + \|\nabla u\|_{L^r(M,w)} + \|u\|_{L^r(M,w)},$$

so we have to compute those three terms.

$$\|\nabla^2 u\|_{L^r(M,w)}^r = \int_M |\nabla^2 u|^r w dv_g \leq \sum_{j \in \mathbb{N}} \|\nabla^2 u\|_{L^r(B_j w)}^r.$$

By (7.25) we get

$$\begin{aligned} \|\nabla^2 u\|_{L^r(B_j w)} &\leq (c_1 R_j^{-2} c_{sw} w_j \|u\|_{L^r(B'_j)} + c_2 c_{sw} w_j \|\Delta u\|_{L^r(B'_j)})^r \leq \\ &\leq 2^{r/r'} c_{sw}^r (c_1 R_j^{-2r} w_j \|u\|_{L^r(B'_j)}^r + c_2 w_j \|\Delta u\|_{L^r(B'_j)}^r). \end{aligned}$$

Hence

$$\|\nabla^2 u\|_{L^r(M,w)}^r \leq 2^{r/r'} c_{sw}^r c_{iw}^{-r} c_1 \sum_{j \in \mathbb{N}} R_j^{-2r} \|u\|_{L^r(B'_j,w)}^r + c_2 w_j \|\Delta u\|_{L^r(B'_j,w)}^r. \quad (7.26)$$

Exactly as in the proof of the RSM we get

$$R_j^{-2r} \int_{B'_j} |u(x)|^r w(x) dv_g(x) \leq 96^{2r} \int_{B'_j} |u(x)|^r R(x)^{-2r} w(x) dv_g(x)$$

hence, because the overlap of the Vitali covering is bounded by  $T$ , even for the double balls  $B'_j$ , we get

$$\sum_{j \in \mathbb{N}} R_j^{-2r} \int_{B'_j} |u(x)|^r w(x) dv_g(x) \leq 96^{2r} T \int_M |u(x)|^r R(x)^{-2r} w(x) dv_g(x).$$

Easier we get

$$\sum_{j \in \mathbb{N}} \int_{B'_j} |\Delta u(x)|^r w(x) dv_g(x) \leq T \int_M |\Delta u(x)|^r w(x) dv_g(x).$$

So, putting in (7.26), we get

$$\|\nabla^2 u\|_{L^r(M,w)} \leq 2^{1/r'} 96^2 T^{1/r} c_1 \|u\|_{L^r(M,ww_0^r)} + 2^{1/r'} T^{1/r} c_2 \|\Delta u\|_{L^r(M,w)}.$$

Exactly the same way we get

$$\|\nabla u\|_{L^r(M,w)} \leq 2^{1/r'} 96^2 T^{1/r} c_1 \|u\|_{L^r(M,ww_0^r)} + 2^{1/r'} T^{1/r} c_2 \|\Delta u\|_{L^r(M,w)}.$$

Hence

$$\|u\|_{W^{2,r}(M,w)} \leq C_1 \|u\|_{L^r(M,ww_0^r)} + C_2 \|\Delta u\|_{L^r(M,w)}$$

with

$$C_1 := 1 + 2^{1/r'} 96^2 T^{1/r} c_1 ; C_2 := 2^{1/r'} T^{1/r} c_2.$$

To get the "moreover" we proceed the same way. By lemma 4.5 (i), we get for the  $\epsilon$  admissible ball  $B_j$ ,

$$t = S_2(r), \quad \forall u \in W^{2,r}(B_j), \quad \|u\|_{L^t(B_j)} \leq C R_j^{-2} \|u\|_{W^{2,r}(B_j)}.$$

So, because  $w$  is relative to the covering,

$$\|u\|_{L^t(M,w)}^t \leq c_{sw} \sum_{j \in \mathbb{N}} w_j \int_{B_j} |u|^t dv_g = c_{sw} \sum_{j \in \mathbb{N}} w_j \|u\|_{L^t(B_j)}^t.$$

But, as above,

$$\int_{B_j} |u|^t dv_g \leq C^t R_j^{-2t} \|u\|_{W^{2,r}(B_j)}^t \leq C^t R_j^{-2t} (\|\nabla^2 u\|_{L^r(B_j)}^t + \|\nabla u\|_{L^r(B_j)}^t + \|u\|_{L^r(B_j)}^t).$$

Hence

$$\|u\|_{L^t(M,w^t)}^t \leq C^t c_{sw} \sum_{j \in \mathbb{N}} w_j^t R_j^{-2t} (\|\nabla^2 u\|_{L^r(B_j)}^t + \|\nabla u\|_{L^r(B_j)}^t + \|u\|_{L^r(B_j)}^t).$$

But

$$\sum_{j \in \mathbb{N}} a_j^t \leq \left( \sum_{j \in \mathbb{N}} a_j^r \right)^{t/r} \text{ because } t \geq r,$$

hence we get

$$A_1 := \sum_{j \in \mathbb{N}} w_j^t R_j^{-2t} \|\nabla^2 u\|_{L^r(B_j)}^t \leq \left( \sum_{j \in \mathbb{N}} w_j^r R_j^{-2r} \|\nabla^2 u\|_{L^r(B_j)}^r \right)^{t/r}.$$

hence, putting the radius and the weight into the integral, which gives the  $w_0^t$ ,

$$w_j^r R_j^{-2r} \|\nabla^2 u\|_{L^r(B_j)}^r \leq c_{iw}^{-r} \int_{B_j} |\nabla^2 u|^r w^r w_0^r dv_g,$$

So

$$A_1 \leq c_{iw}^{-t} \left( \sum_{j \in \mathbb{N}} \int_{B_j} |\nabla^2 u|^r w^r w_0^r dv_g \right)^{t/r}.$$

The overlap of the Vitali covering is bounded by  $T$ , so

$$A_1 \leq c_{iw}^{-t} (T \int_M |\nabla^2 u|^r w^r w_0^r dv_g)^{t/r} = c_{iw}^{-t} T^{t/r} \|\nabla^2 u\|_{L^r(M, w^r w_0^r)}^t.$$

Exactly the same way, we get

$$A_2 := \sum_{j \in \mathbb{N}} w_j^t R_j^{-2t} \|\nabla u\|_{L^r(B_j)}^t \leq c_{iw}^{-t} T^{t/r} \|\nabla u\|_{L^r(M, w^r w_0^r)}^t,$$

and

$$A_3 := \sum_{j \in \mathbb{N}} w_j^t R_j^{-2t} \|u\|_{L^r(B_j)}^t \leq c_{iw}^{-t} T^{t/r} \|u\|_{L^r(M, w^r w_0^r)}^t,$$

Adding we get

$$\begin{aligned} \|u\|_{L^t(M, w^t)}^t &\leq C^t c_{sw} c_{iw}^{-t} (A_1 + A_2 + A_3) \leq \\ &\leq C^t c_{sw} c_{iw}^{-t} T^{t/r} (\|\nabla^2 u\|_{L^r(M, w^r w_0^r)}^t + \|\nabla u\|_{L^r(M, w^r w_0^r)}^t + \|u\|_{L^r(M, w^r w_0^r)}^t) \leq \\ &\leq C^t c_{sw} c_{iw}^{-t} T^{t/r} (\|\nabla^2 u\|_{L^r(M, w^r w_0^r)}^t + \|\nabla u\|_{L^r(M, w^r w_0^r)}^t + \|u\|_{L^r(M, w^r w_0^r)}^t) \leq \\ &\leq C^t c_{sw} c_{iw}^{-t} T^{t/r} (\|u\|_{W^r(M, w^r w_0^r)})^t. \end{aligned}$$

Taking the  $t$  root we get

$$\|u\|_{L^t(M, w^t)} \leq C c_{sw} c_{iw}^{-1} T^{1/r} \|u\|_{W^r(M, w^r w_0^r)}.$$

Which ends the proof of the theorem. ■

**Corollary 7.2.** *Let  $(M, g)$  be a complete riemannian manifold. Set  $w_0 := R(x)^{-2}$ . Let  $u \in L_p^r(M, w_0^r)$  such that  $\Delta u \in L_p^r(M)$ ; then there are constants  $C_1, C_2$  depending only on  $n = \dim_{\mathbb{R}} M$ ,  $r$  and  $\epsilon$  such that :*

$$\|u\|_{W_p^{2,r}(M)} \leq C_1 \|u\|_{L_p^r(M, w_0^r)} + C_2 \|\Delta u\|_{L_p^r(M)}.$$

Moreover we have for  $t = S_2(r)$  that  $u \in L_p^t(M)$  with  $\|u\|_{L_p^t(M)} \leq c \|u\|_{W_p^{2,r}(M, w_0^t)}$ .

Proof.

We choose the weight  $w \equiv 1$ . ■

**Corollary 7.3.** *If the complete riemannian manifold  $(M, g)$  is such that the  $\epsilon_0$  admissible radius is positive, then we get the classical Calderon Zygmund inequalities :*

$$\forall r, 1 < r < \infty, \|u\|_{W^{2,r}(M)} \leq C_1 \|u\|_{L^r(M)} + C_2 \|\Delta u\|_{L^r(M)}.$$

Moreover we have the classical Sobolev inequality :

$$\text{for } t = S_2(r) \text{ we get that } u \in L_p^t(M) \text{ with } \|u\|_{L^t(M)} \leq c \|u\|_{W^{2,r}(M)}.$$

Proof.

If  $\forall x \in M, R(x) \geq \delta > 0$ , then  $w_0(x)^r \simeq 1$  hence the weights disappear. ■

Recall that, by theorem 1.3 in Hebey [12], we have that the harmonic radius  $r_H(1 + \epsilon, 2, 0)$  is bounded below if the Ricci curvature  $Rc$  verifies  $\|Rc\|_{\infty} < \infty$  and the injectivity radius is bounded below. This implies that the  $\epsilon$  admissible radius is also bounded below. Hence we get the conclusion of corollary 7.3 in that case.

## 8 Applications.

**Lemma 8.1.** *Let  $t < 2$ , if the weight  $\alpha \in L^\mu$  with  $\mu := \frac{2t}{2-t}$ , i.e.  $\gamma(\alpha, t) = \|\alpha\|_{L^\mu(M)}^\mu = \int_M \alpha^{\frac{2t}{2-t}} dv_g < \infty$ , we have :*

$$\omega \in L_p^2(M) \Rightarrow \omega \in L_p^t(M, \alpha).$$

Proof.

Young's inequality gives  $\|fg\|_{L^t} \leq \|f\|_{L^2} \|g\|_{L^q}$  with  $\frac{1}{t} = \frac{1}{2} + \frac{1}{q}$ , so let  $\omega \in L_p^2(M)$ , then, with  $t < 2$ , we get

$$\left( \int_M |\omega|^t \alpha dv_g \right)^{1/t} \leq \left( \int_M |\omega|^2 dv_g \right)^{1/2} \left( \int_M \alpha^{\frac{2t}{2-t}} dv_g \right)^{\frac{2-t}{2t}}.$$

So if the weight  $\alpha$  is such that  $\gamma(\alpha, t) = \int_M \alpha^{\frac{2t}{2-t}} dv_g < \infty$ , we are done. ■

For instance take any origin  $0 \in M$ ,  $M$  a complete riemannian manifold, and set  $\rho(x) := d_g(0, x)$ . We can choose a weight  $\alpha$ , function of  $\rho$ ,  $\alpha(x) := f(\rho(x))$ , such that  $\gamma(\alpha, t) < \infty$ , provided that  $\alpha(x)$  goes to 0 quickly enough at infinity.

Recall that  $R(x)$  is the  $\epsilon_0$  admissible radius at  $x \in M$ .

**Corollary 8.2.** *Suppose that  $(M, g)$  is a complete riemannian manifold ; let  $r < 2$  and choose a weight  $\alpha \in L^\infty(M)$  verifying  $\gamma(\alpha, r) < \infty$ . Set  $t := \min(2, S_2(r))$ . If  $t < 2$ , take the weight  $\alpha \in L^\infty(M)$  verifying also  $\gamma(\alpha, t) < \infty$ . Suppose we have condition (HL2,p).*

*Take  $k$  big enough so that the threshold  $S_k(r) \geq 2$ , and set  $w_0(x) := R(x)^{-2k}$ , then for any  $\omega \in L_p^r(M, w_0^r)$  verifying  $H\omega = 0$ , for the orthogonal projection  $H$  defined in corollary 6.8, there is a  $u \in W_p^{2,r}(M, \alpha) \cap L_p^t(M, \alpha)$ , such that  $\Delta u = \omega$ .*

*Moreover the solution  $u$  is given linearly with respect to  $\omega$ .*

Proof.

Take  $\omega \in L_p^r(M, w_0^r)$ , with the choice of  $w \equiv 1$  and  $S_k(r) \geq 2$ , the RSM theorem 6.5, gives linear operators

$$T : L_p^r(M, w_0^r) \rightarrow L_p^r(M) ; A : L_p^r(M, w_0^r) \rightarrow L_p^2(M),$$

such that

$$v := T\omega \in L^r(M) \cap L^s(M) \cap W^{2,r}(M) \text{ verifies } \Delta v = \omega + \tilde{\omega},$$

with  $s = S_2(r)$  and  $\tilde{\omega} := A\omega$ .

But

$$v \in L^t(M) \Rightarrow v \in L^t(M, \alpha) \text{ because } \alpha(x) \in L^\infty(M) \text{ is bounded :}$$

$$\|v\|_{L^t(M, \alpha)}^t = \int_M |v(x)|^t \alpha(x) dv(x) \leq \|\alpha\|_\infty \int_M |v(x)|^t dv(x) = \|\alpha\|_\infty \|v\|_{L^t(M)}^t.$$

And the same  $v \in L^r(M) \Rightarrow v \in L^r(M, \alpha)$ .

By corollary 6.8 if  $H\omega = 0$  then  $H\tilde{\omega} = 0$ .

Now we have  $t_k := S_k(r) \geq 2$  and we use the assumption (HL2,p) :

it gives the existence of a bounded linear operator  $L : L_p^2(M) \rightarrow W_p^{2,2}(M)$  such that

$$\Delta Lg = g, \text{ provided that } Hg = 0,$$

by the spectral theorem (see, for instance, the proof of theorem 5.10, p. 698 in Bueler [5]).

So setting  $f := L\tilde{\omega} \in L_p^2(M)$  we have  $\Delta f = \tilde{\omega} \in L_p^2(M)$ .

We set  $u = v - f$  then  $\Delta u = \omega + \tilde{\omega} - \tilde{\omega} = \omega$ . Let us see the estimates on  $u$ .

Because  $\gamma(\alpha, r) < \infty$ , we have by lemma 8.1,  $f \in L^r(M, \alpha)$ . If  $t < 2$ , we have also  $\gamma(\alpha, t) < \infty$  hence lemma 8.1 gives  $f \in L^t(M, \alpha)$ .

So in this case we have  $u \in L^r(M, \alpha) \cap L^t(M, \alpha)$ .

If  $s \geq 2$ , then we have  $t = 2$ ,  $v \in L^2(M)$  by interpolation between  $L^r(M)$  and  $L^s(M)$ , so now we have  $u \in L^2(M) \subset L^t(M, \alpha)$ .

Because  $v \in W^{2,r}(M)$ , we get that  $\nabla v, \nabla^2 v$  are also in  $L^r(M)$  so, the weight  $\alpha$  being chosen bounded, we get that  $\nabla v, \nabla^2 v$  are in  $L^r(M, \alpha)$  so  $v \in W^{2,r}(M, \alpha)$ . We also have that  $\nabla f, \nabla^2 f$  are in  $L_p^2(M)$ , hence because  $\gamma(\alpha, r) < \infty$ , we get that  $\nabla f, \nabla^2 f$  are in  $L^r(M, \alpha)$ . This gives that  $f \in W^{2,r}(M, \alpha)$  hence  $u = v - f \in W^{2,r}(M, \alpha)$ .  
Hence in any case we get  $u \in W_p^{2,r}(M, \alpha) \cap L_p^t(M, \alpha)$  and  $\Delta u = \omega$ . ■

Now we shall use the linearity of our solution to get, by duality, results for exponents bigger than 2. Take  $r < 2$  and  $r' > 2$  its conjugate.

Let  $T : L_p^r(M, w_0^r) \rightarrow W_p^{2,r}(M) \subset L_p^r(M)$ ,  $A : L_p^r(M, w_0^r) \rightarrow L_p^2(M)$  be the linear operators, given by the RSM, such that

$$\Delta T\omega = \omega + A\omega.$$

The hypothesis (HL2,p) gives the existence of a bounded linear operator  $L : L_p^2(M) \rightarrow W_p^{2,2}(M)$  such that

$$\Delta L\tilde{\omega} = \tilde{\omega}, \text{ provided that } H\tilde{\omega} = 0 \iff H\omega = 0 \text{ by corollary 6.8.}$$

Hence, setting  $C = LA : L_p^r(M, w_0^r) \rightarrow W_p^{2,2}(M)$  we get

$$\forall \omega \in L_p^r(M, w_0^r), \Delta(T - C)\omega = \omega.$$

We notice that  $\forall \psi \in \mathcal{D}_p(M)$ ,

$$\Delta(T - C)\Delta\psi = \Delta\psi,$$

just setting  $\omega = \Delta\psi$ . This is possible because

$$\forall \psi \in \mathcal{D}_p(M), \forall \varphi \in L_p^2(M), \langle H\Delta\psi, \varphi \rangle = \langle \Delta\psi, H\varphi \rangle = \langle \psi, \Delta(H\varphi) \rangle = 0,$$

where we used that  $\Delta$  is essentially self adjoint,  $M$  being complete, and  $\Delta(H\varphi) = 0$  because  $H\varphi$  is harmonic. So  $H\Delta\psi = 0$  and we can set  $\omega = \Delta\psi$  because then  $H\omega = 0$ . Hence

$$(T - C)\Delta\psi = \psi + h, \tag{8.27}$$

with  $h \in \mathcal{H}_p$ .

Now let  $\varphi \in L_p^2(M) \cap L_p^{r'}(M)$  and consider  $u := (T - C)^*\varphi$ , the  $*$  meaning the adjoint operator. This is meaningful because

$$T^* : (W^{2,r}(M))' \supset L^{r'}(M) \rightarrow L^{r'}(M, w_0^r)$$

and

$$C^* : (W^{2,2}(M))' \supset L^2(M) \rightarrow L^{r'}(M, w_0^r)$$

hence  $u \in L^{r'}(M, w_0^r)$ . We get

$$\begin{aligned} \forall \psi \in \mathcal{D}(M) \cap L^r(M, w_0^r), \langle \Delta u, \psi \rangle_{L^2(M, w_0^r)} &= \langle \Delta(T - C)^*\varphi, \psi \rangle_{L^2(M, w_0^r)} = \\ &= \int_M \Delta((T - C)^*\varphi) \psi w_0^r dv_g = \int_M (T - C)^*\varphi \Delta(\psi w_0^r) dv_g = \langle (T - C)^*\varphi, \Delta(\psi w_0^r) \rangle_{L^2(M)}, \end{aligned}$$

because  $\Delta$  is essentially self adjoint and  $\psi w_0^r$  has compact support.

Hence by (8.27)

$$\langle \Delta u, \psi \rangle_{L^2(M, w_0^r)} = \langle \varphi, (T - C)\Delta(\psi w_0^r) \rangle_{L^2(M)} = \langle \varphi, \psi w_0^r + h \rangle_{L^2(M)} = \langle \varphi, \psi w_0^r \rangle_{L^2(M)},$$

provided that  $\varphi \perp \mathcal{H}$ , i.e.  $H\varphi = 0$ . Putting back the weight in the integral, we get

$$\langle \Delta u, \psi \rangle_{L^2(M, w_0^r)} = \langle \varphi, \psi \rangle_{L^2(M, w_0^r)}. \tag{8.28}$$

Now let  $\psi' \in \mathcal{D}_p(M)$  and set  $\psi := \psi' w_0^{-r} = \psi' R(x)^{2kr}$  with  $R(x)$  the  $\epsilon$  admissible radius at the point  $x \in M$ . We have seen that  $\forall x \in M, R(x) > 0$  and we can smooth  $R(x)$  to make it  $\mathcal{C}^\infty(M)$



without changing the properties we used. For instance set  $\tilde{R}(x) := \sum_{j \in \mathbb{N}} \chi_j(x) R_j$  where  $\{\chi_j\}_{j \in \mathbb{N}}$  is a partition of unity subordinated to our Vitali covering  $\mathcal{C}_\epsilon = \{B(x_j, R_j)\}$ ; then the Lipschitz regularity of  $R(x)$  contained in lemma 5.1 gives the existence of a constant  $C > 0$  depending only on  $n, \epsilon$  such that  $\forall x \in M, \frac{1}{C}R(x) \leq \tilde{R}(x) \leq CR(x)$ .

So we have that  $\psi \in \mathcal{D}_p(M)$  and

$$\langle \Delta u, \psi \rangle_{L^2(M, w_0^r)} = \langle \varphi, \psi' \rangle_{L^2(M)} ; \quad \langle \varphi, \psi \rangle_{L^2(M, w_0^r)} = \langle \varphi, \psi' \rangle_{L^2(M)},$$

so (8.28) gives us

$$\langle \Delta u, \psi' \rangle_{L^2(M)} = \langle \varphi, \psi' \rangle_{L^2(M)}.$$

This being true for any  $\psi' \in \mathcal{D}_p(M)$  we get  $\Delta u = \varphi$  in distributions sense, so we proved

**Corollary 8.3.** *Suppose that  $(M, g)$  is a complete riemannian manifold with  $(HL2, p)$ ; suppose we have  $r < 2$  with  $k :: S_k(r) \geq 2$ , setting  $w_0(x) := R(x)^{-2k}$ , for any  $\varphi \in L_p^2(M) \cap L_p^{r'}(M)$ ,  $H\varphi = 0$  we get*

$$u := (T - C)^*\varphi, \quad u \in L_p^{r'}(M, w_0^r) \text{ and } u \text{ verifies } \Delta u = \varphi.$$

Adding the hypothesis that the  $\epsilon_0$  admissible radius is bounded below, we get more.

**Corollary 8.4.** *Suppose that  $(M, g)$  is a complete riemannian manifold and suppose the  $\epsilon_0$  admissible radius verifies  $\forall x \in M, R(x) \geq \delta > 0$ , and suppose also hypothesis  $(HL2, p)$ . Suppose we have  $r < 2$  with  $k :: S_k(r) \geq 2$ , setting  $w_0(x) := R(x)^{-2k}$ , for any  $\varphi \in L_p^2(M) \cap L_p^{r'}(M)$ ,  $H\varphi = 0$  we get*

$$u := (T - C)^*\varphi, \quad u \in W_p^{2, r'}(M) \text{ and } u \text{ verifies } \Delta u = \varphi.$$

Proof.

Because  $w_0^r(x) \geq 1$ , we get that  $L_p^{r'}(M, w_0^r) \subset L_p^{r'}(M)$  hence, applying this to  $u$ , we get that

$$u \in L_p^{r'}(M, w_0^r) \Rightarrow u \in L_p^{r'}(M).$$

Because the  $\epsilon_0$  admissible radius verifies  $\forall x \in M, R(x) \geq \delta > 0$ , we have the classical Calderon Zygmund inequalities, corollary 7.3 :

$$\forall r, 1 < r < \infty, \quad \|u\|_{W^{2, r}(M)} \leq C_1 \|u\|_{L^r(M)} + C_2 \|\Delta u\|_{L^r(M)}.$$

The solution  $u$  given by corollary 8.3,  $u := (T - C)^*\varphi$  is in  $u \in L_p^{r'}(M)$  by (8). Because we have  $\Delta u = \varphi \in L_p^{r'}(M)$ , we get by CZI that  $u \in W_p^{2, r'}(M)$ , with control of the norms. ■

## 9 Non classical strong $L^r$ Hodge decomposition

We shall need :

**Lemma 9.1.** *Let  $r \leq 2$  and  $\gamma \in W_{p+1}^{1, r}(M)$ ;  $\beta \in W_{p-1}^{1, r}(M)$ ,  $h \in \mathcal{H}_p^2(M)$  then*

$$\langle d\gamma, h \rangle = \langle d^*\beta, h \rangle = 0.$$

Proof.

Because  $h \in \mathcal{H}_p^2$ , we have that  $dh = d^*h = 0$  by theorem 5.5, p. 697 in Bueler [5]. By the density of  $\mathcal{D}_k(M)$  in  $W_k^{1, r}(M)$  which is always true in a complete riemannian manifold by theorem 2.7, p. 13 in [12], there is a sequence  $\gamma_k \in \mathcal{D}_{p+1}(M)$  such that  $\|\gamma - \gamma_k\|_{W^{1, r}(M)} \rightarrow 0$  and there is a sequence  $\beta_k \in \mathcal{D}_{p-1}(M)$  such that  $\|\beta - \beta_k\|_{W^{1, r}(M)} \rightarrow 0$ .

By use of corollary 6.9, we have that  $h \in \mathcal{H}_p^2 \Rightarrow h \in \mathcal{H}_p^{r'}$  because  $r' > 2$ , hence, because  $d\gamma \in L_p^r(M)$ ,

$\langle d\gamma, h \rangle = \lim_{k \rightarrow \infty} \langle d\gamma_k, h \rangle = \lim_{k \rightarrow \infty} \langle \gamma_k, d^*h \rangle = 0$ ,  
because  $d^*$  is the formal adjoint of  $d$ ,  $\gamma_k \in \mathcal{D}_{p+1}(M)$  and  $d^*h = 0$ .  
The same way we get  $\langle d^*\beta, h \rangle = 0$ . ■

**Definition 9.2.** Let  $\alpha$  be a weight on  $M$ , we define the space  $\tilde{W}_p^{2,r}(M, \alpha)$  to be

$$\tilde{W}_p^{2,r}(M, \alpha) := \{u \in L_p^r(M, \alpha) :: \Delta u \in L_p^r(M, \alpha)\}$$

with the norm

$$\|u\|_{\tilde{W}_p^{2,r}(M, \alpha)} := \|u\|_{L_p^r(M, \alpha)} + \|\Delta u\|_{L_p^r(M, \alpha)}.$$

With just the hypothesis (HL2p) we get the Hodge decomposition.

**Theorem 9.3.** Let  $(M, g)$  be a complete riemannian manifold. Let  $r \leq 2$  and take a weight  $\alpha \in L^\infty(M)$  such that  $\gamma(\alpha, r) < \infty$ ; with  $k :: S_k(r) \geq 2$ , set  $w_0 = R(x)^{-2k}$ , and suppose we have hypothesis (HL2,p). We have the direct decomposition given by linear operators :

$$L_p^r(M, w_0^r) = \mathcal{H}_p^2 \oplus \Delta(W_p^{2,r}(M, \alpha)).$$

With  $r' > 2$ , the conjugate exponent to  $r$ , we have the weaker decomposition, still given by linear operators :

$$L_p^{r'}(M) \cap L_p^2(M) = \mathcal{H}_p^2 \cap \mathcal{H}_p^{r'} + \Delta(\tilde{W}_p^{2,r'}(M)).$$

Proof.

Let  $\omega \in L_p^r(M, w_0^r)$  the remark 6.7 following the RSM with  $w \equiv 1$ ,  $w_0 = R(x)^{-2k}$ , gives  $u := T\omega \in W_p^{2,r}(M)$ ,  $\tilde{\omega} := A\omega \in L_p^2(M)$  such that  $\Delta u = \omega + \tilde{\omega}$ . So we get

$$\omega = \Delta u - \tilde{\omega} = \Delta u - (\tilde{\omega} - H\tilde{\omega}) - H\tilde{\omega}.$$

This is well defined because  $\tilde{\omega} \in L_p^2(M)$  and  $H$  is the orthogonal projection from  $L_p^2(M)$  on  $\mathcal{H}_p^2$ .

Now  $H(\tilde{\omega} - H\tilde{\omega}) = 0$  hence by (HL2p) we get  $f := L(\tilde{\omega} - H\tilde{\omega})$  solves  $\Delta f = \tilde{\omega} - H\tilde{\omega}$ ,  $f \in W_p^{2,2}(M)$ . So we get

$$\omega = \Delta u - \tilde{\omega} = -H\tilde{\omega} + \Delta u - \Delta f, \tag{9.29}$$

with  $H\tilde{\omega} \in \mathcal{H}_p^2$ .

This gives a first decomposition :

$$\omega = -H\tilde{\omega} + \Delta u - \Delta f, \tag{9.30}$$

with  $H\tilde{\omega} \in \mathcal{H}_p^2(M)$ ,  $u \in W_p^{2,r}(M)$  and  $f \in W_p^{2,2}(M)$ .

With the weight  $\alpha \in L^\infty(M)$  such that  $\gamma(\alpha, r) < \infty$  we have, by lemma 8.1,  $L_p^2(M) \subset L_p^r(M, \alpha)$ , hence the derivatives of  $f$  up to second order are in  $L_p^2(M)$  this implies that  $f \in W_p^{2,r}(M, \alpha)$ . Because  $\alpha$  is bounded, we also have  $u \in W_p^{2,r}(M, \alpha)$ .

It remains to set  $v := u - f \in W_p^{2,r}(M, \alpha)$  to get the decomposition. Because each step is linear, we get that this decomposition can be made linear with respect to  $\omega$ .

To get the uniqueness we consider the first decomposition (9.30) :

$$\omega = h + \Delta(u - f) \text{ with } h \in \mathcal{H}_p^2 \text{ and } u \in W_p^{2,r}(M), f \in W_p^{2,2}(M).$$

If there is another one  $\omega = h' + \Delta(u' - f')$  then  $0 = h - h' + \Delta(u - u' - (f - f'))$ ; so we have to show that

$$0 = h + \Delta(u - f) \text{ with } h \in \mathcal{H}_p^2 \text{ and } u \in W_p^{2,r}(M), f \in W_p^{2,2}(M),$$

implies  $h = 0$  and  $\Delta(u - f) = 0$ .

Now  $\Delta u = d(d^*u) + d^*(du) = d\alpha + d^*\beta$ , with  $\alpha = d^*u \in W_{p+1}^{1,r}(M)$  and  $\beta = du \in W_{p-1}^{1,r}(M)$ . By lemma 9.1 we get  $\langle d\alpha, h \rangle + \langle d^*\beta, h \rangle = 0$ , so  $\langle \Delta u, h \rangle = 0$ . Exactly the same proof with  $r = 2$  gives  $\langle \Delta f, h \rangle = 0$ , so, from  $h + \Delta u - \Delta f = 0$ , we get

$$0 = \langle h, h \rangle + \langle \Delta u, h \rangle + \langle \Delta f, h \rangle = \|h\|_{L^2(M)},$$

which implies  $\Delta(u - f) = 0$  and proves the uniqueness of this decomposition.

Now let  $\omega \in L_p^{r'}(M) \cap L_p^2(M)$ , then we have

$$\omega = H\omega + (\omega - H\omega) \text{ with } H(\omega - H\omega) = 0.$$

We have that  $H\omega \in \mathcal{H}_p^2(M)$  hence, by corollary 6.9 because  $\omega \in L_p^2(M)$ , we get that  $H\omega \in \mathcal{H}_p^{r'}(M)$  so  $\tilde{\omega} := \omega - H\omega \in L_p^{r'}(M) \cap L_p^2(M)$  and  $H\tilde{\omega} = 0$ . Now we have by corollary 8.3 a  $u \in L_p^{r'}(M, w_0^r)$  such that  $\Delta u = \tilde{\omega}$ . Again this implies that  $u \in L_p^{r'}(M)$  hence we have the decomposition

$$\forall \omega \in L_p^{r'}(M) \cap L_p^2(M), \quad \omega = H\omega + \Delta u = h + \Delta u, \quad (9.31)$$

with  $h \in \mathcal{H}_p^2(M) \cap \mathcal{H}_p^{r'}(M)$  and  $u \in \tilde{W}_p^{2,r'}(M)$ .

Because at each step we keep the linearity w.r.t.  $\omega$ , we get that the decomposition is also linear w.r.t.  $\omega$ . ■

There are two extreme cases done in the next corollaries.

**Corollary 9.4.** *Suppose the  $\epsilon_0$  admissible radius verifies  $\forall x \in M, R(x) \geq \delta > 0$ , and suppose also hypothesis (HL2,p). Take  $r \leq 2$  and let the weight  $\alpha \in L^\infty(M)$  be such that  $\gamma(\alpha, r) < \infty$ . Then we have the direct decomposition given by linear operators*

$$L_p^r(M) = \mathcal{H}_p^2 \oplus \Delta(W_p^{2,r}(M, \alpha)).$$

Proof.

In that case we have  $\forall x \in M, 0 < \delta \leq R(x) \leq 1$  hence  $1 \leq w_0^r \leq \frac{1}{\delta^{kr}}$  hence  $L_p^r(M, w_0^r) = L_p^r(M)$ . So we get this decomposition. ■

**Corollary 9.5.** *Suppose the admissible radius verifies  $\forall x \in M, R(x) \geq \delta > 0$ , and suppose also hypothesis (HL2,p). Take  $r' > 2$ , then we have the direct decomposition given by linear operators*

$$L_p^{r'}(M) \cap L_p^2(M) = \mathcal{H}_p^2 \cap \mathcal{H}_p^{r'} \oplus \Delta(W_p^{2,r'}(M)).$$

Proof.

The classical CZI true in this case by corollary 7.3, gives

$$\forall r, 1 < r < \infty, \|u\|_{W^{2,r}(M)} \leq C_1 \|u\|_{L^r(M)} + C_2 \|\Delta u\|_{L^r(M)}.$$

So  $u \in \tilde{W}_p^{2,r'}(M) \Rightarrow u \in W_p^{2,r'}(M)$  and we get the decomposition

$$L_p^{r'}(M) \cap L_p^2(M) = \mathcal{H}_p^2 \cap \mathcal{H}_p^{r'} + \Delta(W_p^{2,r'}(M)).$$

Now let us prove the uniqueness.

We have the decomposition (9.31)

$$\forall \omega \in L_p^{r'}(M) \cap L_p^2(M), \quad \omega = h + \Delta u,$$

with  $h \in \mathcal{H}_p^2(M) \cap \mathcal{H}_p^{r'}(M)$  and  $u \in W_p^{2,r'}(M)$ .

By (HL2,p) we have

$$\exists v \in W_p^{2,2}(M) :: \Delta v = \tilde{\omega} := \omega - h.$$

But  $\Delta v = \Delta u = \tilde{\omega}$ , so if there is another such decomposition

$$\omega = h' + \Delta u' = h' + \Delta v'$$

then

$$0 = h - h' + \Delta(u - u') = h - h' + \Delta(v - v'),$$

Still with  $v - v' \in W_p^{2,2}(M)$ . So changing names we have

$$0 = h + \Delta u = h + \Delta v \tag{9.32}$$

with  $h \in \mathcal{H}_p^2(M)$  and  $v \in W_p^{2,2}(M)$ .

Again  $\Delta v = d\alpha + d^*\beta$  with  $\alpha = d^*v \in W_{p-1}^{1,2}(M)$  and  $\beta = d^*v \in W_{p+1}^{1,2}(M)$  and by lemma 9.1 we get  $\langle d\alpha, h \rangle + \langle d^*\beta, h \rangle = 0$ , so  $\langle \Delta v, h \rangle = 0$ .

Hence  $\langle \Delta u, h \rangle = \langle \Delta v, h \rangle = 0$ . But by (9.32) we have

$$0 = \langle h, h \rangle + \langle \Delta u, h \rangle \text{ so } \|h\|_{L^2(M)} = 0 \Rightarrow h = 0$$

which ends the proof of uniqueness. ■

The admissible radius verifies  $\forall x \in M, R(x) \geq \delta > 0$ , if, for instance, the Ricci curvature of  $M$  is bounded and the injectivity radius is strictly positive [13].

We also have

**Corollary 9.6.** *Let  $r \leq 2$ , and, with  $k :: S_k(r) \geq 2$ , set  $w_0 = R(x)^{-k}$  and suppose the riemannian volume is finite and hypothesis (HL2,p). We have the direct decomposition given by linear operators :*

$$L_p^r(M, w_0^r) = \mathcal{H}_p^2 \oplus \Delta(W_p^{2,r}(M)).$$

Here the weight  $\alpha$  is no longer necessary because the volume being finite, if a form is in  $L^2(M)$  then it is already in  $L^r(M)$ . ■

**Corollary 9.7.** *Let  $r \leq 2$  and choose a weight  $\alpha \in L^\infty(M)$  such that  $\gamma(\alpha, r) < \infty$  ; with  $k :: S_k(r) \geq 2$ , set  $w_0 = R(x)^{-k}$ , and suppose we have hypothesis (HL2,p). We have the direct decompositions given by linear operators*

$$L_p^r(M, w_0^r) = \mathcal{H}_p^2 \oplus d(W_p^{1,r}(M, \alpha)) \oplus d^*(W_p^{1,r}(M, \alpha)).$$

With  $r' > 2$  the conjugate exponent of  $r$ , and adding the hypothesis that the  $\epsilon_0$  admissible radius is bounded below, we get

$$L_p^{r'}(M) \cap L_p^2(M) = \mathcal{H}_p^2 \cap \mathcal{H}_p^{r'} \oplus d(W_p^{1,r'}(M)) \oplus d^*(W_p^{1,r'}(M)).$$

Proof.

For the first part, we have, by (9.30) :  $\forall \omega \in L_p^r(M, w_0^r)$ ,

$$\omega = -H\tilde{\omega} + \Delta u - \Delta f,$$

with  $H\tilde{\omega} \in \mathcal{H}_p^2(M)$ ,  $u \in W_p^{2,r}(M)$  and  $f \in W_p^{2,2}(M)$ . Again

$$\Delta u = d\gamma + d^*\beta, \text{ with } \gamma \in W_{p-1}^{1,r}(M) \text{ and } \beta \in W_{p+1}^{1,r}(M),$$

and

$$\Delta f = d\gamma' + d^*\beta', \text{ with } \gamma' \in W_{p-1}^{1,2}(M) \text{ and } \beta' \in W_{p+1}^{1,2}(M),$$

Hence

$$\omega = h + d(\gamma - \gamma') + d^*(\beta - \beta').$$

With the weight  $\alpha$  we get  $\gamma \in W_{p-1}^{1,r}(M) \Rightarrow \gamma \in W_{p-1}^{1,r}(M, \alpha)$  and the same for  $\beta$ . And also

$\gamma' \in W_{p-1}^{1,2}(M) \Rightarrow \gamma' \in W_{p-1}^{1,r}(M, \alpha)$  and the same for  $\beta'$ . So, setting  $\mu := \gamma - \gamma'$ ,  $\delta = \beta - \beta'$ , we have the decomposition

$\omega \in L_p^r(M, w_0^r) \Rightarrow \omega = h + d\mu + d^*\delta$ ,  
with  $h \in \mathcal{H}_p^2(M) \cap \mathcal{H}_p^r(M, \alpha)$ ,  $\mu \in W_{p-1}^{1,r}(M, \alpha)$ ,  $\delta \in W_{p+1}^{1,r}(M, \alpha)$ .

For the uniqueness, suppose that

$$0 = h + d(\gamma - \gamma') + d^*(\beta - \beta'),$$

by use of lemma 9.1, we get  $\langle d\gamma, h \rangle + \langle d^*\beta, h \rangle = 0$  and also  $\langle d\gamma', h \rangle + \langle d^*\beta', h \rangle = 0$ , so  $h = 0$ . So we have

$$0 = d(\gamma - \gamma') + d^*(\beta - \beta').$$

This implies that

$$d\gamma + d^*\beta = d\gamma' + d^*\beta', \quad (9.33)$$

hence

$$d\gamma + d^*\beta \in L_p^r(M) \cap L_p^2(M) ; d\gamma' + d^*\beta' \in L_p^r(M) \cap L_p^2(M),$$

because

$$d\gamma + d^*\beta \in L_p^r(M) \text{ and } d\gamma' + d^*\beta' \in L_p^2(M).$$

Now take  $\varphi \in \mathcal{D}_p(M)$ , because (HL2,p) is true we have the  $L^2$  decomposition :

$$\varphi = H\varphi + d\gamma\mu + d^*\delta \text{ with } \mu, \delta \in W^{1,2}(M).$$

We have

$$\langle d(\gamma - \gamma'), \varphi \rangle = \langle d(\gamma - \gamma'), H\varphi + d\mu + d^*\delta \rangle ;$$

by use of lemma 9.1, we get  $\langle d(\gamma - \gamma'), H\varphi \rangle = 0$ . By density we have  $\mu = \lim_{k \rightarrow \infty} \mu_k$ ,  $\gamma_k \in \mathcal{D}_{p-1}$  and  $\delta = \lim_{k \rightarrow \infty} \delta_k$ ,  $\delta_k \in \mathcal{D}_{p+1}$ , the convergence being in  $W^{1,2}(M)$ , so  $d\mu = \lim_{k \rightarrow \infty} d\mu_k$  and  $d^*\delta = \lim_{k \rightarrow \infty} d^*\delta_k$  in  $L_p^2(M)$ . So we get

$$\langle d(\gamma - \gamma'), d\mu + d^*\delta \rangle = \lim_{k \rightarrow \infty} \langle d(\gamma - \gamma'), d\mu_k + d^*\delta_k \rangle.$$

But

$$\forall k \in \mathbb{N}, \langle d(\gamma - \gamma'), d^*\delta_k \rangle = \langle (\gamma - \gamma'), d^{*2}\delta_k \rangle = 0$$

because  $d^*$  is the formal adjoint of  $d$  and  $d^*\delta_k$  has compact support and  $d^{*2} = 0$ . So

$$\langle d(\gamma - \gamma'), \varphi \rangle = \lim_{k \rightarrow \infty} \langle d(\gamma - \gamma'), d\mu_k \rangle.$$

With (9.33) we get

$$\forall k \in \mathbb{N}, \langle d(\gamma - \gamma'), d\mu_k \rangle - \langle d^*(\beta - \beta'), d\mu_k \rangle = 0,$$

and

$$\forall k \in \mathbb{N}, \langle d^*(\beta - \beta'), d\mu_k \rangle = 0,$$

because  $d^*$  is the formal adjoint of  $d$ ,  $d\gamma_k$  has compact support and  $d^2 = 0$ . So

$$\forall k \in \mathbb{N}, \langle d(\gamma - \gamma'), d\mu_k \rangle = 0,$$

which gives

$$\langle d(\gamma - \gamma'), \varphi \rangle = \lim_{k \rightarrow \infty} \langle d(\gamma - \gamma'), d\mu_k \rangle = 0,$$

and this being true for any  $\varphi \in \mathcal{D}_p(M)$ , we get  $d(\gamma - \gamma') = 0$  ; this gives with (9.33)  $d^*(\beta - \beta') = 0$ .

For the second case we already have , by theorem 9.3 plus CZI given by corollary 7.3,  $\omega = H\omega + \Delta u$  with  $u \in W^{2,r}(M)$ . Now  $\Delta u = d(d^*u) + d^*(du) = d\gamma + d^*\beta$ , with  $\gamma = d^*u \in W_{p+1}^{1,r'}(M)$  and  $\beta = du \in W_{p-1}^{1,r'}(M)$ . This gives the decomposition.

For the uniqueness the proof is exactly the same as above, so we are done. ■

## 9.1 Non classical weak $L^r$ Hodge decomposition.

Now we shall need another hypothesis :

(HWr) if the space  $\mathcal{D}_p(M)$  is dense in  $W_p^{2,r}(M)$ .

We already know that (HWr) is true if :

- either : the injectivity radius is strictly positive and the Ricci curvature is bounded [[12] theorem 2.8, p. 12].

- or :  $M$  is geodesically complete with a bounded curvature tensor [[11] theorem 1.1 p.3].

We have a non classical weak  $L^r$  Hodge decomposition theorem :

**Theorem 9.8.** *Suppose that  $(M, g)$  is a complete riemannian manifold, fix  $r \leq 2$  and choose a bounded weight  $\alpha$  with  $\gamma(\alpha, r) < \infty$ .*

*Take  $k$  with  $S_k(r) \geq 2$ , and set the weight  $w_0 := R(x)^{-2k}$ . Suppose we have (HL2,p) and (HW2) ; then*

$$L_p^r(M, \alpha) = \mathcal{H}_p^r(M, \alpha) \oplus \overline{\Delta(\mathcal{D}_p(M))},$$

*the closure being taken in  $L^r(M, \alpha)$ .*

Proof.

Take  $\omega \in L_p^r(M, \alpha)$ . By density there is a  $\omega_\epsilon \in \mathcal{D}_p(M)$  such that  $\|\omega - \omega_\epsilon\|_{L^r(M, \alpha)} < \epsilon$ .

Then, because  $\omega_\epsilon \in \mathcal{D}_p(M)$ , we have  $\omega_\epsilon \in L_p^r(M, w_0^s)$  hence by RSM :

$$\forall s \geq r, \exists v_\epsilon \in L_p^r(M) \cap L_p^{s_1}(M) :: \Delta v_\epsilon = \omega_\epsilon + \tilde{\omega}_\epsilon,$$

with  $s_1 := S_2(r)$ ,  $\tilde{\omega}_\epsilon \in L_p^s(M)$ . Moreover, because  $\omega_\epsilon$  is of compact support, so are  $v_\epsilon$  and  $\tilde{\omega}_\epsilon$ .

Taking  $s = 2$ , by (HL2,p) there is a  $f_\epsilon \in W_p^{2,2}(M) :: \Delta f_\epsilon = \tilde{\omega}_\epsilon - H\tilde{\omega}_\epsilon$ .

By (HW2) there is a  $g_\epsilon \in \mathcal{D}_p(M) :: \|f_\epsilon - g_\epsilon\|_{W^{2,2}(M)} < \epsilon$  and this implies

$$\|\Delta f_\epsilon - \Delta g_\epsilon\|_{L^2(M)} < \epsilon.$$

Now we set  $u_\epsilon := v_\epsilon - g_\epsilon$ , then  $u_\epsilon$  is of compact support and we have

$$\Delta u_\epsilon = \Delta v_\epsilon - \Delta g_\epsilon = \Delta v_\epsilon - \Delta f_\epsilon + (\Delta f_\epsilon - \Delta g_\epsilon) = \omega_\epsilon + \tilde{\omega}_\epsilon - \tilde{\omega}_\epsilon + H\tilde{\omega}_\epsilon + E_\epsilon = \omega_\epsilon + H\tilde{\omega}_\epsilon + E_\epsilon,$$

where we set  $E_\epsilon := \Delta f_\epsilon - \Delta g_\epsilon$ .

So we get

$$\omega = -H\tilde{\omega}_\epsilon + \Delta u_\epsilon + (\omega - \omega_\epsilon) + E_\epsilon.$$

Because  $\gamma(\alpha, r) < \infty$ , we get  $\|E_\epsilon\|_{L^r(M, \alpha)} \leq C\|\Delta f_\epsilon - \Delta g_\epsilon\|_{L^2(M)} < C\epsilon$ . For the same reason we have  $H\tilde{\omega}_\epsilon \in \mathcal{H}_p^2(M) \subset \mathcal{H}_p^r(M, \alpha)$ , so we get  $\omega \in \mathcal{H}_p^r(M, \alpha) + \overline{\Delta(\mathcal{D}_p(M))}$ , the closure being taken in  $L^r(M, \alpha)$ .

For the uniqueness we proceed as before. We have to show that if  $0 = \lim_{k \rightarrow \infty} (h_k + \Delta u_k)$  with  $h_k \in \mathcal{H}_p^2(M) \subset \mathcal{H}_p^r(M, \alpha)$  and  $u_k \in \mathcal{D}_p(M)$ , the convergence in  $L^r(M, \alpha)$ , then  $\lim_{k \rightarrow \infty} h_k = 0$  and  $\lim_{k \rightarrow \infty} \Delta u_k = 0$ .

We have  $\Delta u_k = d\gamma_k + d^*\beta_k$ , with  $\gamma_k = d^*u_k \in \mathcal{D}_{p+1}(M)$ , and  $\beta_k = du_k \in \mathcal{D}_{p-1}(M)$ . So we can apply lemma 9.1 to get

$$\forall k, \langle h_k, d\gamma_k \rangle = \langle h_k, d^*\beta_k \rangle = 0,$$

hence

$$\lim_{k \rightarrow \infty} \langle h_k, h_k \rangle = 0 \Rightarrow \lim_{k \rightarrow \infty} h_k = 0 \text{ and hence } \lim_{k \rightarrow \infty} \Delta u_k = 0. \blacksquare$$

We also have a weak  $L^r$  Hodge decomposition without hypothesis (HWr) :

**Theorem 9.9.** *Suppose that  $(M, g)$  is a complete riemannian manifold and suppose we have (HL2,p). Fix  $r < 2$  and take a weight  $\alpha$  verifying  $\gamma(\alpha, r) < \infty$ . Then we have*

$L_p^r(M, \alpha) = \mathcal{H}_p^r(M, \alpha) \oplus \overline{d(\mathcal{D}_{p-1}(M))} \oplus \overline{d^*(\mathcal{D}_{p+1}(M))}$ ,  
the closures being taken in  $L^r(M, \alpha)$ .

Proof.

We start exactly the same way as for theorem 9.8 to have

$$v_\epsilon \in L_p^r(M) \cap L_p^{s_1}(M) :: \Delta v_\epsilon = \omega_\epsilon + \tilde{\omega}_\epsilon,$$

and

$$f_\epsilon \in W_p^{2,2}(M) :: \Delta f_\epsilon = \tilde{\omega}_\epsilon - H\tilde{\omega}_\epsilon.$$

Now we set directly  $u_\epsilon := v_\epsilon - f_\epsilon \Rightarrow \Delta u_\epsilon = \omega_\epsilon + H\tilde{\omega}_\epsilon$ . The point here is that  $u_\epsilon$  is not of compact support because  $f_\epsilon$  is not.

Nevertheless we have :

$$\omega = -H\tilde{\omega}_\epsilon + (H\tilde{\omega}_\epsilon + \omega_\epsilon) + (\omega - \omega_\epsilon) = -H\tilde{\omega}_\epsilon + \Delta u_\epsilon + (\omega - \omega_\epsilon). \quad (9.34)$$

But we can approximate  $d^*u_\epsilon$  by  $\gamma_\epsilon \in \mathcal{D}(M)$  in  $W^{1,2}(M)$ , and  $du_\epsilon$  by  $\beta_\epsilon \in \mathcal{D}(M)$  in  $W^{1,2}(M)$ , and this is always possible by theorem 2.7, p. 13 in [12]. So we have

$$\|d^*u_\epsilon - \gamma_\epsilon\|_{W^{1,2}(M)} < \epsilon, \quad \|du_\epsilon - \beta_\epsilon\|_{W^{1,2}(M)} < \epsilon.$$

And this implies

$$\|\Delta u_\epsilon - d\gamma_\epsilon - d^*\beta_\epsilon\|_{L_p^r(M)} \leq 2\epsilon \Rightarrow \|\Delta u_\epsilon - d\gamma_\epsilon - d^*\beta_\epsilon\|_{L_p^r(M, \alpha)} \leq 2C\epsilon,$$

because  $\gamma(\alpha, r) < \infty$ . As above we have  $H\tilde{\omega}_\epsilon \in \mathcal{H}_p^r(M, \alpha)$  so putting all this in (9.34) we get

$$\omega \in \mathcal{H}_p^r(M, \alpha) + \overline{d(\mathcal{D}_{p-1}(M))} + \overline{d^*(\mathcal{D}_{p+1}(M))},$$

the closure being taken in  $L_p^r(M, \alpha)$ .

The proof of the uniqueness is exactly as in the proof of theorem 9.8, so we are done. ■

**Remark 9.10.** *It seems not "geometrically natural" to take the closure of  $d^*(\mathcal{D}_{p+1}(M))$  with respect to  $L_p^r(M, \alpha)$  because here the adjoint of  $d$ ,  $d^*$ , is taken with respect to the volume measure without any weight. Nevertheless this is "analytically" correct and we get nothing more here. This is why we call the two previous results "non classical".*

For the case  $r > 2$  we need a stronger hypothesis, namely that the  $\epsilon_0$  admissible radius is bounded below. Then we get a *classical weak Hodge decompositions* for  $r \geq 2$ .

**Theorem 9.11.** *Suppose that  $(M, g)$  is a complete riemannian manifold and suppose the  $\epsilon_0$  admissible radius verifies  $\forall x \in M, R(x) \geq \delta > 0$ , suppose (HW $r$ ) and suppose also hypothesis (HL2,p). Fix  $r \geq 2$ , then we have*

$$L_p^r(M) = \mathcal{H}_p^r(M) \oplus \overline{\Delta(\mathcal{D}_p(M))}.$$

Without (HW $r$ ) we still get

$$L_p^r(M) = \mathcal{H}_p^r(M) \oplus \overline{d(\mathcal{D}_{p-1}(M))} \oplus \overline{d^*(\mathcal{D}_{p+1}(M))}.$$

All the closures being taken in  $L^r(M)$ .

Proof.

Take  $\omega \in L_p^r(M)$ , then by density there is a  $\omega_\epsilon \in \mathcal{D}_p(M)$  such that  $\|\omega - \omega_\epsilon\|_{L^r(M)} \leq \epsilon$ . This implies, because  $r > 2$  and  $\omega_\epsilon$  is compactly supported, that  $\omega_\epsilon \in L^r(M) \cap L^2(M)$ . So we have  $H\omega_\epsilon \in L^2(M) \Rightarrow H\omega_\epsilon \in L^r(M)$  by corollary 6.9.

So let  $\varphi_\epsilon := \omega_\epsilon - H\omega_\epsilon \in L^r(M) \cap L^2(M)$ , we have  $H\varphi_\epsilon = 0$  hence by corollary 8.4 we have

$$\exists u_\epsilon \in L^r(M, \omega_0^r) \cap W_p^{2,r}(M) :: \Delta u_\epsilon = \varphi_\epsilon.$$

So if we have (HWr) then  $\exists v_\epsilon \in \mathcal{D}_p(M)$  such that  $\|u_\epsilon - v_\epsilon\|_{W^{2,r}(M)} < \epsilon$  and this implies

$$\|\Delta u_\epsilon - \Delta v_\epsilon\|_{L^r(M)} < \epsilon.$$

Now we can write

$$\omega = H\omega_\epsilon + (\omega - \omega_\epsilon) + (\omega_\epsilon - H\omega_\epsilon) = H\omega_\epsilon + (\omega - \omega_\epsilon) + \Delta u_\epsilon = H\omega_\epsilon + (\omega - \omega_\epsilon) + \Delta v_\epsilon + (\Delta u_\epsilon - \Delta v_\epsilon).$$

The term  $E_\epsilon := (\omega - \omega_\epsilon) + (\Delta u_\epsilon - \Delta v_\epsilon)$  is an error term small in  $L^r(M)$  so we get

$$\omega = H\omega_\epsilon + \Delta v_\epsilon + E_\epsilon, \text{ with } H\omega_\epsilon \in L^r(M) \cap L^2(M), v_\epsilon \in \mathcal{D}_p(M), \|E_\epsilon\|_{L^r(M)} < 2\epsilon.$$

So we have the decomposition :

$$L_p^r(M) = \mathcal{H}_p^r(M) \oplus \overline{\Delta(\mathcal{D}_p(M))}.$$

the closures being taken in  $L^r(M)$ .

Without (HWr) we approximate  $d^*u_\epsilon$  by  $\gamma_\epsilon \in \mathcal{D}(M)$  in  $W^{1,r}(M)$ , and  $du_\epsilon$  by  $\beta_\epsilon \in \mathcal{D}(M)$  in  $W^{1,r}(M)$ , and this is always possible by theorem 2.7, p. 13 in [12]. So we have

$$\|d^*u_\epsilon - \gamma_\epsilon\|_{W^{1,r}(M)} < \epsilon, \|du_\epsilon - \beta_\epsilon\|_{W^{1,r}(M)} < \epsilon.$$

And this implies

$$\|\Delta u_\epsilon - d\gamma_\epsilon - d^*\beta_\epsilon\|_{L_p^r(M)} \leq 2\epsilon.$$

So we have

$$\omega = H\omega_\epsilon + (\omega - \omega_\epsilon) + \Delta u_\epsilon = H\omega_\epsilon + (\omega - \omega_\epsilon) + d\gamma_\epsilon + d^*\beta_\epsilon + (\Delta u_\epsilon - d\gamma_\epsilon - d^*\beta_\epsilon).$$

The term  $E_\epsilon := (\omega - \omega_\epsilon) + (\Delta u_\epsilon - d\gamma_\epsilon - d^*\beta_\epsilon)$  is an error term small in  $L^r(M)$  so we get

$$\omega = H\omega_\epsilon + d\gamma_\epsilon + d^*\beta_\epsilon + E_\epsilon,$$

with

$$H\omega_\epsilon \in L^r(M) \cap L^2(M), \gamma_\epsilon \in \mathcal{D}_{p+1}(M), \beta_\epsilon \in \mathcal{D}_{p-1}(M), \|E_\epsilon\|_{L^r(M)} < 2\epsilon.$$

So we have the decomposition :

$$L_p^r(M) = \mathcal{H}_p^r(M) \oplus \overline{d(\mathcal{D}_{p-1}(M))} \oplus \overline{d^*(\mathcal{D}_{p+1}(M))},$$

the closures being taken in  $L^r(M)$ .

The proof of the uniqueness is a slight modification of the proof of corollary 9.5, so we are done.

■

**Remark 9.12.** *By theorem 1.3 in Hebey [12], we have that the harmonic radius  $r_H(1 + \epsilon, 2, 0)$  is bounded below if the Ricci curvature  $Rc$  verifies  $\|\nabla Rc\|_\infty < \infty$  and the injectivity radius is bounded below. This implies that the  $\epsilon$  admissible radius is also bounded below.*

*Moreover if we add the hypothesis that the Ricci curvature  $Rc$  verifies  $\exists \delta \in \mathbb{R} :: Rc \geq \delta$  then by Proposition 2.10 in Hebey [12], we have hypothesis (HWr).*

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