On the L^r Hodge theory in complete non compact riemannian manifolds.

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Abstract

We study solutions for the Hodge laplace equation $\Delta u = \omega$ on p forms with L^r estimates for $r > 1$. Our main hypothesis is that Δ has a spectral gap in L^2. We use this to get L^r Hodge decomposition theorems. An interesting feature is that our decompositions are completely independent of the boundedness of the Riesz transforms in L^s.

These results are based on a generalisation of the Raising Steps Method to complete non compact riemannian manifolds.

Contents

1 Introduction.

Suppose you are interested by solving an equation $Du = \omega$, in a $k :: S_k(r') \geq 2$, manifold M with estimates of type Lebesgue L^r or Sobolev $W^{d,r}$; you know to solve it globally with estimates $L^s \to L^s$ and locally with estimates $L^r \to L^t$ with a strict increase of the regularity, for instance $\frac{1}{t} = \frac{1}{r} - \delta,\ \delta > 0$ for any $r \leq s$, then the Raising Steps Method (RSM for short) gives a *global* solution v of $Dv = \omega$ which is essentially in $L^t(M)$ for $\omega \in L^r(M)$.

I introduced this method in [1] to get solutions for the $\bar{\partial}$ equation with good estimates in relatively compact domains in Stein manifold. I extend it to linear partial differential operator D of any finite order m in [2] and I apply it to study the Poisson equation for the Hodge laplacian on forms in spaces $L^r(M)$ where (M, g) is a compact riemannian manifold. This gave L^r Hodge decomposition theorems as was done by C. Scott [17], but by an entirely different approach.

The aim of this work is to extend it to the case of complete non compact riemannian manifold.

1.1 Solutions of the Poisson equation for the Hodge laplacian.

Let (M, g) be a $\mathcal{C}^\infty$ smooth complete riemannian manifold with metric tensor g and $n = \dim M \geq 3$; let d be the exterior derivative, d^* its formal adjoint with respect to the Riemannian volume measure $dv_g = \sqrt{\det g} dx$, where dx is the Lebesgue measure in the chart x, and $\Delta = \Delta_p := dd^* + d^*d$ the Hodge laplacian acting on p forms. Let $L^r_p(M)$ be the space of p forms on M in the Lebesgue space $L^r(M)$.

We get the local solution of the Hodge Laplacian $\Delta u = \omega$ in a ball $B(x, R)$ in (M, g) with a radius $R(x)$ small enough to make this ball "conformal" to a ball in the euclidean space $\mathbb{R}^n$; this conformal radius is a special case of the "harmonic radius" of Hebey and Herzlich [12]. If ω is a p form in $L^r(B(x, R))$ then we get a p form solution u in the Sobolev space $W^{2,r}(B(x, r))$ of the ball, hence in $L^t(B(x, R))$ with $\frac{1}{t} = \frac{1}{r} - \frac{2}{n}$. This is done classically by use of the Newtonian potential. So the first assumption for the RSM is true, we have a strict increase of the regularity.

In order to get global solutions we need to cover the manifold M with our conformal balls and for this we use a classical "Vitali type covering" with a uniformly finite overlap. We shall denote it by $\mathcal{C}$.

When comparing non compact M to the compact case treated in [2], we have two important issues :

(i) the conformal radius may go to 0 at infinity, which is the case, for instance, if the canonical volume measure dv_g of (M, g) is finite ;

(ii) if dv_g is not finite, which is the case, for instance, if the conformal radius is bounded below, then p forms in $L^t_p(M)$ are generally not in $L^r_p(M)$ for $r < t$.

We address these problems by use of adapted weights on (M, g). These weights are relative to the covering $\mathcal{C}$: they are positive functions which vary slowly on the balls of the covering $\mathcal{C}$.

To deal with the problem (i) we use a weight $w_0(x) = R(x)^{-k}$ for an adapted integer k, where $R(x)$ is the conformal radius at the point $x \in M$.

To deal with the problem (ii) we use a weight $w(x)$ with the property that, with a $t < 2$,

$$\gamma(w,t) := \int_M w^{\frac{2t}{2-t}} dv_g < \infty.$$

In this case we get $L^2_p(M) \subset L^t_p(M,w)$.

We define the Sobolev spaces $W^{d,r}_p(M)$ of (M,g) following E. Hebey [11], and we set $S_k(r) :: \dfrac{1}{S_k(r)} = \dfrac{1}{r} - \dfrac{k}{n}$. Then our first result is Calderon Zygmund inequalities (CZI) with weight, generalizing results in [10].

Theorem 1.1 *Let w be a weight relative to the covering $\mathcal{C}$ and $u \in L^r_p(M,w)$ such that $\Delta u \in L^r_p(M,w)$ then there are constants c_1, c_2 depending only on $n = \dim_{\mathbb{R}} M$, and r such that if $u,\ \Delta u \in L^r_p(M,w)$ then $u \in W^{2,r}_p(M,w)$ and :*

$$\|u\|_{W^{2,r}(M,w)} \leq c_1 \|u\|_{L^r(M,w)} + c_2 \|\Delta u\|_{L^r(M,w)}.$$

Moreover we have for $t = S_2(r),\ r < \dfrac{n}{2}$, that $u \in L^t_p(M,w)$ with $\|u\|_{L^t(M,w)} \leq c\|u\|_{W^{2,r}(M,w)}$.

We set $\mathcal{H}^r_p(M) := L^r_p(M) \cap \ker\Delta_p$, the space of harmonic p forms in $L^r(M)$.

We set our main hypothesis :

(HL2,p) $\Delta = \Delta_p$ has a spectral gap in $L^2_p(M)$, i.e. there is no spectrum of Δ_p in an interval $(0,\eta)$ with $\eta > 0$.

The (HL2,p) assumption is known to be true in the case of the hyperbolic manifold $\mathbb{H}^{2n}$ of dimension $2n$ for any value of $p \in \{0,\ 2n\}$. For $p \neq n$ the space $\mathcal{H}^2_p$ is reduced to 0. For $\mathbb{H}^{2n+1}$ the (HL2,p) is valid for $p \neq n$ and $p \neq n+1$ and, out of these two cases, the space $\mathcal{H}^2_p$ is reduced to 0 as was proved by Donnelly [6].

When $\mathrm{Ric}(M) \geq -c^2$ and M is open at infinity then $0 \notin \mathrm{Sp}\Delta_0$ by a result of Buser, see Lott [16], proposition 6, p. 353, hence (HL2,0) is true. If M is a normal covering of a compact manifold X with covering group Γ, then $0 \notin \mathrm{Sp}\Delta_0$ iff Γ is not amenable by a result of Brooks, see Lott [16], corollary 3, p. 354, for precise references. Hence (HL2,0) is true if Γ is not amenable.

For $r = 2$, there is the orthogonal projection H from $L^2_p(M)$ on $\mathcal{H}^2_p(M)$; we shall prove that this projection extends to $L^r(M,w_0^r)$, i.e. $H\ :\ L^r(M,w_0^r) \to \mathcal{H}^2_p(M)$ boundedly and we get the following results on solutions of the Poisson equation.

Theorem 1.2 *Suppose that (M,g) is a complete riemannian manifold ; choose a weight $w \in L^\infty(M)$ verifying for $t = S_2(r),\ r < 2,\ \gamma(w,t) < \infty$, and suppose we have the condition (HL2,p).*

Then with $k :: S_k(r) \geq 2$, and $w_0(x) := R(x)^{-k}$, for any $\omega \in L^r(M,w_0^r)$ verifying $H\omega = 0$, there is a $u \in L^r_p(M,w) \cap L^t_p(M,w),\ t = S_2(r)$, such that $\Delta u = \omega$. Moreover the solution u is given linearly with respect to ω.

Setting r' for the conjugate exponent for $r,\ \dfrac{1}{r'} + \dfrac{1}{r} = 1$, by duality from theorem 1.2, we get

Theorem 1.3 *Suppose that (M,g) is a complete riemannian manifold ; suppose we have $r < 2$ and (HL2,p), then with $k :: S_k(r) \geq 2$, and $w_0(x) := R(x)^{-k}$, for any $\varphi \in L^2_p(M) \cap L^{r'}_p(M),\ H\varphi = 0$, there is a $u \in L^{r'}(M,w_0^r)$ such that $\Delta u = \varphi$. This solution is linear with respect to φ.*
Moreover we have $u \in W^{2,r'}_p(M)$ with control of the norm.

1.2 Hodge decomposition in L^r spaces. Known results.

In 1949, Kodaira [13] proved that the L^2 -space of p -forms on (M, g) has the orthogonal decomposition :

$$L^2_p(M) = \mathcal{H}^2_p \oplus \overline{d\mathcal{D}_{p-1}(M)} \oplus \overline{d^*\mathcal{D}_{p+1}(M)},$$

and in 1991 Gromov [9] proved a strong L^2 Hodge decomposition, under the hypothesis (HL2,p) :

$$L^2_p(M) = \mathcal{H}^2_p \oplus dW^{1,2}_{p-1}(M) \oplus d^*W^{1,2}_{p+1}(M).$$

In 1995 Scott [17] proved a strong L^r Hodge decomposition but on *compact* riemannian manifold

$$L^r_p(M) = \mathcal{H}^r_p \oplus dW^{1,r}_{p-1}(M) \oplus d^*W^{1,r}_{p+1}(M).$$

Let d^*_φ be the formal adjoint of d relatively to the measure $d\mu(x) = e^{-\varphi(x)}dv_g(x)$, where $\varphi \in \mathcal{C}^2(M)$, and let $\Delta_{\varphi,p} := dd^*_\varphi + d^*_\varphi d$ acting on p forms. Setting $\Delta = \mathrm{Tr}\nabla^2$ the covariant Laplace Beltrami operator acting on p forms and $L = \Delta - \nabla\varphi \cdot \nabla$, then, in 2009 X-D. Li [14] proved, among other nice results, a strong L^r Hodge decomposition on complete non compact riemannian manifold :

Theorem 1.4 *(X-D. Li) Let $r > 1$, $r' = \dfrac{r}{r-1}$. Let (M, g) be a complete riemannian manifold, $\varphi \in \mathcal{C}^2(M)$, and $d\mu(x) = e^{-\varphi(x)}dv_g(x)$. Suppose that the Riesz transforms $d\Delta^{-1/2}_{\varphi,p}$ and $d^*\Delta^{-1/2}_{\varphi,p}$ are bounded in L^r and $L^{r'}$, and the Riesz potential is bounded in L^r. Suppose also that (M, g) is L stochastically complete, then the strong L^r Hodge direct sum decomposition holds on p forms :*

$$L^r_p(M, \mu) = \mathcal{H}^r_p(M, \mu) \oplus dW^{1,r}_{p-1}(M, \mu) \oplus d^*_\varphi W^{1,r}_{p+1}(M, \mu).$$

These results are valid for the family of weights $\varphi \in \mathcal{C}^2(M)$ and for the Hodge laplacian associated to them, in the Witten sense [20].

1.3 Hodge decomposition in L^r spaces. Main results.

The results of X-D. Li are based on the boundedness of the Riesz transforms in L^r and $L^{r'}$ and the results we get use only the spectral gap hypothesis (HL2,p). In this way they may appear to be the natural generalisation of Gromov results from L^2 to L^r. On the other hand our results are proved only in the case $\varphi = 0$.

To get these decomposition theorems we shall apply our results on solutions of the Poisson equation. Recall that $H : L^r(M, w^r_0) \to \mathcal{H}^2_p(M)$ boundedly, where H is the orthogonal projection from $L^2_p(M)$ on $\mathcal{H}^2_p(M)$, and this explain the appearance of $\mathcal{H}^2_p(M)$ in the following theorem.

Theorem 1.5 *Let $r < 2$ and choose a weight $w \in L^\infty(M)$ such that $\gamma(w, r) < \infty$; with $k :: S_k(r) \geq 2$, set $w_0 = R(x)^{-k}$, and suppose we have hypothesis (HL2,p). We have the decompositions given by linear operators*

$$L^r_p(M, w^r_0) = \mathcal{H}^2_p \oplus \Delta(W^{2,r}_p(M, w)).$$

With $r' > 2$ the conjugate exponent of r,

$$L^{r'}_p(M) \cap L^2_p(M) = \mathcal{H}^2_p \cap \mathcal{H}^{r'}_p \oplus \Delta(W^{2,r'}_p(M)).$$

The $\oplus$ means that we have uniqueness in these decompositions, i.e. they are direct decompositions.

As a corollary we get

Corollary 1.6 *Let $r < 2$ and choose a weight $w \in L^\infty(M)$ such that $\gamma(w, r) < \infty$; with $k :: S_k(r) \geq 2$, set $w_0 = R(x)^{-k}$, and suppose we have hypothesis (HL2,p). We have the direct decompositions given by linear operators*

$$L^r_p(M, w^r_0) = \mathcal{H}^2_p \oplus d(W^{1,r}_p(M, w)) \oplus d^*(W^{1,r}_p(M, w)).$$

With $r' > 2$ the conjugate exponent of r,

$$L^{r'}_p(M) \cap L^2_p(M) = \mathcal{H}^2_p \cap \mathcal{H}^{r'}_p \oplus d(W^{1,r'}_p(M)) \oplus d^*(W^{1,r'}_p(M)).$$

We also have a weak L^r Hodge decomposition.

Theorem 1.7 *Suppose that (M, g) is a complete riemannian manifold and suppose we have (HL2,p). Take a weight w verifying $\gamma(w, r) < \infty$ for $r < 2$.*

Then we have

$$L^r_p(M) = \mathcal{H}^r_p \oplus \overline{d(\mathcal{D}_p(M))} \oplus \overline{d^*(\mathcal{D}_p(M))},$$

the closure being taken in $L^r(M, w)$.

Remark 1.8 *In the case $s > 2$ we get the decomposition*

$$L^s_p(M) \cap L^2_p(M) = \mathcal{H}^2_p \cap \mathcal{H}^s_p \oplus d(W^{1,s}_p(M)) \oplus d^*(W^{1,s}_p(M)),$$

which applies for $L^s_p(M) \cap L^2_p(M)$ and the intersection with $L^2_p(M)$ may seem strange but in fact it is necessary by a result of N. Lohoué [15] : in the hyperbolic space $\mathbb{H}^4$, which fulfils all our assumptions i.e. for any $p = 0, ..., 4$, Δ_p has a spectral gap, he proved that, for s big enough, there is not even a weak decomposition of $L^s_p(M)$. He also proved that the Riesz transforms are not bounded on $L^s(M)$ for s big enough and, because our decomposition is still valid, this proves that there is a priori no link between our results and the boundedness of the Riesz transforms.

So a striking point in this work is the independence of our L^r Hodge decompositions with respect to the boundedness of the Riesz transforms.

This work will be presented in the following way.

In the section 2 we define the conformal balls and the conformal radius and the basic facts relative to them.

In section 3 we use a Vitali type covering lemma with our conformal balls and we prove that its overlap is finite.

In section 4 we define the Sobolev spaces, following E. Hebey [11].

In section 5 we prove the local estimates for the Hodge Laplacian. This is essentially standard by use of classical results from Gilbarg and Trudinger [8].

In section 6 we develop the Raising Steps Method in the non compact case. The useful weights are defined here.

We prove Calderon Zygmund inequalities with weights and we use them to get the RSM theorem which is at the basis of our results.

In section 7 we deduce the applications to the Poisson equation associated to the Hodge Laplacian. We use these solutions to get strong L^r Hodge decomposition theorems. We also get a weak L^r Hodge decomposition theorem.

2 Basic facts.

Definition 2.1 *Let (M,g) be a riemannian manifold and $x \in M$. We shall say that the geodesic ball $B(x,R)$ is ϵ* **conformal** *if there is a chart $\varphi : (x_1,...,x_n)$ defined on it with*

1) $(1-\epsilon)\delta_{ij} \leq g_{ij} \leq (1+\epsilon)\delta_{ij}$ *in* $B(x,R)$ *as bilinear forms,*

2) $\sum_{|\beta|=1} \sup_{i,j=1,...,n,\ y\in B_x(R)} \left|\partial^\beta g_{ij}(y)\right| \leq \epsilon.$

Definition 2.2 *Let $x \in M$, we set $R'(x) = \sup\{R > 0 :: B(x,R)$ is ϵ conformal$\}$. We shall say that $R_\epsilon(x) := \min(1, R'(x))$ is the ϵ conformal radius at x.*

This is a special case of the harmonic radius of Hebey and Herzlich [12].

Of course for any $\epsilon > 0$, $x \in M$, taking $g_{ij}(x) = \delta_{ij}$ in a chart on $B(x,R)$ and the radius R small enough, the ball $B(x,R)$ is ϵ conformal.
We shall use the following lemma.

Lemma 2.3 *Let (M,g) be a riemannian manifold then with $R(x) = R_\epsilon(x)$ and $d(x,y)$ the riemannian distance on (M,g) :*

$$d(x,y) \leq \frac{1}{4}(R(x)+R(y)) \Rightarrow R(x) \leq 4R(y).$$

Proof.
Let $x,y \in M :: d(x,y) \leq \frac{1}{4}(R(x)+R(y))$ and suppose for instance that $R(x) \geq R(y)$. Then $y \in B(x,R(x)/2)$ hence the ball $B(y,R(x)/4) \subset B(x,\frac{3}{4}R(x))$. But by definition of $R(x)$, the ball $B(x,\frac{3}{4}R(x))$ is conformal and this implies that the ball $B(y,R(x)/4)$ is also conformal for exactly the same constants and the same chart ; this implies that $R(y) \geq R(x)/4$. ■

3 Vitali covering.

Lemma 3.1 *Let $\mathcal{F}$ be a collection of balls $\{B(x,r(x))\}$ in a metric space, with $\forall B(x,r(x)) \in \mathcal{F}$, $0 < r(x) \leq R$. There exists a disjoint subcollection $\mathcal{G}$ of $\mathcal{F}$ with the following property :*
every ball B in $\mathcal{F}$ intersects a ball C in $\mathcal{G}$ and $B \subset 5C$.

This is a well known lemma, see for instance [7], section 1.5.1.

So fix $\epsilon > 0$ and let $\forall x \in M$, $r(x) := R_\epsilon(x)/120$, where $R_\epsilon(x)$ is the conformal radius at x, we built a Vitali covering with the collection $\mathcal{F} := \{B(x,r(x))\}_{x\in M}$. So lemma 3.1 gives a disjoint subcollection $\mathcal{G}$ such that every ball B in $\mathcal{F}$ intersects a ball C in $\mathcal{G}$ and we have $B \subset 5C$. We set $\mathcal{G}' := \{x_j \in M :: B(x_j,r(x_j)) \in \mathcal{G}\}$ and $\mathcal{C}_\epsilon := \{B(x,5r(x)),\ x \in \mathcal{G}'\}$: we shall call $\mathcal{C}_\epsilon$ the ϵ conformal covering of (M,g).
Then we have :

Proposition 3.2 *Let (M,g) be a riemannian manifold, then the overlap of the ϵ conformal covering $\mathcal{C}_\epsilon$ is less than $T = \frac{(1+\epsilon)^{n/2}}{(1-\epsilon)^{n/2}}(120)^n$, i.e.*

$\forall x \in M,\ x \in B(y, 5r(y))$ *where* $B(y, r(y)) \in \mathcal{G}$ *for at most* T *such balls.*

Proof.

Let $B_j := B(x_j, r(x_j)) \in \mathcal{G}$ and suppose that $x \in \bigcap_{j=1}^{k} B(x_j, 5r(x_j))$. Then we have

$\forall j = 1, ..., k,\ d(x, x_j) \leq 5r(x_j)$

hence

$$d(x_j, x_l) \leq d(x_j, x) + d(x, x_l) \leq 5(r(x_j) + r(x_l)) \leq \frac{1}{4}(R(x_j) + R(x_l)) \Rightarrow R(x_j) \leq 4R(x_l).$$

So we get

$\forall j, l = 1, ..., k,\ r(x_j) \leq 4r(x_l)$.

Now the ball $B(x_j, 5r(x_j) + 5r(x_l))$ contains x_l hence the ball $B(x_j, 5r(x_j) + 6r(x_l))$ contains the ball $B(x_l, r(x_l))$. But, because $r(x_l) \leq 4r(x_j)$, we get

$B(x_j, 5r(x_j) + 6{\times}4r(x_j)) = B(x_j, r(x_j)(5 + 6{\times}4)) \supset B(x_l, r(x_l))$.

The balls in $\mathcal{G}$ being disjoint, we get

$$\sum_{j=1}^{k} \mathrm{Vol}(B_l) \leq \mathrm{Vol}(B(x_j, (5 + 6{\times}4)r(x_j))).$$

The Lebesgue measure read in the chart φ and the canonical measure dv_g on $B(x, R_\epsilon(x))$ are equivalent ; precisely because of condition 1) in the conformal radius definition, we get that :

$(1-\epsilon)^n \leq |\det g| \leq (1+\epsilon)^n$,

and the measure dv_g read in the chart φ is $dv_g = \sqrt{|\det g_{ij}|}d\xi$, where $d\xi$ is the Lebesgue measure in $\mathbb{R}^n$. In particular :

$\forall x \in M,\ \mathrm{Vol}(B(x,\ R_\epsilon(x))) \leq (1+\epsilon)^{n/2}\nu_n R^n$,

where ν_n is the euclidean volume of the unit ball in $\mathbb{R}^n$.

Now because $R(x_j)$ is the conformal radius and $4(5 + 6{\times}4)r(x_j) < R(x_j)$, we have

$\mathrm{Vol}(B(x_j, (5 + 6{\times}4)r(x_j))) \leq (1+\epsilon)^{n/2}v_n(5 + 6{\times}4)^n r(x_j)^n$.

On the other hand we have also

$\mathrm{Vol}(B_l) \geq v_n(1-\epsilon)^{n/2}r(x_l)^n \geq v_n(1-\epsilon)^{n/2}4^{-n}r(x_j)^n$,

hence

$$\sum_{j=1}^{k} (1-\epsilon)^{n/2}4^{-n}r(x_j)^n \leq (1+\epsilon)^{n/2}(5 + 6{\times}4)^n r(x_j)^n,$$

so finally

$$k \leq \frac{(1+\epsilon)^{n/2}}{(1-\epsilon)^{n/2}}((5 + 6{\times}4)4)^n,$$

which means that $T \leq \frac{(1+\epsilon)^{n/2}}{(1-\epsilon)^{n/2}}(120)^n$. ■

4 Sobolev spaces.

We have to define the Sobolev spaces in our setting, following E. Hebey [11], p. 10.

First define the covariant derivatives by $(\nabla u)_j := \partial_j u$ in local coordinates, while the components of $\nabla^2 u$ are given by

$$(\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma_{ij}^k \partial_k u,$$

with the convention that we sum over repeated index. The Christoffel Γ_{ij}^k verify [3] :

$$\Gamma_{ij}^k = \frac{1}{2} g^{il} (\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}).$$

If $k \in \mathbb{N}$ and $r \geq 1$ are given, we denote by $\mathcal{C}_k^r(M)$ the space of smooth functions $u \in \mathcal{C}^\infty(M)$ such that $\left|\nabla^j u\right| \in L^r(M)$ for $j = 0, ..., k$. Hence

$$\mathcal{C}_k^r(M) := \{u \in \mathcal{C}^\infty(M),\ \forall j = 0, ..., k,\ \int_M \left|\nabla^j u\right|^r dv_g < \infty\}$$

Now we have [11]

Definition 4.1 *The Sobolev space $W^{k,r}(M)$ is the completion of $\mathcal{C}_k^r(M)$ with respect to the norm :*

$$\|u\|_{W^{k,r}(M)} = \sum_{j=0}^k \left(\int_M \left|\nabla^j u\right|^r dv_g \right)^{1/r}.$$

We shall be interested only by $k \leq 2$ and we extend in a natural way this definition to the case of p forms.

Definition 4.2 *We shall define the Sobolev exponents $S_k(r)$ by* $\frac{1}{S_k(r)} := \frac{1}{r} - \frac{k}{n}$.

The k th Sobolev embedding is true if we have

$$\forall u \in W^{k,r}(M),\ u \in L^{S_k(r)}(M).$$

This is the case in $\mathbb{R}^n$, or if M is compact, or if M has a Ricci curvature bounded from below and $\inf_{x \in M} v_g(B_x(1)) \geq \delta > 0$, due to Varopoulos [19], see [11] theorem 3.14, p. 31.

5 Local estimates for the laplacian.

Lemma 5.1 *Let U be a domain in $\mathbb{R}^n$ and suppose that $D = \Delta + A$, where Δ is the standard laplacian in U acting on p forms and A is a second order partial differential (system) operator such that :*

$$\forall c > 0,\ \exists V \subset U,\ \forall u \in W_p^{2,r}(V),\ \|Au\|_{L^r(V)} \leq c(\|\Delta u\|_{L^r(V)} + \|\nabla u\|_{L^r(V)}).$$

Then there is a $V \subset U$ and a $C > 0$ depending only on n and r such that :

$$\forall \gamma \text{ a } p \text{ form in } L_p^r(V),\ \exists v \in W_p^{2,r}(V) :: Dv = \gamma \text{ and } \|v\|_{W^{2,r}} \leq C\|\gamma\|_{L^r(V)},$$

and the constant C depending only on n and r.

Moreover there is a bounded linear operator $T : L_p^r(V) \to W_p^{2,r}(V)$ such that $v = T\gamma$.

Proof.

We know that Δ operates component-wise on the p form $\gamma \in L_p^r(U)$, so we have

$$\forall \gamma \in L_p^r(U),\ \exists v_0 \in W^{2,r}(U) :: \Delta v_0 = \gamma,\ \|v_0\|_{W^{2,r}(U)} \leq C\|\gamma\|_{L^r(U)},$$

simply setting the component of v_0 to be the Newtonian potential of the corresponding component of γ in U, these non trivial estimates coming from Gilbarg and Trudinger [8], Th 9.9, p. 230 and the constant $C = C(n, r)$ depends only on n and r.

Clearly v_0 is linear in γ.

We start with $\gamma \in L^r_p(U)$ and we solve Δ :

$$\exists v_0 \in W^{2,r}(U) :: \Delta v_0 = \gamma,\ \|v_0\|_{W^{2,r}(U)} \leq C\|\gamma\|_{L^r(U)}\ ;$$

so we get $\Delta v_0 + Av_0 = \gamma + \gamma_1$, with

$$\gamma_1 = Av_0 \Rightarrow \|\gamma_1\|_{L^r(U)} \leq c(\|\Delta_{\mathbb{R}} v_0\|_{L^r(U)} + \|\nabla v_0\|_{L^r(U)}).$$

Because $\|v_0\|_{W^{2,r}(U)} \leq C\|\gamma\|_{L^r(U)}$, we have $\|v_0\|_{W^{1,r}(U)} \leq C\|\gamma\|_{L^r(U)}$, with the same constant C hence $\|\nabla v_0\|_{L^r(U)} \leq C\|\gamma\|_{L^r(U)}$. So :

$$\|\gamma_1\|_{L^r(U)} \leq c(\|\Delta v_0\|_{L^r(U)} + \|\nabla v_0\|_{L^r(U)}) \leq c(1+C)\|\gamma\|_{L^r(U)}.$$

Set $\eta := c(1+C)$, then we get $\|\gamma_1\|_{L^r(U)} \leq \eta\|\gamma\|_{L^r(U)}$.

We solve Δ again, still linearly in γ_1,

$$\exists v_1 \in W^{2,r}_p(U) :: \Delta v_1 = \gamma_1,\ \|v_1\|_{W^{2,r}(U)} \leq C\|\gamma_1\|_{L^r(U)} = C\eta\|\gamma\|_{L^r(U)},$$

and we set

$$\gamma_2 := Av_1 \Rightarrow \|\gamma_2\|_{L^r(U)} \leq c(\|\Delta v_1\|_{L^r(U)} + \|\nabla v_1\|_{L^r(U)}) = \eta\|\gamma_1\|_{L^r(U)} \leq \eta^2\|\gamma\|_{L^r(U)}.$$

And by induction :

$$\forall k \in \mathbb{N},\ \gamma_k := Av_{k-1} \Rightarrow \|\gamma_k\|_{L^r(U)} \leq \eta\|\gamma_{k-1}\|_{L^r(U)} \leq \eta^k\|\gamma\|_{L^r(U)}.$$

and

$$\exists v_k \in W^{2,r}_p(U) :: \Delta_{\mathbb{R}} v_k = \gamma_k,\ \|v_k\|_{W^{2,r}(U)} \leq C\|\gamma_k\|_{L^r(B)} \leq C\eta^k\|\gamma\|_{L^r(B)}.$$

Now we take c such that $\eta = c(1+C) = 1/2$ and the associated open set $V \subset U$ and we set

$$v := \sum_{j\in\mathbb{N}} (-1)^j v_j,$$

this series converges in norm $W^{2,r}_p(V)$ and we have

$$Dv = \Delta v + Av = \textstyle\sum_{j\in\mathbb{N}} (-1)^j(\Delta v_j + Av_j) = \gamma,$$

the last series converging in $L^r_p(V)$.

All the steps are linear, hence we proved the lemma. ■

Lemma 5.2 *For $x \in M,\ \epsilon > 0$, we suppose that we have a ϵ conformal ball $B_x(R)$.*
Then there is a $\epsilon_0 > 0$, hence a $R > 0$, and a constant C depending only on $n = \dim_{\mathbb{R}} M,\ r$ and ϵ_0 such that :

$$\forall \omega \in L^r(B_x(R)),\ \exists u \in W^{2,r}(B_x(R)) :: \Delta u = \omega,\ \|u\|_{W^{2,r}(B_x(R))} \leq C\|\omega\|_{L^r(B_x(R))}.$$

Moreover u is linear in ω.
We also have a Calderon Zygmund inequality : there are constants c_1, c_2 depending only on $n = \dim_{\mathbb{R}} M,\ r$ and ϵ_0 such that

$$R' = R/2,\ \forall u \in W^{2,r}(B_x(R)),\ \|u\|_{W^{2,r}(B_x(R'))} \leq c_1\|u\|_{L^r(B_x(R))} + c_2\|\Delta u\|_{L^r(B_x(R))}.$$

Proof.

For $x \in M$ we take $\epsilon > 0$, the ϵ conformal ball $B_x(R)$ and we take the chart φ : $(x_1, ..., x_n)$ such that

1) $(1-\epsilon)\delta_{ij} \leq g_{ij} \leq (1+\epsilon)\delta_{ij}$ in $B_x(R)$ as bilinear forms,

2) $\displaystyle\sum_{|\beta|=1} \sup_{i,j=1,...,n,\ y\in B_x(R)} \left|\partial^\beta g_{ij}(y)\right| \leq \epsilon.$

Of course the operator d on p forms is local and so is d^* as a first order differential operator.

So the Hodge laplacian Δ_φ read by φ in $U := \varphi(B_x(R))$ is still a second order partial differential system of operators and with $\Delta_{\mathbb{R}}$ the usual laplacian in $\mathbb{R}^n$ acting on forms in U, we set :

$$A\omega_\varphi := \Delta_\varphi\omega_\varphi - \Delta_{\mathbb{R}}\omega_\varphi,$$

where ω_φ is the p form ω read in the chart $(B_x(R), \varphi)$ and A is a matrix valued second order operator with $\mathcal{C}^\infty$ smooth coefficients such that $A := \Delta_\varphi - \Delta_{\mathbb{R}} :\ W^{2,r}(U) \to L^r(U)$.

This difference A is controlled by the derivatives of the metric tensor up to order 1 :
for instance for function f we have in the chart φ :

$$\Delta_\varphi f = \frac{1}{\sqrt{\det(g_{ij})}}\partial_i(g^{ij}\sqrt{\det(g_{ij})}\partial_j f) = g^{ij}\partial^2_{ij}f + Y_0 f,$$

where Y_0 is a first order differential operator depending on g and its first derivatives ;
more generally for a k form u, still in the chart φ, [18] formula 21.23, p. 169, gives

$$\Delta_\varphi u = g^{ij}(x)\partial^2_{ij}u + Y_k u,$$

where Y_k is a first order differential operator.
So Δ_φ depends on first order derivatives of g hence the difference $A := \Delta - \Delta_{\mathbb{R}}$, where $\Delta_{\mathbb{R}}u(y) = \delta^{ij}\partial^2_{ij}u(y)$, is controlled by the first order derivatives of g.
So we have

$$|A(u)(y)| \leq \left|(g^{ij}(y) - \delta^{ij})\partial^2_{ij}u(y)\right| + |E(u)(y)|\,, \tag{5.1}$$

where E is a first order partial differential operator whose coefficients depend on the first order derivatives of g, and are 0 for $y = x$. So

$$\forall y \in B_x(R),\ |E(u)(y)| \leq \eta\,|\nabla u(y)|$$

where η is a function of the metric g and ∇g only, hence, because $\left|g^{ij}(x) - \delta^{ij}\right| \leq 2\epsilon$, η may be chosen to depend on $\epsilon > 0$ only and $\eta(0) = 0$.
Hence, integrating (5.1), we get

$$\|Au\|_{L^r(B_x(R))} \leq \|\nabla g\|_{L^\infty(B_x(R))}\|\Delta_{\mathbb{R}}u\|_{L^r(B_x(R))} + \eta(\epsilon)\|\nabla u\|_{L^r(B_x(R))}.$$

So there is a $0 \leq c(\epsilon),\ c(0) = 0$ and c continuous at 0, such that $\|A\| \leq c(\epsilon)(\|\Delta_{\mathbb{R}}\| + \|\nabla\|)$, the norms being the norms as operator $W^{2,r}_p(B_x(R)) \to L^r_p(B_x(R))$.

We can apply lemma 5.1 with $U := \varphi(B_x(R)),\ D := \Delta_\varphi,\ \Delta = \Delta_{\mathbb{R}}$ to get that there is a positive ϵ_0 such that :

$$\forall \omega_\varphi \in L^r_p(U),\ \exists u_\varphi \in W^{2,r}_p(U) :: \Delta_\varphi u_\varphi = \omega_\varphi,\ \|u_\varphi\|_{W^{2,r}(U)} \leq C\|\omega_\varphi\|_{L^r(U)},$$

the constant C depending only on n and r. Moreover u_φ is linear in ω_φ by lemma 5.1.
Now we fix $\epsilon = \epsilon_0$.
The Lebesgue measure on U and the canonical measure dv_g on $B_x(R)$ are equivalent ; precisely because of condition 1) we get that :

$$(1-\epsilon)^n \leq |\det g| \leq (1+\epsilon)^n,$$

and the measure dv_g read in the chart φ is $dv_g = \sqrt{|\det g_{ij}|}d\xi$, where $d\xi$ is the Lebesgue measure in $\mathbb{R}^n$. So the Lebesgue estimates and the Sobolev estimates on U are valid in $B_x(R)$ up to a constant depending only on n and ϵ. In particular :

$$\forall x \in M,\ \mathrm{Vol}(B_x(R)) \leq (1+\epsilon)^{n/2}\nu_n R^n, \tag{5.2}$$

where ν_n is the euclidean volume of the unit ball in $\mathbb{R}^n$.

So going back to the manifold M we get the right estimates :

$$\exists u \in W^{2,r}_p(B_x(R)) :: \Delta_M u = \omega \ in\ B_x(R),\ \|u\|_{W^{2,r}(B_x(R))} \leq C\|\omega\|_{L^r(B_x(R))},$$

where C depends only on $n,\ r$ and ϵ_0 and u is linear in ω.

For the Calderon Zygmund inequality (CZI) we proceed the same way.
By the condition 2) in the lemma, we have that the Christoffel Γ_{ij}^k can be made as small as we wish, so the second covariant derivatives are uniformly near to the usual second derivatives in the chart (U, φ).

By the classical CZI for the usual laplacian $\Delta_{\mathbb{R}}$ in $\mathbb{R}^n$, [8], Th. 9.11, p. 235, we have, with a $R' = R/2,\ U' = \varphi(B_x(R')) \Subset U$,

$$\forall u \in W_p^{2,r}(U),\ \|u\|_{W^{2,r}(U')} \leq c_1\|u\|_{L^r(U)} + c_2'\|\Delta_{\mathbb{R}}u\|_{L^r(U)}$$

because the laplacian on forms in $\mathbb{R}^n$ is diagonal. Because $\Delta_\varphi = \Delta_{\mathbb{R}} + A$ we get

$$\forall u \in W_p^{2,r}(U),\ \|\Delta_{\mathbb{R}}u\|_{L^r(U)} \leq \|\Delta_\varphi u - Au\|_{L^r(U)} \leq \|\Delta_\varphi\|_{L^r(U)} + \|Au\|_{L^r(U)} \leq \|\Delta_\varphi u\|_{L^r(U)} + c\|\Delta_{\mathbb{R}}u\|_{L^r(U)}.$$

So

$$\forall u \in W_p^{2,r}(U),\ (1-c)\|\Delta_{\mathbb{R}}u\|_{L^r(U)} \leq \|\Delta_\varphi u\|_{L^r(U)}$$

and finally

$$\forall u \in W_p^{2,r}(U),\ \|u\|_{W^{2,r}(U')} \leq c_1\|u\|_{L^r(U)} + \frac{c_2'}{1-c}\|\Delta_\varphi u\|_{L^r(U)}.$$

It remains to set $c_2 := \dfrac{c_2'}{1-c}$ to get the CZI for Δ_φ. So passing back to M, we get the CZ interior local inequality on $B_x(R') \subset B_x(R) \subset M$ because the Sobolev are the same up to a constant depending only on $n,\ r$ and ϵ. ■

6 The raising steps method.

From now on we take $\epsilon = \epsilon_0$ with ϵ_0 given by lemma 5.2 and we take the ϵ_0 conformal radius and the Vitali covering $\{B(x_j, 5r(x_j))\}_{j\in\mathbb{N}}$ associated to it.

Definition 6.1 *A weight relative to the covering $\{B(x_j, 5r(x_j))\}_{j\in\mathbb{N}}$ is a function $w(x) > 0$ on M such that :*

there are two constants $0 < c_{iw} \leq 1 \leq c_{sw}$ such that, setting

$$\forall j \in \mathbb{N},\ B_j := B(x_j, 5r(x_j)),\ w_j := \frac{1}{v_g(B_j)}\int_{B_j} w(x)dv_g(x),$$

we have $\forall j \in \mathbb{N},\ \forall x \in B_j,\ c_{iw}w_j \leq w(x) \leq c_{sw}w_j$. By smoothing w if necessary, we shall also suppose that $w \in \mathcal{C}^\infty(M)$.

This means that w varies slowly on B_j.

So let $w(x) > 0$ be a weight we say that $\omega \in L_p^r(M, w)$, if :

$$\|\omega\|_{L^r(M,w)}^r := \int_M |\omega(x)|^r\, w(x)dv_g(x) < \infty.$$

6.1 Weighted Calderon Zygmund inequalities.

The aim is to generalise the Calderon Zygmund inequalities done in [10] by adding weights.

Theorem 6.2 *Let w be a weight relative to the covering $\{B(x_j, 5r(x_j))\}_{j\in\mathbb{N}}$ and $u \in L_p^r(M, w^r)$ such that $\Delta u \in L_p^r(M, w)$ then there are constants c_1, c_2 depending only on $n = \dim_{\mathbb{R}} M,\ r$ and ϵ_0 such that if $u,\ \Delta u \in L_p^r(M, w)$ then $u \in W_p^{2,r}(M, w)$ and :*

$$\|u\|_{W^{2,r}(M,w)} \leq c_1\|u\|_{L^r(M,w)} + c_2\|\Delta u\|_{L^r(M,w)}.$$
Moreover we have for $t = S_2(r)$ *that* $u \in L^t_p(M,w)$ *with* $\|u\|_{L^t(M,w)} \leq c\|u\|_{W^{2,r}(M,w)}$.

Proof.
Let $u \in L^r(M,w),\ \Delta u \in L^r(M,w)$. Set $R_j := 5r(x_j),\ B_j := B(x_j,R_j),\ B'_j = B(x_j,2R_j)$ and apply lemma 5.2 to get :
there are constants c_1, c_2 depending only on $n = \dim_{\mathbb{R}} M,\ r$ and ϵ_0 such that
$$\|u\|_{W^{2,r}(B_j)} \leq c_1\|u\|_{L^r(B'_j)} + c_2\|\Delta u\|_{L^r(B'_j)}.$$
We have
$$\|u\|^r_{W^{2,r}(M,w)} \leq c_{ws}\sum_{j\in\mathbb{N}} w_j\|u\|^r_{W^{2,r}(B_j)}\ ;$$
to see this we just take the L^r norm of covariant derivatives of u :
$$\int_M \left|\nabla^k u(x)\right|^r w(x)dv_g(x) \leq c_{ws}\sum_{j\in\mathbb{N}}\int_{B_j} w_j\left|\nabla^k u(x)\right|^r dv_g(x).$$
But on B_j we have $\left|\nabla^k u(x)\right| \leq C\left|\partial^k u(x)\right|$ where the constant depends only on $n,\ r$ and ϵ_0 and $\partial^k u$ are the k th usual derivatives in the local coordinates, $k \leq 2$.
With the CZ part of lemma 5.2 we get :
$$\int_{B_j}\left|\partial^k u(x)\right|^r dv_g(x))^{1/r} \leq c_1\|u\|_{L^r(B'_j)} + c_2\|\Delta u\|_{L^r(B'_j)},$$
hence
$$\left\|\nabla^k u\right\|^r_{L^r(M,w)} \leq Cc_{ws}c_1\sum_{j\in\mathbb{N}} w_j\|u\|^r_{L^r(B'_j)} + Cc_{ws}c_2\sum_{j\in\mathbb{N}} w_j\|\Delta u\|^r_{L^r(B'_j)}.$$
Because the overlap is less that T we get
$$\sum_{j\in\mathbb{N}} w_j\|u\|^r_{L^r(B'_j)} \leq c_{wi}^{-1}T\|u\|^r_{L^r(M,w)}$$
and
$$\sum_{j\in\mathbb{N}} w_j\|\Delta u\|^r_{L^r(B'_j)} \leq c_{wi}^{-1}T\|\Delta u\|^r_{L^r(M,w)}.$$
So finally, with $C_j = c_j c_{wi}^{-1},\ j = 1,2$,
$$\left\|\nabla^k u\right\|_{L^r(M,w)} \leq C_1 T^{1/r}\|u\|_{L^r(M,w)} + C_2 T^{1/r}\|\Delta u\|_{L^r(M,w)}.$$
Taking successively $k = 0,1,2$ we prove the first part of the theorem.
To prove the "moreover" we follow the same line : because $\forall j \in \mathbb{N},\ u \in W^{2,r}_p(B_j)$ and using the Sobolev embeddings, true in B_j, we have, with $t = S_2(r)$,
$$u \in L^t_p(B_j),\ \|u\|_{L^t(B_j)} \leq c\|u\|_{W^{2,r}(B_j)}.$$
Now
$$\|u\|^t_{L^t(M,w)} \leq c_{ws}\sum_{j\in\mathbb{N}} w_j\|u\|^t_{L^t(B_j)}\ ;$$
hence, because the overlap is smaller than T
$$\sum_{j\in\mathbb{N}} w_j\|u\|^t_{L^t(B_j)} \leq c_{wi}^{-1}T\|u\|^t_{L^t(M,w)},$$
so $\|u\|^t_{L^t(M,w)} \leq c_{ws}c_{wi}^{-1}T\|u\|^t_{L^t(M,w)}$ which proves the theorem. ■

Corollary 6.3 *Let* $h \in \mathcal{H}^r_p(M,w)$ *then* $\forall s \geq r,\ h \in \mathcal{H}^s_p(M,w)$.

Proof.

Let $h \in \mathcal{H}^r_p(M,w)$ then $\Delta h = 0 \in L^r_p(M,w)$ then theorem 6.2 gives $h \in W^{2,r}_p(M,w)$ and $h \in L^{t_1}_p(M,w)$ with $t_1 = S_2(r)$. By interpolation between $L^r_p(M,w)$ and $L^{t_1}_p(M,w)$ we get that $h \in L^{t'}_p(M,w)$ for any $t' \in [r,t_1]$. If $t_1 \geq s$ we are done, if not we have that $h \in \mathcal{H}^{t_1}_p(M,w)$ and we play the same game with t_1 in place of r and this proves the corollary by induction. ■

6.2 The raising steps method.

With $R(x)$ the ϵ_0 conformal radius, we shall prove now :

Theorem 6.4 *(Raising Steps Method) Let (M,g) be a complete riemannian manifold and take w a weight relative to the Vitali covering $\{B(x_j,5r(x_j))\}_{j\in\mathbb{N}}$.*
For any $r \leq 2,\ \forall s \geq r$, there is a $k \in \mathbb{N}$ such that, with $w_0(x) := w(x)R(x)^{-k}$,
$$\forall \omega \in L^r(M,w_0^r),\ \exists v \in L^r(M,w^r) \cap L^{s_1}(M,w^{s_1}) :: \Delta v = \omega + \tilde{\omega}$$
with $s_1 = S_2(r),\ t_k = S_{k-1}(r) \geq s,\ \tilde{\omega} \in L^s(M,w^s)$ and control of the norms.
Moreover v and $\tilde{\omega}$ are linear in ω.
If ω is of compact support, so are v and $\tilde{\omega}$.

Proof.
Set $R_j := 5r(x_j),\ B_j := B(x_j,R_j)$ and apply lemma 5.2 to get :
$$\exists u_j \in W^{2,r}(B_j) :: \Delta u_j = \omega,\ \|u_j\|_{W^{2,r}(B_j)} \leq c_l \|\omega\|_{L^r(B_j)},$$
with u_j linear in $\omega_{|B_j}$.
So by Sobolev embeddings, true in B_j, we get, with $s = S_2(r)$,
$$u_j \in L^s(B_j),\ \|u_j\|_{L^s(B_j)} \leq c_l \|\omega\|_{L^r(B_j)}.$$
Hence, because $u_j \in L^r(B_j)$, we have by interpolation [4], that $\forall s' \in [r,s],\ u_j \in L^{s'}(B_j)$ with control of the norms.

Let $\{\chi_j\}_{j\in\mathbb{N}}$ be a partition of unity associated to $\{B(x_j,R_j)\}_{j\in\mathbb{N}}$ then we set
$$v_0 := \sum_{j\in\mathbb{N}} \chi_j u_j.$$
Because the u_j are linear in $\omega_{|B_j},\ v_0$ is linear in ω.
We have
$$\|\chi_j u_j\|_{L^s(B_j)} \leq c_l \|\omega\|_{L^r(B_j)},$$
and multiplying by w_j
$$w_j \|\chi_j u_j\|_{L^s(B_j)} \leq w_j c_l \|\omega\|_{L^r(B_j)}.$$
So
$$\|v_0\|^s_{L^s(M,w^s)} \leq c^s_{sw} \sum_{j\in\mathbb{N}} w^s_j \|\chi_j u_j\|^s_{L^s(B_j)} \leq c^s_{sw} \sum_{j\in\mathbb{N}} w^s_j \|u_j\|^s_{L^s(B_j)}$$
$$\leq c^s_{sw} c^s_l \sum_{j\in\mathbb{N}} w^s_j \|\omega\|^s_{L^r(B_j)} \leq c^s_{sw} c^s_l \left(\sum_{j\in\mathbb{N}} w^r_j \|\omega\|^r_{L^r(B_j)} \right)^{s/r}$$
because $s \geq r$.
Because the overlap is less that T we get
$$\|v_0\|^s_{L^s(M,w^s)} \leq c^s_{sw} c^s_l \left(\sum_{j\in\mathbb{N}} w^r_j \|\omega\|^r_{L^r(B_j)} \right)^{s/r} \leq c^{-1}_{iw} c^s_{sw} c^s_l (T \|\omega\|^r_{L^r(M,w^r)})^{s/r},$$
so

$$\|v_0\|_{L^s(M,w^s)} \leq c_{iw}^{-1} c_{sw} c_l T^{1/r} \|\omega\|_{L^r(M,w^r)}.$$

We also have $v_0 \in L^r(M, w^r)$ because $u_j \in W^{2,r}(B_j) \Rightarrow u_j \in L^r(B_j)$ as well, this means that $v_0 w \in L^r(M) \cap L^s(M)$ hence by interpolation we have that

$v_0 w \in L^{t'}(M) \Rightarrow v_0 \in L^{t'}(M, w^{t'})$ for any $t' \in [r, s]$ with the same control of the norms.

If ω is of compact support, then we can cover Supp ω by a finite set $\{B_j\}_{j=1,...,N}$ and then add a layer $\{B_j\}_{j=N+1,...,N_0}$ to cover $\overline{\bigcup_{j=1}^{N} B_j}$. Then by linearity we get $\forall j = N+1, ..., N_0,\ u_j = 0$ and setting now $v_0 := \sum_{j=1}^{N_0} \chi_j u_j$ we can extend v_0 as 0 outside $\bigcup_{j=1}^{N_0} B_j$ hence we get that v_0 is compactly supported.

Now, for $\chi \in \mathcal{D}(M)$ and $u \in W_p^{2,r}(M)$, set $B(\chi, u) := \Delta(\chi u) - \chi \Delta(u)$. Exactly as for Proposition G.III.6 in [3] we have in an exponential chart at a point $x \in M$,

$$u = \sum_{J,|J|=p} u_J dx^J,\ g^{ij}(x) = \delta_{ij} \text{ and the basis } \{\frac{\partial}{\partial x_j}\}_{j=1,...,n} \text{ is orthogonal.}$$

In this chart and *at the point* x we have that the laplacian is diagonal so

$$\Delta u(x) = \sum_{J,|J|=p} \frac{\partial^2 u_J}{\partial x_j^2}(x) dx^J$$

hence, for any $x \in M$,

$$B(\chi, u)(x) = \Delta\chi(x) u(x) - 2 \sum_{J,\ |J|=p} (\sum_{j=1}^{n} \frac{\partial u_J}{\partial x_j} \frac{\partial \chi}{\partial x_j}) dx^J.$$

So we get

$$|B(\chi, u)| \leq |\Delta\chi|\,|u| + 2\,|\nabla\chi|\,|\nabla u|\,.$$

Now consider Δv_0, we get

$$\Delta v_0 = \sum_{j\in\mathbb{N}} \Delta(\chi_j u_j) = \sum_{j\in\mathbb{N}} \chi_j \Delta u_j + \sum_{j\in\mathbb{N}} B(\chi_j, u_j) = \omega + \omega_1,$$

with $\omega_1 := \sum_{j\in\mathbb{N}} B(\chi_j, u_j)$.

Clearly Δv_0 is linear in ω so is ω_1.

We have $|\nabla\chi_j| \lesssim \frac{1}{R_j}$ and $|\Delta\chi_j| \lesssim \frac{1}{R_j^2}$. We also have

$$\|\nabla u_j\|_{L^t(B_j)} \leq c_l \|\omega\|_{L^r(B_j)} \text{ and } \|u_j\|_{L^s(B_j)} \leq c_l \|\omega\|_{L^r(B_j)},$$

with $t = S_1(r),\ s = S_2(r)$. Let $q \in [r, t]$.

By Young's inequality we get

$$\|u_j\|_{L^t(B_j)} = \left\|\mathbb{1}_{B_j} u_j\right\|_{L^t(B_j)} \leq \|u_j\|_{L^s(B_j)} \left\|\mathbb{1}_{B_j}\right\|_{L^n(B_j)} = \|u_j\|_{L^t(B_j)} R_j,$$

because $\mathrm{Vol}(B_x(R)) \leq (1+\epsilon)^{n/2} \nu_n R^n$ by equation (5.2), we get, with $c_v = c_l \sqrt[n]{\nu_n (1+\epsilon)^{n/2}}$,

$$\|u_j\|_{L^t(B_j)} \leq |B_j|^{1/n}\ \|u_j\|_{L^s(B_j)} \leq c_v R_j \|\omega\|_{L^r(B_j)}.$$

Hence a fortiori $\|u_j\|_{L^q(B_j)} \leq c_v R_j \|\omega\|_{L^r(B_j)}$.

So we get

$$\|B(\chi_j,u_j)\|_{L^q(B_j)}\leq\|\nabla\chi_j\|_\infty\|\nabla u_j\|_{L^q(B_j)}+\|\Delta\chi_j\|_\infty\|u_j\|_{L^q(B_j)}\leq c_vR_j^{-1}\|\omega\|_{L^r(B_j)}.$$
Multiplying by w_j we get
$$w_j\|B(\chi_j,u_j)\|_{L^q(B_j)}\leq R_j^{-1}w_jc_v\|\omega\|_{L^r(B_j)}.$$
We take the power q and we add
$$\sum_{j\in\mathbb{N}}w_j^q\|B(\chi_j,u_j)\|^q_{L^q(B_j)}\leq c_v^q\sum_{j\in\mathbb{N}}R_j^{-q}w_j^q\|\omega\|^q_{L^r(B_j)}.$$
But
$$\sum_{j\in\mathbb{N}}a_j^q\leq(\sum_{j\in\mathbb{N}}a_j^r)^{q/r}\text{ because }q\geq r,\text{ hence}$$
$$\sum_{j\in\mathbb{N}}R_j^{-q}w_j^q\|\omega\|^q_{L^r(B_j)}\leq c_v^q(\sum_{j\in\mathbb{N}}R_j^{-r}w_j^r\|\omega\|^r_{L^r(B_j)})^{q/r}.$$
By lemma 2.3 we have
$$\forall x\in B_j,\ d(x,x_j)<R_j=5r(x_j)\leq\frac{1}{4}R(x_j)\leq\frac{1}{4}(R(x_j)+R(x))\Rightarrow R(x)\leq 4R(x_j),$$
hence
$$\forall x\in B_j,\ R(x)\leq 4R_j\Rightarrow R_j^{-r}\leq 96^rR(x)^{-r},$$
so, because $r(x_j)=\dfrac{R(x_j)}{120}$ and $R_j=5r(x_j)$,
$$R_j^{-r}w_j^r\|\omega\|^r_{L^r(B_j)}\leq 96^rc_{sw}^r\int_{B_j}|\omega(x)|^r\,R(x)^{-r}w^r(x)dv_g(x).$$
Because the overlap is smaller than T by proposition 3.2 we get
$$\sum_{j\in\mathbb{N}}R_j^{-r}w_j^r\|\omega\|^r_{L^r(B_j)}\leq 96^rc_{sw}^rT\int_M R(x)^{-r}w(x)^r\,|\omega(x)|^r\,dv_g(x).$$
Now, with
$$\omega_1:=\sum_{j\in\mathbb{N}}B(\chi_j,u_j),$$
we have
$$\int_M w(x)^q\,|\omega_1(x)|^q\,dv_g(x)\leq c_{iw}^{-r}c_v^q\sum_{j\in\mathbb{N}}R_j^{-q}w_j^q\|B(\chi_j,u_j)\|^q_{L^q(B_j)}$$
hence
$$\left(\int_M w(x)^q\,|\omega_1(x)|^q\,dv_g(x)\right)^{1/q}\leq c_{iw}^{-1}c_{sw}c_v\left(T\int_M R(x)^{-r}w(x)^r\,|\omega(x)|^r\,dv_g(x)\right)^{1/r}.$$
Set $t_1=t=S_1(r)$, we have, with $w_1(x)=w(x),\ w_0(x)=w(x)R(x)^{-1},\ \forall q\in[r,t_1]$
$$\|\omega_1\|_{L^q(M,w_1^q)}\lesssim\|\omega\|_{L^r(M,w_0^r)}.$$
If ω is of compact support, we have seen that v_0 is also of compact support hence so is $\Delta v_0=\omega+\omega_1$. Which means that ω_1 is also of compact support.

Now we do the same game starting with ω_1 in place of ω and we get, with
$$s_2=S_2(t_1),\ t_2=S_1(t_1)=S_2(r),\ w_2(x)=w(x),\ w_1(x)=w(x)R(x)^{-1},\ w_0(x)=w(x)R(x)^{-2},$$
that
$$\exists v_1\in L^q(M,w_2^q)::\Delta v_1=\omega_1+\omega_2\text{ for any }q\in[r,s_1],$$
and
$$\omega_2\in L^q(M,w_2^q),\ q\in[r,t_2],\ \|\omega_2\|_{L^q(M,w_2^q)}\lesssim\|\omega_1\|_{L^{t_1}(M,w_1^{t_1})}\lesssim\|\omega\|_{L^r(M,w_0^r)}.$$
We keep the linearity of v_1 w.r.t. to ω_1 hence to ω. So ω_2 is still linear w.r.t. ω.

So by induction we have, with

$$t_k = S_k(r),\ w_k(x) := w(x),\ w_{k-1}(x) = w(x)R(x)^{-1}, ...,\ w_0(x) = w(x)R(x)^{-k},$$

and

$$j = 0, ...,\ k-1,\ v_j \in L^q(M, w_{j+1}^q),\ s_{j+1} = S_2(t_j), \text{ for any } q \in [r, s_{j+1}],$$

and

$$\omega_k \in L^q(M, w_k^q),\ q \in [r, t_k],\ \|\omega_k\|_{L^q(M,w_k^q)} \lesssim \cdots \lesssim \|\omega_1\|_{L^{t_1}(M,w_1^{t_1})} \lesssim \|\omega\|_{L^r(M,w_0^r)}. \tag{6.3}$$

Setting now $v := \sum_{j=0}^{k-1} (-1)^j v_j$ and $\tilde{\omega} := (-1)^k \omega_k$, we have that $\Delta v = \omega + \tilde{\omega}$ and

$$\forall q \in [r, s_1],\ v_j \in L^q(M, w_{j+1}^q),\ s_1 = S_2(t_1),\ w_{j+1} = w(x)R(x)^{(j+1-k)},$$

this implies, because $w_k = w \leq w_{j+1}$,

$$\forall q \in [r, s_1],\ v_j \in L^q(M, w^q),\ \|v_j\|_{L^q(M,w^q)} \leq c_l T^{1/r} \|\omega\|_{L^r(M,w_0^r)}.$$

So we have also for $v := \sum_{j=0}^{k-1} (-1)^j v_j$:

$$\forall q \in [r, s_1],\ v \in L^q(M, w^q),\ \|v\|_{L^q(M,w^q)} \leq k c_l T^{1/r} \|\omega\|_{L^r(M,w_0^r)}. \tag{6.4}$$

We cannot go beyond s_1 because of v_0.
And

$$\forall q \in [r, t_k],\ \tilde{\omega} \in L^q(M, w_k^q),\ \|\omega_k\|_{L^q(M,w_k^q)} \lesssim \|\omega\|_{L^r(M,w_0^r)}. \tag{6.5}$$

Clearly the linearity is kept along the induction.

If ω is of compact support, we have seen that v_0 and ω_1 also and by induction all the v_j and ω_j are also of compact support. ■

We shall refer to this theorem as RSM for short.

Remark 6.5 *We have, by inequalities (6.3), that $\forall q \in [r, t_k],\ \omega_k \in L^q(M, w_k^q)$. With the choice of $w \equiv 1$ we get that $w_k^q = R(x)^{-qk} \geq 1$ hence $\forall q \in [r, t_k],\ \omega_k \in L^q(M)$. This implies, with the notations of the RSM, that $\forall q \in [r, s],\ \tilde{\omega} \in L^q(M)$.*

Corollary 6.6 *Let (M, g) be a complete riemannian manifold and choose as a weight $w = 1$. For any $r \leq 2,\ \omega \in L^r(M, w_0^r),\ k \in \mathbb{N},\ w_0(x) := R(x)^{-k}$ and any sequence $\omega^l \in \mathcal{D}_p(M),\ \omega^l \to \omega$ in $L_p^r(M, w_0^r)$, we have two sequences :*

$$v^l \in \mathcal{D}_p(M),\ v^l \to v \text{ in } L^{s_1}(M) \text{ and } \tilde{\omega}^l \in \mathcal{D}_p(M),\ \tilde{\omega}^l \to \tilde{\omega} \text{ in } L^{t_k}(M)$$

such that $\Delta v^l = \omega^l + \tilde{\omega}^l$; $\Delta v = \omega + \tilde{\omega}$.
Moreover if $t_k \geq 2 > t_{k-1}$ then with $H : L_p^2(M) \to \mathcal{H}_p^2(M)$ the orthogonal projection, H extends boundedly from $L_p^r(M, w_0^r)$ to $\mathcal{H}_p^2(M)$.

Proof.
By theorem 6.4 we have that $v^l,\ \tilde{\omega}^l$ have compact support and by linearity with (6.4)

$$(v - v^l) \in L^r(M) \cap L^{s_1}(M),\ \left\|v - v^l\right\|_{L^{s_1}(M)} \leq k c_l T^{1/r} \left\|\omega - \omega^l\right\|_{L^r(M,w_0^r)}$$

so $\left\|v-v^l\right\|_{L^{s_1}(M,w^{s_1})}\to 0$ and the same way with (6.5) we get

$(\tilde{\omega}-\tilde{\omega}^l)\in L^{t_k}(M),\ \left\|\tilde{\omega}-\tilde{\omega}^l\right\|_{L^{t_k}(M)}\lesssim\left\|\omega-\omega^l\right\|_{L^r(M,w_0^r)}$

hence $\left\|\tilde{\omega}-\tilde{\omega}^l\right\|_{L^{t_k}(M)}\to 0$.

It remains the "moreover" to prove. So let $\omega\in L^r(M,w_0^r)$ and take a sequence $\omega^l\in\mathcal{D}_p(M),\ \omega^l\to\omega$ in $L^r_p(M,w_0^r)$, then H is well defined on $v^l,\ \Delta v^l,\ \omega^l$ and $\tilde{\omega}^l$ and we have

$\Delta v^l=\omega^l+\tilde{\omega}^l\Rightarrow H\Delta v^l=H\omega^l+H\tilde{\omega}^l.$

Take $t_k\geq 2,\ h\in\mathcal{H}^2_p(M)$ then $\left\langle H\Delta v^l,h\right\rangle_{L^2(M)}=\left\langle\Delta v^l,h\right\rangle_{L^2(M)}$ because $Hh=h$ and H is self adjoint. But because Δ is essentially self adjoint in $L^2(M)$ and v^l has compact support, we have

$\left\langle\Delta v^l,h\right\rangle_{L^2(M)}=\left\langle v^l,\Delta h\right\rangle_{L^2(M)}=0.$

So we have $\forall l\in\mathbb{N},\ H\Delta v^l=0$ and this implies

$\forall l\in\mathbb{N},\ H\omega^l+H\tilde{\omega}^l=0.$

Now $t_k\geq 2>t_{k-1}$ implies that we can choose, by (6.5), $\tilde{\omega}\in L^2_p(M)$ and the convergence $\left\|\tilde{\omega}-\tilde{\omega}^l\right\|_{L^2(M)}\to 0$. So, because H is bounded on $L^2_p(M)$, we get $H\tilde{\omega}^l\to H\tilde{\omega}$ in $L^2_p(M)$, and this means $H\omega^l\to -H\tilde{\omega}$ also in $L^2_p(M)$. So we define, for any sequence $\omega^l\in\mathcal{D}_p(M),\ \omega^l\to\omega$ in $L^r_p(M,w_0)$,

$H\omega:=\lim_{\ l\to\infty}H\omega^l,$

and we just proved that $H\omega^l$ converges in $L^2_p(M)$ to $-H\tilde{\omega}$, with $\tilde{\omega}$ given by the the Raising Steps Theorem. ■

7 Applications.

Lemma 7.1 *Let $t<2$, if the weight w is such that : $\gamma(w,t):=\int_M w^{\frac{2t}{2-t}}dv_g<\infty$, we have :*

$\omega\in L^2_p(M)\Rightarrow\omega\in L^t_p(M,w).$

Proof.

Young's inequality gives $\|fg\|_{L^t}\leq\|f\|_{L^2}\|g\|_{L^q}$ with $\dfrac{1}{t}=\dfrac{1}{2}+\dfrac{1}{q}$, so let $\omega\in L^2_p(M)$, then, with $t<2$, we get

$$\left(\int_M|\omega|^t\,w dv_g\right)^{1/t}\leq\left(\int_M|\omega|^2\,dv_g.\right)^{1/2}\left(\int_M w^{\frac{2t}{2-t}}dv_g\right)^{\frac{2-t}{2t}}.$$

So if the weight w is such that $\gamma(w,t):=\int_M w^{\frac{2t}{2-t}}dv_g<\infty$, we are done. ■

For instance take any origin $0\in M,\ M$ a complete riemannian manifold, and set $\rho(x):=d_g(0,x)$. We can choose a weight w, function of $\rho,\ w(x):=f(\rho(x))$, such that $\gamma(w,t)<\infty$, provided that $w(x)$ goes to 0 quickly enough at infinity.

Recall that $R(x)$ is the ϵ_0 conformal radius at $x\in M$.

Corollary 7.2 *Suppose that (M,g) is a complete riemannian manifold ; choose a weight $w\in L^\infty(M)$ verifying lemma 7.1 for $r<2,\ t=S_2(r)$, i.e. $\gamma(w,t)<\infty$, and suppose we have the condition (HL2,p).*

With $k::S_k(r)\geq 2$, setting $w_0(x):=R(x)^{-k}$, then for any $\omega\in L^r(M,w_0^r)$ verifying $H\omega=0$, for the orthogonal projection H defined in corollary 6.6, there is a $u\in L^r_p(M,w)\cap L^t_p(M,w)$, such that $\Delta u=\omega$.

Moreover the solution u is given linearly with respect to ω.

Proof.
We have $t_k := S_k(r) \geq 2$ and we use the assumption (HL2,p) :
it gives the existence of a bounded linear operator $L \ : \ L^2_p(M) \to L^2_p(M)$ such that
$\Delta Lg = g$, provided that $Hg = 0$,
by the spectral theorem, see, for instance, the proof of theorem 5.10, p. 698 in Bueler [5].
Take $\omega \in L^r(M, w_0^r)$, the theorem 6.4 gives linear operators
$T \ : \ L^r(M, w_0^r) \to L^r(M) \ ; \ A \ : \ L^r(M, w_0^r) \to L^2(M)$,
such that
$v := T\omega \in L^r(M) \cap L^t(M)$ verifies $\Delta v = \omega + \tilde{\omega}$,
with $t = S_2(r)$ and $\tilde{\omega} := A\omega$.
But
$v \in L^t(M) \Rightarrow v \in L^t(M, w)$ because $w(x) \in L^\infty(M)$ is bounded :
$$\|v\|^t_{L^t(M,w)} = \int_M |v(x)|^t \, w(x) dv(x) \leq \|w\|_\infty \int_M |v(x)|^t \, dv(x) = \|w\|_\infty \|v\|^t_{L^t(M)}.$$
And the same $v \in L^r(M) \Rightarrow v \in L^r(M, w)$.
By corollary 6.6 if $H\omega = 0$ then $H\tilde{\omega} = 0$. So setting $f := L\tilde{\omega} \in L^2_p(M)$ we have $\Delta f = \tilde{\omega} \in L^2_p(M)$. Then lemma 7.1 gives $f \in L^t(M, w)$. But then w verifies also lemma 7.1 for $\tau = r$, because $\dfrac{2t}{2-t} \geq \dfrac{2r}{2-r}$ so we have also $f \in L^r(M, w)$.
Now we set $u = v - f = T\omega - LA\omega$, and we get $u \in L^r(M, w) \cap L^t(M, w)$ and $\Delta u = \omega + \tilde{\omega} - \tilde{\omega} = \omega$.
■

Now we shall use the linearity of our solution to get, by duality, results for exponents bigger than 2. Take $r < 2$ and $r' > 2$ its conjugate.
Let $T \ : \ L^r(M, w_0^r) \to L^r(M)$, $A \ : \ L^r(M, w_0^r) \to L^2(M)$ be the linear operators, given by the RSM, such that
$\Delta T\omega = \omega + A\omega$.
The hypothesis (HL2,p) gives the existence of a bounded linear operator $L \ : \ L^2_p(M) \to L^2_p(M)$ such that
$\Delta L\tilde{\omega} = \tilde{\omega}$, provided that $H\tilde{\omega} = 0 \iff H\omega = 0$.
Hence, setting $C = LA \ : \ L^r(M, w_0^r) \to L^2(M)$ we get
$\forall \omega \in L^r(M, w_0^r), \ \Delta(T - C)\omega = \omega$.
We notice that
$\Delta(T - C)\Delta\psi = \Delta\psi$,
just setting $\omega = \Delta\psi$. This is possible because
$\forall \psi \in \mathcal{D}_p(M), \ \forall \varphi \in L^2_p(M), \ \langle H\Delta\psi, \varphi\rangle = \langle \Delta\psi, H\varphi\rangle = \langle \psi, \Delta(H\varphi)\rangle = 0$,
and we use that Δ is essentially self adjoint and $\Delta(H\varphi) = 0$ because $H\varphi$ is harmonic. So $H\Delta\psi = 0$ and we can set $\omega = \Delta\psi$. Hence

$$(T - C)\Delta\psi = \psi + h, \tag{7.6}$$

with $h \in \mathcal{H}$.
Now let $\varphi \in L^2_p(M) \cap L^{r'}_p(M)$ and consider $u := (T - C)^*\varphi$.

This is meaningful because $T^* : L^{r'}(M) \to L^{r'}(M, w_0^r)$ and $C^* : L^2(M) \to L^{r'}(M, w_0^r)$ hence $u \in L^{r'}(M, w_0^r)$. We get

$$\forall \psi \in \mathcal{D}(M) \cap L^r(M, w_0^r),\ \langle \Delta u, \psi \rangle_{L^2(M,w_0^r)} = \langle \Delta(T-C)^*\varphi, \psi \rangle_{L^2(M,w_0^r)} =$$

$$= \int_M \Delta((T-C)^*\varphi)\psi w_0^r dv_g = \int_M (T-C)^*\varphi \Delta(\psi w_0^r) dv_g = \langle (T-C)^*\varphi, \Delta(\psi w_0^r) \rangle_{L^2(M)},$$

because Δ is essentially self adjoint and ψw_0^r has compact support.

Hence by (7.6)

$$\langle \Delta u, \psi \rangle_{L^2(M,w_0^r)} = \langle \varphi, (T-C)\Delta(\psi w_0^r) \rangle_{L^2(M)} = \langle \varphi, \psi w_0^r + h \rangle_{L^2(M)} = \langle \varphi, \psi w_0^r \rangle_{L^2(M)},$$

provided that $\varphi \perp \mathcal{H}$. Putting back the weight in the integral, we get

$$\langle \Delta u, \psi \rangle_{L^2(M,w_0^r)} = \langle \varphi, \psi \rangle_{L^2(M,w_0^r)}.$$

This being true for any $\psi \in \mathcal{D}(M)$ we get

$\Delta u = \varphi$ in distributions sense, so we proved

Corollary 7.3 *Suppose that (M, g) is a complete riemannian manifold ; suppose we have $r < 2$ and (HL2,p), with $k :: S_k(r) \geq 2$, setting $w_0(x) := R(x)^{-k}$, then for any $\varphi \in L^2_p(M) \cap L^{r'}_p(M),\ H\varphi = 0$,*

$$u := (T-C)^*\varphi,\ u \in L^{r'}(M, w_0^r) \text{ and } u \text{ verifies } \Delta u = \varphi.$$

Moreover we have $u \in W_p^{2,r'}(M)$ with control of the norm.

It remains to prove the "moreover" and for it we notice that, because $R(x) \leq 1 \Rightarrow w_0 \geq 1$, we have

$$u \in L^{r'}(M, w_0^r) \Rightarrow \int_M |u|^{r'} dv_g \leq \int_M |u|^{r'} w_0^r dv_g < \infty,$$

hence $u \in L^{r'}(M)$; but $\Delta u = \varphi \in L^{r'}_p(M)$, so, by the CZI, theorem 6.2, we get $u \in W_p^{2,r'}(M)$. ■

7.1 Strong L^r Hodge decomposition.

We shall need :

Lemma 7.4 *Let $r \leq 2$ and $\alpha \in W^{1,r}_{p+1}(M)$; $\beta \in W^{1,r}_{p-1}(M),\ h \in \mathcal{H}^2_p(M)$ then*

$$\langle d\alpha, h \rangle = \langle d^*\beta, h \rangle = 0.$$

Proof.

Because $h \in \mathcal{H}^2_p$, we have that $dh = d^*h = 0$ by theorem 5.5, p. 697 in Bueler [5]. By the density of $\mathcal{D}_k(M)$ in $W^{1,r}_k(M)$ which is always true in a complete riemannian manifold by theorem 2.7, p. 13 in [11], there is a sequence $\alpha_k \in \mathcal{D}_{p+1}(M)$ such that $\|\alpha - \alpha_k\|_{W^{1,r}(M)} \to 0$ and there is a sequence $\beta_k \in \mathcal{D}_{p-1}(M)$ such that $\|\beta - \beta_k\|_{W^{1,r}(M)} \to 0$.

By use of corollary 6.3, we have that $h \in \mathcal{H}^2_p \Rightarrow h \in \mathcal{H}^{r'}_p$ hence, because $d\alpha \in L^r_p(M)$,

$$\langle d\alpha, h \rangle = \lim_{k\to\infty} \langle d\alpha_k, h \rangle = \lim_{k\to\infty} \langle \alpha_k, d^*h \rangle = 0,$$

because d^* is the formal adjoint of d, $\alpha_k \in \mathcal{D}_{p+1}(M)$ and $d^*h = 0$.

The same way we get $\langle d^*\beta, h \rangle = 0$. ■

Theorem 7.5 *Let the weight $w \in L^\infty(M)$ be such that $\gamma(w, r) < \infty$ with $r < 2$ with $k :: S_k(r) \geq 2$, set $w_0 = R(x)^{-k}$, and suppose we have hypothesis (HL2,p). We have the direct decomposition given by linear operators :*

$$\forall s,\ 1 \leq s \leq 2,\ L^s_p(M, w_0^s) = \mathcal{H}^2_p \oplus \Delta(W^{2,s}_p(M, w)).$$

$$\forall s,\ s > 2,\ L^s_p(M) \cap L^2_p(M) = \mathcal{H}^2_p \cap \mathcal{H}^s_p \oplus \Delta(W^{2,s}_p(M)).$$

Proof.
If $s < 2$ we set $r = s$. Let $\omega \in L^r_p(M, w_0^r)$ the RSM with $w = 1,\ w_0 = R(x)^{-k}$, gives $u := T\omega \in L^r_p(M),\ \tilde{\omega} := A\omega \in L^2_p(M)$ such that $\Delta u = \omega + \tilde{\omega}$. So we get

$$\omega = \Delta u - \tilde{\omega} = \Delta u - (\tilde{\omega} - H\tilde{\omega}) - H\tilde{\omega}.$$

This is well defined because $\tilde{\omega} \in L^2_p(M)$ and H is the orthogonal projection from $L^2_p(M)$ on $\mathcal{H}^2_p$.

Now $H(\tilde{\omega} - H\tilde{\omega}) = 0$ hence $f := L(\tilde{\omega} - H\tilde{\omega})$ solves $\Delta f = (\tilde{\omega} - H\tilde{\omega}),\ f \in L^2_p(M)$. So we get

$$\omega = \Delta u - \tilde{\omega} = -H\tilde{\omega} + \Delta u - \Delta f, \tag{7.7}$$

with $H\tilde{\omega} \in \mathcal{H}^2_p$. We have $\tilde{\omega} \in L^r_p(M) \cap L^2_p(M)$ by remark 6.5 and $\omega \in L^r(M, w_0^r) \subset L^r(M)$ because $w_0^r \geq 1$, so $u \in L^r_p(M),\ \Delta u = \omega + \tilde{\omega} \in L^r_p(M)$ hence, by CZI, $u \in W^{2,r}_p(M)$. The same way we have that $f \in W^{2,2}_p(M)$.

This gives a first decomposition :

$$\omega = -H\tilde{\omega} + \Delta u - \Delta f, \tag{7.8}$$

with $H\tilde{\omega} \in \mathcal{H}^2_p(M),\ u \in W^{2,r}_p(M)$ and $f \in W^{2,2}_p(M)$.
With the weight $w \in L^\infty(M)$ such that $\gamma(w, r) < \infty$ we have that

$$f \in L^2_p(M) \subset L^r_p(M, w), \text{ and } u \in L^r_p(M) \subset L^r_p(M, w).$$

Moreover $\Delta u = \omega \in L^r_p(M, w_0^r)$. But $\omega \in L^r_p(M, w_0^r) \Rightarrow \omega \in L^r_p(M, w)$ hence we can apply the weighted CZI 6.2 :

$$u \in W^{2,r}_p(M, w),\ \|u\|_{W^{2,r}(M,w)} \leq c_1\|u\|_{L^r(M,w)} + c_2\|\Delta u\|_{L^r(M,w)}.$$

This gives the case $s < 2$.

If $s > 2$ we set $s = r'$, the conjugate exponent of r. So let $\omega \in L^{r'}_p(M) \cap L^2_p(M)$, then we have

$$\omega = H\omega + (\omega - H\omega) \text{ with } H(\omega - H\omega) = 0.$$

We have that $H\omega \in \mathcal{H}^2_p(M)$ hence by corollary 6.3 we get that $H\omega \in \mathcal{H}^{r'}_p(M)$ so $\tilde{\omega} := \omega - H\omega \in L^{r'}_p(M) \cap L^2_p(M)$ and $H\tilde{\omega} = 0$. Now we have by corollary 7.3 a $u \in L^{r'}_p(M, w_0^r)$ such that $\Delta u = \tilde{\omega}$. Again this implies that $u \in L^{r'}_p(M)$ hence, by CZI 6.2 :

$$u \in W^{2,r'}_p(M),\ \|u\|_{W^{2,r'}(M)} \leq c_1\|u\|_{L^{r'}(M)} + c_2\|\Delta u\|_{L^{r'}(M)}.$$

Hence we have the decomposition

$$\forall \omega \in L^{r'}_p(M) \cap L^2_p(M),\ \omega = H\omega + \Delta u = h + \Delta u, \tag{7.9}$$

with $h \in \mathcal{H}^2_p(M) \cap \mathcal{H}^{r'}_p(M)$ and $u \in W^{2,r'}_p(M)$.
Because at each step we keep the linearity w.r.t. ω, we get that that the decomposition is also linear w.r.t. ω.

To get the uniqueness in the case $s < 2$, we consider the first decomposition (7.7) :

$$\omega = h + \Delta(u - f) \text{ with } h \in \mathcal{H}^2_p \text{ and } u \in W^{2,r}_p(M),\ f \in W^{2,2}_p(M).$$

If there is another one $\omega = h' + \Delta(u' - f')$ then $0 = h - h' + \Delta(u - u' - (f - f'))$; so we have to show that

$$0 = h + \Delta(u - f) \text{ with } h \in \mathcal{H}^2_p \text{ and } u \in W^{2,r}_p(M),\ f \in W^{2,2}_p(M),$$

implies $h = 0$ and $\Delta(u - f) = 0$.

Now $\Delta u = d(d^*u) + d^*(du) = d\alpha + d^*\beta$, with $\alpha = d^*u \in W^{1,r}_{p+1}(M)$ and $\beta = du \in W^{1,r}_{p-1}(M)$. By lemma 7.4 we get $\langle d\alpha, h\rangle + \langle d^*\beta, h\rangle = 0$, so $\langle \Delta u, h\rangle = 0$. Exactly the same proof with $r = 2$ gives $\langle \Delta f, h\rangle = 0$, so, from $h + \Delta u - \Delta f = 0$, we get

$$0 = \langle h, h\rangle + \langle \Delta u, h\rangle + \langle \Delta f, h\rangle = \|h\|_{L^2(M)},$$

which implies $\Delta(u - f) = 0$ and proves the uniqueness of the decomposition for $s < 2$.

For $s > 2$ we have the decomposition (7.9)

$$\forall \omega \in L^{r'}_p(M) \cap L^2_p(M),\ \omega = h + \Delta u,$$

with $h \in \mathcal{H}^2_p(M) \cap \mathcal{H}^{r'}_p(M)$ and $u \in W^{2,r'}_p(M)$.

By (HL2,p) we have

$$\exists v \in L^2_p(M) :: \Delta v = \tilde{\omega} := \omega - h \text{ because } \tilde{\omega} \in L^2_p(M), \text{ hence, by CZI, } v \in W^{2,2}_p(M).$$

But $\Delta v = \Delta u = \tilde{\omega}$, so if there is another such decomposition

$$\omega = h' + \Delta u' = h' + \Delta v'$$

then

$$0 = h - h' + \Delta(u - u') = h - h' + \Delta(v - v'),$$

Still with $v - v' \in W^{2,2}_p(M)$. So changing names we have

$$0 = h + \Delta u = h + \Delta v \tag{7.10}$$

with $h \in \mathcal{H}^2_p(M)$ and $v \in W^{2,2}_p(M)$.

Again $\Delta v = d\alpha + d^*\beta$ with $\alpha = d^*v \in W^{1,2}_{p-1}(M)$ and $\beta = d^*v \in W^{1,2}_{p+1}(M)$ and by lemma 7.4 we get $\langle d\alpha, h\rangle + \langle d^*\beta, h\rangle = 0$, so $\langle \Delta v, h\rangle = 0$.

Hence $\langle \Delta u, h\rangle = \langle \Delta v, h\rangle = 0$. But by (7.10) we have

$$0 = \langle h, h\rangle + \langle \Delta u, h\rangle \text{ so } \|h\|_{L^2(M)} = 0 \Rightarrow h = 0$$

which ends the proof of uniqueness of the Hodge decomposition. ■

There are two extreme cases done in the next corollaries.

Corollary 7.6 *Let the weight $w \in L^\infty(M)$ be such that $\gamma(w, r) < \infty$ with $r < 2$. Suppose the conformal radius verifies $\forall x \in M,\ R(x) \geq \delta > 0$, and suppose also hypothesis (HL2,p). Then we have the direct decomposition given by linear operators*

$$\forall s,\ 1 \leq s \leq 2,\ L^s_p(M) = \mathcal{H}^2_p \oplus \Delta(W^{2,s}_p(M, w)).$$

Proof.

We have just to set $w_0 \equiv 1$. ■

The conformal radius verifies $\forall x \in M,\ R(x) \geq \delta > 0$, if, for instance, the Ricci curvature of M is bounded and the injectivity radius is strictly positive [12].

We also have

Corollary 7.7 *Let $r < 2$, with $k :: S_k(r) \geq 2$ set $w_0 = R(x)^{-k}$ and suppose the riemannian volume is finite and hypothesis (HL2,p). We have the direct decomposition given by linear operators :*

$$L^r_p(M, w^r_0) = \mathcal{H}^2_p \oplus \Delta(W^{2,r}_p(M)).$$

Here the weight w is no longer necessary because the volume being finite, if a form is in $L^2(M)$ then it is already in $L^s(M)$.

Corollary 7.8 *Let the weight $w \in L^\infty(M)$ be such that $\gamma(w,r) < \infty$ with $r < 2$, with $k :: S_k(r) \geq 2$ set $w_0 = R(x)^{-k}$ and suppose we have hypothesis (HL2,p). We have the direct decomposition given by linear operators :*

$$L^r_p(M, w_0^r) = \mathcal{H}^2_p \oplus d(W^{1,r}_p(M,w)) \oplus d^*(W^{1,r}_p(M,w)).$$

With $r' > 2$ the conjugate exponent for r,

$$L^{r'}_p(M) \cap L^2_p(M) = \mathcal{H}^2_p \cap \mathcal{H}^{r'}_p \oplus d(W^{1,r'}_p(M)) \oplus d^*(W^{1,r'}_p(M)).$$

Proof.

The case r' follows exactly the same line. We already have $\omega = H\omega + \Delta u$ and $\Delta u = d(d^*u) + d^*(du) = d\alpha + d^*\beta$, with $\alpha = d^*u \in W^{1,r'}_{p+1}(M)$ and $\beta = du \in W^{1,r'}_{p-1}(M)$. This gives the decomposition.

For the uniqueness, suppose that

$$0 = h + d\alpha + d^*\beta, \text{ with } \alpha \in W^{1,2}_{p-1}(M) \text{ and } \beta \in W^{1,2}_{p+1}(M),$$

then, by use of lemma 7.4, we get $\langle d\alpha, h\rangle + \langle d^*\beta, h\rangle = 0$, hence $h = 0$. So it remains

$$0 = d\alpha + d^*\beta.$$

Take $\varphi \in \mathcal{D}_{p+1}(M)$ and consider $\langle d\alpha, d^*\varphi\rangle = \langle \alpha, d^{*2}\varphi\rangle = 0$ because $d^{*2} = 0$. By the density of $\mathcal{D}_k(M)$ in $W^{1,2}_k(M)$ there is a sequence $\beta_k \in \mathcal{D}_{p-1}(M)$ such that $\|\beta - \beta_k\|_{W^{1,2}(M)} \to 0$ hence

$$\forall k \in \mathbb{N}, \quad \langle d\alpha, d^*\beta_k\rangle = \langle \alpha, d^{*2}\beta_k\rangle = 0 \Rightarrow \langle d\alpha, d^*\beta\rangle = 0,$$

hence $\|d^*\beta\|_{L^2(M)} = 0$ so $d^*\beta = 0$ and also $d\alpha = 0$.

For r, we have, by (7.8) : $\forall \omega \in L^r_p(M, w_0^r)$,

$$\omega = -H\tilde{\omega} + \Delta u - \Delta f,$$

with $H\tilde{\omega} \in \mathcal{H}^2_p(M)$, $u \in W^{2,r}_p(M)$ and $f \in W^{2,2}_p(M)$. Again

$$\Delta u = d\alpha + d^*\beta, \text{ with } \alpha \in W^{1,r}_{p-1}(M) \text{ and } \beta \in W^{1,r}_{p+1}(M),$$

and

$$\Delta f = d\alpha' + d^*\beta', \text{ with } \alpha' \in W^{1,2}_{p-1}(M) \text{ and } \beta' \in W^{1,2}_{p+1}(M),$$

Hence

$$\omega = h + d(\alpha - \alpha') + d^*(\beta - \beta').$$

With the weight w we get $\alpha \in W^{1,r}_{p-1}(M) \Rightarrow \alpha \in W^{1,r}_{p-1}(M,w)$ and the same for β. And also $\alpha' \in W^{1,2}_{p-1}(M) \Rightarrow \alpha' \in W^{1,r}_{p-1}(M,w)$ and the same for β'. So, setting $\gamma := \alpha - \alpha'$, $\delta = \beta - \beta'$, we have the decomposition

$$\omega \in L^r_p(M, w_0^r) \Rightarrow \omega = h + d\gamma + d^*\delta,$$

with $h \in \mathcal{H}^2_p(M) \cap \mathcal{H}^r_p(M,w)$, $\gamma \in W^{1,r}_{p-1}(M,w)$, $\delta \in W^{1,r}_{p+1}(M,w)$.

For the uniqueness, suppose that

$$0 = h + d(\alpha - \alpha') + d^*(\beta - \beta'),$$

by use of lemma 7.4, we get $\langle d\alpha, h\rangle + \langle d^*\beta, h\rangle = 0$ and also $\langle d\alpha', h\rangle + \langle d^*\beta', h\rangle = 0$, so $h = 0$. So we have

$$0 = d(\alpha - \alpha') + d^*(\beta - \beta').$$

This implies that

$$d\alpha + d^*\beta = d\alpha' + d^*\beta', \tag{7.11}$$

hence

$$d\alpha + d^*\beta \in L^r_p(M) \cap L^2_p(M)\ ;\ d\alpha' + d^*\beta' \in L^r_p(M) \cap L^2_p(M),$$

because

$$d\alpha + d^*\beta \in L^r_p(M) \text{ and } d\alpha' + d^*\beta' \in L^2_p(M).$$

Now take $\varphi \in \mathcal{D}_p(M)$, because (HL2,p) is true we have the L^2 decomposition :

$$\varphi = H\varphi + d\gamma + d^*\delta \text{ with } \gamma, \delta \in W^{1,2}(M).$$

We have

$$\langle d(\alpha - \alpha'), \varphi\rangle = \langle d(\alpha - \alpha'), H\varphi + d\gamma + d^*\delta\rangle\ ;$$

by use of lemma 7.4, we get $\langle d(\alpha - \alpha'), H\varphi\rangle = 0$. By density we have $\gamma = \lim_{k\to\infty}\gamma_k,\ \gamma_k \in \mathcal{D}_{p-1}$ and $\delta = \lim_{k\to\infty}\delta_k,\ \delta_k \in \mathcal{D}_{p+1}$, the convergence being in $W^{1,2}(M)$, so $d\gamma = \lim_{k\to\infty} d\gamma_k$ and $d^*\delta = \lim_{k\to\infty} d^*\delta_k$ in $L^2_p(M)$. So we get

$$\langle d(\alpha - \alpha'), d\gamma + d^*\delta\rangle = \lim_{k\to\infty}\langle d(\alpha - \alpha'), d\gamma_k + d^*\delta_k\rangle.$$

But

$$\forall k \in \mathbb{N},\ \langle d(\alpha - \alpha'), d^*\delta_k\rangle = \left\langle (\alpha - \alpha'), d^{*2}\delta_k\right\rangle = 0$$

because d^* is the formal adjoint of d, $d^*\delta_k$ has compact support and $d^{*2} = 0$. So

$$\langle d(\alpha - \alpha'), \varphi\rangle = \lim_{k\to\infty}\langle d(\alpha - \alpha'), d\gamma_k\rangle.$$

With (7.11) we get

$$\forall k \in \mathbb{N},\ \langle d(\alpha - \alpha'), d\gamma_k\rangle - \langle d^*(\beta - \beta'), d\gamma_k\rangle = 0,$$

and

$$\forall k \in \mathbb{N},\ \langle d^*(\beta - \beta'), d\gamma_k\rangle = 0,$$

because d^* is the formal adjoint of d, $d\gamma_k$ has compact support and $d^2 = 0$. So

$$\forall k \in \mathbb{N},\ \langle d(\alpha - \alpha'), d\gamma_k\rangle = 0,$$

which gives

$$\langle d(\alpha - \alpha'), \varphi\rangle = \lim_{k\to\infty}\langle d(\alpha - \alpha'), d\gamma_k\rangle = 0,$$

and this being true for any $\varphi \in \mathcal{D}_p(M)$, we get $d(\alpha - \alpha') = 0$; this gives with (7.11) $d^*(\beta - \beta')$. ■

Let us set the Sobolev embeddings hypothesis :
(SE,p) provided that $\forall r > 1,\ u \in W^{2,r}_p(M) \Rightarrow u \in L^t_p(M),\ t = S_2(r)$.

This property is true for instance if M has a Ricci curvature bounded from below and $\inf_{x\in M} v_g(B_x(1)) \geq \delta > 0$, due to Varopoulos [19], see [11] theorem 3.14, p. 31.
Then we have a direct proof of the L^r Hodge decomposition for $r > 2$ with this extra hypothesis.

Theorem 7.9 *We have, with the hypotheses (HL2,p) and (SE,p), the decomposition*

$$\forall r,\ r > 2,\ L^r_p(M) \cap L^2_p(M) = \mathcal{H}^r_p \oplus \Delta(W^{2,r}_p(M)).$$

Proof.
Take $\omega \in L^r_p(M) \cap L^2_p(M),\ H\omega = 0$; by interpolation we have $\forall t \in [2, r],\ \omega \in L^t_p(M)$. By (HL2,p) we can solve Δ :

$$\exists u \in L^2_p(M) :: \Delta u = \omega.$$

By CZI we have then $u \in W^{2,2}_p(M)$. By (SE,p) $u \in L^{t_1}_p(M),\ t_1 = S_2(2) > 2$. If $t_1 < r$ then $\Delta u = \omega \in L^{t_1}_p(M)$ hence by CZI we get $u \in W^{2,t_1}_p(M)$, and again with (SE,p) we get $u \in L^{t_2}_p(M),\ t_2 = S_2(t_1) = S_4(2) > t_1$. And by induction up to the fact that $t_k = S_{2k}(2) \geq r$. Then we have $u \in L^{t_k}_p(M) \cap L^2_p(M)$ hence still by interpolation, $u \in L^r_p(M)$. But $\Delta u = \omega \in L^r_p(M)$ so by CZI we get $u \in W^{2,r}_p(M)$. ■

Remark 7.10 *The Sobolev embeddings property is not true if M has a Ricci curvature bounded from below and* $\inf_{x\in M} v_g(B_x(1)) = 0$, *see [11] theorem 3.18, p. 37. Nevertheless theorem 7.5 is still valid.*

The corollary 7.3 which gives the theorem 7.5 in fact is stronger : if $r > 2$, hence $r' < 2$, $\omega \in L^r_p(M) \cap L^2_p(M)$, $H\omega = 0$, set $k :: S_k(r') \geq 2$, then there is a $u \in L^r_p(M, w_0^{r'})$ with $w_0 := R(x)^{-k}$; because $w_0 \leq 1$, the space $L^r_p(M, w_0^{r'})$ is in general much smaller than $L^r_p(M)$ which we use in theorem 7.5.

7.2 Weak L^r Hodge decomposition.

Now we shall need another hypothesis :
(HWr) the space $\mathcal{D}_p(M)$ is dense in $W^{2,r}_p(M)$.
We already know that (HWr) is true if :

• either : the injectivity radius is strictly positive and the Ricci curvature is bounded ([11] theorem 2.8, p. 12).

• or : M is geodesically complete with a bounded curvature tensor([10] theorem 1.1 p.3).

We have a weak L^r Hodge decomposition theorem :

Theorem 7.11 *Suppose that (M, g) is a complete riemannian manifold, choose a weight w verifying lemma 7.1 for $r < 2$, with $k :: S_k(r) \geq 2$, take the weight $w_0 := R(x)^{-k}$. Suppose we have (HL2,p) and (HWr) ; then*

$$L^r_p(M, w) = \mathcal{H}^r_p \oplus \overline{\Delta(\mathcal{D}_p(M))},$$

the closure being taken in $L^r(M, w)$.

Proof.
Take $\omega \in L^r_p(M, w)$, $H\omega = 0$, by corollary 6.6. By density there is a $\omega_\epsilon \in \mathcal{D}_p(M)$ such that $\|\omega - \omega_\epsilon\|_{L^r(M,w)} < \epsilon$.
Then we have $\omega_\epsilon \in L^r_p(M, w_0^r)$ hence by RSM :

$$\exists v_\epsilon \in L^r_p(M) :: \Delta v_\epsilon = \omega_\epsilon + \tilde{\omega}_\epsilon,$$

with $\tilde{\omega}_\epsilon \in L^2_p(M)$ of compact support. Because $\|\omega - \omega_\epsilon\|_{L^r(M,w)} < \epsilon$ we have, by the continuity of $H : L^r_p(M, w_0^r) \to L^2_p(M)$ given by corollary 6.6, that $\|H\omega - H\omega_\epsilon\|_{L^2(M)} < C\epsilon$. Hence, still by corollary 6.6, we have $\|H\tilde{\omega}_\epsilon\| = \|H\omega_\epsilon\|_{L^2(M)} = \|H\omega - H\omega_\epsilon\|_{L^2(M)} < C\epsilon$, because by assumption $H\omega = 0$.

Hence by (HL2,p) there is a $f_\epsilon \in W^{2,2}_p(M) :: \Delta f_\epsilon = \tilde{\omega}_\epsilon - H\tilde{\omega}_\epsilon$.
By (HWr) there is a $g_\epsilon \in \mathcal{D}_p(M) :: \|f_\epsilon - g_\epsilon\|_{W^{2,2}(M)} < \epsilon$ and this implies

$$\|\tilde{\omega}_\epsilon - H\tilde{\omega}_\epsilon - \Delta g_\epsilon\|_{L^2(M)} < \epsilon ;$$

but by the choice of w we get

$$\|\tilde{\omega}_\epsilon - \Delta g_\epsilon\|_{L^r(M,w)} \leq \|\tilde{\omega}_\epsilon - H\tilde{\omega}_\epsilon - \Delta g_\epsilon\|_{L^2(M)} + \|H\tilde{\omega}_\epsilon\|_{L^2(M)} < (1 + C)\epsilon.$$

On the other hand we have $v_\epsilon \in L^r_p(M) \Rightarrow v_\epsilon \in L^r_p(M, w)$ hence setting $u_\epsilon := v_\epsilon - g_\epsilon$ we get $\Delta u_\epsilon = \omega_\epsilon + \tilde{\omega}_\epsilon - \Delta g_\epsilon \Rightarrow \|\omega - \Delta u_\epsilon\|_{L^r(M,w)} \leq \|\omega - \omega_\epsilon\|_{L^r(M,w)} + \|\tilde{\omega}_\epsilon - \Delta g_\epsilon\|_{L^r(M,w)} < (2 + C)\epsilon$,
and the proof is complete. ■

We also have a weaker L^r Hodge decomposition without hypothesis (HWr) :

Theorem 7.12 *Suppose that (M, g) is a complete riemannian manifold and suppose we have (HL2,p). Take a weight w verifying $\gamma(w, r) < \infty$, for $r < 2$. Then we have*

$$L^r_p(M) = \mathcal{H}^r_p \oplus \overline{d(\mathcal{D}_p(M))} \oplus \overline{d^*(\mathcal{D}_p(M))},$$

the closure being taken in $L^r(M, w^r)$.

Proof.
For any $2 \geq r \geq 1$, $\omega \in L^r_p(M) \cap \mathcal{H}^\perp_p$, we proved that there is a $u \in L^r_p(M, w^r)$ such that $\|\omega - \Delta u\|_{L^r(M,w^r)} < \epsilon$. The point here is that $u \notin \mathcal{D}_p(M)$. Nevertheless we have :

$$\|\omega - dd^*u - d^*du\|_{L^r(M,w^r)} < \epsilon$$

and now we approximate d^*u by $\alpha \in \mathcal{D}(M)$ in $W^{1,r}(M)$, and du by $\beta \in \mathcal{D}(M)$ in $W^{1,r}(M)$, and this is always possible by theorem 2.7, p. 13 in [11].
So we get

$$\|\omega - d\alpha - d^*\beta\|_{L^r(M,w^r)} < 2\epsilon$$

and we are done. ■

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