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Connecting walks and connecting dart sequences in n -D combinatorial pyramids

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Abstract. Combinatorial maps define a general framework which allows to encode any subdivision of an n -D orientable quasi-manifold with or without boundaries. Combinatorial pyramids are defined as stacks of successively reduced combinatorial maps. Such pyramids provide a rich framework which allows to encode fine properties of objects (either shapes or partitions). Combinatorial pyramids have first been defined in 2D. This first work has later been extended to pyramids of n -D generalized combinatorial maps. Such pyramids allow to encode stacks of non orientable partitions but at the price of a twice bigger pyramid. These pyramids are also not designed to capture efficiently the properties connected with orientation. This work presents the design of pyramids of n -D combinatorial maps and important notions for their encoding and processing.

Keywords: Combinatorial maps, combinatorial pyramids, hierarchical models.

1 Introduction

Pyramids of combinatorial maps have first been defined in 2D [1], and later extended to pyramids of n -dimensional generalized maps by Grasset et al. [8]. These last pyramids extend the irregular pyramid [13] framework made of simple graphs to the encoding of subdivisions of orientable but also non-orientable quasi-manifolds [11] at the expense of twice the data size of the one required for combinatorial maps. For practical use (for example in image segmentation), this may have an impact on the efficiency of the associated algorithms or may even prevent their use. Furthermore, properties and constraints linked to the notion of orientation may be expressed in a more natural way with the formalism of combinatorial maps. For these reasons, we are interested here in the definition of pyramids of n -dimensional combinatorial maps. Note that Illetschko [9, 10] achieved a first study of the encoding of 3D combinatorial pyramids, as well as the associated memory requirement issues.

The key notion for the definition of pyramids of maps is the operation of simultaneous removal or contraction of cells. These two notions have been defined in [5] (see also [6]) where the definitions have been related to the ones given in

[3] for generalized maps. Their validity was indeed proved using the link between maps and generalized maps established by Lienhardt [11].

The notions of *reduction window* and *receptive field* are two basic concepts within the regular pyramid framework [2]. A reduction window relates a pixel of a pyramid to a set of pixels in the level below while the receptive field corresponds to the transitive closure of the father child relationship induced by the reduction window. After recalling some preliminaries about combinatorial maps and the main results obtained in [5], we define the objects which generalize the notions of reduction window and receptive field within the combinatorial pyramid framework: connecting walks and connecting darts sequences. These objects have interesting properties that should allow us to derive, in future works, efficient encoding schemes and operations on pyramids of n -D maps the same way Brun and Kropatsch did for 2-dimensional combinatorial pyramids [1]. Connecting walks which are introduced in Section 4, somehow fill the gap between two consecutive levels of the pyramid. We first provide a definition of connecting walks in generalized maps and establish a link (Proposition 5) with the definition we give for such walks in combinatorial maps. On the other hand, connecting dart sequences (Section 5) link a level of a pyramid of maps to any of its lower levels. The definition of the latter sequence as well as a discussion of its expected use are given in Section 5.

2 Maps and generalized maps in dimension n

An n -dimensional generalized map (or n -G-map) is defined by a set of basic abstract elements called *darts* connected by $(n + 1)$ involutions. More formally:

Definition 1 (n -G-map [11]) *Let $n \geq 0$, an n -G-map is defined as an $(n + 2)$ -tuple $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ where:*

- \mathcal{D} is a finite non-empty set of darts;
- $\alpha_0, \dots, \alpha_n$ are involutions on \mathcal{D} (i.e. $\forall i \in \{0, \dots, n\}, \alpha_i^2(b) = b$) such that:
 - $\forall i \in \{0, \dots, n - 1\}, \alpha_i$ is an involution without fixed point (i.e. $\forall b \in \mathcal{D}, \alpha_i(b) \neq b$);
 - $\forall i \in \{0, \dots, n - 2\}, \forall j \in \{i + 2, \dots, n\}, \alpha_i \alpha_j$ is an involution¹.

The *dual* of G , denoted by \overline{G} , is the n -G-map $\overline{G} = (\mathcal{D}, \alpha_n, \dots, \alpha_0)$. If α_n is an involution without fixed point, G is said to be *without boundaries* or *closed*. In the following we only consider closed n -G-maps with $n \geq 2$.

Figure 2(a) shows a 2-G-map $G = (\mathcal{D}, \alpha_0, \alpha_1, \alpha_2)$ whose set of darts \mathcal{D} is $\{1, 2, 3, 4, -1, -2, -3, -4\}$, with the involutions $\alpha_0 = (1, -1)(2, -2)(3, -3)(4, -4)$, $\alpha_1 = (1, 2)(-1, 3)(-2, -3)(4, -4)$, and $\alpha_2 = (1, 2)(-1, -2)(3, 4)(-3, -4)$.

Let $\Phi = \{\phi_1, \dots, \phi_k\}$ be a set of permutations on a set \mathcal{D} . We denote by $\langle \Phi \rangle$ the permutation group generated by Φ , i.e. the set of permutations obtained by any composition and inversion of permutations contained in Φ . The *orbit* of

¹ Given two involutions α_i, α_j and one dart d , the expression $d\alpha_i\alpha_j$ denotes $\alpha_j \circ \alpha_i(d)$.

$d \in \mathcal{D}$ relatively to Φ is defined by $\langle \Phi \rangle(d) = \{\phi(d) \mid \phi \in \langle \Phi \rangle\}$. Furthermore, we extend this notation to the empty set by defining $\langle \emptyset \rangle$ as the identity map. If $\Psi = \{\psi_1, \dots, \psi_h\} \subset \Phi$ we denote $\langle \psi_1, \dots, \hat{\psi}_j, \dots, \psi_h \rangle(d) = \langle \Psi \setminus \{\psi_j\} \rangle(d)$. Moreover, when there will be no ambiguity about the reference set Φ we will denote by $\langle \hat{\psi}_1, \hat{\psi}_2, \dots, \hat{\psi}_h \rangle(d)$ the orbit $\langle \Phi \setminus \Psi \rangle(d)$.

Definition 2 (Cells in n -G-maps [11]) Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an n -G-map, $n \geq 1$. Let us consider $d \in \mathcal{D}$. The i -cell (or cell of dimension i) that contains d is denoted by $\mathcal{C}_i(d)$ and defined by the orbit: $\mathcal{C}_i(d) = \langle \alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_n \rangle(d)$.

Thus, the 2-G-map of Fig. 2(a) counts 2 vertices ($v_1 = \langle \alpha_1, \alpha_2 \rangle(1) = \{1, 2\}$ and $v_2 = \{-1, 3, 4, -4, -3, -2\}$), 2 edges ($e_1 = \langle \alpha_0, \alpha_2 \rangle(1) = \{1, -1, 2, -2\}$ and $e_2 = \{3, 4, -3, -4\}$), and 2 faces (the one bounded by e_2 and the outer one).

Definition 3 (n -map [11]) An n -map ($n \geq 1$) is defined as an $(n+1)$ -tuple $M = (\mathcal{D}, \gamma_0, \dots, \gamma_{n-1})$ such that:

- \mathcal{D} is a finite non-empty set of darts;
- $\gamma_0, \dots, \gamma_{n-2}$ are involutions on \mathcal{D} and γ_{n-1} is a permutation on \mathcal{D} such that:
 $\forall i \in \{0, \dots, n-2\}, \forall j \in \{i+2, \dots, n\}, \gamma_i \gamma_j$ is an involution.

The dual of M , denoted by \overline{M} , is the n -map $\overline{M} = (\mathcal{D}, \gamma_0, \gamma_0 \gamma_{n-1}, \dots, \gamma_0 \gamma_1)$. The inverse of M , denoted by M^{-1} is defined by $M^{-1} = (\mathcal{D}, \gamma_0, \dots, \gamma_{n-2}, \gamma_{n-1}^{-1})$. Note that Damiand and Lienhardt introduced a definition of n -map as an $(n+1)$ -tuple $(\mathcal{D}, \beta_n, \dots, \beta_1)$ defined as the inverse of the dual of our map M . If we forget the inverse relationships (which only reverses the orientation), we have $\gamma_0 = \beta_n$ and $\beta_i = \gamma_0 \gamma_i$ for $i \in \{1, \dots, n-1\}$. The application β_1 is the permutation of the map while $(\beta_i)_{i \in \{2, \dots, n\}}$ defines its involutions.

Definition 4 (Cells in n -maps [11]) Let $M = (\mathcal{D}, \gamma_0, \dots, \gamma_{n-1})$ be an n -map, $n \geq 1$. The i -cell (or cell of dimension i) of M that owns a given dart $d \in \mathcal{D}$ is denoted by $\mathcal{C}_i(d)$ and defined by the orbits:

$$\begin{aligned} \forall i \in \{0, \dots, n-1\} \quad \mathcal{C}_i(d) &= \langle \gamma_0, \dots, \hat{\gamma}_i, \dots, \gamma_{n-1} \rangle(d) \\ \text{For } i = n \quad \mathcal{C}_n(d) &= \langle \gamma_0 \gamma_1, \dots, \gamma_0 \gamma_{n-1} \rangle(d) \end{aligned}$$

In both an n -map and an n -G-map, two cells \mathcal{C} and \mathcal{C}' with different dimensions will be called *incident* if $\mathcal{C} \cap \mathcal{C}' \neq \emptyset$. Moreover, the *degree* of an i -cell \mathcal{C} is the number of $(i+1)$ -cells incident to \mathcal{C} , whereas the *dual degree* of \mathcal{C} is the number of $(i-1)$ -cells incident to \mathcal{C} . An n -cell (resp. a 0-cell) has a degree (resp. dual degree) equal to 0.

An n -map may be associated to an n -G-map, as stated by the next definition. This direct link between the two structures has been used in [5] to show that the removal operation in maps which we present in Section 3 is properly defined.

Definition 5 (Map of the hypervolumes) Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an n -G-map, $n \geq 1$. The n -map $HV(G) = (\mathcal{D}, \delta_0 = \alpha_n \alpha_0, \dots, \delta_{n-1} = \alpha_n \alpha_{n-1})$ is called the map of the hypervolumes of G .

A connected component of a map $(\mathcal{D}, \gamma_0, \dots, \gamma_{n-1})$ is a set $\langle \gamma_0, \dots, \gamma_{n-1} \rangle (d)$ for some $d \in \mathcal{D}$. Lienhardt [12] proved that if an n -G-map G is orientable, $HV(G)$ has two connected components. In the following we only consider orientable n -G-maps.

3 Cells removal in maps and G-maps

We recall here the main definitions and results about the simultaneous removal of cells in (G-)maps that have been presented in [5].

3.1 Cells removal in G-maps

As the number of $(i + 1)$ -cells that are incident to it, the degree of an i -cell \mathcal{C} in an n -G-map $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ is the number of orbits in the set $\Delta = \{ \langle \hat{\alpha}_{i+1} \rangle (d) \mid d \in \mathcal{C} \}$. As part of a criterion for cells that may be removed from a G-map, we need a notion of degree that better reflects the local configuration of a cell: the local degree. A detailed justification for the following definition may be found in [6].

Definition 6 (Local degree in G-maps) *The local degree of an i -cell \mathcal{C} in an n -G-map, $0 \leq i < n$, is the number $|\{ \langle \hat{\alpha}_i, \hat{\alpha}_{i+1} \rangle (b) \mid b \in \mathcal{C} \}|$. The local degree of an n -cell is 0.*

Intuitively, the local degree of an i -cell \mathcal{C} is the number of $(i + 1)$ -cells that locally appear to be incident to \mathcal{C} . It is called *local* because it may be different from the degree since an $(i + 1)$ -cell may be incident more than once to an i -cell, as illustrated in Fig. 2 where the 1-cell e_2 is multi-incident to the 0-cell v_2 , hence the cell v_2 has a degree 2 and a local degree 3.

It is known since [3] that cells that may be removed or contracted in a G-map must satisfy a criterion which, although correct, was mistakenly called “having a local degree 2”. In [6, 5], the notion of *regularity*, recalled below, was introduced in order to state a new criterion based on the correct definition of the local degree (Definitions 6 and 10).

Definition 7 (Regular cell) *An i -cell ($i \leq n - 2$) in an n -G-map is said to be regular if it satisfies the two following conditions:*

- a) $\forall d \in \mathcal{C}, d\alpha_{i+1}\alpha_{i+2} = d\alpha_{i+2}\alpha_{i+1}$ or $d\alpha_{i+1}\alpha_{i+2} \notin \langle \hat{\alpha}_i, \hat{\alpha}_{i+1} \rangle (d\alpha_{i+2}\alpha_{i+1})$,
and
- b) $\forall b \in \mathcal{C}, b\alpha_{i+1} \notin \langle \hat{\alpha}_i, \hat{\alpha}_{i+1} \rangle (b)$

Any $(n - 1)$ -cell is said to be regular.

The following theorem shows that the criterion given by Damiand et al. (which corresponds to condition *ii*) is more restrictive than the actual notion of local degree. (Condition *i*) merely excludes cells with local degree 1.)

Theorem 1 For any $i \in \{0, \dots, n-2\}$, an i -cell \mathcal{C} is a regular cell with local degree 2 if and only if

- i) $\exists b \in \mathcal{C}, b\alpha_{i+1} \notin \langle \hat{\alpha}_i, \hat{\alpha}_{i+1} \rangle (b)$, and
- ii) $\forall b \in \mathcal{C}, b\alpha_{i+1}\alpha_{i+2} = b\alpha_{i+2}\alpha_{i+1}$

An illustration of Definition 7 and Theorem 1 is provided in [6]. Figure 1 depicts a vertex with local degree 2 in a 3-G-map, vertex which is not regular according to Definition 7. Following Grasset et al.'s criterion, such a vertex is not considered as removable because it does not have a local degree two (according to their definition of the local degree). In our case, this vertex actually has a local degree two but is still excluded because it is not regular.

We may now describe families of sets of cells to be removed, which we call removal kernels, and for which the simultaneous removal operation is properly defined.

Definition 8 (Removal kernel) Let G be an n -G-map. A removal kernel K_r in G is a family of sets $\{R_i\}_{0 \leq i \leq n}$ where $R_i, 0 \leq i \leq n$, is a set of regular i -cells (Definition 7) with local degree 2 (Definition 6), $R_n = \emptyset$, and all cells of $R = \cup_{i=0}^n R_i$ are disjoint. We denote by $R^* = \cup_{\mathcal{C} \in R} \mathcal{C}$, the set of all darts in K_r .

The following definition for the simultaneous removal of cells is slightly simpler and was proved to be equivalent ([6, Proposition 10]) to the one used in [3, 8].

Definition 9 (Cells removal in n -G-maps [5, 3]) Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an n -G-map and $K_r = \{R_i\}_{0 \leq i \leq n-1}$ be a removal kernel in G . The n -G-map resulting of the removal of the cells of R is $G' = (\mathcal{D}', \alpha'_0, \dots, \alpha'_n)$ where:

1. $\mathcal{D}' = \mathcal{D} \setminus R^*$;
2. $\forall d \in \mathcal{D}', d\alpha'_n = d\alpha_n$;
3. $\forall i, 0 \leq i < n, \forall d \in \mathcal{D}', d\alpha'_i = d' = d(\alpha_i\alpha_{i+1})^k \alpha_i$ where k is the smallest integer such that $d' \in \mathcal{D}'$.

We denote $G' = G \setminus K_r$ or $G' = G \setminus R^*$.

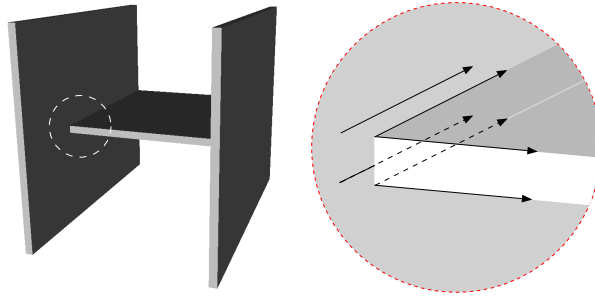


Fig. 1. A solid representation of a part of a 3-G-map where a vertex has a local degree 2 but is not regular. (The vertex is made of all the depicted darts.)

3.2 Cells removal in n -maps

We recall here the definition of the simultaneous removal of cells in an n -map, which was proved to be valid as it actually defines a map [5, Theorem 6]. As for G -maps, we need a notion of local degree in a map.

Definition 10 (Local degree in maps) *Let \mathcal{C} be an i -cell in an n -map. The local degree of \mathcal{C} is the number*

$$\begin{aligned} & |\{ \langle \hat{\gamma}_i, \hat{\gamma}_{i+1} \rangle (b) \mid b \in \mathcal{C} \}| && \text{if } i \in \{0, \dots, n-2\} \\ & |\{ \langle \gamma_0 \gamma_1, \dots, \gamma_0 \gamma_{n-2} \rangle (b) \mid b \in \mathcal{C} \}| && \text{if } i = n-1 \end{aligned}$$

The local degree of an n -cell is 0.

A notion of regular cell in an n -map which derives from the same notion in G -maps (Definition 7) has also been defined ([6, Definition 16]). With Definition 10 for the local degree, it allows us to define removal kernels in maps the same way we did for G -maps (see Definition 8), i.e. families of sets of non-intersecting regular cells with local degree 2.

Definition 11 (Cells removal in n -maps [5]) *Let $M = (\mathcal{D}, \gamma_0, \dots, \gamma_{n-1})$ be an n -map and $S_r = \{R_i\}_{0 \leq i \leq n-1}$ a removal kernel in M . We define the $(n-1)$ -tuple $M \setminus S_r = (\mathcal{D}', \gamma'_0, \dots, \gamma'_{n-1})$ obtained after removal of the cells of S_r by:*

- $\mathcal{D}' = \mathcal{D} \setminus R^*$;
- $\forall i \in \{0, \dots, n-2\}, \forall d \in \mathcal{D}', d\gamma'_i = d(\gamma_i \gamma_{i+1}^{-1})^k \gamma_i$, where k is the smallest integer such that $d(\gamma_i \gamma_{i+1}^{-1})^k \gamma_i \in \mathcal{D}'$.
- For $i = n-1, \forall d \in \mathcal{D}', d\gamma'_{n-1} = d\gamma_{n-1}^{k+1}$ where k is the smallest integer such that $d\gamma_{n-1}^{k+1} \in \mathcal{D}'$.

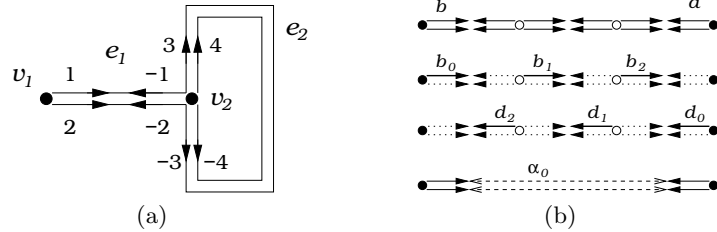


Fig. 2. (a) A 2- G -map. (b) A 2- G -map G (top row) from which the two white vertices are to be removed, yielding a map G' (bottom row). The connecting walks $\text{CW}_{G,G'}^0(b) = (b = b_0, b_1, b_2)$ (second row) and $\text{CW}_{G,G'}^0(d) = (d = d_0, d_1, d_2)$ (third row).

4 Connecting walks

The permutations or involutions which define the map resulting from a removal operation are obtained by somehow following a path in the original map until a surviving dart has been found (Definitions 9 and 11). This leads to the notion of the so called *connecting walks* which we define here and whose main properties are described. Proof of the results presented in this section may be found in [6].

In the sequel, if $S = (d_1, d_2, \dots, d_p)$ and $S' = (b_1, b_2, \dots, b_q)$ are sequences of darts in a (G-)map for $\{p, q\} \subset \mathbb{N}$, we denote by S° the sequence (d_2, \dots, d_p) (i.e. S without its first dart), and by $reverse(S)$ the sequence $(d_p, d_{p-1}, \dots, d_1)$. Furthermore, we denote $S \cdot S' = (d_1, \dots, d_p, b_1, \dots, b_q)$. We also denote by S^* the set $\{d_1, d_2, \dots, d_p\}$ and by $last(S)$ the last dart of S .

4.1 Connecting walks in generalized maps

Definition 12 (Connecting walk in n -G-maps) *Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an n -G-map and $K_r = \{R_i\}_{0 \leq i \leq n}$ be a removal kernel in G . Let $G' = G \setminus K_r = (\mathcal{D}', \alpha'_0, \dots, \alpha'_n)$. The i -connecting walk associated to a dart $d \in \mathcal{D}'$ for $i \in \{0, \dots, n\}$, denoted by $CW_{G, G'}^i(d)$, is the sequence of darts of \mathcal{D} defined by:*

$$CW_{G, G'}^i(d) = (d_0 = d, d_1, \dots, d_p)$$

where $d_u = d(\alpha_i \alpha_{i+1})^u$ for all $u \in \{0, \dots, p\}$ and $p = \text{Min}\{k \in \mathbb{N} | d_k \alpha_i \in \mathcal{D}'\}$.

The above definition is clearly linked to the one of the removal operation (Definition 9). To make this link explicit, we may first prove the following property which states that darts of an i -connecting walk are, except for the first one, darts of i -cells that have been removed ([4]). This property as well as the next one is illustrated by Figure 2(b), in the 2D case for the ease of visualization.

Property 1 *With the notations of Definition 12, for all $d \in \mathcal{D}'$ such that $CW_{G, G'}^i(d) = (d_0, d_1, \dots, d_p)$ we have:*

$$\forall k \in \{1, \dots, p\}, d_{k-1} \alpha_i \in R_i^* \text{ and } d_k \in R_i^*$$

Using Property 1, it is clear from Definition 9 that we also have the following property, which relates i -connecting walks to the corresponding involution α'_i in the resulting map.

Property 2 *Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an n -G-map, K_r be a removal kernel in G , $G' = G \setminus K_r = (\mathcal{D}', \alpha'_0, \dots, \alpha'_n)$ and $d \in \mathcal{D}'$. For all $i \in \{0, \dots, n\}$ we have*

$$d\alpha'_i = last(CW_{G, G' \setminus K_r}^i(d))\alpha_i$$

In [7], Grasset defines connecting walks in G-maps in a slightly different way. A first difference is that in Grasset's definition, d does not appear at the beginning of the sequence that defines $CW_{G, G'}^i(d)$, whereas the dart $d_p \alpha_i$ of Definition 12 is added at the end of the sequence. On the other hand, consecutive

darts in a connecting walk as defined by Grasset are linked by alternately either an α_i or an α_{i+1} involution when they are always linked by the permutation $\alpha_i\alpha_{i+1}$ in our definition. Thus, a connecting walk for a given dart and a given dimension counts $((k-1)/2)+1$ darts when the corresponding one with Grasset's definition has k ones.

Following the definition of [7], connecting walks that are distinct (up to reverse ordering and after removal of their last dart) are always disjoint [7, Proposition 22]. With our definition the property simply becomes that connecting walks are either equal or disjoint. In other words, a removed dart belongs to at most one connecting walk for some $i \in \{0, \dots, n\}$. This property, stated by the following proposition, induces a father-child relationship between darts of consecutive levels similar to the *reduction windows* in the context of regular pyramids.

Proposition 1 *Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an n - G -map, K_r be a removal kernel in G , and d be a dart of R_i^* for $0 \leq i \leq n$. The dart d belongs to at most one connecting walk. In other words, the two following properties hold:*

- i) $d \in \bigcup_{b \in \mathcal{D}'} \text{CW}_{G, G \setminus K_r}^i(b)^* \Rightarrow \exists! b \in \mathcal{D}', d \in \text{CW}_{G, G \setminus K_r}^i(b)^{o*}$
- ii) $\forall j \in \{0, \dots, n\} \setminus \{i\}, \forall b \in \mathcal{D}', d \notin \text{CW}_{G, G \setminus K_r}^j(b)^{o*}$

Where $\bigcup_{b \in \mathcal{D}'} \text{CW}_{G, G \setminus K_r}^i(b)^*$ represents the set of darts belonging to at least one connecting walk.

Furthermore, there exists a one-to-one correspondence between connecting walks, as any i -connecting walk associated with a dart $d \in \mathcal{D}'$ may be built from the connecting walk associated with $d\alpha'_i$ (with the notations of Definition 9). This is illustrated on Figure 2(b). In fact, the above mentioned correspondence coincides with the application of an involution ; it is therefore itself an involution on the set of connecting walks.

Property 3 *Let G be an n - G -map and K_r be a removal kernel in G . Let $G' = G \setminus K_r = (\mathcal{D}', \alpha'_0, \dots, \alpha'_n)$. For all $i \in \{0, \dots, n-1\}$ and all $d \in \mathcal{D}'$; if $\text{CW}_{G, G'}^i(d) = (d_0 = d, d_1, \dots, d_p)$ we have:*

$$\text{CW}_{G, G'}^i(d\alpha'_i) = (b_0 = d\alpha'_i, b_1, \dots, b_p) \text{ where } b_k = d_{p-k}\alpha_i \text{ for } 0 \leq k \leq p$$

Since Property 1 does not guarantee that a dart always belong to a connecting walk, all darts that have been removed may not be traversed by following all the connecting walks. Hence we say that a removal kernel K_r is *simple* if the following property holds:

$$\forall i \in \{0, \dots, n-1\}, \forall d \in R_i, \exists s \in \mathcal{D}' \mid d \in \text{CW}_{G, G'}^i(s)^{o*}$$

By Proposition 1 the dart s is necessarily unique and we deduce the following property.

Property 4 If G is an n - G -map and K_r is a simple removal kernel in G , then we have

$$\mathcal{D} = \mathcal{D}' \sqcup \left[\bigsqcup_{d \in \mathcal{D}', 0 \leq i \leq n-1} \text{CW}_{G,G'}^i(d)^{\circ*} \right]$$

where \sqcup denotes the union of disjoint sets.

When Property 4 applies, the traversal of the connecting walks of all the darts of \mathcal{D}' is guaranteed to visit all darts of \mathcal{D} . In a pyramid, this means that a level may be rebuilt with no hole from the level above it ; in other words there is no loss of information when reducing a map using a simple kernel. Simple removal kernels may be characterized, in a computationally more efficient way, using the following proposition:

Proposition 2 A removal kernel $K_r = \{R_i\}_{i=0,\dots,n}$ in an n - G -map G is simple if and only if:

$$\forall i \in \{0, \dots, n-1\}, \forall d \in R_i^*, \langle \alpha_i \alpha_{i+1} \rangle (d) \cap \mathcal{D}' \neq \emptyset$$

where \mathcal{D}' is the set of darts of $G \setminus K_r$.

Not all removal kernel may be decomposed into simple ones. However, using Proposition 2 some removal operations may be delayed in order to obtain a simple kernel between two specified levels.

4.2 Connecting walks in maps

Definition 13 (Connecting walk in n -maps) Let $M = (\mathcal{D}, \gamma_0, \dots, \gamma_{n-1})$ be an n -map and $K_r = \{R_i\}_{0 \leq i \leq n}$ be a removal kernel in M . Let $M' = M \setminus K_r = (\mathcal{D}', \gamma'_0, \dots, \gamma'_{n-1})$. The i -connecting walk associated to a dart $d \in \mathcal{D}'$ for $i \in \{0, \dots, n-1\}$, denoted by $\text{CW}_{M,M'}^i(d)$, is the sequence of darts of \mathcal{D} defined by

$$\text{CW}_{M,M'}^i(d) = (d_0 = d, d_1, \dots, d_p)$$

where

- For $i \in \{0, \dots, n-2\}$,
 $\forall u, 0 \leq u \leq p, d_u = d(\gamma_i \gamma_{i+1}^{-1})^u$ and $p = \text{Min}\{k \in \mathbb{N} \mid d_k \gamma_i \in \mathcal{D}'\}$
- For $i = n-1$,
 $\forall u, 0 \leq u \leq p, d_u = d\gamma_{n-1}^u$ and $p = \text{Min}\{k \in \mathbb{N} \mid d_k \gamma_{n-1} \in \mathcal{D}'\}$

Again, we have the two following properties which link the definition of the removal operation of cells with the one of connecting walks.

Property 5 With the notations of Definition 13, for all $d \in \mathcal{D}'$ such that $\text{CW}_{M,M'}^i(d) = (d_0, d_1, \dots, d_p)$ we have:

$$\forall k \in \{1, \dots, p\}, d_{k-1} \gamma_i \in R_i^* \text{ and } d_k \in R_i^*$$

Property 6 Let $M = (\mathcal{D}, \gamma_0, \dots, \gamma_{n-1})$ be an n -map, K_r be a removal kernel in M , $M' = M \setminus K_r = (\mathcal{D}', \gamma_0, \dots, \gamma'_{n-1})$ and $d \in \mathcal{D}'$. For all $i \in \{0, \dots, n\}$ we have $d\gamma'_i = \text{last}(\text{CW}_{M,M \setminus K_r}^i(d))\gamma_i$.

As for G-map, connecting walks within maps also provide a father-child relationship, as stated by the following proposition whose precise statement follows the one of Proposition 1 (see also [4, Proposition 27]).

Proposition 3 *Let M be an n -map, K_r be a removal kernel in M , and d be a dart of R_i^* for $0 \leq i < n$. The dart d belongs to at most one connecting walk.*

This property together with Proposition 1 shows that the time required to compute a reduced map given a removal kernel and a set of surviving darts is bounded by the size of the reduced map or G-map. The time required to compute a reduced map or G-map is thus bounded by twice the number of initial darts. As we claimed in our introduction, generalized maps do not allow to manipulate easily notions related with the orientation over the underlying quasi-manifold, when the latter is orientable. This in due, in part, to the fact that in this case a G-map, by using twice as many darts as really needed, actually encodes the two possible orientations at the same time. A connecting walk in a G-map G , as defined in this paper, uses a fixed orientation by skipping darts. Indeed, all darts of the walk thus belong to a single connected component of the map of the hypervolumes $HV(G)$ associated to the G-map G (Definition 5). It is therefore consistent with respect to the orientation property since each component of $HV(G)$ corresponds to one orientation of G . These remarks are based on Proposition 5, for which Proposition 4 is an important intermediary result.

Proposition 4 *With the notations of Definition 13, for all $d \in \mathcal{D}'$ and all $i \in \{0, \dots, n-2\}$, if $CW_{M,M'}^i(d) = (d_0, d_1, \dots, d_p)$ we have*

$$\forall r \in \{0, \dots, p\}, d_r = d_0(\gamma_i \gamma_{i+1})^r$$

Compared to Definition 13, Proposition 4 states that both permutations $\gamma_i \gamma_{i+1}^{-1}$ and $\gamma_i \gamma_{i+1}$ may be used to define a connecting walk.

Proposition 5 *Let $G = (\mathcal{D}, \alpha_0, \dots, \alpha_n)$ be an n -G-map and $M = HV(G)$ be its n -map of the hypervolumes. Let K_r be a removal kernel in G , let $G' = G \setminus K_r$ and $M' = M \setminus HV(K_r) = (\mathcal{D}', \gamma'_0, \dots, \gamma'_{n-1})$. For any dart $d \in \mathcal{D}$ and any $i \in \{0, \dots, n-2\}$. The i -connecting walks of d respectively in G and M (with respect to K_r and $HV(K_r)$) satisfy*

$$CW_{G,G'}^i(d) = CW_{M,M'}^i(d)$$

Furthermore, we have $CW_{G,G'}^{(n-1)}(d)^\circ = \text{reverse}(CW_{M,M'}^{(n-1)}(d\gamma'_{n-1})^\circ)$.

As shown by the the next property, we also proved that an involution may be defined on the set of i -connecting walks in a map, for $i < n-1$.

Property 7 *Let M be an n -map and K_r be a removal kernel in M . Let $M' = M \setminus K_r = (\mathcal{D}', \gamma'_0, \dots, \gamma'_{n-1})$. For $i \in \{0, \dots, n-2\}$, $d \in \mathcal{D}'$, and $b = d\gamma'_i$; if $CW_{M,M'}^i(d) = (d_0 = d, d_1, \dots, d_p)$ we have $CW_{M,M'}^i(b) = (b_0 = b, b_1, \dots, b_p)$ where $b_k = d_{p-k}\gamma'_i$ for $0 \leq k \leq p$.*

5 n -D Combinatorial pyramids

In this section we define pyramids of combinatorial n -maps and introduce the connecting dart sequences which will be used to derive a concise encoding of pyramids.

Definition 14 (Pyramid of n -maps) *A pyramid of n -maps with height $h \in \mathbb{N}$ is an h -tuple (M_0, K_1, \dots, K_h) where M_0 is an n -map and K_l , $l \in \{1, \dots, h\}$, is a removal kernel for the map M_{l-1} , which is defined by $M_l = M_{l-1} \setminus K_l$ for $l \in \{1, \dots, h\}$.*

When dealing with a pyramid of n -maps (M_0, K_1, \dots, K_h) , $h \in \mathbb{N}^*$, we usually denote $M_l = (\mathcal{D}_l, \gamma_{l,0}, \dots, \gamma_{l,n-1})$ for $l \in \{0, \dots, h\}$, and when no confusion may arise we simply refer to a permutation of M_l as $\gamma_{l,i}$ for $i \in \{0, \dots, n-1\}$ without mentioning the map M_l . We also shorten $\gamma_{0,i}$ as γ_i for all $i \in \{0, \dots, n-1\}$. Eventually, we denote $K_l = \{R_{l,i}\}_{i=1, \dots, n}$.

We may now give the definition of a connected dart sequence which makes the link, as shown by two propositions given further on, between any two levels of a pyramid the same way a connecting walk does between two consecutive levels.

Definition 15 (Connecting dart sequence) *Let (M_0, K_1, \dots, K_h) be a pyramid of n -maps and d be a dart of \mathcal{D}_l for $l \in \{0, \dots, h\}$. If $CW_{M_{l-1}, M_l}^i(d) = (d = d_0, \dots, d_p)$ for $i \in \{0, \dots, n-1\}$, we define the i -connecting dart sequence associated to d at level l , denoted by $CDS_l^i(d)$, as follows:*

- For $l = 0$, $CDS_0^i(d) = (d)$, and
 - for $l \in \{1, \dots, h\}$
 - If $i \leq n - 2$, $CDS_l^i(d) = GL_{l-1}^i(d_0) \cdot GL_{l-1}^i(d_1) \cdot \dots \cdot GL_{l-1}^i(d_p)$
- where $\begin{cases} \forall r \in \{0, \dots, p-1\}, GL_{l-1}^i(d_r) = CDS_{l-1}^i(d_r) \cdot CDS_{l-1}^{i+1}(d_r \gamma_{l-1,i}) \\ GL_{l-1}^i(d_p) = CDS_{l-1}^i(d_p) \end{cases}$
- If $i = n - 1$, $CDS_l^{n-1}(d) = CDS_{l-1}^{n-1}(d_0) \cdot CDS_{l-1}^{n-1}(d_1) \cdot \dots \cdot CDS_{l-1}^{n-1}(d_p)$.

Note that for any $d \in D_1$, such that $CW_{M_0, M_1}^i(d) = (d_0, \dots, d_i, \dots, d_p)$, $CDS_1^i(d) = (d_0, d_0 \gamma_{0,i}, \dots, d_i, d_i \gamma_{0,i}, \dots, d_p)$. The sequence $CDS_1^i(d)$ has thus twice as many darts as $CW_{M_0, M_1}^i(d)$. This is a major difference with the 2D case, due to the fact that the straightforward extension of 2D connecting dart sequences to the n D case may induce important gaps within such sequences.

One may obviously not expect the darts of a such defined connecting dart sequence to belong to removed cells of a single dimension, as it is the case for connecting walks (Propositions 1 and 5). For example, darts of the connecting dart sequence $CDS_2^0(b)$ in Figure 3 belong to both 1-cells and 0-cells which have been removed from M_0 and M_1 , respectively. Still, the first dart of a connecting dart sequence at level l is the only dart belonging to \mathcal{D}_l . Indeed, we have the following proposition.

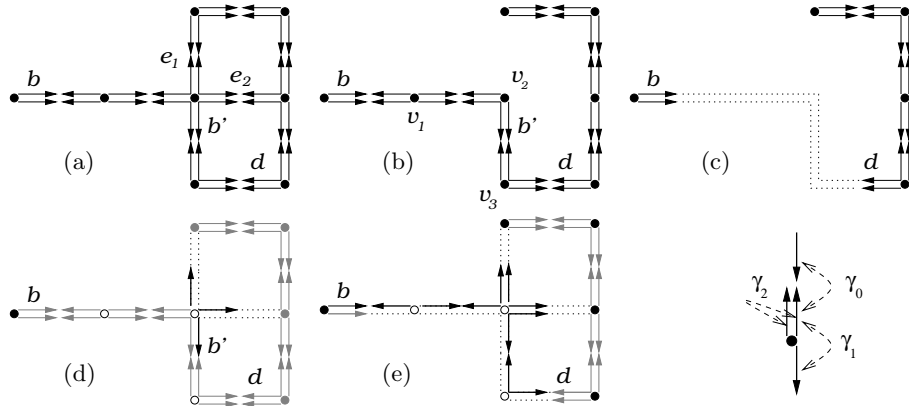


Fig. 3. A 3D combinatorial pyramid (M_0, K_1, K_2) . (a) The 3-map M_0 . (b) The 3-map M_1 obtained after removal of the edges e_1 and e_2 from M_0 . (c) The map M_2 obtained after removal of the vertices v_1 , v_2 , and v_3 from M_1 . Four involutions γ_0 are materialized by two dotted lines. (d) The connecting walk $CW_{M_0, M_1}^1(b')$ (black darts). (e) The connecting dart sequence $CDS_2^0(b)$ (black darts).

Proposition 6 Let (M_0, K_1, \dots, K_h) be a pyramid of n -maps and $l \in \{1, \dots, h\}$. For all dart $d \in \mathcal{D}_l$ and $i \in \{0, \dots, n-1\}$ we have $CDS_l^i(d)^{\circ*} \cap \mathcal{D}_l = \emptyset$.

Connecting dart sequences also share with connecting walks the property that the last dart of an i -connecting dart sequence associated with a dart d at level l is linked with the dart $d\gamma_{l,i}$ by the permutation γ_i .

Proposition 7 Let (M_0, K_1, \dots, K_h) be a pyramid of n -maps for $h \in \mathbb{N}^*$, with the notations of Definition 15. Let $d \in \mathcal{D}_l$ for $l \in \{1, \dots, h\}$. We have

$$\text{last}(CDS_l^i(d))\gamma_{0,i} = d\gamma_{l,i}$$

6 Conclusion

Using the definition given in [6] for the simultaneous removal of cells in an n -map, we have defined here n -dimensional combinatorial pyramids the way Brun and Kropatsch did in the two-dimensional case ([1]) and following the works of Grasset et al. about pyramids of generalized maps ([8]). We have defined connecting walks in both maps and G-maps, and established a link between the two definitions. Such walks are analogous to the *reduction windows* of regular pyramids. Connecting dart sequences, which are analogous to the *receptive fields* within regular pyramids, have also been defined.

The next step of this work consists in the definition of an implicit encoding of n -dimensional combinatorial pyramids (see [1]). This last result will allow us to study several application fields such as 3D hierarchical segmentation of medical images given an initial segmentation to reduce the amount of data and video analysis using time as a third or fourth dimension.

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