CELLS AND CACTI

by

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Abstract. — Let (W,S) be a Coxeter system, let φ be a weight function on S and let Cact_W denote the associated *cactus group*. Following an idea of I. Losev, we construct an action of $\mathrm{Cact}_W \times \mathrm{Cact}_W$ on W which has nice properties with respect to the partition of W into left, right or two-sided cells (under some hypothesis, which hold for instance if φ is constant).It must be noticed that the action depends heavily on φ.

Let (W,S) be a Coxeter system with S finite and let φ be a positive *weight function* on S as defined by Lusztig [**Lu2**]. We denote by Cact_W the *Cactus group* associated with W, as defined for instance in [**Lo**] (see also Section 5). In [**Lo**], I. Losev has constructed, whenever W is a finite Weyl group and φ is constant, an action of $\mathrm{Cact}_W \times \mathrm{Cact}_W$ on W which satisfies some good properties with respect to the partition of W into cells. His construction is realized as the combinatorial shadow of wall-crossing functors on the category $\mathcal{O}$.

In [**Lo**, §5.1], I. Losev suggested that this action could be obtained without any reference to some category $\mathcal{O}$, and thus extended to other types of Coxeter groups and general weight functions φ, using some recent results of Lusztig [**Lu3**]. This is the aim of this paper to show that Losev's idea works, by using slight extensions of results from [**BoGe**] and assuming that some of Lusztig's Conjectures in [**Lu2**, §14.2] hold, as in [**Lu3**]. Note that, if φ is constant, then these Conjectures hold, so this provides at least an action in the equal parameter case: if moreover W is a Weyl group, this action coincides with the one constructed by Losev [**Lo**].

Let us now state our main result. If $I \subset S$, we denote by W_I the subgroup generated by I and by φ_I the restriction of φ to I. If C is a left (respectively right) cell, then $\mathscr{H}^L[C]$ (respectively $\mathscr{H}^R[C]$) denotes the associated left (respectively right) $\mathscr{H}$-module and c_w^L (respectively c_w^R) denotes the image of the Kazhdan-Lusztig basis element C_w in this module (see §1.A). Finally, we set $\boldsymbol{\mu}_2 = \{1,-1\}$.

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Theorem.— *Assume that* ***Lusztig's Conjectures P1, P4, P8 and P9 in*** [**Lu2**, §14] ***hold for all triples*** (W_I, I, φ_I) ***such that*** W_I ***is finite****. Then there exists an action of* $\mathrm{Cact}_W \times \mathrm{Cact}_W$ *on the set* W *such that, if we denote by* τ_φ^L *(respectively* τ_φ^R*) the permutation of* W *obtained through the action of* $(\tau, 1) \in \mathrm{Cact}_W \times \mathrm{Cact}_W$ *(respectively* $(1, \tau) \in \mathrm{Cact}_W \times \mathrm{Cact}_W$*), then:*

(a) *If* C *is a left cell, then* $\tau_\varphi^L(C)$ *is also a left cell. Moreover, there exists a sign map* $\eta_L^{\tau,\varphi} : W \to \boldsymbol{\mu}_2$ *such that the A-linear map* $\mathscr{H}^L[C] \xrightarrow{\sim} \mathscr{H}^L[\tau_\varphi^L(C)]$, $c_w^L \mapsto \eta_{L,w}^{\tau,\varphi} c_{\tau_\varphi^L(w)}^L$ *is an isomorphism of left* $\mathscr{H}$*-modules.*

(a′) *If* C *is a right cell, then* $\tau_\varphi^R(C)$ *is a also right cell. Moreover, there exists a sign map* $\eta_R^{\tau,\varphi} : W \to \boldsymbol{\mu}_2$ *such that the A-linear map* $\mathscr{H}^R[C] \xrightarrow{\sim} \mathscr{H}^R[\tau_\varphi^R(C)]$, $c_w^R \mapsto \eta_{R,w}^{\tau,\varphi} c_{\tau_\varphi^R(w)}^R$ *is an isomorphism of right* $\mathscr{H}$*-modules.*

(b) *If* $w \in W$*, then* $\tau_\varphi^L(w) \sim_R w$ *and* $\tau_\varphi^R(w) \sim_L w$*.*

Commentary.— Lusztig [**Lu2**, §14.2] proposed several Conjectures relating the so-called *Lusztig's* **a***-function* and the partition of W into cells. Throughout this paper, the expression *Lusztig's Conjecture Pi* will refer to [**Lu2**, §14.2, Conjecture Pi] (for $1 \leqslant i \leqslant 15$). For instance, they all hold if φ is constant [**Lu2**, §15]. ■

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1. Notation

Set-up. *We fix a Coxeter system* (W, S)*, whose length function is denoted by* $\ell : W \to \mathbb{N}$*. We also fix a totally ordered abelian group* $\mathscr{A}$ *and we denote by* A *the group algebra* $\mathbb{Z}[\mathscr{A}]$*. We use an exponential notation for* A*:*

$$A = \oplus_{a \in \mathscr{A}} \mathbb{Z} v^a \quad \text{where} \quad v^a v^{a'} = v^{a+a'} \quad \text{for all } a,\ a' \in \mathscr{A}.$$

If $a_0 \in \mathscr{A}$*, we write* $\mathscr{A}_{\leqslant a_0} = \{a \in \mathscr{A} \mid a \leqslant a_0\}$ *and* $A_{\leqslant a_0} = \oplus_{a \in \mathscr{A}_{\leqslant a_0}} \mathbb{Z} v^a$*; we define similary* $A_{<a_0}$*,* $A_{\geqslant a_0}$*,* $A_{>a_0}$*. We denote by* $\overline{} : A \to A$ *the involutive automorphism such that* $\overline{v^a} = v^{-a}$ *for all* $a \in \mathscr{A}$*. Since* $\mathscr{A}$ *is totally ordered,* A *inherits two maps* $\deg : A \to \mathscr{A} \cup \{-\infty\}$ *and* $\mathrm{val} : A \to \mathscr{A} \cup \{+\infty\}$ *respectively called* ***degree*** *and* ***valuation****, and which are defined as usual.*

We also fix a ***weight function*** $\varphi : S \to \mathscr{A}_{>0}$ *(that is,* $\varphi(s) = \varphi(t)$ *for all* $s,\ t \in S$ *which are conjugate in* W*) and, if* $I \subset S$*, we denote by* $\varphi_I : I \to \mathscr{A}_{>0}$ *the restriction of* φ*.*

1.A. Cells. — Let $\mathcal{H} = \mathcal{H}(W,S,\varphi)$ denote the Iwahori-Hecke algebra associated with the triple (W,S,φ). This A-algebra is free as an A-module, with a standard basis denoted by $(T_w)_{w\in W}$. The multiplication is completely determined by the following two rules:

$$\begin{cases} T_w T_{w'} = T_{ww'} & \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ (T_s - v^{\varphi(s)})(T_s + v^{-\varphi(s)}) = 0 & \text{if } s \in S. \end{cases}$$

The involution $\overline{}$ on A can be extended to an A-semilinear involutive automorphism $\overline{} : \mathcal{H} \to \mathcal{H}$ by setting $\overline{T}_w = T_{w^{-1}}^{-1}$. Let

$$\mathcal{H}_{<0} = \mathop{\oplus}_{w\in W} A_{<0} T_w.$$

If $w \in W$, there exists **[Lu2]** a unique $C_w \in \mathcal{H}$ such that

$$\begin{cases} \overline{C}_w = C_w, \\ C_w \equiv T_w \mod \mathcal{H}_{<0}. \end{cases}$$

It is well-known **[Lu2]** that $(C_w)_{w\in W}$ is an A-basis of $\mathcal{H}$ (called the *Kazhdan-Lusztig basis*) and we will denote by $h_{x,y,z} \in A$ the structure constants, defined by

$$C_x C_y = \sum_{z\in W} h_{x,y,z} C_z.$$

We also write

$$C_y = \sum_{x\in W} p^*_{x,y} T_x,$$

with $p^*_{x,y} \in A$. Recall that $p^*_{y,y} = 1$ and $p^*_{x,y} \in A_{<0}$ if $x \neq y$.

We will denote by $\leqslant_L$, $\leqslant_R$, $\leqslant_{LR}$, $<_L$, $<_R$, $<_{LR}$, $\sim_L$, $\sim_R$ and $\sim_{LR}$ the relations defined in **[Lu2]** and associated with the triple (W,S,φ): the relation $\leqslant_L$ is the finest preorder on W such that, for any $w \in W$, $\oplus_{x\leqslant_L w} AC_x$ is a left ideal of $\mathcal{H}$, while $\sim_L$ is the associated equivalence relation associated (the other relations are defined similarly, by replacing left ideal by right or two-sided ideal). Also, we will call left, right and two-sided cells the equivalence classes for the relations $\sim_L$, $\sim_R$ and $\sim_{LR}$ respectively. If C is a left cell, we set

$$\mathcal{H}^{\leqslant_L C} = \mathop{\oplus}_{w\leqslant_L C} A\, C_w, \quad \mathcal{H}^{<_L C} = \mathop{\oplus}_{w<_L C} A\, C_w \quad \text{and} \quad \mathcal{H}^L[C] = \mathcal{H}^{\leqslant_L C}/\mathcal{H}^{<_L C}.$$

These are left $\mathcal{H}$-modules. If $w \in C$, we denote by c^L_w the image of C_w in the quotient $\mathcal{H}^L[C]$ and $\mathcal{H}^{\leqslant_L C}$ and $\mathcal{H}^{<_L C}$ might be also denoted by $\mathcal{H}^{\leqslant_L w}$ and $\mathcal{H}^{<_L w}$ respectively: it is clear that $(c^L_w)_{w\in C}$ is an A-basis of $\mathcal{H}^L[C]$. If C is a right (respectively two-sided) cell, we define similarly $\mathcal{H}^{\leqslant_R C}$, $\mathcal{H}^{<_R C}$ and $\mathcal{H}^R[C]$ (respectively $\mathcal{H}^{\leqslant_{LR} C}$, $\mathcal{H}^{<_{LR} C}$ and $\mathcal{H}^{LR}[C]$), as well as c^R_w (respectively c^{LR}_w).

1.B. Parabolic subgroups. — We denote by $\mathscr{P}(S)$ the set of subsets of S. If $I \subset S$, we denote by W_I the standard parabolic subgroup generated by I and by X_I the set of elements $x \in W$ which have minimal length in xW_I. We also define $\mathrm{pr}_L^I : W \to W_I$ and $\mathrm{pr}_R^I : W \to W_I$ by the following formulas:

$$\forall\ x \in X_I,\ \forall\ w \in W_I, \quad \mathrm{pr}_L^I(xw) = w \quad \text{and} \quad \mathrm{pr}_R^I(wx^{-1}) = w.$$

If $\delta : W_I \to W_I$ is any map, we denote by $\delta^L : W \to W$ and $\delta^R : W \to W$ the maps defined by

$$\delta^L(xw) = x\delta(w) \qquad \text{and} \qquad \delta^R(wx^{-1}) = \delta(w)x^{-1}$$

for all $x \in X_I$ and $w \in W_I$ (see [**BoGe**, §6]). We denote by $\delta^{\mathrm{op}} : W_I \to W_I$ the map defined by

$$\delta^{\mathrm{op}}(w) = \delta(w^{-1})^{-1}$$

for all $w \in W$. Note that $\delta^R = ((\delta^{\mathrm{op}})^L)^{\mathrm{op}}$. If $\sigma : W \to W$ is any automorphism such that $\sigma(S) = S$, then

$$\sigma \circ \mathrm{pr}_L^I = \mathrm{pr}_L^{\sigma(I)} \circ \sigma \quad \text{and} \quad \sigma \circ \mathrm{pr}_R^I = \mathrm{pr}_R^{\sigma(I)} \circ \sigma. \tag{1.1}$$

If $\mathscr{E}$ is a set and $\mu : W_I \to \mathscr{E}$ is any map, we define $\mu_L : W \to \mathscr{E}$ (respectively $\mu_R : W \to \mathscr{E}$) by

$$\mu_L = \mu \circ \mathrm{pr}_L^I \qquad \text{(respectively } \mu_R = \mu \circ \mathrm{pr}_R^I(w).$$

For instance, $\mathrm{pr}_L^I = (\mathrm{Id}_{W_I})_L$ and $\mathrm{pr}_R^I = (\mathrm{Id}_{W_I})_R$.

The Hecke algebra $\mathscr{H}(W_I, I, \varphi_I)$ will be denoted by $\mathscr{H}_I$ and will be viewed as a subalgebra of $\mathscr{H}$ in the natural way. It follows from the multiplication rules in the Hecke algebra that the right $\mathscr{H}_I$-module $\mathscr{H}$ is free (hence flat) with basis $(T_x)_{x \in X_I}$. This remark has the following consequence (in the next lemma, if E is a subset of $\mathscr{H}$, then $\mathscr{H}E$ denotes the left ideal generated by E):

Lemma 1.2. — *If $\mathfrak{I}$ and $\mathfrak{I}'$ are left ideals of $\mathscr{H}_I$ such that $\mathfrak{I} \subset \mathfrak{I}'$, then:*

(a) $\mathscr{H}\mathfrak{I} = \oplus_{x \in X_I} T_x \mathfrak{I}$.

(b) *The natural map $\mathscr{H} \otimes_{\mathscr{H}_I} \mathfrak{I} \to \mathscr{H}\mathfrak{I}$ is an isomorphism of left $\mathscr{H}$-modules.*

(c) *The natural map $\mathscr{H} \otimes_{\mathscr{H}_I} (\mathfrak{I}'/\mathfrak{I}) \to \mathscr{H}\mathfrak{I}'/\mathscr{H}\mathfrak{I}$ is an isomorphism of left $\mathscr{H}$-modules.*

Let $\mathscr{P}_{\mathrm{f}}(S)$ (respectively $\mathscr{P}_{\mathrm{ir,f}}(S)$) denote the set of subsets I of S such that W_I is finite (respectively such that W_I is finite and the Coxeter graph of (W_I, I) is connected). If $I \in \mathscr{P}_{\mathrm{f}}(S)$, we denote by w_I the longest element of W_I and we set

$$\begin{array}{rccc} \omega_I : & W_I & \longrightarrow & W_I \\ & w & \longmapsto & w_I w w_I. \end{array}$$

It is an automorphism of W_I which satisfies $\omega_I(I) = I$. If W is finite, then w_S will be denoted by w_0, according to the tradition. Also, ω_S will be denoted by ω_0.

If $I \in \mathscr{P}_{\mathrm{f}}(S)$, we denote by $\mathbf{a}_I : W_I \to \mathscr{A}$ the *Lusztig's* **a***-function* defined by

$$\mathbf{a}_I(z) = \max_{x,y \in W_I} \deg(h_{x,y,z})$$

for all $z \in W_I$. We also set $\boldsymbol{\alpha}_I(z) = \mathbf{a}_I(w_I z) - \mathbf{a}_I(z)$. If W itself is finite, then $\mathbf{a}_S$ and $\boldsymbol{\alpha}_S$ will be simply denoted by $\mathbf{a}$ and $\boldsymbol{\alpha}$ respectively.

1.C. Descent sets. — If $w \in W$, we set

$$\mathscr{L}(w) = \{s \in S \mid sw < w\} \qquad \text{and} \qquad \mathscr{R}(w) = \{s \in S \mid ws < w\}.$$

Then $\mathscr{L}(w)$ (respectively $\mathscr{R}(w)$) is called the *left descent set* (respectively *right descent set*) of w: it is easy to see that they both belong to $\mathscr{P}_{\mathrm{f}}(S)$. It is also well-known [**Lu2**, Lemma 8.6] that the map $\mathscr{L} : W \to \mathscr{P}_{\mathrm{f}}(S)$ (respectively $\mathscr{R} : W \to \mathscr{P}_{\mathrm{f}}(S)$) is constant on right (respectively left) cells.

1.D. Cells and parabolic subgroups. — We will now recall Geck's Theorem about the parabolic induction of cells [**Ge1**]. First, it is clear that $(C_w)_{w \in W_I}$ is the Kazhdan-Lusztig basis of $\mathscr{H}_I$. We can then define a preorder $\leqslant_L^I$ and its associated equivalence class $\sim_L^I$ on W_I in the same way as $\leqslant_L$ and $\sim_L$ are defined for W. We define similarly $\leqslant_R^I$, $\sim_R^I$, $\leqslant_{LR}^I$ and $\sim_{LR}^I$. If $w \in W$, then there exists a unique $a \in X_I$ and a unique $x \in W_I$ such that $w = ax$: we then set

$$G_w^I = T_a C_x.$$

It is easily seen that $(G_w^I)_{w \in W}$ is an A-basis of $\mathscr{H}$ so that we can write, for $b \in X_I$ and $y \in W_I$,

$$C_{by} = \sum_{\substack{a \in X_I \\ x \in W_I}} p_{a,x,b,y}^I \, T_a C_x,$$

where $p_{a,x,b,y}^I \in A$.

***Theorem 1.3* (Geck).** — *Let E be a subset of W_I such that, if $x \in E$ and if $y \in W_I$ is such that $y \leqslant_L^I x$, then $y \in E$. Let $\mathfrak{I} = \oplus_{w \in E} A\, C_w$. Then*

$$\mathscr{H}\mathfrak{I} = \bigoplus_{w \in X_I \cdot E} A\, G_w^I = \bigoplus_{w \in X_I \cdot E} A\, C_w.$$

In particular, if w, w' are elements of W are such that $w \leqslant_L w'$ (respectively $w \sim_L w'$), then $\mathrm{pr}_L^I(w) \leqslant_L^I \mathrm{pr}_L^I(w')$ (respectively $\mathrm{pr}_L^I(w) \sim_L^I \mathrm{pr}_L^I(w')$).

Moreover, if a, $b \in X_I$ and x, $y \in W_I$, then:

(a) $p_{b,y,b,y}^I = 1$.

(b) *If $ax \neq by$, then $p_{a,x,b,y}^I \in A_{<0}$.*

(c) *If $ax \neq by$ and $p_{a,x,b,y}^I \neq 0$, then $a < b$, $ax \leqslant by$ and $x \leqslant_L^I y$.*

***Corollary 1.4* (Geck).** — *We have:*

(a) $\leqslant_L^I$ *and* $\sim_L^I$ *are just the restriction of* $\leqslant_L$ *and* $\sim_L$ *to* W_I *(and so we will use only the notation* $\leqslant_L$ *and* $\sim_L$*).*

(b) *If* C *is a left cell in* W_I*, then* $X_I \cdot C$ *is a union of left cells of* W*.*

2. Preliminaries

Hypothesis and notation. *In this section, and only in this section, we fix an* A*-module* $\mathcal{M}$ *and we assume that:*

(I1) $\mathcal{M}$ *admits an* A*-basis* $(m_x)_{x\in X}$*, where* X *is a poset. We set*

$$\mathcal{M}_{<0} = \oplus_{x\in X} A_{<0}\, m_x.$$

(I2) $\mathcal{M}$ *admits a semilinear involution* $\overline{} : \mathcal{M} \to \mathcal{M}$*. We set*

$$\mathcal{M}_{\mathrm{skew}} = \{m \in \mathcal{M} \mid m + \overline{m} = 0\}.$$

(I3) *If* $x \in X$*, then* $\overline{m}_x \equiv m_x \mod \Big(\underset{y<x}{\oplus} A m_y\Big)$

(I4) *If* $x \in X$*, then the set* $\{y \in X \mid y \leqslant x\}$ *is finite.*

***Proposition 2.1*.** — *The* $\mathbb{Z}$*-linear map*

$$\begin{array}{ccc} \mathcal{M}_{<0} & \longrightarrow & \mathcal{M}_{\mathrm{skew}} \\ m & \longmapsto & m - \overline{m} \end{array}$$

is an isomorphism.

Proof. — First, note that the corresponding result for the A-module A itself holds. In other words,

(2.2) *The map* $A_{<0} \to A_{\mathrm{skew}}$*,* $a \mapsto a - \overline{a}$ *is an isomorphism.*

Indeed, if $a \in A_{\mathrm{skew}}$, write $a = \sum_{\gamma\in\Gamma} r_\gamma v^\gamma$, with $r_\gamma \in \mathbb{Z}$. Now, if we set $a_- = \sum_{\gamma<0} r_\gamma v^\gamma \in A_{<0}$, then $a = a_- - \overline{a}_-$. This shows the surjectivity, while the injectivity is trivial.

Now, let $\Lambda : \mathcal{M}_{<0} \to \mathcal{M}_{\mathrm{skew}}$, $m \mapsto m - \overline{m}$. For $\mathcal{X} \subset X$, we set $\mathcal{M}^{\mathcal{X}} = \oplus_{x\in\mathcal{X}} A\, m_x$ and $\mathcal{M}^{\mathcal{X}}_{<0} = \oplus_{x\in\mathcal{X}} A_{<0}\, m_x$. Assume that, for all $x \in \mathcal{X}$ and all $y \in X$ such that $y \leqslant x$, then $y \in \mathcal{X}$. By (I3), $\mathcal{M}^{\mathcal{X}}$ is stabilized by the involution $\overline{}$. Since X is the union of such finite $\mathcal{X}$ (by (I4)), it shows that we may, and we will, assume that X is finite. Let us write $X = \{x_0, x_1, \dots, x_n\}$ in such a way that, if $x_i \leqslant x_j$, then $i \leqslant j$ (this is always possible). For simplifying notation, we set $m_{x_i} = m_i$. Note that, by (I3),

$$(*) \qquad \overline{m}_i \in m_i + \Big(\underset{0\leqslant j<i}{\oplus} A\, m_j\Big).$$

In particular, $\overline{m}_0 = m_0$.

Now, let $m \in \mathcal{M}_{<0}$ be such that $\overline{m} = m$ and assume that $m \neq 0$. Write $m = \sum_{i=0}^{r} a_i m_i$, with $r \leqslant n$, $a_i \in A_{<0}$ and $a_r \neq 0$. Then, by (I2),

$$\overline{m} \equiv \overline{a}_r m_r \mod \Big(\underset{0 \leqslant j<i}{\oplus} A m_j\Big).$$

Since $\overline{m} = m$, this forces $\overline{a}_r = a_r$, which is impossible (because $a_r \in A_{<0}$ and $a_r \neq 0$). So Λ is injective.

Let us now show that Λ is surjective. So, let $m \in \mathcal{M}_{\mathrm{skew}}$, and assume that $m \neq 0$ (for otherwise there is nothing to prove). Write $m = \sum_{i=0}^{r} a_i m_i$, with $r \leqslant n$, $a_i \in A$ and $a_r \neq 0$. We shall prove by induction on r that there exists $\mu \in \mathcal{M}_{<0}$ such that $m = \mu - \overline{\mu}$. If $r = 0$, then the result follows from (2.2) and the fact that $\overline{m}_0 = m_0$. So assume that $r > 0$. Then

$$m + \overline{m} \equiv (a_r + \overline{a}_r) m_r \mod \mathcal{M}^{\mathcal{X}_{r-1}},$$

where $\mathcal{X}_j = \{x_0, x_1, \dots, x_j\}$. Since $m + \overline{m} = 0$, this forces $a_r \in A_{\mathrm{skew}}$. So, by (2.2), there exists $a \in A_{<0}$ such that $a - \overline{a} = a_r$. Now, let $m' = m - a m_r + \overline{a m}_r$. Then $m' + \overline{m}' = 0$ and $m' \in \oplus_{0 \leqslant j<r} A m_j$. So, by the induction hypothesis, there exists $\mu' \in \mathcal{M}_{<0}$ such that $m' = \mu' - \overline{\mu}'$. Now, set $\mu = a m_r + \mu'$. Then $\mu \in \mathcal{M}_{<0}$ and $m = \mu - \overline{\mu} = \Lambda(\mu)$, as desired. $\square$

Corollary 2.3. — *Let $m \in \mathcal{M}$. Then there exists a unique $M \in \mathcal{M}$ such that*

$$\begin{cases} \overline{M} = M, \\ M \equiv m \mod \mathcal{M}_{<0}. \end{cases}$$

Proof. — Setting $M = m + \mu$, the problem is equivalent to find $\mu \in \mathcal{M}_{<0}$ such that $\overline{m + \mu} = m + \mu$. This is equivalent to find $\mu \in \mathcal{M}_{<0}$ such that $\mu - \overline{\mu} = \overline{m} - m$: since $\overline{m} - m \in \mathcal{M}_{\mathrm{skew}}$, this problem admits a unique solution, thanks to Proposition 2.1. $\square$

The Corollary 2.3 can be applied to the A-module A itself. However, in this case, its proof becomes obvious: if $a_\circ = \sum_{\gamma \in \Gamma} a_\gamma v^\gamma$, then $a = \sum_{\gamma \leqslant 0} a_\gamma v^\gamma + \sum_{\gamma > 0} a_{-\gamma} v^\gamma$ is the unique element of A such that $\overline{a} = a$ and $a \equiv a_\circ \mod A_{<0}$.

Corollary 2.4. — *Let $\mathcal{X}$ be a subset of X such that, if $x \leqslant y$ and $y \in \mathcal{X}$, then $x \in \mathcal{X}$. Let $M \in \mathcal{M}$ be such that $\overline{M} = M$ and $M \in \mathcal{M}^{\mathcal{X}} + \mathcal{M}_{<0}$. Then $M \in \mathcal{M}^{\mathcal{X}}$.*

Proof. — Let $M_0 \in \mathcal{M}^{\mathcal{X}}$ be such that $M \equiv M_0 \mod \mathcal{M}_{<0}$. From the existence statement of Corollary 2.3 applied to $\mathcal{M}^{\mathcal{X}}$, there exists $M' \in \mathcal{M}^{\mathcal{X}}$ such that $\overline{M}' = M'$ and $M' \equiv M_0 \mod \mathcal{M}^{\mathcal{X}}_{<0}$. The fact that $M = M' \in \mathcal{M}^{\mathcal{X}}$ now follows from the uniquenes statement of Corollary 2.3. $\square$

Corollary 2.5. — *Let $x \in X$. Then there exists a unique element $M_x \in \mathscr{M}$ such that*

$$\begin{cases} \overline{M}_x = M_x, \\ M_x \equiv m_x \mod \mathscr{M}_{<0}. \end{cases}$$

Moreover, $M_x \equiv m_x \mod \oplus_{y<x} A_{<0} m_y$ and $(M_x)_{x\in X}$ is an A-basis of $\mathscr{M}$.

Proof. — The existence and uniqueness of M_x follow from Corollary 2.3. The statement about the base change follows by applying this existence and uniqueness to $\mathscr{M}^{X_x}$, where $X_x = \{y \in X \mid y \leqslant x\}$.

Finally, the fact that $(M_x)_{x\in X}$ is an A-basis of $\mathscr{M}$ follows from the fact that the base change from $(m_x)_{x\in X}$ to $(M_x)_{x\in X}$ is unitriangular. □

3. Cellular pairs

We set $\boldsymbol{\mu}_2 = \{1, -1\}$. The following definition extends slightly [**BoGe**, Definition 4.1]:

Definition 3.1. — *Let $\delta : W \to W$ and $\mu : W \to \boldsymbol{\mu}_2$, $w \mapsto \mu_w$ be two maps. Then the pair (δ, μ) is called **left cellular** if the following conditions are satisfied for every left cell C of W:*

(LC1) *$\delta(C)$ is also is a left cell.*

(LC2) *The A-linear map $(\delta, \mu)_C : \mathscr{H}^L[C] \to \mathscr{H}^L[\delta(C)]$, $c_w^L \mapsto \mu_w c_{\delta(w)}^L$ is an isomorphism of left $\mathscr{H}$-modules.*

*It is called **strongly left cellular** if it is left cellular and if satisfies moreover the following condition:*

(LC3) *If $w \in W$, then $\delta(w) \sim_R w$.*

If μ is constant and δ satisfies (LC1) *and* (LC2) *(respectively* (LC1), (LC2) *and* (LC3)*), then we say that δ is a **left cellular** map (respectively a **strongly left cellular** map).*

*We define similarly the notions of **right cellular** and **strongly right cellular** pair or map, as well as the notion of **two-sided cellular** pair or map.*

The case where μ is constant corresponds to [**BoGe**, Definition 4.1]. We will see in the next section that there exist left cellular pairs (δ, μ) such that μ is not constant.

3.A. Strongness. — It is unclear if there exist left cellular pairs or maps which are not strongly left cellular. At least, we are able to show that this probably cannot happen in finite Coxeter groups:

Proposition 3.2. — *Assume that W is* ***finite*** *and that* ***Lusztig's Conjectures P4 and P9 hold for*** (W,S,φ)*. Then any left (respectively right) cellular pair is strongly left (respectively right) cellular.*

Proof. — Assume that W is finite. Let (δ,μ) be a left cellular pair and let C be a left cell of W. Let K denote the fraction field of A. Since the algebra $K\mathscr{H} = K\otimes_A\mathscr{H}$ is semisimple, there exist two idempotents e and f of $K\mathscr{H}$ such that

$$K\mathscr{H}^{\leqslant_L C} = K\mathscr{H}e \oplus K\mathscr{H}^{<_L C} \qquad \text{and} \qquad K\mathscr{H}^{\leqslant_L \delta(C)} = K\mathscr{H}f \oplus K\mathscr{H}^{<_L \delta(C)}.$$

If $w\in C$ (respectively $w\in\delta(C)$), we write $C_w = c_w^e + d_w^e$ (respectively $C_w = c_w^f + d_w^f$) where $c_w^e \in K\mathscr{H}e$ and $d_w^e \in K\mathscr{H}^{<_L C}$ (respectively $c_w^f \in K\mathscr{H}f$ and $d_w^f \in K\mathscr{H}^{<_L\delta(C)}$). Then, by hypothesis, the K-linear map $\delta^* : K\mathscr{H}e \xrightarrow{\sim} K\mathscr{H}f$ such that $\delta^*(c_w^e) = \mu_w c_{\delta(w)}^f$ for all $w\in C$ is an isomorphism of $K\mathscr{H}$-modules.

Recall that any morphism of left $K\mathscr{H}$-modules $K\mathscr{H}e \to K\mathscr{H}f$ is of the form $m \mapsto mh$ for some $h \in eK\mathscr{H}f$. So there exists $h\in eK\mathscr{H}f$ such that, for all $w\in C$, $c_w^e h = \mu_w c_{\delta(w)}^f$. In other words,

$$C_w h - \mu_w C_{\delta(w)} = d_w^e h - \mu_w d_{\delta(w)}^f.$$

Now, let Γ denote the two-sided cell containing C. By the semisimplicity of $K\mathscr{H}$ and the fact that $\mathscr{H}^L[C] \simeq \mathscr{H}^L[\delta(C)]$, this forces $\delta(C)$ to be contained in Γ. By P4 and P9, we then have d_w^e, $d_{\delta(w)}^f \in K\mathscr{H}^{<_{LR}\Gamma}$, and so

$$C_w h - \mu_w C_{\delta(w)} \in K\mathscr{H}^{<_{LR}\Gamma}.$$

In particular, $\delta(w) \leqslant_R w$. Similarly, $w \leqslant_R \delta(w)$ and so $\delta(w) \sim_R w$, as desired. □

Note also the following result:

Proposition 3.3. — *Let (δ,μ) be a left (respectively right) cellular pair and let $w\in W$. Then $\mathscr{L}(\delta(w)) = \mathscr{L}(w)$ (respectively $\mathscr{R}(\delta(w)) = \mathscr{R}(w)$).*

Proof. — Let C denote the left cell of w and let $s\in S$. Then $s\in\mathscr{L}(w)$ if and only if $C_s c_w^L = (v^{\varphi(s)} + v^{-\varphi(s)})c_w^L$. So the result follows from the fact that the map $(\delta,\mu)_C$ is an isomorphism of left $\mathscr{H}$-modules. □

3.B. Induction of cellular pairs. — The next result extends slightly [**BoGe**, Theorem 6.2]. We present here a somewhat different proof, based on the results of Section 2.

Theorem 3.4. — *Let I be a subset of S and let (δ,μ) be a left cellular pair for (W_I, I, φ_I). Then (δ^L,μ_L) is a left cellular pair for (W,S,φ). If moreover (δ,μ) is strongly left cellular, then (δ^L,μ_L) is strongly left cellular.*

Proof. — The proof is divided in several steps:

• *First step: construction and properties of an isomorphism of left $\mathcal{H}$-modules.* Let C be a left cell of W_I. We denote by $\mathcal{E}$ (respectively $\mathcal{E}^\#$) the set of elements w in W_I such that $w \leqslant_L C$ (respectively $w <_L C$). By Lemma 1.2 and Theorem 1.3, the families $(G_w^I)_{w\in X_I\cdot\mathcal{E}}$ and $(C_w)_{w\in X_I\cdot\mathcal{E}}$ are A-basis of $\mathcal{H}\mathcal{H}_I^{\leqslant_L C}$. Similarly, the families $(G_w^I)_{w\in X_I\cdot\mathcal{E}^\#}$ and $(C_w)_{w\in X_I\cdot\mathcal{E}^\#}$ are A-basis of $\mathcal{H}\mathcal{H}_I^{<_L C}$.

If $w \in X_I\cdot C$, we denote by $\mathbf{g}_w^I$ (respectively $\mathbf{c}_w^I$) the image of G_w^I (respectively C_w) in $\mathcal{H}\mathcal{H}_I^{\leqslant_L C}/\mathcal{H}\mathcal{H}_I^{<_L C}$. Again by Lemma 1.2,

$$\mathcal{H}\otimes_{\mathcal{H}_I}\mathcal{H}_I^L[C] \simeq \mathcal{H}\mathcal{H}_I^{\leqslant_L C}/\mathcal{H}\mathcal{H}_I^{<_L C}.$$

Therefore, $(\mathbf{g}_w^I)_{w\in X_I\cdot C}$ and $(\mathbf{c}_w^I)_{w\in X_I\cdot C}$ can be viewed as A-bases of $\mathcal{H}\otimes_{\mathcal{H}_I}\mathcal{H}_I^L[C]$.

Since the pair (δ,μ) is left cellular, the A-linear map $\mathcal{H}_I^L[C]\to\mathcal{H}_I^L[\delta(C)]$, $c_w^L \mapsto \mu_w c_{\delta(w)}^L$ is an isomorphism of left $\mathcal{H}_I$-modules. Therefore, the A-linear map

$$\begin{array}{rccc}\theta: & \mathcal{H}\otimes_{\mathcal{H}_I}\mathcal{H}_I^L[C] & \longrightarrow & \mathcal{H}\otimes_{\mathcal{H}_I}\mathcal{H}_I^L[\delta(C)]\\ & \mathbf{g}_w^I & \longmapsto & \mu_{L,w}\mathbf{g}_{\delta(w)}^I\end{array}$$

is an isomorphism of left $\mathcal{H}$-modules.

Now, the left $\mathcal{H}$-modules $\mathcal{H}\mathcal{H}_I^{\leqslant_L C}$ and $\mathcal{H}\mathcal{H}_I^{<_L C}$ are stable under the involution $\overline{}$. So $\mathcal{H}\otimes_{\mathcal{H}_I}\mathcal{H}_I^L[C]$ inherits an action of the involution $\overline{}$. Similarly, $\mathcal{H}\otimes_{\mathcal{H}_I}\mathcal{H}_I^L[\delta(C)]$ inherits an action of the involution $\overline{}$. Moreover, these two A-modules (endowed with $\overline{}$) satisfy the hypotheses (I1), (I2), (I3) and (I4) of Section 2 (by Theorem 1.3).

Also, it follows from the definition that the isomorphism θ commutes with this involution. Therefore, $\overline{\theta(\mathbf{c}_w^I)} = \theta(\mathbf{c}_w^I)$ for all $w\in X_I\cdot C$. Moreover, it follows from Theorem 1.3 that

$$\theta(\mathbf{c}_w^I) \equiv \mu_{L,w}\mathbf{g}_{\delta(w)}^I \mod \oplus_{x\in X_I\cdot\delta(C)} A_{<0}\mathbf{g}_x^I.$$

But the element $\mathbf{c}_{\delta(w)}^I$ is stable under the involution $\overline{}$ and, again by Theorem 1.3, it satisfies

$$\mathbf{c}_{\delta(w)}^I \equiv \mathbf{g}_{\delta(w)}^I \mod \oplus_{x\in X_I\cdot\delta(C)} A_{<0}\mathbf{g}_x^I.$$

Therefore, by Proposition 2.1,

$$\theta(\mathbf{c}_w^I) = \mu_{L,w}\mathbf{c}_{\delta(w)}^I. \tag{3.5}$$

• *Second step: partition into left cells.* Now, assume that $w \sim_L w'$. According to Corollary 1.4(b), there exists a unique cell C in W_I such that $w,\ w' \in X_I \cdot C$. By the definition of $\leqslant_L$ and $\sim_L$, there exist four sequences $x_1,\dots,x_m$, $y_1,\dots,y_n$, $w_1,\dots,w_m$, $w_1',\dots,w_n'$ such that:

$$\begin{cases} w_1 = w,\ w_m = w', \\ w_1' = w',\ w_n' = w, \\ \forall\ i \in \{1,2,\dots,m-1\},\ h_{x_i,w_i,w_{i+1}} \neq 0, \\ \forall\ j \in \{1,2,\dots,n-1\},\ h_{y_j,w_j',w_{j+1}'} \neq 0. \end{cases}$$

Therefore, we have $w' = w_m \leqslant_L \cdots \leqslant_L w_2 \leqslant_L w_1 = w = w_n' \leqslant_L \cdots \leqslant_L w_2' \leqslant_L w_1' = w$ and so $w = w_1 \sim_L w_2 \sim_L \cdots \sim_L w_m = w' = w_1' \sim_L w_2' \sim_L \cdots \sim_L w_n' = w$. Again by Corollary 1.4(b), $w_i,\ w_j' \in X_I \cdot C$. So it follows from (3.5) that $h_{x,\delta^L(w_i),\delta^L(w_{i+1})} = \mu_{L,w_i}\mu_{L,w_{i+1}}h_{x,w_i,w_{i+1}}$ and $h_{x,\delta^L(w_j'),\delta^L(w_{j+1}')} = \mu_{L,w_j'}\mu_{L,w_{j+1}'}h_{y_j,w_j',w_{j+1}'}$ for all $x \in W$. Therefore,

$$\begin{cases} \forall\ i \in \{1,2,\dots,m-1\},\ h_{x_i,\delta^L(w_i),\delta^L(w_{i+1})} \neq 0, \\ \forall\ j \in \{1,2,\dots,n-1\},\ h_{y_j,\delta^L(w_j'),\delta^L(w_{j+1}')} \neq 0. \end{cases}$$

It then follows that

$$\begin{aligned} \delta^L(w') = \delta^L(w_m) \leqslant_L \cdots \leqslant_L \delta^L(w_2) &\leqslant_L \delta^L(w_1) = \delta^L(w) = \delta^L(w_n') \\ &\leqslant_L \cdots \leqslant_L \delta^L(w_2') \leqslant_L \delta^L(w_1') = \delta^L(w'), \end{aligned}$$

and so $\delta^L(w) \sim_L \delta^L(w')$, as expected. So we have proved that

$$\textit{if } w \sim_L w',\ \textit{then } \delta^L(w) \sim_L \delta^L(w'). \tag{$*$}$$

Now, let $\delta_1 : W_I \to W_I$ be the map defined by $\delta_1(x) = x$ if $x \notin \delta(C)$ and $\delta_1(\delta(x)) = x$ if $x \in C$. Let $\mu_1 : W \to \boldsymbol{\mu}_2$ be defined by $\mu_{1,x} = 1$ if $x \notin \delta(C)$ and $\mu_{1,\delta(x)} = \mu_x$ if $x \in C$. Since left cellular maps can be defined "locally" (i.e. left cells by left cells), it is easily checked that (δ_1, μ_1) is left cellular. So, applying $(*)$ to the pair (δ_1, μ_1) with w and w' replaced by $\delta^L(w)$ and $\delta^L(w')$, we obtain

$$w \sim_L w' \textit{ if and only if } \delta^L(w) \sim_L \delta^L(w'). \tag{3.6}$$

• *Third step: left cellularity.* Now, let C' be a left cell in W. It follows from (3.6) that $\delta^L(C')$ is also a left cell and it follows from (3.5) that the A-linear map $\mathscr{H}^L[C'] \to \mathscr{H}^L[\delta(C')]$, $c_w^L \mapsto \mu_{L,w} c_{\delta^L(w)}^L$ is an isomorphism of left $\mathscr{H}$-modules. In other words, (δ^L, μ_L) is left cellular.

• *Fourth step: strongness.* Assume moreover that (δ,μ) is strongly left cellular. Let $w \in W$. Let us write $w = ax$ with $a \in X_I$ and $x \in W_I$. Then $\delta(x) \sim_R x$ by (LC3) and so $\delta^L(w) = a\delta(x) \sim_R ax = w$ by [**Lu2**, Proposition 9.11]. □

The next result extends slightly [**Ge2**, Lemma 3.8].

Corollary 3.7. — *Let (δ,μ) be a left cellular pair for (W_I, I, φ_I) and let a, $b \in X_I$ and x, $y \in W_I$ be such that $x \sim_L y$. Then*

$$p^I_{a,x,b,y} = \mu_x \mu_y p^I_{a,\delta(x),b,\delta(y)}.$$

Proof. — This follows from (3.5). □

4. Action of the longest element

Hypothesis. *We fix in this section a subset $I \in \mathscr{P}_{\mathrm{f}}(S)$ such that **Lusztig's Conjectures P1, P4, P8 and P9 hold** for the triple (W_I, I, φ_I).*

Example 4.1. — Recall from [**Lu2**, §15] that, if the weight function φ_I is constant, then Lusztig's Conjectures P1, P2, P3,..., P15 hold for (W_I, I, φ_I). ■

4.A. The following result (which is crucial for our purpose) has been proved by Mathas [**Ma**] in the equal parameter case and extended by Lusztig [**Lu3**, Theorem 2.3] in the unequal parameter case:

***Theorem 4.2* (Mathas, Lusztig)**. — *Let $I \in \mathscr{P}_{\mathrm{f}}(S)$ be such that **Lusztig's Conjectures P1, P4, P8 and P9 hold** for the triple (W_I, I, φ_I). Then there exists a (unique) sign map $\eta^I : W_I \to \boldsymbol{\mu}_2$, $w \mapsto \eta^I_w$ and two (unique) involutions ρ_I and λ_I of the set W_I such that, for all $w \in W_I$,*

$$v^{\boldsymbol{a}_I(w)} T_{w_I} C_w \equiv \eta^I_w C_{\rho_I(w)} \mod \mathscr{H}_I^{<^I_{LR} w}$$

and

$$v^{\boldsymbol{a}_I(w)} C_w T_{w_I} \equiv \eta^I_w C_{\lambda_I(w)} \mod \mathscr{H}_I^{<^I_{LR} w}.$$

Note that $\lambda_I = \rho_I^{\mathrm{op}}$, that $\rho_I = \lambda_I \circ \omega_I$ and that

$$\rho_I(w) \sim_L w \qquad \text{and} \qquad \lambda_I(w) \sim_R w.$$

If W itself is finite and if Lusztig's Conjectures P1, P4, P8 and P9 hold for (W,S,φ), then λ_S, ρ_S and η^S will simply be denoted by λ, ρ and η respectively.

Remark 4.3. — We will explain here why we only need to assume that Lusztig's Conjectures P1, P4, P8 and P9 hold for the above Theorem to hold (in [**Lu3**, Theorem 2.5], Lusztig assumed that P1, P2,..., P14 and P15 hold). This will be a consequence of a simplification of the proof of [**Lu3**, Lemma 1.13], based on the ideas of [**Bo1**]. In particular, we avoid the use of the difficult Lusztig's Conjecture P15 and the construction/properties of the asymptotic algebra.

So assume that Lusztig's Conjectures P1, P4, P8 and P9 hold. We may, and we will, assume that $I=S$ (for simplifying notation). Let us write

$$T_{w_0}C_y = \sum_{x \leqslant_L y} \lambda_{x,y}\ C_x,$$

with $\lambda_{x,y}\in A$. Note that

$$T_{w_0}^{-1}C_y = \sum_{x \leqslant_L y} \overline{\lambda}_{x,y}\ C_x.$$

By [**Bo1**, Proposition 1.4(a)],

$$\deg(\lambda_{x,y})\leqslant -\boldsymbol{\alpha}(x) \text{ with equality only if } x\sim_L y.$$

By [**Bo1**, Proposition 1.4(b)],

$$\deg(\overline{\lambda}_{x,y})\leqslant \boldsymbol{\alpha}(y) \text{ with equality only if } x\sim_L y.$$

Assume now that $x\sim_L y$. Then $\boldsymbol{\alpha}(x)=\boldsymbol{\alpha}(y)$ by P4 and [**Lu2**, Corollary 11.7], so

$$\deg(\lambda_{x,y})\leqslant -\boldsymbol{\alpha}(y)\leqslant \mathrm{val}(\lambda_{x,y}).$$

So

$$\text{if } x\sim_L y, \text{ then } v^{\boldsymbol{\alpha}(y)}\lambda_{x,y}\in\mathbb{Z},$$

Thanks to P9, this is exactly the statement in [**Lu3**, Lemma 1.13(a)]. Note also that [**Lu3**, Lemma 1.13(b)] is already proved in [**Bo1**, Proposition 1.4(c)].

One can then check that, once [**Lu3**, Lemma 1.13] is proved, the argument developed in [**Lu3**, Proof of Theorem 2.3] to obtain Theorem 4.2 does not make use any more of Lusztig's Conjectures. ■

Remark 4.4. — In the equal parameter case, Mathas proved moreover that the sign map $w\mapsto \eta_w^I$ is constant on two-sided cells. However, this property does not hold in general, as it can be seen from direct computations whenever W is of type B_3 (and φ is given by $\varphi(t)=2$ and $\varphi(s_1)=\varphi(s_2)=1$, where $S=\{t,s_1,s_2\}$ and s_1s_2 has order 3). ■

Example 4.5. — Assume here that W is *finite*. Since $\{1\}$ and $\{w_0\}$ are two-sided cells, we have $\lambda(1)=\rho(1)=1$ and $\lambda(w_0)=\rho(w_0)=w_0$. Moreover, $\eta_1=(-1)^{\ell(w_0)}$ and $\eta_{w_0}=1$. ■

4.B. Cellularity. — One of the key results towards a construction of an action of the cactus group is the following:

Theorem 4.6. — *Let $I \in \mathscr{P}_{\mathrm{f}}(S)$ be such that* ***Lusztig's Conjectures P1, P4, P8 and P9 hold*** *for the triple (W_I, I, φ_I). Then the pair (λ_I, η^I) (respectively (ρ_I, η^I)) is strongly left (respectively right) cellular.*

Proof. — For simplifying notation, we may, and we will, assume that W is finite and $I=S$. It is sufficient to prove that λ is strongly left cellular. First, (LC3) holds by Theorem 4.2.

Let x and y be two elements of W such that $x \sim_L y$. Let Γ (respectively C) denote the two-sided (respectively left) cell containing x and y. Then there exists $x=x_0, x_1,\dots, x_m=y=y_0, y_1,\dots, y_n=x$ in W and elements $h_1,\dots, h_m$, $h'_1,\dots, h'_n$ of $\mathscr{H}$ such that C_{x_i} (respectively C_{y_j}) appears with a non-zero coefficient in the expression of $h_i C_{x_{i-1}}$ (respectively $h'_j C_{y_{j-1}}$) in the Kazhdan-Lusztig basis for $1 \leqslant i \leqslant m$ (respectively $1 \leqslant j \leqslant n$). Therefore, $y=x_m \leqslant_L \cdots \leqslant_L x_2 \leqslant_L x_1 = x = y'_n \leqslant_L \cdots \leqslant_L y'_2 \leqslant_L y'_1 = y$ and so $x_i, y_j \in C$. Hence, if we write

$$h_i C_{x_{i-1}} \equiv \sum_{u\in\Gamma} \beta_{i,u} C_u \mod \mathscr{H}^{<_{LR}\Gamma},$$

then $\beta_{i,x_i} \neq 0$ and

$$v^{\boldsymbol{\alpha}(\Gamma)} h_i C_{x_{i-1}} T_{w_0} \equiv \sum_{u\in\Gamma} v^{\boldsymbol{\alpha}(\Gamma)} \beta_{i,u} C_u T_{w_0} \mod \mathscr{H}^{<_{LR}\Gamma},$$

Therefore, by Theorem 4.2,

$$\eta_{x_{i-1}} h_i C_{\lambda(x_{i-1})} \equiv \sum_{u\in\Gamma} \eta_u \beta_{i,u} C_{\lambda(u)} \mod \mathscr{H}^{<_{LR}\Gamma},$$

and so $\lambda(x_i) \leqslant_L \lambda(x_{i-1})$. This shows that $\lambda(y) \leqslant_L \lambda(x)$ and we can prove similarly that $\lambda(x) \leqslant_L \lambda(y)$. Therefore, $\lambda(C)$ is contained in a unique left cell C'. But, similarly, $\lambda(C')$ is contained in a unique left cell, and contains C. So $\lambda(C)=C'$ is a left cell. This shows (LC1).

Finally the map $(\lambda,\eta)_C : \mathscr{H}^L[C] \to \mathscr{H}^L[\lambda(C)]$, $c_w^L \mapsto \eta_w c_{\lambda(w)}^L$ is obtained through the *right* multiplication by $v^{\boldsymbol{\alpha}(\Gamma)} T_{w_0}$. Since this right multiplication commutes with the left action of $\mathscr{H}$, this implies (LC2). □

Corollary 4.7. — *Let $I \in \mathscr{P}_{\mathrm{f}}(S)$ be such that* ***Lusztig's Conjectures P1, P4, P8 and P9 hold*** *for the triple (W_I, I, φ_I). Then the pair (λ_I^L, η_L^I) (respectively (ρ_I^R, η_R^I)) is strongly left (respectively right) cellular.*

Proof. — This follows from Theorems 3.4 and 4.6. □

It must be noticed that the maps λ_I^L and ρ_I^R depend on the weight function φ, even if it is not clear from the notation. The canonicity of their construction shows that, if $\sigma : W \to W$ is an automorphism such that $\sigma(S) = S$ and $\varphi \circ \sigma = \varphi$, then

$$\sigma \circ \lambda_I^L = \lambda_{\sigma(I)}^L \circ \sigma \qquad \text{and} \qquad \sigma \circ \rho_I^R = \rho_{\sigma(I)}^R \circ \sigma. \tag{4.8}$$

For instance, *if W is finite*, then $\omega_0 : W \to W$ satisfies the above properties and so

$$\omega_0 \circ \lambda_I^L = \lambda_{\omega_0(I)}^L \circ \omega_0 \qquad \text{and} \qquad \omega_0 \circ \rho_I^R = \rho_{\omega_0(I)}^R \circ \omega_0. \tag{4.9}$$

Corollary 4.10. — *Let $I \in \mathscr{P}_{\mathrm{f}}(S)$ be such that* ***Lusztig's Conjectures P1, P4, P8 and P9 hold*** *for the triple (W_I, I, φ_I) and let $w \in W$. Then*

$$\eta_{R,w}^I v^{\boldsymbol{a}_{I,R}(w)} T_{w_I} C_w \equiv C_{\rho_I^R(w)} \mod \mathscr{H}_I^{<_R \omega_{I,R}(w)} \mathscr{H}$$

and

$$\eta_{L,w}^I v^{\boldsymbol{a}_{I,L}(w)} C_w T_{w_I} \equiv C_{\lambda_I^L(w)} \mod \mathscr{H} \mathscr{H}_I^{<_L \omega_{I,L}(w)}.$$

Proof. — It is sufficient to prove the second congruence. Let $b \in X_I$ and $y \in W_I$ be such that $w = by$ (so that $y = \mathrm{pr}_L^I(w)$). By Theorem 1.3,

$$C_{by} \equiv \sum_{\substack{(a,x) \in X_I \times W_I \\ \text{such that } a \leqslant b \\ \text{and } x \sim_L y}} p_{a,x,b,y}^I T_a C_x \mod \mathscr{H} \mathscr{H}_I^{<_L y}.$$

If $x \sim_L y$, then $\mathbf{a}_I(x) = \mathbf{a}_I(y)$ and $\mathbf{a}_I(w_I x) = \mathbf{a}_I(w_I y)$ by P4, so it follows from Theorem 4.2 that

$$\eta_y^I v^{\boldsymbol{a}_I(y)} C_{by} T_{w_I} \equiv \sum_{\substack{(a,x) \in X_I \times W_I \\ \text{such that } a \leqslant b \\ \text{and } x \sim_L y}} \eta_y^I \eta_x^I p_{a,x,b,y}^I T_a C_{\lambda_I(x)} \mod \mathscr{H} \mathscr{H}_I^{<_L y}.$$

But, by Corollary 3.7, $p_{a,x,b,y}^I = \eta_y^I \eta_x^I p_{a,\lambda_I(x),b,\lambda_I(y)}$, so

$$\eta_{L,w}^I v^{\boldsymbol{a}_{I,L}(w)} C_w T_{w_I} \equiv C_{\lambda_I^L(w)} \mod \mathscr{H} \mathscr{H}_I^{<_L y} T_{w_I}.$$

It then remains to notice that $T_{w_I}^{-1} C_x T_{w_I} = C_{\omega_I(x)}$ for all $x \in W_I$, so that $\mathscr{H}_I^{<_L y} T_{w_I} = T_{w_I} \mathscr{H}_I^{<_L \omega_I(y)} = \mathscr{H}_I^{<_L \omega_I(y)}$ and the result follows. □

An important consequence of the previous characterization is the following:

Theorem 4.11. — *Let $I \in \mathscr{P}_{\mathrm{f}}(S)$ be such that* ***Lusztig's Conjectures P1, P4, P8 and P9 hold*** *for the triple (W_I, I, φ_I) and let (δ,μ) be a strongly left (respectively right) cellular pair. Then $\delta\circ\rho_I^R=\rho_I^R\circ\delta$ (respectively $\delta\circ\lambda_I^L=\lambda_I^L\circ\delta$). Moreover, $\eta^I_{R,\delta(w)}=\mu_w\mu_{\rho_I^R(w)}\eta^I_{R,w}$ (respectively $\eta^I_{L,w}=\eta^I_{L,\delta(w)}$) for all $w\in W$.*

Proof. — Assume that (δ,μ) is strongly left cellular. Let $w\in W$. By (LC3), we have $\delta(w)\sim_R w$ and so [**Ge1**, Theorem 1]

$$(*)\qquad \mathrm{pr}_I^R(\delta(w))\sim_R \mathrm{pr}_I^R(w).$$

Now, let us write

$$\eta^I_{R,w}v^{\boldsymbol{\alpha}_{I,R}(w)}T_{w_I}C_w\equiv\sum_{u\sim_L w}\beta_u C_u \mod \mathscr{H}^{<_L w},$$

with $\beta_u\in A$. Since (δ,μ) is left cellular, we get

$$\mu_w\eta^I_{R,w}v^{\boldsymbol{\alpha}_{I,R}(w)}T_{w_I}C_{\delta(w)}\equiv\sum_{u\sim_L w}\beta_u\mu_u C_{\delta(u)} \mod \mathscr{H}^{<_L\delta(w)}.$$

But, by Corollary 4.10, we have

$$\eta^I_{R,w}v^{\boldsymbol{\alpha}_{I,R}(w)}T_{w_I}C_w\equiv C_{\rho_I^R(w)} \mod \mathscr{H}_I^{<_R\omega_I(\mathrm{pr}_I^R(w))}\mathscr{H}$$

and $\rho_I^R(w)\sim_L w$ (because ρ_I^R is strongly right cellular by Corollary 4.7). Therefore, $\beta_{\rho_I^R(w)}=\mu_w\mu_{\rho_I^R(w)}$. Again by Corollary 4.10, we get

$$\eta^I_{R,w}v^{\boldsymbol{\alpha}_{I,R}(w)}T_{w_I}C_{\delta(w)}\equiv\eta^I_{R,w}\eta^I_{R,\delta(w)}C_{\rho_I^R(\delta(w))} \mod \mathscr{H}_I^{<_R\omega_I(\mathrm{pr}_I^R(w))}\mathscr{H}$$

(by using also $(*)$). Combining these results, we get

$$C_{\rho_I^R(\delta(w))}-\eta^I_{R,w}\eta^I_{R,\delta(w)}\mu_w\mu_{\rho_I^R(w)}C_{\delta(\rho_I^R(w))}\in\bigoplus_{z\in\mathscr{E}_1\cup\mathscr{E}_2\cup\mathscr{E}_3}AC_z,$$

where

$$\mathscr{E}_1=\{\delta(u)\mid u\sim_L w \text{ and } u\neq\rho_I^R(w)\},$$
$$\mathscr{E}_2=\{u\in W\mid u<_L\delta(w)\}$$

and

$$\mathscr{E}_3=\{u\in W\mid \mathrm{pr}_I^R(u)<_R\omega_I(\mathrm{pr}_I^R(w))\}$$

(we have used the fact that $\mathscr{H}_I^{<_R v}\mathscr{H}=\oplus_{\mathrm{pr}_I^R(u)<_R v}AC_u$ for all $v\in W_I$: this result is due to Geck [**Ge1**], see Theorem 1.3). So, in order to prove that $\rho_I^R(\delta(w))=\delta(\rho_I^R(w))$ and $\eta^I_{R,\delta(w)}=\mu_w\mu_{\rho_I^R(w)}\eta^I_{R,w}$, we only need to show that $\delta(\rho_I^R(w))\notin\mathscr{E}_1\cup\mathscr{E}_2\cup\mathscr{E}_3$.

First, by definition, $\delta(\rho_I^R(w))\notin\mathscr{E}_1$. Also, since ρ_I^R is strongly right cellular, we get that $\rho_I^R(w)\sim_L w$ by (LC3) and so $\delta(\rho_I^R(w))\sim_L\delta(w)$ because δ is left cellular (see (LC1)). So $\delta(\rho_I^R(w))\notin\mathscr{E}_2$. Finally, $\delta(\rho_I^R(w))\sim_R\rho_I^R(w)$ because δ is strongly left cellular (see (LC3)). So $\mathrm{pr}_I^R(\delta(\rho_I^R(w))\sim_R\mathrm{pr}_I^R(\rho_I^R(w))$ by [**Ge1**]. Since $\mathrm{pr}_I^R(\rho_I^R(w))=\rho_I(\mathrm{pr}_I^R(w))\sim_{LR}\mathrm{pr}_I^R(w)\sim_{LR}\omega_I(\mathrm{pr}_I^R(w))$ (see [**Lu3**, Lemma 1.2]) and so $\delta(\rho_I^R(w))\notin\mathscr{E}_3$ by P4, P8 and P9. $\square$

5. Action of the cactus group

We recall here the definition of the *cactus group* Cact_W associated with W. The group Cact_W is the group with the following presentation:

- Generators: $(\tau_I)_{I \in \mathscr{P}_{\mathrm{ir,f}}(S)}$;
- Relations: for all $I, J \in \mathscr{P}_{\mathrm{ir,f}}(S)$, we have:

$$\begin{cases} \text{(C1)} & \tau_I^2 = 1, \\ \text{(C2)} & [\tau_I, \tau_J] = 1 \quad \text{if } W_{I\cup J} = W_I \times W_J, \\ \text{(C3)} & \tau_I \tau_J = \tau_J \tau_{\omega_J(I)} \quad \text{if } I \subset J. \end{cases}$$

By construction, the map $\tau_I \mapsto w_I$ extends to a surjective morphism of groups $\mathrm{Cact}_W \twoheadrightarrow W$ which will not be used in this paper. The main result of this paper is the following:

Theorem 5.1. — *Let $I, J \in \mathscr{P}_{\mathrm{ir,f}}(S)$ be such that* (H_I) ***and*** (H_J) ***hold***. *Then:*

(a) $[\lambda_I^L, \rho_J^R] = \mathrm{Id}_W$.
(b) $(\lambda_I^L)^2 = (\rho_I^R)^2 = \mathrm{Id}_W$.
(c) *If $W_{I\cup J} = W_I \times W_J$, then $[\lambda_I^L, \lambda_J^L] = [\rho_I^R, \rho_J^R] = \mathrm{Id}_W$.*
(d) *If $I \subset J$, then $\lambda_I^L \lambda_J^L = \lambda_J^L \lambda_{\omega_J(I)}^L$ and $\rho_I^R \rho_J^R = \rho_J^R \rho_{\omega_J(I)}^R$.*

Proof. — (a) follows from Theorem 4.11, while (b) is obvious.

(c) Assume that $W_{I\cup J} = W_I \times W_J$. We only need to prove that $[\lambda_I^L, \lambda_J^L] = \mathrm{Id}_W$, the proof of the other equality being similar. Let $w \in W$ and write $w = xw'$, with $x \in X_{I\cup J}$ and $w' \in W_{I\cup J}$. Since $W_{I\cup J} = W_I \times W_J$ and so there exists $w_1 \in W_I$ and $w_2 \in W_J$ such that $w' = w_1 w_2 = w_2 w_1$. Note also that $xw_1 \in X_J$, $x\lambda_I(w_1) \in X_J$, $xw_2 \in X_I$ and $x\lambda_J(w_2) \in X_I$. Therefore,

$$\lambda_I^L(\lambda_J^L(w)) = \lambda_I^L(xw_1\lambda_J(w_2)) = \lambda_I^L(x\lambda_J(w_2)w_1) = x\lambda_J(w_2)\lambda_I(w_1)$$

and, similarly,

$$\lambda_J^L(\lambda_I^L(w)) = x\lambda_I(w_1)\lambda_J(w_2).$$

So $[\lambda_I^L, \lambda_J^L] = \mathrm{Id}_W$, as desired.

(d) Assume here that $I \subset J$. It is easily checked that we may assume that W is finite and $J = S$. Let $w \in W$. Then

$$\begin{aligned} \lambda_S^L(\lambda_I^L(w)) &= \rho_S^L(\omega_0(\lambda_I^L(w))) && \text{by Theorem 4.2.} \\ &= \rho_S \lambda_{\omega_0(I)}^L(\omega_0(w)) && \text{by (4.9),} \\ &= \lambda_{\omega_0(I)}^L(\rho_S(\omega_0(w)) && \text{by (a),} \\ &= \lambda_{\omega_0(I)}^L(\lambda_S(w)) && \text{by Theorem 4.2.} \end{aligned}$$

This proves the first equality and the second follows from a similar argument. □

Let $\mathfrak{S}_W$ denote the symmetric group on the set W and assume until the end of this section that *Lusztig's Conjectures P1, P4, P8 and P9 hold for the triple* (W_I, I, φ_I) *for all* $I \in \mathscr{P}_{\mathrm{f}}(S)$. The statements (b), (c) and (d) of the previous Theorem 5.1 show that there exists a unique morphism of groups

$$\begin{array}{ccc} \mathrm{Cact}_W & \longrightarrow & \mathfrak{S}_W \\ \tau & \longmapsto & \tau_\varphi^L \end{array}$$

such that

$$\tau_{I,\varphi}^L = \lambda_I^L$$

for all $I \in \mathscr{P}_{\mathrm{ir,f}}(S)$. Note that we have here emphasized the fact that the map depends on φ. The same statements also show that there exists a unique morphism of groups

$$\begin{array}{ccc} \mathrm{Cact}_W & \longrightarrow & \mathfrak{S}_W \\ \tau & \longmapsto & \tau_\varphi^R \end{array}$$

such that

$$\tau_{I,\varphi}^R = \rho_I^R$$

for all $I \in \mathscr{P}_{\mathrm{ir,f}}(S)$. Moreover, Theorem 5.1(a) shows that both actions commute or, in other words, that the map

$$\begin{array}{ccc} \mathrm{Cact}_W \times \mathrm{Cact}_W & \longrightarrow & \mathfrak{S}_W \\ (\tau_1, \tau_2) & \longmapsto & \tau_{1,\varphi}^L \tau_{2,\varphi}^R \end{array} \tag{5.2}$$

is a morphism of groups. Let us summarize the properties of this morphism which are proved in this paper:

Theorem 5.3. — *Assume that* ***Lusztig's Conjectures P1, P4, P8 and P9 hold for the triple*** (W_I, I, φ_I) ***for all*** $I \in \mathscr{P}_{\mathrm{f}}(S)$. *Let* $\tau \in \mathrm{Cact}_W$. *Then there exist two sign maps* $\eta_L^{\tau,\varphi} : W \to \boldsymbol{\mu}_2$ *and* $\eta_R^{\tau,\varphi} : W \to \boldsymbol{\mu}_2$ *such that the pairs* $(\tau_\varphi^L, \eta_L^{\tau,\varphi})$ *and* $(\tau_\varphi^R, \eta_R^{\tau,\varphi})$ *are respectively strongly left cellular and strongly right cellular.*

Moreover, if $\tau' \in \mathrm{Cact}_W$, *then* $[\tau_\varphi^L, \tau_\varphi'^R] = \mathrm{Id}_W$.

Note that we do not claim that the sign maps in the above theorem are unique. They are obtained by decomposing τ as a product of the generators and then compose the cellular pairs according to this decomposition: the resulting sign map might depend on the chosen decomposition. It must be added that the maps τ_φ^L and τ_φ^R depend heavily on φ (see for instance the case where $|S| = 2$ in Section 6).

Corollary 5.4. — *If W is a finite Weyl group and φ is constant, then the above action of* $\mathrm{Cact}_W \times \mathrm{Cact}_W$ *coincides with the one constructed by Losev* [**Lo**, Theorem 1.1].

Proof. — This follows from [**Lo**, Theorem 1.1 and Lemma 4.7]. □

6. The example of dihedral groups

Hypothesis. *In this section, and only in this section, we assume that* $|S| = 2$ *and we write* $S = \{s, t\}$*. We denote by* m *the order of* st *and we assume that* $3 \leqslant m < \infty$*. We denote by* $\sigma_{s,t} : W \to W$ *the unique involutive automorphism of* W *which exchanges* s *and* t*.*

Recall [**Ge3**, Proposition 5.1] that Lusztig's Conjectures P1, P2,..., P15 hold in this case, so that the maps λ and ρ are well-defined. We aim to compute explicitly the maps λ and ρ. As we will see, the maps λ and ρ depend on the weight function φ. We will also compute the sign map η and get the following result:

Proposition 6.1. — *If* $|S| = 2$ *and* W *is finite, then the sign map* η *is constant on two-sided cells.*

We will need the following notation:

$$\Gamma = W \setminus \{1, w_0\}, \quad \Gamma_s = \{w \in \Gamma \mid ws < s\} \quad \text{and} \quad \Gamma_t = \{w \in \Gamma \mid wt < t\}.$$

Note that $\Gamma = \Gamma_s \dot\cup \Gamma_t$, where $\dot\cup$ means disjoint union.

Remark 6.2. — Let

$$\mathscr{D} = \{w \in W \mid \mathbf{a}(w) = -\mathrm{val}(p^*_{1,w})\}.$$

From P13, there exists a unique map

$$\mathbf{d} : W \longrightarrow \mathscr{D}$$

such that $w \sim_L \mathbf{d}_w$ for all $w \in W$. Its fibers are the left cells. Finally, it follows from [**Lu3**, §2.6] that

$$\rho(d) = w_0 \mathbf{d}_{w_0 d} \qquad \text{and} \qquad \lambda(d) = \mathbf{d}_{w_0 d}\, w_0. \tag{6.3}$$

for all $d \in \mathscr{D}$. ■

We define inductively two sequences $(s_i)_{i \geqslant 0}$ and $(t_i)_{i \geqslant 0}$ as follows:

$$\begin{cases} s_0 = t_0 = 1, \\ s_{i+1} = t_i s \quad \text{and} \quad t_{i+1} = s_i t, \quad \text{if } i \geqslant 0. \end{cases}$$

Note that $s_1 = s$, $t_1 = t$ and $s_m = t_m = w_0$. Then

$$\Gamma_s = \{s_1, s_2, \dots, s_{m-1}\} \qquad \text{and} \qquad \Gamma_t = \{t_1, t_2, \dots, t_{m-1}\}. \tag{6.4}$$

6.A. The equal parameter case. — We assume here, and only here, that $\varphi(s) = \varphi(t)$ and we may also assume that $\mathscr{A} = \mathbb{Z}$ and $\varphi(s) = \varphi(t) = 1$ (see for instance [**Bo2**, Proposition 2.2]). Then [**Lu2**, §8.7] the two-sided cells of W are

$$\{1\}, \quad \Gamma \quad \text{and} \quad \{w_0\}$$

while the left cells are

$$\{1\}, \quad \Gamma_s, \quad \Gamma_t \quad \text{and} \quad \{w_0\}.$$

Note that $w_0\Gamma = \Gamma w_0 = \Gamma$.

Proposition 6.5. — *Assume that φ is constant. Then*

$$\begin{cases} \lambda(w) = \rho(w) = w, & \text{if } w \in \{1, w_0\} \\ \lambda(w) = \sigma_{s,t}(w)w_0 & \text{if } w \notin \{1, w_0\}. \\ \rho(w) = w_0\sigma_{s,t}(w) & \text{if } w \notin \{1, w_0\}. \end{cases}$$

Moreover,

$$\eta_w = \begin{cases} (-1)^m & \text{if } w = 1, \\ 1 & \text{if } w = w_0, \\ -1 & \text{if } w \notin \{1, w_0\}, \end{cases}$$

Remark 6.6. — More concretely, the (non-trivial parts of) the maps λ and ρ are given as follows. If $1 \leqslant i \leqslant m-1$, then:

(a) $\rho(s_i) = s_{m-i}$ and $\rho(t_i) = t_{m-i}$.
(b) If m is even, then $\lambda = \rho$.
(b′) If m is odd, then $\lambda(s_i) = t_{m-i}$ and $\lambda(t_i) = s_{m-i}$.

In particular, if m is even, then λ stabilizes all the left cells (but nevertheless induces a non-trivial left cellular map) while, if m is odd, then λ exchanges the left cells Γ_s and Γ_t (and stabilizes all the others). ■

Proof. — By Theorem 4.2, we only need to compute ρ. It follows from Example 4.5 that $\lambda(1) = \rho(1) = 1$, that $\lambda(w_0) = \rho(w_0) = w_0$ and that $\eta_1 = (-1)^m$ and $\eta_{w_0} = 1$. Let us also recall the following result from [**Lu2**, §7]:

$$(*) \qquad C_{t_i}C_s = \begin{cases} C_{s_2} & \text{if } i = 1, \\ C_{s_{i+1}} + C_{s_{i-1}} & \text{if } 2 \leqslant i \leqslant m-1. \end{cases}$$

We will use $(*)$ to show by induction on i that $\rho(s_i) = s_{m-i}$, that $\rho(t_i) = t_{m-i}$ and that $\eta_{s_i} = \eta_{t_i} = -1$ (for $1 \leqslant i \leqslant m-1$). Let us first prove it for $i = 1$. Note that $\mathscr{D} \cap \Gamma_s = \{s\}$ and $\mathscr{D} \cap \Gamma_t = \{t\}$ (see [**Lu2**, §8.7]). So it follows from (6.3) that $\rho(s) = w_0\mathbf{d}_{w_0 s}$. But $w_0 s \in \Gamma_t$, so $\mathbf{d}_{w_0 s} = t = \sigma_{s,t}(s)$. Therefore, $\rho(s_1) = \rho(s) = w_0\sigma_{s,t}(s) = s_{m-1}$, as desired. Applying the automorphism $\sigma_{s,t}$, we get $\rho(t_1) = t_{m-1}$. Note also that

$\mathbf{a}(w) = \mathbf{a}(w_0 w)$ for all $w \in \Gamma$ (because $w_0\Gamma = \Gamma$). Moreover, $\eta_s = \eta_t = -1$ by [**Lu3**, Theorem 2.5].

Now, by using $(*)$, we get $C_{s_2} = C_{ts} = C_t C_s$ and

$$\begin{aligned} T_{w_0} C_{s_2} = T_{w_0} C_t C_s &\equiv -C_{t_{m-1}} C_s \mod \mathscr{H}^{<_{LR}\Gamma} \\ &\equiv -C_{s_{m-2}} \mod \mathscr{H}^{<_{LR}\Gamma}. \end{aligned}$$

So $\rho(s_2) = s_{m-2}$ and $\eta_{s_2} = \eta_{s_1} = -1$. Applying the automorphism $\sigma_{s,t}$, we get $\rho(t_2) = t_{m-2}$ and $\eta_{t_2} = -1$.

Now, assume that $2 \leqslant i \leqslant m-2$ and that $\rho(s_i) = s_{m-i}$, that $\rho(t_i) = t_{m-i}$ and that $\eta_{s_i} = \eta_{t_i} = -1$. Then, by using $(*)$, we get

$$\begin{aligned} T_{w_0} C_{s_{i+1}} = T_{w_0}(C_{t_i} C_s - C_{s_{i-1}}) &\equiv -C_{t_{m-i}} C_s + C_{s_{m+1-i}} \mod \mathscr{H}^{<_{LR}\Gamma} \\ &\equiv -C_{s_{m-1-i}} \mod \mathscr{H}^{<_{LR}\Gamma}. \end{aligned}$$

So $\rho(s_{i+1}) = s_{m-1-i}$ and $\eta_{s_{i+1}} = \eta_{s_i} = -1$. Applying the automorphism $\sigma_{s,t}$, we get $\rho(t_{i+1}) = t_{m-1-i}$ and $\eta_{t_{i+1}} = -1$. This completes the computation of ρ. □

Remark 6.7. — Note that the left cellular map λ obtained here is exactly the left cellular map $w \mapsto \tilde{w}$ defined by Lusztig [**Lu1**, §10]. If $m = 3$, this is the $*$-*operation* defined by Kazhdan and Lusztig [**KaLu**]. See also [**BoGe**, Remark 4.3 and Example 6.3]. ■

6.B. The unequal parameter case. — Assume here, and only here, that $\varphi(s) < \varphi(t)$. Note that this forces m to be even (and $m \geqslant 4$). We write $a = \varphi(s)$ and $b = \varphi(t)$. We set

$$\Gamma_s^< = \Gamma_s \setminus \{s\}, \quad \Gamma_t^< = \Gamma_t \setminus \{w_0 s\} \quad \text{and} \quad \Gamma^< = \Gamma_s^< \dot\cup \Gamma_t^<.$$

Then [**Lu2**, §8.8] the two-sided cells of W are

$$\{1\}, \quad \{s\}, \quad \Gamma^<, \quad \{w_0 s\} \quad \text{and} \quad \{w_0\}.$$

The left cells are

$$\{1\}, \quad \{s\}, \quad \Gamma_s^<, \quad \Gamma_t^<, \quad \{w_0 s\} \quad \text{and} \quad \{w_0\}.$$

Note that

$$\Gamma_s^< = \{s_2, s_3, \ldots, s_{m-1}\} \qquad \text{and} \qquad \Gamma_t^< = \{t_1, t_2, \ldots, t_{m-2}\}.$$

Proposition 6.8. — *Assume that $\varphi(s)<\varphi(t)$. Let $m'=m/2$. Then*

$$\begin{cases} \lambda(w)=\rho(w)=w, & \text{if } w\in\{1,s,w_0s,w_0\},\\ \lambda(s_{2i})=\rho(s_{2i})=s_{m-2i} & \text{if } 1\leqslant i\leqslant m'-1,\\ \lambda(s_{2i+1})=\rho(s_{2i+1})=s_{m+1-2i} & \text{if } 1\leqslant i\leqslant m'-1,\\ \lambda(t_{2i})=\rho(t_{2i})=t_{m-2i} & \text{if } 1\leqslant i\leqslant m'-1,\\ \lambda(t_{2i-1})=\rho(t_{2i-1})=t_{m-1-2i} & \text{if } 1\leqslant i\leqslant m'-1. \end{cases}$$

Moreover,

$$\eta_w=\begin{cases} 1 & \text{if } w\in\{1,w_0\},\\ (-1)^{m'} & \text{if } w\in\{s,w_0s\},\\ -1 & \text{if } w\notin\{1,s,w_0s,w_0\}. \end{cases}$$

Proof. — First, note that $\lambda=\rho$ because w_0 is central in W. The facts that $\lambda(w)=w$ if $w\in\{1,s,w_0s,w_0\}$, that $\eta_1=\eta_{w_0}=1$ and that $\eta_s=\eta_{w_0s}=(-1)^{m'}$ are obvious. Also, let $\delta_{s,t}^{<}:W\to W$ be the map defined by

$$\delta_{s,t}^{<}(w)=\begin{cases} w & \text{if } w\in\{1,s,w_0s,w_0\},\\ ws & \text{if } w\notin\{1,s,w_0s,w_0\}. \end{cases}$$

Then $\delta_{s,t}^{<}$ is strongly left cellular [**BoGe**, Example 6.5] so, by Theorem 4.11, it commutes with ρ. In other words,

$$(*)\qquad \forall\ w\in\Gamma^{<},\ \rho(ws)=\rho(w)s.$$

Recall from [**Lu2**, Proposition 7.6 and §8.8] that $\mathscr{D}\cap\Gamma_s^{<}=\{s_3\}$ and $\mathscr{D}\cap\Gamma_t^{<}=\{t\}$. Note also that $\mathbf{a}(w)=\mathbf{a}(w_0w)$ for all $w\in\Gamma^{<}$ (because $\Gamma^{<}=w_0\Gamma^{<}$). It follows from (6.3) that $\rho(t)=w_0s_3=t_{m-3}$, and it follows from [**Lu3**, Theorem 2.5] that $\eta_t=-1$. Similarly, $\rho(s_3)=w_0t=s_{m-1}$ and $\eta_{s_3}=-1$. Using $(*)$, we get that $\rho(t_2)=\rho(s_3s)=s_{m-1}s=t_{m-2}$ and $\rho(s_2)=\rho(t_1s)=t_{m-3}s=s_{m-2}$, as desired. So we have proved that

$$\rho(s_2)=s_{m-2},\quad \rho(s_3)=s_{m-1},\quad \rho(t_1)=t_{m-3}\quad\text{and}\quad \rho(t_2)=t_{m-2}$$

and that

$$\eta_{s_2}=\eta_{s_3}=\eta_{t_1}=\eta_{t_2}=-1.$$

Now, let $\zeta=v^{a-b}+v^{b-a}$. It follows from [**Lu2**, Lemma 7.5 and Proposition 7.6] that

$$C_{t_i}C_{st}=\begin{cases} C_{t_{i+2}}+\zeta C_{t_i} & \text{if } i\in\{1,2\},\\ C_{t_{i+2}}+\zeta C_{t_i}+C_{t_{i-2}} & \text{if } 3\leqslant i\leqslant m-1. \end{cases}$$

Using this multiplication rule and the same induction argument as in Proposition 6.5, we get the desired result. □

Remark 6.9. — Assume here, and only here, that $\varphi(s) > \varphi(t)$. Using the automorphism $\sigma_{s,t}$ which exchanges s and t, we deduce from Proposition 6.8 that:

$$\begin{cases} \lambda(w) = \rho(w) = w, & \text{if } w \in \{1, t, w_0 t, w_0\}, \\ \lambda(s_{2i}) = \rho(s_{2i}) = s_{m-2i} & \text{if } 1 \leqslant i \leqslant m'-1, \\ \lambda(s_{2i-1}) = \rho(s_{2i-1}) = s_{m-1-2i} & \text{if } 1 \leqslant i \leqslant m'-1, \\ \lambda(t_{2i}) = \rho(t_{2i}) = t_{m-2i} & \text{if } 1 \leqslant i \leqslant m'-1, \\ \lambda(t_{2i+1}) = \rho(t_{2i+1}) = t_{m+1-2i} & \text{if } 1 \leqslant i \leqslant m'-1. \end{cases}$$

Moreover,
$$\eta_w = \begin{cases} 1 & \text{if } w \in \{1, w_0\}, \\ (-1)^{m'} & \text{if } w \in \{t, w_0 t\}, \\ -1 & \text{if } w \notin \{1, t, w_0 t, w_0\}. \end{cases}$$

This completes the proof of Proposition 6.1. ■

References

[Bo1] C. BONNAFÉ, Two-sided cells in type B (asymptotic case), *J. Algebra* **304** (2006), 216-236.

[Bo2] C. BONNAFÉ, Semicontinuity properties of Kazhdan-Lusztig cells, *New Zealand J. of Math.* **39** (2009), 171-192.

[BoGe] C. BONNAFÉ & M. GECK, Hecke algebras with unequal parameters and Vogan's left cell invariants, *Festchrift Vogan's 60th birthday*, Birkhaüser, 2015 (to appear). Preprint (2015), `arXiv:1502.01661`.

[Ge1] M. GECK, On the induction of Kazhdan-Lusztig cells, *Bull. London Math. Soc.* **35** (2003), 608-614.

[Ge2] M. GECK, Relative Kazhdan-Lusztig cells, *Repres. Theory* **10** (2006), 481-524.

[Ge3] M. GECK, On Iwahori-Hecke algebras with unequal parameters and Lusztig's isomorphism theorem, *Pure Appl. Math. Quart.* **7** (2011), 587-620.

[KaLu] D. KAZHDAN & G. LUSZTIG, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165-184.

[Lo] I. LOSEV, Cacti and cells, preprint (2015), `arxiv:1506.04400`.

[Lu1] G. LUSZTIG, Cells in affine Weyl groups, *Advanced Studies in Pure Math.* **6**, *Algebraic groups and related topics*, Kinokuniya and North-Holland, 1985, 255-287.

[Lu2] G. LUSZTIG, *Hecke algebras with unequal parameters*, CRM Monograph Series **18**, American Mathematical Society, Providence, RI (2003), 136 pp.

[Lu3] G. LUSZTIG, Action of the longest element on a Hecke algebra cell module, preprint (2014), `arxiv:1406.0452`, to appear in Pacific J. Math.

[Ma] A. MATHAS, On the left cell representations of Iwahori-Hecke algebras of finite Coxeter groups, *J. London Math. Soc.* **54** (1996), 475-488.

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