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CELLS AND CACTI

by

CÉDRIC BONNAFÉ

Abstract. — Let (W, S) be a Coxeter system, let φ be a weight function on S and let Cact_W denote the associated *cactus group*. Following an idea of I. Losev, we construct an action of $\text{Cact}_W \times \text{Cact}_W$ on W which has nice properties with respect to the partition of W into left, right or two-sided cells (under some hypothesis, which hold for instance if φ is constant or if W is finite of rank ≤ 4). It must be noticed that the action depends heavily on φ .

Let (W, S) be a Coxeter system with S finite and let φ be a positive *weight function* on S as defined by Lusztig [Lu1]. We denote by Cact_W the *Cactus group* associated with W , as defined for instance in [Lo]. In [Lo], I. Losev has constructed, whenever W is a finite Weyl group and φ is constant, an action of $\text{Cact}_W \times \text{Cact}_W$ on W which satisfies some good properties with respect to the partition of W into cells. His construction is realized as the combinatorial shadow of wall-crossing functors on the category \mathcal{O} .

In [Lo, §5.1], I. Losev suggested that the construction of this action could be extended to other types of Coxeter groups and general weight function φ , using some recent results of Lusztig [Lu2]. This is the aim of this paper to show that Losev's idea works, by using extensively the results of [BoGe] and assuming that some of Lusztig's Conjectures in [Lu1, §14.2] hold, as in [Lu2], as well as an hypothesis on a sign function (this will be made more precise in §3.B). Note that, if φ is constant, then these Conjectures hold (as well as the hypothesis on the sign function), so this provides at least an action in the equal parameter case: if moreover W is a Weyl group, this action coincides with the one constructed by Losev [Lo].

Let us state our main result (in this Theorem, if $I \subset S$, we denote by W_I the subgroup generated by I and by φ_I the restriction of φ to I).

Theorem.— *Assume that the hypotheses of §3.B hold. There exists an action of $\text{Cact}_W \times \text{Cact}_W$ on the set W such that, if we denote by τ_φ^L (respectively τ_φ^R) the permutation of W obtained through the action of $(\tau, 1) \in \text{Cact}_W \times \text{Cact}_W$ (respectively $(1, \tau) \in \text{Cact}_W \times \text{Cact}_W$), then:*

- (a) *If C is a left cell, then $\tau_\varphi^L(C)$ is a left cell and τ_φ^L induces an isomorphism of left \mathcal{H} -modules $\mathcal{H}^L[C] \xrightarrow{\sim} \mathcal{H}^L[\tau_\varphi^L(C)]$.*
- (a') *If C is a right cell, then $\tau_\varphi^R(C)$ is a right cell and τ_φ^R induces an isomorphism of right \mathcal{H} -modules $\mathcal{H}^R[C] \xrightarrow{\sim} \mathcal{H}^R[\tau_\varphi^R(C)]$.*
- (b) *If $w \in W$, then $\tau_\varphi^L(w) \sim_R w$ and $\tau_\varphi^R(w) \sim_L w$.*

In this Theorem, if C is a left (respectively right) cell, then $\mathcal{H}^L[C]$ (respectively $\mathcal{H}^R[C]$) denotes the associated left (respectively right) \mathcal{H} -module (see §1.A).

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1. Notation

Set-up. We fix a Coxeter system (W, S) , whose length function is denoted by $\ell : W \rightarrow \mathbb{N}$. We also fix a totally ordered abelian \mathcal{A} and we denote by A the group algebra $\mathbb{Z}[\mathcal{A}]$. We use an exponential notation for A :

$$A = \bigoplus_{a \in \mathcal{A}} \mathbb{Z}v^a \quad \text{where} \quad v^a v^{a'} = v^{a+a'} \quad \text{for all } a, a' \in \mathcal{A}.$$

If $a_0 \in \mathcal{A}$, we write $\mathcal{A}_{\leq a_0} = \{a \in \mathcal{A} \mid a \leq a_0\}$ and $A_{\leq a_0} = \bigoplus_{a \in \mathcal{A}_{\leq a_0}} \mathbb{Z}v^a$; we define similar $A_{< a_0}$, $A_{\geq a_0}$, $A_{> a_0}$, ... We denote by $\bar{} : A \rightarrow A$ the involutive automorphism such that $\bar{v}^a = v^{-a}$ for all $a \in \mathcal{A}$. Since \mathcal{A} is totally ordered, A inherits two maps $\text{deg} : A \rightarrow \mathcal{A} \cup \{-\infty\}$ and $\text{val} : A \rightarrow \mathcal{A} \cup \{+\infty\}$ respectively called **degree** and **valuation**, and which are defined as usual.

We also fix a **weight function** $\varphi : S \rightarrow \mathcal{A}_{>0}$ (that is, $\varphi(s) = \varphi(t)$ or all $s, t \in S$ which are conjugate in W) and, if $I \subset S$, we denote by $\varphi_I : I \rightarrow \mathcal{A}_{>0}$ the restriction of φ .

Commentary.— In [Lu1, §14.2], Lusztig states several Conjectures relating the so-called *Lusztig's \mathbf{a} -function* and the partition of W into cells. Throughout this paper, if $1 \leq i \leq 15$, the expression *Lusztig's Conjecture Pi* will refer to [Lu1, §14.2, Conjecture Pi].

Most of the results of this paper only hold if Lusztig's Conjectures P1, P4, P8 and P9 hold for all finite standard parabolic subgroups of W . In particular, they hold if φ is constant [Lu1, §15]. ■

1.A. Cells. — Let $\mathcal{H} = \mathcal{H}(W, S, \varphi)$ denote the Iwahori-Hecke algebra associated with the triple (W, S, φ) . This A -algebra is free as an A -module, with a standard basis denoted by $(T_w)_{w \in W}$. The multiplication is completely determined by the following two rules:

$$\begin{cases} T_w T_{w'} = T_{ww'} & \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ (T_s - v^{\varphi(s)})(T_s + v^{-\varphi(s)}) = 0 & \text{if } s \in S. \end{cases}$$

The involution $\bar{}$ on A can be extended to an A -semilinear involutive automorphism $\bar{} : \mathcal{H} \rightarrow \mathcal{H}$ by setting $\bar{T}_w = T_{w^{-1}}$. Let

$$\mathcal{H}_{<0} = \bigoplus_{w \in W} A_{<0} T_w.$$

If $w \in W$, there exists [Lu1] a unique $C_w \in \mathcal{H}$ such that

$$\begin{cases} \bar{C}_w = C_w, \\ C_w \equiv T_w \pmod{\mathcal{H}_{<0}}. \end{cases}$$

It is well-known [Lu1] that $(C_w)_{w \in W}$ is an A -basis of \mathcal{H} (called the *Kazhdan-Lusztig basis*) and we will denote by $h_{x,y,z}$ the structure constants, defined by

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z.$$

We also set

$$C_y = \sum_{x \in W} p_{x,y}^* T_x,$$

and recall that $p_{y,y}^* = 1$ and $p_{x,y}^* \in A_{<0}$ if $x \neq y$.

We will denote by $\leq_L, \leq_R, \leq_{LR}, <_L, <_R, <_{LR}, \sim_L, \sim_R$ and \sim_{LR} the relations defined in [Lu1] and associated with the triple (W, S, φ) . Also, we will call left, right and two-sided cells the equivalence classes for the relations \sim_L, \sim_R and \sim_{LR} respectively. If C is a left cell, we set

$$\mathcal{H}^{\leq_L C} = \bigoplus_{w \leq_L C} A C_w, \quad \mathcal{H}^{<_L C} = \bigoplus_{w <_L C} A C_w \quad \text{and} \quad \mathcal{H}^L[C] = \mathcal{H}^{\leq_L C} / \mathcal{H}^{<_L C}.$$

These are left \mathcal{H} -modules. If $w \in C$, we denote by c_w^L the image of C_w in the quotient $\mathcal{H}^L[C]$ and $\mathcal{H}^{\leq_L C}$ and $\mathcal{H}^{<_L C}$ might be also denoted by $\mathcal{H}^{\leq_L w}$ and $\mathcal{H}^{<_L w}$ respectively: it is clear that $(c_w^L)_{w \in C}$ is an A -basis of $\mathcal{H}^L[C]$. If C is a right (respectively two-sided) cell, we define similarly $\mathcal{H}^{\leq_R C}, \mathcal{H}^{<_R C}$ and $\mathcal{H}^R[C]$ (respectively $\mathcal{H}^{\leq_{LR} C}, \mathcal{H}^{<_{LR} C}$ and $\mathcal{H}^{LR}[C]$), as well as c_w^R (respectively c_w^{LR}).

1.B. Parabolic subgroups. — We denote by $\mathcal{P}(S)$ the set of subsets of S . If $I \subset S$, we denote by W_I the standard parabolic subgroup generated by I and by X_I the set of elements $x \in W$ which have minimal length in xW_I . We also define $\text{pr}_I^L : W \rightarrow W_I$ and $\text{pr}_I^R : W \rightarrow W_I$ by the following formulas:

$$\forall x \in X_I, \forall w \in W_I, \quad \text{pr}_I^L(xw) = w \quad \text{and} \quad \text{pr}_I^R(wx^{-1}) = w.$$

If $\delta : W_I \rightarrow W_I$ is any map, we denote by $\delta^L : W \rightarrow W$ and $\delta^R : W \rightarrow W$ the maps defined by

$$\delta^L(xw) = x\delta(w) \quad \text{and} \quad \delta^R(wx^{-1}) = \delta(w)x^{-1}$$

for all $x \in X_I$ and $w \in W_I$ (see [BoGe, §6]). Note that

$$(1.1) \quad \text{pr}_I^L \circ \delta^L = \delta \circ \text{pr}_I^L \quad \text{and} \quad \text{pr}_I^R \circ \delta^R = \delta \circ \text{pr}_I^R.$$

We also denote by $\delta^{\text{op}} : W_I \rightarrow W_I$ the map defined by

$$\delta^{\text{op}}(w) = \delta(w^{-1})^{-1}$$

for all $w \in W$. Note that $\delta^R = ((\delta^{\text{op}})^L)^{\text{op}}$. If $\sigma : W \rightarrow W$ is any automorphism such that $\sigma(S) = S$, then

$$(1.2) \quad \sigma \circ \text{pr}_I^L = \text{pr}_{\sigma(I)}^L \circ \sigma \quad \text{and} \quad \sigma \circ \text{pr}_I^R = \text{pr}_{\sigma(I)}^R \circ \sigma.$$

The Hecke algebra $\mathcal{H}(W_I, I, \varphi_I)$ will be denoted by \mathcal{H}_I and will be viewed as a subalgebra of \mathcal{H} in the natural way. Since \mathcal{H} is free as a left or as a right \mathcal{H}_I -module, we will identify, for \mathfrak{H} a left (respectively right) ideal of \mathcal{H}_I , the left (respectively right) ideal $\mathcal{H}\mathfrak{H}$ (respectively $\mathfrak{H}\mathcal{H}$) with $\mathcal{H} \otimes_{\mathcal{H}_I} \mathfrak{H}$ (respectively $\mathfrak{H} \otimes_{\mathcal{H}_I} \mathcal{H}$).

Let $\mathcal{P}_f(S)$ (respectively $\mathcal{P}_{\text{ir},f}(S)$) denote the set of subsets I of S such that W_I is finite (respectively such that W_I is finite and the Coxeter graph of (W_I, I) is connected). If $I \in \mathcal{P}_f(S)$, we denote by w_I the longest element of W_I and we set

$$\begin{aligned} \omega_I : W_I &\longrightarrow W_I \\ w &\longmapsto w_I w w_I. \end{aligned}$$

It is an automorphism of W_I and it satisfies $\omega_I(I) = I$. If W is finite, then w_S will be denoted by w_0 , according to the tradition. Also, ω_S will be denoted by ω_0 .

If $I \in \mathcal{P}_f(S)$, we denote by $\mathbf{a}_I : W \rightarrow \mathcal{A}$ the Lusztig's **a**-function defined by

$$\mathbf{a}_I(z) = \max_{x,y \in W_I} \deg(h_{x,y,z})$$

for all $z \in W_I$. If W itself is finite, then \mathbf{a}_S will be simply denoted by \mathbf{a} .

1.C. Cactus group. — We define the *cactus group* associated with W , and we denote by Cact_W , the group with the following presentation:

- Generators: $(\tau_I)_{I \in \mathcal{D}_{\text{ir},f}(S)}$;
- Relations: for all $I, J \in \mathcal{D}_{\text{ir},f}(S)$, we have:

$$\begin{cases} \text{(C1)} & \tau_I^2 = 1, \\ \text{(C2)} & [\tau_I, \tau_J] = 1 & \text{if } I \cup J \text{ is disconnected,} \\ \text{(C3)} & \tau_I \tau_J = \tau_J \tau_{\omega_J(I)} & \text{if } I \subset J. \end{cases}$$

By construction, the map $\tau_I \mapsto w_I$ extends to a surjective morphism

$$\text{Cact}_W \longrightarrow W$$

which will not be used in this paper.

2. Action of the longest element

In [Lu2, Theorem 2.3], Lusztig proves the following result (which generalizes to the unequal parameter case a result of Mathas [Ma]):

Theorem 2.1 (Mathas, Lusztig). — *Assume that W is finite and that Lusztig's Conjectures P1, P4, P8 and P9 hold for (W, S, φ) . Then there exists a unique involution ρ of the set W such that, for all $w \in W$,*

$$v^{\mathbf{a}(w_0 w) - \mathbf{a}(w)} T_{w_0} C_w \equiv \eta_w^R C_{\rho(w)} \pmod{\mathcal{H}^{<LR} w}$$

for some $\eta_w^R \in \{1, -1\}$. Similarly, there exists a unique involution λ of the set W such that, for all $w \in W$,

$$v^{\mathbf{a}(w_0 w) - \mathbf{a}(w)} C_w T_{w_0} \equiv \eta_w^L C_{\lambda(w)} \pmod{\mathcal{H}^{<LR} w}.$$

Note that $\lambda = \rho^{\text{op}}$ and $\rho = \lambda \circ \omega_0$, and that $\eta_w^L = \eta_{w^{-1}}^R$.

In the equal parameter case, Mathas proved moreover that the map $w \mapsto \eta_w^R$ is constant on two-sided cells.

Remark 2.2. — We will explain here why we only need to assume that Lusztig's Conjectures P1, P4, P8 and P9 hold for the above Theorem to hold (in [Lu2, Theorem 2.5], Lusztig assumed that P1, P2, . . . , P14 and P15 hold). This will be a consequence of a simplification of the proof of [Lu2, Lemma 1.13], based on the ideas of [Bo]. In particular, we avoid the use of the difficult Lusztig's Conjecture P15 and the construction/properties of the asymptotic algebra.

So assume that Lusztig's Conjectures P1, P4, P8 and P9 hold. Let us write

$$T_{w_0} C_y = \sum_{x \leq_L y} \lambda_{x,y} C_x,$$

with $\lambda_{x,y} \in A$. Note that

$$T_{w_0}^{-1} C_y = \sum_{x \leq_L y} \bar{\lambda}_{x,y} C_x.$$

By [Bo, Proposition 1.4(a)],

$$\deg(\lambda_{x,y}) \leq \mathbf{a}(x) - \mathbf{a}(w_0 x) \text{ with equality only if } x \sim_L y.$$

By [Bo, Proposition 1.4(b)],

$$\deg(\bar{\lambda}_{x,y}) \leq \mathbf{a}(w_0 y) - \mathbf{a}(y) \text{ with equality only if } x \sim_L y.$$

Assume now that $x \sim_L y$. Then $\mathbf{a}(x) = \mathbf{a}(y)$ and $\mathbf{a}(w_0 x) = \mathbf{a}(w_0 y)$ by P4 and [Lu1, Corollary 11.7], so

$$\deg(\lambda_{x,y}) \leq \mathbf{a}(y) - \mathbf{a}(w_0 y) \leq \text{val}(\lambda_{x,y}).$$

So

$$\text{if } x \sim_L y, \text{ then } v^{\mathbf{a}(w_0 y) - \mathbf{a}(y)} \lambda_{x,y} \in \mathbb{Z},$$

Thanks to P9, this is exactly the statement in [Lu2, Lemma 1.13(a)]. Note also that [Lu2, Lemma 1.13(b)] is already proved in [Bo, Proposition 1.4(c)].

One can then check that, once [Lu2, Lemma 1.13] is proved, the argument developed in [Lu2, Proof of Theorem 2.3] to obtain Theorem 2.1 does not make use any more of Lusztig's Conjectures. ■

Notation.— If $h, h' \in \mathcal{H}$ and \mathfrak{h} is an A -submodule of \mathcal{H} , we will write

$$h \approx h' \pmod{\mathfrak{h}}$$

if there exists $a \in A^\times$ such that

$$h \equiv ah' \pmod{\mathfrak{h}}.$$

For instance, Theorem 2.1 can be rewritten

$$T_{w_0} C_w \approx C_{\rho(w)} \pmod{\mathcal{H}^{<LR} w}$$

and this property determines the map ρ . ■

3. Action of the cactus group

3.A. Cellular maps. — We recall [BoGe, Definition 4.1]:

Definition 3.1. — A map $\delta : W \rightarrow W$ is called **left cellular** if the following conditions are satisfied for every left cell C of W :

(IC1) $\delta(C)$ is also is a left cell.

(IC2) The A -linear map $\mathcal{H}^L[C] \rightarrow \mathcal{H}^L[\delta(C)]$, $c_w^L \mapsto c_{\delta(w)}^L$ is an isomorphism of left \mathcal{H} -modules.

It is called **strongly left cellular** if it is left cellular and if moreover

(IC3) $\delta(w) \sim_R w$ for all $w \in W$.

We define similarly the notions of **right cellular** and **strongly right cellular** maps.

Of course, $\delta : W \rightarrow W$ is left cellular (respectively strongly left cellular) if and only if δ^{op} is right cellular (respectively strongly right cellular).

3.B. Longest element of finite parabolic subgroups. — We will denote by (H) the following property:

(H) The sign map $w \mapsto \eta_w^R$ defined in Theorem 2.1 is constant on right cells.

We will now work under the following hypothesis:

Hypothesis. From now on, and until the end of this paper, we assume that Lusztig's Conjectures P1, P4, P8 and P9 and Hypothesis (H) hold for all triples (W_I, I, φ_I) , where $I \in \mathcal{P}_f(S)$.

Remark.— If φ is constant, then Lusztig's Conjectures P1, P2, ..., P15 hold [Lu1, §15], as well as Hypothesis (H): in fact, the map $w \mapsto \eta_w^R$ is constant on two-sided cells in this case [Ma]. ■

Let $I \in \mathcal{P}_{\text{ir},f}(S)$. We denote by λ_I (respectively ρ_I) the map $W_I \rightarrow W_I$ denoted by λ (respectively ρ) in Theorem 2.1 in the case where $I = S$.

Proposition 3.2. — The map $\lambda_I : W_I \rightarrow W_I$ (respectively $\rho_I : W_I \rightarrow W_I$) is strongly left (respectively right) cellular.

Proof. — We may, and we will, assume that W is finite and $I = S$, and we set $\rho = \rho_I$ and $\lambda = \lambda_I$. It is sufficient to prove that λ is strongly left cellular. By Theorem 2.1,

$$\eta_w^L v^{\mathbf{a}(w_0 w) - \mathbf{a}(w)} C_w T_{w_0} \equiv C_{\lambda(w)} \pmod{\mathcal{H}^{<LR} w},$$

so $\lambda(w) \leq_R w$. Similarly, $w = \lambda(\lambda(w)) \leq_R \lambda(w)$, so (LC3) holds.

Let x and y be two elements of W such that $x \sim_L y$. Let Γ (respectively C) denote the two-sided (respectively left) cell containing x and y . For simplifying the notation, we set $\alpha_\Gamma = \mathbf{a}(\Gamma) - \mathbf{a}(w_0 \Gamma)$. Then there exists $x = x_0, x_1, \dots, x_m = y = y_0, y_1, \dots, y_n = x$ in W and elements $h_1, \dots, h_m, h'_1, \dots, h'_n$ of \mathcal{H} such that C_{x_i} (respectively C_{y_j}) appears with a non-zero coefficient in the expression of $h_i C_{x_{i-1}}$ (respectively $h'_j C_{y_{j-1}}$) in the Kazhdan-Lusztig basis for $1 \leq i \leq m$ (respectively $1 \leq j \leq n$). By Lusztig's Conjectures P4 and P9, $x_i, y_j \in C$. So, if we write

$$h_i C_{x_{i-1}} \equiv \sum_{u \in \Gamma} \alpha_{i,u} C_u \pmod{\mathcal{H}^{<LR} \Gamma},$$

then $\alpha_{i,x_i} \neq 0$ and

$$h_i C_{x_{i-1}} T_{w_0} \equiv \sum_{u \in \Gamma} \alpha_{i,u} C_u T_{w_0} \pmod{\mathcal{H}^{<LR} \Gamma},$$

Therefore, by Theorem 2.1 and Hypothesis (H),

$$h_i v^{\alpha_\Gamma} C_{\lambda(x_{i-1})} \equiv \sum_{u \in \Gamma} \alpha_{i,u} v^{\alpha_\Gamma} C_{\lambda(u)} \pmod{\mathcal{H}^{<LR} \Gamma},$$

and so $\lambda(x_i) \leq_L \lambda(x_{i-1})$. This shows that $\lambda(y) \leq_L \lambda(x)$ and we can prove similarly that $\lambda(x) \leq_L \lambda(y)$. Therefore, $\lambda(C)$ is contained in a unique left cell C' . But, similarly, $\lambda(C')$ is contained in a unique left cell, and contains C . So $\lambda(C) = C'$ is a left cell. This shows (LC1).

Finally the map λ is obtained through the *right* multiplication by $\eta v^{\alpha_\Gamma} T_{w_0}$. Since this right multiplication commutes with the left action of \mathcal{H} , this implies (LC2). \square

Corollary 3.3. — *The map $\lambda_I^L : W \rightarrow W$ (respectively $\rho_I^R : W \rightarrow W$) is strongly left (respectively right) cellular.*

Proof. — This follows from Proposition 3.2 and [BoGe, Theorem 6.2]. \square

It must be noticed that the maps λ_I^L and ρ_I^R depend on the weight function φ , even if it is not clear from the notation. The canonicity of their construction shows that, if $\sigma : W \rightarrow W$ is an automorphism such that $\sigma(S) = S$ and $\varphi \circ \sigma = \varphi$, then

$$(3.4) \quad \sigma \circ \lambda_I^L = \lambda_{\sigma(I)}^L \circ \sigma \quad \text{and} \quad \sigma \circ \rho_I^R = \rho_{\sigma(I)}^R \circ \sigma.$$

For instance, if W is finite, then $\omega_0 : W \rightarrow W$ satisfies the above properties and so

$$(3.5) \quad \omega_0 \circ \lambda_I^L = \lambda_{\omega_0(I)}^L \circ \omega_0 \quad \text{and} \quad \omega_0 \circ \rho_I^R = \rho_{\omega_0(I)}^R \circ \omega_0.$$

Corollary 3.6. — *Let $w \in W$. Then*

$$\nu^{\mathbf{a}_I(w_I \text{pr}_I^R(w)) - \mathbf{a}_I(\text{pr}_I^R(w))} T_{w_I} C_w \equiv \eta_{I,w}^R C_{\rho_I^R(w)} \pmod{\mathcal{H}_I^{<R \text{pr}_I^R(w)} \mathcal{H}}$$

where $\eta_{I,w}^R$ is the sign associated with the two-sided cell of $\text{pr}_I^R(w)$ in W_I through the Theorem 2.1 for W_I . Similarly,

$$\nu^{\mathbf{a}_I(w_I \text{pr}_I^R(w)) - \mathbf{a}_I(\text{pr}_I^R(w))} C_w T_{w_I} \equiv \eta_{I,w}^L C_{\lambda_I^L(w)} \pmod{\mathcal{H} \mathcal{H}_I^{<L \text{pr}_I^L(w)}},$$

where $\eta_{I,w}^W = \eta_{I,w^{-1}}^R$.

Proof. — It is sufficient to prove the first congruence. Write $w = w'x$, with $w' \in W_I$ and $x \in X_I^{-1}$. By [Ge1], there exists a family $(p_{w',x,u,a}^I)_{w',x \in W_I, u, a \in X_I^{-1}}$ such that

$$C_w = C_{w'} T_x + \sum_{\substack{(u,a) \in W_I \times X_I^{-1} \\ \text{such that } a < x \\ \text{and } u \sim_R w'}} p_{w',x,u,a}^I C_u T_a.$$

Therefore,

$$C_w \equiv C_{w'} T_x + \sum_{\substack{(u,a) \in W_I \times X_I^{-1} \\ \text{such that } a < x \\ \text{and } u \sim_R w'}} p_{w',x,u,a}^I C_u T_a \pmod{\mathcal{H}_I^{<R w'} \mathcal{H}}.$$

If $u \sim_R w'$, then $\mathbf{a}_I(u) = \mathbf{a}_I(w')$ and $\mathbf{a}_I(w_I u) = \mathbf{a}_I(w_I w')$ by P4, so it follows from Theorem 2.1 and Hypothesis (H) for W_I that

$$\eta_{I,w}^R \nu^{\mathbf{a}_I(w_I \text{pr}_I^R(w)) - \mathbf{a}_I(\text{pr}_I^R(w))} T_{w_I} C_w \equiv C_{\rho_I(w')} T_a + \sum_{\substack{(u,a) \in W_I \times X_I^{-1} \\ \text{such that } a < x \\ \text{and } u \sim_R w'}} p_{w',x,u,a}^I C_{\rho_I(u)} T_a \pmod{\mathcal{H}_I^{<R w'} \mathcal{H}}.$$

But, by [Ge2, Lemma 3.8], $p_{w',x,u,a}^I = p_{\rho_I(w'),x,\rho_I(u),a}^I$ so

$$\eta_{I,w}^R \nu^{\mathbf{a}_I(w_I \text{pr}_I^R(w)) - \mathbf{a}_I(\text{pr}_I^R(w))} T_{w_I} C_w \equiv C_{\rho_I(w')x} \pmod{\mathcal{H}_I^{<R w'} \mathcal{H}},$$

as desired. □

3.C. Cactus group. — The main result of this section is the following:

Theorem 3.7. — *Let $I, J \in \mathcal{P}_{\text{ir},f}(S)$. Then:*

- (a) $[\lambda_I^L, \rho_J^R] = \text{Id}_W$.
- (b) $(\lambda_I^L)^2 = (\rho_I^R)^2 = \text{Id}_W$.
- (c) *If $I \cup J$ is disconnected, then $[\lambda_I^L, \lambda_J^L] = [\rho_I^R, \rho_J^R] = \text{Id}_W$.*
- (d) *If $I \subset J$, then $\lambda_I^L \lambda_J^L = \lambda_J^L \lambda_{\omega_J(I)}^L$ and $\rho_I^R \rho_J^R = \rho_J^R \rho_{\omega_J(I)}^R$.*

Proof. — (a) It follows from Corollary 3.6 that

$$T_{w_J} C_w T_{w_I} \cong \eta_{J,w}^R C_{\rho_J^R(w)} T_{w_I} \pmod{\mathcal{H}_J^{<_L \text{pr}_J^R(w)} \mathcal{H}}$$

and so

$$T_{w_J} C_w T_{w_I} \cong \eta_{I,\rho_J^R(w)}^L \eta_{J,w}^R C_{\lambda_I^L(\rho_J^R(w))} \pmod{(\mathcal{H}_J^{<_R \text{pr}_J^R(w)} \mathcal{H} + \mathcal{H} \mathcal{H}_I^{<_L \text{pr}_I^L(\rho_J^R(w))})}.$$

It then follows from Corollary 3.3 that $\rho_J^R(w) \sim_L w$, and so $\text{pr}_I^L(\rho_J^R(w)) \sim_L \text{pr}_I^L(w)$ by [Ge1]. Therefore, $\eta_{I,\rho_J^R(w)}^L = \eta_{I,w}^L$ and

$$T_{w_J} C_w T_{w_I} \cong \eta_{I,w}^L \eta_{J,w}^R C_{\lambda_I^L(\rho_J^R(w))} \pmod{(\mathcal{H}_J^{<_R \text{pr}_J^R(w)} \mathcal{H} + \mathcal{H} \mathcal{H}_I^{<_L \text{pr}_I^L(w)})}.$$

Symmetrically,

$$T_{w_J} C_w T_{w_I} \cong \eta_{I,w}^L \eta_{J,w}^R C_{\rho_J^R(\lambda_I^L(w))} \pmod{(\mathcal{H}_J^{<_R \text{pr}_J^R(w)} \mathcal{H} + \mathcal{H} \mathcal{H}_I^{<_L \text{pr}_I^L(w)})}.$$

Let \mathcal{E} be the set

$$\mathcal{E} = \{u \in W \mid \text{pr}_I^L(u) <_L \text{pr}_I^L(w)\} \cup \{u \in W \mid \text{pr}_J^R(u) <_R \text{pr}_J^R(w)\}.$$

By [Ge1], the previous congruences imply

$$C_{\lambda_I^L(\rho_J^R(w))} \cong C_{\rho_J^R(\lambda_I^L(w))} \pmod{\bigoplus_{u \in \mathcal{E}} A C_u}.$$

So it is sufficient to prove that $\lambda_I^L(\rho_J^R(w)) \notin \mathcal{E}$.

- Let us first assume that $\text{pr}_I^L(\lambda_I^L(\rho_J^R(w))) <_L \text{pr}_I^L(w)$. Since, by (1.1), $\text{pr}_I^L \circ \lambda_I^L = \lambda_I \circ \text{pr}_I^L$, we have $\lambda_I(\text{pr}_I^L(\rho_J^R(w))) <_L \text{pr}_I^L(w)$. But again, $\rho_J^R(w) \sim_L w$ by Corollary 3.3 and, by [Ge1], $\text{pr}_I^L(\rho_J^R(w)) \sim_L \text{pr}_I^L(w)$. This shows that $\lambda_I(\text{pr}_I^L(w)) <_L \text{pr}_I^L(w)$, which contradicts (thanks to P9) the fact that $\lambda_I(u) \sim_{LR} u$ for all $u \in W_I$.
- Let us now assume that $\text{pr}_J^R(\lambda_I^L(\rho_J^R(w))) <_R \text{pr}_J^R(w)$. Still by Corollary 3.3, we have $\lambda_I^L(\rho_J^R(w)) \sim_R \rho_J^R$ and so, by [Ge1], $\text{pr}_J^R(\rho_J^R(w)) <_R \text{pr}_J^R(w)$. Using now (1.1), we get $\text{pr}_J^R(\rho_J^R(w)) = \rho_J(\text{pr}_J^R(w))$, which contradicts (thanks to P9) the fact that $\rho_J(u) \sim_{LR} u$ for all $u \in W_J$.

The proof of (a) is now complete.

(b) is obvious.

(c) Assume that $I \cup J$ is disconnected. We only need to prove that $[\lambda_I^L, \lambda_J^L] = \text{Id}_W$, the proof of the other equality is similar. Let $w \in W$ and write $w = xw'$, with $x \in X_{I \cup J}$ and $w' \in W_{I \cup J}$. Since $I \cup J$ is disconnected, we have $W_{I \cup J} = W_I \times W_J$ and so there exists $w_1 \in W_I$ and $w_2 \in W_J$ such that $w' = w_1 w_2 = w_2 w_1$. Note also that $x w_1 \in X_J$, $x \lambda_I(w_1) \in X_J$, $x w_2 \in X_I$ and $x \lambda_J(w_2) \in X_I$. Therefore,

$$\lambda_I^L(\lambda_J^L(w)) = \lambda_I^L(x w_1 \lambda_J(w_2)) = \lambda_I^L(x \lambda_J(w_2) w_1) = x \lambda_J(w_2) \lambda_I(w_1)$$

and, similarly,

$$\lambda_J^L(\lambda_I^L(w)) = x \lambda_I(w_1) \lambda_J(w_2).$$

So $[\lambda_I^L, \lambda_J^L] = \text{Id}_W$, as desired.

(d) Assume here that $I \subset J$. It is easily checked that we may assume that W is finite and $J = S$. Let $w \in W$. Then

$$\begin{aligned} \lambda_S^L(\lambda_I^L(w)) &= \rho_S^L(\omega_0(\lambda_I^L(w))) && \text{by Theorem 2.1.} \\ &= \rho_S \lambda_{\omega_0(I)}^L(\omega_0(w)) && \text{by (3.5),} \\ &= \lambda_{\omega_0(I)}^L(\rho_S(\omega_0(w))) && \text{by (a),} \\ &= \lambda_{\omega_0(I)}^L(\lambda_S(w)) && \text{by Theorem 2.1.} \end{aligned}$$

This proves the first equality and the second follows from a similar argument. \square

Let \mathfrak{S}_W denote the symmetric group on the set W . The statements (b), (c) and (d) of the previous Theorem 3.7 show that there exists a unique morphism of groups

$$\begin{array}{ccc} \text{Cact}_W & \longrightarrow & \mathfrak{S}_W \\ \tau & \longmapsto & \tau_\varphi^L \end{array}$$

such that

$$\tau_{I,\varphi}^L = \lambda_I^L$$

for all $I \in \mathcal{P}_{\text{ir,f}}(S)$. Note that we have here emphasized the fact that the map depends on φ . The same statements also show that there exists a unique morphism of groups

$$\begin{array}{ccc} \text{Cact}_W & \longrightarrow & \mathfrak{S}_W \\ \tau & \longmapsto & \tau_\varphi^R \end{array}$$

such that

$$\tau_{I,\varphi}^R = \rho_I^R$$

for all $I \in \mathcal{P}_{\text{ir},f}(S)$. Moreover, Theorem 3.7(a) shows that both actions commute or, in other words, that the map

$$(3.8) \quad \begin{array}{ccc} \text{Cact}_W \times \text{Cact}_W & \longrightarrow & \mathfrak{S}_W \\ (\tau_1, \tau_2) & \longmapsto & \tau_{1,\varphi}^L \tau_{2,\varphi}^R \end{array}$$

is a morphism of groups. Let us summarize the properties of this morphism which are proved in this paper:

Theorem 3.9. — *Assume that the hypotheses of §3.B hold. Let $\tau \in \text{Cact}_W$. Then:*

- (a) *If C is a left cell, then $\tau_\varphi^L(C)$ is a left cell and the A -linear map $\mathcal{H}^L[C] \rightarrow \mathcal{H}^L[\tau_\varphi^L(C)]$, $c_w^L \mapsto c_{\tau_\varphi^L(w)}^L$ is an isomorphism of left \mathcal{H} -modules.*
- (a') *If C is a right cell, then $\tau_\varphi^R(C)$ is a right cell and the A -linear map $\mathcal{H}^R[C] \rightarrow \mathcal{H}^R[\tau_\varphi^R(C)]$, $c_w^R \mapsto c_{\tau_\varphi^R(w)}^R$ is an isomorphism of right \mathcal{H} -modules.*
- (b) *If $w \in W$, then $\tau_\varphi^L(w) \sim_R w$ and $\tau_\varphi^R(w) \sim_L w$.*
- (c) *If $\tau' \in \text{Cact}_W$, then $[\tau_\varphi^L, \tau_\varphi'^R] = \text{Id}_W$.*

References

- [Bo] C. BONNAFÉ, Two-sided cells in type B (asymptotic case), *J. Algebra* **304** (2006), 216-236.
- [BoGe] C. BONNAFÉ & M. GECK, Hecke algebras with unequal parameters and Vogan's left cell invariants, *Festschrift Vogan's 60th birthday*, Birkhäuser, 2015 (to appear). preprint (2012), arXiv:1205.4092.
- [Ge1] M. GECK, On the induction of Kazhdan-Lusztig cells, *Bull. London Math. Soc.* **35** (2003), 608-614.
- [Ge2] M. GECK, Relative Kazhdan-Lusztig cells, *Repres. Theory* **10** (2006), 481-524.
- [Lo] I. LOSEV, Cacti and cells, preprint (2015), arxiv:1506.04400.
- [Lu1] G. LUSZTIG, *Hecke algebras with unequal parameters*, CRM Monograph Series **18**, American Mathematical Society, Providence, RI (2003), 136 pp.
- [Lu2] G. LUSZTIG, Action of the longest element on a Hecke algebra cell module, preprint (2014), arxiv:1406.0452.
- [Ma] A. MATHAS, On the left cell representations of Iwahori-Hecke algebras of finite Coxeter groups, *L. London Math. Soc.* **54** (1996), 475-488.

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