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# CELLS AND CACTI

*by*

CÉDRIC BONNAFÉ

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**Abstract.** — Let  $(W, S)$  be a Coxeter system, let  $\varphi$  be a weight function on  $S$  and let  $\text{Cact}_W$  denote the associated *cactus group*. Following an idea of I. Losev, we construct an action of  $\text{Cact}_W \times \text{Cact}_W$  on  $W$  which has nice properties with respect to the partition of  $W$  into left, right or two-sided cells (under some hypothesis, which hold for instance if  $\varphi$  is constant or if  $W$  is finite of rank  $\leq 4$ ). It must be noticed that the action depends heavily on  $\varphi$ .

Let  $(W, S)$  be a Coxeter system with  $S$  finite and let  $\varphi$  be a positive *weight function* on  $S$  as defined by Lusztig [Lu1]. We denote by  $\text{Cact}_W$  the *Cactus group* associated with  $W$ , as defined for instance in [Lo]. In [Lo], I. Losev has constructed, whenever  $W$  is a finite Weyl group and  $\varphi$  is constant, an action of  $\text{Cact}_W \times \text{Cact}_W$  on  $W$  which satisfies some good properties with respect to the partition of  $W$  into cells. His construction is realized as the combinatorial shadow of wall-crossing functors on the category  $\mathcal{O}$ .

In [Lo, §5.1], I. Losev suggested that the construction of this action could be extended to other types of Coxeter groups and general weight function  $\varphi$ , using some recent results of Lusztig [Lu2]. This is the aim of this paper to show that Losev's idea works, by using extensively the results of [BoGe] and assuming that some of Lusztig's Conjectures in [Lu1, §14.2] hold, as in [Lu2], as well as an hypothesis on a sign function (this will be made more precise in §3.B). Note that, if  $\varphi$  is constant, then these Conjectures hold (as well as the hypothesis on the sign function), so this provides at least an action in the equal parameter case: if moreover  $W$  is a Weyl group, this action coincides with the one constructed by Losev [Lo].

Let us state our main result (in this Theorem, if  $I \subset S$ , we denote by  $W_I$  the subgroup generated by  $I$  and by  $\varphi_I$  the restriction of  $\varphi$  to  $I$ ).

**Theorem.**— *Assume that the hypotheses of §3.B hold. There exists an action of  $\text{Cact}_W \times \text{Cact}_W$  on the set  $W$  such that, if we denote by  $\tau_\varphi^L$  (respectively  $\tau_\varphi^R$ ) the permutation of  $W$  obtained through the action of  $(\tau, 1) \in \text{Cact}_W \times \text{Cact}_W$  (respectively  $(1, \tau) \in \text{Cact}_W \times \text{Cact}_W$ ), then:*

- (a) *If  $C$  is a left cell, then  $\tau_\varphi^L(C)$  is a left cell and  $\tau_\varphi^L$  induces an isomorphism of left  $\mathcal{H}$ -modules  $\mathcal{H}^L[C] \xrightarrow{\sim} \mathcal{H}^L[\tau_\varphi^L(C)]$ .*
- (a') *If  $C$  is a right cell, then  $\tau_\varphi^R(C)$  is a right cell and  $\tau_\varphi^R$  induces an isomorphism of right  $\mathcal{H}$ -modules  $\mathcal{H}^R[C] \xrightarrow{\sim} \mathcal{H}^R[\tau_\varphi^R(C)]$ .*
- (b) *If  $w \in W$ , then  $\tau_\varphi^L(w) \sim_R w$  and  $\tau_\varphi^R(w) \sim_L w$ .*

In this Theorem, if  $C$  is a left (respectively right) cell, then  $\mathcal{H}^L[C]$  (respectively  $\mathcal{H}^R[C]$ ) denotes the associated left (respectively right)  $\mathcal{H}$ -module (see §1.A).

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## 1. Notation

**Set-up.** We fix a Coxeter system  $(W, S)$ , whose length function is denoted by  $\ell : W \rightarrow \mathbb{N}$ . We also fix a totally ordered abelian  $\mathcal{A}$  and we denote by  $A$  the group algebra  $\mathbb{Z}[\mathcal{A}]$ . We use an exponential notation for  $A$ :

$$A = \bigoplus_{a \in \mathcal{A}} \mathbb{Z}v^a \quad \text{where} \quad v^a v^{a'} = v^{a+a'} \quad \text{for all } a, a' \in \mathcal{A}.$$

If  $a_0 \in \mathcal{A}$ , we write  $\mathcal{A}_{\leq a_0} = \{a \in \mathcal{A} \mid a \leq a_0\}$  and  $A_{\leq a_0} = \bigoplus_{a \in \mathcal{A}_{\leq a_0}} \mathbb{Z}v^a$ ; we define similar  $A_{< a_0}$ ,  $A_{\geq a_0}$ ,  $A_{> a_0}$ , ... We denote by  $\bar{\phantom{a}} : A \rightarrow A$  the involutive automorphism such that  $\bar{v}^a = v^{-a}$  for all  $a \in \mathcal{A}$ . Since  $\mathcal{A}$  is totally ordered,  $A$  inherits two maps  $\text{deg} : A \rightarrow \mathcal{A} \cup \{-\infty\}$  and  $\text{val} : A \rightarrow \mathcal{A} \cup \{+\infty\}$  respectively called **degree** and **valuation**, and which are defined as usual.

We also fix a **weight function**  $\varphi : S \rightarrow \mathcal{A}_{>0}$  (that is,  $\varphi(s) = \varphi(t)$  or all  $s, t \in S$  which are conjugate in  $W$ ) and, if  $I \subset S$ , we denote by  $\varphi_I : I \rightarrow \mathcal{A}_{>0}$  the restriction of  $\varphi$ .

**Commentary.**— In [Lu1, §14.2], Lusztig states several Conjectures relating the so-called Lusztig's **a**-function and the partition of  $W$  into cells. Throughout this paper, if  $1 \leq i \leq 15$ , the expression *Lusztig's Conjecture Pi* will refer to [Lu1, §14.2, Conjecture Pi].

Most of the results of this paper only hold if Lusztig's Conjectures P1, P4, P8 and P9 hold for all finite standard parabolic subgroups of  $W$ . In particular, they hold if  $\varphi$  is constant [Lu1, §15]. ■

**1.A. Cells.** — Let  $\mathcal{H} = \mathcal{H}(W, S, \varphi)$  denote the Iwahori-Hecke algebra associated with the triple  $(W, S, \varphi)$ . This  $A$ -algebra is free as an  $A$ -module, with a standard basis denoted by  $(T_w)_{w \in W}$ . The multiplication is completely determined by the following two rules:

$$\begin{cases} T_w T_{w'} = T_{ww'} & \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ (T_s - v^{\varphi(s)})(T_s + v^{-\varphi(s)}) = 0 & \text{if } s \in S. \end{cases}$$

The involution  $\bar{\phantom{x}}$  on  $A$  can be extended to an  $A$ -semilinear involutive automorphism  $\bar{\phantom{x}} : \mathcal{H} \rightarrow \mathcal{H}$  by setting  $\bar{T}_w = T_{w^{-1}}$ . Let

$$\mathcal{H}_{<0} = \bigoplus_{w \in W} A_{<0} T_w.$$

If  $w \in W$ , there exists [Lu1] a unique  $C_w \in \mathcal{H}$  such that

$$\begin{cases} \bar{C}_w = C_w, \\ C_w \equiv T_w \pmod{\mathcal{H}_{<0}}. \end{cases}$$

It is well-known [Lu1] that  $(C_w)_{w \in W}$  is an  $A$ -basis of  $\mathcal{H}$  (called the *Kazhdan-Lusztig basis*) and we will denote by  $h_{x,y,z}$  the structure constants, defined by

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z.$$

We also set

$$C_y = \sum_{x \in W} p_{x,y}^* T_x,$$

and recall that  $p_{y,y}^* = 1$  and  $p_{x,y}^* \in A_{<0}$  if  $x \neq y$ .

We will denote by  $\leq_L, \leq_R, \leq_{LR}, <_L, <_R, <_{LR}, \sim_L, \sim_R$  and  $\sim_{LR}$  the relations defined in [Lu1] and associated with the triple  $(W, S, \varphi)$ . Also, we will call left, right and two-sided cells the equivalence classes for the relations  $\sim_L, \sim_R$  and  $\sim_{LR}$  respectively. If  $C$  is a left cell, we set

$$\mathcal{H}^{\leq_L C} = \bigoplus_{w \leq_L C} A C_w, \quad \mathcal{H}^{<_L C} = \bigoplus_{w <_L C} A C_w \quad \text{and} \quad \mathcal{H}^L[C] = \mathcal{H}^{\leq_L C} / \mathcal{H}^{<_L C}.$$

These are left  $\mathcal{H}$ -modules. If  $w \in C$ , we denote by  $c_w^L$  the image of  $C_w$  in the quotient  $\mathcal{H}^L[C]$  and  $\mathcal{H}^{\leq_L C}$  and  $\mathcal{H}^{<_L C}$  might be also denoted by  $\mathcal{H}^{\leq_L w}$  and  $\mathcal{H}^{<_L w}$  respectively: it is clear that  $(c_w^L)_{w \in C}$  is an  $A$ -basis of  $\mathcal{H}^L[C]$ . If  $C$  is a right (respectively two-sided) cell, we define similarly  $\mathcal{H}^{\leq_R C}, \mathcal{H}^{<_R C}$  and  $\mathcal{H}^R[C]$  (respectively  $\mathcal{H}^{\leq_{LR} C}, \mathcal{H}^{<_{LR} C}$  and  $\mathcal{H}^{LR}[C]$ ), as well as  $c_w^R$  (respectively  $c_w^{LR}$ ).

**1.B. Parabolic subgroups.** — We denote by  $\mathcal{P}(S)$  the set of subsets of  $S$ . If  $I \subset S$ , we denote by  $W_I$  the standard parabolic subgroup generated by  $I$  and by  $X_I$  the set of elements  $x \in W$  which have minimal length in  $xW_I$ . We also define  $\text{pr}_I^L : W \rightarrow W_I$  and  $\text{pr}_I^R : W \rightarrow W_I$  by the following formulas:

$$\forall x \in X_I, \forall w \in W_I, \quad \text{pr}_I^L(xw) = w \quad \text{and} \quad \text{pr}_I^R(wx^{-1}) = w.$$

If  $\delta : W_I \rightarrow W_I$  is any map, we denote by  $\delta^L : W \rightarrow W$  and  $\delta^R : W \rightarrow W$  the maps defined by

$$\delta^L(xw) = x\delta(w) \quad \text{and} \quad \delta^R(wx^{-1}) = \delta(w)x^{-1}$$

for all  $x \in X_I$  and  $w \in W_I$  (see [BoGe, §6]). Note that

$$(1.1) \quad \text{pr}_I^L \circ \delta^L = \delta \circ \text{pr}_I^L \quad \text{and} \quad \text{pr}_I^R \circ \delta^R = \delta \circ \text{pr}_I^R.$$

We also denote by  $\delta^{\text{op}} : W_I \rightarrow W_I$  the map defined by

$$\delta^{\text{op}}(w) = \delta(w^{-1})^{-1}$$

for all  $w \in W$ . Note that  $\delta^R = ((\delta^{\text{op}})^L)^{\text{op}}$ . If  $\sigma : W \rightarrow W$  is any automorphism such that  $\sigma(S) = S$ , then

$$(1.2) \quad \sigma \circ \text{pr}_I^L = \text{pr}_{\sigma(I)}^L \circ \sigma \quad \text{and} \quad \sigma \circ \text{pr}_I^R = \text{pr}_{\sigma(I)}^R \circ \sigma.$$

The Hecke algebra  $\mathcal{H}(W_I, I, \varphi_I)$  will be denoted by  $\mathcal{H}_I$  and will be viewed as a subalgebra of  $\mathcal{H}$  in the natural way. Since  $\mathcal{H}$  is free as a left or as a right  $\mathcal{H}_I$ -module, we will identify, for  $\mathfrak{H}$  a left (respectively right) ideal of  $\mathcal{H}_I$ , the left (respectively right) ideal  $\mathcal{H}\mathfrak{H}$  (respectively  $\mathfrak{H}\mathcal{H}$ ) with  $\mathcal{H} \otimes_{\mathcal{H}_I} \mathfrak{H}$  (respectively  $\mathfrak{H} \otimes_{\mathcal{H}_I} \mathcal{H}$ ).

Let  $\mathcal{P}_f(S)$  (respectively  $\mathcal{P}_{\text{ir},f}(S)$ ) denote the set of subsets  $I$  of  $S$  such that  $W_I$  is finite (respectively such that  $W_I$  is finite and the Coxeter graph of  $(W_I, I)$  is connected). If  $I \in \mathcal{P}_f(S)$ , we denote by  $w_I$  the longest element of  $W_I$  and we set

$$\begin{aligned} \omega_I : W_I &\longrightarrow W_I \\ w &\longmapsto w_I w w_I. \end{aligned}$$

It is an automorphism of  $W_I$  and it satisfies  $\omega_I(I) = I$ . If  $W$  is finite, then  $w_S$  will be denoted by  $w_0$ , according to the tradition. Also,  $\omega_S$  will be denoted by  $\omega_0$ .

If  $I \in \mathcal{P}_f(S)$ , we denote by  $\mathbf{a}_I : W \rightarrow \mathcal{A}$  the Lusztig's  $\mathbf{a}$ -function defined by

$$\mathbf{a}_I(z) = \max_{x, y \in W_I} \deg(h_{x, y, z})$$

for all  $z \in W_I$ . If  $W$  itself is finite, then  $\mathbf{a}_S$  will be simply denoted by  $\mathbf{a}$ .

**1.C. Cactus group.** — We define the *cactus group* associated with  $W$ , and we denote by  $\text{Cact}_W$ , the group with the following presentation:

- Generators:  $(\tau_I)_{I \in \mathcal{D}_{\text{ir},f}(S)}$ ;
- Relations: for all  $I, J \in \mathcal{D}_{\text{ir},f}(S)$ , we have:

$$\begin{cases} \text{(C1)} & \tau_I^2 = 1, \\ \text{(C2)} & [\tau_I, \tau_J] = 1 & \text{if } I \cup J \text{ is disconnected,} \\ \text{(C3)} & \tau_I \tau_J = \tau_J \tau_{\omega_J(I)} & \text{if } I \subset J. \end{cases}$$

By construction, the map  $\tau_I \mapsto w_I$  extends to a surjective morphism

$$\text{Cact}_W \longrightarrow W$$

which will not be used in this paper.

## 2. Action of the longest element

In [Lu2, Theorem 2.3], Lusztig proves the following result (which generalizes to the unequal parameter case a result of Mathas [Ma]):

**Theorem 2.1 (Mathas, Lusztig).** — *Assume that  $W$  is finite and that Lusztig's Conjectures P1, P4, P8 and P9 hold for  $(W, S, \varphi)$ . Then there exists a unique involution  $\rho$  of the set  $W$  such that, for all  $w \in W$ ,*

$$v^{\mathbf{a}(w_0 w) - \mathbf{a}(w)} T_{w_0} C_w \equiv \eta_w^R C_{\rho(w)} \pmod{\mathcal{H}^{<LR} w}$$

for some  $\eta_w^R \in \{1, -1\}$ . Similarly, there exists a unique involution  $\lambda$  of the set  $W$  such that, for all  $w \in W$ ,

$$v^{\mathbf{a}(w_0 w) - \mathbf{a}(w)} C_w T_{w_0} \equiv \eta_w^L C_{\lambda(w)} \pmod{\mathcal{H}^{<LR} w}.$$

Note that  $\lambda = \rho^{\text{op}}$  and  $\rho = \lambda \circ \omega_0$ , and that  $\eta_w^L = \eta_{w^{-1}}^R$ .

In the equal parameter case, Mathas proved moreover that the map  $w \mapsto \eta_w^R$  is constant on two-sided cells.

**Remark 2.2.** — We will explain here why we only need to assume that Lusztig's Conjectures P1, P4, P8 and P9 hold for the above Theorem to hold (in [Lu2, Theorem 2.5], Lusztig assumed that P1, P2, . . . , P14 and P15 hold). This will be a consequence of a simplification of the proof of [Lu2, Lemma 1.13], based on the ideas of [Bo]. In particular, we avoid the use of the difficult Lusztig's Conjecture P15 and the construction/properties of the asymptotic algebra.

So assume that Lusztig's Conjectures P1, P4, P8 and P9 hold. Let us write

$$T_{w_0} C_y = \sum_{x \leq_L y} \lambda_{x,y} C_x,$$

with  $\lambda_{x,y} \in A$ . Note that

$$T_{w_0}^{-1} C_y = \sum_{x \leq_L y} \bar{\lambda}_{x,y} C_x.$$

By [Bo, Proposition 1.4(a)],

$$\deg(\lambda_{x,y}) \leq \mathbf{a}(x) - \mathbf{a}(w_0 x) \text{ with equality only if } x \sim_L y.$$

By [Bo, Proposition 1.4(b)],

$$\deg(\bar{\lambda}_{x,y}) \leq \mathbf{a}(w_0 y) - \mathbf{a}(y) \text{ with equality only if } x \sim_L y.$$

Assume now that  $x \sim_L y$ . Then  $\mathbf{a}(x) = \mathbf{a}(y)$  and  $\mathbf{a}(w_0 x) = \mathbf{a}(w_0 y)$  by P4 and [Lu1, Corollary 11.7], so

$$\deg(\lambda_{x,y}) \leq \mathbf{a}(y) - \mathbf{a}(w_0 y) \leq \text{val}(\lambda_{x,y}).$$

So

$$\text{if } x \sim_L y, \text{ then } v^{\mathbf{a}(w_0 y) - \mathbf{a}(y)} \lambda_{x,y} \in \mathbb{Z},$$

Thanks to P9, this is exactly the statement in [Lu2, Lemma 1.13(a)]. Note also that [Lu2, Lemma 1.13(b)] is already proved in [Bo, Proposition 1.4(c)].

One can then check that, once [Lu2, Lemma 1.13] is proved, the argument developed in [Lu2, Proof of Theorem 2.3] to obtain Theorem 2.1 does not make use any more of Lusztig's Conjectures. ■

*Notation.*— If  $h, h' \in \mathcal{H}$  and  $\mathfrak{h}$  is an  $A$ -submodule of  $\mathcal{H}$ , we will write

$$h \approx h' \pmod{\mathfrak{h}}$$

if there exists  $a \in A^\times$  such that

$$h \equiv ah' \pmod{\mathfrak{h}}.$$

For instance, Theorem 2.1 can be rewritten

$$T_{w_0} C_w \approx C_{\rho(w)} \pmod{\mathcal{H}^{<LR} w}$$

and this property determines the map  $\rho$ . ■

### 3. Action of the cactus group

**3.A. Cellular maps.** — We recall [BoGe, Definition 4.1]:

**Definition 3.1.** — A map  $\delta : W \rightarrow W$  is called **left cellular** if the following conditions are satisfied for every left cell  $C$  of  $W$ :

(IC1)  $\delta(C)$  is also is a left cell.

(IC2) The  $A$ -linear map  $\mathcal{H}^L[C] \rightarrow \mathcal{H}^L[\delta(C)]$ ,  $c_w^L \mapsto c_{\delta(w)}^L$  is an isomorphism of left  $\mathcal{H}$ -modules.

It is called **strongly left cellular** if it is left cellular and if moreover

(IC3)  $\delta(w) \sim_R w$  for all  $w \in W$ .

We define similarly the notions of **right cellular** and **strongly right cellular** maps.

Of course,  $\delta : W \rightarrow W$  is left cellular (respectively strongly left cellular) if and only if  $\delta^{\text{op}}$  is right cellular (respectively strongly right cellular).

**3.B. Longest element of finite parabolic subgroups.** — We will denote by (H) the following property:

(H) The sign map  $w \mapsto \eta_w^R$  defined in Theorem 2.1 is constant on right cells.

We will now work under the following hypothesis:

**Hypothesis.** From now on, and until the end of this paper, we assume that Lusztig's Conjectures P1, P4, P8 and P9 and Hypothesis (H) hold for all triples  $(W_I, I, \varphi_I)$ , where  $I \in \mathcal{P}_f(S)$ .

**Remark.**— If  $\varphi$  is constant, then Lusztig's Conjectures P1, P2, ..., P15 hold [Lu1, §15], as well as Hypothesis (H): in fact, the map  $w \mapsto \eta_w^R$  is constant on two-sided cells in this case [Ma]. ■

Let  $I \in \mathcal{P}_{\text{ir},f}(S)$ . We denote by  $\lambda_I$  (respectively  $\rho_I$ ) the map  $W_I \rightarrow W_I$  denoted by  $\lambda$  (respectively  $\rho$ ) in Theorem 2.1 in the case where  $I = S$ .

**Proposition 3.2.** — The map  $\lambda_I : W_I \rightarrow W_I$  (respectively  $\rho_I : W_I \rightarrow W_I$ ) is strongly left (respectively right) cellular.



*Proof.* — We may, and we will, assume that  $W$  is finite and  $I = S$ , and we set  $\rho = \rho_I$  and  $\lambda = \lambda_I$ . It is sufficient to prove that  $\lambda$  is strongly left cellular. By Theorem 2.1,

$$\eta_w^L v^{\mathbf{a}(w_0 w) - \mathbf{a}(w)} C_w T_{w_0} \equiv C_{\lambda(w)} \pmod{\mathcal{H}^{<LR} w},$$

so  $\lambda(w) \leq_R w$ . Similarly,  $w = \lambda(\lambda(w)) \leq_R \lambda(w)$ , so (LC3) holds.

Let  $x$  and  $y$  be two elements of  $W$  such that  $x \sim_L y$ . Let  $\Gamma$  (respectively  $C$ ) denote the two-sided (respectively left) cell containing  $x$  and  $y$ . For simplifying the notation, we set  $\alpha_\Gamma = \mathbf{a}(\Gamma) - \mathbf{a}(w_0 \Gamma)$ . Then there exists  $x = x_0, x_1, \dots, x_m = y = y_0, y_1, \dots, y_n = x$  in  $W$  and elements  $h_1, \dots, h_m, h'_1, \dots, h'_n$  of  $\mathcal{H}$  such that  $C_{x_i}$  (respectively  $C_{y_j}$ ) appears with a non-zero coefficient in the expression of  $h_i C_{x_{i-1}}$  (respectively  $h'_j C_{y_{j-1}}$ ) in the Kazhdan-Lusztig basis for  $1 \leq i \leq m$  (respectively  $1 \leq j \leq n$ ). By Lusztig's Conjectures P4 and P9,  $x_i, y_j \in C$ . So, if we write

$$h_i C_{x_{i-1}} \equiv \sum_{u \in \Gamma} \alpha_{i,u} C_u \pmod{\mathcal{H}^{<LR} \Gamma},$$

then  $\alpha_{i,x_i} \neq 0$  and

$$h_i C_{x_{i-1}} T_{w_0} \equiv \sum_{u \in \Gamma} \alpha_{i,u} C_u T_{w_0} \pmod{\mathcal{H}^{<LR} \Gamma},$$

Therefore, by Theorem 2.1 and Hypothesis (H),

$$h_i v^{\alpha_\Gamma} C_{\lambda(x_{i-1})} \equiv \sum_{u \in \Gamma} \alpha_{i,u} v^{\alpha_\Gamma} C_{\lambda(u)} \pmod{\mathcal{H}^{<LR} \Gamma},$$

and so  $\lambda(x_i) \leq_L \lambda(x_{i-1})$ . This shows that  $\lambda(y) \leq_L \lambda(x)$  and we can prove similarly that  $\lambda(x) \leq_L \lambda(y)$ . Therefore,  $\lambda(C)$  is contained in a unique left cell  $C'$ . But, similarly,  $\lambda(C')$  is contained in a unique left cell, and contains  $C$ . So  $\lambda(C) = C'$  is a left cell. This shows (LC1).

Finally the map  $\lambda$  is obtained through the *right* multiplication by  $\eta v^{\alpha_\Gamma} T_{w_0}$ . Since this right multiplication commutes with the left action of  $\mathcal{H}$ , this implies (LC2).  $\square$

**Corollary 3.3.** — *The map  $\lambda_I^L : W \rightarrow W$  (respectively  $\rho_I^R : W \rightarrow W$ ) is strongly left (respectively right) cellular.*

*Proof.* — This follows from Proposition 3.2 and [BoGe, Theorem 6.2].  $\square$

It must be noticed that the maps  $\lambda_I^L$  and  $\rho_I^R$  depend on the weight function  $\varphi$ , even if it is not clear from the notation. The canonicity of their construction shows that, if  $\sigma : W \rightarrow W$  is an automorphism such that  $\sigma(S) = S$  and  $\varphi \circ \sigma = \varphi$ , then

$$(3.4) \quad \sigma \circ \lambda_I^L = \lambda_{\sigma(I)}^L \circ \sigma \quad \text{and} \quad \sigma \circ \rho_I^R = \rho_{\sigma(I)}^R \circ \sigma.$$

For instance, if  $W$  is finite, then  $\omega_0 : W \rightarrow W$  satisfies the above properties and so

$$(3.5) \quad \omega_0 \circ \lambda_I^L = \lambda_{\omega_0(I)}^L \circ \omega_0 \quad \text{and} \quad \omega_0 \circ \rho_I^R = \rho_{\omega_0(I)}^R \circ \omega_0.$$

**Corollary 3.6.** — *Let  $w \in W$ . Then*

$$\nu^{\mathbf{a}_I(w_I \text{pr}_I^R(w)) - \mathbf{a}_I(\text{pr}_I^R(w))} T_{w_I} C_w \equiv \eta_{I,w}^R C_{\rho_I^R(w)} \pmod{\mathcal{H}_I^{<R \text{pr}_I^R(w)} \mathcal{H}}$$

where  $\eta_{I,w}^R$  is the sign associated with the two-sided cell of  $\text{pr}_I^R(w)$  in  $W_I$  through the Theorem 2.1 for  $W_I$ . Similarly,

$$\nu^{\mathbf{a}_I(w_I \text{pr}_I^R(w)) - \mathbf{a}_I(\text{pr}_I^R(w))} C_w T_{w_I} \equiv \eta_{I,w}^L C_{\lambda_I^L(w)} \pmod{\mathcal{H} \mathcal{H}_I^{<L \text{pr}_I^L(w)}},$$

where  $\eta_{I,w}^W = \eta_{I,w^{-1}}^R$ .

*Proof.* — It is sufficient to prove the first congruence. Write  $w = w'x$ , with  $w' \in W_I$  and  $x \in X_I^{-1}$ . By [Ge1], there exists a family  $(p_{w',x,u,a}^I)_{w',x \in W_I, u, a \in X_I^{-1}}$  such that

$$C_w = C_{w'} T_x + \sum_{\substack{(u,a) \in W_I \times X_I^{-1} \\ \text{such that } a < x \\ \text{and } u \sim_R w'}} p_{w',x,u,a}^I C_u T_a.$$

Therefore,

$$C_w \equiv C_{w'} T_x + \sum_{\substack{(u,a) \in W_I \times X_I^{-1} \\ \text{such that } a < x \\ \text{and } u \sim_R w'}} p_{w',x,u,a}^I C_u T_a \pmod{\mathcal{H}_I^{<R w'} \mathcal{H}}.$$

If  $u \sim_R w'$ , then  $\mathbf{a}_I(u) = \mathbf{a}_I(w')$  and  $\mathbf{a}_I(w_I u) = \mathbf{a}_I(w_I w')$  by P4, so it follows from Theorem 2.1 and Hypothesis (H) for  $W_I$  that

$$\eta_{I,w}^R \nu^{\mathbf{a}_I(w_I \text{pr}_I^R(w)) - \mathbf{a}_I(\text{pr}_I^R(w))} T_{w_I} C_w \equiv C_{\rho_I(w')} T_a + \sum_{\substack{(u,a) \in W_I \times X_I^{-1} \\ \text{such that } a < x \\ \text{and } u \sim_R w'}} p_{w',x,u,a}^I C_{\rho_I(u)} T_a \pmod{\mathcal{H}_I^{<R w'} \mathcal{H}}.$$

But, by [Ge2, Lemma 3.8],  $p_{w',x,u,a}^I = p_{\rho_I(w'),x,\rho_I(u),a}^I$  so

$$\eta_{I,w}^R \nu^{\mathbf{a}_I(w_I \text{pr}_I^R(w)) - \mathbf{a}_I(\text{pr}_I^R(w))} T_{w_I} C_w \equiv C_{\rho_I(w')x} \pmod{\mathcal{H}_I^{<R w'} \mathcal{H}},$$

as desired.  $\square$

### 3.C. Cactus group. — The main result of this section is the following:

**Theorem 3.7.** — *Let  $I, J \in \mathcal{P}_{\text{ir},f}(S)$ . Then:*

- (a)  $[\lambda_I^L, \rho_J^R] = \text{Id}_W$ .
- (b)  $(\lambda_I^L)^2 = (\rho_I^R)^2 = \text{Id}_W$ .
- (c) *If  $I \cup J$  is disconnected, then  $[\lambda_I^L, \lambda_J^L] = [\rho_I^R, \rho_J^R] = \text{Id}_W$ .*
- (d) *If  $I \subset J$ , then  $\lambda_I^L \lambda_J^L = \lambda_J^L \lambda_{\omega_J(I)}^L$  and  $\rho_I^R \rho_J^R = \rho_J^R \rho_{\omega_J(I)}^R$ .*

*Proof.* — (a) It follows from Corollary 3.6 that

$$T_{w_J} C_w T_{w_I} \cong \eta_{J,w}^R C_{\rho_J^R(w)} T_{w_I} \pmod{\mathcal{H}_J^{<_L \text{pr}_J^R(w)} \mathcal{H}}$$

and so

$$T_{w_J} C_w T_{w_I} \cong \eta_{I,\rho_J^R(w)}^L \eta_{J,w}^R C_{\lambda_I^L(\rho_J^R(w))} \pmod{(\mathcal{H}_J^{<_R \text{pr}_J^R(w)} \mathcal{H} + \mathcal{H} \mathcal{H}_I^{<_L \text{pr}_I^L(\rho_J^R(w))})}.$$

It then follows from Corollary 3.3 that  $\rho_J^R(w) \sim_L w$ , and so  $\text{pr}_I^L(\rho_J^R(w)) \sim_L \text{pr}_I^L(w)$  by [Ge1]. Therefore,  $\eta_{I,\rho_J^R(w)}^L = \eta_{I,w}^L$  and

$$T_{w_J} C_w T_{w_I} \cong \eta_{I,w}^L \eta_{J,w}^R C_{\lambda_I^L(\rho_J^R(w))} \pmod{(\mathcal{H}_J^{<_R \text{pr}_J^R(w)} \mathcal{H} + \mathcal{H} \mathcal{H}_I^{<_L \text{pr}_I^L(w)})}.$$

Symmetrically,

$$T_{w_J} C_w T_{w_I} \cong \eta_{I,w}^L \eta_{J,w}^R C_{\rho_J^R(\lambda_I^L(w))} \pmod{(\mathcal{H}_J^{<_R \text{pr}_J^R(w)} \mathcal{H} + \mathcal{H} \mathcal{H}_I^{<_L \text{pr}_I^L(w)})}.$$

Let  $\mathcal{E}$  be the set

$$\mathcal{E} = \{u \in W \mid \text{pr}_I^L(u) <_L \text{pr}_I^L(w)\} \cup \{u \in W \mid \text{pr}_J^R(u) <_R \text{pr}_J^R(w)\}.$$

By [Ge1], the previous congruences imply

$$C_{\lambda_I^L(\rho_J^R(w))} \cong C_{\rho_J^R(\lambda_I^L(w))} \pmod{\bigoplus_{u \in \mathcal{E}} A C_u}.$$

So it is sufficient to prove that  $\lambda_I^L(\rho_J^R(w)) \notin \mathcal{E}$ .

- Let us first assume that  $\text{pr}_I^L(\lambda_I^L(\rho_J^R(w))) <_L \text{pr}_I^L(w)$ . Since, by (1.1),  $\text{pr}_I^L \circ \lambda_I^L = \lambda_I \circ \text{pr}_I^L$ , we have  $\lambda_I(\text{pr}_I^L(\rho_J^R(w))) <_L \text{pr}_I^L(w)$ . But again,  $\rho_J^R(w) \sim_L w$  by Corollary 3.3 and, by [Ge1],  $\text{pr}_I^L(\rho_J^R(w)) \sim_L \text{pr}_I^L(w)$ . This shows that  $\lambda_I(\text{pr}_I^L(w)) <_L \text{pr}_I^L(w)$ , which contradicts (thanks to P9) the fact that  $\lambda_I(u) \sim_{LR} u$  for all  $u \in W_I$ .
- Let us now assume that  $\text{pr}_J^R(\lambda_I^L(\rho_J^R(w))) <_R \text{pr}_J^R(w)$ . Still by Corollary 3.3, we have  $\lambda_I^L(\rho_J^R(w)) \sim_R \rho_J^R$  and so, by [Ge1],  $\text{pr}_J^R(\rho_J^R(w)) <_R \text{pr}_J^R(w)$ . Using now (1.1), we get  $\text{pr}_J^R(\rho_J^R(w)) = \rho_J(\text{pr}_J^R(w))$ , which contradicts (thanks to P9) the fact that  $\rho_J(u) \sim_{LR} u$  for all  $u \in W_J$ .

The proof of (a) is now complete.

(b) is obvious.

(c) Assume that  $I \cup J$  is disconnected. We only need to prove that  $[\lambda_I^L, \lambda_J^L] = \text{Id}_W$ , the proof of the other equality is similar. Let  $w \in W$  and write  $w = xw'$ , with  $x \in X_{I \cup J}$  and  $w' \in W_{I \cup J}$ . Since  $I \cup J$  is disconnected, we have  $W_{I \cup J} = W_I \times W_J$  and so there exists  $w_1 \in W_I$  and  $w_2 \in W_J$  such that  $w' = w_1 w_2 = w_2 w_1$ . Note also that  $xw_1 \in X_J$ ,  $x\lambda_I(w_1) \in X_J$ ,  $xw_2 \in X_I$  and  $x\lambda_J(w_2) \in X_I$ . Therefore,

$$\lambda_I^L(\lambda_J^L(w)) = \lambda_I^L(xw_1\lambda_J(w_2)) = \lambda_I^L(x\lambda_J(w_2)w_1) = x\lambda_J(w_2)\lambda_I(w_1)$$

and, similarly,

$$\lambda_J^L(\lambda_I^L(w)) = x\lambda_I(w_1)\lambda_J(w_2).$$

So  $[\lambda_I^L, \lambda_J^L] = \text{Id}_W$ , as desired.

(d) Assume here that  $I \subset J$ . It is easily checked that we may assume that  $W$  is finite and  $J = S$ . Let  $w \in W$ . Then

$$\begin{aligned} \lambda_S^L(\lambda_I^L(w)) &= \rho_S^L(\omega_0(\lambda_I^L(w))) && \text{by Theorem 2.1.} \\ &= \rho_S \lambda_{\omega_0(I)}^L(\omega_0(w)) && \text{by (3.5),} \\ &= \lambda_{\omega_0(I)}^L(\rho_S(\omega_0(w))) && \text{by (a),} \\ &= \lambda_{\omega_0(I)}^L(\lambda_S(w)) && \text{by Theorem 2.1.} \end{aligned}$$

This proves the first equality and the second follows from a similar argument.  $\square$

Let  $\mathfrak{S}_W$  denote the symmetric group on the set  $W$ . The statements (b), (c) and (d) of the previous Theorem 3.7 show that there exists a unique morphism of groups

$$\begin{array}{ccc} \text{Cact}_W & \longrightarrow & \mathfrak{S}_W \\ \tau & \longmapsto & \tau_\varphi^L \end{array}$$

such that

$$\tau_{I,\varphi}^L = \lambda_I^L$$

for all  $I \in \mathcal{P}_{\text{ir,f}}(S)$ . Note that we have here emphasized the fact that the map depends on  $\varphi$ . The same statements also show that there exists a unique morphism of groups

$$\begin{array}{ccc} \text{Cact}_W & \longrightarrow & \mathfrak{S}_W \\ \tau & \longmapsto & \tau_\varphi^R \end{array}$$

such that

$$\tau_{I,\varphi}^R = \rho_I^R$$

for all  $I \in \mathcal{D}_{\text{ir},f}(S)$ . Moreover, Theorem 3.7(a) shows that both actions commute or, in other words, that the map

$$(3.8) \quad \begin{array}{ccc} \text{Cact}_W \times \text{Cact}_W & \longrightarrow & \mathfrak{S}_W \\ (\tau_1, \tau_2) & \longmapsto & \tau_{1,\varphi}^L \tau_{2,\varphi}^R \end{array}$$

is a morphism of groups. Let us summarize the properties of this morphism which are proved in this paper:

**Theorem 3.9.** — *Assume that the hypotheses of §3.B hold. Let  $\tau \in \text{Cact}_W$ . Then:*

- (a) *If  $C$  is a left cell, then  $\tau_\varphi^L(C)$  is a left cell and the  $A$ -linear map  $\mathcal{H}^L[C] \rightarrow \mathcal{H}^L[\tau_\varphi^L(C)]$ ,  $c_w^L \mapsto c_{\tau_\varphi^L(w)}^L$  is an isomorphism of left  $\mathcal{H}$ -modules.*
- (a') *If  $C$  is a right cell, then  $\tau_\varphi^R(C)$  is a right cell and the  $A$ -linear map  $\mathcal{H}^R[C] \rightarrow \mathcal{H}^R[\tau_\varphi^R(C)]$ ,  $c_w^R \mapsto c_{\tau_\varphi^R(w)}^R$  is an isomorphism of right  $\mathcal{H}$ -modules.*
- (b) *If  $w \in W$ , then  $\tau_\varphi^L(w) \sim_R w$  and  $\tau_\varphi^R(w) \sim_L w$ .*
- (c) *If  $\tau' \in \text{Cact}_W$ , then  $[\tau_\varphi^L, \tau_\varphi'^R] = \text{Id}_W$ .*

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