On the reflected random walk on R_+.

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ON THE REFLECTED RANDOM WALK ON $\mathbb{R}_+$.

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ABSTRACT. Let ρ be a borelian probability measure on $\mathbb{R}$ having a moment of order 1 and a drift $\lambda = \int_{\mathbb{R}} y \mathrm{d}\rho(y) < 0$.

Consider the random walk on $\mathbb{R}_+$ starting at $x \in \mathbb{R}_+$ and defined for any $n \in \mathbb{N}$ by

$$\left\{ \begin{array}{rl} X_0 & = x \\ X_{n+1} & = |X_n + Y_{n+1}| \end{array} \right.$$

where (Y_n) is an iid sequence of law ρ.

We note P the Markov operator associated to this random walk. This is the operator defined for any borelian and bounded function f on $\mathbb{R}_+$ and any $x \in \mathbb{R}_+$ by

$$Pf(x) = \int_{\mathbb{R}} f(|x+y|) \mathrm{d}\rho(y)$$

For a borelian bounded function f on $\mathbb{R}_+$, we call Poisson's equation the equation $f = g - Pg$ with unknown function g.

In this paper, we prove that under a regularity condition on ρ, for any directly Riemann-integrable function, there is a solution to Poisson's equation and using the renewal theorem, we prove that this solution has a limit at infinity.

Then, we use this result to prove the law of large numbers, the large deviation principle, the central limit theorem and the law of the iterated logarithm.

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Introduction

Let ρ be a borelian probability measure on $\mathbb{R}$ having a moment of order 1 and a drift $\lambda = \int_{\mathbb{R}} y \mathrm{d}\rho(y) < 0$.

Consider the random walk on $\mathbb{R}_+$ starting at $x \in \mathbb{R}_+$ and defined for any $n \in \mathbb{N}$ by

$$\left\{ \begin{array}{rl} X_0 & = x \\ X_{n+1} & = |X_n + Y_{n+1}| \end{array} \right.$$

where (Y_n) is an iid sequence of law ρ.

We note P the Markov operator associated to this random walk. This is the operator defined for any borelian and bounded function f on $\mathbb{R}_+$ and any $x \in \mathbb{R}_+$ by

$$Pf(x) = \int_{\mathbb{R}} f(|x+y|) \mathrm{d}\rho(y)$$

The aim of this article is to study this Markov chain and to do so, we will use a standard technique (known as Gordin's method) which consists in finding a solution g to the *"Poisson equation"* $g - Pg = f$ for f in a certain Banach space.

Using results of Glynn and Meyn, we will prove that under a regularity assumption on the measure ρ, there always is a solution to this equation if f is borelian and bounded but this solution will not be bounded in general and this prevents us from studying the large deviation principle and complicates the study of the central limit theorem and of the law of the iterated logarithm.

However, we will see in section 1 that the solution to Poisson's equation satisfies some equations (see proposition 1.3) and using a "stopped" renewal theorem that we will state in section 2 (see corollary 2.6) we will prove the following

Corollary (3.7)**.** *Let ρ be an absolutely continuous probability measure on $\mathbb{R}$ having a moment of order 1, a negative drift $\lambda = \int_{\mathbb{R}} y \mathrm{d}\rho(y) < 0$ and such that $\rho(\mathbb{R}_+^*) > 0$. Let (X_n) be the reflected random walk on $\mathbb{R}_+$ defined by ρ.*

Let ν be the unique stationary probability measure on $\mathbb{R}_+$ (it exists according to [Leg89] *or* [PW06]*).*

Then, for any directly Riemann-integrable function f on $\mathbb{R}_+$ such that $\int f \mathrm{d}\nu = 0$, there is a bounded and a.e. continuous function g on $\mathbb{R}_+$ such that

$$f = g - Pg \text{ and } \lim_{x \to +\infty} g(x) = 0$$

This will allow us to prove the following

Proposition (3.8 and 3.11)**.** *Let ρ be an absolutely continuous probability measure on $\mathbb{R}$ having a moment of order 1, a negative drift $\lambda = \int_{\mathbb{R}} y \mathrm{d}\rho(y) < 0$ and such that $\rho(\mathbb{R}_+^*) > 0$.*

Let (X_n) be the reflected random walk on $\mathbb{R}_+$ defined by ρ.

For any directly Riemann-integrable function f on $\mathbb{R}_+$ such that $\int f \mathrm{d}\nu = 0$, there are constants $C_1, C_2 \in \mathbb{R}_+^*$ such that for any $\varepsilon \in]0,1]$, any $x \in \mathbb{R}_+$ and any $n \in \mathbb{N}^*$,

$$\mathbb{P}_x \left(\left\{ \left| \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) \right| \geqslant \varepsilon \right\} \right) \leqslant C_1 e^{-C_2 \varepsilon^2 n}$$

In particular, for any $x \in \mathbb{R}_+$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \to 0 \ \mathbb{P}_x\text{-a.e. and in } \mathrm{L}^1(\mathbb{P}_x)$$

Moreover, is ρ *as a moment of order* $2+\varepsilon$ *for some* $\varepsilon \in \mathbb{R}_+^*$, *then, for any directly Riemann-integrable function* f *on* $\mathbb{R}_+$ *with* $\int f \mathrm{d}\nu = 0$ *and any* $x \in \mathbb{R}_+$,

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(X_k) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma^2(f)\right)$$

Where we noted $\mathcal{N}(0,0)$ *the Dirac mass at* 0 *and*

$$\sigma^2(f) = \int_{\mathbb{R}_+} g^2 - (Pg)^2 \mathrm{d}\nu$$

with g *the bounded function given by corollary 3.7 and such that* $f = g - Pg$.

Moreover, if $\sigma^2(f) \neq 0$, *then*

$$\limsup \frac{\sum_{k=0}^{n-1} f(X_k)}{\sqrt{2n\sigma^2(f)\ln\ln(n)}} = 1 \text{ a.e. and } \liminf \frac{\sum_{k=0}^{n-1} f(X_k)}{\sqrt{2n\sigma^2(f)\ln\ln(n)}} = -1 \text{ a.e.}$$

1. Induced Markov chains

The aim of this section is to study the process of induction of Markov chains by stopping times and to link the induced chain to the original one. We study it in a general case as it doesn't use any particular property of the reflected random walk.

1.1. **Definitions.** Let (X_n) be a Markov chain on a standard Borel space $\mathbf{X}$. We define a Markov operator on $\mathbf{X}$ setting, for a borelian function f and $x \in \mathbf{X}$,

$$Pf(x) = \mathbb{E}[f(X_1)|X_0 = x]$$

Given a stopping time τ, we can study the Markov chain $(X_{\tau^n})_{n\in\mathbb{N}}$ where τ^n is defined by

$$\begin{cases} \tau^0((X_n)) & = 0 \\ \tau^{k+1}((X_n)) & = \tau^k((X_n)) + \tau(\theta^{\tau^k((X_n))}(X_n)) \end{cases}$$

where θ stands for the shift on $\mathbf{X}^{\mathbb{N}}$.

We note Q the sub-Markov operator associated to (X_{τ^n}), that is, for a borelian function g on $\mathbf{X}$ and $x \in \mathbf{X}$,

$$Q(g)(x) = \int_{\{\tau<+\infty\}} g(X_\tau)\mathrm{d}\mathbb{P}_x((X_n))$$

If, for any $x \in \mathbf{X}$, $\mathbb{P}_x(\{\tau < +\infty\}) = 1$, then Q is a Markov operator.

Finally, we define two other operator on $\mathbf{X}$ setting, for a borelian non negative function f and $x \in \mathbf{X}$,

$$Sf(x) = \int_{\{\tau=1\}} f(X_1)\mathrm{d}\mathbb{P}_x((X_n)) \tag{1.1}$$

$$Rf(x) = \int_{\{\tau<+\infty\}} f(X_0) + \cdots + f(X_{\tau-1})\mathrm{d}\mathbb{P}_x((X_n)) \tag{1.2}$$

Definition 1.1 ($\theta-$compatible stopping times)**.**
We say that a stopping time τ is $\theta-$compatible if for all $x \in \mathbf{X}$, $\mathbb{P}_x(\{\tau = 0\}) = 0$ and for $\mathbb{P}_x-$a.e. $(X_n) \in \mathbf{X}^{\mathbb{N}}$, $\tau((X_n)) \geqslant 2$ implies that $\tau(\theta(X_n)) = \tau((X_n)) - 1$.

Example 1.2. Let $\mathbf{Y}$ be a borelian subset of $\mathbf{X}$ and $\tau_{\mathbf{Y}}$ the time of first return in $\mathbf{Y}$:

$$\tau_{\mathbf{Y}}((X_n)) = \inf\{n \in \mathbb{N}^*;\ X_n \in \mathbf{Y}\}$$

Then, $\tau_{\mathbf{Y}}$ is $\theta-$compatible.

Moreover, $\tau^n_{\mathbf{Y}}$ as we defined it coresponds to the time of n-th return to $\mathbf{Y}$.

For $x \in \mathbf{X}$, we set $u(x) = \mathbb{E}_x\tau_{\mathbf{Y}}$ and we call $\mathbf{Y}$ strongly Harris-recurrent if u is finite on $\mathbf{X}$. This imply in particular that for any x in $\mathbf{X}$, $\tau_{\mathbf{Y}}$ is $\mathbb{P}_x-$a.e. finite.

Indeed for any borelian non negative function f and any $x \in \mathbf{X}$, we have that

$$\begin{aligned}
Qf(x) &= \int_{\{\tau<+\infty\}} f(X_\tau)\mathrm{d}\mathbb{P}_x = \sum_{n=1}^{+\infty} \mathbb{E}_x f(X_n)\mathbf{1}_{\{\tau=n\}} \\
&= \sum_{n=1}^{+\infty} \mathbb{E}_x f(X_n)\mathbf{1}_{\mathbf{Y}^c}(X_1)\ldots\mathbf{1}_{\mathbf{Y}^c}(X_{n-1})\mathbf{1}_{\mathbf{Y}}(X_n) \\
&= \sum_{n=1}^{+\infty} (P\mathbf{1}_{\mathbf{Y}^c})^{n-1}P(f\mathbf{1}_{\mathbf{Y}}) = \sum_{n=0}^{+\infty} (P\mathbf{1}_{\mathbf{Y}^c})^{n}P(f\mathbf{1}_{\mathbf{Y}})
\end{aligned}$$

$$\begin{aligned}
Rf(x) &= \int_{\{\tau<+\infty\}} f(X_0) + \cdots + f(X_{\tau-1})\mathrm{d}\mathbb{P}_x = \sum_{n=0}^{+\infty} \mathbb{E}_x f(X_n)\mathbf{1}_{\{\tau\geqslant n+1\}} \\
&= f(x) + \sum_{n=1}^{+\infty} \mathbb{E}_x f(X_n)\mathbf{1}_{\mathbf{Y}^c}(X_1)\ldots\mathbf{1}_{\mathbf{Y}^c}(X_n) \\
&= \left(f(x) + \sum_{n=1}^{+\infty} (P\mathbf{1}_{\mathbf{Y}^c})^n(f)(x)\right) = \sum_{n=0}^{+\infty} (P\mathbf{1}_{\mathbf{Y}^c})^n(f)(x)
\end{aligned}$$

$$Sf(x) = \int_{\{\tau=1\}} f(X_1)\mathrm{d}\mathbb{P}_x = \int f(X_1)\mathbf{1}_{\mathbf{Y}}(X_1)\mathrm{d}\mathbb{P}_x = P(f\mathbf{1}_{\mathbf{Y}})$$

Thus, we have that $(R+Q)f = (I_d + RP)f$, $RSf = Qf$, $(P-S)Qf = P(\mathbf{1}_{\mathbf{Y}^c}Qf) = Qf - Sf$ and $(P-S)Rf = P(\mathbf{1}_{\mathbf{Y}^c}Rf) = Rf - f$.

Note that $P, Q, R, S, P-S, Q-S$ and $R - I_d$ are positive operators and so the computations we made make sense for any non negative borelian function f.

Next lemma generalizes those relations for any $\theta-$compatible stopping time.

Proposition 1.3. *Let τ be a $\theta-$compatible stopping time such that for any $x \in \mathbf{X}$, τ is $\mathbb{P}_x-$a.e. finite.*

For any non negative borelian function f on $\mathbf{X}$, we have :

$$\begin{aligned}(R+Q)f &= (I_d + RP)f \\ (I_d + PR)f &= (I_d + S)Rf \\ (I_d + S)Qf &= (S + PQ)f \\ RSf &= Qf\end{aligned}$$

Proof. Let f be a borelian non negative function on $\mathbf{X}$ and $x \in \mathbf{X}$.

Using the Markov property and τ being a $\theta-$compatible stopping time, we have that for any $n \in \mathbb{N}^*$,

$$\mathbb{E}_x f(X_n)\mathbf{1}_{\{\tau \geqslant n\}} = \mathbb{E}_x Pf(X_{n-1})\mathbf{1}_{\{\tau \geqslant n\}}$$

And so,

$$\begin{aligned}(R+Q)f(x) = \mathbb{E}_x f(X_0) + \cdots + f(X_\tau)\mathrm{d}\mathbb{P}_x &= \mathbb{E}_x \sum_{n=0}^{+\infty} f(X_n)\mathbf{1}_{\{\tau \geqslant n\}} \\ &= f(x) + \sum_{n=1}^{+\infty} \mathbb{E}_x Pf(X_{n-1})\mathbf{1}_{\{\tau \geqslant n\}} = f(x) + RPf(x)\end{aligned}$$

Moreover, as τ is $\theta-$compatible,

$$\int_{\{\tau \geqslant 2\}} Rf(X_1)\mathrm{d}\mathbb{P}_x((X_n)) = \int_{\{\tau \geqslant 2\}} f(X_1) + \cdots + f(X_{\tau-1})\mathrm{d}\mathbb{P}_x((X_n))$$

Thus,

$$\begin{aligned}f(x)+PRf(x) &= f(x) + \int_{\{\tau=1\}} Rf(X_1)\mathrm{d}\mathbb{P}_x((X_n)) + \int_{\{\tau \geqslant 2\}} Rf(X_1)\mathrm{d}\mathbb{P}_x((X_n)) \\ &= f(x) + SRf(x) + \int_{\{\tau \geqslant 2\}} f(X_1) + \cdots + f(X_{\tau-1})\mathrm{d}\mathbb{P}_x((X_n)) \\ &= SRf(x) + \int f(X_0) + \cdots + f(X_{\tau-1})\mathrm{d}\mathbb{P}_x((X_n)) = SRf(x) + Rf(x)\end{aligned}$$

Then, by definition of S, $\int_{\{\tau=1\}} f(X_1)\mathrm{d}\mathbb{P}_x((X_n)) = \int_{\{\tau=1\}} f(X_\tau)\mathrm{d}\mathbb{P}_x((X_n))$, so,

$$\begin{aligned}Sf(x) + PQf(x) &= \int \mathbf{1}_{\{\tau=1\}} (f(X_1) + Qf(X_1)) + \mathbf{1}_{\{\tau \geqslant 2\}} f(X_\tau)\mathrm{d}\mathbb{P}_x((X_n)) \\ &= SQf(x) + Qf(x)\end{aligned}$$

Finally, for any $n \in \mathbb{N}^*$,

$$\mathbb{E}_x Sf(X_{n-1})\mathbf{1}_{\{\tau \geqslant n\}} = \int_{\{\tau=n+1\}} f(X_{n+1})$$

therefore,

$$RSf(x) = \mathbb{E}_x \sum_{n=1}^{+\infty} Sf(X_{n-1})\mathbf{1}_{\{\tau \geqslant n\}} = \sum_{n=1}^{+\infty} \mathbb{E}_x Sf(X_{n-1})\mathbf{1}_{\{\tau \geqslant n\}}$$
$$= \sum_{n=1}^{\infty} \mathbb{E}_x f(X_{n+1})\mathbf{1}_{\{\tau = n+1\}} = Qf(x)$$

And this finishes the proof of this proposition. □

Lemma 1.4. *Let (X_n) be a Markov chain on a standard borelian space $\mathbf{X}$.*

Let ν be a finite $P-$invariant measure on $\mathbf{X}$ and τ a $\theta-$compatible stopping time such that for ν-a.e $x \in \mathbf{X}$, $\lim_{n\to+\infty} \mathbb{P}_x(\tau \geqslant n) = 0$.

Then, for any non negative borelian function f on $\mathbf{X}$, we have

$$\int_{\mathbf{X}} f \mathrm{d}\nu = \int_{\mathbf{X}} SRf \mathrm{d}\nu$$

Proof. According to proposition 1.3, $f + PRf = Rf + SRf$. So, if $Rf \in \mathrm{L}^1(\mathbf{X}, \nu)$, as ν is $P-$invariant, we get the lemma.

If $f \notin \mathrm{L}^1(\mathbf{X}, \nu)$, we will get the lemma by approximation.

First, we assume that f is bounded. In general, $Rf \notin \mathrm{L}^1(\mathbf{X}, \nu)$ so, we approximate it with a sequence of integrable functions.

More precisely, for $n \in \mathbb{N}^*$, we note R_n the operator defined like R but associated to the stopping time $\min(n, \tau)$ (which is not $\theta-$compatible).

That is to say, for a borelian non negative function f and any $x \in \mathbf{X}$,

$$R_n f(x) = \mathbb{E}_x \sum_{k=0}^{\min(\tau,n)-1} f(X_k)$$

As $\{\min(\tau, n) = 1\} = \{\tau = 1\}$ for $n \geqslant 2$, the operator S associated to $\min(\tau, n)$ does not depend on n for $n \geqslant 2$.

As $\min(\tau, n)$ is not $\theta-$compatible, we can't use proposition 1.3, but we have for $n \geqslant 2$, that

$$PR_n f(x) = \mathbb{E}_x \sum_{k=0}^{\min(\tau\circ\theta,n)-1} f(X_{k+1})$$
$$= SR_n f(x) + \int_{\{\tau\geqslant 2\}} \sum_{k=0}^{\min(\tau-1,n)-1} f(X_{k+1}) \mathrm{d}\mathbb{P}_x$$
$$= SR_n f(x) + \int_{\{\tau\geqslant 2\}} \sum_{k=1}^{\min(\tau,n+1)-1} f(X_k) \mathrm{d}\mathbb{P}_x = SR_n f + R_{n+1} f - f$$

And, as f is bounded, for any $x \in \mathbf{X}$, $|R_n f(x)| \leqslant n\|f\|_\infty$ and so $R_n f$ is integrable since μ is a finite measure and,

$$\begin{aligned}\int SR_n f - f \mathrm{d}\nu &= \int PR_n f - R_{n+1} f \mathrm{d}\nu = \int R_n f - R_{n+1} f \mathrm{d}\nu \\ &= \int f(X_n) \mathbf{1}_{\{\tau \geqslant n\}} \mathrm{d}\mathbb{P}_x((X_n)) \mathrm{d}\nu(x) \\ &= \int Pf(X_{n-1}) \mathbf{1}_{\{\tau \geqslant n\}} \mathrm{d}\mathbb{P}_x((X_n)) \mathrm{d}\nu(x)\end{aligned}$$

So,

$$\left| \int SR_n f - f \mathrm{d}\nu \right| \leqslant \|f\|_\infty \int_X \mathbb{P}_x(\{\tau \geqslant n\}) \mathrm{d}\nu(x) \to 0 \text{ (by monotone convergence)}$$

and using the monotone convergence theorem, we get the expected result for borelian bounded functions.

If f is not bounded and non-negative, we take an increasing sequence (f_n) of bounded positive functions which converges to f and we get the expected result by monotone convergence. □

Example 1.5. If τ is the return time to some strongly Harris-recurrent set $\mathbf{Y}$, then $Sf(x) = P(f\mathbf{1}_{\mathbf{Y}})(x)$. Moreover for every $P-$invariant measure ν and every $f \in \mathrm{L}^1(\mathbf{X}, \nu)$, such that Rf is $\nu-$a.e. finite, $\int_{\mathbf{X}} SRf \mathrm{d}\nu = \int_{\mathbf{Y}} Rf \mathrm{d}\nu$.

In particular, with $f = 1$, we have that, $\int_{\mathbf{Y}} \mathbb{E}\tau \mathrm{d}\nu = \nu(\mathbf{X})$. This is Kac's lemma for dynamical systems.

1.2. **Application to the study of invariant measures.** In this subsection, X is a complete separable metric space endowed with it's Borel tribe and "measure" stands for "borelian measure". We assume that there exist (at least) a $P-$invariant probability measure on $\mathbf{X}$.

We also fix a $\theta-$compatible stopping time τ such that for any x in $\mathbf{X}$, $\mathbb{E}_x\tau$ is finite.

Lemma 1.6. *Let μ be a finite non-zero $P-$invariant borelian measure on $\mathbf{X}$. Then, $S^*\mu$ is a finite non-zero $Q-$invariant measure on $\mathbf{X}$.*

Moreover, $R^*S^*\mu = \mu$ and $S^*\mu$ is absolutely continuous with respect to μ.

Proof. First, for all non negative $f \in \mathcal{B}(\mathbf{X})$ and all $x \in \mathbf{X}$, $Sf(x) \leqslant Pf(x)$.

So, $\int Sf \mathrm{d}\mu \leqslant \int Pf \mathrm{d}\mu = \int f \mathrm{d}\mu$ since μ is $P-$invariant and f is bounded. And this proves that $S^*\mu$ is absolutely continuous with respect to μ. So, as Fubuni's theorem proves that it is $\sigma-$additive, $S^*\mu$ is a finite measure on $\mathbf{X}$.

Moreover, we saw in lemma 1.4 that for all non negative borelian function f on $\mathbf{X}$, $\int SRf \mathrm{d}\mu = \int f \mathrm{d}\mu$ and this proves that $R^*S^*\mu = \mu$.

Then, we need to prove that $S^*\mu(\mathbf{X}) > 0$. But, for all $x \in \mathbf{X}$,

$$P^k S(1)(x) = \mathbb{E}_x S1(X_k) \geqslant \mathbb{P}_x(\{\tau = k+1\})$$

So, $\sum_{k=0}^{n-1} P^k S(1) \geqslant \mathbb{P}_x(\{\tau \leqslant n+1\})$. And, as μ is $P-$invariant, taking the integral on both sides, we get that,

$$nS^*\mu(X) \geqslant \int_{x \in \mathbf{X}} \mathbb{P}_x(\{\tau \leqslant n\}) \mathrm{d}\mu(x)$$

Finally, we use that for μ-a.e. $x \in \mathbf{X}$, $\lim_n \mathbb{P}_x(\tau \leqslant n) = 1$ and the dominated convergence theorem, tells us that $0 < \mu(\mathbf{X}) \leqslant \lim nS^*\mu(\mathbf{X})$, so $S^*\mu(\mathbf{X}) > 0$. □

Lemma 1.7. *Let ν be a non-zero $Q-$invariant borelian measure on $\mathbf{X}$. Then, $R^*\nu$ is a non zero $P-$invariant measure on $\mathbf{X}$.*

Moreover, $S^*R^*\nu = \nu$ and ν is absolutely continuous with respect to $R^*\nu$.

Finally, if $QR(1)$ is bounded on $\mathbf{X}$, then $R^*\nu$ is a finite measure if and only if ν is.

Remark 1.8. The technical assumption $QR1$ bounded on $\mathbf{X}$ is reasonable.

More specifically, using the same notations as in remark 1.2, we call $\mathbf{Y}$ linearily recurrent if $\sup_{y\in\mathbf{Y}} \mathbb{E}_y\tau_{\mathbf{Y}}$ is finite.

In this case, $R1(x) = \mathbb{E}_x\tau_{\mathbf{Y}}$ and $QR1(x) = \mathbb{E}_xR1(X_{\tau_{\mathbf{Y}}}) \leqslant \sup_{y\in\mathbf{Y}} \mathbb{E}_y\tau_{\mathbf{Y}}$ since for any $x \in \mathbf{X}$, $\mathbb{P}_x(X_{\tau_{\mathbf{Y}}} \in \mathbf{Y}) = 1$ be definition of $\tau_{\mathbf{Y}}$.

Proof. To prove that $R^*\mu$ is a measure, one just have to prove that it is $\sigma-$additive.

Let (A_n) be a sequence of pairwise disjoint borelian subsets of $\mathbf{X}$ and $n \in \mathbb{N}$. As R is a linear operator, we have that $\int R(\mathbf{1}_{\cup_{k=0}^n A_k})\mathrm{d}\nu = \sum_{k=0}^n \int R\mathbf{1}_{A_k}\mathrm{d}\nu$, thus, $R^*\nu$ is finitely additive. But, according to the monotone convergence theorem, the left side of this equation converges to $\int R(\mathbf{1}_{\cup A_n})\mathrm{d}\nu$ and this finishes the proof that $R^*\nu$ is $\sigma-$additive.

Moreover, for all non negative $f \in \mathcal{B}(\mathbf{X})$, $f \leqslant Rf$, so $\nu(f) \leqslant \nu(Rf)$ and ν is absolutely continuous with respect to $R^*\nu$ and $R^*\nu(X) > 0$.

Then, proposition 1.3 shows that for any positive borelian function f, $Rf + Qf = f + RPf$. Applying this to $f = \mathbf{1}_A$ for some borelian set A, and taking the integral over ν, we get that $\int R\mathbf{1}_A + Q\mathbf{1}_A\mathrm{d}\nu = \int \mathbf{1}_A + RP\mathbf{1}_A\mathrm{d}\nu$. But, ν is $Q-$invariant so if $\nu(A)$ is finite, we get that $\int R\mathbf{1}_A\mathrm{d}\nu = \int RP\mathbf{1}_A\mathrm{d}\nu$. If $\nu(A)$ is infinite, the result still holds since in this case, $\int R\mathbf{1}_A\mathrm{d}\nu = \nu(A) = Q^*\nu(A) = \int RP\mathbf{1}_A\mathrm{d}\nu = +\infty$. Thus, for any borelian set A, $R^*\nu(A) = P^*R^*\nu(A)$ that is to say, $R^*\nu$ is $P-$invariant.

As $RS = Q$ and ν is $Q-$invariant, we directly have that $S^*R^*\nu = \nu$.

For the last point, assume that $QR(1)$ is a bounded function on $\mathbf{X}$.

If $R^*\nu$ is finite, then so is ν since $\nu(\mathbf{X}) \leqslant R^*\nu(\mathbf{X})$.

Assume that ν is finite. Then according to Chacon-Ornstein's ergodic theorem (see chapter 3 theorem 3.4 in [Kre85]), there exist a $Q-$invariant non negative borelian function g^* such that $\int g^*\mathrm{d}\nu = \int R1\mathrm{d}\nu$ and for $\nu-$almost every $x \in \mathbf{X}$,

$$\frac{1}{n}\sum_{k=0}^{n-1} Q^kR1(x) \to g^*(x)$$

And, since QR is bounded on $\mathbf{X}$ and $R1(x) = \mathbb{E}_x\tau$ is finite, we get that $g^*(x) \leqslant \|QR\|_\infty$ for $\nu-$a.e $x \in \mathbf{X}$. So, $g \in \mathrm{L}^\infty(\mathbf{X},\nu) \subset \mathrm{L}^1(\mathbf{X},\nu)$ since $\nu(\mathbf{X}) < +\infty$ and $\int R1\mathrm{d}\nu \leqslant \|QR\|_\infty\nu(X) < +\infty$. □

We saw in the previous lemmas that R ans S act on invariant measure. As they are linear operators and the set of invariant measures is convex, next proposition shows that they also preserve the ergodic measures (in some sense since they do not preserve probability measures).

Corollary 1.9. *Let (X_n) be a Markov chain on a complete separable metric space $\mathbf{X}$ and τ a $\theta-$compatible stopping time such that for any $x \in \mathbf{X}$, $\mathbb{E}_x\tau$ is finite. Define P, Q, R and S as previously and assume that $QR1$ is bounded on $\mathbf{X}$.*

Then, S^* and R^* are reciproqual linear bijections between the $P-$invariant finite measures and the $Q-$invariant ones which preserve ergodicity.

Proof. We already saw in lemma 1.6 and 1.7 that S^* (resp R^*) maps the $P-$invariant (resp. $Q-$invariant) finite non zero measures onto the $Q-$invariant (resp. $P-$invariant) ones and that they are reciproqual to each-other.

Thus, it remains to prove that the image by S^* or R^* of an ergodic measure still is ergodic. To do so, we use the linearity of S^* and R^* and that ergodic probability measures are extreme points of the set of invariant probability measures for a Markov chain in a complete separable metric space.

Let μ be a $P-$ergodic finite non zero measure. We assume without any loss of generality that μ is a probability measure. We saw in lemma 1.6 that $S^*\mu$ is a $Q-$invariant non zero finite measure.

Assume that $S^*\mu = S^*\mu(\mathbf{X})(t\nu_1 + (1-t)\nu_2)$ where ν_1 and ν_2 are two $Q-$invariant probability measures and $t \in [0,1]$.

Then, we get that $\mu = R^*S^*\mu = S^*\mu(X)(tR^*\nu_1 + (1-t)R^*\nu_2)$. But μ is ergodic, so $\frac{1}{R^*\nu_1(\mathbf{X})}R^*\nu_1 = \frac{1}{R^*\nu_2(\mathbf{X})}R^*\nu_2$. And applyting S^* again, we obtain that $\nu_1 = \nu_2$, hence, $S^*\mu$ is $Q-$ergodic.

The same proof holds to show that if ν is $Q-$ergodic, then $R^*\nu$ is $P-$ergodic. □

2. Induction and the renewal theorem

In this section, we use the renewal theorem on $\mathbb{R}$ to prove a "stopped renewal theorem" in corollary 2.6.

Let ρ be a borelian probability measure on $\mathbb{R}$ and define a random walk on $\mathbb{R}$ starting at $x \in \mathbb{R}$ by

$$\left\{ \begin{array}{ccc} X_0 & = & x \\ X_{n+1} & = & X_n + Y_{n+1} \end{array} \right. \tag{2.1}$$

where (Y_n) has law $\rho^{\otimes\mathbb{N}}$.

We assume that ρ has a moment of order 1 and a negative drift $\lambda = \int_{\mathbb{R}} y\mathrm{d}\rho(y) < 0$.

In particular, for $\rho^{\otimes\mathbb{N}}-$a.e $(Y_n) \in \mathbb{R}^{\mathbb{N}}$, $\sum_{k=0}^{n} Y_k$ converges to $-\infty$.

We note P the Markov operator associated to ρ. This is the operator defined for any bounded borelian function f on $\mathbb{R}$ and any $x \in \mathbb{R}$ by

$$Pf(x) = \int_{\mathbb{R}} f(x+y)\mathrm{d}\rho(y)$$

We note τ the time of first return to $]-\infty, 0]$:

$$\tau((X_n)) = \inf\{n \in \mathbb{N}^*,\ X_n \in]-\infty, 0]\}$$

This is a $\vartheta-$compatible stopping time and our assumption on ρ implies (see P1 in section 18 of [Spi64]) that for any $x \in \mathbb{R}$,

$$\mathbb{E}_x\tau < +\infty$$

In this section, we are interested in the operator R defined as in section 1 for any non negative borelian function f on $\mathbb{R}$ and any $x \in \mathbb{R}$ by

$$Rf(x) := \mathbb{E}_x \sum_{k=0}^{\tau-1} f(X_k)$$

The study of the operator is very close from renewal theory : indeed, if $\rho(\mathbb{R}_+) = 0$ and f is null on $\mathbb{R}_-^*$, then for any $x \in \mathbb{R}$,

$$Rf(x) = \sum_{n=0}^{+\infty} P^n f(x)$$

Therefore, we make the following definition that is usual in renewal theory :

Definition 2.1. Let f be a borelian function on $\mathbb{R}$. We say that f is directly Riemann-integrable if

$$\lim_{h\to 0^+} h \sum_{n\in\mathbb{Z}} \inf_{x\in[nh,(n+1)h]} |f(x)| = \lim_{h\to 0^+} h \sum_{n\in\mathbb{Z}} \sup_{x\in[nh,(n+1)h]} |f(x)| < +\infty$$

In the sequel, we will use the following characterisation

Lemma 2.2 (Lebesgue's criterion for Riemann-integrability)**.** *Let f be a bounded function on $\mathbb{R}$.*

Then, f is directly Riemann integrable if and only if it is a.e. continuous and for some $h \in \mathbb{R}_+^*$,

$$\sum_{n\in\mathbb{Z}} \sup_{x\in[nh,(n+1)h]} |f(x)| < +\infty$$

In the next three lemmas, we are going to prove that, noting R, S the operators defined as in section 1 and associated to τ, then for any directly Riemann-integrable function f, SRf is also directly Riemann-integrable.

Lemma 2.3. *Let ρ be a borelian probability measure on $\mathbb{R}$ having a moment of order 1 and a negative drift $\lambda = \int_{\mathbb{R}} y \mathrm{d}\rho(y) < 0$.*

Note τ the time of first return to $]-\infty, 0]$ and R the associated operator defined as in section 1.

Then, for any directly Riemann-integrable function f on $\mathbb{R}$, the function Rf is bounded on $\mathbb{R}$.

Proof. To prove this proposition, we are going to use the classical renewal theorem. Indeed, for any $x \in \mathbb{R}$ we have that

$$|Rf(x)| \leqslant R|f|(x) = \mathbb{E}_x \sum_{k=0}^{\tau-1} |f(X_k)| \leqslant \mathbb{E}_x \sum_{n=0}^{+\infty} |f(X_n)| = \sum_{n=0}^{+\infty} P^n|f|(x)$$

But, if the measure is non-lattice[1], according to the renewal theorem (see [Fel71]), we have that

$$\lim_{x\to-\infty} \sum_{n=0}^{+\infty} P^n|f|(x) = 0 \text{ and } \lim_{x\to+\infty} \sum_{n=0}^{+\infty} P^n|f|(x) = \frac{-1}{\lambda} \int_{\mathbb{R}} |f| \mathrm{d}\nu$$

[1]For any $\alpha \in \mathbb{R}$, $\rho(\alpha\mathbb{Z}) < 1$

and this proves our lemma in the non-lattice case. The same kind of arguments holds in the lattice case and allows us to conclude. □

Lemma 2.4. *Let ρ be a borelian probability measure on $\mathbb{R}$ having a moment of order 1 and a negative drift $\lambda = \int_{\mathbb{R}} y\mathrm{d}\rho(y) < 0$.*

Then, for any directly-Riemann integrable function f on $\mathbb{R}$ that is null on $\mathbb{R}_-$, we have that

$$\sum_{n\in\mathbb{Z}} \sup_{x\in[n,(n+1)]} |SRf(x)| < +\infty$$

Proof. First, according to the lemma 2.3, Rf is bounded and for any $x \in \mathbb{R}$,

$$|SRf(x)| \leqslant \|Rf\|_\infty S1(x) = \|Rf\|_\infty \rho(]-\infty, -x])$$

So, as the function $(x \mapsto \rho(]-\infty, -x]))$ is decreasing, we have that

$$\sum_{n\in\mathbb{N}} \sup_{x\in[n,(n+1)]} |SRf(x)| \leqslant \|Rf\|_\infty \sum_{n\in\mathbb{N}} \rho(]-\infty, -n])$$

Moreover, using that

$$\rho(]-\infty, -n]) = \sum_{m=n}^{+\infty} \rho(]-m-1, -m])$$

We have that

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \sum_{m=n}^{+\infty} \rho(]-(m+1), -m]) &= \sum_{m\in\mathbb{N}} (m+1)\rho(]-(m+1), -m]) \\
&= \rho(\mathbb{R}_-) + \sum_{m\in\mathbb{N}} m\rho(]-(m+1), -m]) \\
&\leqslant 1 + \sum_{m\in\mathbb{N}} \int_{-(m+1)}^{-m} |y|\mathrm{d}\rho(y) \\
&\leqslant 1 + \int_{\mathbb{R}} |y|\mathrm{d}\rho(y)
\end{aligned}$$

And this proves that

$$\sum_{n\in\mathbb{N}} \sup_{x\in[n,n+1]} |SRf(x)| < +\infty$$

We now have to control the sum over $\mathbb{Z}_-$.

Since for any $x \in \mathbb{R}_-$, $f(x) = 0$, we have, using Markov's property, that for any $x \in \mathbb{R}$,

$$SRf(x) = \mathbb{E}_x \mathbf{1}_{\{\tau=1\}} \sum_{k=1}^{\tau^2-1} f(X_k) = \mathbb{E}_x \mathbf{1}_{\{\tau=1,\tau^2>2\}} \sum_{k=2}^{\tau^2-1} f(X_k) = \mathbb{E}_x \mathbf{1}_{\{\tau=1,\tau^2>2\}} Rf(X_2)$$

So,

$$\begin{aligned}
|SRf(x)| &\leqslant \mathbb{E}_x \mathbf{1}_{\{\tau=1\}\cap\{\tau^2>2\}} |Rf(X_2)| \\
&\leqslant \|Rf\|_\infty \mathbb{P}_x(\tau = 1, \tau^2 > 2)
\end{aligned}$$

But,

$$\mathbb{P}_x(\tau = 1, \tau^2 > 2) = \int_{\mathbb{R}^2} \mathbf{1}_{\{x+y_1 \leqslant 0, x+y_1+y_2 > 0\}} \mathrm{d}\rho(y_1)\mathrm{d}\rho(y_2) \leqslant \rho^{*2}(]-x, +\infty[)$$

So, using the fact that the function $(x \mapsto \rho^{*2}(]-x, +\infty[))$ is non-decreasing, we have that

$$\begin{aligned}\sum_{n=-\infty}^{0} \sup_{x\in[n-1,n]} |SRf(x)| &\leqslant \|Rf\|_\infty \sum_{n=0}^{+\infty} \rho^{*2}(]n, +\infty[) = \|Rf\|_\infty \sum_{n=0}^{+\infty}\sum_{m=n}^{+\infty} \rho^{*2}(]m, m+1]) \\ &\leqslant \|Rf\|_\infty \sum_{m=0}^{+\infty} (m+1)\rho^{*2}(]m, m+1]) \leqslant \|Rf\|_\infty \left(1 + \int_0^{+\infty} y\mathrm{d}\rho(y)\right)\end{aligned}$$

And this finishes the proof of the lemma. □

Lemma 2.5. *Let ρ be a borelian probability measure that is the sum of an absolutely continuous measure ρ_1 and a discrete measure ρ_2.*

Then, for any bounded and a.e. continuous function f on $\mathbb{R}$, Rf and Sf are also a.e. continuous.

Proof. For any $x \in \mathbb{R}$, we have that

$$\begin{aligned}Sf(x) &= \int_{\mathbb{R}} f(x+y)\mathbf{1}_{\{x+y\leqslant 0\}}\mathrm{d}\rho(y) \\ &= \int_{\mathbb{R}} f(x+y)\mathbf{1}_{\{x+y\leqslant 0\}}\varphi(y)\mathrm{d}y + \sum_{y\in\operatorname{supp}\rho_2} f(x+y)\mathbf{1}_{\{x+y\leqslant 0\}}\rho_2(y)\end{aligned}$$

Where we noted φ the density of ρ_1 and we used the fact that ρ_2 is atomic.

But, $x \mapsto f(x)\mathbf{1}_{\{x\leqslant 0\}}$ is bounded and φ is integrable so the function

$$x \mapsto \int_{\mathbb{R}} f(x+y)\mathbf{1}_{\{x+y\leqslant 0\}}\varphi(y)\mathrm{d}y$$

is continuous on $\mathbb{R}$ (as a convolution product of an integrable and a bounded function).

And, as $\operatorname{supp}\rho_2$ is denumerable and f is a.e. continuous, the function

$$x \mapsto \sum_{y\in\operatorname{supp}\rho_2} f(x+y)\mathbf{1}_{\{x+y\leqslant 0\}}\rho_2(y)$$

still is a.e. continuous.

This proves that Sf is a.e. continuous.

To prove that Rf is a.e. continuous, note that for any $x \in \mathbb{R}$, we can write

$$Rf(x) = \sum_{n=0}^{+\infty} \mathbb{E}_x f(X_n)\mathbf{1}_{\{\tau>n\}}$$

Moreover, we have that

$$\left|\mathbb{E}_x f(X_n)\mathbf{1}_{\{\tau>n\}}\right| \leqslant \|f\|_\infty \mathbb{P}_x(\tau > n)$$

so, using the fact that the function $(x \mapsto \mathbb{P}_x(\tau > n))$ is non-decreasing, we have that for any $x_0 \in \mathbb{R}$ and any $x \in]-\infty, x_0]$,

$$\left|\mathbb{E}_x f(X_n)\mathbf{1}_{\{\tau>n\}}\right| \leqslant \|f\|_\infty \mathbb{P}_{x_0}(\tau > n)$$

So the convergence in the series is uniform on every compact subset of $\mathbb{R}$ since we already saw that

$$\sum_n \mathbb{P}_{x_0}(\tau > n) = \mathbb{E}_{x_0}\tau < +\infty$$

Therefore, the set of continuity points of Rf contains the intersection of the sets of continuity points of the functions $\left(x \mapsto \mathbb{E}_x f(X_n)\mathbf{1}_{\{\tau>n\}}\right)$ for $n \in \mathbb{N}$. Moreover, for any $x \in \mathbb{R}$ and any $n \in \mathbb{N}$,

$$\mathbb{E}_x f(X_n)\mathbf{1}_{\{\tau>n\}} = \int_{\mathbb{R}^n} f(x + y_1 + \cdots + y_n)\mathbf{1}_{\forall k\in[1,n],\, x+y_1+\cdots+y_k>0}\mathrm{d}\rho^{\otimes n}((y_i))$$

And we can see that this function is a.e. continuous by using the same kind of arguments than in the proof of the a.e. continuity of Sf. □

Corollary 2.6. *Let ρ be a borelian probability measure on $\mathbb{R}$ having a moment of order 1 and a negative drift $\lambda = \int_{\mathbb{R}} y\mathrm{d}\rho(y) < 0$. Assume that ρ is non lattice[2] and the sum of an absolutely continuous measure and of a discrete one.*

Note τ the time of first return to $]-\infty, 0]$ and note R and S the Markov operators defined for any borelian bounded function f on $\mathbb{R}$ and any $x \in \mathbb{R}$ by

$$Rf(x) = \mathbb{E}_x \sum_{k=0}^{\tau-1} f(X_k) \text{ and } Sf(x) = \int_{\{\tau=1\}} f(X_1)\mathrm{d}\mathbb{P}_x((X_n))$$

Then, for any directly Riemann-integrable function f on $\mathbb{R}$,

$$\lim_{x\to+\infty} Rf(x) = \frac{-1}{\lambda}\int_{\mathbb{R}} (I_d - SR)f(u)\mathrm{d}u$$

Proof. By definition of the operator R, we shall assume without any loss of generality that $f = 0$ on $\mathbb{R}_-$.

According to the previous lemmas (2.2, 2.3, 2.4 and 2.5) we have that $f - SRf$ is directly Riemann-integrable. Thus, we can apply the renewal theorem to get that

$$\lim_{x\to+\infty} \sum_{n=0}^{+\infty} P^n(I_d - SR)f(x) = \frac{-1}{\lambda}\int_{\mathbb{R}} (I_d - SR)f(u)\mathrm{d}u$$

But, for any $n \in \mathbb{N}$,

$$\sum_{k=0}^{n-1} P^k(I_d - SR)f(x) = \sum_{k=0}^{n-1} P^k(I_d - P)Rf(x) = Rf(x) - P^nRf(x)$$

and, as f is null on $\mathbb{R}_-$ and Rf is bounded according to lemma 2.3, we have that

$$\lim_{n\to+\infty} P^nRf(x) = 0$$

[2]For any $\alpha \in \mathbb{R}$, $\rho(\alpha\mathbb{Z}) < 1$.

Thus, for any $x \in \mathbb{R}$,

$$Rf(x) = \sum_{n=0}^{+\infty} P^n(I_d - SR)f(x)$$

Which is what we intended to prove. □

3. Application to the relfected random walk on $\mathbb{R}_+$

In this section, we use the previous results for the stopped renewal theorem to study the regularity at infinity of the solution of Poisson's equation for the reflected random walk on $\mathbb{R}_+$.

Let ρ be a probability measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} |y|\rho(\mathrm{d}y)$ is finite, $\int_{\mathbb{R}} y\rho(\mathrm{d}y) < 0$ and $\rho(\mathbb{R}_+^*) \neq 0$.

These last two asumptions means that for $\rho^{\otimes\mathbb{N}}-$a.e $(Y_n) \in \mathbb{R}^{\mathbb{N}}$, $\sum_{k=0}^{n} Y_k$ converges to $-\infty$ but for any fixed $M \in \mathbb{R}$, $\rho^{\otimes\mathbb{N}}(\{(Y_n) \in \mathbb{R}^{\mathbb{N}};\ \exists n \in \mathbb{N}\ \sum_{k=0}^{n-1} Y_k \geqslant M\}) > 0$.

Let $(Y_n)_{n\in\mathbb{N}}$ be an iid seqence of random variables of law ρ. We define the reflected random walk starting at x on $\mathbb{R}_+$ by

$$\begin{cases} X_0 &= x \\ \text{for all } n \in \mathbb{N},\ X_{n+1} &= |X_n + Y_{n+1}| \end{cases}$$

Defined like this, $(X_n, Y_n)_{n\in\mathbb{N}}$ is a Markov chain on $\mathbb{R}_+ \times \mathbb{R}$. As Peigné and Woess in [PW06], we define a stopping time which we call the time of first reflection by

$$\tau((X_n, Y_n)) = \inf\{n \in \mathbb{N}^*;\ X_{n-1} + Y_n < 0\} \tag{3.1}$$

We see that τ is $\theta-$compatible since it is the time of first return in $\mathbb{R}_-^*$ for the unreflected random walk on $\mathbb{R}$ driven by ρ.

Since $\int y\rho(y)dy < 0$, $\sum_{k=0}^{n-1} Y_k \xrightarrow[n\to\infty]{} -\infty$ *a.e.*, and τ_x^n is well defined (finite almost everywhere for all positive real number x).

We define the operator P, Q, R and S as in section 1.

Remark 3.1. For a borelian non negative function f on $\mathbb{R}_+ \times \mathbb{R}$ we defined $P(f)(x, y) = \mathbb{E}_{(x,y)} f(X_1, Y_1)$. But, since X_1 and Y_1 are independant of Y_0, we have that $P(f)(x, y) = \mathbb{E}_x f(X_1, Y_1) = \int_{\mathbb{R}} f(|x + y|, y)\rho(\mathrm{d}y)$. In particular, if f itself does not depend of it's second argument, we have $P(f)(x, y) = \mathbb{E}_{(x,y)} f(X_1) = \mathbb{E}_x f(X_1)$ and we find the usual Markov operator associated to (X_n).

The same argument applies to Q, R and S (defined as in section 1)

Those considerations are just made to prove that τ is $\theta-$compatible so we can apply the results of the previous sections, but we can anyway "forget" about the second variable.

From now on, we identify functions on $\mathbb{R}_+$ and functions on $\mathbb{R}_+ \times \mathbb{R}$ which don't depend on their second variable and we make the abuse of notations that come with this identification.

As we will need some regularity assumption on ρ, we make the following

Definition 3.2 (Spread-out probability measure on $\mathbb{R}$)**.** We say that a probability measure ρ on $\mathbb{R}$ is spread-out if there exist $m \in \mathbb{N}^*$ such that ρ^{*m} is not singular with respect to Lebesgue's measure on $\mathbb{R}$.

First, we have the following

Theorem 3.3 (Leguesdron [Leg89], Peigné-Woess [PW06])**.** *Let ρ be a spread-out probability measure on $\mathbb{R}$ having an moment of order 1 and a negative drift $\lambda = \int_{\mathbb{R}} y \mathrm{d}\rho(y)$. Consider (X_n), the reflected random walk associated to ρ.*

There exist a unique $P-$invariant probability measure ν on $\mathbb{R}_+$. Moreover, if $\rho(\mathbb{R}_+^*) = 0$, then $\operatorname{supp}\nu = [0, M]$, else $\operatorname{supp}\nu = \mathbb{R}_+$ where $M = -\inf \operatorname{supp}(\rho)$ (which may be infinite).

Finally, the reflected random walk is topologically irreducible on $\operatorname{supp}\nu$.

To solve Poisson's equation, we are going to use the theory of petite sets developed by Glynn, Meyn and Tweedie (see [GM96] and [MT93]).

Definition 3.4. Let a be a probability measure on $\mathbb{N}$. A set $C \subset \mathbf{X}$ is called $\nu-$*petite* where ν is a non trivial measure on $\mathbf{X}$ if for any borelian subset A of $\mathbf{X}$ and any $x \in C$,

$$\sum_{n\in\mathbb{N}} a_n P^n(\mathbf{1}_A)(x) \geqslant \nu(A)$$

Proposition 3.5. *Let ρ be a spread out probability measure on $\mathbb{R}$ having a moment of order 1 and such that $\rho(\mathbb{R}_+^*) > 0$.*

Then the reflected random walk on $\mathbb{R}_+$ defined by ρ is irreducible and every compact set is petite.

Proof. As we already saw in theorem 3.3, if $\int |y|\rho(\mathrm{d}y)$ is finite, reflected random walk is open-set irreducible on $\mathbb{R}_+$.

Therefore, is we prove that (X_n) is a T-chain (see [MT93]), using the first point of Theorem 6.0.1 in [MT93] we will get that the chain is $m-$irreducible and then, using the second point of this theorem, we will get that every compact set is petite.

We need to find $(a_n) \in [0,1]^{\mathbb{N}}$ such that $\sum_n a_n = 1$ and a substochastic transition kernel T such that $\forall x \in \mathbf{X}\ T(1)(x) > 0$, for any borelian set A, $T(\mathbf{1}_A)$ is lower semicontinuous and

$$\forall x \in \mathbf{X}\ \forall A \in \mathcal{B}(\mathbf{X})\ \sum_{n=0}^{+\infty} a_n P^n(\mathbf{1}_A)(x) \geqslant T(\mathbf{1}_A)(x)$$

We assume without any loss of generality that ρ is compactly supported and we note $M \in \mathbb{R}_+^*$ such that $\operatorname{supp}\rho \subset [-M, +\infty[$)

Let m be such that ρ^{*m} is not singular with respect to Lebesgue's measure. We note ψ it's Radon-Nikodym's derivative.

Let $\varepsilon \in \mathbb{R}_+^*$ such that $\rho([\varepsilon, +\infty[) > 0$ (such ε exists since $\rho(\mathbb{R}_+^*) > 0$). We note ρ_ε the measure defined by $\rho_\varepsilon(A) = \rho(A \cap [\varepsilon, +\infty[)$ X_n^ε the random walk assoicated to ρ_ε and P_ε the submarkov operator associated to ρ_ε that is to say : $P_\varepsilon(f)(x) = \int_{\mathbb{R}} f(|x+y|)\mathrm{d}\rho_\varepsilon(y) = \int_\varepsilon^{+\infty} f(x+y)\mathrm{d}\rho(y)$.

The main idea of this proof is that, using ρ_ε, we can "escape" any compact set, and in particular, if we walk for a long enough time N with ρ_ε (N such that $N\varepsilon \geqslant mM$), we can be sure that the time of first reflection for the walk starting at X_N^ε is greater than m. And thus, we can use the hypothesis that ρ is spread out.

More precisely, if f is a non negative borelian function on $\mathbb{R}_+$ and $N \in \mathbb{N}$ is such that $N\varepsilon \geqslant mM$, then,

$$\begin{aligned}
P^{m+N}f(x) &\geqslant \int_{\mathbb{R}^{m+N}} f(||x+y_1|+\cdots+y_{m+N}|)\mathrm{d}\rho_\varepsilon^{\otimes N}\otimes \mathrm{d}\rho^{\otimes m}((y_i)) \\
&\geqslant \int_{\mathbb{R}^{m+N}} f(x+y_1+\cdots+y_{m+N})\mathrm{d}\rho_\varepsilon^{\otimes N}\otimes \mathrm{d}\rho^{\otimes m}((y_i)) \\
&\geqslant \int_{\mathbb{R}^2} f(x+v+u)\mathrm{d}\rho_\varepsilon^{*N}(v)\mathrm{d}\rho^{*m}(u) \geqslant \int_{\mathbb{R}^2} f(x+v+u)\mathrm{d}\rho_\varepsilon^{*N}(v)\psi(u)\mathrm{d}u \\
&\geqslant \int_{\mathbb{R}} f(x+u)\int_{\mathbb{R}}\psi(u-v)\mathrm{d}\rho_\varepsilon^{*N}(v)\mathrm{d}u = \int_{\mathbb{R}} f(x+u)\psi_1(u)\mathrm{d}u
\end{aligned}$$

Where, for $u\in\mathbb{R}$, $\psi_1(u)=\int_{\mathbb{R}}\psi(u-v)\rho_\varepsilon^{*N}(\mathrm{d}v)$.

For $n\in\mathbb{N}$, let $\psi_2^n(u)=\min(n,\psi_1(u))\in \mathrm{L}^1\cap\mathrm{L}^\infty$. By definition, ψ_2^n is a non decreasing sequence of positive functions and using the monotone convergence theorem, we have that $\int\psi_2^n(u)\mathrm{d}u \xrightarrow[n\to+\infty]{} \int\psi_1\mathrm{d}u=\rho_\varepsilon(\mathbb{R})^N=\rho([\varepsilon,+\infty[)^N\in]0,1]$. So there exist $n_0\in\mathbb{N}$ such that $\int\psi_2^{n_0}(u)\mathrm{d}u\neq 0$.

We note $\psi_2^{n_0}=\psi_2$ and we resume our computations.

By construction,

$$\begin{aligned}
P^{2m+2N}f(x) &\geqslant \int_{\mathbb{R}} P^{m+N}f(x+u)\psi_2(u)\mathrm{d}u \geqslant \int_{\mathbb{R}^2} f(x+u+v)\psi_2(v)\psi_2(u)\mathrm{d}u\mathrm{d}v \\
&\geqslant \int_{\mathbb{R}} f(x+u)\left(\int_{\mathbb{R}}\psi_2(u-v)\psi_2(v)\mathrm{d}v\right)\mathrm{d}u = \int_{\mathbb{R}} f(x+u)\psi_3(u)\mathrm{d}u
\end{aligned}$$

Where $\psi_3(u)=\int_{\mathbb{R}}\psi_2(u-v)\psi_2(v)\mathrm{d}v$.

But, ψ_3 is the convolution product is of an integrable and a bounded function so it is continuous and $\int_{\mathbb{R}}\psi_3(u)\mathrm{d}u=\left(\int_{\mathbb{R}}\psi_2(u)\mathrm{d}u\right)^2>0$.

Let ψ_4 be a non zero non negative continuous function on $\mathbb{R}$ such that $\psi_4\leqslant\psi_3$ and $\operatorname{supp}\psi_4$ is compact.

We note T the operator defined for any borelian bounded function f on $\mathbb{R}$ and any $x\in\mathbb{R}$ by

$$Tf(x)=\int_{\mathbb{R}} f(x+u)\psi_4(u)\mathrm{d}u$$

Using the dominated convergence theorem we have that for any borelian and bounded function f, Tf is continuous. Moreover, for all $x\in\mathbb{R}_+$, $1\geqslant T1(x)=\int_{\mathbb{R}}\psi_4\mathrm{d}\lambda>0$.

And, we get that for every borelian non negative function f and every x in $\mathbb{R}_+$,

$$P^{2N+2m}f(x)\geqslant Tf(x)$$

and this finishes the proof of the proposition. □

Proposition 3.6. *Let ρ be a spread out probability measure on $\mathbb{R}$ having a moment of order 1 and such that $0<\int_{\mathbb{R}_+} y\mathrm{d}\rho(y)<\int_{\mathbb{R}_-}(-y)\mathrm{d}\rho(y)$.*

Then, there is a constant C such that for any borelian and bounded fonction f on $\mathbb{R}_+$, there is a function g on $\mathbb{R}_+$ such that

$$f=g-Pg+\int f\mathrm{d}\nu$$

And,

$$\sup_{x\in\mathbb{R}} \frac{|g(x)|}{1+|x|} \leqslant C \left\| f - \int f \mathrm{d}\nu \right\|_\infty$$

Moreover, if ρ is the sum of an absolutely continuous measure and of a discrete one and if f is a.e. continuous, then so is g.

Proof. For any $x \in \mathbb{R}_+$, let $u(x) = 1 + \frac{x}{-\lambda}$ where we noted $\lambda = \int_{\mathbb{R}} y \mathrm{d}\rho(y) < 0$.

Then,

$$\begin{aligned} Pu(x) &= 1 + \int_{-\infty}^{-x} \frac{x+y}{\lambda} \mathrm{d}\rho(y) + \int_{-x}^{+\infty} \frac{-x-y}{\lambda} \mathrm{d}\rho(y) \\ &= 1 + \frac{x}{\lambda} \left(\rho(]-\infty,-x]) - \rho(]-x,+\infty])\right) - \frac{1}{\lambda}\left(\int_{-x}^{+\infty} y \mathrm{d}\rho(y) - \int_{-\infty}^{-x} y \mathrm{d}\rho(y)\right) \\ &= 1 + \frac{x}{-\lambda}) \left(1 - 2\rho(]-\infty,-x])\right) - 1 + \frac{2}{\lambda}\int_{-\infty}^{-x} y \mathrm{d}\rho(y) \\ &= u(x) - 1 + 2\frac{x}{\lambda}\rho(]-\infty,-x]) + \frac{2}{\lambda}\int_{-\infty}^{-x} y \mathrm{d}\rho(y) \end{aligned}$$

Moreover,

$$x\rho(]-\infty,-x]) = \int_{-\infty}^{-x} x \mathrm{d}\rho(y) \leqslant \int_{-\infty}^{-x} |y| \mathrm{d}\rho(y)$$

So, as ρ has a moment of order 1,

$$\lim_{x\to+\infty} Pu(x) - u(x) = -1$$

Thus, there are $x_0 \in \mathbb{R}_+$ and $b \in \mathbb{R}$ such that

$$Pu \leqslant u - \frac{1}{2} + b\mathbf{1}_{[0,x_0]}$$

Therefore, using proposition 3.5 we can apply the theorem 2.3 in [GM96] and get a constant C such that for any borelian bounded function f on $\mathbb{R}_+$ there is a borelian function g such that

$$\sup_x \frac{|g(x)|}{u(x)} \leqslant C \left\| f - \int f \mathrm{d}\nu \right\|_\infty$$

and

$$f = g - Pg + \int f \mathrm{d}\nu$$

The fact that g is a.e. continuous when f is a.e. continuous is also proved since we have an explicit formula for the function g given in [GM96] and using the same ideas as in the proof of lemma 2.5. □

From now on, the assumption on ρ being only spread-out is not enough (since the stopped renewal theorem we have in this case doesn't hold for these probability measures) so we are going to ask that it is absolutely continuous instead.

Corollary 3.7. *Let ρ be an absolutely continuous probability measure on $\mathbb{R}$ having a moment of order 1, a negative drift $\lambda = \int_{\mathbb{R}} y\mathrm{d}\rho(y) < 0$ and that $\rho(\mathbb{R}_+^*) \neq 0$.*

Then, for any directly Riemman-integrable function f on $\mathbb{R}_+$ such that $\int f\mathrm{d}\nu = 0$, there is a bounded and a.e. continuous function g on $\mathbb{R}_+$ such that

$$f = g - Pg \text{ and } \lim_{x\to+\infty} g(x) = 0$$

Proof. According to the previous lemma, there is an a.e. continuous function g on $\mathbb{R}_+$ such that

$$f = g - Pg \text{ and } \sup_{x\in\mathbb{R}_+} \frac{|g(x)|}{1+|x|}$$

We note τ the time of first reflection. This is the stopping time defined by

$$\tau((X_n)) = \inf\{n \in \mathbb{N} \mid X_{n+1} = -X_n - Y_n\}$$

Moreover, we note R and S the Markov operators associated to τ and defined as in section 1.

Note that for $x \geqslant 0$ and before the reflection, the walk is the same as the unreflected random walk. Therefore, as in section 2 we have that for any $x \in \mathbb{R}_+$,

$$\mathbb{E}_x\tau < +\infty$$

The stopping time τ is $\vartheta-$compatible so, we can use the relations of proposition 1.3 to get that for any $x \in \mathbb{R}_+$,

$$Rf(x) = R(g - Pg)(x) = g(x) - Qg(x)$$

Moreover, we also have that

$$Qg(x) = RSg(x)$$

so, we get that

$$g(x) = R(f + Sg)(x)$$

Moreover,

$$|Qg(x)| = |\mathbb{E}g(X_\tau)| \leqslant C\mathbb{E}_x 1 + X_\tau$$

But, using that $0 \leqslant X_\tau \leqslant -Y_\tau$, we have that

$$\mathbb{E}_x X_\tau \leqslant -\mathbb{E}_x Y_\tau \leqslant \int_{\mathbb{R}} |y|\mathrm{d}\rho(y)$$

So the function Qg is bounded on $\mathbb{R}_+$ and as Rf is bounded on $\mathbb{R}_+$ (according to lemma 2.3 that we can use for $x \geqslant 0$ since the operator R for the reflected random walk and for the unreflected one are the same), this proves that g is bounded on $\mathbb{R}_+$.

Thus, using lemma 2.4 and lemma 2.5, we get that Sg is directly Riemann-integrable.

Therefore, we can apply corollary 2.6 to the function $f + Sg$ to get that $g = R(f + Sg)$ has a limit l at infinity. Noting $g_1 = g - l$ we have that g_1 is a.e. continuous, bounded,

$$g_1 - Pg_1 = f \text{ and } \lim_{x\to+\infty} g_1(x) = 0$$

And this is finally what we intended to prove. □

Proposition 3.8 (Large deviations inequality)**.** *Under the assumptions of corollary 3.7.*

For any directly Riemann-integrable function f on $\mathbb{R}_+$ such that $\int f\mathrm{d}\nu = 0$, there are constant $C_1, C_2 \in \mathbb{R}_+^*$ such that for any $\varepsilon \in]0,1]$, any $x \in \mathbb{R}_+$ and any $n \in \mathbb{N}^*$,

$$\mathbb{P}_x\left(\left\{\left|\frac{1}{n}\sum_{k=0}^{n-1} f(X_k)\right| \geqslant \varepsilon\right\}\right) \leqslant C_1 e^{-C_2\varepsilon^2 n}$$

In particular, for any $x \in \mathbb{R}_+$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \to 0 \ \mathbb{P}_x\text{-a.e.}$$

Proof. To prove the result, we are going to use Asuma-Hoeffding's inequality.

First, we write $f = g - Pg$ where g is the function given by corollary 3.7. Write, for any $n \in \mathbb{N}$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) = \frac{1}{n}\sum_{k=0}^{n-1} g(X_{k+1}) - Pg(X_k) + \frac{1}{n}\left(g(X_0) - g(X_n)\right)$$

Thus,

$$\frac{1}{n}\left|\sum_{k=0}^{n-1} g(X_{k+1}) - Pg(X_k)\right| \geqslant \frac{1}{n}\left|\sum_{k=0}^{n-1} f(X_k)\right| - \frac{2}{n}\|g\|_\infty$$

and so, using Azuma-Hoeffding's inequality, if $n\varepsilon > 2\|g\|_\infty$, we have that

$$\begin{aligned} I_n(x) :&= \mathbb{P}_x\left(\frac{1}{n}\left|\sum_{k=0}^{n-1} f(X_k)\right| > \varepsilon\right) \\ &\leqslant \mathbb{P}_x\left(\left|\sum_{k=0}^{n-1} g(X_{k+1}) - Pg(X_k)\right| \geqslant n\varepsilon - 2\|g\|_\infty\right) \\ &\leqslant 2\exp\left(\frac{-(n\varepsilon - 2\|g\|_\infty)^2}{2n(2\|g\|_\infty)^2}\right) \end{aligned}$$

and this last inequality is what we intended to prove. The law of large numbers now comes from Borel-Cantelli's lemma. □

To prove the central limit theorem and the law of the iterated logarithm, we will need a weaker version of the law of large numbers (to show that the variance converges). This will be the following

Lemma 3.9. *Let ρ be an absolutely continuous probability measure on $\mathbb{R}$ having an moment of order $2+\varepsilon$ for some $\varepsilon \in \mathbb{R}_+^*$ and such that $0 < \int_{\mathbb{R}_+} y\mathrm{d}\rho(y) < \int_{\mathbb{R}_-}(-y)\mathrm{d}\rho(y)$.*

For any borelian and bounded function f on $\mathbb{R}_+$ and any $x \in \mathbb{R}_+$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \to \int f\mathrm{d}\nu \ \mathbb{P}_x - a.e.$$

Remark 3.10. In this lemma, we don't ask the function f to be directly Riemann-integrable but the price we have to pay is a stronger moment hypothesis on ρ.

Proof. Write $f = g - Pg + \int f \mathrm{d}\nu$ with g the function given by 3.6 such that

$$C(g) = \sup_x \frac{|g(x)|}{1+|x|} < +\infty$$

Then,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) = \int f \mathrm{d}\nu + \frac{1}{n}(g(X_0) - g(X_n)) + \frac{1}{n}\sum_{k=0}^{n-1} g(X_{k+1}) - Pg(X_k)$$

Moreover, for any $s \in [1, +\infty[$,

$$\mathbb{E}_x|g(X_{k+1}) - Pg(X_k)|^s \leqslant 2^s P^{k+1}|g|^s(x) \leqslant 2^s C(g)^s P^{k+1} u_s(x)$$

with $u_s(x) = (1+x)^s$.

But, doing the same computations as in the proof of lemma 3.6, we see that if ρ has a moment of order $s+1$, then there are $C, B \in \mathbb{R}_+^*$ such that

$$u_s \leqslant C\left(u_{s+1} - Pu_{s+1} + B\right)$$

so,

$$\begin{aligned}\sum_{n=1}^{+\infty} \frac{1}{n^s}\mathbb{E}_x\left|g(X_{k+1}) - Pg(X_k)\right|^s &\leqslant CC(g)^s 2^s \sum_{n=1}^{+\infty} \frac{1}{n^s} P^{n+1}(u_{s+1} - Pu_{s+1} + B) \\ &\leqslant C2^s C(g)^s \left(\sum_{n=1}^{+\infty} \frac{P^{n+1}(I_d - P)u_{s+1}}{n^s} + B\sum_{n=1}^{+\infty}\frac{1}{n^s}\right) \\ &\leqslant C2^s C(g)^s \left(u_{s+1} + B\sum_{n=1}^{+\infty}\frac{1}{n^s}\right) < +\infty\end{aligned}$$

and this proves, using the law of large numbers for martingales (see theorem 2.18 in [HH80]) that if ρ has a moment of order $2+\varepsilon$ for some $\varepsilon \in \mathbb{R}_+^*$, then for any $x \in \mathbb{R}$,

$$\frac{1}{n}\sum_{k=0}^{n-1} g(X_{k+1}) - Pg(X_k) \to 0 \ \mathbb{P}_x\text{-a.e. and in } \mathrm{L}^1(\mathbb{P}_x)$$

Doing the same kind of computations, we also prove that $\frac{1}{n}g(X_n)$ converges to 0 and this proves the expected result. □

Corollary 3.11. *Let ρ be an absolutely continuous probability measure on $\mathbb{R}$ having an moment of order $2+\varepsilon$ for some $\varepsilon \in \mathbb{R}_+^*$ and such that $0 < \int_{\mathbb{R}_+} y\mathrm{d}\rho(y) < \int_{\mathbb{R}_-}(-y)\mathrm{d}\rho(y)$.*

Then, for any directly Riemann-integrable function f on $\mathbb{R}_+$ with $\int f \mathrm{d}\nu = 0$ and any $x \in \mathbb{R}_+$,

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(X_k) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma^2(f)\right)$$

Where

$$\sigma^2(f) = \int_{\mathbb{R}_+} g^2 - (Pg)^2 \mathrm{d}\nu$$

with g the bounded function given by corollary 3.7 and such that $f = g - Pg$.

Moreover, if $\sigma^2(f) \neq 0$, *then*

$$\limsup \frac{\sum_{k=0}^{n-1} f(X_k)}{\sqrt{2n\sigma^2(f)\ln\ln(n)}} = 1 \text{ a.e. and } \liminf \frac{\sum_{k=0}^{n-1} f(X_k)}{\sqrt{2n\sigma^2(f)\ln\ln(n)}} = -1 \text{ a.e.}$$

Proof. First, we use that

$$\sum_{k=0}^{n-1} f(X_k) = g(X_0) - g(X_n) + \sum_{k=0}^{n-1} g(X_{k+1}) - Pg(X_k)$$

Let $M_n = \sum_{k=0}^{n-1} g(X_{k+1}) - Pg(X_k)$. Then, M_n is a martingale with bounded difference sequence.

Moreover, noting $(\mathcal{F}_n)$ the filtration associated to this martingale, we have that

$$\begin{aligned}\mathbb{E}_x\left[(M_{n+1}-M_n)^2|\mathcal{F}_n\right] &= \mathbb{E}_x\left[g(X_{n+1})^2 - g(X_{n+1})Pg(X_n) + (Pg(X_n))^2|\mathcal{F}_n\right] \\ &= P(g^2)(X_n) - (Pg)^2(X_n)\end{aligned}$$

And so,

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}_x\left[(M_{k+1}-M_k)^2|\mathcal{F}_k\right] = \frac{1}{n}\sum_{k=0}^{n-1} Pg^2(X_k) + \frac{1}{n}\sum_{k=0}^{n-1}(Pg)^2(X_k)$$

But, the function g is bounded on $\mathbb{R}$, so, according to lemma 3.9,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}_x\left[(M_{k+1}-M_k)^2|\mathcal{F}_k\right] = \int g^2 \mathrm{d}\nu - \int (Pg)^2 \mathrm{d}\nu \; \mathbb{P}_x - \text{a.e. and in } \mathrm{L}^1(\mathbb{P}_x)$$

Thus, using the central limit theorem and the law of the iterated logarithm for martingales with bounded increments (cf. [Bro71] and theorem 4.8 and corollary 4.2 in [HH80]), we get the expected result. □

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