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# MULTIPLICATIVE ERGODICITY OF LAPLACE TRANSFORMS FOR ADDITIVE FUNCTIONAL OF MARKOV CHAINS

LOÏC HERVÉ AND FRANÇOISE PÈNE

ABSTRACT. We study properties of the Laplace transforms of non-negative additive functionals of Markov chains. We are namely interested in a multiplicative ergodicity property used in [16] to study bifurcating processes with ancestral dependence. We develop a general approach based on the use of the operator perturbation method. We apply our general results to two examples of Markov chains, including a linear autoregressive model. In these two examples the operator-type assumptions reduce to some expected finite moment conditions on the functional (no exponential moment conditions are assumed in this work).

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## 1. INTRODUCTION

In this work we study the Laplace transforms of non-negative additive functionals of Markov chains by the use of the method of perturbation of operators. This method, introduced by Nagaev [18, 19] and by Le Page and Guivarc'h [14, 7] to prove a wide class of limit theorems (central limit theorem, local limit theorem, large and moderate deviations principles), has known an impressive development in the past decades (e.g. see [3, 10] and the references therein), and more recently due to the Keller and Liverani perturbation theorem [13, 1] allowing working with several spaces instead of a single one (e.g. see [12] and the references therein). This new approach, which involves much weaker moment assumptions on the functional, is used here. We point out that, to the best of our knowledge, this paper contains the first investigation on the Laplace transform for non-negative additive functional of Markov chains without any exponential moment assumption.

Actually we study different properties of the Laplace transforms of nonnegative additive functionals of Markov chains, namely their multiplicative ergodicity and the continuity and derivability of the radius of convergence of the Laplace-generating function, together with their spectral counterparts. We emphasize also on some applications of our result in the study of the bifurcating processes developed in [20]. The present work provides examples coming from a markovian context satisfying some assumptions of [16]. We investigate in particular a multiplicative ergodicity property and its spectral analogus.

This paper is organized as follows. In Section 2, we introduce the notion of multiplicative ergodicity we are interested in, our notations and we state our main results. We namely state general conditions ensuring the multiplicative ergodicity of an additive functional of a geometrically ergodic Markov chain. We illustrate our general result by two examples of Markov chains: the Knudsen gas model and some linear autoregressive models. The proofs for these examples are given in Sections 3 and 4. The more technical proofs of our general results are postponed in Appendix A together with some other facts.

## 2. NOTATIONS AND MAIN RESULTS

**2.1. Multiplicative ergodicity, examples.** Given a sequence  $Y = (Y_n)_{n \geq 0}$  of non-negative valued random variables, we consider the generating function of the Laplace transforms of the partial sums of  $Y$ , that will be named *Laplace-generating function* of  $Y$ . We assume that the random variables  $Y_n$  are not identically zero.

**Definition 2.1.** *The Laplace transforms of the partial sums of  $Y$  are denoted by  $L_Y^{(n)}$ :*

$$\forall \gamma \in \mathbb{R}_+, \quad L_Y^{(n)}(\gamma) := \mathbb{E}[\exp(-\gamma S_n)] , \quad (1)$$

with  $S_n := \sum_{k=0}^n Y_k$ . *The Laplace-generating function of  $Y$ , denoted by  $g_Y$ , is the generating function of the  $L_Y^{(n)}(\gamma)$ 's. For  $\gamma, \lambda \in \mathbb{R}^+$ ,*

$$g_Y(\gamma, \lambda) = \sum_{n=0}^{+\infty} \lambda^n L_Y^{(n)}(\gamma) . \quad (2)$$

Observe that for all  $\lambda \in [0, 1)$ ,  $g_Y(\cdot, \lambda)$  is non-increasing on  $[0, +\infty)$ , decreasing on  $\{\gamma \geq 0 : g_Y(\gamma, \lambda) < \infty\}$ , starting at  $g_Y(0, \lambda) = 1/(1 - \lambda)$ . Hence the radius of convergence  $R_Y(\gamma)$

of  $g_Y(\gamma, \cdot)$  is non-decreasing in  $\gamma$  from  $R_Y(0) = 1$ . We are namely interested in the following properties:

$$\nu := \inf\{\gamma > 0 : g_Y(\gamma, 2) < \infty\} < \infty \quad (3)$$

and

$$C_\nu := \lim_{\gamma \rightarrow \nu^+} \frac{\gamma - \nu}{\gamma} g_Y(\gamma, 2) < \infty. \quad (4)$$

In [16], it has been shown that these two properties imply the convergence in average of  $e^{-\nu t} \mathbb{E}[N_t]$  where  $N_t$  is the number of cells at time  $t$  in a mitosis process such that the life duration of the successive individuals of a same line has the distribution of  $(Y_k)_k$ . To prove (3) and (4), we will use the following notion of multiplicative ergodicity (see [16]). Let us precise that the terminology "multiplicative ergodicity" is used in the litterature with different levels of sharpness.

**Definition 2.2.** *Let  $\gamma_1 > 0$ . We say that  $(S_n)_n$  is **multiplicatively ergodic** on  $J = [0, \gamma_1)$  if there exist two continuous maps  $A$  and  $\rho$  from  $J$  to  $(0, +\infty)$  such that, for every compact subset  $K$  of  $(0, \gamma_1)$ , there exist  $M_K > 0$  and  $\theta_K \in (0, 1)$  such that, for every  $n \geq 1$ , we have*

$$\sup_{\gamma \in K} |L_Y^{(n)}(\gamma) - A(\gamma)(\rho(\gamma))^n| \leq M_K(\rho(\gamma)\theta_K)^n.$$

Observe that if  $(S_n)_n$  is multiplicatively ergodic on  $J = [0, \gamma_1)$ , then  $\rho \equiv 1/R_Y$  and

$$\forall \gamma \in J, \forall \lambda > 0, \quad g_Y(\gamma, \lambda) < \infty \quad \Leftrightarrow \quad \lambda < \frac{1}{\rho(\gamma)},$$

and, for every compact subset  $K$  of  $J$ , we have

$$\forall \gamma \in K, \forall \lambda \in \left(0, \frac{1}{\rho(\gamma)}\right), \quad \left\| g_Y(\gamma, \lambda) - \frac{A(\gamma)}{1 - \lambda\rho(\gamma)} \right\| \leq \frac{M_K}{1 - \lambda\rho(\gamma)\theta_K}.$$

**Remark 2.3.** *If  $(S_n)_n$  is multiplicatively ergodic, then  $\nu < \gamma_1$  means that*

$$\nu = \inf\{\gamma \in J : \rho(\gamma) < 1/2\} < \gamma_1. \quad (5)$$

*If moreover  $\rho$  is differentiable at  $\nu$  with  $\rho'(\nu) \neq 0$ , then (4) will follow with  $C_\nu = -\frac{A(\nu)}{2\nu\rho'(\nu)}$ . Actually, to obtain (5), we can relax the continuity assumptions on  $A$  and  $\rho$  on  $J = [0, \gamma_1)$ . For (4), we just need the continuity of  $A$  and the differentiability of  $\rho$  at  $\nu$ .*

We focus our study on the three following properties:

- the geometric ergodicity on some maximal interval  $[0, \gamma_1)$ ,
- (3) and more generally the study of  $\lim_{\gamma \rightarrow \gamma_1} \rho(\gamma)$ ,
- (4) and more generally the differentiability of  $\rho$  on  $(0, \gamma_1)$  and the fact that  $\rho' < 0$  on this interval.

We investigate these properties in the context of additional functional of Markov chains. Let  $(\mathbb{X}, \mathcal{X})$  be a measurable space, let  $(X_n)_n$  be a Markov chain on  $(\mathbb{X}, \mathcal{X})$  with Markov kernel  $P(x, dy)$  and invariant probability  $\pi$ , and let  $\xi : \mathbb{X} \rightarrow [0, +\infty)$  be a measurable function. Recall that  $\xi$  is said to be coercive if  $\lim_{|x| \rightarrow +\infty} \xi(x) = +\infty$ , i.e. if, for every  $\beta$ ,  $[\xi \leq \beta]$  is bounded. Moreover we consider

$$S_n := \sum_{k=0}^n \xi(X_k).$$

We identify  $X$  with the canonical Markov chain and write  $\mathbb{P}_\mu$  for the probability measure corresponding to the case when the initial probability distribution (i.e. the distribution of  $X_0$ ) is  $\mu$ . For every  $x \in \mathbb{X}$ , we simply write  $\mathbb{P}_x$  when  $\mu = \delta_x$ . We write  $\mathbb{E}_\mu[\cdot]$  and  $\mathbb{E}_x[\cdot]$  for the corresponding expectations. We then write

$$\rho_{Y,\mu}(\gamma) := \limsup_{n \rightarrow +\infty} (\mathbb{E}_\mu[e^{-\lambda S_n}])^{\frac{1}{n}} \quad \text{and} \quad \rho_{Y,x}(\gamma) := \limsup_{n \rightarrow +\infty} (\mathbb{E}_x[e^{-\lambda S_n}])^{\frac{1}{n}}.$$

Moreover we simply write  $\rho_Y$  for  $\rho_{Y,\mu}$  in the case when  $\rho_{Y,\mu}$  does not depend on the initial distribution. In this context we develop a general method to prove the multiple ergodicity and even a spectral version of this property. As a consequence, we prove the following result.

**Theorem 2.4** (Linear autoregressive model). *Assume that  $\mathbb{X} := \mathbb{R}$  and  $(X_n)_{n \in \mathbb{N}}$  is the linear autoregressive model defined by  $X_n = \alpha X_{n-1} + \vartheta_n$  for  $n \geq 1$ , where  $X_0$  is a real-valued random variable,  $\alpha \in (-1, 1)$ , and  $(\vartheta_n)_{n \geq 1}$  is a sequence of i.i.d. real-valued random variables, admitting a moment of order  $r_0 > 0$ , independent of  $X_0$ . Assume that  $\vartheta_1$  has a continuous Lebesgue probability density function  $p > 0$  on  $\mathbb{X}$  such that*

$$\forall x \in \mathbb{R}, \quad \exists \varepsilon_x > 0, \quad \int_{\mathbb{R}} \sup_{|z| < \varepsilon_x} p(y + x + z) dy < \infty.$$

*Assume moreover that  $\xi$  is continuous and coercive, that  $\xi(x) > 0$  for Lebesgue almost every  $x \in \mathbb{R}$  and that  $\sup_{x \in \mathbb{R}} \frac{\xi(x)}{(1+|x|)^{r_0}} < \infty$ .*

*Then,  $\rho_{Y,x}(\gamma) = \rho_{Y,\pi}(\gamma)$ ,  $(S_n)_n$  is multiplicatively ergodic on  $(0, +\infty)$  with respect to  $\mathbb{P}_\pi$  and to  $\mathbb{P}_x$  for any  $x \in \mathbb{X}$ . Furthermore  $\lim_{\gamma \rightarrow +\infty} \rho_{Y,\pi}(\gamma) = 0$ . Hence (3) holds true under  $\mathbb{P}_\pi$  or  $\mathbb{P}_x$  for any  $x \in \mathbb{X}$ .*

*If moreover there exists  $\tau > 0$  such that  $\sup_{x \in \mathbb{R}} \frac{\xi(x)^{1+\tau}}{(1+|x|)^{r_0}} < \infty$ , then  $\rho_{Y,\pi}$  is differentiable and admits a negative derivative on  $(0, +\infty)$  and so (4) holds also true under  $\mathbb{P}_\pi$  or  $\mathbb{P}_x$  for any  $x \in \mathbb{X}$ .*

We also prove the following result for the simple example of Knudsen gas.

**Theorem 2.5.** *Let  $\mathbb{X} := \mathbb{R}^d$ ,  $\pi$  be some Borel probability measure on  $\mathbb{X}$ . Given  $\alpha \in (0, 1)$  and a Markov kernel  $U$  on  $\mathbb{R}^d$  with stationary measure  $\pi$ , we consider the canonical Markov chain  $X$  with transition kernel  $P$  given by  $P = \alpha\pi + (1 - \alpha)U$ .*

*Then  $(S_n)_n$  is multiplicatively ergodic on the interval  $J_0 = \{\gamma > 0 : r(\gamma) > 1 - \alpha\}$  with respect to  $\mathbb{P}_\mu$  for any probability distribution  $\mu$  on  $\mathbb{X}$  absolutely continuous with respect to  $\pi$ , with density in  $\mathbb{L}^p(\pi)$  for some  $p > 1$ .*

*Assume  $\alpha > 1/2$  and  $2\alpha \sum_{n \geq 0} (2(1 - \alpha))^n \mathbb{P}_\pi(\sum_{k=0}^n Z_k = 0) < 1$ , where  $(Z_n)_n$  is a Markov process with transition  $U$ , then (3) holds with respect to  $\mathbb{P}_\pi$  and to  $\mathbb{P}_\mu$  for every probability distribution  $\mu$  in  $\mathbb{X}$  satisfying the previous conditions (this is true in particular if  $\alpha > 1/2$  and  $\pi(\xi = 0) = 0$ ).*

*Assume moreover that  $\pi(\xi^\tau) < \infty$  for some  $\tau > 1$ . Then (4) holds also true with respect to  $\mathbb{P}_\pi$  and to  $\mathbb{P}_\mu$  for every probability distribution  $\mu$  admitting a density with respect to  $\pi$  which belongs to  $\mathbb{L}^p(\pi)$  for some  $p > \frac{\tau}{\tau-1}$ .*

Now let us present our general approach. The use of the spectral study of perturbed operators to establish probabilistic limit theorems goes back to [18, 7] (see also [10] and the

references therein). This method has recently been improved namely by the Keller-Liverani perturbation theorem [13]. We will see that this result is appropriate to prove the spectral multiplicative ergodicity of additive functionals of Markov chain.

**2.2. Notations.** For any normed complex vector spaces  $(\mathcal{B}_0, \|\cdot\|_{\mathcal{B}_0})$  and  $(\mathcal{B}_1, \|\cdot\|_{\mathcal{B}_1})$ , the set of continuous  $\mathbb{C}$ -linear operators from  $\mathcal{B}_0$  to  $\mathcal{B}_1$  will be written  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$ . This set is endowed with the operator norm  $\|\cdot\|_{\mathcal{B}_0, \mathcal{B}_1}$  given by

$$\forall Q \in \mathcal{L}(\mathcal{B}_0, \mathcal{B}_1), \quad \|Q\|_{\mathcal{B}_0, \mathcal{B}_1} = \sup_{f \in \mathcal{B}_0, \|f\|_{\mathcal{B}_0}=1} \|Qf\|_{\mathcal{B}_1}.$$

The notation  $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1$  means that  $\mathcal{B}_0$  is continuously injected in  $\mathcal{B}_1$ .

If  $\mathcal{B}$  is a complex Banach space, we will simply write  $(\mathcal{B}^*, \|\cdot\|_{\mathcal{B}^*})$  for the topological dual space  $(\mathcal{L}(\mathcal{B}, \mathbb{C}), \|\cdot\|_{\mathcal{B}, \mathbb{C}})$  of  $\mathcal{B}$  and  $(\mathcal{L}(\mathcal{B}), \|\cdot\|_{\mathcal{B}})$  for  $(\mathcal{L}(\mathcal{B}, \mathcal{B}), \|\cdot\|_{\mathcal{B}, \mathcal{B}})$ . For any  $Q \in \mathcal{L}(\mathcal{B})$ , we denote by  $Q^*$  its adjoint operator. We write  $\sigma(Q) = \sigma(Q|_{\mathcal{B}})$  for the spectrum of  $Q$ :

$$\sigma(Q) := \{\lambda \in \mathbb{C} : (Q - \lambda I) \text{ is non invertible}\},$$

where  $I$  denotes the identity operator on  $\mathcal{B}$ . Recall that  $Q$  and  $Q^*$  have the same norm in  $\mathcal{L}(\mathcal{B})$  and  $\mathcal{L}(\mathcal{B}^*)$  respectively, as well as the same spectrum. We write  $r(Q) = r(Q|_{\mathcal{B}})$  for the spectral radius of  $Q$ :

$$r(Q|_{\mathcal{B}}) := \sup\{|\lambda|, \lambda \in \sigma(Q)\} = \lim_n \|Q^n\|_{\mathcal{B}}^{1/n}$$

and  $r_{ess}(Q) = r_{ess}(Q|_{\mathcal{B}})$  for its essential spectral radius:

$$r_{ess}(Q) := \lim_n \inf_{F \in \mathcal{L}(\mathcal{B}) \text{ compact}} \|Q^n - F\|_{\mathcal{B}}^{1/n}.$$

Recall that we also have

$$r_{ess}(Q) := \sup\{|\lambda| : \lambda \in \mathbb{C} \text{ and } (Q - \lambda I) \text{ is non Fredholm}\}.$$

Let  $(\mathbb{X}, \mathcal{X})$  be a measurable space, let  $X = (X_n)_n$  be a Markov chain on  $(\mathbb{X}, \mathcal{X})$  with Markov kernel  $P(x, dy)$  and invariant probability  $\pi$ , and let  $\xi : \mathbb{X} \rightarrow [0, +\infty)$  be a measurable function. We then consider

$$Y_k := \xi(X_k) \quad \text{and} \quad S_n := \sum_{k=0}^n Y_k.$$

We identify  $X$  with the canonical Markov chain. We consider the nonnegative kernels  $P_\gamma(x, dy)$  defined by

$$\forall \gamma \in [0, +\infty), \quad P_\gamma(x, dy) := e^{-\gamma \xi(y)} P(x, dy) \quad \text{and} \quad P_\infty(x, dy) := \mathbf{1}_{\{\xi=0\}}(y) P(x, dy). \quad (6)$$

We use the same notations  $P_\gamma$  for the linear operators associated with these kernels:

$$\forall x \in \mathbb{X}, \quad (P_\gamma f)(x) := \int_{\mathbb{X}} f(y) P_\gamma(x, dy).$$

In the sequel  $P_\gamma$  will be assumed to continuously act on a (or several) Banach space  $\mathcal{B}$ . Such a space will contain  $\mathbf{1}_{\mathbb{X}}$  and  $\pi$  will be in its topological dual space. Moreover we write  $r(\gamma) := r(P_\gamma|_{\mathcal{B}})$  for the spectral radius of  $P_\gamma$ . With these notations, we have

$$\mathbb{E}_\mu[e^{-\gamma S_n}] = \mu(e^{-\gamma \xi} P_\gamma^n \mathbf{1}_{\mathbb{X}}) \quad \text{and} \quad g_Y(\gamma, \lambda) = \mu(e^{-\gamma \xi} (I - \lambda P_\gamma)^{-1} \mathbf{1}_{\mathbb{X}}), \quad (7)$$

for any  $\lambda < \frac{1}{r(\gamma)}$  and for any initial distribution  $\mu$  on  $\mathbb{X}$  such that  $f \mapsto \mu(e^{-\gamma\xi}f)$  belongs to  $\mathcal{B}^*$ .

Let us now recall the definition of Banach lattice spaces of functions (or classes of functions modulo  $\pi$ ), which will be used in our examples to obtain the expected nonincreasingness of  $r(\cdot)$  and some suitable spectral properties for  $P_\gamma$ .

**Definition 2.6.** *A complex Banach space  $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$  of functions  $f : \mathbb{X} \rightarrow \mathbb{C}$  (or of classes of such functions modulo  $\pi$ ) is said to be a **complex Banach lattice** if it is stable by  $|\cdot|$ , by real part and if*

$$\begin{aligned} \forall f, g \in \mathcal{B}, \quad f(\mathbb{X}) \cup g(\mathbb{X}) \subset \mathbb{R} &\Rightarrow \min(f, g), \max(f, g) \in \mathcal{B}, \\ \forall f, g \in \mathcal{B}, \quad |f| \leq |g| &\Rightarrow \| |f| \|_{\mathcal{B}} = \|f\|_{\mathcal{B}} \leq \|g\|_{\mathcal{B}} = \| |g| \|_{\mathcal{B}}. \end{aligned}$$

Classical instances of Banach lattices of functions are the spaces  $(\mathbb{L}^p(\pi), \|\cdot\|_p)$  and  $(\mathcal{B}_V, \|\cdot\|_V)$  (see (13) and (21)), as well as the space  $(\mathcal{L}^\infty(\mathbb{X}), \|\cdot\|_\infty)$  composed of all the bounded measurable  $\mathbb{C}$ -valued functions on  $\mathbb{X}$ , and equipped with its usual norm  $\|f\|_\infty := \sup_{x \in \mathbb{X}} |f(x)|$ .

**2.3. General results.** We first prove that the monotonicity of  $\gamma \mapsto r(\gamma) := r(P_{\gamma|\mathcal{B}})$  is easy to establish when  $\mathcal{B}$  is a Banach lattice of functions.

**Lemma 2.7.** *If  $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$  is a complex Banach lattice of functions  $f : \mathbb{X} \rightarrow \mathbb{C}$  (or of classes of functions modulo  $\pi$ ), then the map  $\gamma \mapsto r(\gamma)$  is non increasing on  $[0, +\infty)$ .*

*Proof.* For any  $0 \leq \gamma < \gamma' \leq \infty$  and for any  $f, g \in \mathcal{B}$  such that  $|f| \leq |g|$ , we have  $e^{-\gamma'\xi}|f| \leq e^{-\gamma\xi}|g|$  and so  $P_{\gamma'}|f| \leq P_\gamma|g|$ , which implies by induction that  $P_{\gamma'}^n|f| \leq P_\gamma^n|g|$  for every integer  $n \geq 1$ . We conclude that  $\|P_{\gamma'}^n\|_{\mathcal{B}} \leq \|P_\gamma^n\|_{\mathcal{B}}$  since  $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$  is a Banach lattice. This implies that  $r(\gamma') \leq r(\gamma)$  and so the desired statement.  $\square$

However it can happen that  $r(\gamma) = 1$  for every  $\gamma \in [0, +\infty)$  (see Appendix A.3). To study the multiplicative ergodicity as well as regularity properties of  $\gamma \mapsto r(\gamma) := r(P_{\gamma|\mathcal{B}})$ , where  $\mathcal{B}$  is a Banach space on which  $P_\gamma$  continuously acts, we use the Keller-Liverani perturbation theorem [13]. This result brings a significative improvement to the classical Nagaev-Guivarc'h perturbation method [18, 19, 7, 8]. Indeed, it enables the study of spectral properties of family of operators  $(Q(t))_{t \in J}$  acting on  $\mathcal{B}_i$  such that  $t \mapsto Q(t)$  fails to be continuous from  $J$  to  $\mathcal{L}(\mathcal{B}_i)$  but is continuous from  $J$  to  $\mathcal{L}(\mathcal{B}_i, \mathcal{B}_{i+1})$ .

**Hypothesis 2.8.** *Let  $\mathcal{B}_0$  and  $\mathcal{B}_1$  be two Banach spaces, let  $J$  be a subinterval of  $[-\infty, +\infty]$ , and let  $(Q(t))_t$  be a family of operators. We will say that  $((Q(t))_t, J, \mathcal{B}_0, \mathcal{B}_1)$  satisfies Hypothesis 2.8 if*

- $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1$ ,
- for every  $t \in J$ ,  $Q(t) \in \mathcal{L}(\mathcal{B}_0) \cap \mathcal{L}(\mathcal{B}_1)$ ,
- the map  $t \mapsto Q(t)$  is continuous from  $J$  to  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$ ,
- there exist  $c_0 > 0$ ,  $\delta_0 > 0$ ,  $M > 0$  such that

$$\forall t \in J, \quad r_{\text{ess}}(Q(t)|_{\mathcal{B}_0}) \leq \delta_0 \tag{8a}$$

$$\forall t \in J, \quad \forall n \geq 1, \quad \forall f \in \mathcal{B}_0, \quad \|Q(t)^n f\|_{\mathcal{B}_0} \leq c_0(\delta_0^n \|f\|_{\mathcal{B}_0} + M^n \|f\|_{\mathcal{B}_1}) \tag{8b}$$

**Hypothesis 2.8\*.**  $((Q(t))_t, J, \mathcal{B}_0, \mathcal{B}_1)$  satisfies all the conditions of Hypothesis 2.8, except for (8a) and (8b) which are replaced by the following ones:

$$\forall t \in J, \quad r_{\text{ess}}(Q(t)^*|_{\mathcal{B}_1^*}) \leq \delta_0 \quad (9a)$$

$$\forall t \in J, \forall n \geq 1, \forall f^* \in \mathcal{B}_1^*, \quad \|(Q(t)^*)^n f^*\|_{\mathcal{B}_1^*} \leq c_0(\delta_0^n \|f^*\|_{\mathcal{B}_1^*} + M^n \|f^*\|_{\mathcal{B}_0^*}) \quad (9b)$$

**Remark 2.9.** Hypothesis 2.8 contains the conditions of the Keller-Liverani perturbation theorem [13] when applied to the family  $\{Q(t), t \in J\}$  with respect to the spaces  $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1$ . Hypothesis 2.8\* contains the conditions of the Keller-Liverani theorem when applied to the family  $\{Q(t)^*, t \in J\}$  with respect to  $\mathcal{B}_1^* \hookrightarrow \mathcal{B}_0^*$ . Indeed observe that the three first conditions of Hypothesis 2.8, which are assumed in Hypothesis 2.8\*, are equivalent to the following ones:  $\mathcal{B}_1^* \hookrightarrow \mathcal{B}_0^*$ , for every  $t \in J$  we have  $Q(t)^* \in \mathcal{L}(\mathcal{B}_0^*) \cap \mathcal{L}(\mathcal{B}_1^*)$ , and finally  $t \mapsto Q(t)^*$  is continuous from  $J$  to  $\mathcal{L}(\mathcal{B}_1^*, \mathcal{B}_0^*)$ . But it is worth noticing that the conditions (9a)-(9b) cannot be deduced from (8a)-(8b) (and conversely).

Let us now state the Keller-Liverani perturbation theorem in our context.

**Theorem 2.10** ([13]). Under Hypothesis 2.8 (respectively under Hypothesis 2.8\*) the function  $t \mapsto r(t) := r((Q(t))|_{\mathcal{B}_0})$  (respectively  $t \mapsto r(t) := r((Q(t))|_{\mathcal{B}_1})$ ) is continuous on the set  $\{t \in J : r(Q(t)|_{\mathcal{B}_0}) > \delta_0\}$  (respectively on  $\{t \in J : r(Q(t)^*|_{\mathcal{B}_1^*}) > \delta_0\}$ ). Moreover, in both cases, the following inequality holds:

$$\forall t_0 \in J, \quad \limsup_{t \rightarrow t_0} r(t) \leq \max(\delta_0, r(t_0)).$$

Given a Banach space  $\mathcal{B}$  of functions on  $\mathbb{X}$  (or of classes of such functions modulo  $\pi$ ), recall that  $\psi \in \mathcal{B}^*$  is said to be non-negative if  $\psi(f) \geq 0$  for every  $f \in \mathcal{B}$ ,  $f \geq 0$ . Let us introduce another assumption.

**Hypothesis 2.11.** Let  $\gamma \in [0, +\infty]$  and  $\mathcal{B}$  be a Banach lattice of functions on  $\mathbb{X}$  (or of classes of such functions modulo  $\pi$ ). Assume that  $1_{\mathbb{X}} \in \mathcal{B} \subset \mathbb{L}^1(\pi)$ , that  $P_\gamma$  is quasi-compact on  $\mathcal{B}$  with spectral radius  $r(\gamma) := r(P_\gamma|_{\mathcal{B}}) > 0$ , and that

- if  $\phi \in \mathcal{B}$  is non-null and non-negative, then  $P_\gamma \phi > 0$  (modulo  $\pi$ ) and, for every non-null non-negative  $\psi \in \mathcal{B}^* \cap \text{Ker}(P_\gamma - r(\gamma)I)$ , we have  $\psi(P_\gamma \phi) > 0$ .
- for every  $f, g \in \mathcal{B}$  with  $f > 0$ ,  $P_\gamma f = r(\gamma)f$  and  $P_\gamma g = r(\gamma)g$ , we have  $g \in \mathbb{C} \cdot f$ ,
- 1 is the only complex number  $\lambda$  of modulus 1 such that  $P(h/|h|) = \lambda h/|h|$  in  $\mathbb{L}^1(\pi)$  for some  $h \in \mathcal{B}$ ,  $|h| > 0$ , modulo  $\pi$ .

Now we state general conditions ensuring namely the multiplicative ergodicity and the needed regularity properties of  $r$ . Under Hypothesis 2.8 or 2.8\* we define the following set:

$$J_0 := \{t \in J : r(\gamma) > \delta_0\}.$$

**Theorem 2.12.** Let  $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1 \hookrightarrow \mathbb{L}^1(\pi)$  be two Banach spaces and let  $J$  be a subinterval of  $[0, +\infty]$ . Assume that  $(P_\gamma, J, \mathcal{B}_0, \mathcal{B}_1)$  satisfies Hypothesis 2.8 or 2.8\* and that

- Hypothesis 2.11 holds with  $J$  and  $\mathcal{B} := \mathcal{B}_0$  under Hypothesis 2.8
- Hypothesis 2.11 holds with  $J$  and  $\mathcal{B} := \mathcal{B}_1$  under Hypothesis 2.8\*.



Then  $\gamma \mapsto r(\gamma) := r(P_{\gamma|\mathcal{B}})$  is continuous on  $J_0$ , and there exists a map  $\gamma \mapsto \Pi_\gamma$  from  $J_0$  to  $\mathcal{L}(\mathcal{B})$  which is continuous from  $J_0$  to  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$  such that, for every compact subset  $K$  of  $J_0$ , there exist  $\theta_K \in (0, 1)$  and  $M_K \in (0, +\infty)$  such that

$$\forall \gamma \in K, \forall f \in \mathcal{B}, \quad \|(P_\gamma^n(f)) - r(\gamma)^n \Pi_\gamma f\|_{\mathcal{B}} \leq M_K (\theta_K r(\gamma))^n \|f\|_{\mathcal{B}}. \quad (10)$$

Under the assumptions of Theorem 2.12, we obtain for every  $f \in \mathcal{B}_0$  and for every  $\psi \in \mathcal{B}_1^*$ :

$$\forall \gamma \in K, \quad |\psi(P_\gamma^n(f)) - r(\gamma)^n \psi(\Pi_\gamma f)| \leq M_K (\theta_K r(\gamma))^n \|\psi\|_{\mathcal{B}_1^*} \|f\|_{\mathcal{B}_0}. \quad (11)$$

with  $\gamma \mapsto r(\gamma)$  and  $\gamma \mapsto \psi(\Pi_\gamma f)$  continuous from  $J_0$  to  $[0, 1]$  and  $\mathbb{C}$  respectively. Then (10) and (11) can be interpreted as **spectral multiplicative ergodicity properties**.

Theorem 2.12 is established in Appendix A.2. Since  $1_{\mathbb{X}} \in \mathcal{B}$  (see Hypothesis 2.11) with  $\mathcal{B} := \mathcal{B}_0$  or  $\mathcal{B} := \mathcal{B}_1$  according that Hypothesis 2.8 or 2.8\* is assumed, the following corollary easily follows from Theorem 2.12.

**Corollary 2.13.** *Assume that the assumptions of Theorem 2.12 hold and that  $f \mapsto e^{-\gamma\xi} f$  is in  $\mathcal{L}(\mathcal{B}_1)$ . Let  $\mu$  be a probability measure on  $\mathbb{X}$  belonging to  $\mathcal{B}_1^*$  and satisfying:  $\forall \gamma \in J_0$ ,  $\mu(e^{-\gamma\xi} \Pi_\gamma \mathbf{1}_{\mathbb{X}}) > 0$ . Then, under  $\mathbb{P}_\mu$ ,  $\rho_Y(\gamma) = r(\gamma)$  and the sequence  $(S_n)_n$  is multiplicatively ergodic on  $J_0$  (with  $A(\gamma) = \mu(e^{-\gamma\xi} \Pi_\gamma \mathbf{1}_{\mathbb{X}})$ ). If moreover  $\inf r(J_0) < 1/2 < \sup r(J_0)$ , then  $\nu$  is finite and is given by*

$$\nu = \inf\{\gamma > 0 : r(\gamma) < 1/2\}. \quad (12)$$

Now we are interesting in the differentiability of  $r$  and in the sign of its derivative under assumptions analogous to those of Theorem 2.12.

**Theorem 2.14.** *Assume  $\pi(\xi > 0) > 0$ . Let  $\mathcal{B}_0 \hookrightarrow \mathcal{B}_1 \hookrightarrow \mathcal{B}_2 \hookrightarrow \mathcal{B}_3 \hookrightarrow \mathbb{L}^1(\pi)$  be Banach spaces and let  $J$  be a subinterval of  $[0, +\infty]$ . Assume that one of the two following conditions holds*

- (a) *Either: for  $i = 0, 1, 2$ ,  $(P_\gamma, J, \mathcal{B}_i, \mathcal{B}_{i+1})$  satisfies Hypothesis 2.8, and Hypothesis 2.11 holds with  $(J, \mathcal{B}_i)$ ; in this case we set  $\mathcal{B} := \mathcal{B}_0$ .*
- (b) *Or: for  $i = 0, 1, 2$ ,  $(P_\gamma, J, \mathcal{B}_i, \mathcal{B}_{i+1})$  satisfies Hypothesis 2.8\*, and Hypothesis 2.11 holds with  $(J, \mathcal{B}_{i+1})$ ; in this case we set  $\mathcal{B} := \mathcal{B}_3$ .*

Moreover assume that  $\gamma \mapsto P_\gamma$  is continuous from  $J$  to  $\mathcal{L}(\mathcal{B}_i, \mathcal{B}_{i+1})$  for  $i \in \{0, 2\}$  and  $C^1$  from  $J$  to  $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$  with derivative  $P'_\gamma f = P_\gamma(-\xi f)$  ( $f \mapsto \xi f$  being in  $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ ).

Then  $\gamma \mapsto r(\gamma) := r(P_{\gamma|\mathcal{B}})$  is  $C^1$  on  $J_0 := \{t \in J : r(\gamma) > \delta_0\}$  with negative derivative,  $\gamma \mapsto \Pi_\gamma$  is well defined from  $J_0$  to  $\mathcal{L}(\mathcal{B})$  and is  $C^1$  from  $J_0$  to  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$ .

Theorem 2.14 is proved in Appendix A.2. The next corollary follows from Theorem 2.14.

**Corollary 2.15.** *Assume that the assumptions of Theorem 2.14 hold and that  $f \mapsto e^{-\gamma\xi} f$  is in  $\mathcal{L}(\mathcal{B}_3)$ . Let  $\mu$  be a probability measure on  $\mathbb{X}$  belonging to  $\mathcal{B}_3^*$  and satisfying:  $\forall \gamma \in J_0$ ,  $\mu(e^{-\gamma\xi} \Pi_\gamma \mathbf{1}_{\mathbb{X}}) > 0$ . then  $(S_n)_n$  is multiplicatively ergodic on  $J_0$  with respect to  $\mathbb{P}_\mu$ . If moreover  $\inf r(J_0) < 1/2 < \sup r(J_0)$ , then  $\nu$  is finite and is given by (12) and  $C_\nu$  of (4) is well defined and finite.*

**Remark 2.16.**

- *In the previous statements, the Banach lattice assumption in Hypothesis 2.11 can be replaced by:  $r(\gamma)$  is a pole of finite order of  $P_\gamma$ .*

- As already mentioned, the (expected) nonincreasingness of  $r(\cdot)$  is guaranteed since our spaces are assumed to be Banach lattices (see Lemma 2.7). Note that this implies that  $J_0$  is an interval.
- We will see in Appendix A that  $P_\gamma(\Pi_\gamma \mathbf{1}_\mathbb{X}) = r(\gamma)\Pi_\gamma \mathbf{1}_\mathbb{X}$ . Then we deduce from the first condition in Hypothesis 2.11 that  $\Pi_\gamma \mathbf{1}_\mathbb{X} > 0$   $\pi$ -a.s., where  $\pi$  is the stationary distribution of  $(X_n)_n$ . Consequently, if the assumptions of Corollary 2.13 (respectively Corollary 2.15) are fulfilled, then its conclusions hold true with  $\mu = \pi$  since  $\pi \in \mathcal{B}_1^*$  since  $\mathcal{B}_1 \hookrightarrow \mathbb{L}^1(\pi)$  (resp.  $\pi \in \mathcal{B}_3^*$  since  $\mathcal{B}_3 \hookrightarrow \mathbb{L}^1(\pi)$ ) and  $\pi(e^{-\gamma\xi}\Pi_\gamma \mathbf{1}_\mathbb{X}) > 0$ . This is also true for any probability measure  $\mu \in \mathcal{B}_1^*$  (respectively  $\mu \in \mathcal{B}_3^*$ ) which is absolutely continuous with respect to  $\pi$ . If  $\Pi_\gamma \mathbf{1}_\mathbb{X} > 0$  everywhere, then the conclusions of Corollary 2.13 (respectively Corollary 2.15) hold for any  $\mu \in \mathcal{B}_1^*$  (respectively  $\mu \in \mathcal{B}_3^*$ ).
- In Case (a) of Theorem 2.14 we will prove in Appendix that, for every  $\gamma \in J_0$ , the spectral radius  $r(P_{\gamma|\mathcal{B}_i})$  does not depend on  $i \in \{0, 1, 2\}$ , and that (10) holds on  $\mathcal{B}_i$  for every  $i = 0, 1, 2$ . In Case (b) the same properties hold for  $i = 1, 2, 3$ .

We conclude this section with some complementary results which may be useful.

**Corollary 2.17.** *Assume that the assumptions of Theorem 2.12 hold and that  $f \mapsto e^{-\gamma\xi}f$  is in  $\mathcal{L}(\mathcal{B}_1)$ . Let  $\mu$  be a probability measure on  $\mathbb{X}$  belonging to  $\mathcal{B}_1^*$  and satisfying:  $\forall \gamma \in J_0$ ,  $\mu(e^{-\gamma\xi}\Pi_\gamma \mathbf{1}_\mathbb{X}) > 0$ .*

- (i) *If  $J = [0, +\infty]$  and if  $\alpha_0$  is a positive real number such that  $\alpha_0 < 1/\delta_0$ , then under  $\mathbb{P}_\mu$ , for every  $\gamma \in [0, +\infty]$ , we have*

$$g_{Y,\mu}(\gamma, \alpha_0) < \infty \Leftrightarrow r(\gamma) < 1/\alpha_0.$$

- (ii) *If  $J = [0, +\infty]$  and  $\delta_0 < 1/2$ , then  $\nu < \infty \Leftrightarrow r(\infty) < 1/2$ .*

*Proof.* Observe first that, for every  $\gamma \in J_0$ , we have

$$g_{Y,\mu}(\gamma, \alpha_0) < \infty \Leftrightarrow r(\gamma) < \frac{1}{\alpha_0}$$

due to Theorem 2.12 and to Remark 2.3. Now let us consider the first case ( $J = [0, +\infty]$  and  $\delta_0 < 1/\alpha_0$ ). If  $\gamma \in [0, +\infty] \setminus J_0$ , then  $r(\gamma) \leq \delta_0 < 1/\alpha_0$  and

$$g_{Y,\mu}(\gamma, \alpha_0) = \sum_{n=0}^{+\infty} \alpha_0^n L_{Y,\mu}^{(n)}(\gamma) \leq \sum_{n=0}^{+\infty} \alpha_0^n C \delta_0^n < \infty.$$

Hence Assertion (i) is fulfilled. Now, due to Lemma 2.7 and to Theorem 2.10, we have  $\lim_{t \rightarrow +\infty} r(t) \leq \max(\delta_0, r(\infty))$  and even  $\lim_{t \rightarrow +\infty} r(t) = r(\infty)$  if  $r(\infty) > \delta_0$ . So if  $\delta_0 < 1/2$ , we have (3)  $\Leftrightarrow r(\infty) < 1/2$ .  $\square$

**Lemma 2.18.** *Under the assumptions of Theorem 2.12, if  $J_0 \neq \{0\}$ , then we have  $r(\gamma) > 0$  for every  $\gamma \in (0, +\infty)$ .*

*Proof.* Let  $\gamma_0 \in J_0$ . Due to Lemma 2.7,  $r(\gamma) \geq r(\gamma_0) > 0$  for every  $\gamma \leq \gamma_0$ . Next let  $\gamma > \gamma_0$  and set  $p := \gamma/\gamma_0 > 1$ . We have

$$0 < r(\gamma_0) = r\left(\frac{\gamma}{p}\right) = \lim_{n \rightarrow +\infty} (\pi(P_{\frac{\gamma}{p}}^n \mathbf{1}_\mathbb{X}))^{\frac{1}{n}} = \lim_{n \rightarrow +\infty} (\mathbb{E}_\pi[e^{-\frac{\gamma}{p}S_n}])^{\frac{1}{n}}$$

due to (10) since  $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}$  and since  $\pi \in \mathcal{B}'$ . Moreover, due to the Hölder inequality we obtain

$$0 < r(\gamma_0) = \lim_{n \rightarrow +\infty} (\pi(e^{-\frac{\gamma}{p} S_n}))^{\frac{1}{n}} \leq \limsup_{n \rightarrow +\infty} (E_{\pi}[e^{-\gamma S_n}])^{\frac{1}{pn}} \leq (r(\gamma))^{\frac{1}{p}},$$

since  $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}$  and  $\pi \in \mathcal{B}'$ , which implies the positivity of  $r(\gamma)$ .  $\square$

### 3. KNUDSEN GAS: PROOF OF THEOREM 2.5

In this section, we apply our general results for the Knudsen gas and more precisely to  $P_{\gamma}$  acting on the usual Lebesgue space  $(\mathbb{L}^a(\pi), \|\cdot\|_a)$  for some suitable  $a \in [1, +\infty)$ , where

$$\|f\|_a := \left( \int_{\mathbb{X}} |f(x)|^a d\pi(x) \right)^{\frac{1}{a}}. \quad (13)$$

The multiplicative ergodicity follows from Corollary 2.13 together with the following lemma.

**Lemma 3.1.** *Let  $1 \leq b < a$ .*

- (i) *For every  $\gamma \geq 0$ ,  $r_{ess}(P_{\gamma|\mathbb{L}^a(\pi)}) \leq 1 - \alpha$ .*
- (ii) *The function  $\gamma \rightarrow P_{\gamma}$  is continuous from  $(0, +\infty]$  to  $\mathcal{L}(\mathbb{L}^a(\pi), \mathbb{L}^b(\pi))$ .*
- (iii) *For any  $\gamma \in [0, +\infty]$  and any  $f \in \mathbb{L}^a(\pi)$ ,  $\|P_{\gamma}f\|_a \leq (1 - \alpha)\|f\|_a + \alpha\|f\|_1$ .*
- (iv) *For any  $\gamma > 0$ , for any non-null non-negative  $f \in \mathbb{L}^a(\pi)$  and every non-null non-negative  $g \in \mathbb{L}^{a'}(\pi)$  with  $a' = \frac{a}{a-1}$ , we have  $\pi(gP_{\gamma}f) > 0$  and  $P_{\gamma}f > 0$ .*
- (v) *If  $r(\gamma) > 1 - \alpha$ , for every  $f, g \in \mathbb{L}^a(\pi)$  with  $f > 0$ ,  $P_{\gamma}f = r(\gamma)f$  and  $P_{\gamma}g = r(\gamma)g$ , then we have  $g \in \mathbb{C} \cdot f$ .*
- (vi) *1 is the only complex number  $\lambda$  of modulus 1 such that  $P(h/|h|) = \lambda h/|h|$  in  $\mathbb{L}^1(\pi)$  for some  $h \in \mathcal{B}$ ,  $|h| > 0$  (modulo  $\pi$ ).*

*Proof.*

(i) Observe that  $P_{\gamma} = \alpha\pi(e^{-\gamma\xi}) + (1 - \alpha)U_{\gamma}$  with  $U_{\gamma} := U(e^{-\gamma\xi})$ . Since the sum of a Fredholm operator with a compact operator is Fredholm, we directly obtain  $r_{ess}(P_{\gamma}) = (1 - \alpha)r_{ess}(U_{\gamma}) \leq 1 - \alpha$ .

(ii) For every  $0 \leq \gamma, \gamma' < \infty$  and every  $f \in \mathcal{B}$  such that  $\|f\|_a = 1$ , we have

$$\begin{aligned} \|P_{\gamma}f - P_{\gamma'}f\|_b &= \|P((e^{-\gamma\xi} - e^{-\gamma'\xi})f)\|_b \\ &\leq \|(e^{-\gamma\xi} - e^{-\gamma'\xi})f\|_b \leq \|e^{-\gamma\xi} - e^{-\gamma'\xi}\|_c, \end{aligned}$$

where  $c$  is such that  $\frac{1}{a} + \frac{1}{c} = \frac{1}{b}$ . Hence  $\|P_{\gamma} - P_{\gamma'}\|_{\mathbb{L}^a(\pi), \mathbb{L}^b(\pi)} \leq \|e^{-\gamma\xi} - e^{-\gamma'\xi}\|_c$ , which converges to 0 as  $\gamma'$  goes to  $\gamma$ , by the dominated convergence theorem. In the same way, we prove that  $\|P_{\gamma} - P_{\infty}\|_{\mathbb{L}^a(\pi), \mathbb{L}^b(\pi)} \leq \|e^{-\gamma\xi}\|_c$  and hence the continuity of  $\gamma \mapsto P_{\gamma}$  at infinity.

(iii) For every  $\gamma \in [0, +\infty]$  and every  $f \in \mathbb{L}^a(\pi)$ ,  $\|P_{\gamma}f\|_a \leq \|Pf\|_a \leq (1 - \alpha)\|f\|_a + \alpha\|f\|_1$  since  $\|Uf\|_a \leq \|f\|_a$ . This gives the Doeblin-Fortet inequality.

(iv) For any non-null non-negative  $f \in \mathbb{L}^a(\pi)$ , we have  $P_{\gamma}f \geq \alpha\pi(e^{-\gamma\xi}f)\mathbf{1}_{\mathbb{X}} > 0$ . The other assertion of (4) is then obvious.

(v) Let  $f, g \in \mathbb{L}^a(\pi)$  such that  $f > 0$ ,  $P_{\gamma}f = r(\gamma)f$  and  $P_{\gamma}g = r(\gamma)g$  in  $\mathbb{L}^a(\pi)$ . Set  $\beta := \frac{\pi(e^{-\gamma\xi}g)}{\pi(e^{-\gamma\xi}f)}$  and  $h := g - \beta f$ . Then  $\pi(e^{-\gamma\xi}h) = 0$  and  $P_{\gamma}h = r(\gamma)h$ , which gives  $r(\gamma)h = (1 - \alpha)U(e^{-\gamma\xi}h)$ , so that  $r(\gamma)|h| \leq (1 - \alpha)U(|h|)$ . Since  $\pi U = \pi$ , we obtain:  $r(\gamma)\pi(|h|) \leq (1 - \alpha)\pi(|h|)$ . Finally we conclude that  $\pi(|h|) = 0$  because  $r(\gamma) > 1 - \alpha$  and so  $g = \beta f$  in

$\mathbb{L}^a(\pi)$ .

(vi) Let  $k \in \mathbb{L}^1(\pi)$  and  $\lambda \in \mathbb{C}$  be such that  $|\lambda| = 1$ ,  $|k| \equiv \mathbf{1}_{\mathbb{X}}$  and  $P(k) = \lambda k$ . Then  $\lambda k = \alpha\pi(k) + (1 - \alpha)U(k)$ . Taking the modulus, we obtain  $1 \leq \alpha|\pi(k)| + (1 - \alpha)U(\mathbf{1}_{\mathbb{X}}) \leq 1$ . By convexity we conclude that  $|\pi(k)| = 1$  and that  $k$  is constant modulo  $\pi$ , so that  $\lambda = 1$ .  $\square$

*Proof of the multiplicative ergodicity.* Let  $b := \frac{p}{p-1}$  and  $a > b$ . From Lemma 3.1,  $P_\gamma$  satisfies the assumptions of Theorem 2.12 with  $\mathcal{B}_0 = \mathbb{L}^a(\pi)$  and  $\mathcal{B}_1 = \mathbb{L}^b(\pi)$ . Moreover  $f \mapsto e^{-\gamma\xi}f$  is in  $\mathcal{L}(\mathbb{L}^b(\pi))$ . Thus  $(S_n)_n$  is multiplicatively ergodic on  $\{\gamma > 0 : r(\gamma) > 1 - \alpha\}$  with respect to  $\mathbb{P}_\mu$ , provided that  $\mu$  defines a continuous linear form on  $\mathbb{L}^b(\pi)$ , that is when  $\mu$  is absolutely continuous with respect to  $\pi$  with density in  $\mathbb{L}^p(\pi)$ .  $\square$

Below, in view of (3), we study the the spectral radius  $r(\gamma)$  of  $P_\gamma$ . First observe that the nonincreasingness of  $r(\cdot)$  follows from Lemma 2.7 since  $\mathbb{L}^a(\pi)$  is a Banach lattice. Consequently the set  $J_0 := \{\gamma > 0 : r(\gamma) > 1 - \alpha\}$  is an interval with  $\min J_0 = 0$  since  $r(0) = 1$ . Next set  $h_\gamma := e^{-\gamma\xi}$  for  $\gamma \geq 0$  and  $h_\infty := \mathbf{1}_{\{\xi=0\}}$ . Recall that  $P_\gamma f = \alpha\pi(f h_\gamma) + (1 - \alpha)U(f h_\gamma)$ . We set  $\tilde{U}_\gamma(\cdot) := h_\gamma U(\cdot)$ .

**Lemma 3.2.** *Let  $\gamma \in [0, \infty]$  and  $a \in (1, +\infty)$ . Let  $\lambda$  be an eigenvalue of  $(P_\gamma)_{|\mathbb{L}^a(\pi)}$  such that  $\lambda > (1 - \alpha)\rho(\tilde{U}_\gamma)$ . Then*

$$\lambda = \alpha \sum_{n \geq 0} \frac{(1 - \alpha)^n}{\lambda^n} \pi(\tilde{U}_\gamma^n(h_\gamma)). \quad (14)$$

*In particular if  $r(\gamma) > 1 - \alpha$ , then  $\lambda = r(\gamma)$  satisfies (14).*

*Proof.* Let  $\gamma \in [0, \infty]$ . Let  $\lambda \in \mathbb{C}$  and  $f \in \mathbb{L}^a(\pi)$ ,  $f \neq 0$ , be such that  $P_\gamma f = \lambda f$  in  $\mathbb{L}^a(\pi)$ , i.e.  $\lambda f = \alpha\pi(f h_\gamma) + (1 - \alpha)U(f h_\gamma)$  that can be rewritten

$$\lambda f h_\gamma = \alpha\pi(f h_\gamma) h_\gamma + (1 - \alpha)\tilde{U}_\gamma(f h_\gamma).$$

Observe that  $\pi(f h_\gamma) \neq 0$ . Indeed  $\pi(f h_\gamma) = 0$  would imply  $\lambda f h_\gamma = (1 - \alpha)\tilde{U}_\gamma(f h_\gamma)$ , which contradicts the fact that  $\lambda/(1 - \alpha)$  is not in the spectrum of  $\tilde{U}_\gamma$ . Now setting  $g := f h_\gamma / \pi(f h_\gamma)$ , we have

$$\lambda g = \alpha h_\gamma + (1 - \alpha)\tilde{U}_\gamma(g) \quad (15)$$

and so

$$\left[ id - \frac{1 - \alpha}{\lambda} \tilde{U}_\gamma \right] (g) = \frac{\alpha}{\lambda} h_\gamma.$$

Hence

$$g = \frac{\alpha}{\lambda} \left[ id - \frac{1 - \alpha}{\lambda} \tilde{U}_\gamma \right]^{-1} (h_\gamma) = \frac{\alpha}{\lambda} \sum_{n \geq 0} \frac{(1 - \alpha)^n}{\lambda^n} \tilde{U}_\gamma^n h_\gamma$$

and so

$$\lambda = \lambda\pi(g) = \alpha \sum_{n \geq 0} \frac{(1 - \alpha)^n}{\lambda^n} \pi(\tilde{U}_\gamma^n h_\gamma).$$

$\square$

Let  $(Z_n)_n$  be a Markov process with transition  $U$ . We observe that  $\pi(\tilde{U}_\gamma^n(h_\gamma)) = \mathbb{E}_\pi[e^{-\gamma \sum_{k=0}^n Z_k}]$  if  $\gamma \in [0, \infty)$  and  $\pi(\tilde{U}_\infty^n(h_\infty)) = P_\pi[\sum_{k=0}^n Z_k = 0]$ . Hence (14) can be rewritten

$$\lambda = \alpha g_{Z, \pi}(\gamma, (1 - \alpha)/\lambda), \quad (16)$$

where  $g_{Z,\pi}$  denotes the Laplace-generating function of  $(Z_n)_n$  with respect to  $\mathbb{P}_\pi$ , i.e.  $g_{Z,\pi}(\gamma, x) := \sum_{n=0}^{+\infty} x^n \mathbb{E}_\pi[e^{-\gamma \sum_{k=0}^n Z_k}]$ . Now (3) will follow from the following result, coming from Corollary 2.15.

**Corollary 3.3.** *Assume  $\alpha > 1/2$  and  $p > 1$ . Let  $\mu \in \mathbb{L}^p(\pi)$ . Then, with respect to  $\mathbb{P}_\mu$ ,  $\nu$  satisfies*

$$\nu = \inf\{\gamma > 0 : 2\alpha g_{Z,\pi}(\gamma, 2(1-\alpha)) < 1\}$$

and

$$(3) \Leftrightarrow 2\alpha \sum_{n \geq 0} (2(1-\alpha))^n \mathbb{P}_\pi \left( \sum_{k=0}^n Z_k = 0 \right) < 1.$$

In particular (3) holds if the random variables  $Z_n$  are positive.

*Proof.* Let  $b := \frac{p}{p-1}$  and  $a > b$ . Let  $r(\gamma)$  be the spectral radius of  $(P_\gamma)_{\mathbb{L}^a(\pi)}$ . Let us prove that, for every  $\gamma \in [0, +\infty]$ ,  $r(\gamma) < 1/2 \Leftrightarrow 2\alpha g_{Z,\pi}(\gamma, 2(1-\alpha)) < 1$ .

We know that this holds true on  $J_0$  due to Lemma 3.2 and to (16) (since  $g_{Z,\pi}(\gamma, \cdot)$  is increasing). Now if  $\gamma \in [0, +\infty] \setminus J_0$  then  $r(\gamma) \leq 1 - \alpha < 1/2$  and  $2\alpha g_{Z,\pi}(\gamma, 2(1-\alpha)) \leq 2\alpha g_{Z,\pi}(0, 2(1-\alpha)) < \infty$ . We conclude due to Corollary 2.17.  $\square$

*Proof of (4).* Assume that  $\alpha > 1/2$ , that  $2\alpha \sum_{n \geq 0} (2(1-\alpha))^n \mathbb{P}_\pi(\sum_{k=0}^n Z_k = 0) < 1$  and  $\pi(\xi^\tau) < \infty$  for some  $\tau > 1$ . Let  $p > \frac{\tau}{\tau-1}$  and set  $a_3 := \frac{p}{p-1}$  (ie.  $1/p + 1/a_3 = 1$ ). Note that  $a_3 < \tau$ . Let  $a_2$  be such that  $a_3 < a_2 < \tau$ . Since  $\lim_{a \rightarrow +\infty} \frac{\tau a}{\tau+a} = \tau$ , we can chose  $a_1 > a_2$  such that  $a_2 < \frac{\tau a_1}{\tau+a_1}$ . Next let  $a_0 > a_1$ . From Lemma 3.1 we deduce that Theorem 2.14 applies with the spaces  $\mathcal{B}_i = \mathbb{L}^{a_i}(\pi)$  for  $i = 0, 1, 2, 3$ . We conclude that  $r$  is  $C^1$  on  $[0, \theta_1)$  with  $r' < 0$ , and so that (4) also holds due to Remark 2.3, provided that the initial probability measure  $\mu$  defines a continuous linear form on  $\mathcal{B}_3 = \mathbb{L}^{a_3}(\pi)$ , that is (equivalently) when  $\mu = h \cdot \pi$  with  $h \in L^p(\pi)$ .  $\square$

#### 4. LINEAR AUTOREGRESSIVE MODEL: PROOF OF THEOREM 2.4

Assume that  $\mathbb{X} := \mathbb{R}$  and  $(X_n)_{n \in \mathbb{N}}$  is the linear autoregressive model defined by

$$n \in \mathbb{N}^*, \quad X_n = \alpha X_{n-1} + \vartheta_n \tag{17}$$

where  $X_0$  is a real-valued random variable,  $\alpha \in (-1, 1)$ , and  $(\vartheta_n)_{n \geq 1}$  is a sequence of i.i.d. real-valued random variables, independent of  $X_0$ . Assume that  $\vartheta_1$  has a positive Lebesgue probability density function on  $\mathbb{X}$ , say  $p(\cdot)$ , having a moment of order  $r_0$  for some  $r_0 \geq 1$ , that is

$$\int |x|^{r_0} p(x) dx < \infty. \tag{18}$$

$(X_n)_{n \in \mathbb{N}}$  is a Markov chain with transition kernel

$$P(x, A) = \int_{\mathbb{R}} \mathbf{1}_A(\alpha x + y) p(y) dy = \int_{\mathbb{R}} \mathbf{1}_A(y) p(y - \alpha x) dy.$$

Set  $V(x) := (1 + |x|)^{r_0}$ ,  $x \in \mathbb{R}$ . Recall that, under Assumption (18),  $P$  satisfies the following drift condition (see [17])

$$\forall \delta > |\alpha|^{r_0}, \exists L \equiv L(\delta) > 0, \quad PV \leq \delta V + L \mathbf{1}_{\mathbb{X}}. \tag{19}$$

Moreover it is well-known that  $(X_n)_{n \in \mathbb{N}}$  is  $V$ -geometrically ergodic, see [17]. We also assume that, for all  $x_0 \in \mathbb{R}$ , there exist a neighborhood  $V_{x_0}$  of  $x_0$  and a non-negative Lebesgue-integrable function  $q_{x_0}(\cdot)$  such that

$$\forall y \in \mathbb{R}, \forall v \in V_{x_0}, p(y+v) \leq q_{x_0}(y). \quad (20)$$

Let  $(\mathcal{B}_V, \|\cdot\|_V)$  be the weighted-supremum Banach space

$$\mathcal{B}_V := \left\{ f : \mathbb{X} \rightarrow \mathbb{C}, \text{ measurable} : \|f\|_V := \sup_{x \in \mathbb{X}} |f(x)|V(x)^{-1} < \infty \right\}. \quad (21)$$

Let  $(\mathcal{C}_V, \|\cdot\|_V)$  denote the following subspace of  $\mathcal{B}_V$ :

$$\mathcal{C}_V := \left\{ f \in \mathcal{B}_V : f \text{ is continuous and } \lim_{|x| \rightarrow \infty} \frac{f(x)}{V(x)} \text{ exists in } \mathbb{C} \right\},$$

where the symbol  $\lim_{|x| \rightarrow \infty}$  means that the limits when  $x \rightarrow \pm\infty$  exist and are equal. Note that  $V \in \mathcal{C}_V$  and that  $\mathcal{C}_V$  is a closed subspace of  $(\mathcal{B}_V, \|\cdot\|_V)$ . For every  $f \in \mathcal{C}_V$  we define

$$\ell_V(f) := \lim_{|x| \rightarrow \infty} \frac{f(x)}{V(x)}.$$

Let  $\mathcal{C}_{0,V} := \{f \in \mathcal{C}_V : \ell_V(f) = 0\}$ . Finally we denote by  $(\mathcal{C}_b, \|\cdot\|_\infty)$  the space of bounded continuous complex-valued functions on  $\mathbb{R}$  endowed with the supremum norm  $\|\cdot\|_\infty$ .

We will see below that, for every  $\gamma \in [0, +\infty]$ ,  $P_\gamma$  continuously acts on  $\mathcal{C}_V$  (see Lemma 4.2). We denote by  $r(\gamma)$  the spectral radius of  $P_\gamma$  on  $\mathcal{C}_V$ , that is:

$$r(\gamma) \equiv r(P_\gamma) := \lim_n \|P_\gamma^n\|_V^{1/n} = \lim_n \|P_\gamma^n V\|_V^{1/n}$$

where  $\|\cdot\|_V$  also denotes the operator norm on  $\mathcal{C}_V$ . We have  $r(0) = r(P) = 1$  (see below).

Recall that  $\xi : \mathbb{X} \rightarrow [0, +\infty)$  is a measurable function and that  $S_n = \sum_{k=0}^n \xi(X_k)$ . Theorem 4.1 below applied with  $\mu = \delta_x$  or  $\mu = \pi$  directly provides Theorem 2.4.

**Theorem 4.1.** *Assume that the previous assumptions hold. Let  $\mu$  be a probability distribution on  $\mathbb{R}$  belonging to  $\mathcal{C}_V^*$ , namely satisfying  $\mu(V) < \infty$ . Assume moreover that  $\xi$  is continuous, coercive, that  $p$  is continuous, and that  $\sup_{\mathbb{R}} \xi/V < \infty$ . Then*

- (1)  $\rho_Y = r$  on  $[0, +\infty)$  and  $(S_n)_n$  is multiplicatively ergodic on  $[0, +\infty)$  with respect to  $\mathbb{P}_\mu$ .
- (2) If moreover the Lebesgue measure of the set  $[\xi = 0]$  is zero, then  $\lim_{\gamma \rightarrow +\infty} r(\gamma) = 0$ . Hence (3) holds true under  $\mathbb{P}_\mu$ .
- (3) Moreover, if there exists  $\tau > 0$  such that  $\sup_{\mathbb{R}} \xi^{1+\tau}/V < \infty$ , then  $\gamma \mapsto r(\gamma)$  admits a negative derivative on  $[0, +\infty)$ . Hence (4) holds also true under  $\mathbb{P}_\mu$ .

**4.1. Quasi-compactness of  $P_\gamma$ .** We start this section with the following useful lemma.

**Lemma 4.2.** *Assume that Assumption (18) holds, that  $p$  is continuous, that  $\xi$  is continuous and coercive. Then, for every  $\gamma \in [0, +\infty)$ ,  $P_\gamma$  continuously acts on  $\mathcal{C}_V$ . Moreover, for every  $\gamma \in (0, +\infty)$ , we have  $P_\gamma(\mathcal{C}_V) \subset \mathcal{C}_{0,V}$  and  $P_\gamma$  is compact from  $\mathcal{C}_b$  into  $\mathcal{C}_V$ .*

*Proof.* Let  $\gamma \in [0, +\infty)$ . From (19) it easily follows that  $P_\gamma V \leq PV \leq (\delta + L)V$ , so that  $P_\gamma$  continuously acts on  $\mathcal{B}_V$ . Now let  $f \in \mathcal{C}_V$ . Then

$$\forall x \in \mathbb{R}, \quad \frac{(P_\gamma f)(x)}{V(x)} = \int_{\mathbb{R}} \chi(x, y) dy \quad \text{with } \chi(x, y) := e^{-\gamma \xi(\alpha x + y)} \frac{f(\alpha x + y)}{V(x)} p(y).$$

We have for every  $(x, y) \in \mathbb{R}^2$

$$|\chi(x, y)| \leq \|f\|_V \left( \frac{1 + |x| + |y|}{1 + |x|} \right)^{r_0} p(y) \leq \|f\|_V (1 + |y|)^{r_0} p(y)$$

Since  $\chi(\cdot, y)$  is continuous for every  $y \in \mathbb{R}$ , we deduce from (18) and Lebesgue's theorem that the function  $P_\gamma f/V$  is continuous on  $\mathbb{R}$ , thus so is  $P_\gamma f$ . This proves that  $P_\gamma(\mathcal{C}_V) \subset \mathcal{C}_V$ , thus  $P_\gamma$  continuously acts on  $\mathcal{C}_V$ .

Now let us consider  $\gamma \in (0, +\infty)$ . Since

$$|\chi(x, y)| \leq \|f\|_V e^{-\gamma\xi(\alpha x + y)} (1 + |y|)^{r_0} p(y)$$

and  $\lim_{|x| \rightarrow +\infty} e^{-\gamma\xi(\alpha x + y)} = 0$ , it follows again from Lebesgue's theorem that

$$\lim_{|x| \rightarrow +\infty} \frac{(P_\gamma f)(x)}{V(x)} = 0,$$

thus  $P_\gamma f \in \mathcal{C}_{0,V}$ .

To prove the last assertion, observe that, since  $p$  is continuous, the image by  $P$  of the unit ball  $\{f \in \mathcal{C}_b : \|f\|_\infty \leq 1\}$  in  $\mathcal{C}_b$  is equicontinuous from Scheffé's lemma. Then  $P$  is compact from  $\mathcal{C}_b$  into  $\mathcal{C}_V$  from Ascoli's theorem and from  $\lim_{|x| \rightarrow \infty} V(x) = +\infty$ . Next, for every  $\gamma > 0$ , we have  $P_\gamma = P \circ M_\gamma$  with  $M_\gamma f = e^{-\gamma\xi} f$ . Thus  $P_\gamma$  is compact from  $\mathcal{C}_b$  into  $\mathcal{C}_V$  since  $M_\gamma$  is a bounded linear operator on  $\mathcal{C}_b$  and  $P$  is compact from  $\mathcal{C}_b$  into  $\mathcal{C}_V$ .  $\square$

Here we use the duality arguments of [11, prop. 5.4] to prove the quasi-compactness of  $P_\gamma$  on  $\mathcal{C}_V$ . The topological dual spaces of  $\mathcal{C}_V$  and  $\mathcal{C}_b$  are denoted by  $(\mathcal{C}_V^*, \|\cdot\|_V)$  and  $(\mathcal{C}_b^*, \|\cdot\|_\infty)$  respectively (for the sake of simplicity we use the same notation for the dual norms). For any  $\gamma > 0$ , we denote by  $P_\gamma^*$  the adjoint operator of  $P_\gamma$  on  $\mathcal{C}_V$ . Note that each  $P_\gamma^*$  is a contraction with respect to the dual norm  $\|\cdot\|_\infty$  because so is  $P_\gamma$  on  $\mathcal{C}_b$ .

In the sequel,  $\delta > |\alpha|^{r_0}$  is fixed, as well as the associated constant  $L \equiv L(\delta)$  in (19). Iterating Inequality (19) proves that  $P$  is power-bounded on  $\mathcal{C}_V$  (i.e.  $\sup_{n \geq 1} \|P^n V\|_V < \infty$ ), thus  $r(0) = r(P) = 1$  since  $P$  is Markov. Moreover (19) rewrites as the following (dual) Doebelin-Fortet inequality (see the proof in [6, p. 190]):

$$\forall f^* \in \mathcal{C}_V^*, \quad \|P^* f^*\|_V \leq \delta \|f^*\|_V + L \|f^*\|_\infty. \quad (22)$$

Since  $P$  is compact from  $\mathcal{C}_b$  into  $\mathcal{C}_V$  (Lemma 4.2), so is  $P^*$  from  $\mathcal{C}_V^*$  into  $\mathcal{C}_b^*$ . Then we deduce from [9] that, under Assumption (18),  $P$  is a quasi-compact operator on  $\mathcal{C}_V$  and its essential spectral radius  $r_{ess}(P)$  satisfies the following bound (see also [20, Sect. 8]):

$$r_{ess}(P) \leq \delta \quad (23)$$

The next lemma extends Inequality (19) to the operators  $P_\gamma$ .

**Lemma 4.3.** *Assume that Assumption (18) holds true and that  $\xi$  is coercive. Then, for every  $\gamma > 0$  and for every  $\beta > 0$ , there exists a positive constant  $L_\beta$  such that*

$$P_\gamma V \leq e^{-\gamma\beta} \delta V + L_\beta \mathbf{1}_\mathbb{X} \quad (24)$$

Moreover

$$P_\infty V \leq \left( \sup_{[\xi=0]} V \right) \mathbf{1}_\mathbb{X}. \quad (25)$$

*Proof.* We have for every  $\gamma > 0$  and for every  $\beta > 0$

$$\begin{aligned} P_\gamma V &= P(e^{-\gamma\xi}V) = P(e^{-\gamma\xi}\mathbf{1}_{[\xi>\beta]}V) + P(e^{-\gamma\xi}\mathbf{1}_{[\xi\leq\beta]}V) \\ &\leq e^{-\gamma\beta}(\delta V + L\mathbf{1}_{\mathbb{X}}) + \int_{[\xi\leq\beta]} V(y)P(\cdot, dy) \quad (\text{from (19)}) \\ &\leq e^{-\gamma\beta}\delta V + (L + \sup_{[\xi\leq\beta]} V)\mathbf{1}_{\mathbb{X}} \end{aligned}$$

from which we deduce the first desired statement. For  $P_\infty$ , we have

$$P_\infty V = P(\mathbf{1}_{\{\xi=0\}}V) \leq \left( \sup_{[\xi=0]} V \right) P(\mathbf{1}_{\mathbb{X}}) = \left( \sup_{[\xi=0]} V \right) \mathbf{1}_{\mathbb{X}}.$$

□

**Corollary 4.4.** *Assume that Assumption (18) holds true and that  $\xi$  is coercive. Then, for every  $\gamma \in (0, +\infty]$ ,  $P_\gamma$  is a quasi-compact operator on  $\mathcal{C}_V$  and its essential spectral radius  $r_{ess}(P_\gamma)$  is zero.*

*Proof.* Consider any  $\gamma > 0$ . For any  $\varepsilon > 0$ , choose  $\beta = \beta(\gamma, \varepsilon) > 0$  such that  $e^{-\gamma\beta}\delta < \varepsilon$ . Then we deduce from Lemma 4.3 that  $P_\gamma V \leq \varepsilon V + D\mathbf{1}_{\mathbb{X}}$ , where  $D \equiv D(L, \gamma, \varepsilon)$  is a positive constant. This inequality rewrites as (see the proof in [6, p. 190]):

$$\forall f^* \in \mathcal{C}_V^*, \quad \|P_\gamma^* f^*\|_V \leq \varepsilon \|f^*\|_V + D \|f^*\|_\infty. \quad (26)$$

Since  $P_\gamma^*$  is compact from  $\mathcal{C}_V^*$  into  $\mathcal{C}_b^*$  (Lemma 4.2), we deduce from [9] that  $P_\gamma$  is quasi-compact on  $\mathcal{C}_V$  with  $r_{ess}(P_\gamma) \leq \varepsilon$ . We obtain  $r_{ess}(P_\gamma) = 0$  because  $\varepsilon$  is arbitrary. □

With the usual convention  $V^0 := 1$ , we have the identification  $\mathcal{C}_{V^0} = \mathcal{C}_b$ .

**Remark 4.5.** *Let  $\varepsilon > 0$ ,  $0 \leq a \leq a+b \leq 1$ . Observe that Corollary 4.4 holds also if we replace  $V$  by  $V^{a+b}$  (since  $\vartheta_1$  admits a moment of order  $r_0(a+b)$ ). Moreover notice that (26) with  $V^{a+b}$  instead of  $V$  directly gives*

$$\exists D_{\varepsilon, a+b} > 0, \quad \forall f^* \in \mathcal{C}_{V^{a+b}}^*, \quad \|P_\gamma^* f^*\|_{V^{a+b}} \leq \varepsilon \|f^*\|_{V^{a+b}} + D_{\varepsilon, a+b} \|f^*\|_{V^a} \quad (27)$$

since  $\|f^*\|_\infty \leq \|f^*\|_{V^a}$ .

#### 4.2. Continuity of the function $\gamma \mapsto r(\gamma)$ .

**Proposition 4.6.** *Assume that Assumption (18) holds true, that  $\xi$  is coercive and finally that the function  $\xi/V$  is bounded on  $\mathbb{R}$ . Then the function  $\gamma \mapsto r(\gamma)$  is continuous on  $[0, +\infty]$ .*

The continuity of  $\gamma \mapsto r(\gamma)$  at some  $\gamma_0 \in [0, +\infty]$  directly follows from the continuity of  $\gamma \mapsto P_\gamma$  from  $[0, +\infty]$  to  $\mathcal{L}(\mathcal{C}_b, \mathcal{C}_V)$  due to Theorem 2.10 (applied with any  $\delta_0 \in (0, 1)$  if  $\gamma_0 \neq 0$  and with any  $\delta_0 > |\alpha|^{r_0}$  if  $\gamma_0 = 0$ ), and due to (22), to (26) and to Corollary 4.4. Hence Proposition 4.6 comes from the following lemmas.

**Lemma 4.7.** *Let  $0 \leq a < a+b \leq 1$ . Assume that  $\xi \leq cV$  for some positive constant  $c$ . Then the following operator-norm inequality holds for every  $(\gamma, \gamma') \in [0, +\infty)^2$*

$$\|P_\gamma - P_{\gamma'}\|_{\mathcal{C}_{V^a}, \mathcal{C}_{V^{a+b}}} := \sup_{f \in \mathcal{C}_{V^a}, \|f\|_{V^a} \leq 1} \|P_\gamma f - P_{\gamma'} f\|_{V^{a+b}} \leq (c|\gamma - \gamma'|)^b \|P\|_{V^{a+b}}.$$



*Proof.* Let  $(\gamma, \gamma') \in [0, +\infty)^2$ . For all  $(u, v) \in [0, +\infty)^2$ , we have  $|e^{-u} - e^{-v}| \leq |e^{-u} - e^{-v}|^b \leq |u - v|^b$  from Taylor's inequality. Thus we obtain for any  $f \in \mathcal{C}_{V^a}$

$$\begin{aligned} |(P_\gamma f)(x) - (P_{\gamma'} f)(x)| &\leq \|f\|_{V^a} \int_{\mathbb{R}} |e^{-\gamma\xi(y)} - e^{-\gamma'\xi(y)}| (V(y))^a p(y - \alpha x) dy \\ &\leq \|f\|_{V^a} (c|\gamma - \gamma'|)^b \int_{\mathbb{R}} (V(y))^{a+b} p(y - \alpha x) dy \\ &\leq \|f\|_{V^a} (c|\gamma - \gamma'|)^b P V^{a+b}(x), \end{aligned}$$

from which we deduce the desired inequality.  $\square$

**Lemma 4.8.** *Assume that Assumptions (18) and (20) hold true, that  $\xi$  is coercive. Then*

$$\|P_\gamma - P_\infty\|_{\mathcal{C}_b, \mathcal{C}_V} := \sup_{f \in \mathcal{C}_b, \|f\|_\infty \leq 1} \|P_\gamma f - P_\infty f\|_V \rightarrow 0 \quad \text{when } \gamma \rightarrow +\infty.$$

*Proof.* Let  $\varepsilon > 0$ . Let  $f \in \mathcal{C}_b$  be such that  $\|f\|_\infty \leq 1$ . From  $|P_\gamma f| \leq P \mathbf{1}_{\mathbb{X}} = \mathbf{1}_{\mathbb{X}}$  it follows that there exists  $A \equiv A(\varepsilon)$  such that :

$$|x| > A \Rightarrow \forall \gamma \in (0, +\infty), \frac{|(P_\gamma f)(x)|}{V(x)} \leq \varepsilon. \quad (28)$$

Next we deduce from Assumption (20) and a usual compactness argument ( $[-A, A]$  is compact) that there exists a Lebesgue-integrable function  $q \equiv q_A$  such that

$$\forall v \in [-A, A], \forall y \in \mathbb{R}, \quad p(y + v) \leq q(y).$$

Consequently we obtain for any  $\beta > 0$  and  $x \in \mathbb{R}$  such that  $|x| \leq A$

$$\begin{aligned} |(P_\gamma f - P_\infty f)(x)| &\leq e^{-\gamma\beta} \int_{[\xi > \beta]} p(y - \alpha x) dy + \int_{[0 < \xi \leq \beta]} p(y - \alpha x) dy \\ &\leq e^{-\gamma\beta} + \int_{[0 < \xi \leq \beta]} q(y) dy. \end{aligned}$$

Since  $\int_{[0 < \xi \leq \beta]} q(y) dy \rightarrow 0$  when  $\beta \rightarrow 0$ , there exists  $\beta_0 \equiv \beta_0(\varepsilon) > 0$  such that

$$\int_{[0 < \xi \leq \beta_0]} q(y) dy \leq \frac{\varepsilon}{2}.$$

Finally let  $\gamma_0 \equiv \gamma_0(\varepsilon) > 0$  be such that :  $\forall \gamma > \gamma_0, e^{-\gamma\beta_0} \leq \varepsilon/2$ . Then

$$|x| \leq A \Rightarrow \forall \gamma \in (\gamma_0, \infty), \frac{|(P_\gamma f - P_\infty f)(x)|}{V(x)} \leq |(P_\gamma f)(x) - (P_\infty f)(x)| \leq \varepsilon. \quad (29)$$

Inequalities (28) and (29) provides the desired statement.  $\square$

**4.3. Proof of the two first points of Theorem 4.1.** Let  $\theta_1 := \sup\{\gamma > 0 : r(\gamma) > 0\}$ . Since  $r$  is continuous at 0 and  $r(0) = 1$ , we observe that  $\theta_1 > 0$ . Let us prove that the assumptions of Theorem 2.12 hold true on  $J = (0, \theta_1)$  with  $\mathcal{B}_0 := \mathcal{C}_{V^a}$  for some (any)  $a \in (0, 1)$  and  $\mathcal{B}_1 = \mathcal{C}_V$ . Note that  $\mathbf{1}_{\mathbb{X}} \in \mathcal{B}_0$ . The fact that  $(P_\gamma)_\gamma$  satisfies the conditions of Hypothesis 2.8\* on  $J$  with  $\mathcal{B}_0 = \mathcal{C}_b$  and  $\mathcal{B}_1 = \mathcal{C}_V$  comes from (26) and from Lemmas 4.7 and 4.8. Moreover we prove below that Hypothesis 2.11 holds with respect to  $(J, \mathcal{B}_1) = (J, \mathcal{C}_V)$ . Since  $f \mapsto e^{-\gamma\xi} f$  is in  $\mathcal{L}(\mathcal{C}_V)$ , we then deduce from Corollary 2.13 that  $(S_n)_n$  is multiplicatively ergodic on  $(0, \theta_1)$  and so  $\rho_Y(\gamma) = r(\gamma) > 0$  on  $(0, \theta_1)$ . Moreover, since  $\theta_1 > 0$ , it follows from Lemma 2.18 that  $\theta_1 = +\infty$ . We have proved Assertion (1) of Theorem 4.1. For Assertion (2),

observe that  $\text{Leb}(\xi = 0) = 0$  implies that  $P_\infty = 0$ , in particular we have  $r(+\infty) = 0$ . Then Theorem 2.10 gives  $\lim_{\gamma \rightarrow +\infty} r(\gamma) = r(+\infty) = 0$ . Consequently  $\nu$  is finite and satisfies (12), and so (3), with respect to  $\mathbb{P}_\mu$ , provided that  $\mu$  is a probability distribution  $\mu$  belonging to  $\mathcal{C}_V^*$ .

Recall that the previous proof shows that  $r(\gamma) > 0$  for every  $\gamma \geq 0$ . It remains to establish that Hypothesis 2.11 holds with respect to  $(J, \mathcal{B}_1) = (J, \mathcal{C}_V)$ . This is provided by Remark 4.10 and Lemmas 4.11-4.12 below.

**Lemma 4.9.** *For any non-null  $e^* \in \mathcal{C}_V^*$ ,  $e^* \geq 0$ , there exists a nonnegative measure  $\mu \equiv \mu_{e^*}$  on  $(\mathbb{R}, \mathcal{X})$  such that*

$$\forall f \in \mathcal{C}_V, \quad e^*(f) = \mu \left( \frac{f}{V} - \ell_V(f) \mathbf{1}_{\mathbb{R}} \right) + e^*(V) \ell_V(f). \quad (30)$$

**Remark 4.10.** *Due to Lemma 4.9, the first condition of Hypothesis 2.11 is fulfilled with  $J = [0, +\infty)$  and  $\mathcal{B} = \mathcal{C}_V$ . Indeed, let  $\gamma \in [0, +\infty)$  and let  $\phi \in \mathcal{C}_V$  be non-null and non-negative. Then, we have  $P_\gamma \phi > 0$  everywhere from the definition of  $P$  and the strict positivity of the function  $p(\cdot)$ . Moreover, if  $\psi \in \mathcal{B}^* \cap \text{Ker}(P_\gamma^* - r(\gamma)I)$  is non-null and non-negative, then we have  $\psi(P_\gamma \phi) > 0$ . Indeed this property holds for  $\gamma = 0$  since we know that  $\psi = c\pi$  for some  $c > 0$  and that  $P_\gamma \phi > 0$  everywhere. Now let  $\gamma > 0$ . First observe that  $\psi \neq c\ell_V$  for every  $c \in \mathbb{C}$  because  $r(\gamma) > 0$  and  $P_\gamma^*(\ell_V) = 0$  from Lemma 4.2. Second note that  $\mu = 0$  in (30) implies that  $e^* = e^*(V) \ell_V$ . Thus the nonnegative measure  $\mu \equiv \mu_\psi$  associated with  $\psi$  in (30) is non-null. Since  $\ell_V(P_\gamma \phi) = 0$  from Lemma 4.2, we deduce from (30) (applied with  $e^* = \psi$ ) and from  $P_\gamma \phi > 0$  that  $\psi(P_\gamma \phi) = \mu(P_\gamma \phi / V) > 0$ .*

*Proof of Lemma 4.9.* Let  $(\mathcal{C}, \|\cdot\|)$  denote the following space

$$\mathcal{C} := \left\{ g : \mathbb{R} \rightarrow \mathbb{C} \text{ continuous} : \|g\| := \sup_{x \in \mathbb{R}} |g(x)| < \infty \text{ and } \lim_{|x| \rightarrow \infty} g(x) \text{ exists in } \mathbb{C} \right\}.$$

For every  $g \in \mathcal{C}$ , we set:  $\ell(g) := \lim_{|x| \rightarrow \infty} g(x)$ . We denote by  $\mathcal{C}^*$  the topological dual space of  $\mathcal{C}$ . Let  $e^* \in \mathcal{C}_V^*$ ,  $e^* \geq 0$ , and let  $\tilde{e}^* \in \mathcal{C}^*$  be defined by:

$$\forall g \in \mathcal{C}, \quad \tilde{e}^*(g) := e^*(gV).$$

Next let  $\tilde{e}_0^*$  be the restriction of  $\tilde{e}^*$  to  $\mathcal{C}_0 := \{g \in \mathcal{C} : \ell(g) = 0\}$ . From the Riesz representation theorem, there exists a unique positive measure  $\mu$  on  $(\mathbb{R}, \mathcal{X})$  such that

$$\forall g \in \mathcal{C}_0, \quad \tilde{e}_0^*(g) = \mu(g) := \int_{\mathbb{R}} g d\mu.$$

Then, writing  $g = (g - \ell(g) \mathbf{1}_{\mathbb{R}}) + \ell(g) \mathbf{1}_{\mathbb{R}}$  for any  $g \in \mathcal{C}$ , we obtain that

$$\tilde{e}^*(g) = \mu(g - \ell(g) \mathbf{1}_{\mathbb{R}}) + \tilde{e}^*(\mathbf{1}_{\mathbb{R}}) \ell(g).$$

We conclude by observing that, for any  $f \in \mathcal{C}_V$ , we have  $e^*(f) = \tilde{e}^*(f/V)$ .  $\square$

**Lemma 4.11.** *If  $f, g \in \mathcal{C}_V$  are such that  $P_\gamma f = r(\gamma)f$  and  $P_\gamma g = r(\gamma)g$  with  $f > 0$ , then  $g \in \mathbb{C} \cdot f$ .*

*Proof.* Let  $f, g \in \text{Ker}(P_\gamma - r(\gamma)I)$  with  $f > 0$ . Let  $\beta \in \mathbb{C}$  be such that  $h := g - \beta f$  vanishes at 0. Since  $h \in \text{Ker}(P_\gamma - r(\gamma)I)$  we deduce from Proposition A.3 that  $P_\gamma |h| = r(\gamma)|h|$ . Then  $|h|(0) = 0$ , the positivity of  $p(\cdot)$  and finally the continuity of  $|h|$  show that  $h = 0$ .  $\square$

**Lemma 4.12.** *Let  $h \in \mathcal{C}_V$  with  $|h| > 0$  and  $\lambda \in \mathbb{C}$  be such that  $|\lambda| = 1$  and  $P \frac{h}{|h|} = \lambda \frac{h}{|h|}$  in  $\mathbb{L}^1(\pi)$ . Then  $\lambda = 1$ .*

*Proof.* Observe that  $\frac{h}{|h|}$  is in  $\mathcal{C}_b$  so in  $\mathcal{B}_V$ . But it is known from [17] that  $(X_n)_n$  is  $V$ -geometrically ergodic, so  $\lambda=1$ .  $\square$

**4.4. Proof of Part (3) of Theorem 4.1.** We assume now that  $\xi \in \mathcal{B}_{V^{\frac{1}{1+\tau}}}$  for some  $\tau > 0$  and that  $[\xi = 0]$  has Lebesgue measure 0.

Let  $0 < a_0 < a_1 < a_1 + \frac{1}{1+\tau} < a_2 < a_3 = 1$ . Let us prove that the additional assumptions of Theorem 2.14 hold true with  $\mathcal{B}_i := \mathcal{C}_{V^{a_i}}$  for  $i \in \{0, 1, 2, 3\}$ . Let  $i \in \{0, 1, 2\}$ . The fact that  $(P_\gamma)_\gamma$  satisfies the conditions of Hypothesis 2.8\* on  $(J, \mathcal{B}_i, \mathcal{B}_{i+1})$  comes from Lemma 4.7 and Remark 4.5. The fact that Hypothesis 2.11 is satisfied on  $\mathcal{B}_{i+1}$  comes from Remark 4.10 and from Lemmas 4.11 and 4.12 applied with  $V^{a_{i+1}}$  (in place of  $V$ ). Observe that

$$\|\xi f\|_{\mathcal{B}_2} = \sup \frac{\|\xi f\|}{V^{a_2}} \leq \sup \frac{\|\xi\|}{V^{\frac{1}{1+\tau}}} \sup \frac{\|f\|}{V^{a_1}} \leq \|f\|_{\mathcal{B}_1} \sup \frac{\|\xi\|}{V^{\frac{1}{1+\tau}}}.$$

Hence we have proved that  $f \mapsto \xi f$  is in  $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ . The fact that  $\gamma \mapsto P_\gamma$  is  $C^1$  from  $(0, +\infty)$  to  $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$  and that  $P'_\gamma := P_\gamma(-\xi f)$  comes from the proof of [12, Lemma 10.4]. We conclude as explained after Theorem 2.14.

## APPENDIX A. OPERATOR TECHNIQUES

We use the notations of Section 2.2.

**A.1. Decrease of  $r$ .** The non-increasingness of  $r$  was studied in Lemma 2.7. The next result gives a way to prove that  $r' \neq 0$  and so the decrease of  $\gamma \mapsto r(\gamma)$ .

**Proposition A.1.** *Let  $J = (a, b) \subset [0, +\infty)$  and let  $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$  be two Banach spaces such that  $P_\gamma \in \mathcal{L}(\mathcal{B}_1) \cap \mathcal{L}(\mathcal{B}_2)$  for every  $\gamma \in J$ . Assume that  $f \mapsto \xi f \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$  and that, for every  $\gamma \in J$ , there exist  $\phi_\gamma \in \mathcal{B}_1$  and  $\pi_\gamma \in \mathcal{B}_2^*$  such that  $P_\gamma \phi_\gamma = r(\gamma) \phi_\gamma$  and  $P_\gamma^* \pi_\gamma = r(\gamma) \pi_\gamma$  (where  $P_\gamma^*$  is the dual operator of  $P_\gamma$ ). Let  $\gamma_0$  be a point of  $J$  at which the functions  $\gamma \mapsto P_\gamma$  from  $J$  to  $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$  and  $\gamma \mapsto r(\gamma)$  from  $J$  to  $\mathbb{C}$  are differentiable with respective derivatives  $f \mapsto P_{\gamma_0}(-\xi f)$  and  $r'(\gamma_0)$ . We assume moreover that, at  $\gamma_0$ ,  $\gamma \mapsto \phi_\gamma$  is continuous from  $J$  to  $\mathcal{B}_1$  and differentiable from  $J$  to  $\mathcal{B}_2$  with derivative  $\phi'_{\gamma_0}$ .*

*If  $r(\gamma_0) \neq 0$  and  $r'(\gamma_0) = 0$  then  $\pi_{\gamma_0}(\xi \phi_{\gamma_0}) = 0$ .*

*Proof.* We have  $P_\gamma \phi_\gamma = r(\gamma) \phi_\gamma$  in  $\mathcal{B}_2$ . We derive this formula at  $\gamma_0$  by writing  $P_\gamma \phi_\gamma - P_{\gamma_0} \phi_{\gamma_0} = P_{\gamma_0}(\phi_\gamma - \phi_{\gamma_0}) + (P_\gamma - P_{\gamma_0})(\phi_\gamma)$ . Using the fact that  $r'(\gamma_0) = 0$ , we obtain that

$$P_{\gamma_0}(\phi'_{\gamma_0}) + P_{\gamma_0}(-\xi \phi_{\gamma_0}) = r(\gamma_0) \phi'_{\gamma_0} \quad \text{in } \mathcal{B}_2.$$

Composing by  $\pi_{\gamma_0}$ , we obtain  $0 = \pi_{\gamma_0} P_{\gamma_0}(\xi \phi_{\gamma_0}) = r(\gamma_0) \pi_{\gamma_0}(\xi \phi_{\gamma_0})$ , thus  $\pi_{\gamma_0}(\xi \phi_{\gamma_0}) = 0$ .  $\square$

**A.2. Proof of Theorems 2.12 and 2.14.** Let us state, more precisely than in Theorem 2.10, the Keller-Liverani perturbation theorem.

**Theorem A.2** (Keller-Liverani Perturbation Theorem [13, 15, 5]). *Let  $(\mathcal{X}_0, \|\cdot\|_{\mathcal{X}_0})$  be a Banach space and  $(\mathcal{X}_1, \|\cdot\|_{\mathcal{X}_1})$  be a normed space such that  $\mathcal{X}_0 \hookrightarrow \mathcal{X}_1$ . Let  $J \subset [-\infty, +\infty]$  be an interval and let  $(Q(t))_{t \in J}$  be a family of operators. We assume that*

- For every  $t \in J$ ,  $Q(t) \in \mathcal{L}(\mathcal{X}_0) \cap \mathcal{L}(\mathcal{X}_1)$ ,
- $t \mapsto Q(t)$  is a continuous map from  $J$  in  $\mathcal{L}(\mathcal{X}_0, \mathcal{X}_1)$ ,
- There exist  $\delta_0 > 0$ ,  $c_0, M_0 > 0$  such that for every  $t \in J$

$$\forall f \in \mathcal{X}_0, \forall n \in \mathbb{Z}_+, \quad \|(Q(t))^n f\|_{\mathcal{X}_0} \leq c_0(\delta_0^n \|f\|_{\mathcal{X}_0} + M_0^n \|f\|_{\mathcal{X}_1}).$$

Let  $t_0 \in J$ . Then, for every  $\varepsilon > 0$  and every  $\delta > \delta_0$ , there exists  $I_0 \subset J$  containing  $t_0$  such that

$$\sup_{t \in I_0, z \in \mathcal{D}(\delta, \varepsilon)} \|(zI - Q(t))^{-1}\|_{\mathcal{X}_0} < \infty,$$

with  $\mathcal{D}(\delta, \varepsilon) := \{z \in \mathbb{C}, d(z, \sigma(Q(t_0)|_{\mathcal{X}_0})) > \varepsilon, |z| > \delta\}$ .

Furthermore the map  $t \mapsto (zI - Q(t))^{-1}$  from  $J$  to  $\mathcal{L}(\mathcal{X}_0, \mathcal{X}_1)$  is continuous at  $t_0$  in a uniform way with respect to  $z \in \mathcal{D}(\delta, \varepsilon)$ , i.e.

$$\lim_{t \rightarrow t_0, t \in J} \sup \{ \|(zI - Q(t))^{-1} - (zI - Q(t_0))^{-1}\|_{\mathcal{X}_0, \mathcal{X}_1}; z \in \mathcal{D}(\delta, \varepsilon) \} = 0.$$

In particular,  $\limsup_{t \rightarrow t_0} r((Q(t))|_{\mathcal{X}_0}) \leq \max(\delta_0, r((Q(t_0))|_{\mathcal{X}_0}))$ . Moreover the map  $t \mapsto r((Q(t))|_{\mathcal{X}_0})$  is continuous on  $\{t \in J : r((Q(t))|_{\mathcal{X}_0}) > \delta_0 \geq r_{\text{ess}}((Q(t))|_{\mathcal{X}_0})\}$ .

Let  $\mathcal{B}$  be a nonnull complex Banach lattice of functions  $f : \mathbb{X} \rightarrow \mathbb{C}$  (or of classes of such functions modulo a nonnegative nonnull measure  $\mathbf{m}$ ). If  $f \in \mathcal{B}$  is a class of functions, we say that it is nonnegative resp. positive if one of its representant is so and we say that it is nonnull if the null function is not one of its representant. We say that  $\psi \in \mathcal{B}^*$  is nonnegative if for every nonnegative  $f \in \mathcal{B}$ ,  $\psi(f) \geq 0$  and that  $\psi \in \mathcal{B}^*$  is positive if for every nonnegative nonnull  $f \in \mathcal{B}$ ,  $\psi(f) > 0$ .

**Proposition A.3** (First order of the spectral radius). *Let  $\mathcal{B}$  be a non null complex Banach lattice of functions  $f : \mathbb{X} \rightarrow \mathbb{C}$  (or of classes of such functions modulo some nonnegative nonnull measure  $\mathbf{m}$ ). Let  $Q$  be a (nonnull) nonnegative quasicompact operator on  $\mathcal{B}$  such that  $r(Q) \neq 0$  and such that for every nonnull nonnegative  $f \in \mathcal{B}$  and for every nonnull nonnegative  $\psi \in \mathcal{B}^* \cap \text{Ker}(Q^* - r(Q)I)$ , we have  $Qf > 0$  (modulo  $\mathbf{m}$ ) and  $\psi(Qf) > 0$ . Then*

- $r(Q)$  is a first order pole of  $Q$ , and there exists a positive  $\phi \in \mathcal{B}$  and a positive  $\psi \in \mathcal{B}^*$  such that

$$\psi(\phi) = 1, \quad Q\phi = r(Q)\phi \quad \text{and} \quad Q^*\psi = r(Q)\psi.$$

- Let  $\lambda \in \mathbb{C}$  and  $h \in \mathcal{B}$  such that  $|\lambda| = r(Q)$  and  $Qh = \lambda h$ . Then  $Q|h| = r(Q)|h|$  in  $\mathcal{B}$ .
- If moreover  $Q$  is of the form  $Q = P(e^{-\gamma \cdot})$  where  $P$  is the operator associated to a Markov kernel, if  $1_{\mathbb{X}} \in \mathcal{B} \hookrightarrow \mathbb{L}^1(\pi)$ , if  $\text{Ker}(Q - r(Q)I) = \mathbb{C} \cdot \phi$  and if 1 is the only complex number  $\lambda$  of modulus 1 such that  $P(h/|h|) = \lambda h/|h|$  in  $\mathbb{L}^1(\pi)$  for some  $h \in \mathcal{B}$  with  $|h| > 0$ , then  $r(Q)$  is the only eigenvalue of modulus  $r(Q)$  of  $Q$ .

*Proof.* The fact that  $r(Q)$  is a finite pole of  $Q$  is classical for a nonnegative quasi-compact operator  $Q$  on a Banach lattice. Let us just remember the main arguments. From quasi-compactness we know that there exists a finite pole  $\lambda \in \sigma(Q)$  such that  $|\lambda| = r(Q)$ . Thus, setting  $\lambda_n := \lambda(1 + 1/n)$  for any  $n \geq 1$ , we deduce from  $\lambda \in \sigma(Q)$  that

$$\lim_n \|(\lambda_n I - Q)^{-1}\|_{\mathcal{B}} = +\infty.$$

Since  $\mathcal{B}$  is a Banach lattice, we deduce from the Banach-Steinhaus theorem that there exists a nonnegative and nonnull element  $f \in \mathcal{B}$  such that

$$\lim_n \|(\lambda_n I - Q)^{-1} f\|_{\mathcal{B}} = +\infty.$$

Next define  $r_n := r(Q)(1 + 1/n)$  and observe that

$$|(\lambda_n I - Q)^{-1} f| = \left| \sum_{k \geq 0} \lambda_n^{-(k+1)} Q^k f \right| \leq \sum_{k \geq 0} r_n^{-(k+1)} Q^k f.$$

Since  $\mathcal{B}$  is a Banach lattice, the last inequality is true in norm, that is

$$\|(\lambda_n I - Q)^{-1} f\| \leq \left\| \sum_{k \geq 0} r_n^{-(k+1)} Q^k f \right\|$$

from which we deduce that  $\lim_n \|(r_n I - Q)^{-1}\|_{\mathcal{B}} = +\infty$ , thus  $r(Q) \in \sigma(Q)$ . Finally  $r(Q)$  is a finite pole of  $Q$  from quasi-compactness.

Let  $q$  denote the order of the pole  $r(Q)$ , namely  $r(Q)$  is a pole of order  $q$  of the resolvent function  $z \mapsto (zI - Q)^{-1}$ . Then there exists  $\rho > 0$  such that  $(zI - Q)^{-1}$  admits the following Laurent series provided that  $|z - r(Q)| < \rho$  and  $z \neq r(Q)$ :

$$(zI - Q)^{-1} = \sum_{k=-q}^{+\infty} (z - r(Q))^k A_k,$$

where  $A_k$  are bounded linear operators on  $\mathcal{B}$ . By quasi-compactness,  $A_{-1}$  is a projection onto the finite subspace  $\text{Ker}(Q - r(Q)I)^q$ . Moreover we know that

$$A_{-q} = (Q - r(Q)I)^{q-1} \circ A_{-1} = A_{-1} \circ (Q - r(Q)I)^{q-1}. \quad (31)$$

and that, setting  $r_n := r(Q)(1 + 1/n)$ ,

$$\begin{aligned} A_{-q} &= \lim_{n \rightarrow +\infty} (r_n - r(Q))^q (r_n I - Q)^{-1} \\ &= \lim_{n \rightarrow +\infty} (r_n - r(Q))^q \sum_{k \geq 0} r_n^{-(k+1)} Q^k. \end{aligned} \quad (32)$$

Since  $Q$  is a nonnull nonnegative operator on  $\mathcal{B}$ , so is  $A_{-q}$ . Since  $A_{-q} \neq 0$ , we take a nonnegative  $h_0 \in \mathcal{B}$  such that  $\phi := A_{-q} h_0 \neq 0$  in  $\mathcal{B}$ . Moreover we have  $(Q - r(Q)I)A_{-q} = 0$ , so  $r(Q)\phi = Q\phi$ . Similarly there exists a nonnegative  $\psi_0 \in \mathcal{B}^*$  such that  $\psi_1 := A_{-q}^* \psi_0$  is a nonzero and nonnegative element of  $\text{Ker}(Q^* - r(Q)I)$ , where  $A_{-q}^*$  is the adjoint operator of  $A_{-q}$ . We have  $\psi_1(\phi) = \psi_1(Q\phi)/r(Q) > 0$  from our hypotheses, and we set  $\psi := \psi_1/\psi_1(\phi)$ . Now assume that  $q \geq 2$ . Then  $A_{-q}^2 = 0$  from (31) and  $A_{-1}(\mathcal{B}) = \text{Ker}(Q - r(Q)I)^q$ , so that  $\psi_1(\phi) = (A_{-q}^* \psi_0)(A_{-q} h_0) = \psi_0(A_{-q}^2 h_0) = 0$ . This proves by reductio ad absurdum that  $p = 1$ .

Observe that, from our hypotheses, we know that  $\psi(h) = \psi(Qh)/r(Q) > 0$  for every nonnull nonnegative  $h \in \mathcal{B}$ . Now let  $\lambda \in \mathbb{C}$  and  $h \in \mathcal{B}$  be such that  $|\lambda| = r(Q)$  and

$Qh = \lambda h$ . The positivity of  $Q$  gives:  $|\lambda h| = r(Q)|h| = |Qh| \leq Q|h|$ . Moreover we have  $\psi(Q|h| - r(Q)|h|) = 0$ . Thus  $Q|h| = r(Q)|h|$  in  $\mathcal{B}$ .

Now let us prove the last point of Proposition A.3. Recall that the above nonnull non-negative function  $\phi \in \mathcal{B}$  is such that  $Q\phi = r(Q)\phi$ . From our hypotheses we deduce that  $\phi > 0$ . Let  $\lambda \in \mathbb{C}$  and  $h \in \mathcal{B}$  be such that  $|\lambda| = r(Q)$ ,  $h \neq 0$  and  $P_\gamma h = \lambda h$ . Due to the previous point and to our assumptions, we obtain that  $Q|h| = r(Q)|h|$  and  $|h| = \beta\phi$  for some  $\beta > 0$ . One may assume that  $\beta = 1$  for the sake of simplicity. Then there exists a  $\pi$ -full and  $P$ -absorbing  $A \in \mathcal{X}$  (i.e.  $\pi(A) = 1$  and  $P(x, A) = 1, \forall x \in A$ ) such that

$$\forall x \in A, \quad |h(x)| = \phi(x) > 0 \quad (33a)$$

$$\forall x \in A, \quad \lambda h(x) = \int h(y) e^{-\gamma\xi(y)} P(x, dy) \quad (33b)$$

$$\forall x \in A, \quad r(Q)\phi(x) = \int \phi(y) e^{-\gamma\xi(y)} P(x, dy). \quad (33c)$$

Let  $x \in A$  and define the probability measure:  $\eta_x(dy) := (r(Q)\phi(x))^{-1} \phi(y) e^{-\gamma\xi(y)} P(x, dy)$ . We have

$$\int_{\mathbb{R}} \frac{r(Q)\phi(x)h(y)}{\lambda\phi(y)h(x)} \eta_x(dy) = 1.$$

Then a standard convexity argument ensures that the following equality holds for  $P(x, \cdot)$ -almost every  $y \in \mathbb{X}$ :

$$r(Q)\phi(x)h(y) = \lambda\phi(y)h(x). \quad (34)$$

We have  $r(Q)P \frac{h}{|h|} = \lambda \frac{h}{|h|}$  and so  $\lambda = r(Q)$ .  $\square$

From now on, to simplify notations, we write  $R_z(\gamma) := (zI - P_\gamma)^{-1}$  for the resolvent when it is well defined. We first prove Theorems 2.12 and 2.14 under Hypothesis 2.8. Recall that  $J_0 := \{t \in J : r(\gamma) > \delta_0\}$ .

*Proof of Theorem 2.12 under Hypothesis 2.8.* The continuity on  $J_0$  of  $\gamma \mapsto r(\gamma) := r((P_\gamma)|_{\mathcal{B}_0})$  follows from Theorem A.2 since  $(P_\gamma)_\gamma$  satisfies Hypothesis 2.8 with  $(J, \mathcal{B}_0, \mathcal{B}_1)$ . Moreover, due to Proposition A.3 and to Hypothesis 2.11, we know that, for every  $\gamma \in J_0$ ,  $r(\gamma)$  is the only dominating eigenvalue of  $(P_\gamma)|_{\mathcal{B}_0}$  and that it is a simple eigenvalue with multiplicity 1.

Let  $\chi : J_0 \rightarrow (0, +\infty)$  be defined by  $\chi(\gamma) := \max(\delta_0, \lambda(\gamma))$ , where we have set  $\lambda(\gamma) := \max\{|\lambda| : \lambda \in \sigma(P_\gamma|_{\mathcal{B}_0}) \setminus \{r(\gamma)\}\}$ . Due to Theorem A.2,  $\chi$  is continuous on  $J_0$ . Let  $K$  be a compact subset of  $J_0$ . We set  $\theta := \max_K \frac{\chi}{r}$ . Since  $\chi(\gamma) < r(\gamma)$  for every  $\gamma \in K$  and since  $r(\cdot)$  and  $\chi(\cdot)$  are continuous, we conclude that  $\theta \in (0, 1)$ . Next we consider any  $\eta > 0$  such that  $\theta + 2\eta < 1$ .

Let us construct the map  $\gamma \mapsto \Pi_\gamma$ , from  $K$  to  $\mathcal{L}(\mathcal{B}_0)$ , and prove its properties. Let  $\gamma_0 \in K$ . Since  $r$  is continuous on  $K$ , there exists  $\varepsilon > 0$  such that, for every  $\gamma \in K$  such that  $|\gamma - \gamma_0| \leq \varepsilon$ , we have  $|r(\gamma) - r(\gamma_0)| < \eta r(\gamma_0)$ . Let us write  $K(\gamma_0)$  for the set of  $\gamma \in K$  such that  $|\gamma - \gamma_0| \leq \varepsilon$ . Observe that, for any  $\gamma \in K(\gamma_0)$ ,

$$\chi(\gamma) \leq \theta r(\gamma) < \theta(1 + \eta)r(\gamma_0) < (\theta + \eta)r(\gamma_0) < (1 - \eta)r(\gamma_0)$$

and so the eigenprojector  $\Pi_\gamma$  on  $\text{Ker}(P_\gamma - r(\gamma)I)$  can be defined by

$$\Pi_\gamma = \frac{1}{2i\pi} \oint_{\Gamma_1(\gamma_0)} R_z(\gamma) dz, \quad (35)$$

where  $\Gamma_1(\gamma_0)$  is the oriented circle centered on  $r(\gamma_0)$  with radius  $\eta r(\gamma_0)$ . Due to Theorem A.2,  $\gamma \mapsto \Pi_\gamma$  is well defined from  $K(\gamma_0)$  to  $\mathcal{L}(\mathcal{B}_0)$  and is continuous from  $K(\gamma_0)$  to  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$ .

Now, for every  $\gamma \in K$ , we define the oriented circle  $\Gamma_0(\gamma) := \{z \in \mathbb{C} : |z| = (\theta + \eta)r(\gamma)\}$ . By definition of  $\theta$ , for every  $\gamma \in K$ , we have  $\chi(\gamma) \leq \theta r(\gamma)$  and so  $\chi(\gamma) < (\theta + \eta)r(\gamma) < r(\gamma)$ . Hence, by definition of  $\chi(\gamma)$ ,  $R_z(\gamma)$  is well-defined in  $\mathcal{L}(\mathcal{B}_0)$  for every  $\gamma \in K$  and  $z \in \Gamma_0(\gamma)$ . From spectral theory, it comes that

$$N_\gamma^n := P_\gamma^n - r(\gamma)^n \Pi_\gamma = \frac{1}{2i\pi} \oint_{\Gamma_0(\gamma)} z^n R_z(\gamma) dz \quad (36)$$

and so

$$\|P_\gamma^n - r(\gamma)^n \Pi_\gamma\|_{\mathcal{B}_0} \leq B_\gamma ((\theta + \eta)r(\gamma))^{n+1} \quad \text{with} \quad B_\gamma := \sup_{|z|=(\theta+\eta)r(\gamma)} \|R_z(\gamma)\|_{\mathcal{B}_0}. \quad (37)$$

It remains to prove that

$$M_K := \sup_{\gamma \in K} B_\gamma < \infty. \quad (38)$$

Let  $\gamma_0 \in K$ . Since  $\gamma \mapsto r(\gamma)$  is continuous at  $\gamma_0$ , there exists  $\alpha \equiv \alpha(\gamma_0) > 0$  such that, for every  $\gamma \in K$  such that  $|\gamma - \gamma_0| < \alpha$ , we have

$$\frac{\theta + \frac{\eta}{2}}{\theta + \eta} r(\gamma_0) < r(\gamma) < \frac{\theta + \frac{3\eta}{2}}{\theta + \eta} r(\gamma_0).$$

Set  $\delta := \frac{\eta}{2} r(\gamma_0)$ . If  $|\gamma - \gamma_0| < \alpha$  and if  $|z| = (\theta + \eta)r(\gamma)$ , we obtain since  $\delta_0 \leq \chi(\gamma_0) \leq \theta r(\gamma_0)$  and  $\theta + 2\eta < 1$ :

$$\delta_0 + \delta \leq \chi(\gamma_0) + \delta \leq \left(\theta + \frac{\eta}{2}\right) r(\gamma_0) < |z| < \left(\theta + \frac{3\eta}{2}\right) r(\gamma_0) < r(\gamma_0) - \delta.$$

From the previous inequalities, let us just keep in mind that  $\chi(\gamma_0) + \delta < |z| < r(\gamma_0) - \delta$ . Then, by definition of  $\chi(\gamma_0)$ , we conclude that every complex number  $z$  such that  $|z| = (\theta + \eta)r(\gamma)$  satisfies

$$|z| > \delta_0 + \delta \quad \text{and} \quad d(z, \sigma(Q(\gamma_0))) > \delta.$$

Hence, up to a change of  $\alpha$ , due to Theorem A.2, we obtain that

$$\sup_{\gamma > 0 : |\gamma - \gamma_0| < \alpha} B_\gamma = \sup \{ \|R_z(\gamma)\|_{\mathcal{B}_0} : |\gamma - \gamma_0| < \alpha, |z| = (\theta + \eta)r(\gamma) \} < \infty.$$

By a standard compactness argument, we have proved (38). Consequently, with  $\theta_K := \theta + \eta$ , we deduce from (37) that

$$\|P_\gamma^n - r(\gamma)^n \Pi_\gamma\|_{\mathcal{B}_0} \leq M_K (\theta_K r(\gamma))^n$$

from which we derive (10).  $\square$

*Proof of Theorem 2.14 under Hypothesis 2.8.* First we prove the following lemma.

**Lemma A.4.** *For all  $\gamma \in J_0$  and for  $i = 1, 2$ , the spectral radius of  $P_\gamma|_{\mathcal{B}_i}$  is equal to  $r(\gamma) := r(P_\gamma|_{\mathcal{B}_0})$ .*

*Proof.* For  $i = 0, 1, 2$  set  $r_i(\gamma) := r((P_\gamma)|_{\mathcal{B}_i})$ . Due to Theorem 2.12 applied to  $(P_\gamma, J, \mathcal{B}_i, \mathcal{B}_{i+1})$ , there exists  $c_i > 0$  such that  $\pi(P_\gamma^n \mathbf{1}_{\mathbb{X}}) \sim c_i r_i(\gamma)^n$  as  $n$  goes to infinity. This proves the equality of the spectral radius.  $\square$

We define  $\chi_i$  as  $\chi$  in the proof of Theorem 2.12 for each  $\mathcal{B}_i$  ( $i = 0, 1, 2$ ). We define now  $\chi := \max(\chi_0, \chi_1, \chi_2)$ .

Now let us prove the differentiability of  $r$  and  $\Pi$ . Let  $\gamma_0 \in J_0$ . Let  $\eta > 0$  be such that  $r(\gamma_0) > \chi(\gamma_0) + 2\eta$  and let  $\varepsilon > 0$  be such that for every  $\gamma \in J_0$  satisfying  $|\gamma - \gamma_0| < \varepsilon$ , we have  $r(\gamma) > r(\gamma_0) - \eta > \chi(\gamma_0) + \eta > \chi(\gamma)$ . We set  $I_0 := J_0 \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$  and

$$\mathcal{D}_0 := \{z \in \mathbb{C} : \chi(\gamma_0) + \eta < |z| < r(\gamma_0) - \eta\} \cup \{z \in \mathbb{C} : |z - r(\gamma_0)| = \eta\}. \quad (39)$$

Due to the hypotheses of Theorem 2.14 and to an easy adaptation of [12, Lemma A.2] (see Remark A.5), we obtain that, for every  $z \in \mathcal{D}_0$ , the map  $\gamma \mapsto R_z(\gamma)$  is  $C^1$  from  $I_0$  to  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$  with  $R'_z(\gamma) = R_z(\gamma)P'_\gamma R_z(\gamma)$  and

$$\lim_{h \rightarrow 0} \sup_{z \in \mathcal{D}_0} \frac{\|R_z(\gamma_0 + h) - R_z(\gamma_0) - hR'_z(\gamma_0)\|_{\mathcal{B}_0, \mathcal{B}_3}}{|h|} = 0. \quad (40)$$

Moreover, for every  $\gamma \in I_0$ , we deduce from spectral theory that

$$\Pi_\gamma = \frac{1}{2i\pi} \oint_{\Gamma_1} R_z(\gamma) dz \quad \text{and} \quad N_\gamma = \frac{1}{2i\pi} \oint_{\Gamma_0} z R_z(\gamma) dz,$$

where  $\Gamma_1$  is the oriented circle centered at  $r(\gamma_0)$  with radius  $\eta$  and  $\Gamma_0$  is the oriented circle centered at 0 with some radius  $\vartheta_0$  satisfying  $\chi(\gamma_0) + \eta < \vartheta_0 < r(\gamma_0) - \eta$ . Since  $1_{\mathbb{X}} \in \mathcal{B}_0$  by hypothesis this implies the continuous differentiability of  $\gamma \mapsto N_\gamma 1_{\mathbb{X}}$  and of  $\gamma \mapsto \Pi_\gamma 1_{\mathbb{X}}$  from  $J_0$  to  $\mathcal{B}_3$ . Since  $r(\gamma) = \frac{(P_\gamma - N_\gamma)(1_{\mathbb{X}})}{\Pi_\gamma(1_{\mathbb{X}})}$  and  $\gamma \mapsto P_\gamma 1_{\mathbb{X}}$  is  $C^1$  from  $I_0$  to  $\mathcal{B}_3$  by hypothesis, we obtain the continuous differentiability of  $r$  on  $I_0$ . Let us define  $\phi_\gamma = \Pi_\gamma 1_{\mathbb{X}}$  and  $\pi_\gamma = \Pi_\gamma^* \pi$ . To prove that the derivative of  $r$  is negative we now apply Proposition A.1 with  $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ . Indeed  $\pi_\gamma \in \mathcal{B}_2^*$  since  $\pi \in \mathcal{B}_2^*$  and  $\Pi_\gamma^*$  is well defined in  $\mathcal{L}(\mathcal{B}_2^*)$ . Moreover  $\phi_\gamma \in \mathcal{B}_1$  since  $1_{\mathbb{X}} \in \mathcal{B}_1$  and  $\Pi_\gamma$  is well defined in  $\mathcal{L}(\mathcal{B}_1)$ , and  $\gamma \mapsto \phi_\gamma$  is continuous from  $J$  to  $\mathcal{B}_1$  by Theorem 2.12. Finally  $\gamma \mapsto \phi_\gamma$  is differentiable from  $J$  to  $\mathcal{B}_2$  (see the end of Remark A.5).  $\square$

**Remark A.5** (Proof of the differentiability of  $\gamma \mapsto \Pi_\gamma$ ). *We adapt the arguments of [12, Lemma A.2], writing*

$$R_z(\gamma) = R_z(\gamma_0) + R_z(\gamma_0)[P_\gamma - P_{\gamma_0}]R_z(\gamma_0) + \vartheta_z(\gamma),$$

with  $\vartheta_z(\gamma) := R_z(\gamma_0)[P_\gamma - P_{\gamma_0}]R_z(\gamma_0)[P_\gamma - P_{\gamma_0}]R_z(\gamma)$ .

Then

$$\frac{\|\vartheta_z(\gamma)\|_{\mathcal{B}_0, \mathcal{B}_3}}{|\gamma - \gamma_0|} \leq \|R_z(\gamma_0)\|_{\mathcal{B}_2} \left\| \frac{P_\gamma - P_{\gamma_0}}{\gamma - \gamma_0} \right\|_{\mathcal{B}_1, \mathcal{B}_2} \|R_z(\gamma_0)\|_{\mathcal{B}_1} \|P_\gamma - P_{\gamma_0}\|_{\mathcal{B}_0, \mathcal{B}_1} \|R_z(\gamma)\|_{\mathcal{B}_0}. \quad (41)$$

From the hypotheses of Theorem 2.14 and from the resolvent bounds derived from Theorem A.2, the last term goes to 0, uniformly in  $z \in \mathcal{D}$ , when  $\gamma$  goes to  $\gamma_0$ . Similarly we have:

$$\begin{aligned} & \left\| R_z(\gamma_0)(P_\gamma - P_{\gamma_0})R_z(\gamma_0) - (\gamma - \gamma_0)R_z(\gamma_0)P'_{\gamma_0}R_z(\gamma_0) \right\|_{\mathcal{B}_0, \mathcal{B}_3} \\ & \leq M \|P_\gamma - P_{\gamma_0} - (\gamma - \gamma_0)P'_{\gamma_0}\|_{\mathcal{B}_1, \mathcal{B}_2} = o(\gamma - \gamma_0) \end{aligned}$$

when again the finite positive constant  $M$  is derived from the resolvent bounds of Theorem A.2. This shows that  $R'_z(\gamma_0) = R_z(\gamma_0)P'_{\gamma_0}R_z(\gamma_0)$  in  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$ . To prove that  $\gamma \mapsto R'_z(\gamma)$  is continuous from  $J_0$  to  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_3)$  in a uniform way with respect to  $z \in \mathcal{D}$ , observe that  $\gamma \mapsto R_z(\gamma)$  is  $C^0$  from  $J_0$  to  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$  (use Theorem A.2), that  $\gamma \mapsto P'_\gamma$  is  $C^0$  (uniformly in  $z \in \mathcal{D}$ ) from  $J_0$  to  $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$  by hypothesis, and finally that  $\gamma \mapsto R_z(\gamma)$  is  $C^0$  (uniformly in  $z \in \mathcal{D}$ ) from  $J_0$  to  $\mathcal{L}(\mathcal{B}_2, \mathcal{B}_3)$  (again use Theorem A.2). Observe that (41) gives the differentiability at  $\gamma_0$  of the



map  $\gamma \mapsto R_z(\gamma)$  considered from  $J$  to  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_2)$ . The additional space  $\mathcal{B}_3$  is only required to obtain the continuous differentiability.

*Proof of Theorem 2.12 under Hypothesis 2.8\**. Here the Keller-Liverani perturbation theorem must be applied to the dual family  $(P_\gamma^*)_\gamma$ . Actually the hypotheses of Theorem 2.12 are:

- $\mathcal{B}_1^* \hookrightarrow \mathcal{B}_0^*$ ,
- For every  $\gamma \in J$ ,  $P_\gamma^* \in \mathcal{L}(\mathcal{B}_0^*) \cap \mathcal{L}(\mathcal{B}_1^*)$ ,
- $\gamma \mapsto P_\gamma^*$  is a continuous map from  $J$  in  $\mathcal{L}(\mathcal{B}_1^*, \mathcal{B}_0^*)$ ,
- There exist  $\delta_0, c_0, M_0 > 0$  such that, for all  $\gamma \in J$ ,  $r_{ess}((P_\gamma^*)_{|\mathcal{B}_1^*}) \leq \delta_0$  and
 
$$\forall n \geq 1, \forall f^* \in \mathcal{B}_1^*, \quad \|(P_\gamma^*)^n f^*\|_{\mathcal{B}_1^*} \leq c_0(\delta_0^n \|f^*\|_{\mathcal{B}_1^*} + M^n \|f^*\|_{\mathcal{B}_0^*}).$$
- Hypothesis 2.11 holds on  $(J, \mathcal{B}_1)$ .

Under these assumptions it follows from Theorem A.2 applied to  $(P_\gamma^*)_{\gamma \in J}$  with respect to  $(\mathcal{B}_1^*, \mathcal{B}_0^*)$  that, for every  $\varepsilon > 0$  and every  $\delta > \delta_0$ , the map  $t \mapsto (zI - P_\gamma^*)^{-1}$  is well defined from  $J_0$  to  $\mathcal{L}(\mathcal{B}_1^*)$ , provided that  $z \in \mathcal{D}(\delta, \varepsilon)$  with

$$\mathcal{D}(\delta, \varepsilon) := \{z \in \mathbb{C}, d(z, \sigma((P_{\gamma_0}^*)_{|\mathcal{B}_2^*})) > \varepsilon, |z| > \delta\} = \{z \in \mathbb{C}, d(z, \sigma((P_{\gamma_0})_{|\mathcal{B}_2})) > \varepsilon, |z| > \delta\}.$$

In addition, the map  $t \mapsto (zI - P_\gamma^*)^{-1}$ , considered from  $J_0$  to  $\mathcal{L}(\mathcal{B}_1^*, \mathcal{B}_0^*)$ , is continuous at every  $\gamma_0 \in J_0$  in a uniform way with respect to  $z \in \mathcal{D}(\delta, \varepsilon)$ . By duality this implies that  $t \mapsto (zI - P_\gamma)^{-1}$  is well defined from  $J_0$  to  $\mathcal{L}(\mathcal{B}_1)$ . Moreover, when this map is considered from  $J_0$  to  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$ , it is continuous at  $\gamma_0$  in a uniform way with respect to  $z \in \mathcal{D}(\delta, \varepsilon)$ . Finally, since Hypothesis 2.11 is assumed on  $(J, \mathcal{B}_1)$ , Proposition A.3 enables us to identify the spectral elements associated with  $r(\gamma) := r((P_\gamma)_{|\mathcal{B}_1})$ . Consequently one can prove as above that there exists a map  $\gamma \mapsto \Pi_\gamma$  from  $J_0$  to  $\mathcal{L}(\mathcal{B}_1)$ , which is continuous from  $J_0$  to  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$ , such that (10) holds with  $\mathcal{B} := \mathcal{B}_1$ .  $\square$

*Proof of Theorem 2.14 under Hypothesis 2.8\**. When Theorem 2.14 is stated with Hypothesis 2.8\*, then Theorem 2.12 applies on  $(\mathcal{B}_0, \mathcal{B}_1)$ ,  $(\mathcal{B}_1, \mathcal{B}_2)$  and  $(\mathcal{B}_2, \mathcal{B}_3)$  (with Hypothesis 2.8\* in each case). Thus, for every  $\gamma \in J_0$ , the spectral radius  $r_i(\gamma) := r((P_\gamma)_{|\mathcal{B}_i})$  are equal for  $i = 1, 2, 3$  (See the proof of Lemma A.4). Observe that, from our hypotheses, Proposition A.3 applies to  $P_\gamma$  with respect to  $(J, \mathcal{B}_i)$  for  $i = 1, 2, 3$ . Since  $P_\gamma^*$  on  $\mathcal{B}_i^*$  inherits the spectral properties of  $P_\gamma$  on  $\mathcal{B}_i$ , we can prove as above that, for every  $\gamma_0 \in J_0$  and for every  $\varepsilon > 0$  and  $\delta > \delta_0$ , the map  $t \mapsto (zI - P_\gamma^*)^{-1}$  is well defined from some subinterval  $I_0$  of  $J_0$  containing  $\gamma_0$  into  $\mathcal{L}(\mathcal{B}_3^*)$ , provided that  $z \in \mathcal{D}_0$  where the set  $\mathcal{D}_0$  is defined in (39). In addition, by applying Remark A.5 with the adjoint operators  $(P_\gamma^*)_\gamma$  and the spaces  $\mathcal{B}_3^* \hookrightarrow \mathcal{B}_2^* \hookrightarrow \mathcal{B}_1^* \hookrightarrow \mathcal{B}_0^*$ , we can prove that the map  $t \mapsto (zI - P_\gamma^*)^{-1}$ , considered from  $J_0$  to  $\mathcal{L}(\mathcal{B}_3^*, \mathcal{B}_0^*)$ , is  $\mathcal{C}^1$  in a uniform way with respect to  $z \in \mathcal{D}_0$ . By duality, this gives (40). We conclude the differentiability of  $\gamma \mapsto \Pi_\gamma^*$  from  $J_0$  to  $\mathcal{L}(\mathcal{B}_3^*, \mathcal{B}_1^*)$  and so the differentiability of  $\gamma \mapsto \Pi_\gamma$  from  $J_0$  to  $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_3)$ . To prove that the derivative of  $r$  is negative we apply Proposition A.1 with the spaces  $\mathcal{B}_1$  and  $\mathcal{B}_3$ . Note that  $\pi_\gamma := \Pi_\gamma^* \pi \in \mathcal{B}_3^*$  since  $\pi \in \mathcal{B}_3^*$  and  $\Pi_\gamma^*$  is well defined in  $\mathcal{L}(\mathcal{B}_3^*)$ . The function  $\gamma \mapsto P_\gamma$  is differentiable from  $J_0$  to  $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ , thus from  $J_0$  to  $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_3)$ . We have  $\phi_\gamma := \Pi_\gamma \mathbf{1}_\mathbb{X} \in \mathcal{B}_1$  since  $\mathbf{1}_\mathbb{X} \in \mathcal{B}_1$  and  $\Pi_\gamma$  is well defined in  $\mathcal{L}(\mathcal{B}_1)$ . Moreover  $\gamma \mapsto \phi_\gamma$  is continuous from  $J$  to  $\mathcal{B}_1$  since  $\Pi_\gamma$  is well defined in  $\mathcal{L}(\mathcal{B}_1)$ , continuous from  $J_0$  to  $\mathcal{L}(\mathcal{B}_0, \mathcal{B}_1)$ , and  $\mathbf{1}_\mathbb{X} \in \mathcal{B}_0$ . Finally  $\gamma \mapsto \phi_\gamma$  is differentiable from  $J$  to  $\mathcal{B}_3$  since  $\Pi_\gamma$  is well defined in  $\mathcal{L}(\mathcal{B}_3)$  and differentiable from  $J_0$  to  $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_3)$  and  $\mathbf{1}_\mathbb{X} \in \mathcal{B}_1$ .  $\square$

**A.3. A counter-example.** Assume that  $(\mathbb{X}, d)$  is a metric space equipped with its Borel  $\sigma$ -algebra. Let  $\mathcal{L}^\infty$  denote the set of bounded functions  $f : \mathbb{X} \rightarrow \mathbb{C}$ , endowed with the supremum norm.

**Proposition A.6.** *Assume that  $P$  is a Markov kernel satisfying the following condition : there exists  $S \in (0, +\infty)$  such that, for every  $x \in \mathbb{X}$ , the support of  $P(x, dy)$  is contained in the ball  $B(x, S)$  centered at  $x$  with radius  $S$ . Assume that  $\xi(y) \rightarrow 0$  when  $d(y, x_0) \rightarrow +\infty$ , where  $x_0$  is some fixed point in  $\mathbb{X}$ . Then, for every  $\gamma \in [0, +\infty)$ , the kernel  $P_\gamma$  continuously acts on  $\mathcal{L}^\infty$  and its spectral radius  $r(\gamma) = r((P_\gamma)|_{\mathcal{L}^\infty})$  satisfies the following*

$$\forall \gamma \in [0, +\infty), \quad r(\gamma) = 1.$$

*Proof.* We clearly have  $r(\gamma) \leq 1$  since  $P_\gamma \leq P$  and  $P$  is Markov. For any  $\beta > 0$ , we obtain with  $f = \mathbf{1}_{[\xi \leq \beta]}$

$$\forall x \in \mathbb{X}, \quad (P_\gamma f)(x) = \int_{[\xi \leq \beta]} e^{-\gamma \xi(y)} P(x, dy) \geq e^{-\gamma \beta} P(x, [\xi \leq \beta]).$$

The set  $[\xi \leq \beta]$  contains  $\mathbb{X} \setminus B(x_0, R)$  for some  $R > 0$  since  $\xi(y) \rightarrow 0$  when  $d(y, x_0) \rightarrow +\infty$ . Thus, for  $d(x, x_0)$  sufficiently large ( $d(x, x_0) > R + S$ ), we have  $P(x, [\xi \leq \beta]) = 1$ , so that

$$\|P_\gamma\|_{\mathcal{L}^\infty} \geq \|P_\gamma f\|_{\mathcal{L}^\infty} \geq e^{-\gamma \beta}.$$

This gives  $\|P_\gamma\|_{\mathcal{L}^\infty} = 1$  when  $\beta \rightarrow 0$ . Similarly we obtain with  $f = \mathbf{1}_{[\xi \leq \beta]}$

$$\begin{aligned} \forall x \in \mathbb{X} \setminus B(x_0, R + 2S), \quad (P_\gamma^2 f)(x) &= \int e^{-\gamma(\xi(y) + \xi(z))} \mathbf{1}_{[\xi \leq \beta]}(z) P(y, dz) P(x, dy) \\ &\geq e^{-\gamma \beta} \int e^{-\gamma \xi(y)} P(y, [\xi \leq \beta]) P(x, dy) \\ &\geq e^{-\gamma \beta} \int_{\mathbb{X} \setminus B(x_0, R+S)} e^{-\gamma \xi(y)} P(y, [\xi \leq \beta]) P(x, dy) \\ &\geq e^{-2\gamma \beta} \end{aligned}$$

and so

$$\forall \beta > 0, \quad \|P_\gamma^2\|_{\mathcal{L}^\infty} \geq \|P_\gamma^2 f\|_{\mathcal{L}^\infty} \geq e^{-2\gamma \beta}.$$

Again this provides  $\|P_\gamma^2\|_{\mathcal{L}^\infty} = 1$  since  $\beta$  can be taken arbitrarily large. Similarly we can prove that  $\|P_\gamma^n\|_{\mathcal{L}^\infty} = 1$  for every  $n \geq 1$ , thus  $r(\gamma) = 1$ .  $\square$

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