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Riesz transform on graphs under subgaussian estimates

Joseph FENEUIL
Institut Fourier, Grenoble

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Abstract

Let Γ be a doubling graph satisfying some pointwise subgaussian estimates of the Markov kernel. We introduce a space $H^1(\Gamma)$ of functions and a space $H^1(T_\Gamma)$ of 1-forms and give various characterizations of them. We prove the H^1 -boundedness of the Riesz transform, from which we deduce the L^p boundedness of the Riesz transform for any $p \in (1, 2)$. Note that in [19, Theorem 1.40], we showed a H_w^1 -boundedness of the Riesz transform under weaker assumptions, but the L^p boundedness was not established.

Keywords: Graphs - Hardy spaces - Riesz transform - subgaussian estimates.

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We use the following notations. $A(x) \lesssim B(x)$ means that there exists C independent of x such that $A(x) \leq C B(x)$ for all x , while $A(x) \simeq B(x)$ means that $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$. The parameters from which the constant is independent will be either obvious from context or recalled. Furthermore, if E, F are Banach spaces, $E \subset F$ means that E is continuously included in F . In the same way, $E = F$ means that the norms are equivalent.

1 Introduction and statement of the results

Let $d \in \mathbb{N}^*$. In the Euclidean case \mathbb{R}^d , the Riesz transforms are the linear operators $\partial_j(-\Delta)^{-\frac{1}{2}}$. A way to define them is to use the Fourier transform \mathcal{F} : for all $f \in \mathcal{S}(\mathbb{R}^d)$ and all $\xi \in \mathbb{R}^d$, one has

$$\mathcal{F}(\partial_j(-\Delta)^{-\frac{1}{2}}f)(\xi) = i \frac{\xi_j}{|\xi|} \mathcal{F}(f)(\xi).$$

The Riesz transforms have a convolution kernel, that is $\partial_j(-\Delta)^{-\frac{1}{2}}f = k_j * f$ where $k_j = c_d \frac{x_j}{|x|^{d+1}}$ is a tempered distribution. A remarkable property of the Riesz transform is that they are L^p bounded for all $p \in (1, +\infty)$ (see [28, Chapter 2, Theorem 1]).

This result have been extended to other settings. Let M be a complete Riemannian manifold, with ∇ the Riemannian gradient and Δ the Beltrami Laplace operator. Assume M is doubling. Under pointwise Gaussian upper estimate of the heat kernel h_t , the Riesz transform $\nabla \Delta^{-\frac{1}{2}}$ is bounded on $L^p(M)$ for all $p \in (1, 2]$ (see [12]). When $p > 2$, the L^p boundedness of the Riesz transform holds under much stronger condition, expressed in term of Poincaré inequalities on balls and of the domination of the gradient of the semigroup in L^q for some $q > p$ (with L^2 Poincaré inequality, see [13]; with L^q Poincaré inequality, see [6]). Similar results were established in the case of graphs (see [26] when $p < 2$, see [4] when $p > 2$).

We are interested now by the limit case $p = 1$. It appears than the Hardy space H^1 is the proper substitute of L^1 when Riesz transforms are involved. In the Euclidean case, $H^1(\mathbb{R}^d)$ can be defined as the space of functions $f \in L^1(\mathbb{R}^d)$ such that $\partial_j(-\Delta)^{-\frac{1}{2}}f \in L^1(\mathbb{R}^d)$ for all $j \in \llbracket 1, d \rrbracket$ (see [18]). Moreover, the Riesz transforms $\partial_j(-\Delta)^{-\frac{1}{2}}$, that are bounded from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, are actually bounded on $H^1(\mathbb{R}^d)$.

This last result, namely the H^1 boundedness of the Riesz transform, have been extended to complete Riemannian manifolds in [3] (completed in [2]), under the only assumption that the space is doubling. In order to do this, Auscher, Mc Intosh and Russ introduced for functions and forms some Hardy spaces defined by using the Laplacian. In [3], when M is a (complete doubling) Riemannian manifold, the authors deduced then a $H^p(M)$ boundedness of the Riesz transform for $p \in (1, 2)$, where the spaces H^p are defined by means of quadratic functionals. Under pointwise Gaussian upper estimates $H^p(M) = L^p(M)$ and thus they recover $L^p(M)$ boundedness of the Riesz transform obtained in [12].

The problem was also considered on graphs Γ . In the same spirit as [3], the present author established in [19] the $H^1 = H_w^1(\Gamma)$ -boundedness of the Riesz transform. In the present paper, we will assume some pointwise subgaussian upper estimates on the Markov kernel. Under this assumption, we define a new Hardy space $H_s^1(\Gamma)$ which $H_w^1(\Gamma)$ is continuously embedded in. Note that $H_w^1(\Gamma) = H_s^1(\Gamma)$ if we assume some pointwise Gaussian estimates on the Markov kernel. We prove again the $H^1 = H_s^1(\Gamma)$ boundedness of the Riesz transform on graphs. However, those new Hardy spaces $H_s^1(\Gamma)$ satisfy an interpolation property that make us able to get the $L^p(\Gamma)$ boundedness of the Riesz transform on graphs. Hence, for $p \in (1, 2)$, we extend the $L^p(\Gamma)$ boundedness of the Riesz transform to a larger class of graphs than the one considered in [26].

In the context of Riemannian manifolds and graphs, under pointwise subgaussian estimates of the heat (or Markov) kernel, quasi-Riesz transforms and Hardy spaces have been studied by Chen in [9]. We make a comparison in subsection 1.5

1.1 The discrete setting

Let Γ be an infinite set and $\mu_{xy} = \mu_{yx} \geq 0$ a symmetric weight on $\Gamma \times \Gamma$. The couple (Γ, μ) induces a (weighted unoriented) graph structure if we define the set of edges by

$$E = \{(x, y) \in \Gamma \times \Gamma, \mu_{xy} > 0\}.$$

We call then x and y neighbors (or $x \sim y$) if and only if $(x, y) \in E$.

We will assume that the graph is locally uniformly finite, that is there exists $M_0 \in \mathbb{N}$ such that for all $x \in \Gamma$,

$$\#\{y \in \Gamma, y \sim x\} \leq M_0. \tag{1}$$

In other words, the number of neighbors of a vertex is uniformly bounded.

We define the weight $m(x)$ of a vertex $x \in \Gamma$ by $m(x) = \sum_{x \sim y} \mu_{xy}$.

More generally, the volume (or measure) of a subset $E \subset \Gamma$ is defined as $m(E) := \sum_{x \in E} m(x)$.

We define now the $L^p(\Gamma)$ spaces. For all $1 \leq p < +\infty$, we say that a function f on Γ belongs to $L^p(\Gamma, m)$ (or $L^p(\Gamma)$) if

$$\|f\|_p := \left(\sum_{x \in \Gamma} |f(x)|^p m(x) \right)^{\frac{1}{p}} < +\infty,$$

while $L^\infty(\Gamma)$ is the space of functions satisfying

$$\|f\|_\infty := \sup_{x \in \Gamma} |f(x)| < +\infty.$$

Let us define for all $x, y \in \Gamma$ the discrete-time reversible Markov kernel p associated with the measure m by $p(x, y) = \frac{\mu_{xy}}{m(x)m(y)}$. The discrete kernel $p_k(x, y)$ is then defined recursively for all $k \geq 0$ by

$$\begin{cases} p_0(x, y) = \frac{\delta(x, y)}{m(y)} \\ p_{k+1}(x, y) = \sum_{z \in \Gamma} p(x, z) p_k(z, y) m(z). \end{cases} \quad (2)$$

Notice that for all $k \geq 1$, we have

$$\|p_k(x, \cdot)\|_{L^1(\Gamma)} = \sum_{y \in \Gamma} p_k(x, y) m(y) = \sum_{d(x, y) \leq k} p_k(x, y) m(y) = 1 \quad \forall x \in \Gamma, \quad (3)$$

and that the kernel is symmetric:

$$p_k(x, y) = p_k(y, x) \quad \forall x, y \in \Gamma. \quad (4)$$

For all functions f on Γ , we define P as the operator with kernel p , i.e.

$$Pf(x) = \sum_{y \in \Gamma} p(x, y) f(y) m(y) \quad \forall x \in \Gamma. \quad (5)$$

It is easily checked that P^k is the operator with kernel p_k .

Since $p(x, y) \geq 0$ and (3) holds, one has, for all $p \in [1, +\infty]$,

$$\|P\|_{p \rightarrow p} \leq 1. \quad (6)$$

Remark 1.1. Let $1 \leq p < +\infty$. Since, for all $k \geq 0$, $\|P^k\|_{p \rightarrow p} \leq 1$, the operators $(I - P)^\beta$ and $(I + P)^\beta$ are L^p -bounded for all $\beta \geq 0$ (see [15]).

We define a nonnegative Laplacian on Γ by $\Delta = I - P$. One has then

$$\begin{aligned} \langle \Delta f, f \rangle_{L^2(\Gamma)} &= \sum_{x, y \in \Gamma} p(x, y) (f(x) - f(y)) f(x) m(x) m(y) \\ &= \frac{1}{2} \sum_{x, y \in \Gamma} p(x, y) |f(x) - f(y)|^2 m(x) m(y), \end{aligned} \quad (7)$$

where we use (3) for the first equality and (4) for the second one. The last calculus proves that the following operator

$$\nabla f(x) = \left(\frac{1}{2} \sum_{y \in \Gamma} p(x, y) |f(y) - f(x)|^2 m(y) \right)^{\frac{1}{2}},$$

called ‘‘length of the gradient’’ (and the definition of which is taken from [14]), satisfies

$$\|\nabla f\|_{L^2(\Gamma)}^2 = \langle \Delta f, f \rangle_{L^2(\Gamma)} = \|\Delta^{\frac{1}{2}} f\|_{L^2(\Gamma)}^2. \quad (8)$$

We recall now definitions of 1-forms on graphs and their first properties (based on [19]). We define, for all $x \in \Gamma$, the set $T_x = \{(x, y) \in \Gamma^2, y \sim x\}$ and for all set $E \subset \Gamma$,

$$T_E = \bigcup_{x \in E} T_x = \{(x, y) \in E \times \Gamma, y \sim x\}.$$

Definition 1.2. If $x \in \Gamma$, we define, for all functions F_x defined on T_x the norm

$$|F_x|_{T_x} = \left(\frac{1}{2} \sum_{y \sim x} p(x, y) m(y) |F_x(x, y)|^2 \right)^{\frac{1}{2}}.$$

Moreover, a function $F : T_\Gamma \rightarrow \mathbb{R}$ belongs to $L^p(T_\Gamma)$ if

(i) F is antisymmetric, that is $F(x, y) = -F(y, x)$ for all $x \sim y$,

(ii) $\|F\|_{L^p(T_\Gamma)} := \|x \mapsto |F(x, \cdot)|_{T_x}\|_{L^p(\Gamma)} < +\infty$.

The Hilbert space $L^2(T_\Gamma)$ is outfitted with the inner product $\langle \cdot, \cdot \rangle$ defined as

$$\langle F, G \rangle = \frac{1}{2} \sum_{x, y \in \Gamma} p(x, y) F(x, y) G(x, y) m(x) m(y).$$

Definition 1.3. Let f a function on Γ and F an antisymmetric function on T_Γ . Define the operators d and d^* by

$$df(x, y) := f(x) - f(y) \quad \forall (x, y) \in T_\Gamma$$

and

$$d^*F(x) := \sum_{y \sim x} p(x, y) F(x, y) m(y) \quad \forall x \in \Gamma.$$

Remark 1.4. It is plain to see that $d^*d = \Delta$ and $|df(x, \cdot)|_{T_x} = \nabla f(x)$.

As the notation d^* suggests, d^* is the adjoint of d , that is for all $f \in L^2(\Gamma)$ and $G \in L^2(T_\Gamma)$,

$$\langle df, G \rangle_{L^2(T_\Gamma)} = \langle f, d^*G \rangle_{L^2(\Gamma)}. \quad (9)$$

The proof of this fact can be found in [4, Section 8.1].

We introduce a subspace of $L^2(T_\Gamma)$, called $H^2(T_\Gamma)$, defined as the closure in $L^2(T_\Gamma)$ of

$$E^2(T_\Gamma) := \{F \in L^2(T_\Gamma), \exists f \in L^2(\Gamma) : F = df\}.$$

Notice that $d\Delta^{-1}d^* = Id_{E^2(T_\Gamma)}$ (see [19]). The functional $d\Delta^{-1}d^*$ can be extended to a bounded operator on $H^2(T_\Gamma)$ and

$$d\Delta^{-1}d^* = Id_{H^2(T_\Gamma)}. \quad (10)$$

Let us recall Proposition 1.32 in [19].

Proposition 1.5. For all $p \in [1, +\infty]$, the operator d^* is bounded from $L^p(T_\Gamma)$ to $L^p(\Gamma)$.

The operator $d\Delta^{-\frac{1}{2}}$ is an isometry from $L^2(\Gamma)$ to $H^2(T_\Gamma)$, and the operator $\Delta^{-\frac{1}{2}}d^*$ is an isometry from $H^2(T_\Gamma)$ to $L^2(\Gamma)$.

1.2 Assumptions on the graph

Definition 1.6. We say that (Γ, μ) satisfies (LB) if there exists $\epsilon = \epsilon_{LB} > 0$ such that

$$\frac{\mu_{xx}}{m(x)} = p(x, x) m(x) \geq \epsilon \quad \forall x \in \Gamma. \quad (\text{LB})$$

Let us recall a result in [15].

Lemma 1.7. The condition (LB) implies that -1 does not belong to the L^2 -spectrum of P , which implies in turn the analyticity of P in $L^p(\Gamma)$, $1 < p < +\infty$. Namely,

$$\|(I - P)P^n\|_{p \rightarrow p} \lesssim \frac{1}{n}.$$

We will need some assumptions that depends of the metric.

Definition 1.8. Let X be a nonempty set and $\rho : X \rightarrow [0, +\infty)$. Say that ρ is a quasidistance if, and only if there exists $C > 0$ such that, for all $x, y, z \in X$:

(i) $\rho(x, y) = 0$ if, and only if, $x = y$,

(ii) $\rho(x, y) = \rho(y, x)$,

(iii) we have the inequality

$$\rho(x, y) \leq C (\rho(x, z) + \rho(z, y)). \quad (11)$$

Throughout all the paper, whenever ρ is a quasidistance on a graph Γ , we will assume that ρ take values in \mathbb{N} and that $\rho(x, y) = 1$ if $x \sim y$ with $x \neq y$.

Definition 1.9. Let ρ be a quasidistance on Γ .

(i) Define C_ρ as the infimum of the constants $C > 0$ such that (11) holds. Note that (11) is satisfied with C_ρ .

(ii) For $x \in \Gamma$ and $k \in \mathbb{R}_+^*$, the ball $B(x, k)$ (or $B_\rho(x, k)$) is defined by

$$B(x, k) = \{y \in \Gamma, \rho(x, y) < k\}.$$

Conversely, B is a ball (for ρ) if there exists $(x_B, k_B) \in \Gamma \times \mathbb{N}^*$ such that $B = B(x_B, k_B)$. The radius of B is defined as $\sup\{\rho(y, B^c), y \in B\}$. Note that the radius r of the ball $B(x, k)$ satisfies $k \leq r \leq 2C_\rho k$.

(iii) If $j \geq 1$, $C_j(x, k)$ denotes the annulus $B(x, C_\rho 2^{j+1}k) \setminus B(x, C_\rho 2^j k)$. Moreover, $C_0(x, k)$ denotes $B(x, 2C_\rho k)$

(iv) We will use the notation $V_\rho(x, k)$ or only $V(x, k)$ for $m(B(x, k))$.

Definition 1.10. Let ρ be a quasidistance. We say that (Γ, μ, ρ) satisfies (DV) if the measure m is doubling with respect to the quasidistance ρ , that is if there exists $C_{dv} > 0$ such that

$$V(x, 2k) \leq C_{dv} V(x, k) \quad \forall x \in \Gamma, \forall k \in \mathbb{R}_+^*; \quad (DV)$$

Proposition 1.11. Let (Γ, μ, ρ) satisfying (DV). Then there exists $d > 0$ such that

$$V(x, \lambda r) \lesssim \lambda^d V(x, r) \quad \forall x \in \Gamma, r > 0 \text{ and } \lambda \geq 1. \quad (12)$$

Definition 1.12. Let ρ be a quasidistance. We say that (Γ, μ, ρ) satisfies (UE) if there exist three constants $c_{ue}, C_{ue} > 0$ and $\eta \in (0, 1]$ such that p_k satisfies the subgaussian estimates

$$p_{k-1}(x, y) \lesssim \frac{C_{ue}}{V(x, k)} \exp \left[-c_{ue} \left(\frac{\rho(x, y)}{k} \right)^\eta \right], \forall x, y \in \Gamma, \forall k \in \mathbb{N}^*. \quad (UE)$$

Remark 1.13. Notice that when $\rho = d^2$, assumption (UE) is the pointwise Gaussian upper estimate of the Markov kernel. It corresponds then to the estimate (on the Markov kernel) made in [26] or [20]. In particular, when $\rho = d^2$, assumption (UE) is satisfied when Γ is the Cayley graph of finitely generated groups. Other graphs satisfying (UE) are presented in section 5.

1.3 Definition of Hardy spaces

We define two kinds of Hardy spaces. The first one is defined using molecules.

Definition 1.14. Let $M \in \mathbb{N}^*$, $p \in (1, \infty]$ and $\epsilon \in (0, +\infty)$. A function $a \in L^p(\Gamma)$ is called a (M, p, ϵ) -molecule if there exist $x \in \Gamma$, $s \in \mathbb{R}_+^*$ and a function $b \in L^p(\Gamma)$ such that

$$(i) \quad a = [I - (I + s\Delta)^{-1}]^M b,$$

$$(ii) \quad \|b\|_{L^p(C_j(x, s))} \leq 2^{-j\epsilon} V(x, 2^j s)^{\frac{1}{p}-1}, \quad \forall j \geq 0.$$

We say that an (M, p, ϵ) -molecule a is associated with a vertex x and a real s when we want to refer to x and s given by the definition.

Definition 1.15. Let $M \in \mathbb{N}^*$, $p \in (1, \infty]$ and $\epsilon \in (0, +\infty)$. A function $a \in L^p(T_\Gamma)$ is called an $(M - \frac{1}{2}, p, \epsilon)$ -molecule if there exist $x \in \Gamma$, $s \in \mathbb{R}_+^*$ and a function $b \in L^p(\Gamma)$ such that

$$(i) \quad a = s^{M-\frac{1}{2}} d\Delta^{M-1} (I + s\Delta)^{-M+\frac{1}{2}} b;$$

$$(ii) \quad \|b\|_{L^p(C_j(x, s))} \leq 2^{-j\epsilon} V(x, 2^j s)^{\frac{1}{p}-1} \text{ for all } j \geq 0.$$

Remark 1.16. In the particular case $p = 2$, these definitions of molecules can be found in [19].

Remark 1.17. As will be seen in Propositions 2.6 and 2.9 below, when a is a molecule occurring in Definition 1.14 or in Definition 1.15, one has $\|a\|_{L^1} \lesssim 1$.

Definition 1.18. Let $M \in \mathbb{N}^*$, $p \in (1, \infty]$ and $\epsilon \in (0, +\infty)$. We say that a function f on Γ belongs to $H_{mol, M, p, \epsilon}^1(\Gamma)$ if f admits a molecular (M, p, ϵ) -representation, that is if there exist a sequence $(\lambda_i)_{i \in \mathbb{N}} \in \ell^1$ and a sequence $(a_i)_{i \in \mathbb{N}}$ of (M, p, ϵ) -molecules such that

$$f = \sum_{i=0}^{\infty} \lambda_i a_i \quad (13)$$

where the convergence of the series to f holds pointwise. Define, for all $f \in H_{mol, M, p, \epsilon}^1(\Gamma)$,

$$\|f\|_{H_{mol, M, p, \epsilon}^1} = \inf \left\{ \sum_{j=0}^{\infty} |\lambda_j|, \sum_{j=0}^{\infty} \lambda_j a_j, \text{ is a molecular } (M, p, \epsilon)\text{-representation of } f \right\}.$$

Definition 1.19. Let $M \in \mathbb{N}^*$, $p \in (1, \infty]$ and $\epsilon \in (0, +\infty)$. We say that a function f on T_Γ belongs to $H_{mol, M - \frac{1}{2}, p, \epsilon}^1(T_\Gamma)$ if f admits a molecular $(M - \frac{1}{2}, p, \epsilon)$ -representation, that is if there exist a sequence $(\lambda_i)_{i \in \mathbb{N}} \in \ell^1$ and a sequence $(a_i)_{i \in \mathbb{N}}$ of $(M - \frac{1}{2}, p, \epsilon)$ -molecules such that

$$f = \sum_{i=0}^{\infty} \lambda_i a_i \quad (14)$$

where the convergence of the series to f holds pointwise. Define, for all $f \in H_{mol, M - \frac{1}{2}, p, \epsilon}^1(T_\Gamma)$,

$$\|f\|_{H_{mol, M - \frac{1}{2}, p, \epsilon}^1} = \inf \left\{ \sum_{j=0}^{\infty} |\lambda_j|, \sum_{j=0}^{\infty} \lambda_j a_j, \text{ is a molecular } (M - \frac{1}{2}, p, \epsilon)\text{-representation of } f \right\}.$$

Proposition 1.20. Let $M \in \mathbb{N}^*$, $p \in (1, \infty]$ and $\epsilon \in (0, +\infty)$. Then:

- (i) the map $f \mapsto \|f\|_{H_{mol, M, p, \epsilon}^1}$ (resp. $f \mapsto \|f\|_{H_{mol, M - \frac{1}{2}, p, \epsilon}^1}$) is a norm on the space $H_{mol, M, p, \epsilon}^1$ (resp. $H_{mol, M - \frac{1}{2}, p, \epsilon}^1$),
- (ii) the space $H_{mol, M, p, \epsilon}^1(\Gamma)$ (resp. $H_{mol, M - \frac{1}{2}, p, \epsilon}^1(T_\Gamma)$) is complete and continuously embedded in $L^1(\Gamma)$ (resp. $L^1(T_\Gamma)$).

Proof: Remark 1.17 shows that, if f is in H_{mol}^1 the series (13) (or (14)) converges in L^1 , and therefore converges to f in L^1 . This yields (i) and the embeddings in (ii). Moreover, a normed linear vector space X is complete if and only if it has the property

$$\sum_{j=0}^{\infty} \|f_j\|_X < +\infty \implies \sum_{j=0}^{\infty} f_j \text{ converges in } X.$$

Using this criterion, the completeness of the Hardy spaces under consideration is a straightforward consequence of the fact that $\|a\|_{L^1} \lesssim 1$ whenever a is a molecule. See also the argument for the completeness of H_L^1 in [23], p. 48. \square

The second kind of Hardy spaces is defined via quadratic functionals.

Definition 1.21. Define, for $\beta > 0$, the quadratic functional L_β on $L^2(\Gamma)$ by

$$L_\beta f(x) = \left(\sum_{(y, k) \in \gamma(x)} \frac{k^{2\beta-1}}{V(x, k)} |\Delta^\beta P^{k-1} f(y)|^2 m(y) \right)^{\frac{1}{2}}$$

where $\gamma(x) = \{(y, k) \in \Gamma \times \mathbb{N}^*, \rho(x, y) < k\}$.

Definition 1.22. The space $E_{quad, \beta}^1(\Gamma)$ is defined for all $\beta > 0$ by

$$E_{quad, \beta}^1(\Gamma) := \{f \in L^2(\Gamma), \|L_\beta f\|_{L^1} < +\infty\}.$$

It is outfitted with the norm

$$\|f\|_{E_{quad, \beta}^1} := \|L_\beta f\|_{L^1}.$$

The space $E_{quad,\beta}^1(T_\Gamma)$ is defined from $E_{quad,\beta}^1$ as

$$E_{quad,\beta}^1(T_\Gamma) := \left\{ f \in E^2(T_\Gamma), \Delta^{-\frac{1}{2}} d^* f \in E_{quad,\beta}^1(\Gamma) \right\}.$$

It is outfitted with the norm

$$\|f\|_{H_{quad,\beta}^1} := \|L_\beta \Delta^{-\frac{1}{2}} d^* f\|_{L^1}.$$

Remark 1.23. The fact that the map $f \mapsto \|f\|_{H_{quad,\beta}^1}$ is a norm is proven in [19, Remark 1.20].

1.4 Main results

In all the following statements, let (Γ, μ, ρ) be a weighted graph outfitted with a quasidistance and satisfying (LB), (DV) and (UE).

Definition 1.24. Let $(E, \|\cdot\|_E)$ a normed vector space and $(G, \|\cdot\|_G)$ a Banach space such that $E \subset G$. A Banach space $(F, \|\cdot\|_F)$ is the completion of E in G if

(i) we have the continuous embeddings

$$E \subset F \subset G$$

,

(ii) the set E is dense in F ,

(iii) for all $e \in E$, $\|e\|_E = \|e\|_F$.

Remark 1.25. The completion F of E always exists. However, F is defined in an abstract space. Even if E is continuously embedding in a Banach G , F cannot always be identified to a subspace of G .

Theorem 1.26. Let $\beta > 0$. The completion $H_{quad,\beta}^1(\Gamma)$ of $E_{quad,\beta}^1(\Gamma)$ in $L^1(\Gamma)$ exists. Moreover, if $M \in \mathbb{N}^*$, $p \in (1, 2]$ and $\epsilon \in (0, +\infty)$, then the spaces $H_{mol,M,p,\epsilon}^1(\Gamma)$ and $H_{quad,\beta}^1(\Gamma)$ coincide. More precisely, we have

$$E_{quad,\beta}^1(\Gamma) = H_{mol,M,p,\epsilon}^1(\Gamma) \cap L^2(\Gamma).$$

Once the equality $H_{mol,M,p,\epsilon}^1(\Gamma) = H_{quad,\beta}^1(\Gamma)$ is established, this space will be denoted by $H^1(\Gamma)$.

Theorem 1.27. Let $\beta > 0$. The completion $H_{quad,\beta}^1(T_\Gamma)$ of $E_{quad,\beta}^1(T_\Gamma)$ in $L^1(T_\Gamma)$ exists. Moreover, if $M \in \mathbb{N}^*$, $p \in (1, 2]$ and $\epsilon \in (0, +\infty)$, then the spaces $H_{mol,M-\frac{1}{2},p,\epsilon}^1(T_\Gamma)$ and $H_{quad,\beta}^1(T_\Gamma)$ coincide. More precisely, we have

$$E_{quad,\beta}^1(T_\Gamma) = H_{mol,M-\frac{1}{2},p,\epsilon}^1(T_\Gamma) \cap L^2(T_\Gamma).$$

Again, the space $H_{mol,M-\frac{1}{2},p,\epsilon}^1(T_\Gamma) = H_{quad,\beta}^1(T_\Gamma)$ will be denoted by $H^1(T_\Gamma)$.

Remark 1.28. Note that the previous theorem provides that $H_{mol,M,p,\epsilon}^1(\Gamma)$ (resp. $H_{mol,M-\frac{1}{2},p,\epsilon}^1(T_\Gamma)$) is independent of $M \in \mathbb{N}^*$, $\epsilon \in (0, +\infty)$ and $p \in (1, 2]$.

Theorem 1.29. If T is linear operator acting on $L^2(\Gamma)$ such that

1. T is $L^2(\Gamma)$ bounded, that is there exists $C_2 > 0$ such that

$$\|Tf\|_{L^2} \leq C_2 \|f\|_{L^2} \quad \forall f \in L^2(\Gamma),$$

2. T is bounded from $H^1(\Gamma)$ to $L^1(\Gamma)$, that is there exists $C_1 > 0$ such that

$$\|Tf\|_{L^2} \leq C_1 \|f\|_{L^2} \quad \forall f \in H^1(\Gamma) \cap L^2(\Gamma).$$

Then for all $p \in (1, 2]$,

$$\|Tf\|_p \leq C_2^\theta C_1^{1-\theta} \|f\|_p \quad \forall f \in L^p(\Gamma) \cap L^2(\Gamma)$$

where θ is given by

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{1}.$$

Corollary 1.30. *The Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded from $H^1(\Gamma)$ to $H^1(T_\Gamma)$. As a consequence the Riesz transform $\nabla\Delta^{-\frac{1}{2}}$ is bounded from $H^1(\Gamma)$ to $L^1(\Gamma)$, and thus is L^p -bounded for all $p \in (1, 2]$.*

Proof: Theorems 1.26 and 1.27 yields,

$$\|d\Delta^{-\frac{1}{2}}f\|_{H^1(T_\Gamma)} \simeq \|d\Delta^{-\frac{1}{2}}f\|_{H^1_{quad,1}(T_\Gamma)} = \|\Delta^{-\frac{1}{2}}d^*d\Delta^{-\frac{1}{2}}f\|_{H^1_{quad,1}(\Gamma)} = \|f\|_{H^1_{quad,1}(\Gamma)} \simeq \|f\|_{H^1(\Gamma)}.$$

Therefore, $d\Delta^{-\frac{1}{2}}$ is H^1 -bounded. Moreover, by Proposition 1.20,

$$\|\nabla\Delta^{-\frac{1}{2}}f\|_{L^1(\Gamma)} = \|d\Delta^{-\frac{1}{2}}f\|_{L^1(T_\Gamma)} \lesssim \|d\Delta^{-\frac{1}{2}}f\|_{H^1(T_\Gamma)} \lesssim \|f\|_{H^1(\Gamma)}. \quad (15)$$

Define now for any function ϕ on T_Γ the linear operator

$$\nabla_\phi f(x) = \sum_{y \in \Gamma} p(x, y) df(x, y) \phi(x, y) m(y).$$

Then the boundedness (15) yields the estimate

$$\|\nabla_\phi f\|_{L^1} \lesssim \|f\|_{H^1} \sup_{x \in \Gamma} |\phi(x, \cdot)|_{T_x}.$$

Moreover, since $\|\nabla\Delta^{-\frac{1}{2}}f\|_{L^2} = \|f\|_{L^2}$ (cf (8)), one has

$$\|\nabla_\phi f\|_{L^2} \lesssim \|f\|_{L^2} \sup_{x \in \Gamma} |\phi(x, \cdot)|_{T_x}.$$

Define now $\phi_f(x, y) = \frac{df(x, y)}{\nabla f(x)}$. Note that $\sup_{x \in \Gamma} |\phi_f(x, \cdot)|_{T_x} = 1$. With Theorem 1.29, one has then

$$\|\nabla\Delta^{-\frac{1}{2}}f\|_{L^p} = \|\nabla_{\phi_f}\Delta^{-\frac{1}{2}}f\|_{L^p} \lesssim \|f\|_{L^p}$$

□

1.5 Comparison with previous results

For all $x, y \in \Gamma$, let $d(x, y)$ be length of the shortest path linking x to y , where x_0, \dots, x_n is a path of length n if for any $i \in \llbracket 1, n \rrbracket$, $x_{i-1} \simeq x_i$.

Russ established in [26] the following result:

Theorem 1.31. *Let (Γ, μ, d^2) satisfying (LB), (DV) and (UE).*

Then the Riesz transform is of weak type (1, 1), that is

$$\sup_{\lambda > 0} \lambda m(\{\nabla\Delta^{-\frac{1}{2}}f > \lambda\}) \lesssim \|f\|_{L^1} \quad \forall f \in L^1(\Gamma)$$

and of strong type (p, p) (or L^p bounded) for all $p \in (1, 2]$.

In the present paper, as in [26], we established the L^p boundedness of the Riesz transform for $p \in (1, 2]$. However, the class of graphs that satisfies our assumptions (Corollary 1.30) is strictly bigger than the one satisfying the assumptions of Theorem 1.31. Indeed, fractal-type graphs (see Section 5) such as the Sierpinski carpet fit into our theory but not into the one of [26]. Contrary to [26], we do not prove the weak type (1, 1) of the Riesz transform. It was replaced in the present paper by a $H^1(\Gamma)$ boundedness of the Riesz transform.

In [9], Chen stated

Theorem 1.32. *Let (Γ, μ) be a graph satisfying (LB) and the local doubling property*

$$m(x) \simeq m(y) \quad \forall x \sim y.$$

Then, for all $\alpha \in (0, \frac{1}{2})$, the quasi-Riesz transforms $\nabla\Delta^{-\alpha}$ are L^p bounded for all $p \in (1, 2)$.

Let $\beta > 0$ such that (Γ, μ, d^β) satisfy (LB), (DV) and (UE). Then, for all $\alpha \in (0, \frac{1}{2})$, the quasi-Riesz transforms $\nabla\Delta^{-\alpha}$ is of weak type (1, 1), and thus of strong type (p, p) for all $p \in (1, 2]$.

Notice that the L^p boundedness of the Riesz transform $\nabla\Delta^{-\frac{1}{2}}$ implies the L^p boundedness of all quasi-Riesz transforms $\nabla\Delta^{-\frac{\alpha}{2}}$, $\alpha \in (0, \frac{1}{2})$. Therefore, Corollary 1.30 and Theorem 1.32 have a non-empty intersection. Chen proved more general results than us on quasi-Riesz transforms while we succeeded, under subgaussian estimates, to prove the boundedness of the “complete” Riesz transform.

In the context of complete Riemannian manifolds M with the doubling property, Chen also introduced in [9] some results on Hardy spaces “adapted” to some elliptic operators satisfying pointwise subgaussian estimates. She proved $H_{L,s}^1(M) := H_{L,mol}^1(M) = H_{L,quad}^1(M)$, i.e. we can define a Hardy space $H_{L,s}^1(M)$ that have two equivalent characterizations. The first one, $H_{L,mol}^1(M)$ is the space of functions in L^1 that have a molecular decomposition while $H_{L,quad}^1(M)$ is defined by using quadratic functionals of Lusin type. The space $H_{\Delta,s}^1(M)$ is bigger than the one introduced in [3] and, for $p \in (1, 2)$, the “interpolated” spaces $H_{\Delta,s}^p(M)$ are equal to $L^p(M)$.

The main difference between the present paper and [9] is that Chen did not make the same work on graphs and for 1-forms. Working on graphs brings some difficulties that do not appear, for examples, on Riemannian manifolds. Moreover, the present work extends also the notion of subgaussian estimates met in [9]. Besides, by considering all the functionals L_β for $\beta > 0$, we have a difficulty that was not considered in [9], where only the functional L_1 was introduced.

Hardy spaces under subgaussian estimates of the heat kernel was defined previously in [25]. Let X be a space of homogeneous space, L be a injective, non-negative, self-adjoint operator in $L^2(X)$ such that the semigroup generated by $-L$ satisfies Gaffney estimates of arbitrary order m . As in [9], the authors of [25] defined Hardy spaces $H_L^p(X)$ for $p \in [1, 2]$. They showed that $L^p(X) = H^p(X)$. As an application, they proved $H_L^1(X)$ and $L^p(X)$ boundedness of some spectral multipliers of L .

2 Off-diagonal estimates

2.1 Gaffney estimates, first results

Definition 2.1. Let (X, m, ρ) be a measured space equipped with a quasidistance and $p \in [1, +\infty]$. We say that a family of operators $(A_k)_{k \in I}$, $I = \mathbb{N}^*$ or \mathbb{R}_+^* , satisfies $L^p(X, m, \rho)$ (or L^p) Gaffney estimates if there exist $C, c, \eta > 0$ such that, for any sets $E, F \subset X$ and any function $f \in L^p(X, m)$,

$$\|A_k[f\mathbf{1}_F]\|_{L^p(E)} \leq C \exp\left[-c\left(\frac{\rho(E, F)}{k}\right)^\eta\right] \|f\|_p. \quad (16)$$

It is plain to observe that (16) is equivalent to

$$\|A_k(f)\|_{L^p(E)} \leq C \exp\left[-c\left(\frac{\rho(E, F)}{k}\right)^\eta\right] \|f\|_p$$

whenever f is supported in F .

Arguing as in [22, Lemma 2.3], one can establish the following composition property about Gaffney estimates:

Proposition 2.2. If A_s and B_t satisfy L^p Gaffney estimates, then there exist $C, c, \eta > 0$ such that for all subsets $E, F \subset \Gamma$ and all functions $f \in L^p$,

$$\|A_s B_t(f\mathbf{1}_F)\|_{L^p(E)} \leq C \exp\left(-c\left[\frac{\rho(E, F)}{s+t}\right]^\eta\right) \|f\|_{L^p} \quad (17)$$

In particular, $(A_s B_s)_{s \in I}$ satisfies Davies-Gaffney estimates. More precisely, if η_A and η_B are the constants involved in (16) respectively for A_s and B_t , then the constant η that occurs in (17) can be chosen equal to $\min\{\eta_A, \eta_B\}$.

Proposition 2.3. Let (Γ, μ, ρ) be a weighted graph satisfying (DV) and (UE). Then $(P^{k-1})_{k \in \mathbb{N}^*}$ satisfies L^p Gaffney estimates for any $p \in [1, +\infty]$.

Moreover the coefficient η that appears in the Gaffney estimates is the same as the one given by (UE).

Proof: We will prove the cases $p = 1$ and $p = +\infty$. The conclusion can be then deduced from these endpoint estimates by interpolation. Moreover, since P^k is uniformly bounded on $L^1(\Gamma)$ and $L^\infty(\Gamma)$, we can assume without loss of generality that $\rho(E, F) \geq k$.

We begin with $p = 1$. Let $E, F \subset \Gamma$ and $f \in L^1(\Gamma)$. Then, for all $k \geq 1$

$$\begin{aligned}
\|P^{k-1}(f\mathbf{1}_F)\|_{L^1(E)} &= \sum_{x \in E} m(x) \left| \sum_{z \in F} p_{k-1}(x, z) f(z) m(z) \right| \\
&\leq \sum_{z \in F} |f(z)| m(z) \sum_{x \in E} p_{k-1}(x, z) m(x) \\
&\lesssim \sum_{z \in F} |f(z)| m(z) \sum_{x \in E} \frac{m(x)}{V(z, k)} \exp\left(-c \left(\frac{\rho(x, z)}{k}\right)^\eta\right) \\
&\leq \sum_{z \in F} |f(z)| m(z) \sum_{j \geq 0} \sum_{2^j \rho(z, E) \leq \rho(x, z) < 2^{j+1} \rho(z, E)} \frac{m(x)}{V(z, k)} \exp\left(-c \left(\frac{2^j \rho(z, E)}{k}\right)^\eta\right) \\
&\leq \sum_{z \in F} |f(z)| m(z) \sum_{j \geq 0} \frac{V(z, 2^{j+1} \rho(z, E))}{V(z, k)} \exp\left(-c \left(\frac{2^j \rho(z, E)}{k}\right)^\eta\right) \\
&\lesssim \sum_{z \in F} |f(z)| m(z) \sum_{j \geq 0} \left(1 + \frac{2^j \rho(z, E)}{k}\right)^d \exp\left(-c \left(\frac{2^j \rho(z, E)}{k}\right)^\eta\right) \\
&\lesssim \sum_{z \in F} |f(z)| m(z) \sum_{j \geq 0} \exp\left(-\frac{c}{2} \left(\frac{2^j \rho(z, E)}{k}\right)^\eta\right) \\
&\lesssim \|f\|_{L^1} \sum_{l \geq 1} \left[\exp\left(-\frac{c}{2} \left(\frac{\rho(E, F)}{k}\right)^\eta\right) \right]^l \\
&\lesssim \|f\|_{L^1} \frac{1}{1 - \exp(-\frac{c}{2})} \exp\left(-\frac{c}{2} \left(\frac{\rho(E, F)}{k}\right)^\eta\right) \\
&\lesssim \|f\|_{L^1} \exp\left(-\frac{c}{2} \left(\frac{\rho(E, F)}{k}\right)^\eta\right),
\end{aligned}$$

where the third line holds thanks to (UE), the sixth one is a consequence of (12) and the last but one because $\rho(E, F) \geq k$.

We turn to the case $p = +\infty$. One has for all $x \in E$,

$$\begin{aligned}
|P^{k-1}(f\mathbf{1}_F)(x)| &\lesssim \frac{1}{V(x, k)} \sum_{z \in F} |f(z)| m(z) \exp\left(-c \left(\frac{\rho(x, z)}{k}\right)^\eta\right) \\
&\leq \|f\|_{L^\infty} \sum_{j \geq 0} \sum_{2^j \rho(z, E) \leq \rho(x, z) < 2^{j+1} \rho(z, E)} \frac{m(z)}{V(x, k)} \exp\left(-c \left(\frac{2^j \rho(z, E)}{k}\right)^\eta\right) \\
&\lesssim \|f\|_{L^\infty} \exp\left(-\frac{c}{2} \left(\frac{\rho(E, F)}{k}\right)^\eta\right),
\end{aligned}$$

where the first line holds because of (UE) and the last line is obtained as in the case $p = 1$. \square

Proposition 2.4. *Let (Γ, μ, ρ) be a weighted graph satisfying (LB), (DV) and (UE). Let $m \in \mathbb{N}$. Then the family $(k^m \Delta^m P^{k-1})_{k \in \mathbb{N}^*}$ satisfies L^p Gaffney estimates for any $p \in (1, +\infty)$. Moreover the coefficient η that appears in the Gaffney estimates is the same as the one given by (UE).*

Proof: The proof is similar to the one of Theorem 1.2 in [16]. We give it for completeness. First, with Proposition 2.2, we only need to prove the case $m = 1$. Then notice that we have for any $k \in \mathbb{N}$ the following operator identity (see (8) in [16]),

$$I - P = \sum_{l=0}^n 2^{-l-1} (I - P^{2^l})^2 + 2^{-n-1} (I - P^{2^{n+1}}).$$

As a consequence, one has

$$\begin{aligned}
&\|(I - P)P^{k-1}(f\mathbf{1}_F)\|_{L^p(E)} \\
&\leq \sum_{l=0}^n 2^{-l-1} \|(I - P^{2^l})^2 P^{k-1}(f\mathbf{1}_F)\|_{L^p(E)} + 2^{-n-1} \|(I - P^{2^{n+1}})P^{k-1}(f\mathbf{1}_F)\|_{L^p(E)}.
\end{aligned} \tag{18}$$

When $2^n \leq k$, Proposition 2.3 yields

$$\begin{aligned} \|(I - P^{2^{n+1}})P^{k-1}(f\mathbf{1}_F)\|_{L^p(E)} &\leq \|P^{k-1}(f\mathbf{1}_F)\|_{L^p(E)} + \|P^{k-1+2^{n+1}}(f\mathbf{1}_F)\|_{L^p(E)} \\ &\lesssim \exp\left[-c\left(\frac{\rho(E,F)}{3k}\right)^\eta\right]. \end{aligned} \quad (19)$$

Also, Lemma 2.1 in [16] (which is only a consequence of the analyticity of P in L^p) implies

$$\begin{aligned} \sum_{l=0}^n 2^{-l-1} \|(I - P^{2^l})^2 P^{k-1}(f\mathbf{1}_F)\|_{L^p(E)} &\leq \sum_{l=0}^n 2^{-l-1} \|(I - P^{2^l})^2 P^{k-1}\|_{p \rightarrow p} \|f\|_p \\ &\lesssim \sum_{l=0}^n 2^{-l-1} \left(\frac{2^l}{k}\right)^2 \|f\|_p \\ &\leq \frac{2^n}{k^2} \|f\|_p. \end{aligned} \quad (20)$$

In order to end the proof, we only need to choose the right n . If $k \exp\left(-c\left[\frac{\rho(E,F)}{5k}\right]^\eta\right) \geq 1$, we choose $n \in \mathbb{N}$ such that

$$2^n \leq k \exp\left(-c\left[\frac{\rho(E,F)}{5k}\right]^\eta\right) < 2^{n+1}$$

and the Gaffney estimates of $k\Delta P^{k-1}$ is a consequence of (18), (19) and (20). Otherwise $\exp\left(-c\left[\frac{\rho(E,F)}{5k}\right]^\eta\right) \leq \frac{1}{k}$ and the desired result is a consequence of Proposition 2.3. \square

Corollary 2.5. *Let (Γ, μ, ρ) be a weighted graph satisfying (LB), (DV) and (UE). Let $M \in \mathbb{N}^*$. Then $([I - (I + s\Delta)^{-1}]^M)_{k \in \mathbb{R}_+^*}$ satisfies L^p Gaffney estimates for any $p \in (1, +\infty)$.*

Moreover the coefficient η that appears in the Gaffney estimates is the half of the one given by (UE).

Proof: The proof is analogous to the one of [19, Proposition 2.6] once we have Proposition 2.3. \square

Corollary 2.6. *Let (Γ, μ, ρ) be a weighted graph satisfying (LB), (DV) and (UE). Let $M \in \mathbb{N}^*$, $p \in (1, +\infty]$ and $\epsilon > 0$. Then the (M, p, ϵ) -molecules are uniformly bounded in $L^1(\Gamma)$.*

Proof: Let $q \in (1, p] \cap (1, +\infty)$. Then $([I - (I + s\Delta)^{-1}]^M)_{s \in \mathbb{R}_+^*}$ satisfies L^q Gaffney estimates. Thus, if $a = [I - (I + s\Delta)^{-1}]^M b$ is a molecule associated with the vertex $x \in \Gamma$ and the real $s > 0$, Corollary 2.5 yields

$$\begin{aligned} \|a\|_1 &\leq \sum_{j \geq 0} V(x, 2^j s)^{1-\frac{1}{q}} \|[I - (I + s\Delta)^{-1}]^M b\|_{L^q(C_j(x, s))} \\ &\lesssim \sum_{i \geq 0} \sum_{j \geq 0} V(x, 2^j s)^{1-\frac{1}{q}} \|[I - (I + s\Delta)^{-1}]^M [b\mathbf{1}_{C_i(x, s)}]\|_{L^q(C_j(x, s))} \\ &\lesssim \sum_{|i-j| \geq n_\rho} V(x, 2^j s)^{1-\frac{1}{q}} e^{-c2^{\eta \max\{i, j\}}} \|b\|_{L^q(C_i(x, s))} + \sum_{|i-j| < n_\rho} V(x, 2^j s)^{1-\frac{1}{q}} \|b\|_{L^q(C_i(x, s))} \\ &\lesssim \sum_{|i-j| \geq n_\rho} V(x, 2^j s)^{1-\frac{1}{q}} V(x, 2^i s)^{\frac{1}{q}-\frac{1}{p}} e^{-c2^{\eta j}} \|b\|_{L^p(C_i(x, s))} \\ &\quad + \sum_{|i-j| < n_\rho} V(x, 2^j s)^{1-\frac{1}{q}} V(x, 2^i s)^{\frac{1}{q}-\frac{1}{p}} \|b\|_{L^p(C_i(x, s))} \\ &\lesssim \sum_{|i-j| \geq n_\rho} \left(\frac{V(x, 2^j s)}{V(x, 2^i s)}\right)^{1-\frac{1}{q}} e^{-c2^{\eta j}} 2^{-i\epsilon} + \sum_{|i-j| < n_\rho} \left(\frac{V(x, 2^j s)}{V(x, 2^i s)}\right)^{1-\frac{1}{q}} 2^{-i\epsilon} \\ &\lesssim \sum_{|i-j| \geq n_\rho} 2^{jd(1-\frac{1}{q})} e^{-c2^{\eta j}} 2^{-i\epsilon} + \sum_{|i-j| < n_\rho} 2^{-i\epsilon} \\ &\lesssim 1, \end{aligned}$$

where n_ρ denote $2 + \ln_2(C_\rho)$, so that $\rho(C_j(x, s), C_i(x, s)) \gtrsim 2^{\max\{i, j\}} s$ if $j \geq i + n_\rho$. \square

2.2 Gaffney estimates for the gradient

We establish in this paragraph some additional Gaffney estimates.

Proposition 2.7. *Let (Γ, μ, ρ) satisfying (LB), (DV) and (UE). Let $p \in (1, 2)$. There exist $C, c > 0$ such that for all sets $E, F \subset \Gamma$, there holds*

$$\|\sqrt{k}\nabla P^{k-1}[f\mathbf{1}_F]\|_{L^p(E)} \leq C \exp\left[-c\left(\frac{\rho(E, F)}{k}\right)^\eta\right] \|f\|_{L^p} \quad (21)$$

for any function $f \in L^p(\Gamma)$ and any $k \in \mathbb{N}^*$.

Moreover, the value of η occurring in (21) is the same as the one in (UE).

Proof: First, assume that f is nonnegative and in $L^1(\Gamma) \cap L^\infty(\Gamma)$. We define for all $k \in \mathbb{N}^*$ and all $p \in (1, 2)$ a “pseudo-gradient” by

$$N_p(P^{k-1}f) = -(P^{k-1}f)^{2-p}[\partial_k + \Delta][(P^{k-1}f)^p]$$

where for any function u_k defined on $\Gamma \times \mathbb{N}^*$, $\partial_k u_k = u_{k+1} - u_k$.

Moreover we define for any function f defined on Γ the operator A defined by

$$Af(x) = \sum_{y \sim x} f(y).$$

Propositions 4.6 and 4.7 of [20] state the following results.

(i) For all $x \in \Gamma$, $N_p(P^{k-1}f)(x) \geq 0$. That is, for all $x \in \Gamma$,

$$J_k f(x) := -[\partial_k + \Delta][(P^{k-1}f)^p](x) \geq 0. \quad (22)$$

(ii) For all $p \in (1, 2]$, there exists $C = C_p$ such that for all $k \in \mathbb{N}^*$ and all nonnegative function $f \in L^\infty(\Gamma)$, there holds

$$|\nabla P^{k-1}f(x)|^2 \leq C [AN_p(P^{k-1}f)](x). \quad (23)$$

As a consequence of (ii), if $0 \leq f \in L^\infty$ and $E, F \subset \Gamma$, one has

$$\begin{aligned} \|\nabla P^{k-1}f\|_{L^p(E)}^p &\lesssim \|AN_p(P^{k-1}f)\|_{L^{\frac{p}{2}}(E)}^{\frac{p}{2}} \\ &\leq \sum_{x \in E} \left(\sum_{y \sim x} [N_p(P^{k-1}f)(y)] \right)^{\frac{p}{2}} m(x) \\ &\lesssim \sum_{y \in E_{+1}} [N_p(P^{k-1}f)(y)]^{\frac{p}{2}} m(y) \\ &\lesssim \| [N_p(P^{k-1}f)]^{\frac{1}{2}} \|_{L^p(E_{+1})}^p, \end{aligned}$$

where

$$E_{+1} = \{y \in \Gamma, \exists x \in E : x \sim y\} \subset \{y \in \Gamma, \rho(y, E) \leq 1\}.$$

It remains to estimate $\|(N_p(P^{k-1}f))^{\frac{1}{2}}\|_{L^p(E_{+1})}$. Without loss of generality, we can assume that $\rho(E, F) \geq 2C_\rho k \geq 2C_\rho$ (otherwise, [20, Corollary 4.3] or [17, Corollary 1.2] would provide the conclusion of Proposition 2.7). Under this assumption, one has $\rho(E_{+1}, F) \geq \frac{\rho(E, F)}{C_\rho} - 1 \geq \frac{\rho(E, F)}{2C_\rho} \gtrsim \rho(E, F)$. The proof of the case where f is nonnegative will be thus complete if we prove that for all $p \in (1, 2)$ and for all $E, F \subset \Gamma$

$$\|(N_p(P^{k-1}[f\mathbf{1}_F]))^{\frac{1}{2}}\|_{L^p(E)} \leq \frac{C}{\sqrt{k}} \exp\left[-c\left(\frac{\rho(E, F)}{k}\right)^\eta\right] \|f\|_{L^p} \quad \forall f \in L^p, \forall k \in \mathbb{N}^* \quad (24)$$

with some constant $C, c > 0$ independent of E and F .

In order to do this, we follow the idea of the proof of [20, Corollary 4.3]. Let $u_k = P^{k-1}[f\mathbf{1}_F]$, then

$$\begin{aligned}
\|N_p^{\frac{1}{2}}(u_k)\|_{L^p(E)}^p &= \sum_{x \in E} m(x) N_p^{p/2}(u_k)(x) \\
&= \sum_{x \in E} m(x) u_k^{\frac{p(2-p)}{2}} J_k f(x)^{p/2} \\
&\leq \left[\sum_{x \in E} m(x) u_k(x)^p \right]^{\frac{2-p}{2}} \left[\sum_{x \in E} J_k f(x) m(x) \right]^{\frac{p}{2}} \\
&\leq \|u_k\|_{L^p(E)}^{p(1-\frac{p}{2})} \left[\sum_{x \in \Gamma} J_k f(x) m(x) \right]^{\frac{p}{2}}
\end{aligned} \tag{25}$$

where the last but one step follows from Hölder inequality and the last one from (22) stated above. Yet,

$$\begin{aligned}
\sum_{x \in \Gamma} J_k f(x) m(x) &= - \sum_{x \in \Gamma} \partial_k(u_k^p)(x) m(x) \\
&\leq -p \sum_{x \in \Gamma} m(x) u_k^{p-1}(x) \partial_k u_k(x) \\
&\leq p \|u_k\|_p^{p/p'} \|\partial_k u_k\|_p
\end{aligned}$$

where the first line holds because $\sum_{x \in \Gamma} \Delta g(x) m(x) = 0$ if $g \in L^1$, the second line follows from Young inequality, and the third one from Hölder inequality again (with $\frac{1}{p} + \frac{1}{p'} = 1$). Here $\|u_k\|_p \leq \|f\|_p$ while $\|\partial_k u_k\|_p = \|\Delta u_k\|_p \lesssim \frac{1}{k} \|f\|_p$ by the analyticity of P on L^p . Thus

$$\sum_{x \in \Gamma} J_k f(x) m(x) \lesssim \frac{1}{k} \|f\|_p^p.$$

Substitution of the last estimate in (25) gives

$$\|N_p^{\frac{1}{2}}(u_k)\|_{L^p(E)} \lesssim \frac{1}{\sqrt{k}} \|f\|_p^{\frac{p}{2}} \|u_k\|_{L^p(E)}^{1-\frac{p}{2}},$$

which ends the proof of (24) if we replace $\|u_k\|_{L^p(E)}$ by the upper estimate given by Proposition 2.3.

The result for the case where $f \in L^\infty(\Gamma) \cap L^1(\Gamma)$ is deduced by writing $f = f_+ - f_-$, with $f_+ = \max\{0, f\}$ and $f_- = \max\{0, -f\}$. Then

$$\begin{aligned}
\|\nabla P^{k-1}[f\mathbf{1}_F]\|_{L^p(E)} &\leq \|\nabla P^{k-1}[f_+\mathbf{1}_F]\|_{L^p(E)} + \|\nabla P^{k-1}[f_-\mathbf{1}_F]\|_{L^p(E)} \\
&\lesssim \frac{1}{\sqrt{k}} \exp\left[-c \left(\frac{\rho(E, F)}{k}\right)^\eta\right] [\|f_+\|_p + \|f_-\|_p] \\
&\lesssim \frac{1}{\sqrt{k}} \exp\left[-c \left(\frac{\rho(E, F)}{k}\right)^\eta\right] \|f\|_p.
\end{aligned}$$

The result for the general case $f \in L^p(\Gamma)$ is then a consequence of the density of $L^\infty(\Gamma) \cap L^1(\Gamma)$ in $L^p(\Gamma)$. \square

Corollary 2.8. *Let (Γ, μ, ρ) be a weighted graph satisfying (LB), (DV) and (UE). Let $M \in \mathbb{N}^*$. Then $(k^{M-\frac{1}{2}} d\Delta^{M-1} (I + k\Delta)^{-M+\frac{1}{2}})_{k \in \mathbb{N}^*}$ satisfies L^p Gaffney estimates for any $p \in (1, 2)$. Moreover the coefficient η occurring in the Gaffney estimates is the half of the one given by (UE).*

Proof: The proof is analogous to the one of [19, Corollary 2.11], using Proposition 2.7. \square

Corollary 2.9. *Let (Γ, μ, ρ) be a weighted graph satisfying (LB), (DV) and (UE). Let $M \in \mathbb{N}^*$, $p \in (1, +\infty]$ and $\epsilon > 0$. Then the $(M - \frac{1}{2}, p, \epsilon)$ -molecules are uniformly bounded in $L^1(T_\Gamma)$.*

Proof: The proof is analogous to the one of Corollary 2.6, using Corollary 2.8. \square

2.3 L^q - L^p Gaffney estimates

Definition 2.10. Let (X, m) a measured space equipped with a quasidistance ρ and $\alpha > 0$. Let $E, F \subset X$, $x_0 \in X$ and $k \in \mathbb{N}$.

We say that (E, F, x_0) is α -Gaffney suited if it satisfies one of the following conditions

(i) $\sup \{\rho(x_0, y), y \in F\} < \alpha\rho(E, F)$,

(ii) $\sup \{\rho(x_0, x), x \in E\} < \alpha\rho(E, F)$.

Moreover, we say that (E, x_0, k) is α -Gaffney suited if it satisfies

$$\sup \{d(x_0, x), x \in E\} < \alpha k.$$

At last, we say that (E, F, x_0, k) is α -Gaffney suited if either (E, F, x_0) or (E, x_0, k) or (F, x_0, k) is α -Gaffney suited.

Proposition 2.11. Let (Γ, μ, ρ) satisfying (LB), (DV) and (UE). Then for all $a \in \mathbb{N}$, there exists $C_a, c_a > 0$ such that

$$|D^a p_{k-1}(x, y)| \lesssim \frac{C_a}{k^a V(x, k)} \exp \left[-c_a \left(\frac{\rho(x, y)}{k} \right)^\eta \right] \quad \forall x, y \in \Gamma, \forall k \in \mathbb{N}^* \quad (26)$$

where D is the operator acting on sequences as $Da_k = a_k - a_{k+1}$.

Moreover, the value of η occurring in (26) is the same as the one in (UE).

Proof: The proof is similar to [20, Theorem A.1] (see also [16, Theorem 1.1]). □

Proposition 2.12. Let (Γ, μ, ρ) satisfying (LB), (DV) and (UE). Let $j \in \mathbb{N}$, $\alpha > 0$, $p, q \in [1, +\infty]$ such that $q \leq p$.

There exists c_j and $C_{j, \alpha}$ such that, for all sets $E, F \subset \Gamma$, all $x_0 \in \Gamma$, all $k \in \mathbb{N}$ such that (E, F, x_0, k) is α -Gaffney suited, there holds

$$\|(k\Delta)^a P^{k-1}[f \mathbf{1}_F]\|_{L^p(E)} \leq C_{j, \alpha} V(x_0, k)^{\frac{1}{p} - \frac{1}{q}} \exp \left[-c_j \left(\frac{\rho(E, F)}{k} \right)^\eta \right] \|f\|_{L^q} \quad (27)$$

for any function $f \in L^q(X)$.

Moreover, the value of η occurring in (27) is the same as the one in (UE).

Proof: When E, F and x_0 satisfy (i) in Definition 2.10 and when $q = 1$, the proof is inspired by the one of [20, Theorem A.3].

We need the following result. There exist $C'_j, c'_j > 0$ such that

$$I_{c'_j} := \sum_{y \in \Gamma} |(kD)^j p_{k-1}(x, y)|^p e^{c'_j \left(\frac{\rho(x, y)}{k} \right)^\eta} m(y) \leq C'_j V(x, k)^{1-p}. \quad (28)$$

Indeed, the estimate (26) yields, with $c'_j = \frac{c_j}{2}$,

$$\begin{aligned} I_{c'_j} &\lesssim \sum_{y \in \Gamma} \frac{1}{V(x, k)^p} e^{-\frac{c_j}{2} \left(\frac{\rho(x, y)}{k} \right)^\eta} m(y) \\ &\lesssim \frac{1}{V(x, k)^p} \sum_{j=0}^{+\infty} \sum_{y \in C_j(x, k)} e^{-\frac{c_j}{2} 2^{j\eta}} \\ &\leq \sum_{j=0}^{+\infty} \frac{V(x, 2^{j+1}k)}{V(x, k)^p} e^{-\frac{c_j}{2} 2^{j\eta}} \\ &\lesssim \frac{1}{V(x, k)^{p-1}} \sum_{j=0}^{+\infty} 2^{jd} e^{-\frac{c_j}{2} 2^{j\eta}} \\ &\lesssim \frac{1}{V(x, k)^{p-1}}. \end{aligned}$$

We can assume without loss of generality that $\|f\|_1 = 1$. Then, Jensen inequality implies

$$\begin{aligned}
\|(k\Delta)^a P^{k-1}(f\mathbb{1}_F)\|_{L^p(E)}^p &= \sum_{x \in E} m(x) \left(\sum_{z \in F} (kD)^a p_{k-1}(x, z) f(z) m(z) \right)^p \\
&\leq \sum_{x \in E} m(x) \sum_{z \in F} |(kD)^a p_{k-1}(x, z)|^p |f(z)| m(z) \\
&\leq e^{-c'_a \left(\frac{\rho(E, F)}{k}\right)^\eta} \sum_{z \in F} |f(z)| m(z) \sum_{x \in E} m(x) |(kD)^a p_{k-1}(x, z)|^p e^{c'_a \left(\frac{\rho(x, z)}{k}\right)^\eta} \\
&\lesssim \exp\left(-c'_a \left(\frac{\rho(E, F)}{k}\right)^\eta\right) \sum_{z \in F} |f(z)| m(z) \frac{1}{V(z, k)^{p-1}} \\
&\lesssim \frac{1}{V(x_0, k)^{p-1}} \exp\left(-c \left(\frac{\rho(E, F)}{k}\right)^\eta\right)
\end{aligned}$$

where, for the 4th line, we use the estimate (28). The doubling property (12) shows

$$\frac{V(x_0, k)}{V(z, k)} \leq \frac{V(z, C_\rho[k + \alpha\rho(E, F)])}{V(z, k)} \lesssim \left(1 + \frac{\alpha\rho(E, F)}{k}\right)^d \lesssim (1 + \alpha)^d \exp\left(\frac{c}{2} \left(\frac{\rho(E, F)}{k}\right)^\eta\right), \quad (29)$$

then the last line in the previous calculus holds for some $c < c'_a$.

Let us now prove the result when E, F and x_0 still satisfy (i) in Definition 2.10 with $q \in (1, p]$. Without loss of generality, we can assume that f is supported in F and in this case, one has

$$\begin{aligned}
\|(k\Delta)^a P^{k-1}[f\mathbb{1}_F]\|_{L^p(E)} &\lesssim \frac{1}{V(x_0, k)^{1-\frac{1}{p}}} \exp\left(-c \left(\frac{\rho(E, F)}{k}\right)^\eta\right) \|f\|_{L^1(F)} \\
&\lesssim \left(\frac{m(F)}{V(x_0, k)}\right)^{1-\frac{1}{q}} \frac{1}{V(x_0, k)^{\frac{1}{q}-\frac{1}{p}}} \exp\left(-c \left(\frac{\rho(E, F)}{k}\right)^\eta\right) \|f\|_{L^q(F)}.
\end{aligned}$$

Then it remains to check that (12) yields

$$\frac{m(F)}{V(x_0, k)} \leq \frac{V(x_0, \alpha\rho(E, F))}{V(x_0, k)} \lesssim (1 + \alpha)^d \left(1 + \frac{\rho(E, F)}{k}\right)^d \lesssim (1 + \alpha)^d \exp\left(\frac{c}{2} \left(\frac{\rho(E, F)}{k}\right)^\eta\right). \quad (30)$$

The proof when (F, x_0, k) is α -Gaffney suited is the same as the case where (E, F, x_0) satisfies (i) in Definition 2.10, once we replaced (30) and (29) by

$$\frac{m(F)}{V(x_0, k)} \lesssim \alpha^d \quad \text{and} \quad \frac{V(x_0, k)}{V(z, k)} \lesssim \alpha^d \quad \forall z \in F,$$

which are both consequences of (12).

When (E, F, x_0) satisfies (ii) in Definition 2.10 or when (E, x_0, k) is α -Gaffney suited, the proof of Proposition 2.12 can be deduced from the previous cases. A way to do this is to adapt the proof of [20, Corollary A.4]. \square

2.4 Off diagonal decay of Lusin functionals

Proposition 2.13. *Let (Γ, μ, ρ) satisfying (LB), (DV) and (UE), and $\alpha, \beta, M > 0$. Then there exists $C, c > 0$ such that, for all $x_0 \in \Gamma$ and all sets $E, F \subset \Gamma$ such that (E, F, x_0) is α -Gaffney suited, there holds for all $s > 0$ and all $f \in L^1(\Gamma) \cap L^2(\Gamma)$,*

$$\|L_\beta(I - (I + s\Delta)^{-1})^M [f\mathbb{1}_F]\|_{L^2(E)} \leq \frac{C}{V(x_0, \rho(E, F))^{\frac{1}{2}}} \left(\frac{s}{\rho(E, F)}\right)^M \|f\|_{L^1}.$$

The proof of Proposition 2.13 is similar to the one of [20, Lemma 3.5] (based itself on Lemma 3.1 in [4]). We need the following result, whose proof is analogous to the ones of [20, Lemma C.1 and Proposition C.2].

Lemma 2.14. Let $M > 0$ and $\alpha \in [0, 1]$. Define $\mathcal{A} = \{(A_k^{d,u})_{k \in \mathbb{N}^*}, d \in \mathbb{R}_+, u \in \mathbb{N}\}$, where, for all $l \geq 1$,

$$A_k^{d,u} = k^\alpha \frac{\exp\left(-\left(\frac{d}{k+u}\right)^\eta\right)}{(k+u)^{1+M}}.$$

Then there exists $C = C_{M,\alpha}$ such that

$$\left(\sum_{k \in \mathbb{N}^*} \frac{1}{k} a_k^2\right)^{\frac{1}{2}} \leq C \sum_{k \in \mathbb{N}^*} \frac{1}{k} a_k \quad \forall (a_k)_k \in \mathcal{A}.$$

Proof: (Proposition 2.13)

Without loss of generality, we can assume that f is supported in F . We also assume that x_0, E and F satisfy (i) of Definition 2.10 (if they satisfy (ii) instead of (i), the proof is similar).

First, if $\sum_m \tilde{b}_m z^m$ is the Taylor series of the function $(1-z)^{-M}$, one has the identity

$$\begin{aligned} (I - (I + s\Delta)^{-1})^M f &= (s\Delta)^M (I + s\Delta)^{-M} f \\ &= \left(\frac{s\Delta}{1+s}\right)^M \left(I - \frac{s}{1+s}P\right)^{-M} f \\ &= \left(\frac{s\Delta}{1+s}\right)^M \sum_{m=0}^{+\infty} \tilde{b}_m \left(\frac{s}{1+s}\right)^m P^m f \\ &= (s\Delta)^M \sum_{m=0}^{\infty} b_m P^m f, \end{aligned} \tag{31}$$

where $b_m := \tilde{b}_m \frac{s^m}{(1+s)^{m+M}}$ and the series converges in $L^2(\Gamma)$. Notice that

$$\sum_{m=0}^{\infty} b_m = 1. \tag{32}$$

Moreover, let κ be the only integer such that $\kappa < \beta + M \leq \kappa + 1$. Since $\beta > 0$ and both κ and M are integers, notice that

$$M \leq \kappa \tag{33}$$

If $\sum_l a_l z^l$ is the Taylor series of the function $(1-z)^{\beta+M-\kappa-1}$, then one has

$$\Delta^{\beta+M} f = \sum_{l \geq 0} a_l P^l \Delta^{\kappa+1} f \tag{34}$$

where the sum converges in $L^2(\Gamma)$ (see [19, Proposition 2.1] for the proof of the convergence).

The Minkowski inequality together with the identities (31) and (34) yields

$$\begin{aligned} &\|L_\beta(I - (I + s\Delta)^{-1})^M f\|_{L^2(E)} \\ &= \left(\sum_{k \geq 1} k^{2\beta-1} \sum_{x \in E} \frac{m(x)}{V(x,k)} \sum_{y \in B(x,k)} m(y) |\Delta^\beta P^{k-1} (I - (I + s\Delta)^{-1})^M f(y)|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k \geq 1} k^{2\beta-1} \sum_{y \in D_\kappa(E)} m(y) |\Delta^\beta P^{k-1} (I - (I + s\Delta)^{-1})^M f(y)|^2 \sum_{x \in B(y,k)} \frac{m(x)}{V(x,k)}\right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{k \geq 1} k^{2\beta-1} \|\Delta^\beta P^{k-1} (I - (I + s\Delta)^{-1})^M f\|_{L^2(D_\kappa(E))}^2\right)^{\frac{1}{2}} \\ &\leq s^M \sum_{l,m \geq 0} a_l b_m \left(\sum_{k \geq 1} k^{2\beta-1} \|\Delta^{1+\kappa} P^{k+l+m-1} f\|_{L^2(D_\kappa(E))}^2\right)^{\frac{1}{2}} \\ &:= s^M \sum_{l,m \geq 0} a_l b_m \left(\sum_{k \geq 1} \frac{1}{k} I(k,l,m)^2\right)^{\frac{1}{2}}, \end{aligned}$$

where $D_k(E)$ denotes the set $\{y \in \Gamma, \rho(y, E) < k\}$

We want to get the following estimate estimate: there exists $c > 0$ such that

$$I(k, l, m) \lesssim J(k, l, m) := k^{M+\beta-\kappa} \frac{1}{V(x_0, \rho(E, F))} \frac{\exp \left[-c \left(\frac{\rho(E, F)}{k+l+m} \right)^\eta \right]}{(k+l+m)^{1+M}} \|f\|_{L^1}. \quad (35)$$

We will first establish (35) when $k \leq \frac{\rho(E, F)}{2C_\rho}$. In this case, notice that

$$\rho(D_k(E), F) \geq \frac{\rho(E, F)}{C_\rho} - k \geq \frac{\rho(E, F)}{2C_\rho}$$

and thus $(D_k(E), F, x_0)$ are $2C_\rho\alpha$ -Gaffney suited. Proposition 2.12 implies then

$$\begin{aligned} I(k, l, m) &\lesssim \frac{k^\beta}{(k+l+m)^{\kappa+1}} \frac{1}{V(x_0, k+l+m)^{\frac{1}{2}}} \exp \left(-c \left(\frac{\rho(E, F)}{2C_\rho(k+l+m)} \right)^\eta \right) \|f\|_{L^1} \\ &\lesssim \frac{k^{\beta+M-\kappa}}{(k+l+m)^{M+1}} \frac{1}{V(x_0, \rho(E, F))^{\frac{1}{2}}} \exp \left(-\frac{c}{2} \left(\frac{\rho(E, F)}{2C_\rho(k+l+m)} \right)^\eta \right) \|f\|_{L^1} \end{aligned}$$

where the last line holds thanks to estimate (30) and (33).

Otherwise, $k > \frac{\rho(E, F)}{2C_\rho}$ and then $(F, x_0, k+l+m)$ is $2C_\rho\alpha$ -Gaffney suited and Proposition 2.12 yields

$$\begin{aligned} I(k, l, m) &\lesssim \frac{k^\beta}{(k+l+m)^{\kappa+1}} \frac{1}{V(x_0, k+l+m)^{\frac{1}{2}}} \|f\|_{L^1} \\ &\lesssim \frac{k^{\beta+M-\kappa}}{(k+l+m)^{M+1}} \frac{1}{V(x_0, \rho(E, F))^{\frac{1}{2}}} \|f\|_{L^1} \\ &\lesssim \frac{k^{\beta+M-\kappa}}{(k+l+m)^{M+1}} \frac{1}{V(x_0, \rho(E, F))^{\frac{1}{2}}} \exp \left(-\left(\frac{\rho(E, F)}{k+l+m} \right)^\eta \right) \|f\|_{L^1} \end{aligned} \quad (36)$$

where the second line holds thanks to (DV) and (33). This ends the proof of (35).

Recall that $M + \beta - \kappa \in (0, 1]$. Then Lemma 2.14 implies

$$\begin{aligned} \|L_\beta(I - (I + s\Delta)^{-1})^M f\|_{L^2(E)} &\lesssim s^M \sum_{k, l, m \geq 0} \frac{1}{k} a_l b_m J(k, l, m) \\ &= s^M \frac{\|f\|_{L^1}}{V(x_0, \rho(E, F))^{\frac{1}{2}}} \sum_{m \geq 0} b_m \sum_{n \geq 1} \frac{\exp \left[-c \left(\frac{\rho(E, F)}{n+m} \right)^\eta \right]}{(n+m)^{1+M}} \sum_{l=0}^{n-1} a_l (n-l)^{M+\beta-\kappa-1}. \end{aligned} \quad (37)$$

We claim

$$\sum_{l=0}^{n-1} a_l (n-l)^{M+\beta-\kappa-1} \lesssim 1.$$

Indeed, when $M + \beta = \kappa + 1$, one has $a_l = \delta_0(l)$ and the estimate is true. Otherwise, [20, Lemma B.1] yields that $a_l \lesssim (l+1)^{\kappa-M-\beta}$ and therefore

$$\sum_{l=0}^{n-1} a_l (n-l)^{M+\beta-\kappa-1} \lesssim \int_0^1 t^{\kappa-M-\beta} (1-t)^{M+\beta-\kappa-1} dt < +\infty.$$

Besides,

$$\begin{aligned} \sum_{n \geq 1} \frac{\exp \left[-c \left(\frac{\rho(E, F)}{n+m} \right)^\eta \right]}{(n+m)^{1+M}} &= \rho(E, F)^{-M-1} \sum_{n \geq 1} \left(\frac{\rho(E, F)}{n+m} \right)^{1+M} \exp \left[-c \left(\frac{\rho(E, F)}{n+m} \right)^\eta \right] \\ &\lesssim \rho(E, F)^{-M-1} \left[\sum_{n=1}^{\rho(E, F)} 1 + \sum_{n > \rho(E, F)} \left(\frac{\rho(E, F)}{m+n} \right)^{M+1} \right] \\ &\lesssim \rho(E, F)^{-M}. \end{aligned}$$

Consequently, the estimate (37) yields

$$\begin{aligned} \|L_\beta(I - (I + s\Delta)^{-1})^M f\|_{L^2(E)} &\lesssim \frac{1}{V(x_0, \rho(E, F))^{\frac{1}{2}}} \left(\frac{s}{\rho(E, F)}\right)^M \|f\|_{L^1} \sum_{m \geq 0} b_m \\ &= \frac{1}{V(x_0, \rho(E, F))^{\frac{1}{2}}} \left(\frac{s}{\rho(E, F)}\right)^M \|f\|_{L^1} \end{aligned}$$

where the last line is due to (32). \square

Let us define a discrete version of the Littlewood-Paley functionals, that can be found in [20]. For any $\beta > 0$, the functional g_β is defined as

$$g_\beta f(x) = \left(\sum_{k=1}^{\infty} k^{2\beta-1} |\Delta^\beta P^{k-1} f(x)|^2 \right)^{\frac{1}{2}}.$$

Proposition 2.15. *Let (Γ, μ, ρ) satisfying (LB), (DV) and (UE), and $\alpha, \beta > 0$. Then there exists $C > 0$ such that, for all $x_0 \in \Gamma$ and all sets $E, F \subset \Gamma$ such that (E, F, x_0) is α -Gaffney suited, there holds for all $k \in \mathbb{N}^*$ and all $f \in L^1(\Gamma) \cap L^2(\Gamma)$,*

$$\|L_\beta(I - P^k)[f\mathbf{1}_F]\|_{L^2(E)} \leq \frac{C}{V(x_0, \rho(E, F))^{\frac{1}{2}}} \frac{k}{\rho(E, F)} \|f\|_{L^1}$$

and

$$\|g_\beta(I - P^k)[f\mathbf{1}_F]\|_{L^2(E)} \leq \frac{C}{V(x_0, \rho(E, F))^{\frac{1}{2}}} \frac{k}{\rho(E, F)} \|f\|_{L^1}$$

Proposition 2.16. *Let (Γ, μ, ρ) satisfying (LB), (DV) and (UE), $M > 0$, $\beta \in (0, 1]$ and $\alpha > 0$. Then there exists $C > 0$ such that, for all $x_0 \in \Gamma$ and all sets $E, F \subset \Gamma$ such that (E, F, x_0) is α -Gaffney suited, there holds for all $k \in \mathbb{N}^*$ and all $f \in L^1(\Gamma) \cap L^2(\Gamma)$,*

$$\|g_\beta(k\Delta)^M [f\mathbf{1}_F]\|_{L^2(E)} \leq C \frac{C}{V(x_0, \rho(E, F))^{\frac{1}{2}}} \left(\frac{k}{\rho(E, F)}\right)^M \|f\|_{L^1}$$

Proof: The proofs of these two propositions are similar to the one of Proposition 2.13 and is left to the reader. See also [20, Lemma 3.5] and [19, Lemmata 2.14 and 2.18]. \square

2.5 Application to interpolation results

Definition 2.17. *A function $a \in L^2(\Gamma)$ is called an atom if there exist $x \in \Gamma$ and $k \in \mathbb{N}^*$ and a function $b \in L^2(\Gamma)$ supported in $B(x, k)$ such that*

$$(i) \quad a = (I - P^k)b,$$

$$(ii) \quad \|b\|_{L^2} = \|b\|_{L^2(B(x, k))} \leq V(x, k)^{-\frac{1}{2}}.$$

We say that f belongs to $E_0^1(\Gamma)$ if f admits an atomic representation, that is if there exist a finite sequence $(\lambda_i)_{i=0..N}$ and a finite sequence $(a_i)_{i=0..N}$ of atoms such that

$$f = \sum_{i=0}^N \lambda_i a_i. \tag{38}$$

The space is outfitted with the norm

$$\|f\|_{E_0^1} = \inf \left\{ \sum_{j=0}^{\infty} |\lambda_j|, \sum_{j=0}^{\infty} \lambda_j a_j, \text{ is a atomic representation of } f \right\}.$$

Theorem 2.18. *Let (Γ, μ, ρ) satisfying (LB), (DV) and (UE). If T is an $L^2(\Gamma)$ bounded linear operator and if there exists $C > 0$ such that for all atoms*

$$\|Ta\|_{L^1} \leq C,$$

then for all $p \in (1, 2]$, there exists a constant $C = C(p) > 0$ such that

$$\|Tf\|_p \leq C_p \|f\|_p \quad \forall f \in L^p \cap L^2.$$

The next result is an immediate corollary of Theorem 2.18.

Corollary 2.19. *Let (Γ, μ, ρ) satisfying (LB), (DV) and (UE). Let $(H_0^1(\Gamma), \|\cdot\|_{H_0^1})$ a normed vector space that satisfies the continuous embedding*

$$E_0^1(\Gamma) \subset H_0^1(\Gamma) \subset L^1(\Gamma).$$

If T is an $L^2(\Gamma)$ -bounded linear operator that verifies

$$\|Tf\|_{L^1} \lesssim \|f\|_{H_0^1} \quad \forall f \in H_0^1(\Gamma) \cap L^2(\Gamma),$$

then for all $p \in (1, 2]$, there exists a constant $C = C(p) > 0$ such that

$$\|Tf\|_p \leq C_p \|f\|_p \quad \forall f \in L^p \cap L^2.$$

The constant C_p can be chosen equal to $\|T\|_{L^2 \rightarrow L^2}^\theta \|T\|_{H_0^1 \rightarrow L^1}^{1-\theta}$, where θ is defined with

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{1}.$$

Remark 2.20. *The result holds even if the normed vector space H_0^1 is not a Banach space.*

This corollary will be used for the Hardy space $H_0^1 = H^1(\Gamma)$. However, since we did not establish Theorem 1.26 yet, we cannot speak of $H^1(\Gamma)$.

Proof: (Theorem 2.18)

The Vitali lemma, the weak L^1 -boundedness and the L^p -boundedness (for $p > 1$) of the Hardy-Littlewood maximal function, or the Whitney decomposition are classical results from harmonic analysis and are proven in particular when the metric is a quasidistance. These tools are the only ones needed to prove Theorem 5.3 in [7], which thus remains true in our context. That is, Theorem 2.18 is a straightforward consequence of Theorem 5.3 in [7] if we prove that for any $x \in \Gamma$, any $k \in \mathbb{N}^*$ and any $h \in L^2$, we have

$$\sup_{y \in B(x, k)} |P^k h(y)| \lesssim C_M \inf_{z \in B(x, k)} [\mathcal{M}(|h|^2)(z)]^{\frac{1}{2}}. \quad (39)$$

With Proposition 2.12, since $(B(x, k), x, k)$ is 1-Gaffney suited, there holds for any $z \in C_0(x, k) \supset B(x, k)$,

$$\begin{aligned} \sup_{y \in B(x, k)} |P^k h(y)| &\leq \sum_{j \geq 0} \|P^k [h \mathbf{1}_{C_j(x, k)}]\|_{L^\infty(B(x, k))} \\ &\lesssim \sum_{j \geq 0} \frac{1}{V(x, 2^j k)^{\frac{1}{2}}} e^{-c2^{j\eta}} \|h\|_{L^2(C_j(x, k))} \\ &\lesssim \sum_{j \geq 0} \frac{1}{V(x, C_\rho 2^{j+1} k)^{\frac{1}{2}}} e^{-c2^{j\eta}} \|h\|_{L^2(B(x, C_\rho 2^{j+1} k))} \\ &\lesssim \sum_{j \geq 0} e^{-c2^{j\eta}} [\mathcal{M}(|h|^2)(z)]^{\frac{1}{2}} \\ &\lesssim [\mathcal{M}(|h|^2)(z)]^{\frac{1}{2}} \end{aligned}$$

where the third line is due to (DV). Hence, (39) holds true. \square

Let us recall a result on L^p boundedness of Calderón-Zygmund operators (originally due to Blunck and Kunstmann, see Theorem 1.1 in [8], see also Theorem 1.1 in [1]).

Definition 2.21. *A function f on Γ is in $L^{1, \infty}(\Gamma)$ if*

$$\|f\|_{L^{1, \infty}} := \sup_{\lambda > 0} \lambda m(\{x \in \Gamma, |f(x)| > \lambda\}) < +\infty.$$

Theorem 2.22. *For any ball B , let A_B be a linear operator in $L^2(\Gamma)$. Let T be a L^2 -bounded sublinear operator such that for all balls $B = B(x, k)$ and all functions f supported in B*

$$\frac{1}{V(x, 2^j k)^{\frac{1}{2}}} \|T(I - A_B)f\|_{L^2(C_j(x, k))} \leq \alpha_j(B) \frac{1}{V(x, k)} \|f\|_{L^1(B)}$$

for all $j \geq 1$ and

$$\frac{1}{V(x, 2^j k)^{\frac{1}{2}}} \|A_B f\|_{L^2(C_j(x, k))} \leq \alpha_j(B) \frac{1}{V(x, k)} \|f\|_{L^1(B)}$$

for all $j \geq 0$.

If the coefficients $\alpha_j(B)$ satisfy

$$\sup_{B=B(x, k)} \sum_{\text{ball}} \sum_{j \geq 0} \frac{V(x, 2^{j+1} k)}{V(x, k)} \alpha_j(B) < +\infty$$

then there exists a constant C such that

$$\|Tf\|_{L^{1, \infty}} \leq C \|f\|_{L^1} \quad \forall f \in L^2 \cap L^1.$$

So by interpolation, for all $p \in (1, 2]$, there exists a constant $C = C_p$ such that

$$\|Tf\|_{L^p} \leq C_p \|f\|_{L^p} \quad \forall f \in L^2 \cap L^p.$$

As a consequence, we have the following result

Theorem 2.23. *Let (Γ, μ, ρ) be a weighted graph satisfying (LB), (DV) and (UE). Then for all $\beta > 0$, the functional L_β is L^p bounded for any $p \in (1, 2]$ and also bounded from $L^{1, \infty}$ to L^1 .*

Moreover, if g_β is the discrete Littlewood-Paley quadratic functional defined for any $\beta > 0$ as

$$g_\beta f(x) = \left(\sum_{k=1}^{\infty} k^{2\beta-1} |\Delta P^{k-1} f(x)|^2 \right)^{\frac{1}{2}},$$

then g_β is also L^p bounded for any $p \in (1, 2]$ and also bounded from $L^{1, \infty}$ to L^1 .

Proof: We set $A_B = P^{k_B}$. It is then a straightforward consequence of Theorem 2.22, Proposition 2.15 and Proposition 2.12. \square

3 Tent spaces

In all this section, (Γ, μ, ρ) is a weighted graph satisfying (DV). We will prove that in this context, the atomic decomposition in tent spaces still holds (see [11] and [27] for similar results).

Definition 3.1. *We introduce the following sets in $\Gamma \times \mathbb{N}^*$. If $x \in \Gamma$,*

$$\gamma(x) = \{(y, k) \in \Gamma \times \mathbb{N}^*, \rho(x, y) < k\},$$

if $F \subset \Gamma$,

$$\mathcal{R}(F) = \bigcup_{x \in F} \gamma(x) = \{(y, k) \in \Gamma \times \mathbb{N}^*, \rho(y, E) < k\}$$

$$\mathfrak{R}(F) = \{(y, k) \in \Gamma \times \mathbb{N}^*, \rho(y, E) < \frac{k}{2C_\rho}\},$$

and if $O \subset \Gamma$,

$$\mathcal{T}(O) = \{(y, k) \in \Gamma \times \mathbb{N}^*, \rho(y, O^c) \geq k\}$$

$$\mathfrak{T}(O) = \{(y, k) \in \Gamma \times \mathbb{N}^*, \rho(y, O^c) \geq \frac{k}{2C_\rho}\}.$$

We define the functionals \mathcal{A} and \mathcal{C} mapping functions on $\Gamma \times \mathbb{N}^*$ to functions on Γ by

$$\mathcal{A}f(x) = \left(\sum_{(y, k) \in \gamma(x)} \frac{m(y)}{kV(x, k)} |f(y, k)|^2 \right)^{\frac{1}{2}}$$

and

$$\mathcal{C}f(x) = \sup_{x \in B} \left(\frac{1}{V(B)} \sum_{(y, k) \in \mathcal{T}(B)} \frac{m(y)}{k} |f(y, k)|^2 \right)^{\frac{1}{2}}.$$

For any $p \in [1, +\infty)$, $T^p(\Gamma)$ denotes the space of functions f on $\Gamma \times \mathbb{N}^*$ such that $\mathcal{A}f \in L^p(\Gamma)$. Moreover, $T^\infty(\Gamma)$ is the space of functions f on $\Gamma \times \mathbb{N}^*$ such that $\mathcal{C}f \in L^\infty(\Gamma)$. The tent space $T^p(\Gamma)$ is equipped with the norm $\|f\|_{T^p} = \|\mathcal{A}f\|_{L^p}$ (or $\|f\|_{T^p} = \|\mathcal{C}f\|_{L^\infty}$ when $p = \infty$).

Remark 3.2. One has the following equality of sets

$$\mathcal{R}(E^c) = \mathcal{T}(O)^c \quad \text{and} \quad \mathfrak{R}(O^c) = \mathfrak{T}(O)^c.$$

Moreover, $\mathcal{T}(O) \subset \mathfrak{T}(O)$ and $\mathcal{R}(F) \supset \mathfrak{R}(F)$.

Definition 3.3. A function a defined on $\Gamma \times \mathbb{N}^*$ is a T^1 -atom if there exists a ball B such that

(i) a is supported in $\mathcal{T}(B)$,

$$(ii) \quad \sum_{(y,k) \in \mathcal{T}(B)} \frac{m(y)}{k} |a(y,k)|^2 \leq \frac{1}{V(B)}.$$

Lemma 3.4. There exists a constant $C > 0$ such that, for every T^1 -atom a , one has

$$\|a\|_{T^1} \leq C$$

Proof: Indeed, if a is a T^1 -atom, there holds

$$\begin{aligned} \|a\|_{T^1} &= \sum_{x \in \Gamma} m(x) \left(\sum_{(y,k) \in \gamma(x)} \frac{m(y)}{kV(x,k)} |a(y,k)|^2 \right)^{\frac{1}{2}} \\ &= \sum_{x \in B} m(x) \left(\sum_{(y,k) \in \gamma(x)} \frac{m(y)}{kV(x,k)} |a(y,k)|^2 \right)^{\frac{1}{2}} \\ &\leq \left(V(B) \sum_{x \in B} m(x) \sum_{(y,k) \in \gamma(x)} \frac{m(y)}{kV(x,k)} |a(y,k)|^2 \right)^{\frac{1}{2}} \\ &= \left(V(B) \sum_{(y,k) \in \mathcal{T}(B)} \frac{m(y)}{k} |a(y,k)|^2 \sum_{x \in B(y,k)} \frac{m(x)}{V(x,k)} \right)^{\frac{1}{2}} \\ &\lesssim \left(V(B) \sum_{(y,k) \in \mathcal{T}(B)} \frac{m(y)}{k} |a(y,k)|^2 \right)^{\frac{1}{2}} \\ &\lesssim 1, \end{aligned}$$

where the last but one line holds because the doubling property (DV) implies $\sum_{x \in B(y,k)} \frac{m(x)}{V(x,k)} \lesssim 1$. \square

Definition 3.5. Let $F \subset \Gamma$ and $\delta \in (0, 1)$. We say that a point $x_0 \in \Gamma$ has global δ -density with respect to F if, for all $k \in \mathbb{N}^*$,

$$\frac{m(F \cap B(x_0, k))}{V(x_0, k)} \geq \delta.$$

We define F^* as the set made of the points in Γ with δ -density with respect to F . If $O = F^c$, then we define O_* as

$$O_* := (F^*)^c.$$

Proposition 3.6. Let $\delta \in (0, 1)$. One has

(i) $F^* \subset F$,

(ii) there exists $c_\delta > 0$ such that any set $O = F^c$ with finite measure verify $m(O_*) \leq c_\delta m(O)$.

Proof: The point (i) is obvious. For (ii), notice that $O_* \subset \{x \in \Gamma, \mathcal{M}(\mathbb{1}_O) > 1 - \delta\}$, where \mathcal{M} denote the Hardy-Littlewood maximal function. We conclude then by applying the weak L^1 -boundedness of the maximal function. \square

Lemma 3.7. *There exists $\delta \in (0, 1)$ (sufficiently close to 1) so that whenever $F \subset \Gamma$ and ϕ is a non-negative function on $\Gamma \times \mathbb{N}^*$, there holds*

$$\sum_{(y,k) \in \mathfrak{R}(F^*)} \phi(y,k)V(y,k)m(y) \lesssim \sum_{x \in F} m(x) \sum_{(y,k) \in \gamma(x)} \phi(y,k)m(y).$$

Proof: We only need to check that for any $(y,k) \in \mathfrak{R}(F^*)$, one has

$$m\{x \in F, x \in B(y,k)\} \gtrsim V(y,k). \quad (40)$$

Indeed, once (40) is established, Fubini theorem yields

$$\begin{aligned} \sum_{(y,k) \in \mathfrak{R}(F^*)} \phi(y,k)V(y,k)m(y) &\lesssim \sum_{(y,k) \in \mathfrak{R}(F^*)} \phi(y,k)m(y) \sum_{x \in \Gamma} m(x) \mathbf{1}_{F \cap B(y,k)}(x) \\ &= \sum_{x \in F} m(x) \sum_{(y,k) \in \mathfrak{R}(F^*)} \phi(y,k)m(y) \mathbf{1}_{B(x,k)}(y) \\ &\leq \sum_{x \in F} m(x) \sum_{(y,k) \in \gamma(x)} \phi(y,k)m(y). \end{aligned}$$

We turn now to the proof of (40). Since $(y,k) \in \mathfrak{R}(F^*)$, there exists $x_0 \in F^*$ such that $y \in B\left(x_0, \frac{k}{2C_\rho}\right)$. Notice then that $B\left(x_0, \frac{k}{2C_\rho}\right) \subset B(y,k)$. Consequently, assumption (DV) implies

$$m(B(x_0,k) \cap B(y,k)) \geq V\left(x_0, \frac{k}{2C_\rho}\right) \geq cV(x_0,k)$$

for some constant $c \in (0, 1]$ depending only of C_{dv} and C_ρ . Thus

$$m(B(x_0,k) \cap B(y,k)^c) \leq (1-c)V(x_0,k)$$

Therefore,

$$\begin{aligned} m(F \cap B(y,k)) &\geq m(F \cap B(x_0,k)) - m(B(x_0,k) \cap B(y,k)^c) \\ &\geq (\delta + c - 1)V(x_0,k) \\ &\gtrsim (\delta + c - 1)V(y,k) \end{aligned}$$

where the last line holds because the doubling property (DV) implies $V(y,k) \leq V(x_0, 2C_\rho k) \lesssim V(x_0,k)$. As a consequence, (40) holds and the lemma is proven if we choose $\delta \in (1-c, 1)$. \square

Theorem 3.8. (i) *The following inequality holds, whenever $f \in T^1(\Gamma)$ and $g \in T^\infty(\Gamma)$:*

$$\sum_{(y,k) \in \Gamma \times \mathbb{N}^*} \frac{m(y)}{k} |f(y,k)g(y,k)| \lesssim \sum_{y \in \Gamma} \mathcal{A}f(x) \mathcal{C}f(x).$$

(ii) *The pairing*

$$\langle f, g \rangle = \sum_{(y,k) \in \Gamma \times \mathbb{N}^*} \frac{m(y)}{k} f(y,k)g(y,k)$$

realizes $T^\infty(\Gamma)$ as equivalent to the Banach space dual of $T^1(\Gamma)$.

(iii) *Every element $f \in T^1(\Gamma)$ can be written as*

$$f = \sum \lambda_j a_j \quad \text{in } T^1(\Gamma),$$

where a_j are T^1 -atoms, $\lambda_j \in \mathbb{R}$, $\sum |\lambda_j| \lesssim \|f\|_{T^1}$.

(iv) *Moreover, if $f \in T^1(\Gamma) \cap T^2(\Gamma)$, the atomic decomposition can be chosen to be convergent in $T^2(\Gamma)$.*

We use the following Whitney decomposition.

Lemma 3.9. *Let (Γ, μ) a weighted graph equipped with a quasidistance ρ satisfying (DV).*

There exists $C > 0$ such that, for all subset $E \subset \Gamma$ with finite measure, there exists a finite sequence of sets $(Q_i)_{i \in [1, n]}$ such that

$$(i) \quad E = \bigcup_{i=1}^n Q_i,$$

$$(ii) \quad Q_i \cap Q_j = \emptyset \text{ if } i \neq j,$$

$$(iii) \quad \frac{1}{C} \rho(Q_i, E^c) \leq r_i := \max\{\rho(x, y), x, y \in Q_i\} + 1 \leq C \rho(Q_i, E^c).$$

Proof: (Lemma 3.9)

Let M (depending only of C_ρ) be the constant replacing 5 in the version of the lemma ‘5r’ adapted to quasidistances. Define for all $x \in E$, $B_x = B(x, \frac{\rho(x, E^c)}{M})$. We use then Lemma ‘Mr’ and we end the proof as in [10].

Notice that in the particular case of graphs, under assumption (DV), $m(E) < +\infty$ is equivalent to E finite. Thus the sequence $(Q_j)_j$ is necessary finite. \square

Proof: (Theorem 3.8)

(i) Let us define the truncated cone $\gamma^h(x)$ by

$$\gamma^h(x) = \{(y, k) \in \Gamma \times \mathbb{N}^*, \rho(x, y) < k < h\}$$

and define

$$\mathcal{A}(f|h)(x) = \left(\sum_{(y, k) \in \gamma^h(x)} \frac{m(y)}{kV(x, k)} |f(y, k)|^2 \right)^{\frac{1}{2}}.$$

Check that $\mathcal{A}(f|h)(x)$ is nondecreasing in h , and that $\mathcal{A}(f|\infty)(x) = \mathcal{A}f(x)$. For every g , we define the stopping time $h(x)$ as

$$h(x) = \sup\{h \in \mathbb{N}^*, \mathcal{A}(g|h)(x) \leq M \mathcal{C}g(x)\}$$

where M is a large constant that will be fixed later.

Define for $x \in \Gamma$ and $k \in \mathbb{N}^*$ the sets $B_2(x, k)$ and $B_3(x, k)$ by

$$B_2(x, k) := \{a \in \Gamma, \exists y \in \Gamma : \max\{\rho(x, y), \rho(y, a)\} < k\}$$

and

$$B_3(x, k) := \{a \in \Gamma, \exists y, z \in \Gamma : \max\{\rho(x, y), \rho(y, z), \rho(z, a)\} < k\}.$$

Let us prove first that for any $x \in \Gamma$ and $k \in \mathbb{N}^*$, one has

$$\bigcup_{y \in B(x, k)} \gamma^k(y) \subset \mathcal{T}(B_3(x, k)). \quad (41)$$

Indeed, if $(a, l) \in \bigcup_{y \in B(x, k)} \gamma^k(y)$, then there exists $y \in B(x, k)$ such that $\rho(a, y) < l < k$. As a consequence $\max\{\rho(x, y), \rho(y, a)\} < k$ and $a \in B_2(x, k)$. From the definition of $B_3(x, k)$, one get easily that $\rho(B_2(x, k), B_3(x, k)^c) \geq k$. Hence $\rho(a, B_3(x, k)^c) \geq k > l$, which yields that $(a, l) \in \mathcal{T}(B_3(x, k))$.

Let us prove now the following result: whenever $x \in \Gamma$ and $k \in \mathbb{N}^*$, there holds

$$m(\{y \in B(x, k), h(y) \geq k\}) \geq \frac{1}{2} V(x, k). \quad (42)$$

Check that $B(x, k) \subset B_3(x, k) \subset B(x, \alpha k)$ where $\alpha = C_\rho(1 + 2C_\rho) \geq 1$. Together with (41), it yields

$$\begin{aligned} \frac{1}{V(x, k)} \sum_{y \in B(x, k)} [\mathcal{A}(g|k)]^2(y) m(y) &= \frac{1}{V(x, k)} \sum_{y \in B(x, k)} m(y) \sum_{(z, l) \in \gamma^k(y)} \frac{m(z)}{lV(y, l)} |g(z, l)|^2 \\ &\leq \frac{1}{V(x, k)} \sum_{(z, l) \in \mathcal{T}(B(x, \alpha k))} \frac{m(z)}{l} |g(z, l)|^2 \sum_{y \in B(z, l)} \frac{m(y)}{V(y, l)} \\ &\lesssim \frac{1}{V(x, \alpha k)} \sum_{(z, l) \in \mathcal{T}(B(x, \alpha k))} \frac{m(z)}{l} |g(z, l)|^2 \\ &\leq \inf_{y \in B(x, k)} \mathcal{C}g(y)^2 \end{aligned}$$

where the third line is a consequence of the doubling property (DV) and the fact that

$$\sum_{y \in B(z,l)} \frac{m(y)}{V(y,l)} \lesssim 1.$$

Thus, we deduce

$$\begin{aligned} \frac{M^2}{V(x,k)} m(\{y \in B, h(y) < r\}) \inf_{y \in B(x,k)} Cg(y)^2 &\leq \frac{1}{V(x,k)} \sum_{y \in B(x,k)} \mathbb{1}_{\{h(y) < r\}} [MCg(y)]^2 m(y) \\ &\leq \frac{1}{V(x,k)} \sum_{y \in B(x,k)} \mathbb{1}_{\{h(y) < r\}} \mathcal{A}(g|r)^2 m(y) \\ &\leq \frac{1}{V(x,k)} \sum_{y \in B(x,k)} \mathcal{A}(g|r)^2 m(y) \\ &\leq C_{\rho,dv} \inf_{y \in B(x,k)} Cg(y)^2. \end{aligned}$$

where $C_{\rho,dv}$ depends only of C_ρ and C_{dv} . We choose then M such that $\frac{C_{\rho,dv}}{M^2} \leq \frac{1}{2}$ and we obtain (42). From (42) and with Fubini theorem, one has for any non-negative function ϕ on $\Gamma \times \mathbb{N}^*$,

$$\begin{aligned} \sum_{(y,k) \in \Gamma \times \mathbb{N}^*} \phi(y,k) V(y,k) m(y) &\lesssim \sum_{(y,k) \in \Gamma \times \mathbb{N}^*} \phi(y,k) m(y) \sum_{x \in \Gamma} m(x) \mathbb{1}_{\{x \in B(y,k), h(x) \geq k\}} \\ &= \sum_{x \in \Gamma} m(x) \sum_{(y,k) \in \Gamma \times \mathbb{N}^*} \phi(y,k) m(y) \mathbb{1}_{\{y \in B(x,k), k \leq h(x)\}} \\ &= \sum_{x \in \Gamma} m(x) \sum_{(y,k) \in \gamma^{h(x)}(x)} \phi(y,k) m(y). \end{aligned}$$

Take $\phi(y,k) = |f(y,k)| |g(y,k)| \frac{1}{kV(y,k)}$ and by Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_{(y,k) \in \Gamma \times \mathbb{N}^*} \frac{m(y)}{k} |f(y,k)g(y,k)| &\lesssim \sum_{x \in \Gamma} \mathcal{A}(f|h(x))(x) \mathcal{A}(g|h(x))(x) m(x) \\ &\leq M \sum_{x \in \Gamma} \mathcal{A}f(x) Cg(x) m(x) \end{aligned}$$

- (ii) The fact that every element $g \in T^\infty$ induces a linear functional on T^1 is immediate from (i). The converse is obtained by a classical argument. Indeed, if $l \in (T^1(\Gamma))^*$, we construct a function $g \in L_{loc}^2(\Gamma)$ such that $l(f) = \langle f, g \rangle$ for any function f with finite support. We check that $g \in T^\infty$ and we conclude by noticing that the space made of finitely supported functions is dense in $T^1(\Gamma)$. Consequently, $T^\infty(\Gamma)$ is the dual of $T^1(\Gamma)$. See proof of Theorem 1 in [11] for details.
- (iii) Let $\delta \in (0,1)$ sufficiently close to 1 so that Lemma 3.7 is verified. Define for all $i \in \mathbb{Z}$ the set O^i as $O^i = \{x \in \Gamma, \mathcal{A}(f) > 2^i\}$ and let $F_i = (O^i)^c$. We can define then F_i^* and O_*^i and Proposition 3.6 provides $O_*^i \supset O^i$ and $m(O_*^i) \lesssim m(O^i)$. Notice that $\bigcup \mathcal{T}(O^i)$ and then $\bigcup \mathfrak{T}(O_*^i)$ contain the support of f . Indeed if $f(y,k) > 0$, then for all $x \in B(y,k)$, $\mathcal{A}f(x) > \epsilon_{y,k} > 0$. Thus $B(y,k) \subset O_i$ for some $i \in \mathbb{Z}$ and then $(y,k) \in \mathcal{R}(O_i)$.

Remark that $m(O_*^i) \lesssim m(O^i) \lesssim 2^{-i} \|f\|_{T^1} + \infty$. Let $(Q_j^i)_{j \in \mathbb{N}}$ be the Whitney decomposition of O_*^i provided by Lemma 3.9. Then we can write $\mathfrak{T}(O_*^i) \setminus \mathfrak{T}(O_*^{i+1})$ as a disjoint union $\bigcup_j \Delta_j^i$ where

$$\Delta_j^i = (Q_j^i \times \mathbb{N}^*) \cap (\mathfrak{T}(O_*^i) \setminus \mathfrak{T}(O_*^{i+1})).$$

Denote by r_j^i the diameter of Q_j^i and $x_j^i \in Q_j^i$, we claim that there exists $C > 0$ such that any j, i , there holds

$$\Delta_j^i \subset \mathcal{T}(B_j^i) := \mathcal{T}(B(x_j^i, Cr_j^i)). \quad (43)$$

It suffices to check that $\max\{k \in \mathbb{N}^*, \exists x \in \Gamma : (x,k) \in \Delta_j^i\} \lesssim r_j^i$. But for all $y \in Q_j^i$,

$$\begin{aligned} r_j^i &\gtrsim \rho(Q_j^i, (O_*^i)^c) \\ &\gtrsim \rho(y, (O_*^i)^c) \\ &\geq \frac{1}{2C_\rho} \max\{k \in \mathbb{N}^*, (y,k) \in \mathfrak{T}(O_*^i)\}. \end{aligned}$$

And thus

$$\begin{aligned} r_j^i &\gtrsim \max\{k \in \mathbb{N}^*, \exists y \in Q_j^i : (y, k) \in \mathfrak{T}(O_*^i)\} \\ &= \max\{k \in \mathbb{N}^*, \exists y \in \Gamma : (y, k) \in (Q_j^i \times \mathbb{N}^*) \cap \mathfrak{T}(O_*^i)\} \\ &\leq \max\{k \in \mathbb{N}^*, \exists y \in \Gamma : (y, k) \in \Delta_j^i\} \end{aligned}$$

Now write

$$a_j^i = f \mathbb{1}_{\Delta_j^i} V(B_j^i)^{-\frac{1}{2}} (\mu_j^i)^{-\frac{1}{2}}$$

where

$$\mu_j^i = \sum_{(y,k) \in \Delta_j^i} \frac{m(y)}{k} |f(y, k)|^2$$

and set

$$\lambda_j^i = V(B_j^i)^{\frac{1}{2}} (\mu_j^i)^{\frac{1}{2}}.$$

We have then

$$f = \sum_{i,j} \lambda_j^i a_j^i, \quad (44)$$

where the convergence holds pointwise. Observe first that by construction, a_j^i is a T^1 -atom associated with the ball B_j^i . Thus the series in (44) converge in $T^1(\Gamma)$ and provide the desired atomic decomposition if we have that $\sum \lambda_j^i \lesssim \|\mathcal{A}f\|_{L^1}$. However, since $x \notin B_j^i$ implies that $(y, k) \notin \hat{B}_j^i$ for all $(y, k) \in \gamma(x)$, remark that

$$\begin{aligned} \mu_j^i &\leq \sum_{(y,k) \in \mathcal{T}(B_j^i) \cap \mathfrak{R}(F_{i+1}^*)} \frac{m(y)}{k} |f(y, k)|^2 \\ &= \sum_{(y,k) \in \mathfrak{R}(F_{i+1}^*)} \frac{m(y)}{k} |f(y, k)|^2 \mathbb{1}_{\mathcal{T}(B_j^i)}(y, k) \\ &\lesssim \sum_{x \in F_{i+1}} m(x) \sum_{(y,k) \in \gamma(x)} \frac{m(y)}{kV(y, k)} |f(y, k)|^2 \mathbb{1}_{\mathcal{T}(B_j^i)}(y, k) \\ &\leq \sum_{x \in F_{i+1}} m(x) \sum_{(y,k) \in \gamma(x)} \frac{m(y)}{kV(y, k)} |f(y, k)|^2 \mathbb{1}_{B_j^i}(x) \\ &= \sum_{x \in B_j^i \cap F_{i+1}} |\mathcal{A}f(x)|^2 m(x) \\ &\leq V(B_j^i) 4^{i+1}, \end{aligned}$$

where we used Lemma 3.7 for the third line and the definition of F_{i+1}^* for the last one. We conclude by noticing that

$$\begin{aligned} \sum_{i,j} \lambda_j^i &\lesssim \sum_{i,j} V(B_j^i) 2^i \\ &\lesssim \sum_{i,j} m(Q_j^i) 2^i \\ &= \sum_{i \in \mathbb{Z}} |O_i^*| 2^i \\ &\lesssim \sum_{i \in \mathbb{Z}} |O_i| 2^i \\ &\lesssim \|\mathcal{A}f\|_{L^1}. \end{aligned}$$

(iv) For the last point, it suffices to show that if $f \in T^2(\Gamma) \cap T^1(\Gamma)$ the decomposition constructed in (iii) converges in T^2 . Notice that

First, since $\sum_{x \in V(y,k)} \frac{m(x)}{V(x,k)} \lesssim 1$, check that

$$\|f\|_{T^2}^2 \lesssim \sum_{(y,k) \in \Gamma \times \mathbb{N}^*} \frac{m(y)}{k} |f(y, k)|^2.$$

Recall that the sets Δ_j^i form a partition of the support of f . Thus

$$\begin{aligned} \left\| \sum_{|i,j>N} \lambda_j^i a_j^i \right\|_{T^2}^2 &\simeq \sum_{(y,k) \in \Gamma \times \mathbb{N}^*} \frac{m(y)}{k} \left| \sum_{|i,j>N} \mathbb{1}_{\Delta_j^i} f(y,k) \right|^2 \\ &\leq \sum_{|i,j>N} \sum_{(y,k) \in \Delta_j^i} \frac{m(y)}{k} |f(y,k)|^2 \\ &\xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

□

4 Equality of Hardy spaces

In all this section, (Γ, μ) is a weighted graph, equipped with a quasidistance ρ and satisfying (LB) and (DV) and (UE).

4.1 $H_{mol}^1 \cap L^2 \subset E_{quad}^1$

Proposition 4.1. *Let $\epsilon > 0$, $M \in \mathbb{N}^*$, $p \in (1, 2]$ and $\beta > 0$. Then $H_{mol, M, p, \epsilon}^1(\Gamma) \cap L^2(\Gamma) \subset E_{quad, \beta}^1(\Gamma)$ and*

$$\|f\|_{H_{quad, \beta}^1} \lesssim \|f\|_{H_{mol, M, p, \epsilon}^1} \quad \forall f \in H_{mol, M, p, \epsilon}^1(\Gamma) \cap L^2(\Gamma).$$

Proof: The proof is similar to the one of [19, Proposition 4.1]. Let $f \in H_{mol, M, p, \epsilon}^1 \cap L^2(\Gamma)$. Then there exist $(\lambda_i)_{i \in \mathbb{N}} \in \ell^1$ and a sequence $(a_i)_{i \in \mathbb{N}}$ of (M, p, ϵ) -molecules such that $f = \sum \lambda_i a_i$ where the convergence is in $L^1(\Gamma)$ and

$$\sum_{i \in \mathbb{N}} |\lambda_i| \simeq \|f\|_{H_{mol, M, p, \epsilon}^1}.$$

First, since $\|P^k\|_{1 \rightarrow 1} \leq 1$ for all $k \in \mathbb{N}$, the operators Δ^β and then $\Delta^\beta P^{k-1}$ are L^1 -bounded for $\beta > 0$ (see [15]). Consequently,

$$\Delta^\beta P^{l-1} \sum_{i \in \mathbb{N}} \lambda_i a_i = \sum_{i \in \mathbb{N}} \lambda_i \Delta^\beta P^{l-1} a_i.$$

Since the L^1 -convergence in Γ implies the pointwise convergence, that is, for all $x \in \Gamma$,

$$\begin{aligned} \left| \Delta^\beta P^{k-1} \sum_{i \in \mathbb{N}} \lambda_i a_i(x) \right| &= \left| \sum_{i \in \mathbb{N}} \lambda_i \Delta^\beta P^{k-1} a_i(x) \right| \\ &\leq \sum_{i \in \mathbb{N}} |\lambda_i| |\Delta^\beta P^{k-1} a_i(x)|. \end{aligned}$$

From here, the estimate

$$\|L_\beta f\|_{L^1} = \left\| L_\beta \sum_{i \in \mathbb{N}} \lambda_i a_i \right\|_{L^1} \lesssim \sum_{i \in \mathbb{N}} |\lambda_i| \|L_\beta a_i\|_{L^1}$$

is just a consequence of the generalized Minkowski inequality.

It remains to prove that there exists a constant C such that for all (M, p, ϵ) -molecules a , one has

$$\|L_\beta a\|_{L^1} \leq C. \tag{45}$$

Let $x \in \Gamma$ and $s > 0$ associated with the (M, p, ϵ) -molecule a . By Hölder inequality and the doubling property, we may write

$$\|L_\beta a\|_{L^1} \lesssim \sum_{j=0}^{\infty} V(x, 2^j s)^{1-\frac{1}{p}} \|L_\beta a\|_{L^p(C_j(x, s))}. \tag{46}$$

We will now estimate each term of the above sum. We treat first the case $j = 0$. Remark that $\|(I + s\Delta)^{-1}\|_{L^p \rightarrow L^p} \leq 1$. Then $(I - (I + s\Delta)^{-1})^M$ is L^p -bounded by 1. Together with the fact that L_β is L^p -bounded (see Theorem 2.23), one has

$$\|L_\beta a\|_{L^p(C_0(x,s))} \lesssim \|a\|_{L^p} \leq \|b\|_{L^p} \lesssim \frac{1}{V(x,s)^{1-\frac{1}{p}}}.$$

When $j \geq 1$, decompose $\|L_\beta a\|_{L^p(C_j(x,s))}$ as

$$\|L_\beta a\|_{L^p(C_j(x,s))} \leq \sum_{k \geq 0} \|L_\beta (I - (I + s\Delta)^{-1})^M [\mathbb{1}_{C_k(x,s)} b]\|_{L^p(C_j(x,s))}.$$

Notice, if $|j - k| \geq n_\rho$, that

$$\rho(C_k(x,s), C_j(x,s)) \simeq 2^{\max\{j,k\}} s$$

where n_ρ stands for $2 + \ln_2 C_\rho$.

If $|j - k| < n_\rho$, the L^p -boundedness of L_β and the uniform L^p -boundedness of $(I - (I + s\Delta)^{-1})^M$ implies

$$\|L_\beta (I - (I + s\Delta)^{-1})^M [\mathbb{1}_{C_k(x,s)} b]\|_{L^p(C_j(x,s))} \lesssim \|b\|_{L^p(C_k(x,s))}.$$

Besides, there exists $\alpha > 0$ such that $(C_k(x,s), C_j(x,s), x)$ are α -Gaffney suited whenever $|j - k| \geq n_\rho$. Thus, Proposition 2.13 yields

$$\begin{aligned} \|L_\beta a\|_{L^p(C_j(B))} &\lesssim \sum_{|j-k| \geq n_\rho} V(x, 2^j s)^{\frac{1}{p}-\frac{1}{2}} \|L_\beta (I - (I + s\Delta)^{-1})^M [\mathbb{1}_{C_k(x,s)} b]\|_{L^2(C_j(x,s))} \\ &\quad + \sum_{|j-k| < n_\rho} \|b\|_{L^p(C_k(x,s))} \\ &\lesssim \sum_{|j-k| \geq n_\rho} V(x, 2^j s)^{\frac{1}{p}-1} 2^{-\max\{j,k\}M} \|b\|_{L^1(C_k(x,s))} + \sum_{|j-k| < n_\rho} \|b\|_{L^p(C_k(x,s))} \\ &\lesssim \sum_{|j-k| \geq n_\rho} V(x, 2^j s)^{\frac{1}{p}-1} 2^{-\max\{j,k\}M} + \sum_{|j-k| < n_\rho} \frac{2^{-k\epsilon}}{V(x, 2^k s)^{1-\frac{1}{p}}} \\ &\lesssim 2^{-j\bar{\epsilon}} V(x, 2^j s)^{\frac{1}{p}-1} \left(\sum_{|j-k| \geq n_\rho} 2^{-k\frac{M}{2}} + \sum_{|j-k| < n_\rho} 2^{-k\frac{\epsilon}{2}} \right) \\ &\lesssim 2^{-\bar{\epsilon}j} V(x, 2^j s)^{\frac{1}{p}-1} \end{aligned}$$

where $\bar{\epsilon} = \frac{1}{2} \min\{\epsilon, M\}$.

As a consequence, one has

$$\begin{aligned} \|L_\beta a\|_{L^1} &\lesssim \sum_{j \geq 0} 2^{-\bar{\epsilon}j} \left(\frac{V(x, 2^j s)}{V(x, 2^j s)} \right)^{1-\frac{1}{p}} \\ &\lesssim 1. \end{aligned}$$

□

Proposition 4.2. *Let $\epsilon > 0$, $M \in \mathbb{N}^*$, $p \in (1, 2]$. Then $H_{mol, M-\frac{1}{2}, p, \epsilon}^1(T_\Gamma) \cap L^2(T_\Gamma) \subset E_{quad, \frac{1}{2}}^1(T_\Gamma)$ and*

$$\|f\|_{H_{quad, \frac{1}{2}}^1} \lesssim \|f\|_{H_{mol, M-\frac{1}{2}, p, \epsilon}^1} \quad \forall f \in H_{mol, M-\frac{1}{2}, p, \epsilon}^1(T_\Gamma) \cap L^2(T_\Gamma).$$

Proof: The proof is similar to the previous one then we will point out only the main differences. Since d^* and P^k are L^1 -bounded, it is enough to prove the uniform boundedness of $\|L_{\frac{1}{2}} \Delta^{-\frac{1}{2}} d^* a\|_{L^1}$ when a is a (M, p, ϵ) -molecule.

However, notice that if $a = s^{M-\frac{1}{2}} d \Delta^{M-1} (I + s\Delta)^{-M+\frac{1}{2}} b$ (with s, b associated with a), one has

$$\|L_{\frac{1}{2}} \Delta^{-\frac{1}{2}} d^* a\|_{L^1} = \|L_{\frac{1}{2}} (I - (I + s\Delta)^{-1})^{M-\frac{1}{2}} b\|_{L^1}.$$

We conclude then as in Proposition 4.1.

□

Proposition 4.3. *The space $E_0^1(\Gamma)$ is continuously embedded in $E_{quad, \frac{1}{2}}^1(\Gamma)$ and*

$$\|f\|_{H_{quad, \frac{1}{2}}^1} \lesssim \|f\|_{E_0^1} \quad \forall f \in E_0^1(\Gamma).$$

Proof: We refer to subsection 2.5 for the definition of $E_0^1(\Gamma)$ and of atoms.

Due to the definition of $E_0^1(\Gamma)$, we only need to check that the quantity $\|L_\beta a\|_{L^1}$ is uniformly bounded on atoms. The proof is analogous to the one of Proposition 4.1, using Proposition 2.15. \square

4.2 $E_{quad}^1 \subset H_{mol}^1 \cap L^2$

Proposition 4.4. *Let $M \in \mathbb{N}^*$, $\epsilon > 0$ and $\beta > 0$. Then $E_{quad, \beta}^1(\Gamma) \subset H_{mol, M, 2, \epsilon}^1(\Gamma) \cap L^2(\Gamma)$ and*

$$\|f\|_{H_{mol, M, 2, \epsilon}^1} \lesssim \|f\|_{H_{quad, \beta}^1} \quad \forall f \in H_{quad, \beta}^1(\Gamma).$$

Proposition 4.5. *Let $M \in \mathbb{N}^*$, $\epsilon > 0$ and $\beta > 0$. Then $E_{quad, \beta}^1(T_\Gamma) \subset H_{mol, M - \frac{1}{2}, 2, \epsilon}^1(\Gamma) \cap L^2(\Gamma)$ and*

$$\|f\|_{H_{mol, M - \frac{1}{2}, 2, \epsilon}^1} \lesssim \|f\|_{H_{quad, \beta}^1} \quad \forall f \in H_{quad, \beta}^1(T_\Gamma).$$

Proof: The proofs of the two above results are analogous to the ones in subsections 4.3 and 4.4 of [19], using the atomic decomposition in Theorem (3.8) and Propositions 2.16 and 2.12.

The use of L^1 - L^2 off-diagonal estimates instead of the L^2 - L^2 ones enable to simplify the proof (the homogeneous dimension do not need to appear). \square

4.3 Proof of Theorems 1.26, 1.27 and 1.29

Proof: (Theorem 1.26)

Let $\beta > 0$, $M \in \mathbb{N}^*$, $p \in (1, 2]$ and $\epsilon > 0$. Propositions 4.1 and 4.4 yield the continuous embeddings

$$H_{mol, M, p, \epsilon}^1(\Gamma) \cap L^2(\Gamma) \subset E_{quad, \beta}^1(\Gamma) \subset H_{mol, M, 2, \epsilon}^1(\Gamma) \cap L^2(\Gamma) \subset H_{mol, M, p, \epsilon}^1(\Gamma) \cap L^2(\Gamma).$$

Thus, we deduce

$$H_{mol, M, p, \epsilon}^1(\Gamma) \cap L^2(\Gamma) = E_{quad, \beta}^1(\Gamma) \quad (47)$$

with equivalent norms. In particular, $E_{quad, \beta}^1(\Gamma) \subset L^1(\Gamma)$.

Since the space of finite sum of (M, p, ϵ) -molecule is dense in $H_{mol, M, p, \epsilon}^1(\Gamma)$ (see [7, Lemma 4.5] or [19, Lemma 3.5]), the space $H_{mol, M, p, \epsilon}^1(\Gamma)$ is the completion of $H_{mol, M, p, \epsilon}^1(\Gamma) \cap L^2(\Gamma)$ in L^1 . The completion $H_{quad, \beta}^1(\Gamma)$ of $E_{quad, \beta}^1(\Gamma)$ in L^1 exists then too and satisfies

- (i) the two sets $H_{quad, \beta}^1(\Gamma)$ and $H_{mol, M, p, \epsilon}^1(\Gamma)$ are equal,
- (ii) the norms in $H_{quad, \beta}^1(\Gamma)$ and $H_{mol, M, p, \epsilon}^1(\Gamma)$ are equivalent.

\square

Proof: (Theorem 1.27)

Let $M \in \mathbb{N}^*$, $p \in (1, 2]$ and $\epsilon > 0$. Propositions 4.2 and 4.5 yield the continuous embeddings

$$H_{mol, M - \frac{1}{2}, p, \epsilon}^1(T_\Gamma) \cap L^2(T_\Gamma) \subset E_{\frac{1}{2}, \beta}^1(T_\Gamma) \subset H_{mol, M - \frac{1}{2}, 2, \epsilon}^1(T_\Gamma) \cap L^2(T_\Gamma) \subset H_{mol, M - \frac{1}{2}, p, \epsilon}^1(T_\Gamma) \cap L^2(T_\Gamma).$$

from which we deduce the equality of the all the spaces, with equivalent norms.

It follows that the completion of $E_{quad, \frac{1}{2}}^1(T_\Gamma)$ in $L^1(T_\Gamma)$ exists and satisfies, since $H_{mol, M - \frac{1}{2}, p, \epsilon}^1(T_\Gamma)$ is the completion of $H_{mol, M - \frac{1}{2}, p, \epsilon}^1(T_\Gamma) \cap L^2(T_\Gamma)$ in $L^1(T_\Gamma)$,

- (i) the two sets $H_{quad, \frac{1}{2}}^1(T_\Gamma)$ and $H_{mol, M - \frac{1}{2}, p, \epsilon}^1(T_\Gamma)$ are equal,
- (ii) the norms in $H_{quad, \frac{1}{2}}^1(T_\Gamma)$ and $H_{mol, M - \frac{1}{2}, p, \epsilon}^1(T_\Gamma)$ are equivalent.

Moreover, notice that if $F \in H^2(T_\Gamma)$,

$$F \in E_{quad, \beta}^1(T_\Gamma) \iff \Delta^{-\frac{1}{2}} d^* F \in E_{quad, \beta}^1(\Gamma). \quad (48)$$

Indeed, the implication $\Delta^{-\frac{1}{2}} d^* F \in E_{quad, \beta}^1(\Gamma) \Rightarrow F \in E_{quad, \beta}^1(T_\Gamma)$ is obvious, and the converse is due to Proposition 1.5. Theorem 1.26 implies that all the spaces $E_{quad, \beta}^1(\Gamma)$, $\beta > 0$, coincide. Together with (48), for all $\beta > 0$, the spaces $E_{quad, \beta}^1(T_\Gamma)$ coincide with $E_{quad, \frac{1}{2}}^1(T_\Gamma)$. Hence, for all $\beta > 0$, the completion $H_{quad, \beta}^1(T_\Gamma)$ of $E_{quad, \beta}^1(T_\Gamma)$ in $L^1(T_\Gamma)$ exists and satisfies

- (i) the two sets $H_{quad, \beta}^1(T_\Gamma)$ and $H_{quad, \frac{1}{2}}^1(T_\Gamma)$ are equal,
- (ii) the norms in $H_{quad, \beta}^1(T_\Gamma)$ and $H_{quad, \frac{1}{2}}^1(T_\Gamma)$ are equivalent.

□

Proof: (Theorem 1.29)

According to Corollary 2.19, it suffices to show that $H^1(\Gamma) = H_{mol, 1, 2, 3}^1(\Gamma)$ is continuously embedded in $L^1(\Gamma)$ and that $E_0^1(\Gamma)$ is continuously embedded in $H^1(\Gamma) = H_{quad, 1}^1(\Gamma)$. However, the first embedding is stated in Proposition 1.20 and the second one is due to Proposition 4.3. □

5 Examples of graph satisfying (DV) and (UE)

In this paragraph, we need to recall some classical definitions on graphs.

A path joining x to y is a sequence $x = x_0, x_1, \dots, x_n = y$ where for any $i \in \llbracket 1, n \rrbracket$ we have $x_{i-1} \sim x_i$. The length of such path is n . We define then on the graph Γ the “classical” distance $d(x, y)$ as the length of the shortest path joining x to y . We denote by $B_d(x, r)$ and $V_d(x, r)$ respectively the ball and the volume of the ball of center x and of radius r .

Let (Γ, μ) be a weighted graph with the doubling property for the distance d and that satisfy some Gaussian upper estimates of the Markov kernels

$$p_{k-1}(x, y) \lesssim \frac{1}{V_d(x, \sqrt{k})} \exp\left(-c \frac{d^2(x, y)}{k}\right) \quad \forall x, y \in \Gamma, \forall k \in \mathbb{N}^*.$$

It is the case, for example, of the Cayley graph of a discrete group with polynomial growth (see [21]).

We define then $\rho(x, y) = d^2(x, y)$ and one can easily check that both (DV) and (UE) are verified. As a consequence, the Riesz transform $\nabla \Delta^{-\frac{1}{2}}$ is L^p bounded on Γ for all $p \in (1, 2]$ and we find again the main result in [26].

The second example are fractal-like graphs, that is doubling graphs where the Markov kernel satisfies some subgaussian estimates such as

$$p_k(x, y) \lesssim \frac{1}{V_d(x, k^{\frac{1}{\beta}})} \exp\left(-c \left[\frac{d^\beta(x, y)}{k}\right]^{\frac{1}{\beta-1}}\right) \quad \forall x, y \in \Gamma, \forall k \in \mathbb{N}^* \quad (49)$$

for some $\beta > 2$. An example of such graphs is the Sierpinski carpets (see [5], [24]).

In this case, we choose $\rho(x, y) = \lfloor d^\beta(x, y) \rfloor$, where $\lfloor r \rfloor$ denotes the integer part of $r \in \mathbb{R}_+$. Since the collections of balls induced by ρ and by d coincide, one has

$$V_\rho(x, 2r) = V_d(x, s_2) \text{ and } V_d(x, s_1) = V_\rho(x, r)$$

with for all $i \in \{1, 2\}$, s_i such that $\lfloor s_i^\beta \rfloor \leq ir < \lfloor (s_i + 1)^\beta \rfloor$. Therefore, the assumption (DV) is implied by the fact that d is a doubling metric. Besides, assumption (UE) with $\eta = \frac{1}{\beta-1} \in (0, 1]$ is easily deduced from (49).

For our third example, we will present a graph that satisfies (DV) and (UE) for some quasidistance ρ , but where the collections of balls defined with ρ and d do not coincide.

Definition 5.1. Let $(\Gamma_1, \mu^1, \rho_1)$ and $(\Gamma_2, \mu^2, \rho_2)$ be two weighted graphs. The graph (Γ, μ, ρ) is the free product of Γ_1 and Γ_2 if

(i) $\Gamma = \Gamma_1 \times \Gamma_2$,

(ii) for all $x = (x_1, x_2) \in \Gamma$ and $y = (y_1, y_2) \in \Gamma$, $\mu_{xy} = \mu_{x_1 y_1}^1 \mu_{x_2 y_2}^2$,

(iii) for all $x = (x_1, x_2) \in \Gamma$ and $y = (y_1, y_2) \in \Gamma$, $\rho(x, y) = \max\{\rho_1(x_1, y_1), \rho_2(x_2, y_2)\}$.

Remark 5.2. Let (Γ, μ, ρ) be the free product of $(\Gamma_1, \mu^1, \rho_1)$ and $(\Gamma_2, \mu^2, \rho_2)$. Then the following facts are satisfied

(i) $(x_1, x_2) \sim (y_1, y_2)$ if and only if $x_1 \sim y_1$ and $x_2 \sim y_2$.

(ii) If $x \sim y$ and $x \neq y$, then $\rho(x, y) = 1$.

(iii) $d((x_1, x_2), (y_1, y_2)) = \max\{d(x_1, y_1), d(x_2, y_2)\}$.

Proposition 5.3. Let $(\Gamma_1, \mu^1, \rho_1)$ and $(\Gamma_2, \mu^2, \rho_2)$ satisfying (LB), (DV) and (UE). Let (Γ, μ, ρ) be the graph defined as the free product of Γ_1 and Γ_2 . Then the graph (Γ, μ, ρ) satisfies (LB), (DV) and (UE).

Corollary 5.4. There exists a graph (Γ, μ) satisfying (LB) that can be outfitted with a quasidistance ρ such that

- the graph (Γ, μ, ρ) satisfies (DV) and (UE),
- there does not exist any $\beta \geq 2$ such that $\rho \simeq d^\beta$.

Proof: (Corollary)

Take Γ_1 and Γ_2 two infinite graphs satisfying subgaussian estimates (49) with $\beta_1 \neq \beta_2$ (for example $\Gamma_1 = \mathbb{Z}^n$ and Γ_2 is the Sierpinski carpet). Proposition 5.3 implies that $\Gamma = \Gamma_1 \times \Gamma_2$ satisfies (LB), (DV) and (UE). Notice then that $\rho((x, y), (x, z)) \simeq d_2^{\beta_2}(y, z) \simeq d^{\beta_2}((x, y), (x, z))$ and $\rho((y, x), (z, x)) \simeq d_1^{\beta_1}(y, z) \simeq d^{\beta_1}((y, x), (z, x))$ \square

Proof: (Proposition)

The fact that Γ satisfies (LB) if and only if both Γ_1 and Γ_2 satisfies (LB) is immediate.

By construction, one has $B_\rho(x, k) = B_{\rho_1}(x_1, k) \times B_{\rho_2}(x_2, k)$. As a consequence,

$$V_\rho(x, k) = V_{\rho_1}(x_1, k) V_{\rho_2}(x_2, k)$$

and then assertion (DV) follows from the doubling property of the graphs Γ_1 and Γ_2 .

We have by construction $p(x, y) = p^1(x_1, y_1) p^2(x_2, y_2)$. Therefore, by induction, we get the relation $p_k(x, y) = p_k^1(x_1, y_1) p_k^2(x_2, y_2)$. As a consequence,

$$\begin{aligned} p_k(x, y) &\lesssim \frac{1}{V_{\rho_1}(x_1, k) V_{\rho_2}(x_2, k)} \exp \left[-c_1 \left(\frac{\rho_1(x_1, y_1)}{k} \right)^{\eta_1} - c_2 \left(\frac{\rho_2(x_2, y_2)}{k} \right)^{\eta_2} \right] \\ &\lesssim \frac{1}{V_\rho(x, k)} \exp \left[-c \left(\frac{\rho_1(x_1, y_1)}{k} \right)^\eta - c \left(\frac{\rho_2(x_2, y_2)}{k} \right)^\eta \right] \\ &\leq \frac{1}{V_\rho(x, k)} \exp \left[-c \left(\frac{\rho(x, y)}{k} \right)^\eta \right] \end{aligned}$$

where $c = \min\{c_1, c_2\}$ and $\eta = \min\{\eta_1, \eta_2\}$. Thus Γ satisfies (UE). \square

We finish the article with a question:

Question:

Let (Γ, μ, d) be any graph satisfying (LB) (DV). Does there always exist a quasidistance ρ such that (Γ, μ, ρ) satisfies (DV) and (UE)?

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