Algebra properties for Besov spaces on unimodular Lie groups

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Abstract

We consider the Besov space $B_\alpha^{p,q}(G)$ on a unimodular Lie group G equipped with a sublaplacian Δ. Using estimates of the heat kernel associated with Δ, we give several characterizations of Besov spaces, and show an algebra property for $B_\alpha^{p,q}(G) \cap L^\infty(G)$ for $\alpha > 0$, $1 \leq p \leq +\infty$ and $1 \leq q \leq +\infty$. These results hold for polynomial as well as for exponential volume growth of balls.

Keywords: Besov spaces, unimodular Lie groups, algebra property.

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1 Introduction and statement of the results

We use the following notations. $A(x) \lesssim B(x)$ means that there exists C independent of x such that $A(x) \leq C\, B(x)$ for all x. $A(x) \simeq B(x)$ means that $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$. The parameters which the constant is independent to will be either obvious from context or recalled.

1.1 Introduction

Let $d \in \mathbb{N}^*$. In $\mathbb{R}^d$, the Besov spaces $B^{p,q}_\alpha(\mathbb{R}^d)$ are obtained by real interpolation of Sobolev spaces and can be defined, for $p, q \in [1, +\infty]$ and $\alpha \in \mathbb{R}$, as the subset of distributions $\mathcal{S}'(\mathbb{R}^d)$ satisfying

$$\|f\|_{B^{p,q}_\alpha} := \|\psi * f\|_{L^p} + \left(\sum_{k=1}^{\infty} \left[2^{k\alpha} \|\varphi_k * f\|_{L^p} \right]^q \right)^{\frac{1}{q}} < +\infty \tag{1}$$

where, if $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is supported in $B(0,2)\backslash B(0,\frac{1}{2})$, φ_k and ψ are such that $\mathcal{F}\varphi_k(\xi) = \varphi(2^{-k}\xi)$ and $\mathcal{F}\psi(\xi) = 1 - \sum_{k=1}^{\infty} \varphi(2^{-k}\xi)$.

The norm of the Besov space $B^{p,q}_\alpha(\mathbb{R}^d)$ can be also written by using the heat operator. Indeed, Triebel proved in [16, 18, Section 2.12.2] that for all $p, q \in [1, +\infty]$, all $\alpha > 0$ and all integer $m > \frac{\alpha}{2}$,

$$\|f\|_{B^{p,q}_\alpha} \simeq \|f\|_{L^p} + \left(\int_0^\infty t^{(m-\frac{\alpha}{2})q} \left\| \frac{\partial^M H_t}{\partial t^M} f \right\|^q_{L^p} \frac{dt}{t} \right)^{\frac{1}{q}} \tag{2}$$

where $H_t = e^{t\Delta}$ is the heat semigroup (generated by $-\Delta$). Note that we can give a similar characterization by using, instead of the heat semigroup, the harmonic extension or another extensions obtained by convolution (see [19, 12]).

Another characterization in term of functional using differences of functions was done. Define for $M \in \mathbb{N}^*$, $f \in L^p(\mathbb{R}^d)$, $x, h \in \mathbb{R}^d$ the term

$$\nabla^M_h f(x) = \sum_{l=0}^{M} \binom{M}{l} (-1)^{M-l} f(x + lh)$$

and then for $M > \alpha > 0$, $p, q \in [1, +\infty]$

$$S^{p,q}_{\alpha,M} f = \left(\int_{\mathbb{R}^d} |h|^{-\alpha q} \|\nabla^M_h f\|^q_{L^p} \right)^{\frac{1}{q}}. \tag{3}$$

We have then for all $\alpha > 0$, $p, q \in [1, +\infty]$ and $M \in \mathbb{N}$ with $M > \alpha$,

$$\|f\|_{B^{p,q}_\alpha} \simeq \|f\|_{L^p} + S^{p,q}_{\alpha,M} f. \tag{4}$$

One of the remarkable property of Besov spaces (see [7, Proposition 1.4.3], [14, Theorem 2, p. 336], [12, Proposition 6.2]) is that $B^{p,q}_\alpha(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is an algebra for the pointwise product, that is for all $\alpha > 0$, all $p, q \in [1, +\infty]$, one has

$$\|fg\|_{B^{p,q}_\alpha} \lesssim \|f\|_{B^{p,q}_\alpha} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{B^{p,q}_\alpha}. \tag{5}$$

The idea of [7] consists in decomposing the product fg by some paraproducts. The authors of [12] wrote $B^{p,q}_\alpha(\mathbb{R}^d)$ as a trace of some weighted (non fractional) Sobolev spaces, and thus deduced the algebra property $B^{p,q}_\alpha(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ from the one of $W^{p,k}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Notice also that, when $\alpha \in (0,1)$ and $M = 1$, the algebra property of $B^{p,q}_\alpha(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a simple consequence of (4).

The property (5) have also been studied in the more general setting of Besov spaces on Lie groups. Gallagher and Sire stated in [10] an algebra property for Besov spaces on H-type groups, which are a subclass of Carnot groups. In order to do this, they used a some paradifferential calculus and a Fourier transform adapted to H-groups.

Moreover, in the more general case where G is a unimodular Lie group with polynomial growth, they used the definition of Besov spaces obtained using Littlewood-Paley decomposition proved in [9]. When $\alpha \in (0,1)$, they proved a equivalence of the Besov norms with some functionals using differences of functions, in the spirit of (3), and thus they obtained an algebra property for $B^{p,q}_\alpha(G) \cap L^\infty(G)$. They shows a recursive definition of Besov spaces and wanted to use it to extend the property (5) to $\alpha \geq 1$. However, it seems to us that there is a small gap in their proof and they actually proved the property $\|fg\|_{B^{p,q}_\alpha} \lesssim (\|f\|_{B^{p,q}_\alpha} + \|f\|_{L^\infty})(\|g\|_{B^{p,q}_\alpha} + \|g\|_{L^\infty})$.

In our paper, we defined Besov spaces on unimodular Lie group (that can be of exponential growth) for all $\alpha > 0$, and then we proved an algebra property on them. We used two approaches. One with functionals in the spirit of (3) and the other one using paraproducts. We did not state any results on homogeneous Besov spaces because the

definition of these spaces need a particular work (a extension of the work in [5] to $\alpha \notin (-1,1)$ should work). However, we have no doubt that our methods work once we get the proper definition of homogeneous Besov spaces with some good Calderón-Zygmund formulas.

Note that methods used in [10] or in the present paper are similar to the ones in [6] and [1], where fractional Sobolev spaces $L^p_\alpha(G)$ are considered on unimodular Lie groups (and on Riemannian manifolds). In these two last articles, the authors proved the algebra property for $L^p_\alpha(G) \cap L^\infty(G)$ when $p \in (1, +\infty)$ and $\alpha > 0$.

1.2 Lie group structure

In this paper, G is a unimodular connected Lie group endowed with its Haar measure dx. We recall that "unimodular" means that dx is both left- and right-invariant. We denote by $\mathcal{L}$ the Lie algebra of G and we consider a family $\mathbb{X} = \{X_1, \dots, X_k\}$ of left-invariant vector fields on G satisfying the Hörmander condition (which means that the Lie algebra generated by the family $\mathbb{X}$ is $\mathcal{L}$). Denote $\mathcal{I}_\infty(\mathbb{N}) = \bigcup_{l \in \mathbb{N}} \{1, \dots, k\}^l$. Then if $I = (i_1, \dots, i_n) \in \mathcal{I}_\infty(\mathbb{N})$, the length of I will be denoted by $|I|$ and is equal to n, whereas X_I denotes the vector field $X_{i_1} \dots X_{i_n}$.

A standard metric, called the Carnot-Caratheodory metric, is naturally associated with $(G, \mathbb{X})$ and is defined as follows. Let $l : [0,1] \to G$ be an absolutely continuous path. We say that l is admissible if there exist measurable functions $a_1, \dots, a_k : [0,1] \to \mathbb{C}$ such that

$$l'(t) = \sum_{i=1}^{k} a_i(t) X_i(l(t)) \qquad \text{for a.e. } t \in [0,1].$$

If l is admissible, its length is defined by $|l| = \displaystyle\int_0^1 \left(\sum_{i=0}^{k} |a_i(t)|^2 \right)^{\frac{1}{2}} dt$. For any $x, y \in G$, the distance $d(x,y)$ between x and y is then the infimum of the lengths of all admissible curves joining x to y (such a curve exists thanks to the Hörmander condition). The left-invariance of the X_i's implies the left-invariance of d. For short, $|x|$ denotes the distance between the neutral e and x, and therefore $d(x,y) = |y^{-1}x|$ for all x and y in G.

For $r > 0$ and $x \in G$, we denote by $B(x,r)$ the open ball with respect to the Carnot-Caratheodory metric centered at x and of radius r. Define also by $V(r)$ the Haar measure of any ball of radius r.

From now and abusively, we will write G for $(G, \mathbb{X}, d, dx)$. Recall that G has a local dimension (see [13]):

Proposition 1.1. *Let G be a unimodular Lie group and $\mathbb{X}$ be a family of left-invariant vector fields satisfying the Hörmander condition. Then G has the local doubling property, that is there exists $C > 0$ such that*

$$V(2r) \leq C V(r) \qquad \forall 0 < r \leq 1.$$

More precisely, there exist $d \in \mathbb{N}^*$ and $c, C > 0$ such that

$$c r^d \leq V(r) \leq C r^d \qquad \forall 0 < r \leq 1.$$

For balls with radius bigger than 1, we have the result of Guivarc'h (see [11]):

Proposition 1.2. *If G is a unimodular Lie group, only two situations may occur.*

Either G has polynomial growth and there exist $D \in \mathbb{N}^*$ and $c, C > 0$ such that

$$c r^D \leq V(r) \leq C r^D \qquad \forall r \geq 1,$$

or G has exponential growth and there exist $c_1, c_2, C_1, C_2 > 0$ such that

$$c_1 e^{c_2 r} \leq V(r) \leq C_1 e^{C_2 r} \qquad \forall r \geq 1.$$

We consider the positive sublaplacian Δ on G defined by

$$\Delta = -\sum_{i=1}^{k} X_i^2.$$

We will denote by $H_t = e^{-t\Delta}$ the heat semigroup on G associated with Δ.

1.3 Definition of Besov spaces

Definition 1.3. *Let G be a unimodular Lie group. We define the Schwartz space $\mathcal{S}(G)$ as the space of functions $\varphi \in C^\infty(G)$ where all the seminorms*

$$N_{I,c}(\varphi) = \sup_{x\in G} e^{c|x|}|X_I\varphi(x)| \qquad c \in \mathbb{N},\ I \in \mathcal{I}_\infty(\mathbb{N})$$

are finite.

The space $\mathcal{S}'(G)$ is defined as the dual space of $S(G)$.

Remark 1.4. *Note that we have the inclusion $\mathcal{S}(G) \subset L^p(G)$ for any $p \in [1,+\infty]$. As a consequence, $L^p(G) \subset \mathcal{S}'(G)$.*

Definition 1.5. *Let G be a unimodular Lie group and let $\alpha \geq 0$, $p,q \in [1,+\infty]$. The space $f \in B^{p,q}_\alpha(G)$ is defined as the subspace of $\mathcal{S}'(G)$ made of distributions f such that, for all $t \in (0,1)$, $\Delta^m H_t f \in L^p(G)$ and satisfying*

$$\|f\|_{B^{p,q}_\alpha} := \Lambda^{p,q}_\alpha f + \|H_{\frac{1}{2}} f\|_p < +\infty,$$

where

$$\Lambda^{p,q}_\alpha f := \left(\int_0^1 \left(t^{m-\frac{\alpha}{2}} \left\|\Delta^m H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

if $q < +\infty$ (with the usual modification if $q = +\infty$) and m stands for the only integer such that $\frac{\alpha}{2} < m \leq \frac{\alpha}{2} + 1$.

Remark 1.6. *Lemma 2.6 provides that the heat kernel h_t is in $\mathcal{S}(G)$ for all $t > 0$. Thus $H_t\varphi \in \mathcal{S}(G)$ whenever $t > 0$ and $\varphi \in \mathcal{S}(G)$. When $f \in \mathcal{S}'(G)$, the term $X_I H_t f$ denotes the distribution in $\mathcal{S}'(G)$ defined by*

$$\langle X_I H_t f, \varphi\rangle = (-1)^{|I|} \langle f, H_t X_I \varphi\rangle \qquad \forall \varphi \in \mathcal{S}(G).$$

1.4 Statement of the results

Proposition 1.7. *Let G be a unimodular Lie group. The one has for all $p \in [1,+\infty]$, all multi indexes $I \in \mathcal{I}_\infty(\mathbb{N})$ and all $t \in (0,1)$,*

$$\|X_I H_t f\|_p \leq C_I t^{-\frac{|I|}{2}} \|f\|_p \qquad \forall f \in L^p(G)\,.$$

Remark 1.8. *In particular, one has that $\|t\Delta H_t\|_p \lesssim 1$ once $t \in (0,1)$ and for all $p \in [1,+\infty]$. When $p \in (1,+\infty)$, since Δ is analytic on L^2 (and thus on L^p), we actually have $\|t\Delta H_t\|_p \lesssim 1$ for all $t > 0$. The case*

The following result gives equivalent definitions of the Besov spaces $B^{p,q}_\alpha$ only involving the Laplacian.

Theorem 1.9. *Let G be a unimodular Lie group and $p,q \in [1,+\infty]$ and $\alpha \geq 0$.*

If $m > \frac{\alpha}{2}$ and t_0 a real in $\begin{cases} (0,1) & \text{if } \alpha = 0 \\ [0,1) & \text{if } \alpha > 0 \end{cases}$, then the following norms are equivalent to the norm of $B^{p,q}_\alpha(G)$.

(i) $\displaystyle\left(\int_0^1 \left(t^{m-\frac{\alpha}{2}} \left\|\Delta^m H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} + \|H_{t_0} f\|_p.$

(ii) $\displaystyle\|H_{t_0} f\|_p + \left(\sum_{j\leq -1} \left[2^{j(m-\frac{\alpha}{2})} \|\Delta^m H_{2^j} f\|_p\right]^q\right)^{\frac{1}{q}}.$

(iii) $\displaystyle\|H_{t_0} f\|_p + \left(\sum_{j\leq -1} \left[2^{-j\frac{\alpha}{2}} \left\|\int_{2^j}^{2^{j+1}} |(t\Delta)^m H_t f| \frac{dt}{t}\right\|_p\right]^q\right)^{\frac{1}{q}}$

if we assume that $\alpha > 0$.

Remark 1.10. *Here and after, we say that "a norm N is equivalent to the norm in $B^{p,q}_\alpha$" if and only if the space of distributions $f \in \mathcal{S}'$ such that $\Delta^m H_t f$ is a locally integrable function in G for all $t > 0$ and $N(f) < +\infty$ coincides with $B^{p,q}_\alpha$ and the norm N is equivalent to $\|\cdot\|_{B^{p,q}_\alpha}$.*

The previous theorem allows us to recover some well known facts about Besov spaces in $\mathbb{R}^d$.

Corollary 1.11. *[Embeddings] Let G a unimodular Lie group, $p,q,r \in [1,+\infty]$ and $\alpha \geq 0$. We have the following continuous embedding*

$$B_\alpha^{p,q}(G) \subset B_\alpha^{p,r}(G)$$

once $q \leq r$.

Corollary 1.12. *[Interpolation]*

Let G be a unimodular Lie group. Let $s_0, s_1 \geq 0$ and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$.
Define

$$s^* = (1-\theta)s_0 + \theta s_1$$
$$\frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$
$$\frac{1}{q^*} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

The Besov spaces form a scale of interpolation for the complex method, that is, if $s_0 \neq s_1$,

$$(B_{s_0}^{p_0,q_0}, B_{s_1}^{p_1,q_1})_{[\theta]} = B_{s^*}^{p^*,q^*}.$$

The following result is another characterization of Besov spaces, using explicitly the family of vector fields $\mathbb{X}$.

Theorem 1.13. *Let G be a unimodular Lie group, $p,q \in [1,+\infty]$ and $\alpha > 0$. Let $\bar{m}$ be an integer strictly greater than α. Then*

$$\|H_{\frac{1}{2}} f\|_p + \left(\sum_{j \leq -1} \left[2^{j\frac{\bar{m}-\alpha}{2}} \max_{t \in [2^j, 2^{j+1}]} \sup_{|I| \leq \bar{m}} \|X_I H_t f\|_p \right]^q \right)^{\frac{1}{q}} \tag{6}$$

is an equivalent norm in $B_\alpha^{p,q}(G)$.

With the use of paraproducts, we can deduce from Corollary 1.12 and Theorem 1.13 the complete following Leibniz rule.

Theorem 1.14. *Let G be a unimodular Lie group, $0 < \alpha$ and $p, p_1, p_2, p_3, p_4, q \in [1,+\infty]$ such that*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}.$$

Then for all $f \in B_\alpha^{p_1,q} \cap L^{p_3}$ and all $g \in B_\alpha^{p_4,q} \cap L^{p_2}$, one has

$$\|fg\|_{B_\alpha^{p,q}} \lesssim \|f\|_{B_\alpha^{p_1,q}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|g\|_{B_\alpha^{p_4,q}}. \tag{7}$$

Remark 1.15. *The Leibniz rule implies that $B_\alpha^{p,q}(G) \cap L^\infty(G)$ is an algebra under pointwise product, that is*

$$\|fg\|_{B_\alpha^{p,q}} \lesssim \|f\|_{B_\alpha^{p,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{B_\alpha^{p,q}}.$$

Let us state another characterization of $B_\alpha^{p,q}$ in term of functionals using differences of functions.
Define $\nabla_y f(x) = f(xy) - f(x)$ for all functions f on G and all $x, y \in G$. Consider the following sublinear functional

$$L_\alpha^{p,q}(f) = \left(\int_{|y| \leq 1} \left(\frac{\|\nabla_y f\|_p}{|y|^\alpha} \right)^q \frac{dy}{V(|y|)} \right)^{\frac{1}{q}}.$$

Theorem 1.16. *Let G be a unimodular Lie group. Let $p,q \in [1,+\infty]$. Then for all $f \in L^p(G)$,*

$$L_\alpha^{p,q}(f) + \|f\|_p \simeq \Lambda_\alpha^{p,q}(f) + \|f\|_p$$

once $\alpha \in (0,1)$.

Remark 1.17. *When G has polynomial volume growth, Theorem 1.16 is the inhomogeneous counterpart of Theorem 2 in [15]. Note that this statement is new when G has exponential volume growth.*

Remark 1.18. *From Theorem 1.16, we can deduce the Leibniz rule stated in Theorem 1.14 in the case $\alpha \in (0,1)$.*

As Sobolev spaces, Besov spaces can be characterized recursively.

Theorem 1.19. *Let G be a unimodular Lie group. Let $p,q \in [1,+\infty]$ and $\alpha > 0$. Then*

$$f \in B_{\alpha+1}^{p,q}(G) \Leftrightarrow \forall i,\ X_i f \in B_\alpha^{p,q}(G) \text{ and } f \in L^p(G).$$

Remark 1.20. *Note that a similar statement is established in [10]. However, we prove this fact for $p \in [1,+\infty]$ while the authors of [10] used the boundedness of the Riesz transforms and thus are restricted to $p \in (1,+\infty)$.*

2 Estimates of the heat semigroup

2.1 Preliminaries

The following lemma is easily checked:

Lemma 2.1. *Let (A, dx) and (B, dy) be two measured spaces. Let $K(x, y) : A \times B \to \mathbb{R}_+$ be such that*

$$\sup_{x\in A}\int_B K(x,y)dy \le C_B$$

and

$$\sup_{y\in B}\int_A K(x,y)dx \le C_A.$$

Let $q \in [1, +\infty]$. Then for all $f \in L^q(B)$

$$\left(\int_A \left|\int_B K(x,y)f(y)dy\right|^q dx\right)^{\frac{1}{q}} \le C_B^{1-\frac{1}{q}} C_A^{\frac{1}{q}} \|f\|_q,$$

with obvious modifications when $q = +\infty$.

Lemma 2.2. *Let $(a, b) \in (\mathbb{Z} \cup \{\pm\infty\})^2$ such that $a < b$, $0 < \alpha < \beta$ two real numbers and $q \in [1, +\infty]$. Then there exists $C_{\alpha,\beta} > 0$ such that for any sequence $(c_n)_{n\in\mathbb{Z}}$, one has*

$$\sum_{j=a}^{b}\left[2^{j\alpha}\sum_{n=a}^{b}2^{-\max\{n,j\}\beta}c_n\right]^q \lesssim \sum_{n=a}^{b}\left[2^{(\alpha-\beta)n}c_n\right]^q.$$

Proof: We have

$$\sum_{j=a}^{b}\left[2^{j\alpha}\sum_{n=a}^{b}2^{-\max\{n,j\}\beta}c_n\right]^q = \sum_{j=a}^{b}\left[\sum_{n=a}^{b}K(n,j)d_n\right]^q$$

with $d_n = 2^{n(\alpha-\beta)}c_n$ and $K(n, j) = 2^{(j-n)\alpha}2^{(n-\max\{j,n\})\beta}$.

According to Lemma 2.1, one has to check that

$$\sup_{j\in[\![a,b]\!]}\sum_{n=a}^{b}K(n,j) \lesssim 1$$

and

$$\sup_{n\in[\![a,b]\!]}\sum_{j=a}^{b}K(n,j) \lesssim 1.$$

For the first estimate, check that

$$\begin{aligned}\sup_{j\in[\![a,b]\!]}\sum_{n=a}^{b}K(n,j) &= \sup_{j\in[\![a,b]\!]}\left[2^{j(\alpha-\beta)}\sum_{n=a}^{j}2^{n(\beta-\alpha)} + 2^{j\alpha}\sum_{n=j+1}^{b}2^{-n\alpha}\right]\\ &\le \sup_{j\in\mathbb{Z}}\left[2^{j(\alpha-\beta)}\sum_{n=-\infty}^{j}2^{n(\beta-\alpha)} + 2^{j\alpha}\sum_{n=j+1}^{+\infty}2^{-n\alpha}\right]\\ &\lesssim 1,\end{aligned}$$

since $\beta - \alpha > 0$ and $\alpha > 0$.

The second estimate can be checked similarly:

$$\begin{aligned}\sup_{n\in[\![a,b]\!]}\sum_{j=a}^{b}K(n,j) &= \sup_{j\in[\![a,b]\!]}\left[2^{-n\alpha}\sum_{j=a}^{n}2^{j\alpha} + 2^{n(\beta-\alpha)}\sum_{j=n+1}^{b}2^{j(\alpha-\beta)}\right]\\ &\lesssim 1.\end{aligned}$$

□

Proposition 2.3. *Let $s \geq 0$ and $c > 0$. Define, for all $t \in (0,1)$ and all $x, y \in G$,*

$$K_t(x,y) = \left(\frac{|y^{-1}x|^2}{t}\right)^s \frac{1}{V(\sqrt{t})} e^{-c\frac{|y^{-1}x|^2}{t}}.$$

Then, for all $q \in [1, +\infty]$,

$$\left(\int_G \left(\int_G K_t(x,y)g(y)dy\right)^q dx\right)^{\frac{1}{q}} \lesssim \|g\|_q.$$

Proof: Let us check that the assumptions of Lemma 2.1 are satisfied. For all $x \in G$ and all $t \in (0,1)$,

$$\begin{aligned}
\int_G K_t(x,y)dy &= \frac{1}{V(\sqrt{t})}\int_G \left(\frac{|y^{-1}x|^2}{t}\right)^s e^{-c\frac{|y^{-1}x|^2}{t}} dy \\
&= \frac{1}{V(\sqrt{t})}\int_{|y^{-1}x|^2<t} \left(\frac{|y^{-1}x|^2}{t}\right)^s e^{-c\frac{|y^{-1}x|^2}{t}} dy \\
&\quad + \frac{1}{V(\sqrt{t})}\int_{|y^{-1}x|^2\geq t} \left(\frac{|y^{-1}x|^2}{t}\right)^s e^{-c\frac{|y^{-1}x|^2}{t}} dy \\
&= I_1 + I_2.
\end{aligned}$$

The term I_1 is easily dominated by 1. As for I_2, it is estimated as follows:

$$\begin{aligned}
I_2 &= \sum_{j=0}^{\infty} \frac{1}{V(\sqrt{t})}\int_{2^j\sqrt{t}\leq|y^{-1}x|<2^{j+1}\sqrt{t}} \left(\frac{|y^{-1}x|^2}{t}\right)^s e^{-c\frac{|y^{-1}x|^2}{t}} dy \\
&\lesssim \sum_{j=0}^{\infty} \frac{V(2^{j+1}\sqrt{t})}{V(\sqrt{t})} 4^{js} e^{-c4^j}.
\end{aligned}$$

Notice that Propositions 1.1 and 1.2 imply that $\frac{V(2^{j+1}\sqrt{t})}{V(\sqrt{t})} \lesssim 2^{jd}$ if $2^j\sqrt{t} \leq 1$ and

$$\frac{V(2^{j+1}\sqrt{t})}{V(\sqrt{t})} = \frac{V(2^{j+1}\sqrt{t})}{V(1)}\frac{V(1)}{V(\sqrt{t})} \lesssim e^{C2^j} 2^{jd} \tag{8}$$

if $2^j\sqrt{t} \geq 1$. Hence,

$$\sum_{j=0}^{\infty} \frac{V(2^{j+1}\sqrt{t})}{V(\sqrt{t})} 4^{js} e^{-c4^j} \lesssim \sum_{j=0}^{\infty} e^{-c'4^j} \lesssim 1,$$

which yields with the uniform estimate

$$\int_G K_t(x,y)dy \lesssim 1. \tag{9}$$

In the same way, one has

$$\int_G K_t(x,y)dx \lesssim 1.$$

Lemma 2.1 provides then the desired result. □

Proposition 2.4. *Let $s \geq 0$ and $c > 0$. Define*

$$K(t,y) = \left(\frac{|y|^2}{t}\right)^s \frac{V(|y|)}{V(\sqrt{t})} e^{-c\frac{|y|^2}{t}}.$$

Then, for all $q \in [1, +\infty]$,

$$\left(\int_0^1 \left(\int_G K(t,y)g(y)dy\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \lesssim \left(\int_G |g(y)|\frac{dy}{V(|y|)}\right)^{\frac{1}{q}}.$$

Proof: Let us check again that the assumptions of Lemma 2.1 are satisfied, that are in our case

$$\sup_{t\in(0,1)}\int_G K(t,y)dy \leq C_B$$

and

$$\sup_{y\in G}\int_0^1 K(t,y)\frac{dt}{t} \leq C_A.$$

The first one is exactly as the estimate (9).For the second one, check that

$$\begin{aligned}
\int_0^1 K(t,y)\frac{dt}{t} &= \int_0^1 \frac{V(|y|)}{V(\sqrt{t})}\left(\frac{|y|^2}{t}\right)^s e^{-c'\frac{|y|^2}{t}}\frac{dt}{t} \\
&\lesssim \int_0^{|y|^2}\frac{V(|y|)}{V(\sqrt{t})}\left(\frac{|y|^2}{t}\right)^s e^{-c'\frac{|y|^2}{t}}\frac{dt}{t} + \int_{|y|^2}^{\infty}\left(\frac{|y|^2}{t}\right)^s\frac{dt}{t} \\
&\lesssim \sum_{j=0}^{+\infty}\int_{4^{-(j+1)}|y|^2}^{4^{-j}|y|^2}\frac{V(|y|)}{V(\sqrt{t})}\left(\frac{|y|^2}{t}\right)^s e^{-c'\frac{|y|^2}{t}}\frac{dt}{t} + 1 \\
&\lesssim \sum_{j=0}^{+\infty}\frac{V(|y|)}{V(2^{-j+1}|y|)}4^{js}e^{-c4^j} + 1 \\
&\lesssim \sum_{j=0}^{+\infty}2^{j(d+2s)}e^{C2^j}e^{-c4^j} + 1 \\
&\lesssim 1,
\end{aligned}$$

where the last but one line is obtained with the estimate (8). □

2.2 Estimates for the semigroup

Because of left-invariance of Δ and hypoellipticity of $\frac{\partial}{\partial t}+\Delta$, $H_t = e^{-t\Delta}$ has a convolution kernel $h_t \in C^\infty(G)$ satisfying, for all $f \in L^1(G)$ and all $x \in G$,

$$H_t f(x) = \int_G h_t(y^{-1}x)f(y)dy = \int_G h_t(y)f(xy)dy = \int_G h_t(y)f(xy^{-1})dy.$$

The kernel h_t satisfies the following pointwise estimates.

Proposition 2.5. *Let G be a unimodular Lie group. For all $I \in \mathcal{I}_\infty(\mathbb{N})$, there exist$C_I, c_I > 0$ such that for all $x \in G$, all $t \in (0,1]$, one has*

$$|X_I h_t(x)| \leq \frac{C_I}{t^{\frac{|I|}{2}}V(\sqrt{t})}\exp\left(-c_I\frac{|x|^2}{t}\right).$$

Proof: It is a straightforward consequence of Theorems VIII.2.4, VIII.4.3 and V.4.2. in [20]. □

Lemma 2.6. *Let G be a unimodular group. Then $h_t \in \mathcal{S}(G)$ for all $t > 0$.*

Proof: The case $t < 1$ is a consequence of the estimates on h_t. For $t \geq 1$, just notice that $\mathcal{S}(G) * \mathcal{S}(G) \subset \mathcal{S}(G)$. □

Proposition 2.7. *For all $I \in \mathcal{I}_\infty(\mathbb{N})$ and all $p \in [1,+\infty]$, one has*

$$\|X_I H_t f\|_p \lesssim t^{-\frac{|I|}{2}}\|f\|_p \qquad \forall t \in (0,1],\ \forall f \in L^p(G).$$

Proof: Proposition 2.5 yields for any $t \in (0,1]$

$$\|X_I H_t f\|_p \lesssim t^{-\frac{|I|}{2}}\left(\int_G\left|\int_G K_t(x,y)f(y)dy\right|^p dx\right)^{\frac{1}{p}}$$

where $K_t(x,y) = \frac{1}{V(\sqrt{t})}\exp\left(-c\frac{|y^{-1}x|^2}{t}\right)$.

The conclusion of Proposition 2.7 is an immediate consequence of Proposition 2.3. □

3 Littlewood-Paley decomposition

We need a Littlewood-Paley decomposition adapted to this context. In [10], the authors used the Littlewood-Paley decomposition proven in [9, Proposition 4.1], only established in the case of polynomial volume growth. We state here a slightly different version of the Littlewood-Paley decomposition, also valid for the case of exponential volume growth.

Lemma 3.1. *Let G be a unimodular group and let $m \in \mathbb{N}^*$. For any $\varphi \in \mathcal{S}(G)$ and any $f \in \mathcal{S}'(G)$, one has the identities*

$$\varphi = \frac{1}{(m-1)!}\int_0^1 (t\Delta)^m H_t\varphi \frac{dt}{t} + \sum_{k=0}^{m-1}\frac{1}{k!}\Delta^k H_1\varphi$$

where the integral converges in $\mathcal{S}(G)$, and

$$f = \frac{1}{(m-1)!}\int_0^1 (t\Delta)^m H_t f \frac{dt}{t} + \sum_{k=0}^{m-1}\frac{1}{k!}\Delta^k H_1 f$$

where the integral converges in $\mathcal{S}'(G)$.

Proof: We only have to prove the first identity since the second one can be obtained by duality.

Let $\varphi \in \mathcal{S}(G)$. Check first the formula

$$(m-1)! = \int_0^{+\infty}(tu)^m e^{-tu}\frac{dt}{t} = \int_0^1 (tu)^m e^{-tu}\frac{dt}{t} + \sum_{k=0}^{m-1}\frac{(m-1)!}{k!}u^k e^{-u}.$$

Thus by functional calculus, since $\varphi \subset L^2(G)$, one has

$$\varphi = \frac{1}{(m-1)!}\int_0^1 (t\Delta)^m H_t\varphi\frac{dt}{t} + \sum_{k=0}^{m-1}\frac{1}{k!}\Delta^k H_1\varphi, \tag{10}$$

where the integral converges in $L^2(G)$. Since the kernel h_t of H_t is in $\mathcal{S}(G)$ for any $t>0$ (see Lemma 2.6), the formula (10) will be proven if we have for any $c \in \mathbb{N}$ and any $I \in \mathcal{I}_\infty(\mathbb{N})$,

$$\lim_{u\to 0} N_{I,c}\left(\int_0^u (t\Delta)^m H_t\varphi\frac{dt}{t}\right) = 0. \tag{11}$$

Let $n > \frac{|I|}{2}$ be an integer. Similarly to (10), one has for all $x \in G$ and all $t \in (0,1)$,

$$H_t\varphi(x) = \frac{1}{(n-1)!}\int_t^1 (v-t)^{n-1}\Delta^n H_v\varphi(x)dv + \sum_{k=0}^{n-1}\frac{1}{k!}(1-t)^k\Delta^k H_1\varphi(x).$$

Hence, for all $x \in G$ and all $u \in (0,1)$, we have the identity

$$\begin{aligned}\int_0^u (t\Delta)^m H_t\varphi(x)\frac{dt}{t} = \frac{1}{(n-1)!}\int_t^1 \Delta^{n+m}H_v\varphi(x)\left(\int_0^{\min\{u,v\}} t^{m-1}(v-t)^{n-1}dt\right)dv \\ + \sum_{k=0}^{n-1}\frac{1}{k!}\Delta^{k+m}H_1\varphi(x)\int_0^u t^{m-1}(1-t)^k dt.\end{aligned}$$

Note that

$$\int_0^{\min\{u,v\}} t^{m-1}(v-t)^{n-1}dt \lesssim u^m v^{n-1}$$

and

$$\int_0^u t^{m-1}(1-t)^k dt \lesssim u^m.$$

Therefore, the Schwartz seminorms of $\int_0^u (t\Delta)^m H_t\varphi\frac{dt}{t}$ can be estimated by

$$\begin{aligned}N_{I,c}\left(\int_0^u (t\Delta)^m H_t\varphi\frac{dt}{t}\right) \lesssim u^m\int_0^1 v^{n-1}\sup_{x\in G} e^{c|x|}|X_I\Delta^{m+n}H_v\varphi(x)|dv \\ + u^m\sum_{k=0}^{n-1}\sup_{x\in G}e^{c|x|}|X_I\Delta^{k+m}H_1\varphi(x)|.\end{aligned} \tag{12}$$

Check then that for all $w \in (0,1]$ and all $l \in \mathbb{N}$, we have

$$\begin{aligned}\sup_{x\in G} e^{c|x|}|X_I\Delta^l H_w\varphi(x)| &= \sup_{x\in G} e^{c|x|}\left|X_I H_w\Delta^l\varphi(x)\right| \\ &\leq \sup_{x\in G} e^{c|x|}\int_G \left|X_I h_w(y^{-1}x)\right||\Delta^l\varphi(y)|dy \\ &\lesssim \sup_{x\in G}\int_G e^{c|y^{-1}x|}\left|X_I h_w(y^{-1}x)\right| e^{c|y|}|\Delta^l\varphi(y)|dy \\ &\lesssim \left(\sup_{x\in G}\int_G e^{c|y^{-1}x|}\left|X_I h_w(y^{-1}x)\right| dy\right)\sum_{|I|=2l} N_{I,c}(\varphi)\end{aligned} \tag{13}$$

where the third line holds because $|x| \leq \left|y^{-1}x\right| + |x|$.
However, for all $x \in G$ and all $w \in (0,1]$, Proposition 2.5 yields that, for all $x \in G$,

$$\begin{aligned}\int_G e^{c|y^{-1}x|}\left|X_I h_w(y^{-1}x)\right| dy &\lesssim w^{-\frac{|I|}{2}}\frac{1}{V(\sqrt{w})}\int_G e^{c|y^{-1}x|}e^{-c'\frac{|y^{-1}x|^2}{w}}dy \\ &\lesssim w^{-\frac{|I|}{2}}\frac{1}{V(\sqrt{w})}\int_G e^{-c'\frac{|y^{-1}x|^2}{2w}}dy \\ &\lesssim w^{-\frac{|I|}{2}}.\end{aligned} \tag{14}$$

By gathering the estimates (12), (13) and (14), we obtain

$$\begin{aligned}N_{I,c}\left(\int_0^u (t\Delta)^m H_t\varphi\frac{dt}{t}\right) &\lesssim u^m\left[\sum_{|I|\leq 2(m+n)} N_{I,c}(\varphi)\right]\left[\int_0^1 v^{n-1}v^{-\frac{|I|}{2}}dv + \sum_{k=0}^{n-1}1\right] \\ &\lesssim u^m \sum_{|I|\leq 2(m+n)} N_{I,c}(\varphi) \\ &\xrightarrow{u\to 0} 0,\end{aligned}$$

which proves (11) and finishes the proof. □

4 Proof of Theorem 1.9 and of its corollaries

4.1 Proof of Theorem 1.9

In this section, we will always assume that $\alpha \geq 0$, $p, q \in [1, +\infty]$.

Proposition 4.1. *For all $t_1, t_0 \in (0,1)$ and all integers $m > \frac{\alpha}{2}$,*

$$\|f\|_p \lesssim \|H_{t_0}f\|_p + \left(\int_0^1 \left(t^{m-\frac{\alpha}{2}}\left\|\Delta^m H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \qquad \forall f \in \mathcal{S}'(G)$$

when $\alpha > 0$ and

$$\|H_{t_1}f\|_p \lesssim \|H_{t_0}f\|_p + \left(\int_0^1 \left(t^{m-\frac{\alpha}{2}}\left\|\Delta^m H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \qquad \forall f \in \mathcal{S}'(G)$$

when $\alpha \geq 0$ and $q < +\infty$, with the usual modification when $q = +\infty$.

Proof: Lemma 3.1 (recall that $L^p(G) \subset \mathcal{S}'(G)$) yields the estimate

$$\|f\|_p \lesssim \int_0^1 t^m \|\Delta^m H_t f\|_p \frac{dt}{t} + \sum_{k=0}^{m-1}\|\Delta^k H_1 f\|_p.$$

However, for all $k \in \mathbb{N}$, $\|\Delta^k H_1 f\|_p \leq \frac{C}{(1-t_0)^k}\|H_{t_0} f\|_p$. Then, when $\alpha > 0$,

$$\begin{aligned}
\|f\|_p &\lesssim \int_0^1 t^m \left\|\Delta^m H_t f\right\|_p \frac{dt}{t} + \|H_{t_0} f\|_p \\
&\lesssim \left(\int_0^1 \left(t^{m-\frac{\alpha}{2}} \left\|\Delta^m H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \left(\int_0^1 t^{\frac{q'\alpha}{2}} \frac{dt}{t}\right)^{\frac{1}{q'}} + \|H_{t_0} f\|_p \\
&\lesssim \left(\int_0^1 \left(t^{m-\frac{\alpha}{2}} \left\|\Delta^m H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} + \|H_{t_0} f\|_p,
\end{aligned}$$

which prove the case $\alpha > 0$.

If $\alpha = 0$, Lemma 3.1 for the integer $m+1$ implies

$$\begin{aligned}
\|H_{t_1} f\|_p &\lesssim \int_0^1 t^{m+1} \left\|\Delta^{m+1} H_{t+t_1} f\right\|_p \frac{dt}{t} + \sum_{k=0}^{m} \|\Delta^k H_{1+t_1} f\|_p \\
&\lesssim \int_0^1 \frac{t^{m+1}}{t_1} \left\|\Delta^m H_t f\right\|_p \frac{dt}{t} + \|H_{t_0} f\|_p \\
&\lesssim \left(\int_0^1 \left(t^m \left\|\Delta^m H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \left(\int_0^1 \left(\frac{t}{t_1}\right)^{q'} \frac{dt}{t}\right)^{\frac{1}{q'}} + \|H_{t_0} f\|_p \\
&\lesssim \left(\int_0^1 \left(t^m \left\|\Delta^m H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} + \|H_{t_0} f\|_p.
\end{aligned}$$

□

Proposition 4.2. *For all integers $m > \frac{\alpha}{2}$,*

$$\left(\int_0^1 \left(t^{m-\frac{\alpha}{2}} \left\|\Delta^m H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \lesssim \|f\|_p + \left(\int_0^1 \left(t^{m+1-\frac{\alpha}{2}} \left\|\Delta^{m+1} H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

Proof: We use Lemma 3.1 and get

$$\Delta^m H_t f = \int_0^1 s \Delta H_s \Delta^m H_t f \frac{ds}{s} + H_1 \Delta^m H_t f.$$

Thus,

$$\begin{aligned}
&\left(\int_0^1 \left(t^{m-\frac{\alpha}{2}} \left\|\Delta^m H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\quad \leq \left(\int_0^1 \left(t^{m-\frac{\alpha}{2}} \int_0^1 \left\|\Delta^{m+1} H_{t+s} f\right\|_p ds\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} + \left(\int_0^1 \left(t^{m-\frac{\alpha}{2}} \left\|\Delta^m H_{t+1} f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\quad := I_1 + I_2.
\end{aligned}$$

We start with the estimate of I_1. One has $\Delta^{m+1} H_{t+s} f = H_s \Delta^{m+1} H_t f = H_t \Delta^{m+1} H_s f$. Then

$$\begin{aligned}
I_1 &\leq \left(\int_0^1 \left(t^{m-\frac{\alpha}{2}} \int_0^t \left\|\Delta^{m+1} H_t f\right\|_p ds\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} + \left(\int_0^1 \left(t^{m-\frac{\alpha}{2}} \int_t^1 \left\|\Delta^{m+1} H_s f\right\|_p ds\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&:= II_1 + II_2.
\end{aligned}$$

Notice

$$II_1 = \left(\int_0^1 \left(t^{m+1-\frac{\alpha}{2}} \left\|\Delta^{m+1} H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

which is the desired estimate. As far as II_2 is concerned,

$$II_2 = \left(\int_0^1 \left(\int_0^1 K(s,t) g(s) \frac{ds}{s}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

with $g(s) = s^{m+1-\frac{\alpha}{2}} \left\|\Delta^{m+1} H_s f\right\|_p$ and $K(s,t) = \left(\frac{t}{s}\right)^{m-\frac{\alpha}{2}} \mathbb{1}_{s\geq t}$. Since

$$\int_0^1 K(s,t)\frac{ds}{s} \lesssim 1 \qquad \text{and} \qquad \int_0^1 K(s,t)\frac{dt}{t} \lesssim 1,$$

Lemma 2.1 yields then

$$II_2 \lesssim \left(\int_0^1 g(s)^q \frac{ds}{s}\right)^{\frac{1}{q}}$$

which is also the desired estimate.

It remains to estimate I_2. First, verify that Proposition 2.7 of H_t implies $\|\Delta^m H_{t+1} f\|_p \lesssim \|f\|_p$. Then we obtain

$$I_2 \lesssim \|f\|_p$$

since $\int_0^1 t^{q(m-\frac{\alpha}{2})}\frac{dt}{t} < +\infty$. □

Proposition 4.3. *For all integers $\beta \geq \gamma > \frac{\alpha}{2}$,*

$$\left(\int_0^1 \left(t^{\beta-\frac{\alpha}{2}} \left\|\Delta^\beta H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \lesssim \left(\int_0^1 \left(t^{\gamma-\frac{\alpha}{2}} \|\Delta^\gamma H_t f\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

Proof: Proposition 2.7 implies $\left\|\Delta^{\beta-\gamma} H_{\frac{t}{2}} f\right\|_p \lesssim t^{\gamma-\beta}\|f\|_p$. Then

$$\begin{aligned}
\left(\int_0^1 \left(t^{\beta-\frac{\alpha}{2}} \left\|\Delta^\beta H_t f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} &\lesssim \left(\int_0^1 \left(t^{\gamma-\frac{\alpha}{2}} \left\|\Delta^\gamma H_{\frac{t}{2}} f\right\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\lesssim \left(\int_0^{\frac{1}{2}} \left(u^{\gamma-\frac{\alpha}{2}} \|\Delta^\gamma H_u f\|_p\right)^q \frac{du}{u}\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1 \left(t^{\gamma-\frac{\alpha}{2}} \|\Delta^\gamma H_t f\|_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}.
\end{aligned}$$

□

Remark 4.4. *Propositions 4.1, 4.2 and 4.3 imply (i) of Theorem 1.9.*

Proposition 4.5. *Let $m > \frac{\alpha}{2}$. Then*

$$\|f\|_p + \left(\sum_{j\leq -1} \left[2^{j(m-\frac{\alpha}{2})} \|\Delta^m H_{2^j} f\|_p\right]^q\right)^{\frac{1}{q}}$$

is an equivalent norm in $B^{p,q}_\alpha(G)$.

Proof: Assertion (i) in Theorem 1.9 and the following calculus prove the equivalence of norms:

$$\begin{aligned}
\left(\sum_{j\leq -1} \left[2^{j(m-\frac{\alpha}{2})} \|\Delta^m H_{2^j} f\|_p\right]^q\right)^{\frac{1}{q}} &\leq \left(\sum_{j\leq -1} \int_{2^{j-1}}^{2^j} \left[2^{j(m-\frac{\alpha}{2})} \|\Delta^m H_t f\|_p\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\lesssim \left(\int_0^{\frac{1}{2}} \left[t^{m-\frac{\alpha}{2}} \|\Delta^m H_t f\|_p\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1 \left[t^{m-\frac{\alpha}{2}} \|\Delta^m H_t f\|_p\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&= \left(\sum_{j\leq -1} \int_{2^j}^{2^{j+1}} \left[t^{m-\frac{\alpha}{2}} \|\Delta^m H_t f\|_p\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\leq \left(\sum_{j\leq -1} \left[2^{j(m-\frac{\alpha}{2})} \|\Delta^m H_{2^j} f\|_p\right]^q\right)^{\frac{1}{q}}.
\end{aligned}$$

This proves item (ii) in Theorem 1.9. □

Proposition 4.6. *Let $\alpha > 0$ and $l > \frac{\alpha}{2}$. Then*

$$\|H_{\frac{1}{2}} f\|_p + \left(\sum_{j \leq -1} \left[2^{-j\frac{\alpha}{2}} \left\| \int_{2^j}^{2^{j+1}} \left| (t\Delta)^l H_t f \right| \frac{dt}{t} \right\|_p \right]^q \right)^{\frac{1}{q}} \tag{15}$$

is an equivalent norm in $B^{p,q}_\alpha(G)$.

Proof: We denote by $\|.\|_{B^{p,q}_{\alpha,1}}$ the norm defined in (15). It is easy to check, using assertion (i) in Theorem 1.9, the Hölder inequality and the triangle inequality, that

$$\|f\|_{B^{p,q}_{\alpha,1}} \lesssim \|f\|_{B^{p,q}_\alpha}.$$

For the converse inequality, we proceed as follows. Fix an integer $m > \frac{\alpha}{2}$.

1. **Decomposition of f:**
   The first step is to decompose f as in Lemma 3.1

$$f = \frac{1}{(l-1)!} \int_0^1 (t\Delta)^l H_t f \frac{dt}{t} + \sum_{k=0}^{l-1} \frac{1}{k!} \Delta^k H_1 f \text{ in } \mathcal{S}'(G).$$

   We introduce

$$f_n = -\int_{2^n}^{2^{n+1}} (t\Delta)^l H_t f \frac{dt}{t} dt$$

   and

$$c_n = \left\| \int_{2^{n-1}}^{2^n} \left| (t\Delta)^l H_t f \right| \frac{dt}{t} \right\|_p.$$

   Remark then that

$$f = \frac{1}{(l-1)!} \sum_{n=-\infty}^{-1} f_n + \sum_{k=0}^{l-1} \frac{1}{k!} \Delta^k H_1 f \qquad \text{in } \mathcal{S}'(G).$$

2. **Estimates of $\Delta^m H_{2^j} f_n$**
   Note that

$$\begin{aligned}
&\Delta^m H_{2^j} f_n \\
&\quad = -\Delta^m H_{2^{n-1}+2^j} \int_{2^n}^{3.2^{n-1}} (t\Delta)^l H_{t-2^{n-1}} f \frac{dt}{t} - \Delta^m H_{2^n+2^j} \int_{3.2^{n-1}}^{2^{n+1}} (t\Delta)^l H_{t-2^n} f \frac{dt}{t} \\
&\quad = -\Delta^m H_{2^{n-1}+2^j} \int_{2^{n-1}}^{2^n} (t+2^{n-1})^l \Delta^l H_t f \frac{dt}{t} - \Delta^m H_{2^n+2^j} \int_{2^{n-1}}^{2^n} (t+2^n)^l \Delta^l H_t f \frac{dt}{t}.
\end{aligned}$$

   Then Proposition 2.7 implies,

$$\begin{aligned}
\|\Delta^m H_{2^j} f_n\|_p &\lesssim [(2^{(n-1)} + 2^j]^{-m} \left\| \int_{2^{n-1}}^{2^n} (t+2^{n-1})^l \Delta^l H_t f \frac{dt}{t} \right\|_p \\
&\quad + [2^n + 2^j]^{-m} \left\| \int_{2^{n-1}}^{2^n} (t+2^n)^l \Delta^l H_t f \frac{dt}{t} \right\|_p \\
&\lesssim [2^n + 2^j]^{-m} \left\| \int_{2^{n-1}}^{2^n} \left| (t\Delta)^l H_t f \right| \frac{dt}{t} \right\|_p.
\end{aligned} \tag{16}$$

   In other words,

$$\|\Delta^m H_{2^j} f_n\|_p \lesssim \begin{cases} 2^{-nm} c_n & \text{if } j \leq n \\ 2^{-jm} c_n & \text{if } j > n \end{cases}. \tag{17}$$

3. **Estimate of $\Lambda_\alpha^{p,q}(\sum f_n)$**
As a consequence,

$$\sum_{j\leq -1}\left[2^{j(m-\frac{\alpha}{2})}\left\|\Delta^m H_{2^j}\sum_{n\leq -1}f_n\right\|_p\right]^q \lesssim \sum_{j\leq -1}\left[2^{j(m-\frac{\alpha}{2})}\sum_{n\leq -1}2^{-m\max\{j,n\}}c_n\right]^q.$$

According to Lemma 2.2, since $0<m-\frac{\alpha}{2}<m$, one has

$$\sum_{j\leq -1}\left[2^{j(m-\frac{\alpha}{2})}\left\|\Delta^m H_{2^j}\sum f_n\right\|_p\right]^q \lesssim \sum_{n=-\infty}^{-1}\left[2^{-n\frac{\alpha}{2}}c_n\right]^q.$$

4. **Estimate of the remaining term**
Remark that

$$\|f\|_{B_\alpha^{p,q}} \lesssim \|H_{t_0}f\|_p + \Lambda_\alpha^{p,q}\left(\sum f_n\right) + \sum_{k=0}^{l-1}\Lambda_\alpha^{p,q}\left(\Delta^k H_1 f\right).$$

From the previous step and Proposition 4.5, we proved that

$$\Lambda_\alpha^{p,q}\left(\sum f_n\right) \lesssim \|f\|_{B_{\alpha,1}^{p,q}}.$$

In order to conclude the proof of Proposition 4.6, it suffices then to check that for all $k\in[\![0,l-1]\!]$, one has

$$\|\Delta^k H_1 f\|_{B_\alpha^{p,q}} \lesssim \|f\|_{L^p}. \tag{18}$$

Indeed, one has for all $j\leq -1$

$$\begin{aligned}\|\Delta^m H_{2^j}\Delta^k H_1 f\|_p &= \|\Delta^{m+k}H_{1+2^j}f\|_p\\ &\lesssim \left(1+2^j\right)^{-(m+k)}\|f\|_p\\ &\lesssim \|f\|_p.\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{j\leq -1}\left[2^{j(m-\frac{\alpha}{2})}\|\Delta^m H_{2^j}\Delta^k H_1 f\|_p\right]^q &\lesssim \|f\|_p^q\sum_{j\leq -1}2^{jq(m-\frac{\alpha}{2})}\\ &\lesssim \|f\|_p^q.\end{aligned}$$

□

4.2 Proof of Theorem 1.13

Proof: (Theorem 1.13)

We denote by $\|.\|_{B_{\alpha,Xsup}^{p,q}}$ the norm defined in (6). Since

$$\|\Delta^m H_{2^j}f\|_p \leq \max_{t\in[2^j,2^{j+1}]}\sup_{|I|\leq 2m}\|X_I H_t f\|_p,$$

it is easy to check that

$$\|f\|_{B_\alpha^{p,q}} \lesssim \|f\|_{B_{\alpha,Xsup}^{p,q}}.$$

For the converse inequality, it is enough to check that

$$\|f\|_{B_{\alpha,Xsup}^{p,q}} \lesssim \|f\|_{B_{\alpha,1}^{p,q}}.$$

We proceed then as the proof of Proposition 4.6 since Proposition 2.7 yields

$$\max_{t\in[2^j,2^{j+1}]}\sup_{|I|\leq\bar{m}}\|X_I H_t f_n\|_p \lesssim \begin{cases}2^{-n\frac{\bar{m}}{2}}c_n & \text{if } j\leq n\\ 2^{-j\frac{\bar{m}}{2}}c_n & \text{if } j>n\end{cases}$$

with a proof analogous to the one of (17). □

4.3 Embeddings and interpolation

Proof: (of Corollary 1.11) The proof is analogous to the one of Proposition 2.3.2/2 in [17] using Proposition 4.5. It relies on the monotonicity of l_q spaces, see [17, 1.2.2/4]. □

Let us turn to interpolation properties of Besov spaces, that implies in particular Corollary 1.12.

Corollary 4.7. *Let $s_0, s_1, s \geq 0$, $1 \leq p_0, p_1, p, q_0, q_1, r \leq \infty$ and $\theta \in (0,1)$.*
Define

$$s^* = (1-\theta)s_0 + \theta s_1,$$
$$\frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$
$$\frac{1}{q^*} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

i. If $s_0 \neq s_1$ then

$$(B_{s_0}^{p,q_0}, B_{s_1}^{p,q_1})_{\theta,r} = B_{s^*}^{p,r}.$$

ii. In the case where $s_0 = s_1$, we have

$$(B_s^{p,q_0}, B_s^{p,q_1})_{\theta,q^*} = B_s^{p,q^*}.$$

iii. If $p^* = q^* := r$,

$$(B_{s_0}^{p_0,q_0}, B_{s_1}^{p_1,q_1})_{\theta,r} = B_{s^*}^{r,r}.$$

iv. If $s_0 \neq s_1$,

$$(B_{s_0}^{p_0,q_0}, B_{s_1}^{p_1,q_1})_{[\theta]} = B_{s^*}^{p^*,q^*}.$$

Proof: The proof is inspired by [2, Theorem 6.4.3].

Recall (see Definition 6.4.1 in [2]) that a space B is called a retract of A if there exists two bounded linear operators $\mathcal{J} : B \to A$ and $\mathcal{P} : A \to B$ such that $\mathcal{P} \circ \mathcal{J}$ is the identity on B.

Therefore, we just need to prove that the spaces $B_\alpha^{p,q}$ are retracts of $l_q^\alpha(L^p)$ where, for any Banach space A (see paragraph 5.6 in [2]),

$$l_q^\alpha(A) = \left\{ u \in A^{\mathbb{Z}_-},\ \|u\|_{\ell_q^\alpha(A)} := \left(\sum_{j \leq 0} \left[2^{-j\frac{\alpha}{2}} \|u_j\|_A \right]^q \right)^{\frac{1}{q}} < +\infty \right\}.$$

Then interpolation on the spaces $l_q^\alpha(L^p)$ (see [2], Theorems 5.6.1, 5.6.2 and 5.6.3) provides the result. Note the weight appearing $l_q^\alpha(A)$ is $2^{-j\frac{\alpha}{2}}$ (and not $2^{j\frac{\alpha}{2}}$) because we sum on *negative* integers.

Fix $m > \frac{\alpha}{2}$. Define the functional $\mathcal{J}$ by $\mathcal{J}f = ((\mathcal{J}f)_j)_{j \leq 0}$ where

$$(\mathcal{J}f)_j = 2^{jm} \Delta^m H_{2^{j-1}} f$$

if $j \leq -1$ and

$$(\mathcal{J}f)_0 = H_{\frac{1}{2}} f.$$

Moreover, define $\mathcal{P}$ on $l_q^\alpha(L^p)$ by

$$\mathcal{P}u = \sum_{k=0}^{2m-1} \frac{1}{k!} \Delta^k H_{\frac{1}{2}} u_0 + \frac{1}{(2m-1)!} \sum_{j \leq -1} 2^{-jm} \int_{2^j}^{2^{j+1}} t^{2m} \Delta^m H_{t-2^{j-1}} u_j \frac{dt}{t}.$$

We will see below that $\mathcal{P}$ is well-defined on $l_q^\alpha(L^p)$. Proposition 4.5 implies immediately that $\mathcal{J}$ is bounded from $B_\alpha^{p,q}$ to $l_q^\alpha(L^p)$. Moreover, Lemma 3.1 easily provides that

$$\mathcal{P} \circ \mathcal{J} = Id_{B_\alpha^{p,q}}.$$

It remains to verify that $\mathcal{P}$ is a bounded linear operator from $l^\alpha_q(L^p)$ to $B^{p,q}_\alpha$. The proof is similar to the one of Proposition 4.6. Indeed, proceeding as the fourth step of Proposition 4.6, one gets

$$\left\| \sum_{k=0}^{2m-1} \frac{1}{k!} \Delta^k H_{\frac{1}{2}} u_0 \right\|_{B^{p,q}_\alpha} \lesssim \|u_0\|_p.$$

It is plain to see that

$$\begin{aligned}
\left\| H_{\frac{1}{2}} \sum_{j\leq -1} 2^{-jm} \int_{2^j}^{2^{j+1}} t^{2m} \Delta^m H_{t-2^{j-1}} u_j \frac{dt}{t} \right\|_p &\leq \sum_{j\leq -1} 2^{-jm} \int_{2^j}^{2^{j+1}} t^{2m} \left\| H_{\frac{1}{2}} \Delta^m H_{t-2^{j-1}} u_j \right\|_p \frac{dt}{t} \\
&\leq \sum_{j\leq -1} 2^{-jm} \int_{2^j}^{2^{j+1}} t^{2m} \left\| H_{\frac{1}{2}} \Delta^m u_j \right\|_p \frac{dt}{t} \\
&\lesssim \sum_{j\leq -1} 2^{jm} \|u_j\|_p \\
&\lesssim \|u\|_{l^\alpha_q(L^p)}.
\end{aligned}$$

Then the proof of the boundedness of $\mathcal{P}$ is reduced to the one of

$$I := \left(\sum_{k\leq -1} \left(2^{k(m-\frac{\alpha}{2})} \left\| \Delta^m H_{2^k} \sum_{j\leq -1} 2^{-jm} \int_{2^j}^{2^{j+1}} t^{2m} \Delta^m H_{t-2^{j-1}} u_j \frac{dt}{t} \right\|_p \right)^q \right)^{\frac{1}{q}} \lesssim \|u\|_{l^\alpha_q(L^p)}. \tag{19}$$

Indeed,

$$\begin{aligned}
I^q &\lesssim \sum_{k\leq -1} \left(2^{k(m-\frac{\alpha}{2})} \left\| \sum_{j\leq -1} 2^{-jm} \int_{2^j}^{2^{j+1}} (t\Delta)^{2m} H_{t-2^{j-1}+2^k} u_j \frac{dt}{t} \right\|_p \right)^q \\
&\lesssim \sum_{k\leq -1} \left(2^{k(m-\frac{\alpha}{2})} \sum_{j\leq -1} 2^{jm} \left\| \Delta^{2m} H_{2^{j-1}+2^k} u_j \right\|_p \right)^q \\
&\lesssim \sum_{k\leq -1} \left(2^{k(m-\frac{\alpha}{2})} \sum_{j\leq -1} \frac{2^{jm}}{(2^j+2^k)^{2m}} \|u_j\|_p \right)^q \\
&\lesssim \sum_{k\leq -1} \left(2^{k(m-\frac{\alpha}{2})} \sum_{j\leq -1} 2^{-2m\max\{j,k\}} 2^{jm} \|u_j\|_p \right)^q
\end{aligned}$$

Check that $0 < m - \frac{\alpha}{2} < 2m$. Thus, Lemma 2.2 yields

$$I^q \lesssim \sum_{j\leq -1} \left[2^{-j(\frac{\alpha}{2}+m)} 2^{jm} \|u_j\|_p \right]^q \leq \|u\|^q_{l^\alpha_q(L^p)},$$

which proves (19) and thus concludes the proof. □

5 Algebra under pointwise product - Theorem 1.14

We want to introduce some paraproducts. The idea of paraproducts goes back to [4]. The term "paraproducts" is used to a denotes some non-commutative bilinear forms Λ_i such that $fg = \sum \Lambda_i(f,g)$. They are introduced in some cases, where the bilinear forms Λ_i are easier to handle than the pointwise product.
In the context of doubling spaces, a definition of paraproducts is given in [3, 8]. We need to slightly modify the definition in [3] to adapt them to non-doubling spaces.

For all $t > 0$, define

$$\phi_t(\Delta) = -\sum_{k=0}^{m-1} \frac{1}{k!}(t\Delta)^k H_t,$$

and observe that the derivative of $t \mapsto \phi_t(\Delta)$ is given by

$$\phi_t'(\Delta) = \frac{1}{(m-1)!}\frac{1}{t}(t\Delta)^m H_t := \frac{1}{t}\psi_t(\Delta).$$

Remark 5.1. *Even if ϕ_t actually depends on m, we do not indicate this dependence explicitly.*

Recall that Lemma 3.1 provides the identity

$$f = \int_0^1 \psi_t(\Delta) f \frac{dt}{t} - \phi_1(\Delta) f \qquad \text{in } \mathcal{S}'(G). \tag{20}$$

Proposition 5.2. *Let $p, q, r \in [1, +\infty]$ such that $\frac{1}{r} := \frac{1}{p} + \frac{1}{q} \leq 1$. Let $(f, g) \in L^p(G) \times L^q(G)$. One has the formula*

$$fg = \Pi_f(g) + \Pi_g(f) + \Pi(f, g) - \phi_1(\Delta)[\phi_1(\Delta) f \cdot \phi_1(\Delta) g] \qquad \text{in } \mathcal{S}'(G),$$

where

$$\Pi_f(g) = \int_0^1 \phi_t(\Delta)[\psi_t(\Delta) f \cdot \phi_t(\Delta) g] \frac{dt}{t}$$

and

$$\Pi(f, g) = \int_0^1 \psi_t(\Delta)[\phi_t(\Delta) f \cdot \phi_t(\Delta) g] \frac{dt}{t}.$$

Proof: Since $fg \in L^r \subset \mathcal{S}'(G)$, the formula (20) provides in $\mathcal{S}'(G)$

$$[f \cdot g] = \int_0^1 \psi_t(\Delta)[f \cdot g] \frac{dt}{t} - \phi_1(\Delta)[f \cdot g]. \tag{21}$$

We can use again twice (one for f and one for g) the identity (20) to get

$$\begin{aligned}
[f \cdot g] &= \int_0^1 \psi_t(\Delta) \left[\left\{ \int_0^1 \psi_u(\Delta) f \frac{du}{u} - \phi_1(\Delta) f \right\} \cdot \left\{ \int_0^1 \psi_v(\Delta) g \frac{dv}{v} - \phi_1(\Delta) g \right\} \right] \frac{dt}{t} \\
&\quad - \phi_1(\Delta) \left[\left\{ \int_0^1 \psi_u(\Delta) f \frac{du}{u} - \phi_1(\Delta) f \right\} \cdot \left\{ \int_0^1 \psi_v(\Delta) g \frac{dv}{v} - \phi_1(\Delta) g \right\} \right] \\
&= \int_0^1 \int_0^1 \int_0^1 \psi_t(\Delta)[\psi_u(\Delta) f \cdot \psi_v(\Delta) g] \frac{dt\, du\, dv}{tuv} \\
&\quad - \int_0^1 \int_0^1 \psi_t(\Delta)[\phi_1(\Delta) f \cdot \psi_v(\Delta) g] \frac{dt\, dv}{tv} - \int_0^1 \int_0^1 \psi_t(\Delta)[\psi_u(\Delta) f \cdot \phi_1(\Delta) g] \frac{dt\, du}{tu} \\
&\qquad\qquad - \int_0^1 \int_0^1 \phi_1(\Delta)[\psi_u(\Delta) f \cdot \psi_v(\Delta) g] \frac{du\, dv}{uv} \\
&\quad + \int_0^1 \psi_t(\Delta)[\phi_1(\Delta) f \cdot \phi_1(\Delta) g] \frac{dt}{t} + \int_0^1 \phi_1(\Delta)[\psi_u(\Delta) f \cdot \phi_1(\Delta) g] \frac{du}{u} \\
&\qquad\qquad + \int_0^1 \phi_1(\Delta)[\phi_1(\Delta) f \cdot \psi_v(\Delta) g] \frac{dv}{v} \\
&\quad - \phi_1(\Delta)[\phi_1(\Delta) \cdot \phi_1(\Delta)] \\
&:= R(f, g) + \int_0^1 \int_0^1 \int_0^1 \psi_t(\Delta)[\psi_u(\Delta) f \cdot \psi_v(\Delta) g] \frac{dt\, du\, dv}{tuv} - \phi_1(\Delta)[\phi_1(\Delta) \cdot \phi_1(\Delta)].
\end{aligned} \tag{22}$$

The domain $[0, 1]^3$ can be divided in the subsets $D(t, u, v)$, $D(u, t, v)$ and $D(v, u, t)$ where $D(a, b, c) = \{(a, b, c) \in$

$[0,1]^3$, $a < \min\{b,c\}\}$. Consequently,

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1 \psi_t(\Delta)[\psi_u(\Delta)f\cdot\psi_v(\Delta)g]\frac{dt\,du\,dv}{tuv}\\
&\quad=\int_0^1\int_t^1\int_t^1 \psi_t(\Delta)[\psi_u(\Delta)f\cdot\psi_v(\Delta)g]\frac{dt\,du\,dv}{tuv}+\int_0^1\int_u^1\int_u^1 \psi_t(\Delta)[\psi_u(\Delta)f\cdot\psi_v(\Delta)g]\frac{du\,dt\,dv}{utv}\\
&\qquad\qquad+\int_0^1\int_v^1\int_v^1 \psi_t(\Delta)[\psi_u(\Delta)f\cdot\psi_v(\Delta)g]\frac{dv\,du\,dt}{vut}\\
&\quad=\int_0^1 \psi_t(\Delta)[\{\phi_1(\Delta)f-\phi_t(\Delta)f\}\cdot\{\phi_1(\Delta)g-\phi_t(\Delta)g\}]\frac{dt}{t}\\
&\qquad\qquad+\int_0^1\{\phi_1(\Delta)-\phi_u(\Delta)\}[\psi_u(\Delta)f\cdot\{\phi_1(\Delta)g-\phi_u(\Delta)g\}]\frac{du}{u}\\
&\qquad\qquad+\int_0^1\{\phi_1(\Delta)-\phi_v(\Delta)\}[\{\phi_1(\Delta)f-\phi_v(\Delta)f\}\cdot\psi_v(\Delta)g]\frac{dv}{v}\\
&\quad:=S(f,g)+\Pi_f(g)+\Pi_g(f)+\Pi(f,g).
\end{aligned}\tag{23}$$

It remains to check that $R(f,g)+S(f,g)=0$. This identity, that can be proven with similar computations as (23), is left to the reader. □

Proposition 5.3. *Let G be a unimodular Lie group. Let $\alpha>0$ and $p,p_1,p_2,q\in[1,+\infty]$ such that*

$$\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}.$$

Then for all $f\in B_\alpha^{p_1,q}$ and all $g\in L^{p_2}$, one has

$$\Lambda_\alpha^{p,q}[\Pi_f(g)]\lesssim\|f\|_{B_\alpha^{p_1,q}}\|g\|_{L^{p_2}}.$$

Proof: Let $m>\frac{\alpha}{2}$ and $j\le -1$. Notice that, for all $u\in(0,1)$,

$$\|\Delta^m H_u\Pi_f(g)\|_p\le\int_0^1\|\Delta^m H_u\phi_t(\Delta)[\psi_t(\Delta)f\cdot\phi_t(\Delta)g]\|_p\,\frac{dt}{t}.$$

Remark that

$$\|\phi_t(\Delta)h\|_r\lesssim\|H_{\frac{t}{2}}h\|_r$$

for all $r\in[1,+\infty]$ and all $h\in L^r$. As a consequence,

$$\begin{aligned}
\|\Delta^m H_u\phi_t(\Delta)[\psi_t(\Delta)f\cdot\phi_t(\Delta)g]\|_p&=\|\phi_t(\Delta)\Delta^m H_u[\psi_t(\Delta)f\cdot\phi_t(\Delta)g]\|_p\\
&\lesssim\left\|\Delta^m H_{\frac{t}{2}+u}[\psi_t(\Delta)f\cdot\phi_t(\Delta)g]\right\|_p\\
&\lesssim\left(\frac{t}{2}+u\right)^{-m}\|\psi_t(\Delta)f\cdot\phi_t(\Delta)g\|_p\\
&\lesssim\min\left\{t^{-m},u^{-m}\right\}\|\psi_t(\Delta)f\cdot\phi_t(\Delta)g\|_p\\
&\lesssim\min\left\{t^{-m},u^{-m}\right\}\|\psi_t(\Delta)f\|_{p_1}\|\phi_t(\Delta)g\|_{p_2}\\
&\lesssim\min\left\{t^{-m},u^{-m}\right\}\|(t\Delta)^m H_t f\|_{p_1}\|g\|_{p_2}.
\end{aligned}$$

We deduce then

$$\begin{aligned}
&\Lambda_\alpha^{p,q}[\Pi_f(g)]^q\\
&\quad\lesssim\|g\|_{p_2}^q\int_0^1\left(u^{m-\frac{\alpha}{2}}\int_0^1\left(\max\{u,t\}\right)^{-m}\|(t\Delta)^m H_t f\|_{p_1}\frac{dt}{t}\right)^q\frac{du}{u}\\
&\quad\lesssim\|g\|_{p_2}^q\sum_{j\le -1}\left(2^{j(m-\frac{\alpha}{2})}\sum_{n=-\infty}^{-1}2^{-m\max\{j,n\}}\|(2^n\Delta)^m H_{2^n}f\|_{p_1}\right)^q\\
&\quad\lesssim\|g\|_{L^{p_2}}\left(\sum_{n\le -1}2^{-n\frac{\alpha}{2}q}2^{nmq}\|\Delta^m H_{2^n}f\|_{p_1}^q\right)^{\frac{1}{q}},
\end{aligned}$$

where we used Lemma 2.2 for the last line. As a consequence, we obtain if $\alpha \in (0, 2m)$,

$$\begin{aligned}\Lambda_\alpha^{p,q}[\Pi_f(g)] &\lesssim \|g\|_{L^{p_2}} \left(\sum_{n \leq -1} 2^{nq(m-\frac{\alpha}{2})} \|\Delta^m H_{2^n} f\|_{p_1}^q \right)^{\frac{1}{q}} \\ &\lesssim \|g\|_{L^{p_2}} \|f\|_{B_\alpha^{p_1,q}}\end{aligned}$$

where we used Proposition 4.5 for the last line. □

Proposition 5.4. *Let G be a unimodular Lie group. Let $\alpha > 0$ and $p, p_1, p_2, p_3, p_4, q \in [1, +\infty]$ such that*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}.$$

Then for all $f \in B_\alpha^{p_1,q} \cap L^{p_3}$ and all $g \in B_\alpha^{p_4,q} \cap L^{p_2}$, one has

$$\Lambda_\alpha^{p,q}[\Pi(f,g)] \lesssim \|f\|_{B_\alpha^{p_1,q}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|g\|_{B_\alpha^{p_4,q}}.$$

Proof: Notice first that

$$\|\Delta^m H_u \Pi(f,g)\|_p \leq \int_0^1 \|\Delta^m H_u H_t (t\Delta)^m [\phi_t(\Delta) f \cdot \phi_t(\Delta) g]\|_p \frac{dt}{t}.$$

Let us recall then that $X_i(f \cdot g) = f \cdot X_i g + X_i f \cdot g$. Consequently, since $\Delta = \sum_{i=1}^k X_i^2$, one has

$$\|\Delta^m [f \cdot g]\|_p \lesssim \|\Delta^m f \cdot g\|_p + \|f \cdot \Delta^m g\|_p + \sum_{k=1}^{2m-1} \sup_{|I_1|=k} \sup_{|I_2|=2m-k} \|X_{I_1} f \cdot X_{I_2} g\|_p.$$

In the following computations, (Y_{I_1}, Z_{I_2}) denotes the couple (X_{I_1}, X_{I_2}) if $|I_1| \neq 0$ and $|I_2| \neq 0$, $(\Delta^{|I_1|/2}, I)$ if $|I_2| = 0$ and $(I, \Delta^{|I_2|/2})$ if $|I_1| = 0$. With these notations, one has

$$\begin{aligned}&\|\Delta^m H_{u+t} (t\Delta)^m [\phi_t(\Delta) f \cdot \phi_t(\Delta) g]\|_p \\ &\qquad \lesssim \min\left\{t^{-m}, u^{-m}\right\} \|(t\Delta)^m [\phi_t(\Delta) f \cdot \phi_t(\Delta) g]\|_p \\ &\qquad \lesssim \min\left\{t^{-m}, u^{-m}\right\} \sum_{k,l=0}^{m-1} \left\|(t\Delta)^m [(t\Delta)^k H_t f \cdot (t\Delta)^l H_t g]\right\|_p \\ &\qquad \lesssim \min\left\{t^{-m}, u^{-m}\right\} \sum_{k,l=0}^{m-1} \sum_{i=0}^{2m} t^m \sup_{|I_1|=i} \sup_{|I_2|=2m-i} \left\|Y_{I_1} (t\Delta)^k H_t f \cdot Z_{I_2} (t\Delta)^l H_t g\right\|_p \\ &\qquad = \min\left\{t^{-m}, u^{-m}\right\} \sum_{k,l=0}^{m-1} \sum_{i=0}^{2m} t^{m+k+l} \sup_{|I_1|=i} \sup_{|I_2|=2m-i} \left\|Y_{I_1} \Delta^k H_t f \cdot Z_{I_2} \Delta^l H_t g\right\|_p \\ &\qquad \lesssim \min\left\{t^{-m}, u^{-m}\right\} \sum_{k,l=0}^{m-1} \sum_{i=0}^{2m} t^{m+k+l} \sup_{|I_1|=i+2k} \sup_{|I_2|=2m+2l-i} \|Y_{I_1} H_t f \cdot Z_{I_2} H_t g\|_p \\ &\qquad \lesssim \min\left\{t^{-m}, u^{-m}\right\} \sum_{\substack{2m \leq k+l \leq 6m-4 \\ k+l \text{ even}}} t^{\frac{k+l}{2}} \sup_{|I_1|=k} \sup_{|I_2|=l} \|Y_{I_1} H_t f \cdot Z_{I_2} H_t g\|_p.\end{aligned}$$

Setting $c_n = \sum_{\substack{2m\le k+l\le 6m-4\\ k+l \text{ even}}} \int_{2^n}^{2^{n+1}} t^{\frac{k+l}{2}} \sup_{|I_1|=k}\sup_{|I_2|=l} \|Y_{I_1}H_tf\cdot Z_{I_2}H_tg\|_p \frac{dt}{t}$, one has

$$\begin{aligned}
&\Lambda_\alpha^{p,q}[\Pi(f,g)]^q\\
&\quad\lesssim \int_0^1 \left(u^{m-\frac{\alpha}{2}}\int_0^1 \|\Delta^m H_{u+t}(t\Delta)^m[\phi_t(\Delta)f\cdot\phi_t(\Delta)g]\|_p\frac{dt}{t}\right)^q\frac{du}{u}\\
&\quad\lesssim \int_0^1\left(u^{m-\frac{\alpha}{2}}\int_0^1 \min\left\{t^{-m},u^{-m}\right\}\sum_{\substack{2m\le k+l\le 6m-4\\ k+l \text{ even}}} t^{\frac{k+l}{2}}\sup_{|I_1|=k}\sup_{|I_2|=l}\|Y_{I_1}H_tf\cdot Z_{I_2}H_tg\|_p\frac{dt}{t}\right)^q\frac{du}{u}\\
&\quad\lesssim \sum_{j=-\infty}^{-1}\left[2^{j(m-\frac{\alpha}{2})}\sum_{n=-\infty}^{-1}2^{-m\max\{n,j\}}c_n\right]^q\\
&\quad\lesssim \sum_{n\le -1}2^{-nq\frac{\alpha}{2}}c_n^q
\end{aligned}$$

where the last line is a consequence of Lemma 2.2, since $0<m-\frac{\alpha}{2}<m$.

It remains to prove that for any couple $(k,l)\in\mathbb{N}^2$ satisfying $6m-4\ge k+l\ge 2m$ and $k+l$ even, we have

$$\begin{aligned}
T &:= \left(\sum_{n\le -1}2^{-nq\frac{\alpha}{2}}\left(\int_{2^n}^{2^{n+1}} t^{\frac{k+l}{2}}\sup_{|I_1|=k}\sup_{|I_2|=l}\|Y_{I_1}H_tf\cdot Z_{I_2}H_tg\|_p\frac{dt}{t}\right)^q\right)^{\frac{1}{q}}\\
&\lesssim \|f\|_{B_\alpha^{p_1,q}}\|g\|_{L^{p_2}}+\|f\|_{L^{p_3}}\|g\|_{B_\alpha^{p_4,q}}.
\end{aligned}\tag{24}$$

1. **If $k=0$ or $l=0$:**
   Since k and l play symmetric roles, we can assume without loss of generality that $l=0$. In this case, k is even and if $k=2k'$,

$$\begin{aligned}
\sup_{|I_1|=k}\sup_{|I_2|=0}\|Y_{I_1}H_tf\cdot Z_{I_2}H_tg\|_p &= \left\|\Delta^{k'}H_tf\cdot H_tg\right\|_p\\
&\le \left\|\Delta^{k'}H_tf\|_{p_1}\|H_tg\right\|_{p_2}\\
&\le \left\|\Delta^{k'}H_tf\right\|_{p_1}\|g\|_{p_2}.
\end{aligned}$$

   Therefore,

$$\begin{aligned}
T &\le \|g\|_{L^{p_2}}\left(\sum_{n\le -1}2^{-nq\frac{\alpha}{2}}\left(\int_{2^n}^{2^{n+1}}t^{k'}\left\|\Delta^{k'}H_tf\right\|_{p_1}\frac{dt}{t}\right)^q\right)^{\frac{1}{q}}\\
&\lesssim \|g\|_{L^{p_2}}\|f\|_{B_\alpha^{p_1,q}}
\end{aligned}$$

   where the second line is due to the fact that $k'\ge m>\frac{\alpha}{2}$.

2. **If $k\ge 1$ and $l\ge 1$:**
   Define $\alpha_1,\alpha_2,r_1,r_2,q_1$ and q_2 by

$$\begin{aligned}
\alpha_1=\frac{k}{k+l}\alpha \quad &, \quad \alpha_2=\frac{l}{k+l}\alpha,\\
\frac{k+l}{r_1}=\frac{k}{p_1}+\frac{l}{p_3} \quad &, \quad \frac{k+l}{r_2}=\frac{k}{p_2}+\frac{l}{p_4},\\
\frac{k+l}{q_1}=\frac{k}{q} \quad &, \quad \frac{k+l}{q_2}=\frac{l}{q}.
\end{aligned}$$

   In this case, notice that $k>\alpha_1$ and $l>\alpha_2$. One has then

$$\sup_{|I_1|=k}\sup_{|I_2|=l}\|X_{I_1}H_tf\cdot X_{I_2}H_tg\|_p\le \sup_{|I_1|=k}\|X_{I_1}H_tf\|_{r_1}\sup_{|I_2|=l}\|X_{I_2}H_tg\|_{r_2}$$

and thus Hölder inequality provides

$$\begin{aligned} T &\leq \left(\sum_{n\leq -1} \left(2^{n\frac{k-\alpha_1}{2}} \max_{t\in[2^j,2^{j+1}]} \sup_{|I_1|=k} \|X_{I_1} H_t f\|_{r_1} \right)^{q_1} \right)^{\frac{1}{q_1}} \\ &\qquad \left(\sum_{n\leq -1} \left(2^{n\frac{l-\alpha_2}{2}} \max_{t\in[2^j,2^{j+1}]} \sup_{|I_2|=l} \|X_{I_2} H_t g\|_{r_2} \right)^{q_2} \right)^{\frac{1}{q_2}} \\ &\lesssim \|f\|_{B^{r_1,q_1}_{\alpha_1}} \|g\|_{B^{r_2,q_2}_{\alpha_2}} \end{aligned}$$

where the second line is due to Theorem 1.13.
Let $\theta = \frac{k}{k+l}$. Complex interpolation (Corollary 1.12) provides

$$(B_0^{p_3,\infty}, B_\alpha^{p_1,q})_{[\theta]} = B_{\alpha_1}^{r_1,q_1}$$

and

$$(B_\alpha^{p_4,q}, B_0^{p_2,\infty})_{[\theta]} = B_{\alpha_2}^{r_2,q_2}.$$

Remark also that $L^s(G)$ is continuously embedded in $B_0^{s,\infty}(G)$ (this can be easily seen from the definition of Besov spaces). As a consequence,

$$\begin{aligned} T &\lesssim \|f\|_{B^{r_1,q_1}_{\alpha_1}} \|g\|_{B^{r_2,q_2}_{\alpha_2}} \\ &\lesssim \|f\|_{L^{p_3}}^{\theta} \|f\|_{B_\alpha^{p_1,q}}^{1-\theta} \|g\|_{B_\alpha^{p_4,q}}^{\theta} \|g\|_{L^{p_2}}^{1-\theta} \\ &\lesssim \|f\|_{L^{p_3}} \|g\|_{B_\alpha^{p_4,q}} + \|f\|_{B_\alpha^{p_1,q}} \|g\|_{L^{p_2}} \end{aligned}$$

which is the desired conclusion.

□

Let us now prove Theorem 1.14

Proof: With the use of Propositions 5.2, 5.3 and 5.4, it remains to check that

$$\|H_{\frac{1}{2}}[f\cdot g]\|_{L^p} \lesssim \|f\|_{B_\alpha^{p_1,q}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|g\|_{B_\alpha^{p_4,q}} \tag{25}$$

and

$$\|\phi_1(\Delta)[\phi_1(\Delta) f \cdot \phi_1(\Delta) g\|_{B_\alpha^{p,q}} \lesssim \|f\|_{B_\alpha^{p_1,q}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|g\|_{B_\alpha^{p_4,q}}. \tag{26}$$

The inequality (25) is easy to check. By Proposition 4.1, one has

$$\|H_{\frac{1}{2}}[f\cdot g]\|_{L^p} \leq \|f\cdot g\|_p \leq \|f\|_{p_1} \|g\|_{p_2} \leq \|f\|_{B_\alpha^{p_1,q}} \|g\|_{L^{p_2}}.$$

For (26), recall that (18) implies

$$\begin{aligned} \|\phi_1(\Delta)[\phi_1(\Delta) f \cdot \phi_1(\Delta) g]\|_{B_\alpha^{p,q}} &\lesssim \|\phi_1(\Delta) f \cdot \phi_1(\Delta) g\|_{L^p} \\ &\lesssim \|\phi_1(\Delta) f\|_{p_1} \|\phi_1(\Delta) g\|_{p_2} \\ &\lesssim \|f\|_{B_\alpha^{p_1,q}} \|g\|_{L^{p_2}}. \end{aligned}$$

□

6 Other characterizations of Besov spaces

6.1 Characterization by differences of functions - Theorem 1.16

Lemma 6.1. *Let $p, q \in [1, +\infty]$ and $\alpha > 0$. There exists $c > 0$ such that, for all $f \in L^p(G)$,*

$$\Lambda_\alpha^{p,q}(f) \lesssim \left(\int_G \left(\frac{\|\nabla_y f\|_p e^{-c|y|^2}}{|y|^\alpha} \right)^q \frac{dy}{V(|y|)} \right)^{\frac{1}{q}}.$$

Proof: Since $\int_G \frac{\partial h_t}{\partial t}(y)dx = 0$,

$$\begin{aligned}\frac{\partial H_t}{\partial t}f(x) &= \int_G \frac{\partial h_t}{\partial t}(y)f(xy)dy \\ &= \int_G \frac{\partial h_t}{\partial t}(y)[f(xy)-f(x)]dy \\ &= \int_G \frac{\partial h_t}{\partial t}(y)\nabla_y f(x)dy.\end{aligned} \tag{27}$$

Consequently,

$$\left\|\frac{\partial H_t}{\partial t}f\right\|_p \leq \int_G \left|\frac{\partial h_t}{\partial t}(y)\right| \|\nabla_y f\|_p dy.$$

Proposition 2.5 provides

$$\begin{aligned}\Lambda_\alpha^{p,q}(f) &\lesssim \left(\int_0^1 \left(t^{1-\frac{\alpha}{2}}\int_G \frac{1}{tV(\sqrt{t})}e^{-c\frac{|y|^2}{t}}\|\nabla_y f\|_p dy\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^1 \left(t^{1-\frac{\alpha}{2}}\int_G \frac{1}{tV(\sqrt{t})}e^{-c'\frac{|y|^2}{t}}\|\nabla_y f\|_p e^{-c'\frac{|y|^2}{t}} dy\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^1 \left(t^{1-\frac{\alpha}{2}}\int_G \frac{1}{tV(\sqrt{t})}e^{-c'\frac{|y|^2}{t}}\|\nabla_y f\|_p e^{-c'|y|^2} dy\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &= \left(\int_0^1 \left(\int_G K(t,y)g(y)\frac{dy}{V(|y|)}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}\end{aligned}$$

with $c' = \frac{c}{2}$, $g(y) = \frac{\|\nabla_y f\|_p}{|y|^\alpha}e^{-c'|y|^2}$ and $K(t,y) = \frac{V(|y|)}{V(\sqrt{t})}\left(\frac{|y|^2}{t}\right)^{\frac{\alpha}{2}} e^{-c'\frac{|y|^2}{t}}$ (note that we used the fact that $t \in (0,1)$ in the third line). Lemma 2.1 and Proposition 2.4 imply then

$$\begin{aligned}\Lambda_\alpha^{p,q}(f) &\lesssim \left(\int_G |g(y)|^q \frac{dy}{V(|y|)}\right)^{\frac{1}{q}} \\ &= \left(\int_G \left(\frac{\|\nabla_y f\|_p e^{-c|y|^2}}{|y|^\alpha}\right)^q \frac{dy}{V(|y|)}\right)^{\frac{1}{q}}.\end{aligned}$$

□

Proposition 6.2. *Let $p, q \in [1, +\infty]$ and $\alpha > 0$, then*

$$\Lambda_\alpha^{p,q}(f) \lesssim L_\alpha^{p,q}(f) + \|f\|_p.$$

Proof: According to Lemma 6.1, it is sufficient to check that

$$\left(\int_G \left(\frac{\|\nabla_y f\|_p e^{-c|y|^2}}{|y|^\alpha}\right)^q \frac{dy}{V(|y|)}\right)^{\frac{1}{q}} \lesssim L_\alpha^{p,q}(f) + \|f\|_p.$$

Since we obviously have

$$\left(\int_{|y|\leq 1} \left(\frac{\|\nabla_y f\|_p e^{-c|y|^2}}{|y|^\alpha}\right)^q \frac{dy}{V(|y|)}\right)^{\frac{1}{q}} \leq L_\alpha^{p,q}(f),$$

all we need to prove is

$$T = \left(\int_{|y|\geq 1} \left(\frac{\|\nabla_y f\|_p e^{-c|y|^2}}{|y|^\alpha}\right)^q \frac{dy}{V(|y|)}\right)^{\frac{1}{q}} \lesssim \|f\|_p.$$

Indeed, $\|\nabla_y f\|_p \leq 2\|f\|_p$ and thus

$$\begin{aligned} T &\lesssim \|f\|_p \left(\int_{|y|\geq 1} \left(e^{-c|y|^2} \right)^q dy \right)^{\frac{1}{q}} \\ &\lesssim \|f\|_p \left(\sum_{j=0}^{\infty} e^{-cq4^j} V(2^{j+1}) \right)^{\frac{1}{q}} \\ &\lesssim \|f\|_p, \end{aligned}$$

where the last line holds because $V(r)$ have at most exponential growth. □

Proposition 6.3. *Let $p, q \in [1, +\infty]$ and $\alpha \in (0,1)$. Then*

$$L_\alpha^{p,q}(f) \lesssim \Lambda_\alpha^{p,q}(f) + \|f\|_p \qquad \forall f \in B_\alpha^{p,q}(G).$$

Proof: 1. **Decomposition of f:**
The first step is to decompose f as

$$f = (f - H_1 f) + H_1 f.$$

We introduce

$$f_n = -\int_{2^n}^{2^{n+1}} \frac{\partial H_t f}{\partial t} dt = -\int_{2^n}^{2^{n+1}} \Delta H_t f dt$$

and

$$c_n = \int_{2^{n-1}}^{2^n} \left\| \frac{\partial H_t f}{\partial t} \right\|_p dt.$$

Remark then that

$$\|f_n\|_p \leq c_{n+1}$$

and Lemma 3.1 provides

$$f - H_1 f = \sum_{n=-\infty}^{-1} f_n \qquad \text{in } \mathcal{S}'(G).$$

2. **Estimate of $X_i f_n$:**
Let us prove that if $n \leq -1$, one has for all $i \in [\![1, k]\!]$

$$\|X_i f_n\|_p \lesssim 2^{-\frac{n}{2}} c_n \tag{28}$$

Indeed, notice first

$$\begin{aligned} f_n &= -2 \int_{2^{n-1}}^{2^n} \Delta H_{2t} f dt \\ &= -2 H_{2^{n-1}} \int_{2^{n-1}}^{2^n} H_{t-2^{n-1}} \Delta H_t f dt \\ &:= H_{2^{n-1}} g_n. \end{aligned}$$

Proposition 2.7 implies then

$$\begin{aligned} \|X_i f_n\|_p &\lesssim 2^{-\frac{n}{2}} \|g_n\|_p \\ &\lesssim 2^{-\frac{n}{2}} \int_{2^{n-1}}^{2^n} \left\| H_{t-2^{n-1}} \frac{\partial H_t f}{\partial t} \right\|_p dt \\ &\lesssim 2^{-\frac{n}{2}} \int_{2^{n-1}}^{2^n} \left\| \frac{\partial H_t f}{\partial t} \right\|_p dt \\ &= 2^{-\frac{n}{2}} c_n. \end{aligned}$$

If $\varphi : [0,1] \to G$ is an admissible path linking e to y with $l(\varphi) \leq 2|y|$,

$$\begin{aligned}\nabla_y f_n(x) &= \int_0^1 \frac{d}{ds} f_n(x\varphi(s))ds \\ &= \int_0^1 \sum_{i=1}^k c_i(s) X_i f_n(x\varphi(s))ds.\end{aligned}$$

Hence, (28) implies

$$\begin{aligned}\|\nabla_y f_n\|_p &\leq \int_0^1 \sum_{i=1}^k |c_i(s)| \|X_i f_n(.\varphi(s))\|_p ds \\ &= \|X_i f_n\|_p \int_0^1 \sum_{i=1}^k |c_i(s)| ds \\ &\lesssim 2^{-\frac{n}{2}} c_n \int_0^1 \sum_{i=1}^k |c_i(s)| ds \\ &\lesssim |y| 2^{-\frac{n}{2}} c_n\end{aligned}$$

where the second line is a consequence of the right-invariance of the measure and the last one follows from the definition of $l(\varphi)$. Thus, one has

$$\|\nabla_y f_n\|_p \lesssim \begin{cases} |y| 2^{-\frac{n}{2}} c_n & \text{if } |y|^2 < 2^n \\ c_{n+1} & \text{if } |y|^2 \geq 2^n \end{cases} \tag{29}$$

3. **Estimate of $L_\alpha^{p,q}(f - H_1 f)$**
   As a consequence of (29),

$$\begin{aligned}[L_\alpha^{p,q}(f - H_1 f)]^q &= \sum_{j=-\infty}^{-1} \int_{2^j < |y|^2 \leq 2^{j+1}} \left(\frac{\|\nabla_y f\|_p}{|y|^\alpha} \right)^q \frac{dy}{V(|y|)} \\ &\lesssim \sum_{j=-\infty}^{-1} \int_{2^j < |y|^2 \leq 2^{j+1}} \left(\sum_{n=-\infty}^{-1} \frac{\|\nabla_y f_n\|_p}{|y|^\alpha} \right)^q \frac{dy}{V(|y|)} \\ &\lesssim \sum_{j=-\infty}^{-1} 2^{-\frac{j\alpha}{2} q} \left(\sum_{n=-\infty}^{j} c_{n+1} + \sum_{n=j+1}^{-1} 2^{\frac{j-n}{2}} c_n \right)^q \\ &\lesssim \sum_{j=-\infty}^{-1} 2^{jq(1-\frac{\alpha}{2})} \left(\sum_{n=-\infty}^{-1} 2^{-\frac{\max\{j,n\}}{2}} [c_{n+1} + c_n] \right)^q \\ &\lesssim \sum_{n=-\infty}^{-1} \left(2^{-\frac{n\alpha}{2}} [c_{n+1} + c_n] \right)^q \\ &\lesssim \sum_{n=-\infty}^{0} \left(2^{-\frac{n\alpha}{2}} c_n \right)^q\end{aligned}$$

Note that the third line holds since $2^j \leq 1$, so that $V(2^{j+1}) \lesssim V(2^j)$ and the fifth one is obtained with Lemma 2.2, since $\alpha \in (0,1)$.

However

$$\begin{aligned}\sum_{n=-\infty}^{0}\left[2^{-n\frac{\alpha}{2}}c_n\right]^q &= \sum_{n=-\infty}^{0}\left[2^{-n\frac{\alpha}{2}}\int_{2^{n-1}}^{2^n}\left\|\frac{\partial H_t f}{\partial t}\right\|_p dt\right]^q \\ &\lesssim \sum_{n=-\infty}^{0}2^{-nq\frac{\alpha}{2}}2^{n(q-1)}\int_{2^{n-1}}^{2^n}\left\|\frac{\partial H_t f}{\partial t}\right\|_p^q dt \\ &\lesssim \sum_{n=-\infty}^{0}\int_{2^{n-1}}^{2^n}\left(t^{1-\frac{\alpha}{2}}\left\|\frac{\partial H_t f}{\partial t}\right\|_p\right)^q\frac{dt}{t} \\ &= \int_0^1\left(t^{1-\frac{\alpha}{2}}\left\|\frac{\partial H_t f}{\partial t}\right\|_p\right)^q\frac{dt}{t} \\ &= \left(\Lambda_\alpha^{p,q}(f)\right)^q.\end{aligned} \tag{30}$$

4. **Estimate of $L_\alpha^{p,q}(H_1 f)$**
With computations similar to those of the second step of this proof, we find that

$$\|\nabla_y H_1 f\|_p \lesssim |y|\|f\|_p.$$

Consequently,

$$\begin{aligned}L_\alpha^{p,q}(H_1 f) &\leq \|f\|_p\left(\int_{|y|\leq 1}|y|^{q(1-\alpha)}\frac{dy}{V(|y|)}\right)^{\frac{1}{q}} \\ &\leq \|f\|_p\left(\sum_{j\leq -1}\int_{2^j<|y|\leq 2^{j+1}}|y|^{q(1-\alpha)}\frac{dy}{V(|y|)}\right)^{\frac{1}{q}} \\ &\lesssim \|f\|_p\left(\sum_{j\leq -1}2^{qj(1-\alpha)}\right)^{\frac{1}{q}} \\ &\lesssim \|f\|_p\end{aligned}$$

where the third line is a consequence of the local doubling property.

□

Theorem 6.4. *Let G be a unimodular Lie group and $\alpha\in(0,1)$, then we have the following Leibniz rule. If $p_1,p_2,p_3,p_4,p,q\in[1,+\infty]$ are such that*

$$\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}=\frac{1}{p}$$

then for all $f\in B_\alpha^{p_1,q}(G)\cap L^{p_3}(G)$ and all $g\in B_\alpha^{p_4,q}(G)\cap L^{p_2}(G)$, one has

$$\|fg\|_{B_\alpha^{p,q}}\lesssim \|f\|_{B_\alpha^{p_1,q}}\|g\|_{L^{p_2}}+\|f\|_{L^{p_3}}\|g\|_{B_\alpha^{p_4,q}}.$$

Proof: Check that

$$\nabla_y(f\cdot g)(x)=g(xy)\cdot\nabla_y f(x)+f(x)\cdot\nabla_y g(x).$$

Thus, with Hölder inequality,

$$\begin{aligned}\|fg\|_{B_\alpha^{p,q}} &\simeq \|f\cdot g\|_p+L_\alpha^{p,q}(f\cdot g) \\ &\lesssim \|f\|_{p_1}\|g\|_{p_2}+L_\alpha^{p_1,q}(f)\cdot\|g\|_{L^{p_2}}+\|f\|_{L^{p_3}}\cdot L_\alpha^{p_4,q}(g) \\ &\lesssim \|f\|_{B_\alpha^{p_1,q}}\|g\|_{p_2}+\|f\|_{L^{p_3}}\|g\|_{B_\alpha^{p_4,q}}.\end{aligned}$$

□

6.2 Characterization by induction - Theorem 1.19

Proposition 6.5. *Let $p, q \in [1, +\infty]$ and $\alpha > -1$. Let $m > \frac{\alpha}{2}$. One has for all $i \in [\![1, k]\!]$*

$$\Lambda_\alpha^{p,q}(X_i f) \lesssim \Lambda_{\alpha+1}^{p,q} f + \|f\|_p = \|f\|_{B_{\alpha+1}^{p,q}}.$$

Proof: The scheme of the proof is similar to Proposition 4.6.

1. **Decomposition of f:**
   Let M be an integer with $M > \frac{\alpha+1}{2}$. We decompose f as in Lemma 3.1:

$$f = \frac{1}{(M-1)!} \int_0^1 (t\Delta)^M H_t f \frac{dt}{t} + \sum_{k=0}^{M-1} \frac{1}{k!} \Delta^k H_1 f$$

   and we introduce

$$f_n = -\int_{2^n}^{2^{n+1}} (t\Delta)^M H_t f \frac{dt}{t}$$

   and

$$c_n = \int_{2^{n-1}}^{2^n} t^M \left\|\Delta^M H_t f\right\|_p \frac{dt}{t}.$$

   Remark then that

$$\|f_n\|_p \leq c_{n+1}$$

   and

$$f = \frac{1}{(M-1)!} \sum_{n=-\infty}^{-1} f_n + \sum_{k=0}^{M-1} \frac{1}{k!} \Delta^k H_1 f.$$

2. **A first estimate of $\Delta^m H_t X_i f_n$:**
   Let us prove that if $n \leq -1$, one has for all $i \in [\![1, k]\!]$

$$\|\Delta^m X_i f_n\|_p \lesssim 2^{-n(m+\frac{1}{2})} c_n. \tag{31}$$

   Indeed, notice first

$$\begin{aligned} f_n &= -2^M \int_{2^{n-1}}^{2^n} (t\Delta)^M H_{2t} f \frac{dt}{t} \\ &= -2^M H_{2^{n-1}} \int_{2^{n-1}}^{2^n} H_{t-2^{n-1}} (t\Delta)^M H_t f \frac{dt}{t} \\ &:= H_{2^{n-1}} g_n. \end{aligned}$$

   Thus, since $\Delta = -\sum_{i=1}^k X_i^2$ can be written as a polynomial in the X_i's, we obtain with the upper estimate of the heat kernel (Proposition 2.5),

$$\begin{aligned} \|\Delta^m X_i f_n\|_p &\lesssim \left(\int_G \left| \int_G |\Delta^M X_i h_{2^{n-1}}(z^{-1}x)| g_n(z) dz \right|^p dx \right)^{\frac{1}{p}} \\ &\lesssim \frac{2^{-n(m+\frac{1}{2})}}{V(2^{\frac{n}{2}})} \left(\int_G \left| \int_G \exp\left(-c \frac{|z^{-1}x|^2}{2^n} \right) g_n(z) dz \right|^p dx \right)^{\frac{1}{p}} \\ &\lesssim 2^{-n(m+\frac{1}{2})} \|g_n\|_p \\ &\lesssim 2^{-n(m+\frac{1}{2})} c_n \end{aligned}$$

   where the second line is due to the fact that $V(2^{\frac{n}{2}}) \lesssim V(2^{\frac{n-1}{2}})$ and the last two lines are obtained by an argument analogous to the one for (28).

$$\begin{aligned} \left\|\Delta^m H_t X_i f_n\right\|_p &\lesssim \tfrac{1}{t^m} \left\|X_i f_n\right\|_p \\ &\lesssim \tfrac{1}{t^m} \left\|X_i H_{2^{n-1}} g_n\right\|_p \end{aligned}$$

   As a consequence, one has for all $t \in (0, 1]$,

$$\|\Delta^m H_t X_i f_n\|_p = \|H_t \Delta^m X_i f_n\|_p \lesssim 2^{-n(m+\frac{1}{2})} c_n,$$

   since H_t is uniformly bounded.

3. **A second estimate of $\Delta^m H_t X_i f_n$:**
Let us prove that for all $f \in L^p(G)$ and for all $i \in [\![1,k]\!]$, one has

$$\|\Delta^m H_t X_i f\|_p \lesssim t^{-m-\frac{1}{2}} \|f\|_p. \tag{32}$$

First, notice that

$$\begin{aligned}
\Delta^m H_t X_i f(x) &= \int_G \frac{\partial^m}{\partial t^m} h_t(y)(X_i f)(xy)dy \\
&= \int_G \frac{\partial^m}{\partial t^m} h_t(y)[X_i f(x.)](y)dy \\
&= -\int_G X_i \frac{\partial^m}{\partial t^m} h_t(y) f(xy)dy \\
&= -\int_G X_i \Delta^m h_t(x^{-1}y) f(y)dy.
\end{aligned}$$

Then, using the estimates on the heat kernel (Proposition 2.5)and the fact that $\Delta = -\sum X_i^2$, we obtain

$$\begin{aligned}
\|\Delta^m H_t X_i f\|_p &\lesssim \left(\int_G \left| \frac{t^{-m-\frac{1}{2}}}{V(\sqrt{t})} \int_G \exp\left(-c\frac{|x^{-1}y|^2}{t}\right) |f(y)|dy \right|^p dx \right)^{\frac{1}{p}} \\
&:= t^{-m-\frac{1}{2}} \left(\int_G \left| \int_G K(x,y)|f(y)|dy \right|^p dx \right)^{\frac{1}{p}}
\end{aligned}$$

with $K(x,y) = \frac{1}{V(\sqrt{t})} \exp\left(-c\frac{|x^{-1}y|^2}{t}\right)$. Proposition 2.3 yields the estimate (32).

4. **Estimate of $\Lambda_\alpha^{p,q}(\sum f_n)$**
The two previous steps imply

$$\|\Delta^m H_t X_i f_n\|_p \lesssim \begin{cases} 2^{-n(m+\frac{1}{2})} c_n & \text{if } t < 2^n \\ t^{-m-\frac{1}{2}} c_{n+1} & \text{if } t \geq 2^n \end{cases}.$$

As a consequence,

$$\begin{aligned}
&\int_0^1 \left(t^{m-\frac{\alpha}{2}} \left\| \Delta^m H_t X_i \sum_{n=-\infty}^{-1} f_n \right\|_p \right)^q \frac{dt}{t} \\
&\quad\lesssim \sum_{j=-\infty}^{-1} \int_{2^j < t \leq 2^{j+1}} \left(t^{m-\frac{\alpha}{2}} \sum_{n=-\infty}^{-1} \|\Delta^m H_t X_i f_n\|_p \right)^q \frac{dt}{t} \\
&\quad\lesssim \sum_{j=-\infty}^{-1} \left(2^{j(m-\frac{\alpha}{2})} \sum_{n=-\infty}^{j} 2^{-j(m+\frac{1}{2})} c_{n+1} + \sum_{n=j+1}^{-1} 2^{-n(m+\frac{1}{2})} c_n \right)^q \\
&\quad\lesssim \sum_{j=-\infty}^{-1} \left(2^{j(m-\frac{\alpha}{2})} \sum_{n=-\infty}^{-1} 2^{-\max\{j,n\}(m+\frac{1}{2})} [c_n + c_{n+1}] \right)^q \\
&\quad\lesssim \sum_{n=-\infty}^{-1} \left[2^{-n\frac{\alpha+1}{2}} [c_n + c_{n+1}] \right]^q \\
&\quad\lesssim \sum_{n=-\infty}^{0} \left[2^{-n\frac{\alpha+1}{2}} c_n \right]^q
\end{aligned}$$

where we used Lemma 2.2 for the fourth estimate, relevant since $-1 < \frac{\alpha}{2} < m$ by assumption. We get then the domination

$$\int_0^1 \left(t^{m-\frac{\alpha}{2}} \left\| \Delta^m H_t X_i \sum_n f_n \right\|_p \right)^q \frac{dt}{t} \lesssim \sum_{n=-\infty}^{0} \left[2^{-n\frac{\alpha+1}{2}} c_n \right]^q. \tag{33}$$

However computations analogous to those leading to (30) prove that

$$\sum_{n=-\infty}^{0}\left[2^{-n\frac{\alpha+1}{2}}c_n\right]^q \lesssim \int_0^1 \left(t^{M-\frac{\alpha+1}{2}}\left\|\Delta^M f\right\|_p\right)^q \frac{dt}{t} \lesssim \left(\Lambda^{p,q}_{\alpha+1}f\right)^q. \tag{34}$$

5. **Estimate of the remaining term.**
Recall that

$$f = \frac{1}{(M-1)!}\sum_{n=-\infty}^{-1} f_n + \sum_{k=0}^{M-1}\frac{1}{k!}\Delta^k H_1 f.$$

We already estimated $\Lambda^{p,q}_\alpha(\sum f_n)$. What remains to be estimated is $\Lambda^{p,q}_\alpha(\sum \frac{1}{k!}\Delta^k H_1 X_i f)$. Proposition 2.7 provides as well

$$\left\|\Delta^m H_t X_i \Delta^k H_1 f\right\|_p \leq \left\|\Delta^m X_i \Delta^k H_1 f\right\|_p \lesssim \|f\|_p.$$

As a consequence, we get,

$$\int_0^1 \left(t^{m-\frac{\alpha}{2}}\left\|\Delta^m H_t X_i \sum_{k=0}^{M-1}\frac{1}{k!}\Delta^k H_1 f\right\|_p\right)^q \frac{dt}{t} \lesssim \left(\|f\|_p \int_0^1 t^{q(m-\frac{\alpha}{2})}\frac{dt}{t}\right)^q \lesssim \|f\|_p^q.$$

□

Corollary 6.6. *Let $p, q \in [1, +\infty]$ and $\alpha > 0$.*

$$\|f\|_{B^{p,q}_{\alpha+1}} \simeq \|f\|_{L^p} + \sum_{i=1}^k \|X_i f\|_{B^{p,q}_\alpha}.$$

Proof: The main work was done in the previous proposition. Indeed, notice that Proposition 6.5 implies

$$\begin{aligned}\Lambda^{p,q}_{\alpha+1}f &= \Lambda^{p,q}_{\alpha-1}(\Delta f)\\ &\leq \sum_{i=1}^k \Lambda^{p,q}_{\alpha-1}X_i(X_i f)\\ &\lesssim \sum_{i=1}^k \|X_i f\|_{B^{p,q}_\alpha},\end{aligned}$$

which provides the domination of the first term by the second one.

The converse inequality splits into two parts. The first one is the domination of $\Lambda^{p,q}_\alpha(X_i f)$ by $\|f\|_{B^{p,q}_{\alpha+1}}$, which is an immediate application of Proposition 6.5. The second one is the domination of $\|X_i f\|_p$. But recall that Theorem 1.9 states that we can replace $\|X_i f\|_p$ by $\|H_{\frac{1}{2}}X_i f\|_p$ in the Besov norm, and (32) provides that

$$\|H_{\frac{1}{2}}X_i f\|_p \lesssim \|f\|_p \leq \|f\|_{B^{p,q}_\alpha}.$$

□

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