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Rates in the Central Limit Theorem and diffusion approximation via Stein's Method

Thomas Bonis

Télécom Paristech, Department of Computer Science and Networking
thomas.bonis@telecom-paristech.fr

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Abstract

We present a way to use Stein's method in order to bound the Wasserstein distance of order 2 between a measure ν and another measure μ , assumed to be the reversible measure of a diffusion operator, using non-exchangeable pairs of random variables drawn from ν . We then show that, whenever μ is the Gaussian measure γ , one can use exchangeable pairs of random variables drawn from ν to bound the Wasserstein distance of order p , for any $p \geq 2$, between ν and γ . Using our results, we are able to obtain convergence rates for the multi-dimensional Central Limit Theorem in terms of Wasserstein distances of order $p \geq 2$. In a second time, we use our approach to bound the Wasserstein distance of order 2 between the measure of a Markov chain and the reversible measure of a diffusion process satisfying some technical conditions and tackle two problems appearing in the field of data analysis: density estimation for geometric random graphs and sampling via the Langevin Monte Carlo algorithm.

1 Introduction

Stein's method is a general approach to bound distances between two measures and was first introduced by [27] to provide quantitative bounds for normal approximation. This approach relies on the following observation: the Gaussian measure γ is the only measure on \mathbb{R}^d such that, for any compactly supported smooth function (or test function) ϕ ,

$$\int_{\mathbb{R}^d} -x\phi(x) + \nabla\phi(x)d\gamma(x) = 0.$$

Hence, if another measure ν satisfies

$$\int_{\mathbb{R}^d} -x\phi(x) + \nabla\phi(x)d\nu(x) \approx 0$$

on a sufficiently large set of functions, then ν should be close to γ . This idea was later generalized by Barbour in [4] to deal with target measures assumed to be invariant measures of diffusion processes with infinitesimal generators of the form $\mathcal{L}_\mu = b \cdot \nabla + \langle a, Hess \rangle$. Indeed, under such assumptions, one usually has

$$\int_{\mathbb{R}^d} \mathcal{L}_\mu \phi d\mu = 0$$

for any test function ϕ . Thus, similarly to the Gaussian case, if a measure ν satisfies

$$\int_{\mathbb{R}^d} \mathcal{L}_\mu \phi d\nu \approx 0$$

for a sufficiently large set of functions, then ν should be close to μ .

In practice, there are two main approaches to Stein's method. In the first approach, one considers a set of functions \mathcal{U} and solves the Stein equation: for any $u \in \mathcal{U}$ find f_u such that

$$u - \int_{\mathbb{R}^d} u d\mu = \mathcal{L}_\mu f_u.$$

From here, taking the integral over ν yields

$$\int_{\mathbb{R}^d} u d\nu - \int_{\mathbb{R}^d} u d\mu = \int_{\mathbb{R}^d} \mathcal{L}_\mu f_u d\nu.$$

Thus, if

$$\sup_{u \in \mathcal{U}} \left| \int_{\mathbb{R}^d} \mathcal{L}_\mu f_u d\nu \right| \leq \epsilon,$$

then

$$\sup_{u \in \mathcal{U}} \left| \int_{\mathbb{R}^d} u d\nu - \int_{\mathbb{R}^d} u d\mu \right| \leq \epsilon.$$

By choosing some specific set \mathcal{U} , one is then able to obtain a bound on various distances between μ and ν :

- if $U = \{u : \mathbb{R}^d \rightarrow \mathbb{R} \mid \|u\|_\infty \leq 1\}$, then $\sup_{u \in U} \left| \int_{\mathbb{R}^d} u d\nu - \int_{\mathbb{R}^d} u d\mu \right|$ is the total variation distance between μ and ν ;
- in dimension 1, if $U = \{u : \mathbb{R} \rightarrow \mathbb{R} \mid \exists t \in \mathbb{R}, u(x) = 1_{x \leq t}\}$, then $\sup_{u \in U} \left| \int_{\mathbb{R}} u d\nu - \int_{\mathbb{R}} u d\mu \right|$ is the Kolmogorov distance between μ and ν ;
- if $U = \{u : \mathbb{R}^d \rightarrow \mathbb{R} \mid \forall x, y \in \mathbb{R}^d, \|u(y) - u(x)\| \leq \|y - x\|\}$, then $\sup_{u \in U} \left| \int_{\mathbb{R}^d} u d\nu - \int_{\mathbb{R}^d} u d\mu \right|$ is the Wasserstein distance of order 1 between μ and ν .

However, solving the Stein equation can be difficult as it involves computations depending on the target measure μ which are usually difficult to carry out in the multi-dimensional setting. So far, one of the most generic way to solve Stein equation requires μ to be a strictly log-concave measure of \mathbb{R}^d assumed to be

the invariant measure of a diffusion process with generator $\mathcal{L}_\mu = b.\nabla + \Delta$ [19]. Yet, for many applications, these assumptions are not general enough.

The second approach to applying Stein's method consists in finding an operator \mathcal{L}_ν such that, for any test function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\int_{\mathbb{R}^d} \mathcal{L}_\nu \phi d\nu = 0,$$

in which case we say ν is invariant under \mathcal{L}_ν . One then aims at bounding a given distance between ν and μ by a discrepancy between \mathcal{L}_μ and \mathcal{L}_ν . There are many ways to obtain such an operator \mathcal{L}_ν . The most classical approach consists in using an exchangeable pair (X, X') drawn from ν where a pair of random variable (X, X') is said to be exchangeable if (X, X') and (X', X) follow the same law. In this case, ν is invariant under the operator \mathcal{L}_ν where, for any test function ϕ ,

$$\forall x \in \mathbb{R}^d, \mathcal{L}_\nu \phi(x) = \mathbb{E}[(X' - X)(\phi(X') + \phi(X)) \mid X = x].$$

In fact, the exchangeability condition is not always required: in dimension one, one can recover standard results for the Gaussian measure or the Poisson distribution by using any pair of random variables (X, X') drawn from ν [25]. In this case, ν is invariant under the operator \mathcal{L}_ν where, for any test function ϕ ,

$$\forall x \in \mathbb{R}^d, \mathcal{L}_\nu \phi(x) = \mathbb{E} \left[\int_0^{X'} \phi(y) dy - \int_0^X \phi(y) dy \mid X = x \right].$$

The relative ease with which one can build an exchangeable pair is the main strength of this approach. In contrast, other approaches to building a suitable operator \mathcal{L}_ν , such as Stein kernels or biasing techniques, are usually hard to compute in the multi-dimensional setting.

In this work, we adapt the approach from [18], originally designed for Stein kernels, to operators of the form $\mathcal{L}_\nu f(x) = \mathbb{E}[f(X') - f(X) \mid X = x]$ where (X, X') is a pair of random variables drawn from ν . However, we cannot obtain a meaningful bound from a single pair of random variables, hence we use a stochastic process $(X_t)_{t \geq 0}$ such that, for any $t \geq 0$, X_t is drawn from ν . Whenever μ is the Gaussian measure γ , we show in Theorem 1 how to use the stochastic process $(X_t)_{t \geq 0}$ to bound the Wasserstein distance of order 2 between ν and γ . We then generalize this result by considering more general target measures in Theorem 5 and Corollary 6. In order to apply this latter result, we only require μ to be the reversible measure of a diffusion process with a generator of the form $\mathcal{L}_\mu = b.\nabla + \langle a, Hess \rangle$ such that the measure of the diffusion process converges exponentially fast to its invariant measure. Then, still following an approach from [18], we use a to bound Wasserstein distances of any order $p \geq 1$ between the measure of the random variables and the one-dimensional Gaussian measure in Theorem 7 using $(X_t)_{t \geq 0}$. Finally, we extend this result to the multi-dimensional setting in Theorem 9 by requiring that, for any $t \geq 0$, (X_t, X_0) is an exchangeable pair.

Obtaining convergence rates for the Central Limit Theorem is the most classical application of Stein's method. However, when it comes to Wasserstein distances, standard results obtained using Stein's method are usually restricted to a smoothed Wasserstein distance of order 1 (see e.g. Section 12 [11]). While the Stein kernel approach of [18] could be used to derive convergence rates for Wasserstein distances of any order, the resulting bounds would involve properties of the Stein kernel which cannot be easily computed in practice. Using our result, we are able to derive convergence rates in the Central Limit Theorem for Wasserstein distances of order $p \geq 2$ which only involves moments of the random variables considered. More precisely, if we consider i.i.d random variables X_1, \dots, X_n in \mathbb{R}^d with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1 X_1^T] = I_d$ admitting a finite moment of order $p + q$ for $p \geq 2, q \in [0, 2]$, then the Wasserstein distance of order p between the measure of $S_n = n^{-1/2} \sum_{i=1}^n X_i$ and the Gaussian measure is bounded by

$$C_p \left(n^{-1/2 + (2-q)/2p} \mathbb{E}[\|X_1\|^{p+q}]^{1/p} + n^{-m/4} \mathbb{E}[\|X_1\|^{2+m}]^{1/2} + \begin{cases} o(n^{-m/4}) & \text{if } m < 2 \\ d^{1/4} \|\mathbb{E}[X_1 X_1^T \|X_1\|^2]\|^{1/2} & \text{if } m = 2 \end{cases} \right),$$

where $C_p > 0$ is a constant depending only on p and $m = \min(4, q + 2)$. For $p, q = 2$, our result improves on the multi-dimensional result of [33] which requires $\|X_1\|$ to be almost surely bounded and suffers from an additional $\log n$ factor. In fact, the dependency on n obtained in our bound is optimal as it matches optimal results for the one-dimensional setting obtained in [23] for $p = 2$, in [26] for $p > 2, q = 0$ and in [7] for $p > 2, q = 2$. Still, the dependency of our bound with respect to the moments of X_1 , and thus to the dimension, can be suboptimal. Indeed, for $p, q = 2$, our bound scales (at least) linearly with respect to the dimension. Yet, if all the coordinates of X_1 are independent, then one can use the one-dimensional result to obtain a bound which scales with the square root of the dimension. Hence, one can expect tighter bounds can be obtained under stronger assumptions. For instance, if the measure of X_1 satisfies a Poincaré inequality for a constant $C > 0$, it is possible to use an approach based on the Stein kernel to bound the Wasserstein distance of order 2 between the measure of S_n and the Gaussian measure by $n^{-1/2} \sqrt{d(C-1)}$ [12] thus improving on our result whenever the constant C is small with respect to the dimension.

In the last few years, Stein's method was also used to give convergence rates in steady-state diffusion approximation in a series of papers [8, 10, 9] in order to study invariant measures of Markov chains appearing in queuing systems. Seeing these Markov chains as approximations of continuous diffusion processes, one can use Stein's method to bound the distance between the invariant measures of the Markov chains and the invariant measures of the limiting diffusion processes. However, the computations involved in these results are specific to the Markov chains and diffusion processes considered. We generalize this ap-

proach in Corollary 11 in which we bound the Wasserstein distance of order 2 between the invariant measure of a Markov chain and the reversible measure of a diffusion process using quantities appearing in standard results in diffusion approximation. Let us emphasize that using an approach based exchangeable pairs would require the Markov chain considered to be reversible, thus leading to a weaker result. Using our result, we are able to tackle a couple of problems relative to the field of data analysis and involving invariant measures of non-reversible Markov chains. We first provide quantitative bounds for the convergence of invariant measures of random walks on random geometric graphs. In a second time, we evaluate the complexity of a Monte Carlo algorithm for approximate sampling.

2 Notations

For $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$, we denote by $x^{\otimes k} \in (\mathbb{R}^d)^{\otimes k}$ the tensor of order k of x ,

$$\forall j \in \{1, \dots, d\}^k, (x^{\otimes k})_{j_1, \dots, j_k} = \prod_{i=1}^k x_{j_i}.$$

For any $x, y \in (\mathbb{R}^d)^{\otimes k}$ and any symmetric positive-definite $d \times d$ matrix A , we pose

$$\langle x, y \rangle_A = \sum_{l, j \in \{1, \dots, d\}^k} x_l y_j \prod_{i=1}^k A_{j_i, l_i},$$

and, by extension,

$$\|x\|_A^2 = \langle x, x \rangle_A.$$

For simplicity, we denote by $\|\cdot\|$ the traditional Hilbert-Schmidt norm, corresponding to $\|\cdot\|_{I_d}$. For any smooth function ϕ and $x \in \mathbb{R}^d$, we denote by $\nabla^k \phi \in (\mathbb{R}^d)^{\otimes k}$ the k -th gradient of ϕ :

$$\forall j \in \{1, \dots, d\}^k, (\nabla^k \phi(x))_j = \frac{\partial^k \phi}{\partial x_{j_1} \dots \partial x_{j_k}}(x).$$

Finally, the Wasserstein distance of order $p \geq 1$ between two measures μ and ν on \mathbb{R}^d is defined as

$$W_p(\mu, \nu) = \inf_{\pi} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^p \pi(dx, dy) \right)^{1/p},$$

where π has marginals μ and ν .

3 The approach

Let E be a convex domain of \mathbb{R}^d and let ν and μ be two measures with support E . Suppose μ is a reversible measure for the diffusion process with generator

$\mathcal{L}_\mu = b \cdot \nabla + \langle a, Hess \rangle_{HS}$ where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $a : \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ are smooth functions on E and a is symmetric positive-definite on all of E . For any measure $d\eta = h d\mu$, the Fisher information of η with respect to μ and a is given by

$$I_\mu(\eta) = \int_E \frac{\|\nabla h\|_a^2}{h} d\mu.$$

Let $(P_t)_{t \geq 0}$ be the Markov semigroup with infinitesimal generator \mathcal{L}_μ . Let $(Y_t)_{t \geq 0}$ be a diffusion process with infinitesimal generator \mathcal{L}_μ and initial condition Y_0 drawn from ν . We denote by ν_t the measure of Y_t . We assume that, for any $t > 0$, ν_t admits a smooth density h_t with respect to μ and $I_\mu(\nu_t) < \infty$. By construction, for any compactly supported smooth function ϕ ,

$$\int_E \phi d\nu_t = \int_E P_t \phi d\nu.$$

Since μ is the invariant measure of \mathcal{L}_μ then, under reasonable assumptions, ν_t converges to μ as t goes to infinity. We can thus control the distance between μ and ν by controlling the distance between ν_t and ν for any $t > 0$. This can be achieved using Lemma 2 [22]

$$\frac{d^+}{dt} W_2(\nu, \nu_t) \leq I_\mu(\nu_t)^{1/2}, \quad (1)$$

along with a bound on $I_\mu(\nu_t)$. Let us derive such a bound.

We have

$$I_\mu(\nu_t) = \int_E \frac{\|\nabla h_t\|_a^2}{h_t} d\mu = \int_E \langle \nabla h_t, \nabla \log h_t \rangle_a d\mu = \int_E \langle \nabla h_t, \nabla v_t \rangle_a,$$

where $v_t = \log(h_t)$. Since μ is reversible, it satisfies the following integration by parts formula

$$\int_E \langle \nabla f, \nabla g \rangle_a d\mu = - \int_E f \mathcal{L}_\mu g d\mu,$$

for smooth functions $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$. Hence,

$$I_\mu(\nu_t) = \int_E \langle \nabla h_t, \nabla v_t \rangle_a d\mu = - \int_E h_t \mathcal{L}_\mu v_t d\mu = - \int_E \mathcal{L}_\mu v_t d\nu_t.$$

By the definition of ν_t and the commutativity of P_t and \mathcal{L}_μ ,

$$I_\mu(\nu_t) = - \int_E P_t \mathcal{L}_\mu v_t d\nu = - \int_E \mathcal{L}_\mu P_t v_t d\nu.$$

Let us assume there exist two random variables X, X' drawn from ν and let $s > 0$. Let ϕ be a test function, for any $x \in \mathbb{R}^d$, let

$$\mathcal{L}_\nu \phi(x) = \frac{1}{s} \mathbb{E} [\phi(X') - \phi(X) | X = x],$$

we have

$$\int_{\mathbb{R}^d} \mathcal{L}_\nu P_t \phi d\nu = \mathbb{E}[\mathcal{L}_\nu \phi(X)] = \frac{1}{s} \mathbb{E}[\phi(X') - \phi(X)] = 0.$$

Assuming $P_t v_t$ is real analytic on E , we have

$$\mathcal{L}_\nu P_t v_t(x) = \frac{1}{s} \mathbb{E} \left[\sum_{k=1}^{\infty} \left\langle \frac{(X' - X)^{\otimes k}}{k!}, \nabla^k P_t v_t(X) \right\rangle \middle| X = x \right].$$

Therefore,

$$\begin{aligned} I_\mu(\nu_t) &= I_\mu(\nu_t) = \int_E (\mathcal{L}_\nu - \mathcal{L}_\mu) P_t v_t d\nu \\ &= \mathbb{E} \left[\left\langle \mathbb{E} \left[\frac{X' - X}{s} - b(X) \middle| X \right], \nabla P_t v_t(X) \right\rangle \right] \\ &+ \mathbb{E} \left[\left\langle \mathbb{E} \left[\frac{(X' - X)^{\otimes 2}}{2s} - a(X) \middle| X \right], \nabla^2 P_t v_t(X) \right\rangle \right] \\ &+ \mathbb{E} \left[\sum_{k=3}^{\infty} \left\langle \mathbb{E} \left[\frac{(X' - X)^{\otimes k}}{sk!} \middle| X \right], \nabla^k P_t v_t(X) \right\rangle \right]. \end{aligned} \quad (2)$$

The last step of the approach consists in exploiting the regularizing properties of the semigroup P_t in order to bound Equation 2 by a quantity involving $\mathbb{E}[P_t \|\nabla v_t\|_a^2(X)]^{1/2}$ and the moments of $X' - X$. Then, since $I_\mu(\nu_t)$ is finite and

$$\mathbb{E}[P_t \|\nabla v_t\|_a^2(X)] = \int_{\mathbb{R}^d} P_t \|\nabla v_t\|_a^2 d\nu = \int_{\mathbb{R}^d} \|\nabla v_t\|_a^2 d\nu_t = I_\mu(\nu_t),$$

we obtain a bound on $I_\mu(\nu_t)^{1/2}$ which can be turned into a bound on $W_2(\mu, \nu)$ thanks to Equation 1. Let us note that, since a is positive-definite on all of E , the bounds we derive on $\|\nabla^k P_t v_t\|_a$ imply $P_t v_t$ is real analytic on all of E [16].

3.1 Gaussian case

For $x \in \mathbb{R}^d$, let $d\mu(x) = d\gamma(x) = (2\pi)^{-d/2} e^{-\frac{\|x\|^2}{2}} dx$ be the Gaussian measure on \mathbb{R}^d . The measure γ is the reversible measure of the operator $\mathcal{L}_\gamma = -x \cdot \nabla + \Delta$ whose associated semigroup $(P_t)_{t \geq 0}$ is the Ornstein-Uhlenbeck semigroup. Let ϕ be a smooth function with compact support on \mathbb{R}^d . For any $x \in \mathbb{R}^d$, $P_t \phi$ admits the following representation

$$P_t \phi(x) = \int_{\mathbb{R}^d} \phi(xe^{-t} + \sqrt{1 - e^{-2t}}y) d\gamma(y).$$

For any $k > 0$ and any $i \in \{1, \dots, d\}^k$, let H_i be the multivariate Hermite polynomial of index i , defined for any $x \in \mathbb{R}^d$ by

$$H_i(x) = (-1)^k e^{\frac{\|x\|^2}{2}} \frac{\partial^k}{\partial x_{i_1} \dots \partial x_{i_k}} e^{-\frac{\|x\|^2}{2}}. \quad (3)$$

Using an integration by parts, we obtain

$$\begin{aligned}
\nabla P_t \phi(x) &= e^{-t} \int_{\mathbb{R}^d} \nabla \phi(xe^{-t} + \sqrt{1 - e^{-2t}}y) d\gamma(y) \\
&= \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \int_{\mathbb{R}^d} y \phi(xe^{-t} + \sqrt{1 - e^{-2t}}y) d\gamma(y) \\
&= \frac{1}{\sqrt{e^{2t} - 1}} \int_{\mathbb{R}^d} H_1(y) \phi(xe^{-t} + \sqrt{1 - e^{-2t}}y) d\gamma(y).
\end{aligned}$$

More generally, for $k \in \mathbb{N}$, using k integrations by parts yields

$$\forall i \in \{1, \dots, d\}^k, (\nabla^k P_t \phi(x))_i = \frac{1}{(e^{2t} - 1)^{k/2}} \int_{\mathbb{R}^d} H_i(y) \phi(xe^{-t} + \sqrt{1 - e^{-2t}}y) d\gamma(y).$$

Since Hermite polynomials form an orthogonal basis of $L^2(\gamma)$ with norm

$$\forall i \in \{1, \dots, d\}^k, \|H_i\|_\gamma^2 = \int_{\mathbb{R}^d} H_i^2(y) d\gamma(y) = \prod_{j=1}^d \left(\sum_{l=1}^k \delta_{i_l, j} \right)!,$$

we have, for any $x \in \mathbb{R}^d$,

$$\begin{aligned}
&\sum_{j \in \{1, \dots, d\}} \sum_{k=1}^{\infty} \sum_{i \in \{1, \dots, d\}^{k-1}} \frac{e^{2t}(e^{2t} - 1)^{k-1}}{\|H_i\|_\gamma^2} (\nabla^k P_t \phi)_{j, i_1, \dots, i_{k-1}}^2(x) \\
&= \sum_{j \in \{1, \dots, d\}} \sum_{k=1}^{\infty} \sum_{i \in \{1, \dots, d\}^{k-1}} \left(\int_{\mathbb{R}^d} \frac{H_i(y)}{\|H_i\|_\gamma} (\nabla \phi(xe^{-t} + \sqrt{1 - e^{-2t}}y))_j d\gamma(y) \right)^2 \\
&= \sum_{j \in \{1, \dots, d\}} \int_{\mathbb{R}^d} (\nabla \phi(xe^{-t} + \sqrt{1 - e^{-2t}}y))_j^2 d\gamma(y) \\
&= \int_{\mathbb{R}^d} \|\nabla \phi(xe^{-t} + \sqrt{1 - e^{-2t}}y)\|^2 d\gamma(y) \\
&= P_t \|\nabla \phi\|^2(x).
\end{aligned}$$

For any $k \in \mathbb{N}$ and any tensor $M \in (\mathbb{R}^d)^{\otimes k}$, we pose

$$\|M\|_H^2 = \sum_{j \in \{1, \dots, d\}^k} \sum_{i \in \{1, \dots, d\}^{k-1}} \|H_i\|_\gamma^2 M_{j, i_1, \dots, i_{k-1}}$$

and

$$\begin{aligned}
S(t)^2 &= e^{-2t} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{X' - X}{s} + X \mid X \right] \right\|^2 \right] \\
&\quad + \frac{e^{-2t}}{e^{2t} - 1} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{(X' - X)^{\otimes 2}}{2s} - I_d \mid X \right] \right\|^2 \right] \\
&\quad + \sum_{k=3}^{\infty} \frac{e^{-2t}}{(sk!)^2 (e^{2t} - 1)^{k-1}} \mathbb{E} \left[\|\mathbb{E}[(X' - X)^{\otimes k} \mid X]\|_H^2 \right].
\end{aligned}$$

Applying Cauchy-Schwarz's inequality to Equation 2, we obtain

$$\begin{aligned}
I_\gamma(\nu_t) &\leq S(t) \left(\sum_{k=1}^{\infty} \sum_{i \in \{1, \dots, d\}^{k-1}} \frac{e^{2t}(e^{2t}-1)^{k-1}}{\|H_i\|_\gamma^2} \mathbb{E} \left[\sum_{j \in \{1, \dots, d\}} (\nabla^k P_t v_t(X))_{j, i_1, \dots, i_{k-1}}^2 \right] \right)^{1/2} \\
&\leq S(t) \mathbb{E}[P_t \|\nabla v_t(X)\|^2]^{1/2} \\
&= S(t) I_\gamma(\nu_t)^{1/2},
\end{aligned}$$

from which we deduce that

$$I_\gamma(\nu_t)^{1/2} \leq S(t).$$

Now, according to Equation 1, integrating $S(t)$ for $t > 0$ would give us a bound on $W_2(\nu, \gamma)$. However, $S(t)$ suffers from integrability issues for small values of t . To circumvent this issue, we use a stochastic process $(X_t)_{t \geq 0}$ such that, for any $t \geq 0$, X_t has measure ν . Then, for any fixed $t \in \mathbb{R}$, we can replace X by X_0 and X' by X_t in the previous computations to bound $I_\gamma(\nu_t)^{1/2}$ by $S(t)$ where

$$\begin{aligned}
S(t)^2 &= e^{-2t} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{X_t - X_0}{s} + X_0 \mid X_0 \right] \right\|^2 \right] \\
&\quad + \frac{e^{-2t}}{e^{2t}-1} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{(X_t - X_0)^{\otimes 2}}{2s} - I_d \mid X_0 \right] \right\|^2 \right] \\
&\quad + \sum_{k=3}^{\infty} \frac{e^{-2t}}{(sk!)^2 (e^{2t}-1)^{k-1}} \mathbb{E} \left[\left\| \mathbb{E}[(X_t - X_0)^{\otimes k} \mid X_0] \right\|_H^2 \right].
\end{aligned}$$

Then, if $S(t)$ is integrable for $t \geq 0$, we can use Equation 1 to obtain $W_2(\nu, \gamma) \leq \int_0^\infty S(t) dt$.

So far, we have assumed the Fisher information of ν_t with respect to γ is finite for any $t > 0$. By Remark 2.1 [20], this assumption is verified as long as the second moment of ν is finite which is a necessary condition for $W_2(\nu, \gamma)$ to be finite in the first place.

Theorem 1. *Let ν be a measure on \mathbb{R}^d with finite second moment. Let $(X_t)_{t \geq 0}$ be a stochastic process such that, for any $t \geq 0$, X_t has measure ν . For any $s > 0$,*

$$W_2(\nu, \gamma) \leq \int_0^\infty S(t) dt,$$

where

$$\begin{aligned}
S(t)^2 &= e^{-2t} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{X_t - X_0}{s} + X_0 \mid X_0 \right] \right\|^2 \right] \\
&\quad + \frac{e^{-2t}}{e^{2t} - 1} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{(X_t - X_0)^{\otimes 2}}{2s} - I_d \mid X_0 \right] \right\|^2 \right] \\
&\quad + \sum_{k=3}^{\infty} \frac{e^{-2t}}{(sk!)^2 (e^{2t} - 1)^{k-1}} \mathbb{E} \left[\left\| \mathbb{E}[(X_t - X_0)^{\otimes k} \mid X_0] \right\|_H^2 \right].
\end{aligned}$$

Remark 2. For our result to produce non-trivial bounds, the quantity $S(t)$ must be finite and integrable for $t \geq 0$. Since

$$\forall k \in \mathbb{N}, M \in (\mathbb{R}^d)^{\otimes k}, \|M\|_H \leq \max_{i \in \{1, \dots, d\}^{k-1}} \|H_i\|_{\gamma} \|M\| \leq \sqrt{(k-1)!},$$

we have, by Jensen's inequality and the triangle inequality,

$$\begin{aligned}
e^t S(t) &\leq \mathbb{E}[\|X_0\|^2]^{1/2} + \left(\frac{\|I_d\|^2}{e^{2t} - 1} \right)^{1/2} + \frac{1}{s} \left(\mathbb{E} \left[\sum_{k=1}^{\infty} \frac{\|X_t - X_0\|^{2k}}{k! (e^{2t} - 1)^k} \right] \right)^{1/2} \\
&\leq \mathbb{E}[\|X_0\|^2]^{1/2} + \left(\frac{d}{e^{2t} - 1} \right)^{1/2} + \frac{1}{s} \mathbb{E} \left[e^{\frac{\|X_t - X_0\|^2}{e^{2t} - 1}} \right]^{1/2}.
\end{aligned}$$

Hence, since ν is assumed to have a finite second moment, it is sufficient for $e^{-t} \mathbb{E} \left[e^{\frac{\|X_t - X_0\|^2}{e^{2t} - 1}} \right]^{1/2}$ to be integrable for $t \geq 0$ to obtain a non-trivial bound with our result. Our following results, either dealing with more general target measures and with Wasserstein distances of order $p \geq 1$, also rely on a stochastic process $(X_t)_{t \geq 0}$ and can be shown to produce non-trivial bounds under similar conditions.

3.2 General case

Let us first apply Cauchy-Schwarz's inequality to Equation 2 in order to obtain

$$\begin{aligned}
I_{\mu}(\nu_t)^2 &\leq \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{X' - X}{s} - b(X) \mid X \right] \right\|_{a^{-1}(X)}^2 \right] \mathbb{E}[\|\nabla P_t v_t(X)\|_{a(X)}^2] \\
&\quad + \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{(X' - X)^{\otimes 2}}{2s} \mid X \right] - a(X) \right\|_{a^{-1}(X)}^2 \right] \mathbb{E}[\|\nabla^2 P_t v_t(X)\|_{a(X)}^2] \\
&\quad + \sum_{k=3}^{\infty} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{(X' - X)^{\otimes k}}{sk!} \mid X \right] \right\|_{a^{-1}(X)}^2 \right] \mathbb{E}[\|\nabla^k P_t v_t(X)\|_{a(X)}^2].
\end{aligned} \tag{4}$$

Our objective is to bound $\|\nabla^k P_t v_t\|_a^2$ by a quantity involving $P_t \|\nabla v_t\|_a^2$. To obtain such bounds, we use the framework of Γ -calculus described in [3]. This

approach relies on the iterated gradients Γ_i , defined recursively for any smooth functions f, g by

$$\begin{aligned}\Gamma_0(f, g) &= fg; \\ \Gamma_{i+1}(f, g) &= \frac{1}{2} [\mathcal{L}_\mu(\Gamma_i(f, g)) - \Gamma_i(\mathcal{L}_\mu f, g) - \Gamma_i(f, \mathcal{L}_\mu g)].\end{aligned}$$

The triple (E, μ, Γ_1) is called a Markov triple. The operators Γ_1 , also called the Carré du champ operator, and Γ_2 are efficient tools to study the properties of the semigroup $(P_t)_{t \geq 0}$ [3]. In particular if there exists $\rho \in \mathbb{R}$ such that, for any smooth function ϕ ,

$$\Gamma_2(\phi, \phi) \geq \rho \Gamma_1(\phi, \phi),$$

the Markov triple is said to verify a *curvature-dimension* condition, or $CD(\rho, \infty)$ condition, under which the semigroup $(P_t)_{t \geq 0}$ satisfies many interesting properties. For instance, under a $CD(\rho, \infty)$ condition, $(P_t)_{t \geq 0}$ verifies the following gradient bound (see e.g. Theorem 3.2.3 [3])

$$\|\nabla P_t \phi\|_a^2 \leq e^{-2\rho t} P_t(\|\nabla \phi\|_a^2).$$

Hence, a $CD(\rho, \infty)$ condition is sufficient to bound the first term in Equation 4. In the proof of Theorem 4.1 of [18], it is shown that, under a $CD(\rho, \infty)$ condition for $\rho > 0$ and assuming there exists $\kappa, \sigma > 0$ such that for any smooth function f , $\Gamma_3(\phi, \phi) \geq \kappa \Gamma_2(\phi, \phi)$ and $\Gamma_2(\phi, \phi) \geq \sigma \|\nabla^2 \phi\|_a$,

$$\|\nabla^2 P_t \phi\|_a^2 \leq \frac{\kappa}{\sigma(e^{\kappa t} - 1)} P_t \|\nabla \phi\|_a^2. \quad (5)$$

However, while a $CD(\rho, \infty)$ is usually easy to verify, assumptions involving the operators $(\Gamma_k)_{k \geq 3}$ quickly becomes unpractical. Let us consider a simple one-dimensional example for which $\mathcal{L}\phi = -b'\phi' + \phi''$. In this case, a $CD(\rho, \infty)$ condition is verified as long as $b'' \geq \rho$. On the other hand, following the computation of Section 4.4 [18], in order to have $\Gamma_3(f, f) \geq 3c\Gamma_2(f, f)$ and $\Gamma_2(f, f) \geq c\|f''\|_a$ for some $c > 0$, one requires

$$b^{(4)} - b'b^{(3)} + 2(b'')^2 - 6cb'' \geq 0$$

and

$$3(b^{(3)})^2 \leq 2(u'' - c)(b^{(4)} - b'b^{(3)} + 2(b'')^2 - 6cb'').$$

Even in this rather simple case, these conditions are quite strong and obtaining bounds on $\|\nabla^k P_t \phi\|_a$ for $k > 2$ in a similar manner would require even stronger assumptions. In Section 6.1, we derive bounds on $\|\nabla^k P_t \phi\|_a$ under a simple $CD(\rho, \infty)$ condition.

Proposition 3. *If \mathcal{L}_μ satisfies a $CD(\rho, \infty)$ condition for $\rho \in \mathbb{R}$, then, for any integer $k > 0$, any $t > 0$ and any smooth function ϕ ,*

$$\|\nabla^k P_t \phi\|_a \leq f_k(t) \sqrt{P_t \|\nabla \phi\|_a^2},$$

where

$$f_k(t) = \begin{cases} e^{-\rho t \max(1, k/2)} \left(\frac{2\rho d}{e^{2\rho t/(k-1)} - 1} \right)^{(k-1)/2} & \text{if } \rho \neq 0 \\ \left(\frac{d(k-1)}{t} \right)^{(k-1)/2} & \text{if } \rho = 0. \end{cases}$$

Remark 4. The bounds we obtain are not dimension-independent as one could expect from the Gaussian case or from Equation 5. We believe this dependency to be an artifact of the proof.

Injecting these bounds in Equation 4, we obtain a bound on $I_\mu(\nu_t)^{1/2}$. Similarly to the Gaussian case, this bound is not integrable for small values of t . Again, we deal with this issue by using a stochastic process $(X_t)_{t \geq 0}$ such that, for each $t > 0$, X_t is drawn from ν . However, contrary to the Gaussian case, we may also face integration issues when t goes to infinity whenever \mathcal{L}_μ only satisfies a $CD(\rho, \infty)$ condition with $\rho \leq 0$. In this case, we are only able to bound $W_2(\nu, \nu_T)$ for $T > 0$.

So far, we have assumed the Fisher information of ν_t with respect to μ is finite. If a $CD(\rho, \infty)$ condition is satisfied, this assumption can be weakened. Indeed, for any $t, s > 0$ and any measure $\eta = h d\mu$, we have, by Theorem 5.5.2 [3]

$$I_\mu(\nu_{t+s}) \leq \frac{2\rho}{1 - e^{-2\rho t}} \int_E (P_t(h_s \log h_s) - P_t h_s \log(P_t h_s)) d\mu \leq \frac{2\rho}{1 - e^{-2\rho t}} H_\mu(\nu_s),$$

where $H_\mu(\nu_s)$ is the entropy of ν_s with respect to μ .

Theorem 5. *Suppose μ is the reversible measure of a diffusion operator \mathcal{L}_μ satisfying a $CD(\rho, \infty)$ condition for $\rho \in \mathbb{R}$. Let ν be a measure of \mathbb{R}^d and let $(X_t)_{t \geq 0}$ be a stochastic process such that, for any $t \geq 0$, X_t is drawn from ν . If the entropy of ν_t with respect to μ is finite for any $t > 0$, then, for any $s > 0$, $T > 0$,*

$$W_2(\nu, \nu_T) \leq \int_0^T S(t) dt,$$

where

$$\begin{aligned} S(t)^2 = & f_1(t) \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{X_t - X_0}{s} - b(X_0) \mid X_0 \right] \right\|_{a^{-1}(X_0)}^2 \right] \\ & + f_2(t) \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{(X_t - X_0)^{\otimes 2}}{2s} - a(X_0) \mid X_0 \right] \right\|_{a^{-1}(X_0)}^2 \right] \\ & + \sum_{k=3}^{\infty} \frac{f_k(t)}{sk!} \mathbb{E} \left[\left\| \mathbb{E}[(X_t - X_0)^{\otimes k} \mid X_0] \right\|_{a^{-1}(X_0)}^2 \right], \end{aligned}$$

where the functions $(f_k)_{k \geq 1}$ are defined in Proposition 3.

If $\rho > 0$, we can set T to infinity in order to bound $W_2(\nu, \mu)$. On the other hand, if $\rho \leq 0$, it is still possible to bound $W_2(\nu, \mu)$ as long as we can bound $W_2(\nu_T, \mu)$.

Corollary 6. *Suppose μ is the reversible measure of a diffusion operator \mathcal{L}_μ satisfying a $CD(\rho, \infty)$ condition for $\rho \in \mathbb{R}$. Moreover, suppose there exists κ such that, for any measure η and any $t > 0$, we have*

$$W_2(\eta_t, \mu) \leq e^{-\kappa t} W_2(\eta, \nu).$$

Let ν be a measure of \mathbb{R}^d and let $(X_t)_{t \geq 0}$ be a stochastic process such that, for any $t \geq 0$, X_t is drawn from ν . If the entropy of ν_t with respect to μ is finite for any $t > 0$, then, for any $s > 0$, $T > 0$,

$$(1 - e^{-\kappa T}) W_2(\nu, \mu) \leq \int_0^T S_t dt,$$

where S_t is defined in Theorem 5.

Proof. We have

$$\begin{aligned} W_2(\nu, \mu) &\leq W_2(\nu, \nu_T) + W_2(\nu_T, \mu) \\ &\leq W_2(\nu, \nu_T) + e^{-\kappa T} W_2(\nu, \mu). \end{aligned}$$

The result is then obtained by plugging the bound on $W_2(\nu, \nu_T)$ from Theorem 5. \square

Such an exponential convergence to μ can be verified under weaker conditions than a $CD(\rho, \infty)$ inequality for $\rho > 0$. For example, if $a = I_d$ and $b = -\nabla V$, where V is a function from \mathbb{R}^d to \mathbb{R} , this assumption is satisfied whenever V is strongly convex outside a bounded set C with bounded first and second order derivatives on C [14]. An extension of this result for more general functions a and for the manifold setting is proposed in [32].

4 Gaussian measure and Wasserstein distance of any order

Whenever the target is the Gaussian measure, [18] used Stein kernels to obtain an explicit expression of the score function $v_t = \log(h_t)$ which can be turned into a bound for Wasserstein distances of order $p \geq 1$ between the measure ν and the Gaussian measure by using a simple modification of Lemma 2 [22] giving

$$\frac{d_+}{dt} W_p(\nu, \mu) \leq \left(\int_{\mathbb{R}^d} \|\nabla v_t\|^p d\nu_t \right)^{1/p}. \quad (6)$$

Let us show a similar approach can be followed using a stochastic process $(X_t)_{t \geq 0}$. In order to bound $(\int_{\mathbb{R}^d} \|\nabla v_t\|^p d\nu_t)^{1/p}$ we are going to provide a version of ∇v_t . By a simple integration by parts, one can see that ∇v_t is characterized

by the fact that, for any smooth test function ϕ ,

$$\begin{aligned}
\int_{\mathbb{R}^d} \nabla \phi(x) d\nu_t(x) &= \int_{\mathbb{R}^d} \nabla \phi(x) h_t(x) d\gamma(x) \\
&= \int_{\mathbb{R}^d} \nabla(\phi h_t)(x) - \phi(x) \nabla h_t(x) d\gamma(x) \\
&= \int_{\mathbb{R}^d} x \phi(x) h_t(x) - \phi(x) \nabla h_t(x) \gamma(x) \\
&= \int_{\mathbb{R}^d} \phi(x) (x - \nabla v_t(x)) d\nu_t(x).
\end{aligned}$$

4.1 One-dimensional case

Let (X, X') be two random variables drawn from a measure ν on \mathbb{R} and let Z be a one-dimensional Gaussian random variable. For any $k \in \mathbb{N}$, let H_k be the Hermite polynomial of order k . For $t > 0$, let $F_t = e^{-t}X + \sqrt{1 - e^{-2t}}Z$ and for any $s > 0$, let ρ_t be a real-valued function such that

$$\begin{aligned}
\forall x \in \mathbb{R}, \rho_t(x) &= \mathbb{E} \left[e^{-t} \left(\frac{X' - X}{s} + X \right) + \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \left(\frac{(X' - X)^2}{2s} - 1 \right) Z \mid F_t = x \right] \\
&\quad + \mathbb{E} \left[\sum_{k=3}^{\infty} \frac{e^{-kt}}{s \sqrt{1 - e^{-2t}}^{k-1}} \frac{(X' - X)^k}{k!} H_{k-1}(Z) \mid F_t = x \right].
\end{aligned}$$

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a compactly supported smooth function. Since $\phi(F_t) = \phi(e^{-t}X + \sqrt{1 - e^{-2t}}Z)$, we obtain, after successive integrations by parts with respect to Z ,

$$\mathbb{E}[\rho_t(F_t)\phi(F_t)] = \mathbb{E} [e^{-t}X\phi(F_t) - e^{-2t}\phi'(F_t)] + \mathbb{E} \left[\sum_{k=1}^{\infty} e^{-kt} \frac{(X' - X)^k}{sk!} \phi^{(k-1)}(F_t) \right].$$

Let Φ be a primitive function of ϕ . By the results of Section 3.1, the function $x \rightarrow \mathbb{E}[\Phi(F_t) \mid X = x] = P_t\Phi(x)$ is real analytic. Hence, since X' and X have the same measure, we have

$$\begin{aligned}
&\mathbb{E} \left[\sum_{k=1}^{\infty} e^{-kt} \frac{(X' - X)^k}{sk!} \phi^{(k-1)}(F_t) \right] \\
&= \frac{1}{s} \mathbb{E}[\Phi(e^{-t}X' + \sqrt{1 - e^{-2t}}Z) - \Phi(e^{-t}X + \sqrt{1 - e^{-2t}}Z)] \\
&= 0.
\end{aligned}$$

Then, using a last integration by parts with respect to Z ,

$$\begin{aligned}
\mathbb{E}[(\rho_t(F_t) - F_t)\phi(F_t)] &= \mathbb{E}[(e^{-t}X - F_t)\phi(F_t) - e^{-2t}\phi'(F_t)] \\
&= \mathbb{E}[\sqrt{1 - e^{-2t}}Z\phi(F_t) - e^{-2t}\phi'(F_t)] \\
&= \mathbb{E}[(e^{-2t} - 1)\phi'(F_t) - e^{-2t}\phi'(F_t)] \\
&= -\mathbb{E}[\phi'(F_t)].
\end{aligned}$$

Therefore, ρ_t is a version of v'_t and, for $p \geq 1$, we can bound

$$\left(\int_{\mathbb{R}} |v'_t|^p d\nu_t \right)^{1/p} = \mathbb{E}[|\rho_t(F_t)|^p]^{1/p}$$

by

$$\begin{aligned} S_p(t) &\leq \mathbb{E} \left[\left| \mathbb{E} \left[\frac{X_t - X_0}{s} + X_0 \mid X_0 \right] \right|^p \right]^{1/p} \\ &\quad + \frac{e^{-2t} \|H_1\|_{p,\gamma}}{\sqrt{1 - e^{-2t}}} \mathbb{E} \left[\left| \mathbb{E} \left[\frac{(X_t - X_0)^2}{2s} - 1 \mid X_0 \right] \right|^p \right]^{1/p} \\ &\quad + \sum_{k=3}^{\infty} \frac{e^{-kt} \|H_{k-1}\|_{p,\gamma}}{k!s\sqrt{1 - e^{-2t}{}^{k-1}}} \mathbb{E} \left[\left| \mathbb{E}[(X_t - X_0)^k \mid X_0] \right|^p \right]^{1/p}, \end{aligned}$$

where $\|H_k\|_{p,\gamma}^p = \int_{\mathbb{R}} |H_k|^p d\gamma$. By Equation 6, integrating $S_p(t)$ for $t \geq 0$ would give us a bound on the Wasserstein distance of order p between ν and the one-dimensional Gaussian measure γ . Once again, we deal with integration issues by using a stochastic process $(X_t)_{t \geq 0}$ such that X_t is drawn from ν for any $t > 0$.

Theorem 7. *Let ν be a measure on \mathbb{R} and let $(X_t)_{t \geq 0}$ be a stochastic process such that, for any $t \geq 0$, X_t is drawn ν . We have, for any $p \geq 1, s > 0$,*

$$W_p(\nu, \gamma) \leq \int_0^{\infty} S_p(t) dt,$$

where

$$\begin{aligned} S_p(t) &\leq e^{-t} \mathbb{E} \left[\left| \mathbb{E} \left[\frac{X_t - X_0}{s} + X_0 \mid X_0 \right] \right|^p \right]^{1/p} \\ &\quad + \frac{e^{-t} \|H_1\|_{p,\gamma}}{\sqrt{e^{2t} - 1}} \mathbb{E} \left[\left| \mathbb{E} \left[\frac{(X_t - X_0)^2}{2s} - 1 \mid X_0 \right] \right|^p \right]^{1/p} \\ &\quad + \sum_{k=3}^{\infty} \frac{e^{-t} \|H_{k-1}\|_{p,\gamma}}{k!s\sqrt{e^{2t} - 1}{}^{k-1}} \mathbb{E} \left[\left| \mathbb{E}[(X_t - X_0)^k \mid X_0] \right|^p \right]^{1/p}. \end{aligned}$$

4.2 Multi-dimensional case

Let (X, X') be two random variables drawn from a measure ν on \mathbb{R}^d and let Z be a d -dimensional Gaussian random variable. Since there is no notion of primitive function in the multi-dimensional setting, the previous approach cannot be used. Instead, let us assume that (X, X') and (X', X) follow the same law (i.e. (X, X') is an exchangeable pair). For any $k \in \mathbb{N}$, any $M \in (\mathbb{R}^d)^{\otimes k}$ and any $N \in (\mathbb{R}^d)^{\otimes k-1}$, we define the vector $MN \in \mathbb{R}^d$ by

$$\forall i \in \{1, \dots, d\}, (MN)_i = \sum_{j \in \{1, \dots, d\}^{k-1}} M_{i, j_1, \dots, j_{k-1}} N_j.$$

For any $k \in \mathbb{N}$ we denote by $\mathcal{H}_k : \mathbb{R}^d \rightarrow (\mathbb{R}^d)^{\otimes k}$ the tensor of Hermite polynomials of order k where

$$\forall x \in \mathbb{R}^d, \forall i \in \{1, \dots, d\}^k, (\mathcal{H}_k(x))_i = H_i(x),$$

where for any $i \in \{1, \dots, d\}^k$, H_i is the multi-dimensional Hermite polynomial of index i , see Equation 3. For $t > 0$, let $F_t = e^{-t}X + \sqrt{1 - e^{-2t}}Z$ and consider the vector-valued function ρ_t defined by

$$\begin{aligned} \forall x \in \mathbb{R}^d, \rho_t(x) &= \mathbb{E} \left[e^{-t} \left(\frac{X' - X}{s} + X \right) + \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \left(\frac{(X' - X)^{\otimes 2}}{2s} - I_d \right) Z \mid F_t = x \right] \\ &+ \mathbb{E} \left[\sum_{k=3}^{\infty} \frac{e^{-kt}}{2s(k-1)! \sqrt{1 - e^{-2t}}^{k-1}} (X' - X)^{\otimes k} \mathcal{H}_{k-1}(Z) \mid F_t = x \right]. \end{aligned}$$

For any integer k and any $A, B \in (\mathbb{R}^d)^{\otimes k}$, let $\langle A, B \rangle$ be the Hilbert-Schmidt scalar product between A and B . We have

$$\begin{aligned} \rho_t(x) &= \mathbb{E} \left[e^{-t}X - \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}}Z + \frac{e^{-t}}{s}(X' - X) \mid F_t = x \right] \\ &+ \frac{e^{-t}}{2s} \mathbb{E} \left[(X' - X) \sum_{k=1}^{\infty} \frac{1}{k!} \left\langle e^{-kt}(X' - X)^{\otimes k}, \sqrt{1 - e^{-2t}}^{-k} \mathcal{H}_k(Z) \right\rangle \mid F_t = x \right]. \end{aligned}$$

Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a compactly supported smooth function. Since $\phi(F_t) = \phi(e^{-t}X + \sqrt{1 - e^{-2t}}Z)$, successive integrations by parts with respect to Z yield

$$\begin{aligned} \mathbb{E}[\rho_t(F_t)\phi(F_t)] &= \mathbb{E} \left[e^{-t}X\phi(F_t) - \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \nabla \phi(F_t) \right] \\ &+ \frac{e^{-t}}{s} \mathbb{E} \left[(X' - X) \left(\phi(F_t) + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k!} \langle e^{-kt}(X' - X)^{\otimes k}, \nabla^k \phi(F_t) \rangle \right) \right]. \end{aligned}$$

Let us pose $F'_t = e^{-t}X' + \sqrt{1 - e^{-2t}}Z$. Since the function $x \rightarrow \mathbb{E}[\phi(F_t) \mid X = x] = P_t\phi(x)$ is real analytic, we have

$$\begin{aligned} \mathbb{E}[\rho_t(F_t)\phi(F_t)] &= \mathbb{E} \left[e^{-t}X\phi(F_t) - \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \nabla \phi(F_t) \right] + \frac{e^{-t}}{s} \mathbb{E} \left[(X' - X) \left(\phi(F_t) + \frac{\phi(F'_t) - \phi(F_t)}{2} \right) \right] \\ &= \mathbb{E} \left[e^{-t}X\phi(F_t) - \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \nabla \phi(F_t) \right] + \frac{e^{-t}}{2s} \mathbb{E} [(X' - X)(\phi(F'_t) + \phi(F_t))]. \end{aligned}$$

Since (X, X') is an exchangeable pair, $\mathbb{E}[(X' - X)(\phi(F'_t) + \phi(F_t))] = 0$. Therefore,

$$\begin{aligned} \mathbb{E}[(\rho_t(F_t) - F_t)\phi(F_t)] &= \mathbb{E}[(-F_t + e^{-t}X)\phi(F_t) - e^{-2t}\nabla\phi(F_t)] \\ &= \mathbb{E}[\sqrt{1 - e^{-2t}}Z\phi(F_t) - e^{-2t}\nabla\phi(F_t)] \\ &= \mathbb{E}[(e^{-2t} - 1)\nabla\phi(F_t) - e^{-2t}\nabla\phi(F_t)] \\ &= -\mathbb{E}[\nabla\phi(F_t)]. \end{aligned}$$

Thus, ρ_t is a version of ∇v_t . Let us consider some $p \geq 1$. Applying Jensen's inequality and the triangle inequality, we have

$$\begin{aligned} \mathbb{E}[\|\nabla v_t(F_t)\|^p]^{1/p} &\leq e^{-t} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{X' - X}{s} + X \mid X \right] \right\|^p \right]^{1/p} \\ &\quad + \frac{e^{-t}}{\sqrt{e^{2t} - 1}} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{(X' - X)^{\otimes 2}}{2s} - I_d \mid X \right] Z \right\|^p \right]^{1/p} \\ &\quad + \sum_{k=3}^{\infty} \frac{e^{-t}}{2s(k-1)!(e^{2t} - 1)^{(k-1)/2}} \mathbb{E} \left[\left\| \mathbb{E}[(X' - X)^{\otimes k} \mid X] \mathcal{H}_{k-1}(Z) \right\|^p \right]^{1/p}. \end{aligned}$$

In order to refine this bound, we require the following result.

Lemma 8. *Let Z be a Gaussian random variable. For any $k \in \mathbb{N}$, $M \in (\mathbb{R}^d)^{\otimes k}$, we have*

$$\mathbb{E}[\|M\mathcal{H}_{k-1}(Z)\|^p]^{1/p} \leq \begin{cases} \|M\|_H & \text{if } 1 \leq p \leq 2 \\ (p-1)^{(k-1)/2} \|M\|_H & \text{if } p > 2 \end{cases}.$$

Proof. First, if $1 \leq p \leq 2$, then

$$\mathbb{E}[\|M\mathcal{H}_{k-1}(Z)\|^p]^{1/p} \leq \mathbb{E}[\|M\mathcal{H}_{k-1}(Z)\|^2]^{1/2} = \|M\|_H.$$

Let us consider $p > 2$. Since the Ornstein-Uhlenbeck semigroup $(P_t)_{t \geq 0}$ is hypercontractive (see e.g. Theorem 5.2.3 [3]), taking $t = \log(\sqrt{p-1})$, we have

$$\forall \phi \in L^2(\gamma), \mathbb{E}[|P_t \phi(Z)|^p]^{1/p} \leq \mathbb{E}[\phi(Z)^2]^{1/2}.$$

This inequality can be extended to vector-valued function ϕ , in which case we have

$$\forall \phi, \|\phi\| \in L^2(\gamma), \mathbb{E}[\|P_t \phi(Z)\|^p]^{1/p} \leq \mathbb{E}[\|(P_t \phi(Z))\|^2]^{1/2} \leq \mathbb{E}[\|\phi(Z)\|^2]^{1/2}.$$

Let $k \in \mathbb{N}$ and $M \in (\mathbb{R}^d)^{\otimes k}$. For $t = \log(\sqrt{p-1})$, the entries of \mathcal{H}_{k-1} are Hermite polynomials and thus eigenvectors of P_t with eigenvalue $e^{-(k-1)t} = (p-1)^{-(k-1)/2}$. Therefore

$$\begin{aligned} \mathbb{E}[\|M\mathcal{H}_{k-1}(Z)\|^p]^{1/p} &= (p-1)^{(k-1)/2} \mathbb{E}[\|MP_t \mathcal{H}_{k-1}(Z)\|^p]^{1/p} \\ &= (p-1)^{(k-1)/2} \mathbb{E}[\|P_t M \mathcal{H}_{k-1}(Z)\|^p]^{1/p} \\ &\leq (p-1)^{(k-1)/2} \mathbb{E}[\|M\mathcal{H}_{k-1}(Z)\|^2]^{1/2} \\ &\leq (p-1)^{(k-1)/2} \|M\|_H \end{aligned}$$

□

Thanks to this result, we obtain

$$\begin{aligned} \mathbb{E}[\|\nabla v_t(F_t)\|^p]^{1/p} &\leq e^{-t} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{X' - X}{s} + X \mid X \right] \right\|^p \right]^{1/p} \\ &\quad + e^{-t} \sqrt{\frac{\max(1, p-1)}{e^{2t} - 1}} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{(X' - X)^{\otimes 2}}{2s} - I_d \mid X \right] \right\|^p \right]^{1/p} \\ &\quad + \sum_{k=3}^{\infty} \frac{e^{-t}}{2s(k-1)!} \left(\frac{\max(1, p-1)}{e^{2t} - 1} \right)^{(k-1)/2} \mathbb{E} \left[\left\| \mathbb{E}[(X' - X)^{\otimes k} \mid X] \right\|^p \right]^{1/p}. \end{aligned}$$

Again, we deal with integrability issues by using a stochastic process $(X_t)_{t \geq 0}$ such that, for any $t > 0$, X_t is drawn from ν and (X_0, X_t) is an exchangeable pair of random variables.

Theorem 9. *Let ν be a measure on \mathbb{R}^d and let $(X_t)_{t \geq 0}$ be a stochastic process such that, for any $t \geq 0$, X_t is drawn from ν and (X_0, X_t) is an exchangeable pair. For any $p \geq 1$, $s > 0$,*

$$W_p(\nu, \gamma) \leq \int_0^{\infty} S_p(t) dt,$$

where,

$$\begin{aligned} S_p(t) &= \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{X_t - X_0}{s} + X_0 \mid X \right] \right\|^p \right]^{1/p} \\ &\quad + e^{-t} \sqrt{\frac{\max(1, p-1)}{e^{2t} - 1}} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{(X_t - X_0)^{\otimes 2}}{2s} - I_d \mid X_0 \right] \right\|^p \right]^{1/p} \\ &\quad + \sum_{k=3}^{\infty} \frac{e^{-t}}{2s(k-1)!} \left(\frac{\max(1, p-1)}{e^{2t} - 1} \right)^{(k-1)/2} \mathbb{E} \left[\left\| \mathbb{E}[(X_t - X_0)^{\otimes k} \mid X_0] \right\|^p \right]^{1/p}. \end{aligned}$$

5 Applications

5.1 Central Limit Theorem

Let X_1, \dots, X_n be i.i.d. random variables taking values in \mathbb{R}^d such that $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^{\otimes 2}] = I_d$. Let ν_n be the measure of $S_n = n^{-1/2} \sum_{i=1}^n X_i$. According to the Central Limit Theorem, ν_n converges to the Gaussian measure γ . In this Section, we quantify this convergence in terms of Wasserstein distance of order p , for $p \geq 2$. Let X'_1, \dots, X'_n be independent copies of X_1, \dots, X_n and let I be a uniform random variable on $\{1, \dots, n\}$. For any $t > 0$, we pose

$$(S_n)_t = S_n + n^{-1/2} (X'_I - X_I) 1_{\|X'_I\|, \|X_I\| \leq \sqrt{n(e^{2t}-1)}}.$$

By construction, $((S_n)_t)_{t \geq 0}$ is a stochastic process and, for any $t > 0$, $(S_n)_t$ is drawn from ν_n and $((S_n)_0, (S_n)_t)$ is an exchangeable pair. Applying Theorem 7 with $s = \frac{1}{n}$, see Section 6.2 for the detailed computations, we obtain the following result.

Theorem 10. Let X_1, \dots, X_n be i.i.d. random variables in \mathbb{R}^d with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^{\otimes 2}] = I_d$. For any $p \geq 2$, there exists $C_p > 0$ such that, if $\mathbb{E}[\|X_1\|^{p+q}] < \infty$ for some $q \in [0, p]$, then, taking $m = \min(4, p+q) - 2$, we have

$$W_p(\nu_n, \gamma) \leq C_p \left(n^{-1/2+(2-q)/2p} \mathbb{E}[\|X_1\|^{p+q}]^{1/p} + n^{-m/4} \mathbb{E}[\|X_1\|^{2+m}]^{1/2} \right. \\ \left. + \begin{cases} o(n^{-m/4}) & \text{if } m < 2 \\ d^{1/4} \|\mathbb{E}[X_1^{\otimes 2} \|X_1\|^2]\|^{1/2} & \text{if } m = 2 \end{cases} \right).$$

5.2 Invariant measures and diffusion approximation

Let $(M^n)_{n \in \mathbb{N}}$ be a Markov chain with transition kernel K and invariant measure π . Suppose M_0 is drawn from π . Then, by definition of the invariant measure, for any $t > 0, \tau > 0$, let

$$X_t = M^0 + 1_{t \geq \tau} (M^1 - M^0).$$

While, by the definition of the invariant measure, X_t is drawn from ν for any $t \geq 0$, (X_t, X_0) is an exchangeable pair for any $t \geq 0$ if and only if the $(M^n)_{n \geq 0}$ is reversible. Hence, given a suitable target measure μ , we can use Corollary 6 to bound $W_2(\pi, \mu)$ without requiring the $(M^n)_{n \in \mathbb{N}}$ to be reversible.

Corollary 11. Suppose μ and ν satisfy the assumptions of Corollary 6. Let π be the invariant measure of a Markov chain with transition kernel K and let X be a random variable drawn from π . Then, there exists $C(\rho) > 0$ such that, for any $0 < \tau < 1, s > 0$,

$$(1 - e^{-\kappa}) W_2(\pi, \mu) \leq \\ C(\rho) \left(\tau \mathbb{E}[\|b(X)\|_{a^{-1}}^2]^{1/2} + \mathbb{E} \left[\left\| \int_{y \in \mathbb{R}^d} (y - X) K(X, dy) - b(X) \right\|_{a^{-1}(X)}^2 \right]^{1/2} \right) \\ + C(\rho)^2 \sqrt{d} \left(\sqrt{\tau d} + \mathbb{E} \left[\left\| \int_{y \in \mathbb{R}^d} \frac{(y - X)^{\otimes 2}}{2} K(X, dy) - a(X) \right\|_{a^{-1}(X)}^2 \right]^{1/2} \right) \\ + \frac{C(\rho)^3 \log(\tau) d}{3\sqrt{2}s} \mathbb{E} \left[\left\| \int_{y \in \mathbb{R}^d} (y - X)^{\otimes 3} K(X, dy) \right\|_{a^{-1}(X)}^2 \right]^{1/2} \\ + \sum_{k=4}^{\infty} \frac{C(\rho)^k \sqrt{d(k-1)}^{k-1}}{k! \sqrt{\tau}^{k-3} s} \mathbb{E} \left[\left\| \int_{y \in \mathbb{R}^d} (y - X)^{\otimes k} K(X, dy) \right\|_{a^{-1}(X)}^2 \right]^{1/2}.$$

Proof. This result is obtained by applying Corollary 6 with $T = 1$ and remarking

that for any integer $k > 0$ and any $0 \leq t \leq 1$,

$$f_k(t) \leq \begin{cases} e^{\max(1, k/2)\rho} \left(\frac{d(k-1)}{t}\right)^{(k-1)/2} & \text{if } \rho > 0 \\ \left(\frac{d}{t}\right)^{(k-1)/2} & \text{if } \rho = 0 \\ e^{(1+\max(1, k/2))|\rho|} \left(\frac{d(k-1)}{t}\right)^{(k-1)/2} & \text{if } \rho < 0 \end{cases}.$$

□

Let us note that the quantities appearing in our bound are natural as they appear in standard diffusion approximation results (see e.g. Section 11.2 [28]).

5.2.1 Invariant measure of random walks on nearest neighbors graphs

Let X_1, \dots, X_n be i.i.d. random variables on \mathbb{R}^d drawn from a measure μ with smooth density f . Let \mathcal{X}_n be the set of points (X_1, \dots, X_n) and let $r_{\mathcal{X}_n}$ be a function from \mathbb{R}^d to \mathbb{R}^+ . A graph G with vertices \mathcal{X}_n and edges $\{(x, y) \in \mathcal{X}^2 \mid \|x - y\|^2 \leq r_{\mathcal{X}_n}(x)\}$ is called a random geometric graph. These graphs are at the center of many data analysis algorithms, such as Spectral clustering [30], semi-supervised label propagation [6] or dimensionality reduction [5], which use the properties of such graphs to process the data \mathcal{X}_n . However, these algorithms usually consider the sole graph structure while discarding all other information regarding the data, such as the coordinates of the data points, and one may wonder whether critical information regarding the data could be lost in the process. To answer this question, [31] proposed to check whether it is possible to estimate the density f from which the data is drawn using only the structure of a random geometric graph. Indeed, if we can recover f , we should be able to recover most of the statistical information contained in the initial data.

As n gets to infinity, it has been shown by [29] that, if $r_{\mathcal{X}_n}$ converges, after a proper rescaling, to a deterministic function $\tilde{r} : \mathbb{R}^d \rightarrow \mathbb{R}^+$, then random walks on the random geometric graphs G_n with radii $r_{\mathcal{X}_n}$ converge, as n goes to infinity, to diffusion processes with infinitesimal generator

$$\mathcal{L}_{\tilde{\mu}} = \tilde{r}^2 \left(\nabla \log f \cdot \nabla + \frac{1}{2} \Delta \right).$$

Since the density of the invariant measure $\tilde{\mu}$ of the limiting diffusion process has density proportional to $\frac{f^2}{\tilde{r}^2}$, one can derive an estimator of f from an estimator of $\tilde{\mu}$. Since random walks on the graphs G_n converge to diffusion processes with invariant measure $\tilde{\mu}$, it seems natural to use the invariant measures of the random walks to estimate $\tilde{\mu}$. In fact, in this context, [15] proved, under technical assumption on $r_{\mathcal{X}_n}$, that invariant measures of random walks on the graphs G_n converge weakly to $\tilde{\mu}$. Let us show how our results can be used to quantify this convergence in terms of Wasserstein distance of order 2 by tackling the specific case of nearest neighbors graphs.

Nearest neighbors graphs are obtained by picking an integer $k > 0$ and setting

$$r_{\mathcal{X}_n}(x) = \inf \left\{ s \in \mathbb{R}^+ \mid \sum_{i=1}^n \mathbb{1}_{\|X_i - x\| \leq s} \geq k \right\}$$

and are extremely popular in data analysis as they are sparse. If k is correctly chosen, random walks on such graph approximate diffusion processes with infinitesimal generator

$$\mathcal{L}_{\tilde{\mu}} = f^{-2/d}(\nabla \log f \cdot \nabla + \Delta),$$

whose invariant measure $\tilde{\mu}$ has a density proportional to $f^{2+2/d}$. Let us quantify the Wasserstein distance of order 2 between the invariant measure of a k -nearest neighbors graph and $\tilde{\mu}$. To avoid boundary issues, let us assume μ is a measure on the flat torus $T = (\mathbb{R}/\mathbb{Z})^d$ with strictly positive smooth density f . For any integer $k \leq n$, we denote by $\pi_{k,n}$ an invariant measure of a random walk on the k -nearest neighbor graphs with vertices \mathcal{X}_n . We have the following result.

Proposition 12. *There exists $C > 0$ such that, for any positive integers k, n ,*

$$\mathbb{P} \left(W_2(\pi_{k,n}, \tilde{\mu}) \leq C \left(\frac{\sqrt{\log nn^{1/d}}}{k^{1/2} + 1/d} + \left(\frac{k}{n} \right)^{1/d} \right) \right) \geq 1 - \frac{C}{n}.$$

In particular, if $n \gg k \gg \log(n)^{d/(2+d)} n^{2/2+d}$ then $W_2(\pi_{k,n}, \tilde{\mu})$ converges, in terms of Wasserstein distance of order 2, to $\tilde{\mu}$. However, a couple problems still remain. First, we only manage to obtain a convergence for a metric on the space of measures but we would require a point-wise convergence of $\pi_{k,n}$ to use it as an actual estimator of f . The second problem is that our bound is likely to be suboptimal. In practice, we want k to be small as it leads to sparser graphs but according to our result k should be at least of order $\log(n)^{d/(d+2)} n^{2/(2+d)}$. This is counterintuitive since our assumptions on k gets weaker as the dimension increases while we would expect the problem of estimating $\tilde{\mu}$ to be more complex in higher dimensions. In fact, it is conjectured in [15] that it is sufficient for $n \gg k \gg \log(n)$ for $\pi_{k,n}$ to converge weakly to $\tilde{\mu}$ which would imply our bound can be improved.

5.2.2 Analysis of a one-dimensional scheme for the Langevin Monte Carlo algorithm

Quite often in Bayesian statistics, one needs to sample points from a probability measure $d\mu$ on \mathbb{R}^d with density f . To solve this task, multiple sampling algorithms based on the Monte-carlo approach were proposed and analyzed. We want to show how our bounds can be used to study the complexity of a simple Monte-Carlo algorithm.

The measure μ is a reversible measure for the diffusion process Y_t with infinitesimal generator

$$\mathcal{L}_{\mu} = -\nabla u \cdot \nabla + \Delta.$$

Since, under mild assumptions on μ , the measure of Y_t converges to μ as t goes to infinity, one may want to sample points from μ by approximating Y_t . Using the Euler-Maruyama approximation scheme with timestep h , one can discretize Y_t using a Markov chain M with $M^0 = 0$ and transitions given by

$$M^{n+1} = M^n - h\nabla u(M^n) + \sqrt{2h}\mathcal{N}_n,$$

where $\mathcal{N}_1, \dots, \mathcal{N}_n$ is a sequence of independent normal random variables with mean 0 and covariance matrix I_d . If the timestep h is small enough, one can expect the invariant measure π of $(M^n)_{n \in \mathbb{N}}$ to be close to μ . Hence, for n large enough, the measure of M^n should be close to π and thus be close to μ . This approach to sampling is known as the Langevin Monte-Carlo (LMC) algorithm and was first proposed by [24].

One may then wonder how large n should be for a given metric between μ and ν_n , the measure of M^n , to be smaller than ϵ . Answering this question is linked to the choice of the timestep h as this parameter must satisfy some trade-off: large values of h lead to a poor approximation of μ by π but the smaller h is, the larger the number of iterations required for ν^n to be close to π . Recently, [13] proved that, whenever $\log f$ is a strictly concave function (i.e. μ is a strictly log-concave measure) and $\nabla \log f$ is Lipschitz continuous, the LMC algorithm can reach an accuracy ϵ for the Wasserstein distance of order 2 in no more than $O(\epsilon^{-2}d \log(d/\epsilon))$ steps. Since the complexity of each step of the Euler-Maruyama discretization is of order d , the overall complexity of the algorithm is bounded by $O(\epsilon^{-2}d^2 \log(d/\epsilon))$. This rate can be improved whenever $\nabla^2 \log f$ is Lipschitz continuous, in which case one only requires $O(\epsilon^{-1}\sqrt{d} \log(d/\epsilon))$ steps to reach an accuracy ϵ which means the complexity of the algorithm is bounded by $O(\epsilon^{-1}d^{3/2} \log(d/\epsilon))$. One may wonder if other discretization schemes would perform better than the Euler-Maruyama scheme however the approach used to obtain the previous bounds are specific to the Euler-Maruyama scheme. We show how our result can be used to study the efficiency of other discretization schemes. For instance, let $e_1, \dots, e_d \in \mathbb{R}^d$ be the canonical basis of \mathbb{R}^d , $(I_n)_{n \in \mathbb{N}}$ be a uniform random variable on $\{1, \dots, d\}$, $(B_n)_{n \in \mathbb{N}}$ be independent Rademacher random variables and consider the following scheme

$$M^{n+1} = M^n + \left(-h \frac{\partial u}{\partial x_{I_n}}(M^n) + \sqrt{2h}B_n \right) e_{I_n}. \quad (7)$$

Following the computations presented in Section 6.4, we obtain the following result.

Proposition 13. *Let μ be a measure of \mathbb{R}^d with density f and let $u = -\log f$. Let $h > 0$ and let $(M^n)_{n \geq 0}$ be a Markov chain with $M^0 = 0$ and increments given by Equation 7. Suppose $\nabla u(0) = 0$ and assume there exists $\rho > 0, L > 0$ such that for all $i \in \{1, \dots, d\}$ and $x, y \in \mathbb{R}^d$,*

$$\left(\frac{\partial u}{\partial x_i}(y) - \frac{\partial u}{\partial x_i}(x) \right) (y_i - x_i) \leq -\rho(y_i - x_i)^2$$

and

$$\left(\frac{\partial u}{\partial x_i}(y) - \frac{\partial u}{\partial x_i}(x) \right)^2 \leq L(y_i - x_i)^2.$$

There exist constants $C_1, C_2 > 0$ depending on ρ and L such that for any $\epsilon > 0$, if $h = C_1 \epsilon^2 d^{-2}$ and $n = C_2 h^{-1} d \log(d/\epsilon)$ then the measure ν^n of M^n satisfies

$$W_2(\nu^n, \mu) \leq \epsilon.$$

Moreover, if μ is the Gaussian measure then the previous result holds true with $h = C_1 \epsilon^2 d^{-1}$.

Since each step of this one-dimensional discretization has a complexity independent of the dimension, the overall complexity of the LMC algorithm with our discretization scheme is bounded by $O(\epsilon^{-2} d^3 \log(d/\epsilon))$ and by $O(\epsilon^{-2} d^2 \log(d/\epsilon))$ when μ is the Gaussian measure. The discrepancy between the Gaussian case and the more general case is due the dependency on the dimension of the function f_k defined in Proposition 3 which we believe is suboptimal (see Remark 4). Hence we conjecture the correct complexity of the LMC algorithm using the one-dimensional discretization scheme to be bounded by $O(\epsilon^{-2} d^2 \log(d/\epsilon))$ for target measures μ satisfying the assumptions of Proposition 13. Under this conjecture, the complexity of this one-dimensional discretization scheme matches the standard complexity of $O(\epsilon^{-2} d^2 \log(d/\epsilon))$ of the Euler-Maruyama scheme under slightly stronger assumptions. However, we are not able to recover the better complexity of $O(\epsilon^{-1} d^{3/2} \log(d/\epsilon))$ obtained for the Euler-Maruyama scheme whenever $\nabla^2 u$ is Lipschitz continuous. As this stronger assumption should be verified in most practical cases, the Euler-Maruyama scheme should be more efficient than the one-dimensional scheme in practice.

6 Proofs

6.1 Proof of Proposition 3

We are going to prove the result for the case $\rho \neq 0$, the case $\rho = 0$ can be obtained in a similar manner. In order to prove the Proposition, we need to prove an equivalent to the integration by parts used in the Gaussian case.

Lemma 14. *Suppose \mathcal{L}_μ satisfies a $CD(\rho, \infty)$ condition for $\rho \in \mathbb{R}$. Then, for any compactly supported smooth function ϕ , and any $t > 0$,*

$$\|\nabla P_t \phi\|_a^2 \leq \frac{2\rho}{e^{2\rho t} - 1} P_t |\phi|^2.$$

Proof. Consider a compactly supported smooth function ϕ and let $t > 0$. For any $0 \leq s \leq t$ let

$$\Lambda(s) = P_s(\Gamma_0(P_{t-s}\phi, P_{t-s}\phi)).$$

Since \mathcal{L}_μ is the infinitesimal generator of the semi-group $(P_t)_{t \geq 0}$, we have $\frac{d}{dt}P_t\phi = \mathcal{L}_\mu P_t\phi = P_t\mathcal{L}_\mu\phi$. Hence, the derivative of Λ is equal to

$$\begin{aligned}\Lambda'(s) &= \mathcal{L}_\mu P_s(\Gamma_0(P_{t-s}\phi, P_{t-s}\phi)) - 2P_s(\Gamma_0(\mathcal{L}_\mu P_{t-s}\phi, P_{t-s}\phi)) \\ &= P_s(\mathcal{L}_\mu \Gamma_0(P_{t-s}\phi, P_{t-s}\phi) - 2\Gamma_0(\mathcal{L}_\mu P_{t-s}\phi, P_{t-s}\phi)) \\ &= 2P_s(\Gamma_1(P_{t-s}\phi, P_{t-s}\phi)).\end{aligned}$$

Similarly, the second derivative of Λ is

$$\Lambda''(s) = 4P_s(\Gamma_2(P_{t-s}\phi, P_{t-s}\phi)).$$

By our assumption, $\Lambda''(s) \geq 2\rho\Lambda'(s)$. Hence, by Gronwall's Lemma, $\Lambda'(s) \geq e^{2\rho s}\Lambda'(0)$. Thus, we have

$$\begin{aligned}\Gamma_1(P_t\phi, P_t\phi) &= \frac{2\rho}{e^{2\rho t} - 1} \int_0^t e^{2\rho s} \Gamma_1(P_t\phi, P_t\phi) ds \\ &= \frac{2\rho}{e^{2\rho t} - 1} \int_0^t e^{2\rho s} \Lambda'(0) ds \\ &\leq \frac{2\rho}{e^{2\rho t} - 1} \int_0^t \Lambda'(s) ds \\ &\leq \frac{2\rho}{e^{2\rho t} - 1} (P_t(\Gamma_0(\phi, \phi)) - \Gamma_0(P_t\phi, P_t\phi)) \\ &\leq \frac{2\rho P_t(\Gamma_0(\phi, \phi))}{e^{2\rho t} - 1} \\ &\leq \frac{2\rho P_t|\phi|^2}{e^{2\rho t} - 1}.\end{aligned}$$

□

Let us prove Proposition 3 by induction. Let us assume a $CD(\rho, \infty)$ condition is verified for $\rho \neq 0$. Then, by Theorem 3.2.4 [3], for any compactly supported smooth function ϕ ,

$$\|\nabla P_t\psi\|_a \leq e^{-\rho t} P_t \|\nabla\psi\|_a. \quad (8)$$

Now, let $k \in \mathbb{N}$ and suppose that, for any compactly supported smooth function ϕ ,

$$\forall i \leq k, \|\nabla^i P_t\phi\|_a \leq e^{-\rho t \max(1, i/2)} \left(\frac{2\rho d}{e^{(2\rho t)/(k-1)} - 1} \right)^{(i-1)/2}.$$

Let $x \in \mathbb{R}^d$ and let (e_1, \dots, e_d) be an orthonormal basis of \mathbb{R}^d with respect to the $a(x)$ -scalar product $\langle \cdot, \cdot \rangle_{a(x)}$. We have

$$\begin{aligned}\|\nabla^{k+1} P_t\phi(x)\|_{a(x)}^2 &= \sum_{i=1}^d \|\nabla^k \langle \nabla P_t\phi(x), e_i \rangle_{a(x)} \phi\|_{a(x)}^2 \\ &= \sum_{i=1}^d \lim_{\epsilon \rightarrow 0} \|\nabla^k (P_t\phi(x + \epsilon a(x)e_i) - P_t\phi(x))\|_{a(x)}^2.\end{aligned}$$

Let $\epsilon > 0$ and let $(X_t)_{t \geq 0}$ and $(\tilde{X}_t)_{t \geq 0}$ be two diffusion processes with infinitesimal generator \mathcal{L}_μ , started respectively at x and $x + \epsilon a e_1$. Letting $\psi(y) = \mathbb{E}[\phi(\tilde{X}_t) \mid X_t = y]$, we have

$$\begin{aligned} P_t \phi(x + \epsilon a(x) e_i) - P_t \phi(x) &= \mathbb{E}[\phi(\tilde{X}_t) - \phi(X_t)] \\ &= \mathbb{E}[\mathbb{E}[\phi(\tilde{X}_t) \mid X_t] - \phi(X_t)] \\ &= P_t(\psi - \phi)(x). \end{aligned}$$

By our induction hypothesis, we have

$$\begin{aligned} \|\nabla^k P_t(\psi - \phi)(x)\|_{a(x)}^2 &\leq \\ e^{-\rho t \max(2, k) \frac{k-1}{k}} \left(\frac{2\rho d}{e^{2\rho d t/k} - 1} \right)^{k-1} P_t \frac{k-1}{k} \|\nabla P_{t/k}(\psi - \phi)(x)\|_{a(x)}^2. \end{aligned}$$

Then, applying Lemma 14,

$$\|\nabla^k P_t(\psi - \phi)(x)\|_{a(x)}^2 \leq e^{-\rho t(k-1)} d^{k-1} \left(\frac{2\rho}{e^{2\rho d t/k} - 1} \right)^k P_t |\psi - \phi|^2(x).$$

By Theorem 2.2 [17], Equation 8 implies that we can take \tilde{X}_t such that, $\|\tilde{X}_t - X_t\|_{a^{-1}(x)} \leq \epsilon e^{-\rho t}$ almost surely. Using such \tilde{X}_t , we have

$$\begin{aligned} \mathbb{E} \left[\lim_{\epsilon \rightarrow 0} \left| \frac{\psi(X_t) - \phi(X_t)}{\epsilon} \right| \right] &= \mathbb{E} \left[\lim_{\epsilon \rightarrow 0} \left| \mathbb{E} \left[\frac{\psi(X_t) - \phi(X_t)}{\epsilon} \mid X_t \right] \right| \right] \\ &\leq \mathbb{E} \left[\lim_{\epsilon \rightarrow 0} \left| \frac{\psi(X_t) - \phi(X_t)}{\epsilon} \mid X_t \right| \right] \\ &\leq \mathbb{E} \left[\lim_{\epsilon \rightarrow 0} \left| \frac{\|\tilde{X}_t - X_t\|_{a^{-1}(x)}}{\epsilon} \|\nabla \phi(X_t)\|_{a(x)} \mid X_t \right| \right] \\ &\leq e^{-\rho t} \mathbb{E} [\|\nabla \phi(X_t)\|_{a(x)}]. \end{aligned}$$

Since a similar result holds for any e_i , we have shown that

$$\|\nabla^{k+1} P_t \phi\|_a^2 \leq e^{-\rho t(k+1)} \left(\frac{2\rho d}{e^{2\rho d t/k} - 1} \right)^k \left(\sqrt{P_t \|\nabla \phi\|^2} \right)^2$$

which concludes the proof by induction.

6.2 Proof of Theorem 10

We first need to state a multidimensional version of the Rosenthal inequality. Let $k > 0, p \geq 2$ and suppose Y_1, \dots, Y_n are independent random variables taking values in $(\mathbb{R}^d)^{\otimes k}$, then by Theorem 2.1 [1] there exists $C_p > 0$ such that

$$\mathbb{E} \left[\left\| \sum_{i=1}^n Y_i - \mathbb{E} \left[\sum_{i=1}^n Y_i \right] \right\|^p \right]^{1/p} \leq C_p n^{1/2} \mathbb{E}[\|Y_1\|^2]^{1/2} + n^{1/p} \mathbb{E}[\|Y_1\|^p]^{1/p}.$$

Hence, by Jensen's inequality,

$$\begin{aligned} \mathbb{E} \left[\left\| \sum_{i=1}^n Y_i \right\|^p \right]^{1/p} &\leq \mathbb{E} \left[\left\| \sum_{i=1}^n Y_i - \mathbb{E} \left[\left\| \sum_{i=1}^n Y_i \right\| \right] \right\|^p \right]^{1/p} + \mathbb{E} \left[\left\| \sum_{i=1}^n Y_i \right\| \right] \\ &\leq C_p \left(\mathbb{E} \left[\left\| \sum_{i=1}^n Y_i \right\|^2 \right]^{1/2} + n^{1/2} \mathbb{E} \left[\|Y\|^2 \right]^{1/2} + n^{1/p} \mathbb{E} \left[\|Y\|^p \right]^{1/p} \right), \end{aligned}$$

so there exists $C'_p > 0$ such that

$$\mathbb{E} \left[\left\| \sum_{i=1}^n Y_i \right\|^p \right]^{1/p} \leq C'_p \left(n \mathbb{E} \|Y_1\| + n^{1/2} \mathbb{E} \left[\|Y_1\|^2 \right]^{1/2} + n^{1/p} \mathbb{E} \left[\|Y_1\|^p \right]^{1/p} \right). \quad (9)$$

We are now ready to start the proof of Theorem 10. Let Z be a Gaussian random variable and let us pose $X = X_1$, $X' = X'_1$ and $\alpha(t) = e^{2t} - 1$. In the remainder of this proof, we are going to show there exist $C_p > 0$ such that

- $\int_0^\infty e^{-t} \mathbb{E} \left[\mathbb{E} [n((S_n)_t - S_n) \mid S_n] + S_n \right]^p dt$,
- $\int_0^\infty \frac{e^{-2t}}{\sqrt{1-e^{-2t}}} \mathbb{E} \left[\left\| \mathbb{E} \left[n \frac{((S_n)_t - S_n)^{\otimes 2}}{2} - I_d \mid S_n \right] \right\|^p \right]^{1/p} dt$ and
- $\sum_{k=3}^\infty \frac{(p-1)^{k/2}}{(k-1)!} \int_0^\infty \frac{e^{-kt}}{\sqrt{1-e^{-2t}k-1}} \mathbb{E} \left[\mathbb{E} [n((S_n)_t - S_n)^{\otimes k} \mid S_n] \right]^p dt$

are bounded by

$$\begin{aligned} C_p \left(n^{-1/2+(2-q)/2p} \mathbb{E} \left[\|X\|^{p+q} \right]^{1/p} + n^{-m/4} \mathbb{E} \left[\|X\|^{2+m} \right]^{1/2} \right. \\ \left. + \begin{cases} o(n^{-m/4}) & \text{if } m < 2 \\ d^{1/4} \mathbb{E} \left[\|X^{\otimes 2}\| \|X\|^2 \right]^{1/2} & \text{if } m = 2 \end{cases} \right). \end{aligned}$$

Theorem 10 is then obtained using these bounds in Theorem 9.

In the remainder of this proof, C_p denotes a generic constant depending only on p . For any $t \geq 0$, we have, by definition of $(S_n)_t$,

$$(S_n)_t - S_n = \frac{1}{\sqrt{n}} (X'_I - X_I) 1_{\|X_I\|, \|X'_I\| \leq \sqrt{n\alpha(t)}}.$$

Since I and S_n are independent, we have, for any $k \in \mathbb{N}$,

$$\mathbb{E} [n((S_n)_t - S_n)^{\otimes k} \mid S_n] = n^{-k/2} \mathbb{E} \left[\sum_{i=1}^n (X'_i - X_i)^{\otimes k} 1_{\|X_i\|, \|X'_i\| \leq \sqrt{n\alpha(t)}} \mid S_n \right]. \quad (10)$$

In particular, taking $k = 1$, we have

$$\mathbb{E} [n((S_n)_t - S_n) + S_n \mid S_n] = \frac{1}{\sqrt{n}} \mathbb{E} \left[\sum_{i=1}^n (X'_i - X_i) 1_{\|X_i\|, \|X'_i\| \leq \sqrt{n\alpha(t)}} + X_i \mid S_n \right].$$

Since X' is independent from S_n , $\mathbb{E}[X' | S_n] = \mathbb{E}[X'] = 0$. Hence,

$$\mathbb{E}[X' 1_{\|X\|, \|X'\| \leq \sqrt{n\alpha(t)} | S_n}] = -\mathbb{E}[X' 1_{\max\|X\|, \|X'\| \geq \sqrt{n\alpha(t)} | S_n}].$$

Therefore

$$\mathbb{E}[(X' - X) 1_{\|X\|, \|X'\| \leq \sqrt{n\alpha(t)} + X | S_n] = \mathbb{E}[(X - X') 1_{\max\|X\|, \|X'\| \geq \sqrt{n\alpha(t)} | S_n],$$

and

$$\begin{aligned} & \mathbb{E}[\|\mathbb{E}[n((S_n)_t - S_n) + S_n | S_n]\|^p]^{1/p} = \\ & n^{-1/2} \mathbb{E} \left[\left\| \mathbb{E} \left[\sum_{i=1}^n (X_i - X'_i) 1_{\max\|X_i\|, \|X'_i\| \geq \sqrt{n\alpha(t)} | S_n} \right] \right\|^p \right]^{1/p}. \end{aligned}$$

Applying Jensen's inequality to get rid of the conditional expectation,

$$\mathbb{E}[\|\mathbb{E}[n((S_n)_t - S_n) + S_n | S_n]\|^p]^{1/p} \leq n^{-1/2} \mathbb{E} \left[\left\| \sum_{i=1}^n (X_i - X'_i) 1_{\max\|X_i\|, \|X'_i\| \geq \sqrt{n\alpha(t)}} \right\|^p \right]^{1/p}.$$

Let us pose

$$Y = (X - X') 1_{\max\|X\|, \|X'\| \geq \sqrt{n\alpha(t)}}.$$

Since the $(X_i)_{1 \leq i \leq n}$ and the $(X'_i)_{1 \leq i \leq n}$ are i.i.d. random variables, so are the $((X_i - X'_i) 1_{\max\|X_i\|, \|X'_i\| \geq \sqrt{n\alpha(t)}})_{1 \leq i \leq n}$. Hence, we can use Equation 9 to obtain

$$\mathbb{E}[\|\mathbb{E}[n((S_n)_t - S_n) + S_n | S_n]\|^p]^{1/p} \leq C_p (n^{1/2} \|\mathbb{E}[Y]\| + \mathbb{E}[\|Y\|^2]^{1/2} + n^{1/p-1/2} \mathbb{E}[\|Y\|^p]^{1/p}).$$

Since X and X' follow the same law, $\mathbb{E}[Y] = 0$. On the other hand, we have

$$\begin{aligned} \mathbb{E}[\|Y\|^p]^{1/p} &= \mathbb{E}[\|X - X'\|^p 1_{\max\|X\|, \|X'\| \geq \sqrt{n\alpha(t)}}]^{1/p} \\ &\leq \mathbb{E}[(\|X\| + \|X'\|)^p 1_{\max\|X\|, \|X'\| \geq \sqrt{n\alpha(t)}}]^{1/p} \\ &\leq 2 \mathbb{E}[\max(\|X\|, \|X'\|)^p 1_{\max\|X\|, \|X'\| \geq \sqrt{n\alpha(t)}}]^{1/p} \\ &\leq 2(n\alpha(t))^{-q/2p} \mathbb{E}[\max(\|X\|, \|X'\|)^{p+q}]^{1/p} \\ &\leq 4(n\alpha(t))^{-q/2p} \mathbb{E}[\|X\|^{p+q}]^{1/p}, \end{aligned}$$

and, similarly,

$$\mathbb{E}[\|Y\|^2]^{1/2} \leq 4(n\alpha(t))^{-m/4} \mathbb{E}[\|X\|^{2+m}]^{1/2}.$$

Overall, we obtained

$$\begin{aligned} & \mathbb{E}[\|\mathbb{E}[n((S_n)_t - S_n) + S_n | S_n]\|^p]^{1/p} \leq \\ & C_p \left(n^{-m/4} \alpha(t)^{-m/4} \mathbb{E}[\|X\|^{2+m}]^{1/2} + n^{-1/2+(2-q)/2p} \alpha(t)^{-q/2p} \mathbb{E}[\|X\|^{p+q}]^{1/p} \right). \end{aligned}$$

Finally, since $\alpha(t) \geq 2t$,

$$\int_0^\infty e^{-t} \mathbb{E}[\|\mathbb{E}[n((S_n)_t - S_n) + S_n \mid S_n]\|_p^p]^{1/p} dt \leq C_p \left(n^{-m/4} \mathbb{E}[\|X\|^{2+m}]^{1/2} + n^{-1/2+(2-q)/2p} \mathbb{E}[\|X\|^{p+q}]^{1/p} \right).$$

Let us now bound the second order term. By Equation 10 for $k = 2$, we have

$$\begin{aligned} \mathbb{E} \left[\frac{n((S_n)_t - S_n)^{\otimes 2}}{2} - I_d \mid S_n \right] &= \mathbb{E} \left[\frac{(X'_I - X_I)^{\otimes 2}}{2} 1_{\|X_I\|, \|X'_I\| \leq \sqrt{n\alpha(t)}} - I_d \mid S_n \right] \\ &= \frac{1}{n} \mathbb{E} \left[\sum_{i=1}^n \left(\frac{(X'_i - X_i)^{\otimes 2}}{2} 1_{\|X_i\|, \|X'_i\| \leq \sqrt{n\alpha(t)}} - I_d \right) \mid S_n \right]. \end{aligned}$$

Again, taking

$$Y = \left(\frac{(X' - X)^{\otimes 2}}{2} 1_{\|X\|, \|X'\| \leq \sqrt{n\alpha(t)}} - I_d \right)$$

and using a combination of Jensen's inequality and Equation 9, we obtain

$$\begin{aligned} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{n((S_n)_t - S_n)^{\otimes 2}}{2} \mid S_n \right] - I_d \right\|_p^p \right]^{1/p} &\leq \\ C_p \left(\|\mathbb{E}[Y]\| + n^{-1/2} \mathbb{E}[\|Y\|^2]^{1/2} + n^{1/p-1} \mathbb{E}[\|Y\|^p]^{1/p} \right). \end{aligned}$$

First, since $\mathbb{E}[X^{\otimes 2}] = \mathbb{E}[X'^{\otimes 2}] = I_d$,

$$\mathbb{E}[Y] = \mathbb{E} \left[\frac{(X' - X)^{\otimes 2}}{2} 1_{\max\|X\|, \|X'\| \geq \sqrt{n\alpha(t)}} \right].$$

For two $x, y \in (\mathbb{R}^d)^{\otimes k}$, we denote by $\langle x, y \rangle$ the corresponding Hilbert-Schmidt scalar product between x and y . Letting Z and Z' be two random variables such that X, X', Z, Z' are i.i.d. and denoting by C a generic positive

constant, we have

$$\begin{aligned}
\|\mathbb{E}[Y]\| &= \sqrt{\left\langle \mathbb{E}\left[(X' - X)^{\otimes 2} \mathbf{1}_{\max\|X\|, \|X'\| \geq \sqrt{n\alpha(t)}}\right], \mathbb{E}\left[(Z' - Z)^{\otimes 2} \mathbf{1}_{\max\|Z\|, \|Z'\| \geq \sqrt{n\alpha(t)}}\right] \right\rangle} \\
&= \sqrt{\mathbb{E}\left[\langle (X' - X)^{\otimes 2}, (Z' - Z)^{\otimes 2} \rangle \mathbf{1}_{\max\|X\|, \|X'\| \geq \sqrt{n\alpha(t)}} \mathbf{1}_{\max\|Z\|, \|Z'\| \geq \sqrt{n\alpha(t)}}\right]} \\
&= \sqrt{\mathbb{E}\left[\langle X' - X, Z' - Z \rangle^2 \mathbf{1}_{\max\|X\|, \|X'\| \geq \sqrt{n\alpha(t)}} \mathbf{1}_{\max\|Z\|, \|Z'\| \geq \sqrt{n\alpha(t)}}\right]} \\
&\leq C \sqrt{\mathbb{E}\left[\langle X, Z \rangle^2 \mathbf{1}_{\max\|X\|, \|X'\| \geq \sqrt{n\alpha(t)}} \mathbf{1}_{\max\|Z\|, \|Z'\| \geq \sqrt{n\alpha(t)}}\right]} \\
&\leq C(n\alpha(t))^{-m/2} \sqrt{\mathbb{E}\left[\langle X, Z \rangle^2 \max(\|X\|, \|X'\|)^m \max(\|Z\|, \|Z'\|)^m\right]} \\
&\leq C(n\alpha(t))^{-m/2} \sqrt{\mathbb{E}\left[\langle X, Z \rangle^2 (\|X\|^m + \|X'\|^m)(\|Z\|^m + \|Z'\|^m)\right]} \\
&\leq C(n\alpha(t))^{-m/2} \|\mathbb{E}[X^{\otimes 2}(\|X\|^m + \|X'\|^m)]\| \\
&\leq C(n\alpha(t))^{-m/2} (\|\mathbb{E}[X^{\otimes 2}\|X\|^m]\| + \|\mathbb{E}[X^{\otimes 2}\|X'\|^m]\|).
\end{aligned}$$

Since X and X' are independent,

$$\begin{aligned}
\|\mathbb{E}[X^{\otimes 2}\|X'\|^m]\| &= \sqrt{d} \mathbb{E}[\|X\|^m] \\
&\leq d^{-1/2} \mathbb{E}[\|X\|^2] \mathbb{E}[\|X\|^m] \\
&\leq d^{-1/2} \mathbb{E}[\|X\|^{2+m}]^{2/(2+m)} \mathbb{E}[\|X\|^{2+m}]^{m/(2+m)} \\
&\leq d^{-1/2} \mathbb{E}[\|X\|^{2+m}] \\
&\leq d^{-1/2} \sum_{i=1}^d \mathbb{E}[X_i^2 \|X\|^m] \\
&\leq \sqrt{\sum_{i=1}^d \mathbb{E}[X_i^2 \|X\|^m]^2} \\
&\leq \sqrt{\sum_{i,j=1}^d \mathbb{E}[X_i X_j \|X\|^m]^2} \\
&\leq \|\mathbb{E}[X^{\otimes 2}\|X\|^m]\|.
\end{aligned}$$

Therefore, we have

$$\|\mathbb{E}[Y]\| \leq C(n\alpha(t))^{-m/2} \|\mathbb{E}[X^{\otimes 2}\|X\|^m]\|.$$

Let us remark that this bound may cause integration issues when $m \geq 1$ since

$$\frac{e^{-t}}{\sqrt{e^{2t} - 1}} \alpha(t)^{-m/2} = \frac{e^{-t}}{\alpha(t)^{(m+1)/2}} \underset{t \rightarrow 0}{\sim} \frac{1}{(2t)^{(m+1)/2}}.$$

In order to tackle this issue, let us remark that, replacing m by 0 in the previous bound, we obtain

$$\|\mathbb{E}[Y]\| \leq C \|\mathbb{E}[X^{\otimes 2}]\| \leq C\sqrt{d}.$$

Let $0 < t_0 < 1$. Since $\alpha(t) \geq 2t$, there exists $C > 0$ such that

$$\begin{aligned} & \int_0^\infty \frac{e^{-t}}{\sqrt{\alpha(t)}} \|\mathbb{E}[Y]\| dt \\ & \leq C \left(\int_0^{t_0} \frac{\sqrt{d}e^{-t}}{\sqrt{\alpha(t)}} dt + n^{-m/2} \|\mathbb{E}[X^{\otimes 2} \|X\|^m]\| \int_{t_0}^\infty \frac{e^{-t}}{\alpha(t)^{(m+1)/2}} dt \right). \end{aligned}$$

Therefore, taking $t_0 = \frac{1}{n} \left(\frac{\|\mathbb{E}[X^{\otimes 2} \|X\|^m]\|}{\sqrt{d}} \right)^{2/m}$, there exists $C > 0$ such that

$$\int_0^\infty \frac{e^{-t}}{\sqrt{e^{2t}-1}} \|\mathbb{E}[Y]\| dt \leq C \begin{cases} o(n^{-m/4}) & \text{if } m < 2 \\ d^{1/4} \|\mathbb{E}[X^{\otimes 2} \|X\|^2]\|^{1/2} & \text{if } m = 2 \end{cases}.$$

Let us now deal with the higher moments of Y . We have

$$\begin{aligned} \mathbb{E}[\|Y\|^p] & \leq 2^p (\mathbb{E}[\|\frac{(X' - X)^{\otimes 2}}{2}\|^p] + \|I_d\|^p) \\ & \leq 2^p (\mathbb{E}[\|X\|^p + d^{p/2}]) \\ & \leq 2^p (\mathbb{E}[\|X\|^{2p} 1_{\|X\| \leq \sqrt{n\alpha(t)}}] + \mathbb{E}[\|X\|^2]^{p/2}) \\ & \leq 2^p ((n\alpha(t))^{(p-q)/2} + 1) \mathbb{E}[\|X\|^{p+q}]. \end{aligned}$$

and

$$\mathbb{E}[\|Y\|^2] \leq 4((n\alpha(t))^{m-2} + 1) \mathbb{E}[\|X\|^{2+m}].$$

Putting everything together,

$$\begin{aligned} & \int_0^\infty \frac{e^{-t}}{\sqrt{e^{2t}-1}} \mathbb{E} \left[\left\| \mathbb{E} \left[n \frac{((S_n)_t - S_n)^{\otimes 2}}{2} - I_d \mid S_n \right] \right\|^p \right]^{1/p} dt \\ & \leq C_p \left(n^{-1/2+(2-q)/2p} \mathbb{E}[\|X\|^{p+q}] + n^{-m/4} \mathbb{E}[\|X\|^{2+m}]^{1/2} \right. \\ & \quad \left. + \begin{cases} o(n^{-m/4}) & \text{if } m < 2 \\ d^{1/4} \|\mathbb{E}[X^{\otimes 2} \|X\|^2]\|^{1/2} & \text{if } m = 2 \end{cases} \right). \end{aligned}$$

We are now left with dealing with the higher order terms. For $k > 2$, let

$$Y = \mathbb{E}[(X' - X)^{\otimes k} 1_{\|X\|, \|X'\| \leq \sqrt{n\alpha(t)}} \mid S_n].$$

By a combination of Equation 10, Jensen's inequality to get rid of the conditional expectation and Equation 9, we obtain

$$\begin{aligned} & \mathbb{E}[\|\mathbb{E}[n((S_n)_t - S_n)^{\otimes k} \mid S_n]\|_p^p]^{1/p} \leq \\ & n^{1-k/2} \|\mathbb{E}[Y]\| + n^{1/2-k/2} \mathbb{E}[\|Y\|^2]^{1/2} + n^{1/p-k/2} \mathbb{E}[\|Y\|^p]^{1/p}. \end{aligned}$$

First, we have

$$\begin{aligned}\mathbb{E}[\|Y\|^p] &\leq \mathbb{E}[\|X' - X\|^{kp} 1_{\|X\|, \|X'\| \leq \sqrt{n\alpha(t)}}] \\ &\leq 2^{kp} \mathbb{E}[\|X\|^{kp} 1_{\|X\| \leq \sqrt{n\alpha(t)}}] \\ &\leq 2^{kp} (n\alpha(t))^{((k-1)p-q)/2} \mathbb{E}[\|X\|^{p+q}],\end{aligned}$$

and

$$\mathbb{E}[\|Y\|^2] \leq 4^k (n\alpha(t))^{k-1-m/2} \mathbb{E}[\|X\|^{2+m}].$$

Then, since X' and X are i.i.d., $\mathbb{E}[Y] = 0$ for odd values of k . Let us now consider an even integer $k > 2$. Let us denote by Z and Z' two random variables such that X, X', Z, Z' are i.i.d. We have

$$\begin{aligned}\|\mathbb{E}[Y]\| &= \mathbb{E} \left[\langle X' - X, Z' - Z \rangle >^k 1_{\|X\|, \|X'\| \leq \sqrt{n\alpha(t)}} 1_{\|Z\|, \|Z'\| \leq \sqrt{n\alpha(t)}} \right]^{1/2} \\ &\leq 2^k \mathbb{E} \left[\langle X, Z \rangle >^k 1_{\|X\|, \|Z\| \leq \sqrt{n\alpha(t)}} \right]^{1/2} \\ &\leq 2^k \mathbb{E} \left[\langle X, Z \rangle >^2 \|X\|^{k-2} \|Z\|^{k-2} 1_{\|X\|, \|Z\| \leq \sqrt{n\alpha(t)}} \right]^{1/2} \\ &\leq 2^k (n\alpha(t))^{(k-m-2)/2} \mathbb{E} \left[\langle X, Z \rangle >^2 \|X\|^m \|Z\|^m \right]^{1/2} \\ &\leq 2^k (n\alpha(t))^{(k-m-2)/2} \|\mathbb{E}[X^{\otimes 2} \|X\|^m]\|,\end{aligned}$$

and, similarly, replacing m by 0,

$$\|\mathbb{E}[Y]\| \leq 2^k (n\alpha(t))^{(k-2)/2} \|\mathbb{E}[X^{\otimes 2}]\| \leq 2^k (n\alpha(t))^{k/2-1} \sqrt{d}.$$

Then, using the same integration procedure we used to bound the second order term, we obtain

$$\begin{aligned}\int_0^\infty \frac{e^{-kt}}{\sqrt{1-e^{-2t}}^{k-1}} \mathbb{E}[\|\mathbb{E}[n((S_n)_t - S_n)^{\otimes k} | S_n]\|^p]^{1/p} dt \\ \leq C_p 2^k \left(n^{-1/2+(2-q)/2p} \mathbb{E}[\|X\|^{p+q}] + n^{-m/4} \mathbb{E}[\|X\|^{2+m}]^{1/2} \right. \\ \left. + \begin{cases} o(n^{-m/4}) & \text{if } m < 2 \\ d^{1/4} \|\mathbb{E}[X^{\otimes 2} \|X\|^2]\|^{1/2} & \text{if } m = 2 \end{cases} \right).\end{aligned}$$

Finally, since

$$\forall M \in (\mathbb{R}^d)^{\otimes k}, \|M\|_H \leq \max_{i \in \{1, \dots, d\}^{k-1}} \|H_i\|_\gamma \|M\| \leq \sqrt{(k-1)!} \|M\|,$$

we have

$$\begin{aligned} & \sum_{k=3}^{\infty} \frac{(p-1)^{k/2}}{(k-1)!} \int_0^{\infty} \frac{e^{-kt}}{\sqrt{1-e^{-2t}k^{-1}}} \mathbb{E}[\|\mathbb{E}[n((S_n)_t - S_n)^{\otimes k} | S_n]\|_H^p]^{1/p} dt \\ & \leq C_p \left(n^{-1/2+(2-q)/2p} \mathbb{E}[\|X\|^{p+q}] + n^{-m/4} \mathbb{E}[\|X\|^{2+m}]^{1/2} \right. \\ & \quad \left. + \begin{cases} o(n^{-m/4}) & \text{if } m < 2 \\ d^{1/4} \|\mathbb{E}[X^{\otimes 2} \|X\|^2]\|^{1/2} & \text{if } m = 2 \end{cases} \right), \end{aligned}$$

which is the last bound required to conclude the proof.

6.3 Proof of Proposition 12

While T is not a domain of \mathbb{R}^d , the arguments used in the proof of Theorem 5 still hold. Let $\tilde{\mu}$ be the measure with density $\tilde{f} = C f^{2+2/d}$, where $C > 0$ is a renormalization factor. For any two smooth functions ϕ and ψ , we have, using standard integrations by parts,

$$\begin{aligned} \int_T \phi f^{-2/d} (\nabla \log f \cdot \nabla \psi + \delta \psi) d\tilde{\mu} &= C \int_T \phi (\nabla \log f \cdot \nabla \psi + \frac{1}{2} \delta \psi) f^2 d\lambda \\ &= \frac{C}{2} \int_T \phi (\nabla \log f^2 \cdot \nabla \psi + \delta \psi) f^2 d\lambda \\ &= \frac{C}{2} \int_T \phi \div (f^2 \nabla \psi) d\lambda \\ &= -\frac{C}{2} \int_T f^2 \nabla \phi \cdot \nabla \psi d\lambda \\ &= \frac{C}{2} \int_T \psi \div (f^2 \nabla \phi) d\lambda \\ &= \frac{C}{2} \int_T \psi (\nabla \log f^2 \cdot \nabla \phi + \delta \phi) f^2 d\lambda \\ &= \int_T \psi f^{-2/d} (\nabla \log f \cdot \nabla \phi + \delta \phi) d\tilde{\mu}, \end{aligned}$$

hence $\tilde{\mu}$ is a reversible measure for $\mathcal{L}_{\tilde{\mu}} = \tilde{r}^2 (\nabla \log f \cdot \nabla + \delta)$. As T is compact and f is smooth and strictly positive, $f^{-2/d} \nabla \log f$ and $f^{-2/d}$ are smooth. Thus, $\mathcal{L}_{\tilde{\mu}}$ verifies a $CD(\rho, \infty)$ condition for some $\rho \in \mathbb{R}$. Moreover, for any $t > 0$, $(\pi_{k,n})_t$ is a measure with strictly positive smooth density and, since T is compact, finite Fisher information with respect to $\tilde{\mu}$. Finally, thanks to Corollary 2.2 [32], the last assumption of Corollary 11 is verified.

We pose

$$s = \left(\frac{k}{n}\right)^{2/d} \frac{\int_{\|x\| \leq 1} x_1^2 dx}{\left(\int_{\|x\| \leq 1} 1 dx\right)^{1+2/d}}$$

and

$$r_{\mathcal{X}_n}(x) = \inf \left\{ s \in \mathbb{R}^+ \mid \sum_{i=1}^n \mathbb{1}_{\|X_i - x\| \leq s} \geq k \right\}.$$

In this Section, we show that there exists a constant C such that with probability $1 - \frac{C}{n}$,

- (i) $\sup_{i \in \{1, \dots, n\}} \left\| \frac{1}{ks} \sum_{X_j \in \mathcal{B}(X_i, \bar{r})} (X_j - X_i) - f^{-2/d} \nabla \log f(X_i) \right\| \leq C \left(\frac{\sqrt{\log n} n^{1/d}}{k^{1/2+1/d}} + \left(\frac{k}{n}\right)^{2/d} \right);$
- (ii) $\sup_{i \in \{1, \dots, n\}} \left\| \frac{1}{ks} \sum_{X_j \in \mathcal{B}(X_i, \bar{r})} \frac{(X_j - X_i)^{\otimes 2}}{2} - f(X_i)^{-2/d} I_d \right\| \leq C \left(\sqrt{\frac{\log n}{k}} + \left(\frac{k}{n}\right)^{2/d} \right);$
- (iii) $\sup_{i \in \{1, \dots, n\}} \left\| \frac{1}{ks} \sum_{X_j \in \mathcal{B}(X_i, \bar{r})} (X_j - X_i)^{\otimes 3} \right\| \leq C \left(\frac{\sqrt{\log n} k^{1/d}}{n^{1/d} k^{1/2}} + \left(\frac{k}{n}\right)^{2/d} \right);$
- (iv) $\forall m > 3, \sup_{i \in \{1, \dots, n\}} \left\| \frac{1}{ks} \sum_{X_j \in \mathcal{B}(X_i, \bar{r})} (X_j - X_i)^{\otimes m} \right\| \leq C^m \left(\frac{k}{n}\right)^{(m-2)/d}.$

Proposition 12 is then obtained by applying Corollary 11 with s and $\tau = s$.

Let $x \in \mathbb{R}^d$. In the remainder of this proof C denotes a generic constant depending only on d and f . and, for $k > 0$, we pose $V_k = \int_{\mathcal{B}(0,1)} x_1^k dx$ and

$$N_r = \sum_{i=1}^n \mathbb{1}_{X_i \in \mathcal{B}(x, r)}.$$

For any $0 < \epsilon < 1$, Chernoff's bound yields

$$P(|N_r - nP_r| \geq n\epsilon P_r) \leq 2e^{-\frac{\epsilon^2 n P_r}{3}}. \quad (11)$$

Taking $r_M = \left(\frac{2k}{nV_0 \min f} \right)^{1/d}$, we have $P_{r_M} \geq \frac{2k}{n}$. Hence, if $\frac{k}{\log(n)}$ is sufficiently large, then $\mathbb{P}(N_{r_M} \geq k) \geq 1 - \frac{1}{n^2}$. Hence, $\mathbb{P}(r_{\mathcal{X}_n}(x) \geq r_M) \geq 1 - \frac{1}{n^2}$. Using a union-bound, $\mathbb{P}(\sup_{x \in \mathcal{X}_n} r_{\mathcal{X}_n}(x) \geq r_M) \geq 1 - \frac{1}{n}$. Hence, for $\frac{k}{\log(n)}$ sufficiently large,

$$\forall m > 3, \left\| \frac{1}{k} \sum_{X_j \in \mathcal{B}(X_i, \bar{r})} (X_j - X_i)^{\otimes m} \right\| \leq r_M^m$$

with probability $1 - \frac{1}{n}$ which implies (iv).

Let us now prove the first inequality. We have

$$\begin{aligned}
& \mathbb{E}[(X_i - x)1_{X_i \in \mathcal{B}(x,r)}] \\
&= \int_{\mathcal{B}(x,r)} (y - x)\mu(dy) \\
&= \int_{\mathcal{B}(x,r)} (y - x)f(y)dy \\
&= \int_{\mathcal{B}(x,r)} (y - x)f(x) + (y - x)^{\otimes 2}\nabla f(x) + \frac{(y - x)^{\otimes 3}\nabla^2 f(x)}{2} + Cr^4 dy \\
&= V_2 r^{d+2}\nabla f(x) + Cr^{d+4}.
\end{aligned}$$

Let us pose $b_1 = \sum_{X_i \in \mathcal{B}(x,r)} X_i - x$. Since

$$\mathbb{E}[(X_i - x)^2 1_{X_i \in \mathcal{B}(x,r)}] \leq r^2 P_r,$$

applying Bernstein's inequality yields

$$P\left(|b_1 - nV_2 r^{d+2}\nabla f(x)| \geq C\left(r\sqrt{nP_r \log n} + nr^{d+4}\right)\right) \leq \frac{2}{n^2}.$$

For $r = \left(\frac{k}{nV_0 f(x)}\right)^{1/d}$, we have $|P_r - \frac{k}{n}| \leq C\left(\frac{k}{n}\right)^{1+2/d}$. Hence, by Equation 11,

$$\mathbb{P}\left(|N_r - k| \leq C\left(\sqrt{k \log n} + \frac{k^{1+2/d}}{n^{2/d}}\right)\right) \geq 1 - \frac{1}{n^2}.$$

Let $b_2 = \sum_{X_i \in \mathcal{B}(x, r_{X_n}(x))} X_i - x$. With probability greater than $1 - \frac{1}{n^2}$,

$$|b_1 - b_2| \leq Cr_M \left(\sqrt{k \log n} + \frac{k^{1+2/d}}{n^{2/d}}\right).$$

Putting everything together, we have, with probability $1 - \frac{C}{n^2}$

$$\begin{aligned}
\left\| \frac{1}{ks} \sum_{X_i \in \mathcal{B}(x, \bar{r})} (X_i - x) - f^{-2/d}\nabla \log f \right\| &= \left\| \frac{b_2}{ks} - f^{-2/d}\nabla \log f \right\| \\
&\leq \left\| \frac{b_1}{ks} - f^{-2/d}\nabla \log f \right\| + C\left(\frac{\sqrt{\log n} n^{2/d}}{k^{1/2+2/d}} + \frac{1}{k}\right) \\
&\leq C\left(\frac{\sqrt{\log n} n^{1/d}}{k^{1/2+1/d}} + \left(\frac{k}{n}\right)^{2/d}\right).
\end{aligned}$$

We finally obtain (i) using a union-bound and (ii) and (iii) can be obtained in the same way.

6.4 Proof of Proposition 13

Let $n \in \mathbb{N}, h > 0$. We denote by π_h the invariant measure of the Markov chain $(M^n)_{n \geq 0}$. We have, by the triangle inequality,

$$W_2(\nu_n, \mu) \leq W_2(\nu_n, \pi_h) + W_2(\pi_h, \mu).$$

Before bounding these two terms, let us prove a few results on π_h . Let X be a random variable drawn from π_h and let

$$\xi = \left(-h \frac{\partial u}{\partial x_I}(X) + \sqrt{2h}B \right) e_I,$$

where I is a uniform random variable on $\{1, \dots, d\}$ and B is a Rademacher random variable. Let us bound the second moment of X . Since π_h is the invariant measure of $(M^n)_{n \geq 0}$, X and $X + \xi$ follow the same law, hence

$$\begin{aligned} 0 &= \mathbb{E}[\|X + \xi\|^2] - \mathbb{E}[\|X\|^2] \\ &= \mathbb{E}[2 \langle X, \xi \rangle + \|\xi\|^2] \\ &= \mathbb{E} \left[-2h \left\langle X, \frac{\partial u}{\partial x_I}(X) \right\rangle + \|\xi\|^2 \right] \\ &= \frac{1}{d} \mathbb{E} \left[-2h \left\langle X, \frac{\partial u}{\partial x_I}(X) \right\rangle + h^2 \|\nabla(u)(X)\|^2 \right] + 2h. \end{aligned}$$

By our assumption on u , we have

$$\|\nabla u(X)\| \leq L\|X\|$$

and

$$\left\langle X, \frac{\partial u}{\partial x_I}(X) \right\rangle \geq \rho\|X\|^2.$$

Therefore,

$$\frac{1}{d} \mathbb{E}[-h\rho\|X\|^2 + h^2 L^2\|X\|] + 2h \leq 0,$$

from which we deduce that

$$\mathbb{E}[\|X\|^2] \leq \frac{2dh}{\rho(2h - L^2 h^2)} \leq \frac{d}{\rho} + O(dh). \quad (12)$$

In addition to this bound, we can obtain a bound $\|X\|_\infty$. Indeed, for any $n \geq 0$ and any $i \in \{1, \dots, d\}$, we have

$$\begin{aligned} |M_i^{n+1}| &\leq |M_i^n - h \frac{\partial u}{\partial x_i}(M_i^n)| + \sqrt{2h} \\ &\leq \left((M_i^n)^2 - 2M_i^n h \frac{\partial u}{\partial x_i}(M_i^n) + h^2 \left(\frac{\partial u}{\partial x_i}(M_i^n) \right)^2 \right)^{1/2} + \sqrt{2h} \\ &\leq |M_i^n| \sqrt{1 - 2h\rho + h^2 L^2} + \sqrt{2h}. \end{aligned}$$

Thus, since $M^0 = 0$, $\|M^n\|_\infty \leq \frac{\sqrt{2}}{\rho\sqrt{h}}$ for any $n \in \mathbb{N}$ and, since π_h is the invariant measure of $(M^n)_{n \geq 0}$,

$$\|X\|_\infty \leq \frac{\sqrt{2h}}{1 - \sqrt{1 - 2h\rho\sqrt{h} + h^2L^2}} \text{ a.s.}$$

and, if h is sufficiently small,

$$\|X\|_\infty \leq \frac{2\sqrt{h}}{h\rho\sqrt{h}} \text{ a.s.} \quad (13)$$

Let us bound $W_2(\nu_n, \pi_h)$. For $x \in \mathbb{R}^d$, we denote by ν_x the measure of

$$M^x = x + \left(-h \frac{\partial u}{\partial x_I}(x) + \sqrt{2h}B \right) e_I.$$

Let $x, y \in \mathbb{R}^d$, we have, for any $i \in \{1, \dots, d\}$,

$$\begin{aligned} \mathbb{E}[(M_i^x - M_i^y)^2] &= \frac{d-1}{d}(x_i - y_i)^2 + \frac{1}{d} \left(x_i - y_i - h \left(\frac{\partial u}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(y) \right) \right)^2 \\ &= (x_i - y_i)^2 - \frac{2h}{d}(x_i - y_i) \left(\frac{\partial u}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(y) \right) + \frac{h^2}{d} \left(\frac{\partial u}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(y) \right)^2 \\ &\leq \left(1 + \frac{L^2h^2 - 2\rho h}{d} \right) (x_i - y_i)^2. \end{aligned}$$

Hence, by definition of the Wasserstein distance,

$$W_2^2(\nu_x, \nu_y) \leq \mathbb{E}[\|M^x - M^y\|^2] \leq \left(1 + \frac{L^2h^2 - 2\rho h}{d} \right) \|x - y\|^2.$$

Thus, applying Corollary 21 [21] for the Wasserstein distance of order 2 and using Equation 12,

$$\begin{aligned} W_2(\nu_n, \pi_h) &\leq \left(1 - \left(1 + \frac{L^2h^2 - 2\rho h}{d} \right)^{1/2} \right)^n W_2(\nu_0, \pi_h) \\ &\leq \left(\frac{2\rho h - L^2h^2}{d} \right)^{n/2} W_2(\nu_0, \pi_h) \\ &\leq \left(\frac{2\rho h - L^2h^2}{d} \right)^{n/2} \frac{2dh}{\rho(2h - L^2h^2)}. \end{aligned}$$

Hence, there exists a constant $C_1 > 0$ depending only on ρ and L such that, if $n = C_1 \frac{d}{h\rho} \log(d/h\epsilon)$ then

$$W_2(\nu_n, \pi_h) \leq \frac{\epsilon}{2}. \quad (14)$$

Let us now bound $W_2(\pi_h, \mu)$. By Equation 12, π_h has finite second moment and, by assumption, μ is a strictly log-concave measure. Hence, by Theorem 5.1 [2], $(\pi_h)_t$ has finite entropy with respect to μ for $t > 0$. In the remainder of this proof, we are going to show the following bounds.

- (i) $\mathbb{E}[\frac{\xi^d}{h} - \nabla u(X) \mid X] = 0;$
- (ii) $\mathbb{E}[\|\mathbb{E}[\frac{\xi^{\otimes 2}d}{2h} - I_d \mid X]\|^2] \leq \frac{4L^4dh}{\rho^3} + O(dh^2);$
- (iii) $\mathbb{E}[\|\mathbb{E}[\frac{\xi^{\otimes 3}d}{h} \mid X]\|^2] \leq (4 + \frac{16L^4}{\rho^4})\frac{dh^2L^2}{\rho} + O(dh^3);$
- (iv) $\forall k > 3, \mathbb{E}[\|\mathbb{E}[\frac{\xi^{\otimes k}d}{h} \mid X]\|^2] \leq \frac{2^{4k}L^{2k}h^{k-2}d}{\rho^{2k}}.$

Then, using these bounds in Corollary 11 with $s = \frac{h}{d}$ and $\tau = h$, we obtain that

$$W_2(\pi_h, \mu) \leq O(d\sqrt{h}).$$

Moreover, whenever μ is the Gaussian measure, we can use a more refined version of Corollary 11, obtained using Theorem 1, with $s = \frac{h}{d}$ and $\tau = h$ to obtain

$$W_2(\pi_h, \mu) \leq O(\sqrt{dh}).$$

Overall, there exists $C_2 > 0$, depending on ρ, L such that if

$$h = C_2\epsilon^{-2} \begin{cases} d & \text{if } \mu \text{ is the Gaussian measure} \\ d^2 & \text{otherwise} \end{cases},$$

then

$$W_2(\pi_h, \mu) \leq \frac{\epsilon}{2}. \quad (15)$$

Proposition 13 is then obtained by combining Equations 14 and 15. In order to conclude this proof, let us prove (i)-(iv).

Equation (i) is true by construction of ξ . Let us prove (ii). By our assumption on u , we have

$$\begin{aligned} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{\xi^{\otimes 2}d}{2h} - I_d \mid X \right] \right\|^2 \right] &= h^2 \mathbb{E} \left[\sum_{i=1}^d (\nabla u(X))_i^4 \right] \\ &\leq h^2 L^4 \mathbb{E} \left[\sum_{i=1}^d (X_i)^4 \right]. \end{aligned}$$

Hence, by Equation 13 and Equation 12,

$$\mathbb{E} \left[\left\| \mathbb{E} \left[\frac{\xi^{\otimes 2}d}{2h} - I_d \mid X \right] \right\|^2 \right] \leq \frac{4L^4h}{\rho^2} \mathbb{E}[\|X\|^2] \leq \frac{2L^4dh}{\rho^3} + O(dh^2).$$

Let us now deal with (iii). We have,

$$\begin{aligned} \mathbb{E} \left[\left\| \mathbb{E} \left[\frac{\xi^{\otimes 3}d}{h} \mid X \right] \right\|^2 \right] &= 4h^2 \mathbb{E}[\|\nabla u(X)\|^2] + h^3 \mathbb{E} \left[\sum_{i=1}^d (\nabla u(X))_i^6 \right] \\ &\leq 4L^2h^2 \mathbb{E}[\|X\|^2] + h^3 L^6 \mathbb{E} \left[\sum_{i=1}^d (X_i)^6 \right]. \end{aligned}$$

Thus, using Equation 13, our assumption on u and Equation 12,

$$\begin{aligned}\mathbb{E} \left[\left\| \mathbb{E} \left[\frac{\xi^{\otimes 3} d}{h} \mid X \right] \right\|^2 \right] &\leq \left(4 + \frac{16L^4}{\rho^4} \right) h^2 L^2 \mathbb{E}[\|X\|^2] \\ &\leq \left(4 + \frac{16L^4}{\rho^4} \right) \frac{dh^2 L^2}{\rho} + O(dh^3).\end{aligned}$$

Finally, we have

$$\begin{aligned}\mathbb{E} \left[\left\| \mathbb{E} \left[\frac{\xi^{\otimes k} d}{h} \mid X \right] \right\|^2 \right] &= \frac{1}{h^2} \mathbb{E}_X \left[\sum_{i=1}^d \mathbb{E}_B \left[\left(-(h\nabla u(X))_i + \sqrt{2h}B \right)^k \right]^2 \right] \\ &\leq \frac{1}{h^2} \sum_{i=1}^d \mathbb{E} \left[\left(-(h\nabla u(X))_i + \sqrt{2h}B \right)^{2k} \right] \\ &\leq \frac{2^{2k-1}}{h^2} \left(h^{2k} \mathbb{E} \left[\sum_{i=1}^d (\nabla u(X))_i^{2k} \right] + 2^k h^k d \right) \\ &\leq \frac{2^{2k-1}}{h^2} \left(h^{2k} L^{2k} \mathbb{E} \left[\sum_{i=1}^d (X_i)^{2k} \right] + 2^k h^k d \right) \\ &\leq \frac{2^{2k-1}}{h^2} \left(\frac{L^{2k} 2^{2k} h^k d}{\rho^{2k}} + 2^k h^k d \right) \\ &\leq \frac{2^{4k} L^{2k} h^{k-2} d}{\rho^{2k}}.\end{aligned}$$

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