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Stable measures and Stein's method: rates in the Central Limit Theorem and diffusion approximation

Thomas Bonis

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Abstract

We present a way to apply Stein's method in order to bound the Wasserstein distance between a, possibly discrete, measure and the Gaussian measure using what we call stable measures. We apply this construction to obtain convergence rates in terms of Wasserstein distance, for orders $p \geq 2$, in the Central Limit Theorem in dimension 1 under precise moment conditions. We also establish a similar result for the Wasserstein distance of order 2 in the multidimensional setting. In a second time, we generalize our construction to more general target measures and we show how it can be applied to stationary distributions of Markov chains in the context of diffusion approximation.

1 Introduction

Let $d\gamma$ be the Gaussian measure on \mathbb{R}^d . A measurable map τ_ν is said to be a Stein kernel for the probability measure ν of \mathbb{R}^d if, for every smooth test function ϕ ,

$$\int_{\mathbb{R}^d} -x \cdot \nabla \phi(x) d\nu + \int_{\mathbb{R}^d} \langle \tau_\nu(x), \text{Hess}(\phi)(x) \rangle_{HS} d\nu = 0, \quad (1)$$

where $\langle \dots \rangle_{HS}$ and $\|\cdot\|_{HS}$ correspond to the usual Hilbert-Schmidt scalar product and norm respectively. Standard integration by parts formula gives us that $\tau_\gamma = I_d$, the identity matrix in \mathbb{R}^d ; intuitively, the closer τ_ν and τ_γ , the closer ν and γ . Whenever such a kernel exists, this proximity is characterized by a quantity named the Stein discrepancy of ν with respect to γ

$$S(\nu) = \left(\int_{\mathbb{R}^d} \|\tau_\nu - I_d\|_{HS}^2 d\nu \right)^{1/2}.$$

Both the Stein kernel and the Stein discrepancy are at the centre of Stein's method, see for example the books from Nourdin and Peccati [11] or from Chen, Goldstein and Shao [8]. Following the work from Nourdin, Peccati and Swan [12], Ledoux, Nourdin and Peccati [10] have shown how this construction can be used to bound the entropy of the measure ν using both the Stein discrepancy and the Fisher information of the measure. They also derived an inequality involving both the Stein discrepancy and the entropy of a measure to bound the Wasserstein distance of order 2 between ν and γ (the WSH inequality). Remember that the Wasserstein distance of order p between two

measures μ and ν is defined as

$$W_p(\mu, \nu) = \inf_{\pi} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|_2^p d\pi(x, y) \right)^{1/p}$$

where π has marginals μ and ν . One interesting property of the WSH bound is its finiteness even when the Fisher information of ν is infinite (for example when ν is discrete): in this situation, the inequality becomes

$$W_2(\nu, \gamma) \leq S(\nu). \quad (2)$$

However, one of the main restrictions of this approach is the existence of a Stein kernel for ν . Let us consider a measure which does not admit a smooth density with respect to γ such as the Rademacher measure $\nu = \frac{1}{2}(\delta_{-1} + \delta_1)$ in \mathbb{R} ; in order to derive a Stein kernel for this measure, we should find a function τ_ν such that, for any smooth test function ϕ ,

$$\frac{1}{2}(\phi'(1) - \phi'(-1)) = \frac{1}{2}(\tau_\nu(1)\phi''(1) + \tau_\nu(-1)\phi''(-1)).$$

This is impossible as we can find a smooth function ϕ such that $\phi''(1) = \phi''(-1) = 0$ and $\phi'(-1) = -\phi'(1) = 1$. Hence, in general, a measure with atoms does not admit a Stein kernel. One solution proposed by Azmodeeh, Peccati and Poly in [2] to circumvent this issue, in the case of the Rademacher measure, is to compute the Stein kernel for a smoothed version of the measure. Yet, it is unclear whether such a smoothing procedure could be derived for general measures.

In this work, we propose a new approach to apply Stein's method which can deal with discrete measures by introducing a new tool we call *stable measure*.

Definition 1. An application $\tau_\nu(\cdot, dy, \cdot)$ of $\mathbb{R}^d \times \mathbb{R}^+$ into the space of probability measures of \mathbb{R}^d is a stable measure for ν if, for every $t > 0$,

- For all $k > 0$, $x \in \mathbb{R}^d \rightarrow \int_{\mathbb{R}^d} \|y\|_2^k \tau_\nu(x, dy, t)$ is in $L_1(\nu)$;
- For any smooth test function ϕ , $\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x + y) \tau_\nu(x, dy, t) d\nu(x) = \int_{\mathbb{R}^d} \phi(x) d\nu(x)$.

Remark first that, for any measure ν , $\tau_\nu(\cdot, dy, \cdot) = \delta_0(dy)$ is always a stable measure for ν . The role t will be made clear later on. Consider a smooth test function ϕ and some $t > 0$, a simple application of Taylor's theorem on

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (\phi(x + y) - \phi(x)) \tau_\nu(x, dy, t) d\nu(x)$$

yields, for any integer $i \geq 1$,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\sum_{0 < k < i} \left[\langle \frac{y^{\otimes k}}{k!}, \nabla^k \phi(x) \rangle_{HS} \right] + \langle \frac{y^{\otimes i}}{(i-1)!}, D_{x,y}^i \phi \rangle_{HS} \right) \tau_\nu(x, dy, t) d\nu(x) = 0, \quad (3)$$

where $\nabla^k \phi$ is viewed as a map from \mathbb{R}^d to \mathbb{R}^{d^k} , $(y^{\otimes k})_{j_1, \dots, j_k} = \prod_{l=1}^k y_{j_l}$ and

$$(D_{x,y}^i \phi)_{j_1, \dots, j_i} = \int_0^1 (1 - \alpha)^{i-1} \frac{\partial^i \phi}{\partial x_{j_1} \dots \partial x_{j_i}}(x + \alpha y) d\alpha.$$

Thus, as in the case of the Stein kernel, the existence of a stable measure ensures the existence of an operator \mathcal{T} such that, for any smooth test function ϕ ,

$$\mathbb{E}_\nu[\mathcal{T}(\phi)] = 0.$$

Following the spirit of the Stein method, one can expect that if this operator is close to $-x \cdot \nabla + \Delta$, then ν and γ are close. We prove such a result in Section 2 where the proximity between ν and γ is characterized by the Wasserstein distance of order 2. We also show that a similar result for the Wasserstein distance of order $p \geq 1$ holds for measures in \mathbb{R} .

In Theorem 5, as an application of our results, we provide convergence rates for the Central Limit Theorem in terms of Wasserstein distances under various moment conditions. More precisely, if we consider i.i.d random variables X_1, \dots, X_n in \mathbb{R}^d with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1 X_1^T] = I_d$ admitting a finite moment of order $2+m$ for $m \leq 2$, then the Wasserstein distance of order 2 between ν_n , the measure of $S_n = n^{-1/2} \sum_{i=1}^n X_i$, and γ decreases in $n^{-1/2+(2-m)/4}$. Moreover, if $d = 1$ and X_1 admits a finite moment of order $p+m$ for some $m \leq 2 \leq p$, then $W_p(\nu_n, \gamma)$ decreases in $n^{-1/2+(2-m)/2p}$. This completes a result obtained by Rio [14] who considered the case $1 \leq p \leq 2, m = 2$ and generalizes a result obtained by Sakhanenko [15] treating the case $p > 2, m = 0$. More recently, Bobkov [4] recovered the case $p = 2, m = 2$ using an entropic approach. To our knowledge, the multidimensional result is new although the entropic approach from Bobkov is claimed to work in the multidimensional setting but with stronger assumptions on the moment of the variables.

In Section 3, following the approach developed in Section 4 [10], we generalize our construction to more general target measures μ assumed to be the invariant measures of diffusion processes with generators $\mathcal{L}f = b \cdot \nabla f + \langle a, Hess f \rangle_{HS}$. Under suitable conditions on \mathcal{L} , we show that if τ_ν is a stable measure for ν , then the Wasserstein distance of order 2 between ν and μ can be bounded using the proximity between \mathcal{L} and the operator \mathcal{T} defined in Equation (3). We apply this result in the case of diffusion approximation in Section 3.1. Consider a Markov chain M_s with state space S_s , transition kernel K_s and invariant measure π_s which we see as a discrete approximation, with time-step s , of the diffusion process with generator $\mathcal{L}f = b \cdot \nabla f + \langle a, Hess f \rangle_{HS}$: we show how to build a stable measure using K_s and, under suitable conditions on \mathcal{L} , obtain bounds on the convergence of π_s to μ .

Finally, the proofs of our results are displayed in Section 4.

2 The Gaussian case

Let $d\gamma = (2\pi)^{-d/2} e^{-\frac{|x|^2}{2}} dx$ be the Gaussian measure in \mathbb{R}^d and consider a measure $\nu \in \mathbb{R}^d$. Following the approach of [10], we show how one can bound the Wasserstein distance of order 2 between ν and γ using the stable measure framework. As mentioned in the introduction, if ν admits a stable measure τ_ν , then one can expect that if, for some $i > 2$, the quantities

$$\begin{aligned} & \frac{1}{s} \int_{\mathbb{R}^d} y \tau_\nu(x, dy, t) + x; \\ & \frac{1}{s} \int_{\mathbb{R}^d} \frac{y^{\otimes 2}}{2} \tau_\nu(x, dy, t) - I_d; \\ & k \leq i, \frac{1}{s} \int_{\mathbb{R}^d} \frac{y^{\otimes k}}{k!} \tau_\nu(x, dy, t), \end{aligned}$$

are small, then $W_2(\nu, \gamma)$ will be small. Consider a function $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d^i}$ for $i > 0$ and $q \geq 0$ we define

$$\|f\|_{H(q,i,\nu)}^2 = \int_{\mathbb{R}^d} \sum_{k \in [1,d]^i} \|H_k\|_{q,\gamma}^2 |f_k(x)|^2 d\nu(x),$$

where H_k is the multidimensional Hermite polynomial of index k , i.e.

$$\frac{\partial^i \gamma}{\partial x_{k_1} \dots \partial x_{k_i}} = (-1)^i H_k \gamma.$$

In this context, the generalization of Equation (2) to the stable measure framework is given by the following result.

Theorem 2. *Let ν be a measure of \mathbb{R}^d with finite second moment admitting a stable measure τ_ν . Then, for any $s > 0$ and $i > 2$,*

$$\begin{aligned} W_2(\nu, \gamma) &\leq \int_0^\infty e^{-t} \left\| \frac{1}{s} \int_{\mathbb{R}^d} y \tau_\nu(x, dy, t) + x \right\|_{H(2,1,\nu)} dt \\ &+ \int_0^\infty \frac{e^{-2t}}{\sqrt{1-e^{-2t}}} \left\| \frac{1}{s} \int_{\mathbb{R}^d} \frac{y^{\otimes 2}}{2} \tau_\nu(x, dy, t) - I_d \right\|_{H(2,2,\nu)} dt \\ &+ \int_0^\infty \sum_{2 < k < i} \frac{e^{-kt}}{(1-e^{-2t})^{(k-1)/2}} \left\| \frac{1}{s} \int_{\mathbb{R}^d} \frac{y^{\otimes k}}{k!} \tau_\nu(x, dy, t) \right\|_{H(2,k,\nu)} dt \\ &+ \int_0^\infty \frac{e^{-it}}{(1-e^{-2t})^{(i-1)/2}} \left\| \frac{1}{s} \int_{\mathbb{R}^d} \left[\frac{|y|^{\otimes i}}{(i-1)!} \int_0^1 \left((1-\alpha)^{i-1} e^{\frac{3\|\alpha y\|_2^2 e^{-2t}}{4(1-e^{-2t})}} \right) d\alpha \right] \tau_\nu(x, dy, t) \right\|_{H(4,i,\nu)} dt. \end{aligned}$$

As we mentioned in the introduction, $\tau_\nu(\cdot, dy, \cdot) = \delta_0(dy)$ is always a stable measure for any measure ν . Using our result for this (trivial) stable measure yields the (trivial) bound $W_2(\nu, \gamma) \leq \int_{\mathbb{R}^d} \|x\|_2^2 (d\nu + d\gamma)$. Now, if we were to apply our result to a non-trivial stable measure constant with respect to t , we would encounter integrability issues for small t . Hence, one has to build stable measures by interpolating between the trivial stable measure at small t and a more refined one for large t . Finally, the parameter s is a simple renormalization factor.

We now turn ourselves to deriving a bound for the Wasserstein distance of order p between ν and γ . In Section 3.2 [10], it has been shown that the W_2 bound presented in Equation (2) can be generalized to Wasserstein distance of order $p \geq 1$ under a stronger version of the Stein kernel verifying

$$\int_{\mathbb{R}^d} [-x\phi(x) + \tau_\nu(x)\nabla\phi(x)] d\nu = 0.$$

for every smooth function ϕ . With this definition they have shown that, for $p \geq 1$,

$$W_p(\nu, \gamma) \leq \begin{cases} d^{1-1/p} \int_{\mathbb{R}} x d\gamma^1 S_p & \text{if } 1 \leq p \leq 2 \\ d^{1-2/p} \int_{\mathbb{R}} x d\gamma^1 S_p & \text{if } p > 2 \end{cases},$$

where γ_1 is the Gaussian measure on \mathbb{R} and

$$S_p(\nu) = \left(\int_{\mathbb{R}^d} \|\tau_\nu - I_d\|_p^p d\nu \right)^{1/p}.$$

Unfortunately, we cannot show that stable measures satisfy a generalization of the strong Stein kernel assumption. However, in dimension 1, the strong Stein kernel assumption is equivalent to the standard Stein kernel presented in Equation (1). Thus, in this case, one can apply the stable measure framework to obtain the following result.

Theorem 3. *Let $p \geq 1$ and ν be a measure of \mathbb{R} admitting a stable measure τ_ν . Then for any $s > 0$, $i > 2$,*

$$\begin{aligned} W_p(\nu, \gamma) &\leq \int_0^\infty e^{-t} \left\| \frac{1}{s} \int_{\mathbb{R}} y \tau_\nu(x, dy, t) - x \right\|_{p, \nu} dt \\ &\quad + \int_0^\infty \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \left\| \frac{1}{s} \int_{\mathbb{R}} y^2 d\tau_\nu(x, y, t) - I_d \right\|_{p, \nu} dt \\ &\quad + \int_0^\infty \sum_{2 < k < i} \|H_k\|_{p, \gamma} \frac{e^{-kt}}{(1 - e^{-2t})^{(k-1)/2}} \left\| \frac{1}{s} \int_{\mathbb{R}} y^k \tau_\nu(x, dy, t) \right\|_{p, \nu} dt \\ &\quad + \int_0^\infty \frac{e^{-it}}{(1 - e^{-2t})^{(i-1)/2}} \|H_i\|_{2p, \gamma} \left\| \int_{\mathbb{R}} \left[\int_0^y \left(\frac{|z|^{i-1}}{s(i-1)!} e^{\frac{(2p^2-3p+1)z^2 e^{-2t}}{(2p)(1-e^{-2t})}} \right) dz \right] \tau_\nu(x, dy, t) \right\|_{p, \nu} dt, \end{aligned}$$

Remark 4. [9] gives the asymptotic of the p -norm of Hermite polynomials with respect to the Gaussian measure. In particular, in dimension 1, we have that, for any $k \in [1, d]^i$, $\|H_k\|_p = o(i!^{3/4})$. It is thus possible to set i to infinity in both Theorems.

As an application of our previous results, we use the stable measure framework to obtain rates of convergence for the central limit theorem in terms of Wasserstein distance.

Theorem 5. *Let X_1, \dots, X_n be i.i.d random variables in \mathbb{R}^d with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = I_d$. If X_1 admits a moment of order $p + m$ ($\mathbb{E}[\|X_1\|_2^{p+m}] < \infty$) for some $m \in [0, 2]$ then*

$$W_2(\nu_n, \gamma) = O(n^{-1/2+(2-m)/4}).$$

Moreover, if $d = 1$ then for any $p \geq 2$, if X_1 admits a finite moment of order $p + m$ for some $m \in [0, 2]$ then

$$W_p(\nu_n, \gamma) = O(n^{-1/2+(2-m)/2p}).$$

3 General target measures

We now turn ourselves to the case of general measures. Let E be a domain of \mathbb{R}^d and μ be a measure with support E . Assume that μ is the unique stationary distribution of a diffusion process with infinitesimal generator $\mathcal{L} = b \cdot \nabla + \langle a, \text{Hess} \rangle_{HS}$ where b and a are C^∞ on E and a is symmetric positive-definite on all of E . Let $(P_t)_{t \geq 0}$ be the Markov semigroup associated to this diffusion process. For any $k > 0$ and any function $\phi : E^{d^k} \rightarrow \mathbb{R}$ and matrix A , we define

$$\|\phi\|_A^2 = \sum_{0 \leq j_1, \dots, j_k \leq d} \sum_{0 \leq l_1, \dots, l_k \leq d} \phi_{j_1, \dots, j_k} \phi_{l_1, \dots, l_k} \prod_{i=1}^k A_{j_i, l_i}.$$

In this context, and under suitable assumptions on the operator \mathcal{L} , the stable measure framework can be used to bound $W_2(\nu, \mu)$.

Theorem 6. Let ν be a measure in E admitting a stable measure τ_ν , assume that the following holds for some $i > 2$

- (i) There exists $(f_k)_{0 < k < i} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for any smooth test function $\phi : E \rightarrow \mathbb{R}$, $\|\nabla^k P_t \phi\|_a \leq f_k(t) (P_t \|\phi\|_a^2)^{1/2}$.
- (ii) There exists $f_i : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^+$ such that for any smooth test function $\phi : E \rightarrow \mathbb{R}$ and any $x, y \in E \times \mathbb{R}^d$ such that the segment $[x, x + y]$ belongs to E , $\|\nabla^i P_t f \phi\|_a(x + y) \leq f_i(y, t) (P_t \|\phi\|_a^2)^{1/2}(x)$.
- (iii) For any $t > 0, x, y \in E \times \mathbb{R}^d$, if y belongs to the support of $\tau_\nu(x, \cdot, t)$, then the segment $[x, x + y]$ belongs to E .
- (iv) For any $t > 0$, $d\nu^t$ has finite Fisher information with respect to μ (i.e. $\nu^t = h_t d\gamma$ and $\|\nabla h\|_2^2/h \in L_1(d\mu)$).

Then, for any $s \geq 0$,

$$\begin{aligned} W_2(\nu, \mu) &\leq \int_0^\infty f_1(t) \left(\int_E \left\| \frac{1}{s} \int_{\mathbb{R}^d} y d\tau_\nu(x, dy, t) - b(x) \right\|_{a-1}^2 d\nu(x) \right)^{1/2} dt \\ &\quad + \int_0^\infty f_2(t) \left(\int_E \left\| \frac{1}{s} \int_{\mathbb{R}^d} \frac{y^{\otimes 2}}{2} d\tau_\nu(x, dy, t) - a(x) \right\|_{a-1}^2 d\nu(x) \right)^{1/2} dt \\ &\quad + \int_0^\infty \sum_{2 < k < i} f_k(t) \left(\int_E \left\| \frac{1}{s} \int_{\mathbb{R}^d} \frac{y^{\otimes k}}{i!} d\tau_\nu(x, dy, t) \right\|_{a-1}^2 d\nu(x) \right)^{1/2} dt \\ &\quad + \int_0^\infty \left\| \frac{1}{s} \int_{\mathbb{R}^d} f_i(y, t) \frac{\|y^{\otimes i}\|_{a-1}}{i!} d\tau_\nu(\cdot, dy, t) \right\|_{2, \nu} dt \end{aligned}$$

Assumption (iv) of this theorem is linked to the smoothing properties of the diffusion operator \mathcal{L} ; for example, [6, 5] give conditions ensuring the finiteness of the entropy and the Fisher information with respect to the Lebesgues measure of ν^t for $t > 0$; this result along with a proper control of the Fisher information of μ with respect to ν (which corresponds to ν having a finite second moment in the Gaussian case) can be used to prove assumption (iv). In order to verify whether the assumptions (i) and (ii) of this theorem are checked for a specific diffusion operator \mathcal{L} , one can use the framework of Γ -calculus described in [3]. This approach relies on the study of the iterated gradients Γ_i defined recursively for any smooth test functions f, g by

$$\begin{aligned} \Gamma_0(f, g) &= fg; \\ \Gamma_{i+1} &= \frac{1}{2} [\mathcal{L}(\Gamma_i(f, g)) - \Gamma_i(\mathcal{L}f, g) - \Gamma_i(f, \mathcal{L}g)]. \end{aligned}$$

In particular, the diffusion operator \mathcal{L} is said to satisfy a *curvature-dimension inequality* if there exists $\rho \in \mathbb{R}$ such that

$$\Gamma_2 \geq \rho \Gamma_1.$$

In such a case and if $\nu = h d\mu$, according to Theorem 5.5.2 [3],

$$I_\mu(\nu^t) \leq \frac{2\rho}{1 - e^{-2\rho t}} (P_t(h \log h) - P_t h \log(P_t h))$$

Remark 7. Under a curvature-dimension inequality, it is sufficient to show that ν^t has finite entropy with respect to μ for $t > 0$ for assumption (iv) to be verified.

It is worth noting that a result similar to Theorem 6 was derived in [10] using conditions on Γ_1, Γ_2 and Γ_3 which can be too restrictive. Here we propose to show that, in a simple case, that assumptions (i) and (ii) hold using weaker conditions.

Proposition 8. *Let μ be a measure on \mathbb{R} with density $e^{-u/2}$ and suppose that it is the invariant probability measure of the following diffusion operator*

$$\mathcal{L}(f) = f'' - u' \cdot f'.$$

If

$$H_1 \quad \exists \rho > 0, u'' \geq \rho ;$$

$$H_2 \quad \exists A, \forall x \in \mathbb{R}^d, |u^{(3)}(x)|, |u^{(4)}(x)| \leq Au'' ;$$

then for any measure ν with finite second moment, assumption (iv) of Theorem 6 is verified. Moreover, there exists constants C_1, C_2 such that the following functions verify assumption (i) and (ii) of Theorem 6

$$\begin{aligned} f_1(t) &= e^{-\rho t}; \\ f_2(t) &= \frac{C_1}{(e^{\rho t} - 1)^{1/2}}; \\ f_3(t, y) &= \frac{C_2 e^{\rho(-t/2 + y^2/(2(e^{\rho t} - 1)))}}{e^{\rho t/4} - 1}. \end{aligned}$$

3.1 Diffusion approximation

As an application of this extension, we study of the convergence of the stationary measures of a family of Markov chains toward μ , the invariant measure of the diffusion process with infinitesimal generator $\mathcal{L} = b \cdot \nabla + \langle a, Hess \rangle_{HS}$. Let $s > 0$ and M_s be a family of Markov chains with states spaces S_s , transition kernels K_s and invariant measures π_s . The measures π_s admits stable measures which can be expressed using K_s .

Proposition 9. *Suppose that for any $x, k > 0$, $\int_{E \times E} \|y - x\|_2^k K_s(x, dy) d\pi_h(x) < \infty$. Then, for any $T > 0$, the measure*

$$\tau_{\pi_s}(x, dy, t) = 1_{t < T} \delta_0(dy) + 1_{t > T} K_s(x, d(y + x))$$

is a stable measure for π_s . Moreover, if the M_s are reversible (that is if $K_s(x, dy) d\pi(x) = K_s(y, dx) d\pi(y)$) then, for any positive function $r(t)$,

$$\tau_{\pi_s}(x, dy, t) = \left[\int_{\|x-y\|_2 > r(t)} K_s(x, dy) \right] \delta_0(dy) + 1_{\|y-x\|_2 \leq r(t)} K_s(x, d(y + x))$$

is a stable measure for π_s .

Proof. We first deal with the general case. Let ϕ be a smooth function on \mathbb{R}^d , by definition of the stationary measure π_s , we have

$$\int_{S_s \times \mathbb{R}^d} \phi(x+y) K_s(x, d(x+y)) d\pi_s(x) = \int_{S_s^2} \phi(y) K_s(x, dy) d\pi(x) = \int_{S_s} \phi(y) d\pi_s(y).$$

We now deal with the reversible case, recall that M_s is said to be reversible if

$$\forall x, y \in S_s, K_s(x, dy) d\pi(x) = K_s(y, dx) d\pi(y).$$

Let $R > 0$, we have

$$\begin{aligned} \int_{x, y \in S_s, \|x-y\|_2 \leq R} \phi(y) K_s(x, dy) d\pi_s(x) &= \int_{x, y \in S_s, \|x-y\|_2 \leq R} \phi(y) K_s(y, dx) d\pi_s(y) \\ &= \left(1 - \int_{\|x-y\|_2 > R} K_s(x, dy) \right) \int_{y \in S_s} \phi(y) d\pi_s(y). \end{aligned}$$

□

Applying Theorem 6 to this stable measure yields the following result.

Corollary 10. *Under the assumptions of Theorem 6, we have, for $T > 0, s > 0$,*

$$\begin{aligned} W_2(\pi_s, \mu) &\leq \int_0^T f_1(t) \|b\|_{a^{-1}, \pi_s} dt + \int_T^\infty f_1(t) \left(\int_{\mathbb{R}^d} \left\| \frac{1}{s} \int_{y \in S_s} (y-x) K_s(x, dy) - b(x) \right\|_{a^{-1}}^2 d\nu(x) \right)^{1/2} dt \\ &\quad + \int_0^T f_2(t) dt + \int_T^\infty f_2(t) \left(\int_{\mathbb{R}^d} \left\| \frac{1}{s} \int_{y \in S_s} \frac{(y-x)^T (y-x)}{2} K_s(x, dy) - a(x) \right\|_{a^{-1}}^2 d\nu(x) \right)^{1/2} dt \\ &\quad + \int_T^\infty \sum_{2 < k < i} f_k(t) \left(\int_{\mathbb{R}^d} \left\| \frac{1}{s} \int_{y \in S_s} \frac{(y-x)^{\otimes k}}{k!} K_s(x, dy) \right\|_{a^{-1}}^2 d\nu(x) \right)^{1/2} dt \\ &\quad + \int_T^\infty \left(\int_{\mathbb{R}^d} \left[\frac{1}{s} \int_{y \in S_s} f_i(y-x, t) \frac{\|(y-x)^{\otimes i}\|_{a^{-1}}}{k!} K_s(x, dy) \right]^2 d\nu(x) \right)^{1/2} dt. \end{aligned}$$

It is worth noting that the quantities involved in this Corollary are natural and appears in standard results obtaining weak convergence of M_s to the diffusion process. An example of such result can be found in [16] which states that if

$$\begin{aligned} \limsup_{s \rightarrow 0} \frac{1}{s} \int_{y \in S_s} (x-y) K_s(x, dy) &= b; \\ \limsup_{s \rightarrow 0} \frac{1}{2s} \int_{y \in S_s} (x-y)^T (x-y) K_s(x, dy) &= a; \\ \forall r > 0, \limsup_{s \rightarrow 0} \frac{1}{s} \int_{y \in S_s, |x-y| > r} K_s(x, dy) &= 0; \\ \lim_{s \rightarrow 0} M_s^0 &= X_0. \end{aligned}$$

then, for any $T > 0$, M_s converges weakly to X , the diffusion process with generator $\mathcal{L}f = b \cdot \nabla f + a \cdot \text{Hess}(f) >_{HS}$, in $C([0, T], \mathbb{R}^d)$.

4 Proofs

4.1 Proof of Theorem 2

The proof is based on the proofs of Proposition 2.4 and Proposition 3.1 from [10] which rely on the heat flow interpolation along the Ornstein-Uhlenbeck semigroup $(P_t)_{t \geq 0}$ with infinitesimal generator

$$\mathcal{L} = \Delta - x \cdot \nabla.$$

We refer the reader to [3] Section 2.7.1 for a throughout presentation of the Ornstein-Uhlenbeck semi-group properties. Let f be a smooth test function of $\mathbb{R}^d \rightarrow \mathbb{R}$, according to the Mehler's formula we have

$$P_t f(x) = \int_{\mathbb{R}^d} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) d\gamma(y).$$

In particular,

$$\frac{\partial}{\partial x_{k_1} \dots \partial x_{k_i}} (P_t f)(x) = e^{-it} P_t \left(\frac{\partial}{\partial x_{k_1} \dots \partial x_{k_i}} f \right)(x).$$

On the other hand, integrating by parts multiple times by parts with respect to the Gaussian density yields

$$\frac{\partial}{\partial x_{k_1} \dots \partial x_{k_i}} P_t(f)(x) = \frac{e^{-it}}{(1 - e^{-2t})^{\frac{i-1}{2}}} \int_{\mathbb{R}^d} H_{k_1, \dots, k_i}(y) f(e^{-t}x + \sqrt{1 - e^{-2t}}y) d\gamma(y),$$

where H_{k_1, \dots, k_i} is the multidimensional Hermite polynomial with index (k_1, \dots, k_i) defined by $\frac{\partial}{\partial x_{k_1} \dots \partial x_{k_i}} \gamma = (-1)^i H_{k_1, \dots, k_i} \gamma$. Applying Cauchy-Schwartz inequality,

$$\left| \frac{\partial}{\partial x_{k_1} \dots \partial x_{k_i}} P_t(f)(x) \right| \leq \frac{e^{-it}}{(1 - e^{-2t})^{\frac{i-1}{2}}} \|H_{k_1, \dots, k_i}\|_{2, \gamma} (P_t f^2)^{1/2}. \quad (4)$$

Now, consider $u \in \mathbb{R}^d$, we have

$$\frac{\partial}{\partial x_{k_1} \dots \partial x_{k_i}} P_t(f)(x + u) = \frac{e^{-it}}{(1 - e^{-2t})^{\frac{i-1}{2}}} \int_{\mathbb{R}^d} H_{k_1, \dots, k_i}(y) f(e^{-t}(x + u) + \sqrt{1 - e^{-2t}}y) d\gamma(y),$$

let $v = \frac{e^{-t}u}{\sqrt{1 - e^{-2t}}}$, using the change of variable $z = y + v$,

$$\begin{aligned} \frac{\partial}{\partial x_{k_1} \dots \partial x_{k_i}} P_t(f)(x + u) &= \frac{e^{-it}}{(1 - e^{-2t})^{\frac{i-1}{2}}} \int_{\mathbb{R}^d} H_{k_1, \dots, k_i}(z - v) f(e^{-t}x + \sqrt{1 - e^{-2t}}z) d\gamma(z - v) \\ &= \frac{e^{-it}}{(1 - e^{-2t})^{\frac{i-1}{2}}} \int_{\mathbb{R}^d} H_{k_1, \dots, k_i}(z - v) f(e^{-t}x + \sqrt{1 - e^{-2t}}z) e^{v \cdot z - \frac{\|v\|_2^2}{2}} d\gamma(z). \end{aligned}$$

Therefore, applying Cauchy-Schwartz inequality twice yields

$$\begin{aligned}
\left| \frac{\partial}{\partial x_{k_1} \dots \partial x_{k_i}} P_t(f)(x+u) \right| &\leq \frac{e^{-it}}{(1-e^{-2t})^{\frac{i-1}{2}}} \left(P_t f^2(x) \int_{\mathbb{R}^d} H_{k_1, \dots, k_i}^2(z-v) e^{2v \cdot z - \|v\|^2} d\gamma(z) \right)^{1/2} \\
&\leq \frac{e^{-it}}{(1-e^{-2t})^{\frac{i-1}{2}}} \left(P_t f^2(x) \int_{\mathbb{R}^d} H_{k_1, \dots, k_i}^2(z-v) e^{v \cdot z - \frac{\|v\|^2}{2}} d\gamma(z-v) \right)^{1/2} \\
&\leq \frac{e^{-it}}{(1-e^{-2t})^{\frac{i-1}{2}}} (P_t f^2(x))^{1/2} \|H_{k_1, \dots, k_i}\|_{4, \gamma} \left(\int_{\mathbb{R}^d} e^{2v \cdot z - \|v\|^2} d\gamma(z-v) \right)^{1/4} \\
&\leq \frac{e^{-it}}{(1-e^{-2t})^{\frac{i-1}{2}}} (P_t f^2(x))^{1/2} \|H_{k_1, \dots, k_i}\|_{4, \gamma} e^{3\frac{\|v\|_2^2}{4}} \left(\int_{\mathbb{R}^d} d\gamma(z - \frac{3v}{2}) \right)^{1/4}.
\end{aligned}$$

We have thus obtained the following inequality

$$\left| \frac{\partial}{\partial x_{k_1} \dots \partial x_{k_i}} P_t(f)(x+u) \right| \leq e^{3\frac{\|v\|_2^2}{4}} \frac{e^{-it}}{(1-e^{-2t})^{\frac{i-1}{2}}} \|H_{k_1, \dots, k_i}\|_{4, \gamma} (P_t f^2)^{1/2} \quad (5)$$

Finally, remark that \mathcal{L} satisfies the following integration by parts formula for f, g two smooth test functions,

$$\int_{\mathbb{R}^d} f \mathcal{L} g d\gamma = - \int_{\mathbb{R}^d} \nabla f \cdot \nabla g d\gamma.$$

In a first time, we assume that ν admits a strictly positive smooth density with respect to γ and we write $d\nu = h d\gamma$ and $d\nu^t = P_t h d\gamma$, we also assume that ν has finite Fisher information with respect to γ . Our objective is to bound the Fisher information of $d\nu^t$, where

$$I_\gamma(\nu) = \int_{\mathbb{R}^d} \frac{\|\nabla h\|_2^2}{h} d\gamma = \int_{\mathbb{R}^d} \nabla h \cdot \nabla(\log h) d\gamma.$$

If we write $v_t = \log(P_t h)$, applying the previous equations to $d\nu^t$ yields

$$I_\gamma(\nu^t) = \int_{\mathbb{R}^d} \nabla P_t h \cdot \nabla(\log P_t h) d\gamma.$$

Hence, we can use the integration by part formula to obtain

$$I_\gamma(\nu^t) = - \int_{\mathbb{R}^d} \mathcal{L}(\log P_t h) P_t h d\gamma.$$

Finally, using the symmetry of P_t ,

$$\begin{aligned}
I_\gamma(\nu^t) &= - \int_{\mathbb{R}^d} \mathcal{L}(v_t) P_t h d\gamma = - \int_{RR^d} \mathcal{L}(P_t v_t) h d\gamma = - \int_{\mathbb{R}^d} \mathcal{L}(P_t v_t) d\nu \\
&= \int_{\mathbb{R}^d} x \cdot \nabla P_t v_t - \langle I_d, Hess(P_t v_t) \rangle_{HS} d\nu.
\end{aligned}$$

For $t, s > 0$, let us pose

$$\begin{aligned}\tau_{\nu,1}(x, t) &= \frac{1}{s} \int_{\mathbb{R}^d} y \tau_{\nu}(x, dy, t) + x; \\ \tau_{\nu,2}(x, t) &= \frac{1}{s} \int_{\mathbb{R}^d} \frac{y^{\otimes 2}}{2} \tau_{\nu}(x, dy, t) - I_d; \\ \tau_{\nu,k}(x, t) &= \frac{1}{s} \int_{\mathbb{R}^d} \frac{y^{\otimes k}}{k!} \tau_{\nu}(x, dy, t).\end{aligned}$$

By Equation (3), we have

$$\begin{aligned}I_{\gamma}(\nu^t) &= \int_{\mathbb{R}^d} \left[\sum_{0 < k < i} \langle \tau_{\nu,k}(x, t), \nabla^k P_t v_t(x) \rangle_{HS} + \frac{1}{s} \int_{\mathbb{R}^d} \left\langle \frac{y^{\otimes i}}{(i-1)!}, D_{x,y}^i \phi \right\rangle_{HS} \tau_{\nu}(x, dy, t) \right] d\nu(x) \\ &\leq \left(\int_{\mathbb{R}^d} P_t(|\nabla v_t|^2) d\nu \right)^{1/2} \left[\sum_{0 < k < i} \frac{e^{-kt} \|\tau_{\nu,k}(\cdot, t)\|_{H(2,k,\nu)}}{(1 - e^{-2t})^{(k-1)/2}} \right. \\ &\quad \left. + \frac{e^{-it}}{(1 - e^{-2t})^{(i-1)/2}} \left\| \frac{1}{s} \int_{\mathbb{R}^d} \left[\frac{|y|^{\otimes i}}{(i-1)!} \int_0^1 \left((1-\alpha)^{i-1} e^{\alpha^2 \frac{3\|y\|_2^2 e^{-2t}}{4(1-e^{-2t})}} \right) d\alpha \right] d\tau_{\nu}(\cdot, y, t) \right\|_{H(4,i,\nu)} \right],\end{aligned}$$

where the last line is obtained using Equations (4) and (5) on ∇v_t and applying a Cauchy-Schwartz inequality. On the other hand, remark that $I_{\gamma}(\nu^t) = \left(\int_{\mathbb{R}^d} P_t(|\nabla v_t|^2) d\nu \right)^{1/2}$: we have thus bounded $I_{\gamma}(\nu^t)^{1/2}$.

The convergence of ν^t to γ can then be quantified by the following estimate from [17]

$$\frac{d^+}{dt} W_2(\nu, \nu_t) \leq I_{\gamma}(\nu^t)^{1/2}.$$

Since $\nu^t \rightarrow_{t \rightarrow \infty} \gamma$, we can integrate our previous upper bound of $I_{\gamma}(\nu^t)^{1/2}$ for t between 0 and ∞ to conclude the proof whenever ν admits a smooth density with respect to γ .

We now extend the result to discrete measures using a regularization procedure similar to the one employed to derive Proposition 3.1 in [10]. For any $\epsilon > 0$, we define the random variable $F_{\epsilon} = e^{-\epsilon} F + \sqrt{1 - e^{-2\epsilon}} Z$ where Z and F are independent with respective laws ν and γ . The distribution ν^{ϵ} of F_{ϵ} admits a strictly positive smooth density with respect to γ with finite Fisher information with respect to γ since F admits a finite second moment (see Remark 2.1 [12]). Moreover, for any smooth test function ϕ ,

$$\begin{aligned}\int_{\mathbb{R}^d} \sqrt{1 - e^{-2\epsilon}} \nabla \phi(x) + (1 - e^{-2\epsilon}) \Delta \phi(x) + \sum_{0 < k < i} \langle e^{-k\epsilon} \tau_{\nu,k}(x, t, h), \nabla^k P_t v_t^{\epsilon}(x) \rangle_{HS} d\nu^{\epsilon} \\ + \int_{\mathbb{R}^d} \frac{1}{s} \left\langle \frac{(e^{-\epsilon} y)^{\otimes i}}{(i-1)!}, D_{x, y e^{-\epsilon}}^i \phi \right\rangle_{HS} \tau_{\nu}(x, dy, t) d\nu^{\epsilon} = 0,\end{aligned}$$

which implies

$$\begin{aligned}W_2(\nu^{\epsilon}, \gamma) &\leq \int_0^{\infty} \sum_{0 < k < i} \frac{e^{-kt} \|\tau_{\nu,k}(\cdot, t)\|_{H(2,k,\nu)}}{(1 - e^{-2t})^{(k-1)/2}} dt \\ &\quad + \frac{e^{-it}}{(1 - e^{-2t})^{(i-1)/2}} \left\| \int_{\mathbb{R}^d} \frac{|y|^{(i)}}{h(i-1)!} \int_0^1 (1-\alpha)^{i-1} e^{\alpha^2 \frac{3\|y\|_2^2 e^{-2t}}{4(1-e^{-2t})}} d\alpha d\tau_{\nu}(\cdot, y, t) \right\|_{H(4,i,\nu)} dt.\end{aligned}$$

Finally, as $\epsilon \rightarrow 0$, $W_2(\nu^\epsilon, \nu) \rightarrow 0$ and we deduce the result by a simple triangular inequality.

4.2 Proof of Theorem 3

The proof of this Theorem is based on the proofs of Lemma 2.9 from [12] and Proposition 3.4 from [10].

Let us pose

$$\begin{aligned}\tau_{\nu,1}(x,t) &= \int_{\mathbb{R}} \frac{1}{s} y \tau_{\nu}(x, dy, t) + x; \\ \tau_{\nu,2}(x,t) &= \int_{\mathbb{R}} \frac{1}{s} \frac{y^2}{2} \tau_{\nu}(x, dy, t) - I_d; \\ \tau_{\nu,k}(x,t) &= \int_{\mathbb{R}} \frac{1}{s} \frac{y^k}{k!} \tau_{\nu}(x, dy, t).\end{aligned}$$

Similarly to the proof of Theorem 2, we first assume that ν admits a smooth density h with respect to γ . Let $t > 0$, as before we write $v_t = \log(P_t h)$ and for $z \in \mathbb{R}^d$ we note $g(t, z) = \frac{ze^{-t}}{\sqrt{1-e^{-2t}}}$. Consider the function ρ_t defined for any $x \in \mathbb{R}$ as follows

$$\begin{aligned}\rho_t(x) &= \mathbb{E}\left[\sum_{0 < k < i} \frac{e^{-kt}}{\sqrt{1-e^{-2t}k-1}} \tau_{\nu,k}(F) H_{k-1}(Z) \right. \\ &\quad \left. + \frac{e^{-it}}{\sqrt{1-e^{-2t}i-1}} \int_{\mathbb{R}} \int_0^y \frac{z^{i-1}}{(i-1)!} H_{i-1}(Z - g(t, z)) e^{Zg(t, z) - \frac{g(t, z)^2}{2}} dz \tau_{\nu}(F, dy, t) | F_t = x \right]\end{aligned}$$

where F and Z are independent with respective laws ν and γ and $F_t = e^{-t}F + \sqrt{1-e^{-2t}}Z$. Consider a smooth test function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, by successive integrations by parts with respect to Z , we obtain

$$\begin{aligned}\mathbb{E}[(\rho_t(F_t) - F_t)\phi(F_t)] &= -\mathbb{E}[F_t\phi(F_t)] + \mathbb{E}\left[\left(\sum_{0 < k < i} \frac{e^{-kt}}{\sqrt{1-e^{-2t}k-1}} \tau_{\nu,k}(F, t) H_{k-1}(Z) \right. \right. \\ &\quad \left. \left. + \frac{e^{-it}}{\sqrt{1-e^{-2t}i-1}} \int_{\mathbb{R}} \int_0^y \frac{z^{i-1}}{(i-1)!} H_{i-1}(Z - g(t, z)) e^{Zg(t, z) - \frac{g(t, z)^2}{2}} dz \tau_{\nu}(F, dy, t)\right) \phi(F_t)\right] \\ &= -\mathbb{E}[F_t\phi(F_t)] + \mathbb{E}\left[\left(\sum_{0 < k < i} e^{-kt} \tau_{\nu,k}(F, t) \phi^{(k-1)}(F_t) \right. \right. \\ &\quad \left. \left. + e^{-it} \int_{\mathbb{R}} \int_0^y \frac{z^{i-1}}{(i-1)!} \phi^{(k-1)}((F+z)e^{-t} + \sqrt{1-e^{-2t}}Z) dz \tau_{\nu}(F, dy, t)\right)\right] \\ &= -\mathbb{E}[F_t\phi(F_t)] + e^{-t} \int_{\mathbb{R}} \left(\sum_{0 < k < i} \tau_{\nu,k}(x, t) P_t(\phi)^{(k)}(x) \right. \\ &\quad \left. + \int_{\mathbb{R}} \int_0^y \frac{z^{i-1}}{(i-1)!} P_t(\phi)^{(i-1)}(x+z) dz \tau_{\nu}(x, dy, t)\right) d\nu(x).\end{aligned}$$

Since ϕ is a smooth test function, we can Equation (3) to obtain

$$\begin{aligned}
\mathbb{E}[(\rho_t(F_t) - F_t)\phi(F_t)] &= -\mathbb{E}[F_t\phi(F_t)] + e^{-t} \int_{\mathbb{R}^d \times \mathbb{R}^d} xP_t(\phi)(x) - \nabla P_t(\phi)(x) d\nu(x) \\
&= \mathbb{E}[(-F_t + e^{-t}F)\phi(F_t) - e^{-2t}\nabla\phi(F_t)] \\
&= \mathbb{E}[-(1 - e^{-2t})\nabla\phi(F_t) - e^{-2t}\nabla\phi(F_t)] \\
&= -\mathbb{E}[\nabla\phi(F_t)].
\end{aligned}$$

Therefore, ρ_t satisfies the characterization of ∇v_t presented in Equation 2.28 [12]. In other words: ρ_t is a version of ∇v_t .

From here, a simple modification of the proof of Lemma 2 from [13] yields the general estimate

$$\frac{d^+}{dt} W_p(\nu, \nu^t) \leq \left(\int_{\mathbb{R}} |\nabla v_t|^p d\nu^t \right)^{1/p},$$

which, after integration gives

$$W_p(\nu, \gamma) \leq \int_0^\infty \left(\int_{\mathbb{R}} |\nabla v_t|^p d\nu^t \right)^{1/p} dt = \int_0^\infty \mathbb{E}[|\rho_t(F_t)|^p]^{1/p} dt.$$

Applying Jensen's inequality to get rid of the conditional expectation along with a triangular inequality, we obtain

$$\begin{aligned}
W_p(\nu, \gamma) &\leq \int_0^\infty \sum_{0 < k < i} \frac{e^{-kt}}{\sqrt{1 - e^{-2t}k-1}} \|H_{k-1}(Z)\|_{p,\gamma} \|\tau_{\nu,k}(F)\|_{p,\nu} dt \\
&+ \int_0^\infty \frac{e^{-it}}{\sqrt{1 - e^{-2t}i-1}} \mathbb{E} \left[\left| \int_{\mathbb{R}} \int_0^y \frac{z^{i-1}}{(i-1)!} H_{i-1}(Z - g(t,z)) e^{Zg(t,z) - \frac{g(t,z)^2}{2}} dz \tau_{\nu}(F, dy, t) \right|^p \right]^{1/p} dt.
\end{aligned}$$

In order to bound this last quantity, we use an approach similar to the proof of Young's inequality. Suppose that τ_{ν} admits a smooth density with respect to the Lebesgue measure. Let $\eta = \frac{2p^2 - 3p + 1}{2p}$, $\alpha(y, z) = 1_{z \in [0, y]} \frac{z^{i-1}}{(i-1)!} e^{\eta g(t,z)^2} \tau_{\nu}(F, y, t)$ and $\beta(x, z) = H_{i-1}(x - g(t, z)) e^{Zg(t,z) - (\eta + \frac{1}{2})g(t,z)^2}$. Using Hölder inequality,

$$\begin{aligned}
\int_{\mathbb{R}^2} \alpha(y, z) \beta(x - g(t, z)) dy dz &= \int_{\mathbb{R}^2} |\alpha(z)|^{1/p} |\beta(x - g(t, z))| |\alpha(y, z)|^{\frac{p-1}{p}} dy dz \\
&\leq \| |\alpha(y, z)|^{1/p} \beta(x - g(t, z)) \|_p \| \alpha(y, z) \|_1^{\frac{p-1}{p}},
\end{aligned}$$

and we thus have

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} \alpha(y, z) \beta(x - g(t, z)) dy dz \right)^p d\gamma(x) \leq \| \alpha(y, z) \|_1^{p-1} \int_{\mathbb{R}} \| |\alpha(y, z)|^{1/p} \beta(x - g(t, z)) \|_p^p d\gamma(x).$$

We pose $\delta = x - g(t, z)$, using a Cauchy-Schwartz inequality,

$$\begin{aligned}
\int_{\mathbb{R}} \|\alpha(y, z)|^{1/p} \beta(\delta)\|_p^p d\gamma(x) &= \int_{\mathbb{R}^3} |\alpha(y, z) \beta(\delta)|^p dz dy d\gamma(x) \\
&= \int_{\mathbb{R}^3} |\alpha(y, z)| H_{i-1}^p(\delta) e^{(p-1)xg(t, z) + (\frac{1}{2} - p(\eta + \frac{1}{2}))g(t, z)^2} d\gamma(\delta) dy dz \\
&\leq \|H_{i-1}\|_{2p, \gamma}^p \int_{\mathbb{R}^2} |\alpha(y, z)| \left(\int_{\mathbb{R}} e^{2(p-1)xg(t, z) + (1-2p(\eta + \frac{1}{2}))g(t, z)^2} d\gamma(\delta) \right)^{1/2} dy dz \\
&\leq \|H_{i-1}\|_{2p, \gamma}^p \int_{\mathbb{R}^2} |\alpha(y, z)| \left(\int_{\mathbb{R}} e^{-\frac{x^2}{2} + (2p-1)xg(t, z) - \frac{(2p-1)^2}{2}g(t, z)^2} dx \right)^{1/2} dy dz \\
&\leq \|H_{i-1}\|_{2p, \gamma}^p \int_{\mathbb{R}^2} |\alpha(y, z)| \left(\int_{\mathbb{R}} d\gamma(x - (2p-1)g(t, z)) \right)^{1/2} dy dz \\
&\leq \|H_{i-1}\|_{2p, \gamma}^p \int_{\mathbb{R}^2} |\alpha(y, z)| dy dz.
\end{aligned}$$

Therefore, we have shown that

$$\begin{aligned}
\mathbb{E} \left[\left| \int_{\mathbb{R}} \int_0^y \frac{z^{i-1}}{(i-1)!} H_{i-1}(Z - g(t, z)) e^{Zg(t, z) - \frac{g(t, z)^2}{2}} dz \tau_{\nu}(F, dy, t) \right|^p \right]^{1/p} \\
\leq \|H_{i-1}\|_{2p, \gamma} \left\| \int_{\mathbb{R}} \int_0^y \frac{|z|^{i-1}}{(i-1)!} e^{\frac{2p^2-3p+1}{2p}g(t, z)^2} dz \tau_{\nu}(F, dy, t) \right\|_{p, \nu}.
\end{aligned}$$

Simple limit arguments are sufficient to deal with cases where neither τ_{ν} nor ν admit a smooth density with respect to the Lebesgue measure.

4.3 Proof of Theorem 5

We prove the results for $d = 1$ and $p \geq 2$, the proof for $d > 1, p = 2$ is similar.

Let X_1, \dots, X_n be i.i.d random variables such that $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = 1$, let ν be the measure of X_1 . Let τ_{ν} be a stable measure for ν , then

$$\tau_{\nu^n}(x, dy, t) = \mathbb{E} \left[\sum_{j=1}^n \tau_{\nu}(X_j, d\frac{y}{\sqrt{n}}, t) \mid S_n = x \right]$$

is a stable measure for ν^n , the measure of S_n . For a measure μ , we note

$$\begin{aligned}
\tau_{\mu, 1}(x, t) &= \int_{\mathbb{R}} y \tau_{\mu}(x, dy, t) - x; \\
\tau_{\mu, 2}(x, t) &= \int_{\mathbb{R}} \frac{y^2}{2} \tau_{\mu}(x, dy, t) - 1; \\
\tau_{\mu, k}(x, t) &= \int_{\mathbb{R}} \frac{y^k}{k!} \tau_{\mu}(x, dy, t).
\end{aligned}$$

Let $a(t) = 1_{t \geq \frac{1}{n}} \sqrt{tn}$. The following measure is a stable measure for ν

$$\tau_{\nu}(x, dy, t) = \begin{cases} 1_{x+y \in [-a(t), a(t)]} d\nu(x+y) + P(X \in [-a(t), a(t)]^C) \delta_0 & \text{if } x \in [-a(t), a(t)] \\ \delta_0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned}\tau_{\nu,1}(x,t) &= -1_{[-a(t),a(t)]^c}x - 1_{[-a(t),a(t)]} \left(x\mathbb{P}(X \in [-a(t),a(t)]^c) + \int_{[-a(t),a(t)]} yd\nu(y) \right); \\ \tau_{\nu,2}(x,t) &= -1 + 1_{[-a(t),a(t)]} \int_{[-a(t),a(t)]} \frac{1}{2}(y-x)^2d\nu(y); \\ \tau_{\nu,k}(x,t) &= 1_{[-a(t),a(t)]} \int_{[-a(t),a(t)]} \frac{1}{k!}(y-x)^kd\nu(y).\end{aligned}$$

Applying Theorem 3,

$$W_2(\nu^n, \gamma) \leq \int_0^\infty \sum_{k=1}^3 \frac{e^{-kt}}{(1-e^{-2t})^{(k-1)/2}} \|H_{k-1}\|_{p,\gamma} \|\tau_{\nu^n,k}\|_{p,\nu^n} + \frac{e^{\frac{2p^2-3p+1}{2p}e^{-4t}}}{(1-e^{-2t})^{3/2}} \|H_3\|_{2p,\gamma} \|\tau_{\nu^n,4}\|_{p,\nu^n} dt.$$

Hence, we want to bound the $\|\tau_{\nu^n,k}\|_{p,\nu^n}$. By Jensen's inequality,

$$\|\tau_{\nu^n,k}(\cdot, t)\|_{p,\nu^n} = \mathbb{E}[\tau_{\nu^n,k}(S_n, t)^p]^{1/p} \leq \mathbb{E} \left[\left(\sum_{j=1}^n n^{-k/2} \tau_{\nu,k}(X_j, t) \right)^p \right]^{1/p}.$$

Moreover, since the X_i are independent, using Rosenthal's inequality (see [7]), there exists a constant C_p such that,

$$\|\tau_{\nu^n,k}(\cdot, t)\|_{p,\nu^n} \leq C_p \left(n^{1-k/2} |\mathbb{E}[\tau_{\nu,k}(X, t)]| + n^{1/2-k/2} \mathbb{E}[\tau_{\nu,k}(X, t)^2]^{1/2} + n^{1/p-k/2} \mathbb{E}[\tau_{\nu,k}(X, t)^p]^{1/p} \right).$$

Now, all we are left to is bounding $|\mathbb{E}[\tau_{\nu,k}(X, t)]|$, $\mathbb{E}[\tau_{\nu,k}^2(X, t)]$ and $\mathbb{E}[\tau_{\nu,k}^p(X, t)]$ using the moments of order smaller than $p+m$ of ν and a . We pose $l = \min(4-(p+m), 2)$. In the following, X is a random variable with measure ν . Let us start with $\tau_{\nu,1}$: by symmetry, we have $\mathbb{E}[\tau_{\nu,1}(X, t)] = 0$. On the other hand, since $\mathbb{E}[X] = 0$, using Bienaymé-Tchebychev inequality,

$$\begin{aligned}\mathbb{E}[|\tau_{\nu,1}(X, t)|^p] &= \int_{[-a,a]^c} |x|^p d\nu(x) + \int_{[-a,a]} \left| -\mathbb{P}(X \in [-a,a]^c)x + \int_{[-a,a]} yd\nu(y) \right|^p d\nu(x) \\ &\leq \int_{[-a,a]^c} |x|^p d\nu(x) + 2^{p-1} \left[a^{-2p} \int_{[-a,a]} |x|^p d\nu(x) + \left| \int_{[-a,a]} x d\nu(x) \right|^p \right] \\ &\leq \int_{[-a,a]^c} |x|^p d\nu(x) + 2^{p-1} a^{-2p} \mathbb{E}[|X|^p] + 2^{p-1} \left| \int_{[-a,a]} x d\nu(x) \right|^p \\ &\leq 2^p \int_{[-a,a]^c} |x|^p d\nu(x) + 2^{p-1} a^{-2p} \mathbb{E}[|X|^p] \\ &\leq 2^p \left[a^{-m} \int_{[-a,a]^c} x^{p+m} d\nu(x) + a^{-2p} \mathbb{E}[|X|^p] \right] \\ &\leq 2^p [a^{-m} \mathbb{E}[|X|^{p+m}] + a^{-2p} \mathbb{E}[|X|^p]] \\ &\leq 2^{p+1} a^{-m} \mathbb{E}[|X|^{p+m}],\end{aligned}$$

similarly,

$$\mathbb{E}[\tau_{\nu,1}(X, t)^2] \leq 8a^{-l}\mathbb{E}[|X|^{2+l}].$$

Thus, there exists $C_{1,p,\nu}$ such that

$$\begin{aligned} \mathbb{E}[|\tau_{\nu,1}(S_n, t)|^p]^{1/p} &\leq C_{1,p}(\mathbb{E}[|X|^{p+m}]n^{1/p-1/2}a^{-m/p} + \mathbb{E}[|X|^{2+l}]^{1/2}a^{-l/2}) \\ &\leq C_{1,p,\nu}(t^{-m/2p} + t^{-l/4})n^{-1/2+(2-m)/2p}. \end{aligned}$$

Coming back to the quantity of interest, there exists $C'_{1,p,\nu}$ such that

$$\begin{aligned} \int_{\mathbb{R}^+} e^{-t} \|H_0\|_{p,\gamma^1} \mathbb{E}[|\tau_{\nu,1}(\cdot, t)|^p]^{1/p} dt &\leq \frac{1}{n} + \int_{t \geq \frac{1}{n}} e^{-t} C_{1,p,\nu} (t^{-m/2p} + t^{-l/4}) n^{-1/2+(2-m)/2p} dt \\ &\leq C'_{1,p,\nu} n^{-1/2+(2-m)/2p}. \end{aligned}$$

We now deal with $\tau_{\nu,2}$. Since $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] = 1$,

$$\begin{aligned} |\mathbb{E}[\tau_{\nu,2}(X, t)]| &= \left| -1 + \int_{[a,a]^2} \frac{1}{2}(y-x)^2 d\nu_X(x) d\nu_X(y) \right| \\ &= \left| -1 + \int_{[-a,a]} x^2 d\nu_X(x) - \left(\int_{[-a,a]} x d\nu_X(x) \right)^2 \right| \\ &= \left| - \int_{[-a,a]^c} x^2 d\nu_X(x) - \left(- \int_{[-a,a]^c} x d\nu_X(x) \right)^2 \right| \\ &\leq 2 \int_{[-a,a]^c} x^2 d\nu_X(x) \\ &\leq 2a^{-l} \mathbb{E}[|X|^{2+l}]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbb{E}[|\tau_{\nu,2}|^p] &= \mathbb{P}(X \in [-a, a]^c) + \int \left(\int_{[-a,a]} 1 - \frac{1}{2}(y-x)^2 d\nu_X(y) \right)^p d\nu_X(x) \\ &\leq 1 + \int_{[-a,a]^2} \left(1 - \frac{1}{2}(y-x)^2 \right)^p d\nu_X(y) d\nu_X(x) \\ &\leq 1 + \int_{[-a,a]^2} \left(2^{p-1} + \frac{(y-x)^{2p}}{2} \right) d\nu_X(y) d\nu_X(x) \\ &\leq (1 + 2^{p-1}) + 2^{p-1} \int_{[-a,a]} x^{2p} d\nu_X(x) \\ &\leq (1 + 2^{p-1}) + 2^{p-1} a^{p-m} \mathbb{E}[|X|^{p+m}] \\ &\leq 2^{p+1} a^{p-m} \mathbb{E}[|X|^{p+m}], \end{aligned}$$

and,

$$\mathbb{E}[\tau_{\nu,2}(X, t)^2] \leq 8a^{2-l} \mathbb{E}[|X_1|^{2+m}].$$

putting everything together, there exists $C_{2,p,\nu}$ such that,

$$\begin{aligned}\mathbb{E}[|\tau_{\nu^n,2}(S_n,t)|^p]^{1/p} &\leq C_{2,p,\nu}(a^{-l} + n^{1/p-1}a^{1-m/p} + n^{-1/2}a^{1-l/2}) \\ &\leq C_{2,p,\nu}((tn)^{-l/2} + n^{-1/2+(2-m)/2p}(t^{1-m/p} + t^{1-l/2})).\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^+} \frac{e^{-2t}}{\sqrt{1-e^{-2t}}} \|H_1\|_{p,\gamma^1} \mathbb{E}[|\tau_{\nu^n,2}(\cdot,t)|^p]^{1/p} dt &\leq \frac{1}{\sqrt{n}} + \\ &\int_{t \geq \frac{1}{n}} \frac{e^{-2t}}{\sqrt{1-e^{-2t}}} C_{2,p,\nu}((tn)^{-l/2} + n^{-1/2+(2-m)/2p}(t^{1-m/p} + t^{1-l/2})) dt.\end{aligned}$$

Since $\sqrt{1-e^{-2t}} \sim_{t \rightarrow 0} \sqrt{2t}$,

$$\int_{\mathbb{R}^+} \frac{e^{-2t}}{\sqrt{1-e^{-2t}}} \|H_1\|_{p,\gamma^1} \mathbb{E}[|\tau_{\nu^n,2}(\cdot,t)|^p]^{1/p} dt \leq C'_{2,p,\nu} n^{-1/2+(2-m)/2p}.$$

Similarly,

$$\begin{aligned}\mathbb{E}[\tau_{\nu,3}(X,t)] &= 0; \\ \mathbb{E}[|\tau_{\nu,3}(X,t)|^p] &\leq C \mathbb{E}[|X_1|^{p+m}] a^{2p-m}; \\ \mathbb{E}[\tau_{\nu,3}(X,t)^2] &\leq C \mathbb{E}[|X_1|^{2+l}] a^{4-l}; \\ \mathbb{E}[\tau_{\nu,4}(X,t)] &\leq C \mathbb{E}[|X_1|^{2+l}] a^{2-l}; \\ \mathbb{E}[|\tau_{\nu,4}(X,t)|^p] &\leq C \mathbb{E}[|X_1|^{p+m}] a^{3p-m}; \\ \mathbb{E}[\tau_{\nu,4}(X,t)^2] &\leq C \mathbb{E}[|X_1|^{2+l}] a^{6-l}.\end{aligned}$$

Hence, there exists $C'_{3,p,\nu}$ and $C'_{4,p,\nu}$ such that

$$\int_{\mathbb{R}^+} \frac{e^{-3t}}{1-e^{-2t}} \|H_2\|_{p,\gamma^1} \mathbb{E}[|\tau_{\nu^n,3}(S_n,t)|^p]^{1/p} dt \leq C'_{3,p,\nu} n^{-1/2+(2-m)/2p},$$

and

$$\int_{\mathbb{R}^+} \frac{e^{-4t}}{(1-e^{-2t})^{3/2}} \|H_3\|_{p,\gamma} \mathbb{E}[|\tau_{\nu^n,4}(S_n,t)|^p]^{1/p} dt \leq C'_{4,p,\nu} n^{-1/2+(2-m)/2p},$$

which concludes the proof.

4.4 Proof of Theorem 6

The proof of this Theorem is quite similar to the proof of Theorem 2, the main difference being using assumptions (i) and (ii) in order to replace Equations (4) and (5).

Suppose first that $\nu = h d\mu$ and consider some $t > 0$, as before we pose $v_t = \log(P_t h)$. Similarly to the proof of Theorem 2, the Fisher information of ν^t with respect to μ is

$$I_\mu(\nu^t) = \int_E P_t \|v_t\|_a^2 d\mu = \int_E \mathcal{L}(P_t v_t) d\mu,$$

and is finite by assumption. For $s > 0$, we pose

$$\begin{aligned}\tau_{\nu,1}(x, t) &= \frac{1}{s} \int_{\mathbb{R}^d} y \tau_{\nu}(x, y, t) - b(x); \\ \tau_{\nu,2}(x, t) &= \frac{1}{s} \int_{\mathbb{R}^d} \frac{y^{\otimes 2}}{2} \tau_{\nu}(x, y, t) - a(x); \\ \tau_{\nu,k}(x, t) &= \frac{1}{s} \int_{\mathbb{R}^d} \frac{y^{\otimes k}}{k!} \tau_{\nu}(x, y, t).\end{aligned}$$

Using assumption (iii), we can obtain an equivalent on E to Equation (3) and applying Cauchy-Schwartz inequality yields

$$\begin{aligned}I_{\mu}(\nu^t) &= \int_E \sum_{0 < k < i} \langle \nabla^k P_t v_t, \tau_{\nu,k} \rangle d\nu + \frac{1}{s} \langle D_{x,y}^i \phi, \frac{y^{\otimes i}}{(i-1)!} \rangle \\ &\leq I(\nu^t)^{1/2} \left(\sum_{0 < k < i} f_k(t) \|\tau_{\nu,k}\|_{a^{-1}} \|_{2,\nu} + \frac{1}{s} \left\| \int_{\mathbb{R}^d} f_i(y, t) \frac{\|y^{\otimes i}\|_{a^{-1}}}{i!} d\tau_{\nu}(\cdot, dy, t) \right\|_{2,\nu} \right),\end{aligned}$$

where the last line is obtained thanks to assumptions (i) – (ii). Finally, we conclude the proof for the continuous case by integrating $I(\nu^t)^{1/2}$ between 0 and ∞ .

From a limit argument similar to the ones use in Theorem 2, we can generalize this result to the case where ν does not admit a smooth density with respect to μ using assumption (iv).

4.5 Proof of Proposition 8

First, remark that assumption H_1 implies that \mathcal{L} satisfies a curvature-dimension inequality with constant ρ . Hence, according to Theorem 1.3 [1], ν^t has finite Entropy with respect to μ if ν has a finite second moment and, by Remark 7, finite Fisher information. Moreover, the existence of f_1 is directly given by Theorem 3.3.18 [3].

Let us now deal with the definition of f_2 . Using similar notations as previously, for some $M \geq 0$, let us define

$$\Lambda_M(s) = P_s((g'')^2 + M(g')^2).$$

Taking the derivative yields,

$$\begin{aligned}\Lambda'_M(s) &= 2P_s \left((g^{(3)})^2 + (M + u'')(g'')^2 + M u''(g')^2 + u^{(3)} g' g'' \right) \\ &\geq P_s \left((2u'' - \frac{|u^{(3)}|}{A})(g'')^2 + (2M u'' - A|u^{(3)}|)(g')^2 \right).\end{aligned}$$

By assumption H_2 and with an adequate choice of M , we have

$$\Lambda'_M(s) \geq \rho \Lambda_M(s),$$

and thus

$$\Lambda_M(0) \leq e^{-\rho s} \Lambda_M(s).$$

Therefore, for any $t > 0$,

$$\begin{aligned}
((P_t f)'')^2 &= \frac{\rho}{e^{\rho t} - 1} \int_0^t e^{\rho s} ((P_t f)'')^2 ds \\
&\leq \frac{\rho}{e^{\rho t} - 1} \int_0^t e^{\rho s} \Lambda_M(0) ds \\
&\leq \frac{\rho}{e^{\rho t} - 1} \int_0^t \Lambda_M(s) ds \\
&\leq \frac{\rho}{e^{\rho t} - 1} \int_0^t \frac{d}{ds} (P_s((g')^2)) + M P_s((g')^2) ds. \\
&\leq \frac{\rho}{e^{\rho t} - 1} (P_t((f')^2) + M t((P_t g)')^2) ds. \\
&\leq \frac{\rho}{e^{\rho t} - 1} (1 + t M e^{-2\rho t}) P_t((f')^2),
\end{aligned}$$

and we have obtained f_2 . With a similar treatment,

$$((P_t f)^{(3)})^2 \leq \frac{C_2 P_{t/2}((P_{t/2} f)')^2}{(e^{\frac{-\rho t}{4}} - 1)^2}.$$

Now, by Theorem 3.2.4 [3], we have $|(P_{t/2} f)'| \leq e^{\frac{-\rho t}{2}} P_{t/2} |f'|$, thus we can use the Harnack inequality from Theorem 5.6.1 [3], we obtain

$$P_{t/2}((P_{t/2} f)')^2 \leq e^{-\rho t + \frac{\rho y^2}{e^{\rho t} - 1}} P_t(f')^2,$$

which gives us the expression of f_3 and concludes the proof.

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