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# MONOKINETIC SOLUTIONS TO A SINGULAR VLASOV EQUATION FROM A SEMICLASSICAL PERSPECTIVE

RÉMI CARLES AND ANNE NOURI

ABSTRACT. Local in time mono-kinetic solutions to a singular one-dimensional Vlasov equation are obtained as the semiclassical limit of the Wigner transform associated to a logarithmic Schrödinger equation.

## 1. INTRODUCTION AND MAIN RESULT

This paper is concerned by the Cauchy problem for the Vlasov equation

$$(1.1) \quad \partial_t f + \xi \partial_x f - \lambda (\partial_x \ln(\rho)) \partial_\xi f = 0, \quad f(0, x, \xi) = f_0(x, \xi), \quad t > 0, (x, \xi) \in \mathbb{R}^2,$$

where  $\lambda > 0$  and  $\rho(t, x) = \int_{\mathbb{R}} f(t, x, \xi) d\xi$ . It arises in plasma physics, e.g. for quasineutral plasmas in the core or tokamaks when one focuses on the direction of the magnetic lines. There,  $f$  denotes the ionic distribution function and the electrons of the plasma are assumed adiabatic.

Due to the derivative of the density  $\rho$  with respect to space in the force term  $\partial_x \ln(\rho)$ , this equation is highly singular. The Cauchy problem can be proven to be well-posed for very specific initial data like mono-kinetic distribution functions of the form

$$f_0(x, \xi) = \rho_0(x) dx \otimes \delta_{\xi=\Phi'_0(x)},$$

with time-dependent mono-kinetic solutions of the form

$$f(t, x, \xi) = \rho(t, x) dx \otimes \delta_{\xi=\partial_x \Phi(t, x)},$$

with  $(\rho, \partial_x \Phi)$  solution to a system of isothermal Euler equations, with initial datum  $(\rho_0, \Phi'_0)$ . This paper addresses the existence of mono-kinetic solutions to (1.1), as limits of the Wigner transform of the solution to a logarithmic Schrödinger equation.

We construct a solution to (1.1) such that  $\rho_f$  remains bounded away from zero. To do so, we do not consider  $\rho \in H^s(\mathbb{R})$  the standard Sobolev space, or  $\rho \in \rho_* + H^s(\mathbb{R})$  for some  $\rho_* > 0$ , but rather Zhidkov spaces, as introduced in [19, 18], and further analyzed in [8] in the case of Schrödinger equations.

For  $s \geq 1$ , we set

$$X^s(\mathbb{R}) = \{f \in L^\infty(\mathbb{R}), \quad f' \in H^{s-1}(\mathbb{R})\}.$$

Note that since we work in dimension one, we have, for all  $s \geq 1$ ,  $H^s(\mathbb{R}) \subset X^s(\mathbb{R})$ , and  $X^s(\mathbb{R})$  is an algebra.

The main result of the paper is the following theorem.

**Theorem 1.1.** *Let  $\lambda > 0$  and  $s \geq 2$ . Suppose that  $(\rho_0, \Phi_0) \in X^s(\mathbb{R}) \times C(\mathbb{R})$  with  $\Phi'_0 \in X^s(\mathbb{R})$  and  $\rho_0(x) \geq \rho_{0*}$  for some positive constant  $\rho_{0*}$ .*

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There are  $T > 0$ ,  $\alpha > 0$ ,  $\rho \in C([0, T]; X^s(\mathbb{R}))$ ,  $\Phi \in C([0, T] \times \mathbb{R})$ , with

$$\partial_x \Phi \in C([0, T]; X^s(\mathbb{R})) \quad \text{and} \quad \rho(x) \geq \alpha,$$

such that

$$\mu = \rho(t, x)dx \otimes \delta_{\xi = \partial_x \Phi(t, x)} \quad \text{with} \quad (\rho, \Phi) \Big|_{t=0} = (\rho_0, \Phi_0),$$

is a measure solution to (1.1).

*Remark 1.2.* In (1.1), the term  $\partial_x \left( \ln \int \mu d\xi \right) \partial_\xi \mu$  can be considered in a weak sense as  $\partial_\xi \left( \partial_x \left( \ln \int \mu d\xi \right) \mu \right)$ , since  $\int \mu d\xi = \rho \in C([0, T]; C^1(\mathbb{R}))$  and  $\rho > 0$ .

From a fluid dynamics perspective, it is well-known that  $\mu = \rho(t, x)dx \otimes \delta_{\xi = \partial_x \Phi(t, x)}$  is a distributional solution of (1.1) if and only if its moments

$$\rho(t, x) = \int f(x, \xi) d\xi \quad \text{and} \quad \rho(t, x)u(t, x) = \int \xi f(t, x, \xi) d\xi$$

are solutions of the isothermal Euler system

$$(1.2) \quad \begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \rho) = 0. \end{cases}$$

For  $(\rho, u)$  with values in  $]0, +\infty[ \times \mathbb{R}$ , (1.2) is a strictly hyperbolic system. Consequently, for  $(\rho_0, u_0) \in (X^2(\mathbb{R}))^2$  with  $\rho_0 \geq \alpha$  for some  $\alpha > 0$ , there are  $T > 0$  and  $(\rho, u) \in (C^1([0, T] \times \mathbb{R}))^2$  solution to the Cauchy problem associated to (1.2) and the initial datum  $(\rho_0, u_0)$ . Note that the assumption on the initial density  $\rho_0$  is more general than merely  $\rho_0 \in \rho_{0*} + H^s(\mathbb{R})$  for some  $s \geq 2$ , since for instance,  $\rho_0$  may have different limits as  $x \rightarrow -\infty$  and  $x \rightarrow +\infty$ , or, even, no limit at all.

In this paper, we make an extra connection with a logarithmic Schrödinger equation, for which the above system corresponds to a limit system in the semiclassical regime.

The plan of the paper is the following. Section 2 recalls the main steps of the derivation of the model and related mathematical results. Section 3 studies the underlying Schrödinger equation. In Section 4, the convergence of the Wigner transform is performed. A link is made in Section 5 between the limiting system and the Cauchy problem (1.1).

## 2. DERIVATION OF THE MODEL AND RELATED RESULTS

In this section, we recall the main lines of the derivation of the model, used for studying fusion plasmas ([10]). The evolution of the ions in the core of such plasmas is well described by the Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + \frac{Ze}{m_i} (-\nabla_x \Phi + v \wedge B) \cdot \nabla_v f = 0,$$

where  $f$  is the ionic distribution function depending on time, position (in the domain  $\Omega$  of the plasma) and velocity (in  $\mathbb{R}^3$ ),  $Ze$  and  $m_i$  are the ion charge and mass respectively. The electric potential  $\Phi$  and the magnetic field  $B$  should be governed by the Maxwell equations. But a finite Larmor radius approximation is derived in the limit of a large

and uniform external magnetic field. This leads to the following equation for the ionic distribution function  $\bar{f}$  in gyro coordinates,

$$(2.1) \quad \partial_t \bar{f} + v_{\parallel} \partial_{x_{\parallel}} \bar{f} - J_{\rho_L}^0 (\partial_{x_{\parallel}} \Phi) \partial_{v_{\parallel}} \bar{f} - (J_{\rho_L}^0 \nabla_{x_{\perp}} \Phi)^{\perp} \cdot \nabla_{x_{\perp}} \bar{f} = 0.$$

Here, the index  $\parallel$  (resp.  $\perp$ ) refers to the direction parallel (resp. orthogonal) to the external magnetic field. For any vector  $u = (u_i) \in \mathbb{R}^3$ ,  $u^{\perp}$  denotes the vector  $(u_2, -u_1, 0)$ . The operator  $J_{\rho_L}^0$  is a Bessel operator performing averages on circles of Larmor radius  $\rho_L$  in planes orthogonal to the magnetic field. Since it is not used in this paper, we do not enter into more details about it. The electrons move quite more quickly than the ions, so that their density  $n_e$  is given in terms of the electric potential  $\Phi$  by the Maxwell-Boltzmann equation

$$(2.2) \quad n_e = n_0 e^{\frac{e}{T_e} (\Phi - \langle \Phi \rangle)}.$$

Here,  $e$  (resp.  $T_e$ ) is the electronic charge (resp. temperature), and  $\langle \Phi \rangle$  is the average of the potential on a magnetic field line. Due to the electroneutrality of the plasma, the Poisson equation is replaced by the electroneutrality equation

$$n_e = \rho,$$

where  $\rho$  is the ionic density. The operator  $J_{\rho_L}^0$  induces some regularity in the orthogonal direction, but none in the parallel direction. The two-dimensional dynamics in the direction perpendicular to the magnetic field is studied in [12]. In order to analyze the difficulty coming from the highly singular term  $J_{\rho_L}^0 (\partial_{x_{\parallel}} \Phi) \partial_{v_{\parallel}} \bar{f}$ , we restrict to a one-dimensional spatial setting, e.g. by considering ionic distribution functions written in the form

$$f(t, x, v) = f_{\parallel}(t, x_{\parallel}, v_{\parallel}) f_{\perp}(|v_{\perp}|),$$

with

$$\int_0^{+\infty} f_{\perp}(|v_{\perp}|) 2\pi |v_{\perp}| d|v_{\perp}| = 1.$$

Then the term  $f_{\perp}$  has no incidence in equation (2.1) and can be factorized. The equation that  $f_{\parallel}$  should solve is

$$\partial_t f_{\parallel} + v_{\parallel} \partial_{x_{\parallel}} f_{\parallel} - \lambda (\partial_{x_{\parallel}} \ln(\rho_{\parallel})) \partial_{v_{\parallel}} f_{\parallel} = 0, \quad t > 0, (x_{\parallel}, v_{\parallel}) \in \mathbb{R}^2,$$

i.e. the partial differential equation in (1.1) for  $f_{\parallel}$  (resp.  $\rho_{\parallel}, x_{\parallel}, v_{\parallel}$ ) denoted by  $f$  (resp.  $\rho, x, \xi$ ), and  $\lambda = \frac{T_e}{e}$ .

Mathematical results related to (1.1) have been obtained for a system close to equilibrium, i.e. in the case where the departure of the electric potential  $\Phi$  from its average along the magnetic lines  $\langle \Phi \rangle$  is small. Equation (2.2) simplifies into

$$n_e = n_0 \left( 1 + \frac{e}{T_e} (\Phi - \langle \Phi \rangle) \right),$$

so that (1.1) is replaced by

$$(2.3) \quad \partial_t f + \xi \partial_x f - \lambda (\partial_x \rho) \partial_{\xi} f = 0, \quad t > 0, (x, \xi) \in \mathbb{R}^2.$$

The Cauchy problem for (2.3) is locally well-posed for initial analytic data [13] but is ill-posed in the sense of Hadamard for regular initial data in Sobolev spaces and arbitrarily small time [2].

### 3. AN UNDERLYING LOGARITHMIC SCHRÖDINGER EQUATION

For  $\varepsilon > 0$ , consider the Cauchy problem for the Schrödinger equation

$$(3.1) \quad i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \partial_{xx}^2 u^\varepsilon = \lambda \ln(|u^\varepsilon|^2) u^\varepsilon, \quad u^\varepsilon(0, x) = \sqrt{\rho_0(x)} e^{i\Phi_0(x)/\varepsilon}.$$

Following the idea from [11], any function  $u^\varepsilon = a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$ , with  $(t, x) \mapsto a^\varepsilon(t, x) \in \mathbb{C}$  and  $(t, x) \mapsto \Phi^\varepsilon(t, x) \in \mathbb{R}$  solutions to the quasilinear problem

$$(3.2) \quad \partial_t \Phi^\varepsilon + \frac{(\partial_x \Phi^\varepsilon)^2}{2} + \lambda \ln(|a^\varepsilon|^2) = 0, \quad \Phi^\varepsilon(0, x) = \Phi_0(x),$$

$$(3.3) \quad \partial_t a^\varepsilon + \partial_x \Phi^\varepsilon \partial_x a^\varepsilon + \frac{a^\varepsilon}{2} \partial_{xx}^2 \Phi^\varepsilon = i \frac{\varepsilon}{2} \partial_{xx}^2 a^\varepsilon, \quad a^\varepsilon(0, x) = \sqrt{\rho_0(x)} =: a_0(x),$$

is a solution to (3.1). An important remark is that by allowing  $a^\varepsilon$  to be complex-valued (even though its initial datum is real-valued), one gains a degree of freedom to dispatch terms from (3.1) into (3.2)-(3.3), and the choice introduced by Grenier is much more robust than the Madelung transform when semiclassical limit is considered (see [3]).

**3.1. Constructing the solution.** Determining  $\Phi^\varepsilon$  solution to (3.2) turns out to be equivalent to determining  $v^\varepsilon = \partial_x \Phi^\varepsilon$  and  $a^\varepsilon$  solution to

$$(3.4) \quad \begin{cases} \partial_t v^\varepsilon + v^\varepsilon \partial_x v^\varepsilon + \lambda \partial_x \ln(|a^\varepsilon|^2) = 0, & v^\varepsilon(0, x) = \Phi_0'(x), \\ \partial_t a^\varepsilon + v^\varepsilon \partial_x a^\varepsilon + \frac{a^\varepsilon}{2} \partial_x v^\varepsilon = i \frac{\varepsilon}{2} \partial_{xx}^2 a^\varepsilon, & a^\varepsilon(0, x) = a_0(x). \end{cases}$$

Indeed, given  $(v^\varepsilon, a^\varepsilon)$  solution to (3.4), we can define  $\Phi^\varepsilon$  by

$$(3.5) \quad \Phi^\varepsilon(t, x) = \Phi_0(x) - \int_0^t \left( \frac{1}{2} |v^\varepsilon(\tau, x)|^2 + \lambda \ln(|a^\varepsilon(\tau, x)|^2) \right) d\tau.$$

We check that

$$\partial_t (\partial_x \Phi^\varepsilon - v^\varepsilon) = \partial_x \partial_t \Phi^\varepsilon - \partial_t v^\varepsilon = 0,$$

so that  $v^\varepsilon = \partial_x \Phi^\varepsilon$  and  $\Phi^\varepsilon$  solves (3.2), and  $a^\varepsilon$  solves (3.3).

Note that for fixed  $\varepsilon > 0$ , the Cauchy problem for (3.1) has been considered in [5] (see also [4, Section 9.1]), for initial data in the class

$$W = \left\{ f \in H^1(\mathbb{R}), \quad \int_{\mathbb{R}} |f(x)|^2 |\ln |f(x)|| dx < \infty \right\}.$$

This class is not compatible with the assumption  $\rho(x) \geq \rho_{0*} > 0$  from Theorem 1.1, which is equivalent to  $|u^\varepsilon(0, x)|^2 = |a_0(x)|^2 \geq \rho_{0*} > 0$  in the approach that we follow. Therefore, we choose to rather work in Zhidkov spaces  $X^s(\mathbb{R})$ . The system (3.4) has a unique smooth solution as stated in the following proposition, which includes the case  $\varepsilon = 0$ .

**Proposition 3.1.** *Let  $s > 5/2$  and  $\lambda > 0$ . Suppose that  $\rho_0, \Phi_0' \in X^s(\mathbb{R})$ , with*

$$\rho_0(x) \geq \rho_{0*} > 0.$$

*Then there exists  $T$  independent of  $s > 5/2$  and  $\varepsilon \in [0, 1]$ , and a unique solution  $(a^\varepsilon, v^\varepsilon) \in C([0, T]; X^s \times X^s)$  to (3.4).*

*Proof.* This result is a rather direct consequence of [1, Proposition 2.1], whose proof we recall the main idea. Separate real and imaginary parts of  $a^\varepsilon$ ,  $a^\varepsilon = a_1^\varepsilon + ia_2^\varepsilon$ , and introduce

$$\mathbf{u}^\varepsilon = \begin{pmatrix} a_1^\varepsilon \\ a_2^\varepsilon \\ v^\varepsilon \end{pmatrix}, \quad \mathbf{u}_0 = \begin{pmatrix} \sqrt{\rho_0} \\ 0 \\ \Phi_0' \end{pmatrix}, \quad L = \begin{pmatrix} 0 & -\partial_{xx}^2 & 0 \\ \partial_{xx}^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } A(\mathbf{u}) = \begin{pmatrix} v & 0 & \frac{a_1}{2} \\ 0 & v & \frac{a_2}{2} \\ \frac{2\lambda a_1}{a_1^2 + a_2^2} & \frac{2\lambda a_2}{a_1^2 + a_2^2} & v \end{pmatrix}.$$

We now have the system:

$$(3.6) \quad \partial_t \mathbf{u}^\varepsilon + A(\mathbf{u}^\varepsilon) \partial_x \mathbf{u}^\varepsilon = \frac{\varepsilon}{2} L \mathbf{u}^\varepsilon \quad ; \quad \mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}_0.$$

Since  $\rho_0$  is bounded away from zero, its square root is also in  $X^s$ , so that  $\mathbf{u}_0 \in X^s(\mathbb{R})^3$ . The matrix  $A$  is symmetrized by the matrix

$$S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{a_1^2 + a_2^2}{4\lambda} \end{pmatrix},$$

which is symmetric positive if and only if  $a_1^2 + a_2^2 > 0$ , that is, so long as no vacuum appears. By assumption,

$$(a_1^\varepsilon)^2 + (a_2^\varepsilon)^2 \Big|_{t=0} \geq \rho_{0*} > 0.$$

Then the main idea is that the operator  $L$  is skew-symmetric, and so does not appear in  $L^2$ -based energy estimates. Standard tame estimates (see e.g. [16, 17]) do not involve the  $L^2$  norm of  $\mathbf{u}^\varepsilon$ , and so the only aspect remaining is that  $L^\infty$ -estimates can be obtained rather directly. So long as, say,

$$(3.7) \quad (a_1^\varepsilon(t, x))^2 + (a_2^\varepsilon(t, x))^2 \geq \frac{\rho_{0*}}{2}, \quad \forall x \in \mathbb{R},$$

we have:

$$\begin{aligned} \|\mathbf{u}^\varepsilon(t)\|_{L^\infty} &\leq \|\mathbf{u}_0\|_{L^\infty} + \int_0^t \|A(\mathbf{u}^\varepsilon(\tau)) \partial_x \mathbf{u}^\varepsilon(\tau)\|_{L^\infty} d\tau + \int_0^t \|\partial_{xx}^2 a^\varepsilon(\tau)\|_{L^\infty} d\tau \\ &\leq \|\mathbf{u}_0\|_{L^\infty} + C \int_0^t \|\mathbf{u}^\varepsilon(\tau)\|_{L^\infty} \|\partial_x \mathbf{u}^\varepsilon(\tau)\|_{L^\infty} d\tau + \int_0^t \|\partial_{xx}^2 a^\varepsilon(\tau)\|_{L^\infty} d\tau \\ &\leq \|\mathbf{u}_0\|_{L^\infty} + C \int_0^t \|\mathbf{u}^\varepsilon(\tau)\|_{X^s}^2 d\tau + \int_0^t \|\mathbf{u}^\varepsilon(\tau)\|_{X^s} d\tau, \end{aligned}$$

where we have used Sobolev embedding under the assumption  $s > 5/2$ . Indeed, by definition,  $X^s \subset L^\infty$ , and for  $\mathbf{u}^\varepsilon \in X^s$ ,  $\partial_x \mathbf{u}^\varepsilon \in H^{s-1} \subset L^\infty$ , provided that  $s > 3/2$ , and similarly,  $\partial_{xx}^2 \mathbf{u}^\varepsilon \in L^\infty$  for  $s > 5/2$ .

Now we set  $P = (I - \partial_{xx}^2)^{(s-1)/2} \partial_x$ , so that  $\|f\|_{X^s} \approx \|f\|_{L^\infty} + \|Pf\|_{L^2}$ , and denote by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx,$$

the scalar product in  $L^2$ . Since  $L$  is skew-symmetric and  $S$  is real-valued,

$$\begin{aligned} & \frac{d}{dt} \langle SP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t) \rangle \\ &= \langle (\partial_t S)P\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t) \rangle + 2 \operatorname{Re} \langle S\partial_t P\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t) \rangle \\ &= \langle (\partial_t S)P\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t) \rangle + \varepsilon \operatorname{Re} \langle SLP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t) \rangle \\ &\quad - 2 \operatorname{Re} \left\langle SP \left( A(u^\varepsilon(t)) \partial_x u^\varepsilon(t) \right), P\mathbf{u}^\varepsilon(t) \right\rangle. \end{aligned}$$

So long as (3.7) holds, we have the following set of estimates. First,

$$\begin{aligned} \langle (\partial_t S)P\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t) \rangle &\leq \|\partial_t S\|_{L^\infty} \|P\mathbf{u}^\varepsilon(t)\|_{L^2}^2 \\ &\leq C (\|\mathbf{u}^\varepsilon(t)\|_{L^\infty}) \|\partial_t \mathbf{u}^\varepsilon(t)\|_{L^\infty} \|\mathbf{u}^\varepsilon(t)\|_{X^s}^2. \end{aligned}$$

Directly from (3.6), we have:

$$\begin{aligned} \|\partial_t \mathbf{u}^\varepsilon(t)\|_{L^\infty} &\leq C (\|\mathbf{u}^\varepsilon(t)\|_{L^\infty}) \|\partial_x \mathbf{u}^\varepsilon(t)\|_{L^\infty} + \|\partial_{xx}^2 a^\varepsilon(t)\|_{L^\infty} \\ &\leq C (\|\mathbf{u}^\varepsilon(t)\|_{X^s}) \|\mathbf{u}^\varepsilon(t)\|_{X^s}. \end{aligned}$$

Since  $SL$  is skew-symmetric, we have

$$\operatorname{Re} \langle SLP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t) \rangle = 0,$$

which prevents any loss of regularity in the estimates. For the quasi-linear term involving the matrix  $A$ , we note that since  $SA$  is symmetric, commutator estimates (see [14]) yield:

$$\begin{aligned} \langle SP(A(\mathbf{u}^\varepsilon)\partial_x \mathbf{u}^\varepsilon), P\mathbf{u}^\varepsilon(t) \rangle &\leq C (\|\mathbf{u}^\varepsilon(t)\|_{L^\infty}) \|P\mathbf{u}^\varepsilon(t)\|_{L^2}^2 \|\partial_x \mathbf{u}^\varepsilon(t)\|_{L^\infty} \\ &\leq C (\|\mathbf{u}^\varepsilon(t)\|_{X^s}) \|P\mathbf{u}^\varepsilon(t)\|_{L^2}^2. \end{aligned}$$

Finally, we have:

$$\frac{d}{dt} \langle SP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t) \rangle \leq C (\|\mathbf{u}^\varepsilon(t)\|_{X^s}) \|\mathbf{u}^\varepsilon(t)\|_{X^s}^2.$$

This estimate, along with the  $L^\infty$ -estimate, shows that on a sufficiently small time interval  $[0, T]$ , with  $T > 0$  independent of  $\varepsilon \in [0, 1]$ , (3.7) holds, hence the existence of a unique solution. The fact that the local existence time does not depend on  $s > 5/2$  follows from the continuation principle based on Moser's calculus and tame estimates (see e.g. [16, Section 2.2] or [17, Section 16.1]).  $\square$

**Corollary 3.2.** *Under the assumptions of Proposition 3.1, if we suppose in addition that  $\Phi'_0 \in L^2(\mathbb{R})$ , then (3.1) has a unique solution  $u^\varepsilon \in C([0, T]; X^s(\mathbb{R}))$ , where  $T$  is given by Proposition 3.1.*

*Proof.* From Proposition 3.1, (3.4) has a solution  $(v^\varepsilon, a^\varepsilon) \in C([0, T]; X^s \times X^s)$ . Plugging this information into (3.4), we infer

$$\begin{aligned} \|v^\varepsilon(t)\|_{L^2} &\leq \|\Phi'_0\|_{L^2} + \int_0^t \|v^\varepsilon(\tau)\|_{L^\infty} \|\partial_x v^\varepsilon(\tau)\|_{L^2} d\tau \\ &\quad + C \int_0^t \left\| \frac{1}{a^\varepsilon(\tau)} \right\|_{L^\infty} \|\partial_x a^\varepsilon(\tau)\|_{L^2} d\tau. \end{aligned}$$

Therefore,  $v^\varepsilon \in C([0, T]; L^2)$ , and  $\Phi^\varepsilon$ , stemming from  $v^\varepsilon$  via the formula (3.5), satisfies  $\Phi^\varepsilon \in C([0, T]; X^{s+1})$ . The existence part follows readily, since  $X^s(\mathbb{R})$  is an algebra, by setting  $u^\varepsilon = a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$ .

For the uniqueness property, consider two such solutions  $u^\varepsilon, \tilde{u}^\varepsilon \in C([0, T]; X^s)$ , and set  $w^\varepsilon = u^\varepsilon - \tilde{u}^\varepsilon$ . It satisfies

$$(3.8) \quad i\varepsilon \partial_t w^\varepsilon + \frac{\varepsilon^2}{2} \partial_{xx}^2 w^\varepsilon = \lambda \left( \ln(|u^\varepsilon|^2) u^\varepsilon - \ln(|\tilde{u}^\varepsilon|^2) \tilde{u}^\varepsilon \right), \quad w^\varepsilon|_{t=0} = 0.$$

Recall the pointwise estimate from [5] (see also [4, Lemma 9.3.5]),

$$\left| \operatorname{Im} \left( \ln(|u^\varepsilon|^2) u^\varepsilon - \ln(|\tilde{u}^\varepsilon|^2) \tilde{u}^\varepsilon \right) (u^\varepsilon - \tilde{u}^\varepsilon) \right| \leq 4|u^\varepsilon - \tilde{u}^\varepsilon|^2.$$

Multiply (3.8) by  $\overline{w^\varepsilon}$ , integrate on an interval  $I = [M_-, M_+]$ , and take the imaginary part. This yields, along with the above estimate,

$$\frac{\varepsilon}{2} \frac{d}{dt} \int_I |w^\varepsilon(t, x)|^2 dx + \frac{\varepsilon^2}{2} \operatorname{Im} \int_I \overline{w^\varepsilon} \partial_{xx}^2 w^\varepsilon \leq 4\lambda \int_I |w^\varepsilon(t, x)|^2 dx.$$

We have, by integration by parts,

$$\operatorname{Im} \int_I \overline{w^\varepsilon} \partial_{xx}^2 w^\varepsilon = \operatorname{Im} \overline{w^\varepsilon}(t, M_+) \partial_x w^\varepsilon(t, M_+) - \operatorname{Im} \overline{w^\varepsilon}(t, M_-) \partial_x w^\varepsilon(t, M_-).$$

Since  $w^\varepsilon \in C([0, T]; X^1)$ , we can choose sequences  $M_\pm^n \rightarrow \pm\infty$  along which the above term goes to zero, and the Gronwall lemma implies  $\|w^\varepsilon(t)\|_{L^2} \equiv 0$ .  $\square$

**3.2. Asymptotic expansion.** Proposition 3.1 with  $\varepsilon = 0$  yields the existence of a unique solution  $(v, a) \in C([0, T]; (X^s(\mathbb{R}))^2)$  to

$$(3.9) \quad \begin{cases} \partial_t v + v \partial_x v + \lambda \partial_x \ln(|a|^2) = 0, & v|_{t=0} = \Phi'_0, \\ \partial_t a + v \partial_x a + \frac{a}{2} \partial_x v = 0, & a|_{t=0} = \sqrt{\rho_0}. \end{cases}$$

As a direct consequence of Proposition 3.1 and [1, Proposition 3.1], we have:

**Proposition 3.3.** *Under the assumptions of Proposition 3.1, there exists  $C$  independent of  $\varepsilon \in [0, 1]$  such that*

$$\|\partial_x (\Phi^\varepsilon - \Phi)\|_{L^\infty([0, T]; X^{s-2})} + \|a^\varepsilon - a\|_{L^\infty([0, T]; X^{s-2})} \leq C\varepsilon.$$

#### 4. WIGNER TRANSFORM

The Wigner transform of  $u^\varepsilon$ , solution to (3.1), is defined by (see e.g. [9, 15])

$$(4.1) \quad W^\varepsilon(t, x, \xi) = \int_{\mathbb{R}} e^{iy\xi} u^\varepsilon\left(t, x - \frac{\varepsilon}{2}y\right) \overline{u^\varepsilon}\left(t, x + \frac{\varepsilon}{2}y\right) dy, \quad (t, x, \xi) \in [0, T] \times \mathbb{R}^2.$$

For  $u^\varepsilon = a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$ , with  $(\Phi^\varepsilon, a^\varepsilon)$  solution to (3.2)-(3.3), or equivalently  $(\partial_x \Phi^\varepsilon, a^\varepsilon)$  solution to (3.4), we have

$$(4.2) \quad W^\varepsilon(t, x, \xi) = \int e^{i\xi v} a^\varepsilon\left(t, x - \frac{\varepsilon}{2}y\right) \overline{a^\varepsilon}\left(t, x + \frac{\varepsilon}{2}y\right) e^{i\varphi^\varepsilon(t, x, y)/\varepsilon} dy,$$

where

$$\varphi^\varepsilon(t, x, y) = \Phi^\varepsilon\left(t, x - \frac{\varepsilon}{2}y\right) - \Phi^\varepsilon\left(t, x + \frac{\varepsilon}{2}y\right).$$



**Theorem 4.1.** *When  $\varepsilon \rightarrow 0$ , the Wigner transform  $W^\varepsilon$  of  $u^\varepsilon$  weakly converges to the bounded measure*

$$(4.3) \quad \mu(t, dx, d\xi) = |a(t, x)|^2 dx \otimes \delta_{\xi = \partial_x \Phi(t, x)},$$

where  $(\partial_x \Phi, a)$  is a solution of (3.9). Moreover,  $\mu$  is a solution to (1.1).

*Proof.* In view of Proposition 3.3,  $a^\varepsilon = a + r_a^\varepsilon$  and  $\partial_x \Phi^\varepsilon = \partial_x \Phi + r_v^\varepsilon$ , with

$$\|r_a^\varepsilon\|_{L^\infty([0, T]; X^{s-2})} + \|r_v^\varepsilon\|_{L^\infty([0, T]; X^{s-2})} \leq C\varepsilon.$$

Therefore,

$$W^\varepsilon(t, x, \xi) = \int e^{iy(\xi - \partial_x \Phi(t, x))} a\left(t, x - \frac{\varepsilon}{2}y\right) \bar{a}\left(t, x + \frac{\varepsilon}{2}y\right) dy + R_1^\varepsilon + R_2^\varepsilon + R_3^\varepsilon,$$

where

$$\begin{aligned} R_j^\varepsilon(t, x, \xi) &= \int e^{iy(\xi - \partial_x \Phi(t, x))} r_j^\varepsilon(t, x, y) dy, \quad 1 \leq j \leq 3, \\ r_1^\varepsilon(t, x, y) &= a^\varepsilon\left(t, x - \frac{\varepsilon}{2}y\right) \bar{a}^\varepsilon\left(t, x + \frac{\varepsilon}{2}y\right) \left(e^{i(\varphi^\varepsilon(t, x, y) + \varepsilon \partial_x \Phi(t, x))/\varepsilon} - 1\right), \\ r_2^\varepsilon(t, x, y) &= \bar{a}\left(t, x + \frac{\varepsilon}{2}y\right) r_a^\varepsilon\left(t, x - \frac{\varepsilon}{2}y\right) + a\left(t, x - \frac{\varepsilon}{2}y\right) \bar{r}_a^\varepsilon\left(t, x + \frac{\varepsilon}{2}y\right), \\ r_3^\varepsilon(t, x, y) &= r_a^\varepsilon\left(t, x - \frac{\varepsilon}{2}y\right) \bar{r}_a^\varepsilon\left(t, x + \frac{\varepsilon}{2}y\right). \end{aligned}$$

Propositions 3.1 and 3.3 yield, along with Taylor's formula for the term  $r_1^\varepsilon$ ,

$$\|r_j^\varepsilon\|_{L^\infty([0, T] \times \mathbb{R}^2)} \leq C\varepsilon, \quad 1 \leq j \leq 3.$$

Consequently,  $W^\varepsilon$  tends to  $|a|^2 dx \otimes \delta_{\xi = \partial_x \Phi}$  in  $\mathcal{M}_b([0, T] \times \mathbb{R}^2)$  when  $\varepsilon$  tends to zero. Moreover, denote by  $(\cdot, \cdot)$  the duality between bounded measures on  $[0, T] \times \mathbb{R}^2$  and continuous functions with compact support in  $[0, T] \times \mathbb{R}^2$ . For any test function  $\alpha(t, x, \xi) \in C^1$  with compact support in  $[0, T] \times \mathbb{R}^2$ , it holds

$$\begin{aligned} &(\mu, \partial_t \alpha + \xi \partial_x \alpha - \lambda (\partial_x \ln |a|^2) \partial_\xi \alpha) \\ &= \int |a(t, x)|^2 (\partial_t \alpha(t, x, \partial_x \Phi(t, x)) + \partial_x \Phi(t, x) \partial_x \alpha(t, x, \partial_x \Phi(t, x))) dx dt \\ &\quad - \lambda \int |a(t, x)|^2 ((\partial_x \ln |a|^2) \partial_\xi \alpha(t, x, \partial_x \Phi(t, x))) dx dt \\ &= \int |a|^2 (\partial_t (\alpha(t, x, v(t, x))) - \partial_T v \partial_\xi \alpha(t, x, v(t, x))) dx dt \\ &+ \int |a|^2 (\partial_x (v \alpha(t, x, v(t, x))) - \partial_x v \alpha(t, x, v) - v \partial_x v \partial_\xi \alpha(t, x, v)) dx dt \\ &\quad - \lambda \int |a|^2 ((\partial_x \ln |a|^2) \partial_\xi \alpha(t, x, v(t, x))) dx dt \\ &= - \int \alpha(t, x, v) (\partial_t |a|^2 + v \partial_x |a|^2 + |a|^2 \partial_x v) dx dt \\ &\quad - \int |a|^2 \partial_\xi \alpha(t, x, v) (\partial_t v + v \partial_x v + \lambda \partial_x \ln |a|^2) dx dt. \end{aligned}$$

This is zero, in view of (3.9), since

$$\partial_t |a|^2 + \partial_x \Phi \partial_x |a|^2 + |a|^2 \partial_{xx}^2 \Phi = 2 \operatorname{Re} \bar{a} \left( \partial_t a + \partial_x \Phi \partial_x a + \frac{a}{2} \partial_{xx}^2 \Phi \right) = 0.$$

Therefore, any solution to (3.9) yields a solution to (1.1).  $\square$

## 5. THE LIMITING SYSTEM AND GLOBAL CAUCHY PROBLEM

As observed above, to get a solution to (1.1), it suffices to have a solution to (3.9). With data bounded away from vacuum as in Proposition 3.1, [3, Theorem 2.3] remains valid:

**Proposition 5.1** (From Theorem 2.3 in [3]). *Let  $\rho_0, \Phi_0$  as in Proposition 3.1, with  $s > 3/2$ . There exists a unique maximal solution  $(v, a) \in C([0, T_{\max}); X^s)$  to (3.9). In addition,  $T_{\max}$  is independent of  $s > 3/2$  and*

$$T_{\max} < \infty \implies \int_0^{T_{\max}} \|(v, a)(t)\|_{W^{1,\infty}} dt = \infty.$$

The assumption  $s > 5/2$  has been changed to the weaker one  $s > 3/2$ , since for  $\varepsilon = 0$ , it is no longer necessary to control  $\partial_{xx}^2 a^\varepsilon$  in  $L^\infty$  as in the first step of the proof of Proposition 3.1.

We infer that on  $[0, T_{\max})$ , vacuum does not appear, that is the density  $\rho$  remains positive:

**Lemma 5.2.** *Under the assumptions of Proposition 3.1, we have*

$$\rho(t, x) > 0, \quad \forall (t, x) \in [0, T_{\max}) \times \mathbb{R}.$$

*Proof.* Define the characteristics  $x(t, y)$  by

$$\frac{d}{dt}x(t, y) = v(t, x(t, y)), \quad x(0, y) = y,$$

where  $v$  stems from Proposition 3.1 (case  $\varepsilon = 0$ ). For  $0 \leq t < T_{\max}$ , this is a global diffeomorphism of  $\mathbb{R}$ , in view of Proposition 5.1,

$$\frac{d}{dt}\partial_y x(t, y) = \partial_x v(t, x(t, y)) \partial_y x(t, y), \quad \partial_y x(0, y) = 1,$$

and the Gronwall lemma. Denote by  $x \mapsto y(t, x)$  its inverse mapping. We check that in (3.9),  $a$  is given by the formula

$$a(t, z) = \frac{1}{\sqrt{\partial_y x(t, y(t, z))}} a_0(y(t, z)),$$

that is,

$$\rho(t, z) = \frac{1}{\partial_y x(t, y(t, z))} \rho_0(y(t, z)),$$

so vacuum cannot appear for  $t \in [0, T_{\max})$ .  $\square$

*Remark 5.3.* Although the Cauchy problem for the isothermal Euler (1.2) has a global in time entropy weak solution  $(\rho, u) \in L^\infty(\mathbb{R}_+ \times \mathbb{R})^2$  (see [6, 7]), we cannot use it for our purpose. Indeed, the momentum equation in (1.2) is obtained from the kinetic equation (1.1) by multiplying (1.1) by  $\xi$  and integrating the resulting equation with respect to  $\xi$ . This leads to the product

$$\rho \partial_x \ln(\rho).$$

Since it is not under a conservative form, it is well known that there is no rigorous way to give a sense to this product for  $\rho \in L^\infty$ . It is why we have had recourse to regular solutions to the isothermal Euler system, thus restricting to local in time solutions far from vacuum.

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