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# An application of the KMT construction to the pathwise weak error in the Euler approximation of the geometric Brownian motion

Emmanuelle Clément\*      Arnaud Gloter†

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## Abstract

It is well known that the strong error approximation, in the space of continuous paths equipped with the supremum norm, between a diffusion process, with smooth coefficients, and its Euler approximation with step  $1/n$  is  $O(n^{-1/2})$  and that the weak error estimation between the marginal laws, at the terminal time  $T$ , is  $O(n^{-1})$ . An analysis of the weak trajectorial error has been developed by Alfonsi, Jourdain and Kohatsu-Higa [1], through the study of the  $p$ -Wasserstein distance between the two processes. For a one-dimensional diffusion, they obtained an intermediate rate for the pathwise Wasserstein distance of order  $n^{-2/3+\varepsilon}$ . Using the Komlós, Major and Tusnády construction, we improve this bound in the case of the geometric Brownian motion and we obtain a rate of order  $\log n/n$ .

**MSC 2010.** 65C30, 60H35.

**Key words:** Diffusion process, Euler scheme, Wasserstein couplings, Komlós-Major-Tusnády construction.

## 1 Introduction

A classical problem in numerical probabilities is the computation of  $Ef(X)$ , where  $X = (X_t)_{t \in [0,1]}$  is a stochastic process defined on the time interval  $[0, 1]$  and  $f$  a functional which may depend on the whole path of the process  $X$ . This problem appears for instance in finance where  $X$  represents the

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dynamic of a stock price and  $f$  the payoff of an option. The usual way to solve this problem is to approximate  $X$  by a numerical scheme and then to compute the expectation by using a Monte Carlo method.

Due to its implementation easiness, the most popular discretization scheme, when  $X$  is a diffusion process, is the Euler scheme. Denoting by  $X^n$ , the Euler approximation of  $X$  with step  $1/n$ , it is well known that the pathwise strong order of convergence between  $X$  and  $X^n$  is  $n^{-1/2}$ , under regularity assumptions on the coefficients of the diffusion  $X$ . Moreover the weak order of convergence at a fixed time  $t$ , evaluated by the difference  $|Ef(X_t) - Ef(X_t^n)|$ , is  $n^{-1}$ . However, for the pathwise weak approximation of  $X$  (when  $f$  depends on the whole trajectory of  $X$ ), the order of convergence is still unknown, excepted for specific functionals  $f$  such as  $f(X) = \int_0^1 X_s ds$  or  $f(X) = \max_s X_s$ . Recently Alfonsi, Jourdain and Kohatsu-Higa [1], have proposed a general approach to control the pathwise weak approximation of a diffusion by its Euler scheme by considering the Wasserstein distance between the law of  $X$  and the law of  $X^n$ .

For  $X$  and  $\bar{X}$ , two random variables with values in a normed vector space  $(\mathcal{X}, \|\cdot\|)$  the representations of the Wasserstein distance in the Kantorovitch duality ( see for example [10] ) are given by:

$$\mathcal{W}_1(X, \bar{X}) = \sup_{f \in \mathcal{L}(1)} |Ef(X) - Ef(\bar{X})| = \inf_{(Y, \bar{Y}) \in \Pi(Y, \bar{Y})} E\|Y - \bar{Y}\|$$

where  $\mathcal{L}(1)$  is the set of Lipschitz functions with Lipschitz constant less than 1, and  $\Pi(Y, \bar{Y})$  is the set of random variables  $(Y, \bar{Y})$  with values in  $\mathcal{X} \times \mathcal{X}$  with marginal laws respectively  $X$  and  $\bar{X}$ . More generally, for  $p \geq 1$ , the  $p$ -Wasserstein distance between the law of  $X$  and the law of  $\bar{X}$  is defined by :

$$\mathcal{W}_p(X, \bar{X}) = \inf_{(Y, \bar{Y}) \in \Pi(X, \bar{X})} E^{1/p} \|Y - \bar{Y}\|^p. \quad (1)$$

In our context,  $\mathcal{X} = \mathcal{C}([0, 1])$  equipped with the supremum norm or  $\mathcal{X} = \mathbb{R}^n$ .

From these representations and using the strong and weak orders of convergence of the Euler scheme, one can easily deduce the following upper and lower bounds :

$$\frac{c}{n} \leq \mathcal{W}_1(X, X^n) \leq \frac{C}{\sqrt{n}},$$

for some positive constants  $c$  and  $C$ . For a one-dimensional diffusion, the main result of [1], is to construct a coupling between  $X$  and  $X^n$  which improves the preceding upper bound and leads to :

$$\mathcal{W}_1(X, X^n) \leq \frac{C}{n^{\frac{2}{3}-\varepsilon}},$$

for all  $\varepsilon > 0$ . This result gives an intermediate rate, for the pathwise weak approximation, between the strong order rate and the weak marginal rate and raises a natural question : is it possible to construct a coupling between a diffusion and its Euler scheme in such a way that the Wasserstein distance is of order  $1/n$  ?

The aim of this paper is to give an answer to this question in the case of the geometric Brownian motion. Considering respectively  $X$  a geometric Brownian motion and  $X^n$  its Euler approximation, we prove that (see Theorem 2) :

$$\mathcal{W}_1(X, X^n) \leq C \frac{\log n}{n}.$$

This result is obtained by the construction of a sharp discrete time coupling between  $(X_{k/n})_{1 \leq k \leq n}$  and  $(X_{k/n}^n)_{1 \leq k \leq n}$ , following the dyadic construction due to Komlós, Major and Tusnády ([6] [7]). We mention that recently, the KMT construction has been used in a series of papers (Davie [2], [3], Flint and Lyons [5]), to propose an approximation scheme, close to the Milstein scheme and with weak pathwise order of convergence  $1/n$ .

The KMT construction permits essentially to obtain an optimal coupling between a sequence of i.i.d. standard gaussian variables  $(Y_i)_{1 \leq i \leq n}$  and some other i.i.d. variables  $(X_i)_{1 \leq i \leq n}$  with finite Laplace transform in a neighbourhood of zero and such that  $EX_1 = 0$ ,  $VX_1 = 1$ , in such a way that almost surely :

$$\max_{1 \leq k \leq n} \left| \sum_{i=1}^k Y_i - \sum_{i=1}^k X_i \right| \leq C \log n.$$

In Section 2, we improve the KMT result in the case where the variables  $X_i$  are equal in law to  $Y_i - \frac{1}{2\sqrt{n}}(Y_i^2 - 1)$ . In this particular case, we obtain, as a consequence of the Proposition 1 below, that almost surely :

$$\max_{1 \leq k \leq n} \left| \sum_{i=1}^k Y_i - \sum_{i=1}^k X_i \right| \leq C \log n / \sqrt{n}.$$

This is done through refined quantile coupling inequalities, which are established at the end of the paper, in Section 4. These results are applied in Section 3 to construct a coupling between the geometric Brownian motion and its Euler approximation which achieves the pathwise weak order  $\log n/n$ .

## 2 A KMT type result

Let  $(Y_i)_{i \geq 1}$  be a sequence of i.i.d. standard gaussian variables and let us consider the triangular array:

$$\bar{Y}_i^n = Y_i - \frac{1}{2\sqrt{n}}(Y_i^2 - 1), \quad 1 \leq i \leq n. \quad (2)$$

We set :  $S_k = \sum_{i=1}^k Y_i$  and  $\bar{S}_k^n = \sum_{i=1}^k \bar{Y}_i^n$ .

Our aim is to give an upper bound for the Wasserstein distance  $\mathcal{W}_p((S_k)_{1 \leq k \leq n}, (\bar{S}_k^n)_{1 \leq k \leq n})$ . We obtain the following result.

**Theorem 1** *For all  $p \geq 1$ , there exists a positive constant  $C$  such that :*

$$\mathcal{W}_p((S_k)_{1 \leq k \leq n}, (\bar{S}_k^n)_{1 \leq k \leq n}) \leq C \frac{\log n}{\sqrt{n}}.$$

The result of Theorem 1 is a consequence of the Proposition 1 below, which is derived from the KMT construction (see [6] , [7]).

**Proof** Let  $(S_k)_{1 \leq k \leq n}$  and  $(\bar{S}_k^n)_{1 \leq k \leq n}$  be constructed as in Proposition 1. From the definition of the Wasserstein distance, one has :

$$\mathcal{W}_p((S_k)_{1 \leq k \leq n}, (\bar{S}_k^n)_{1 \leq k \leq n}) \leq E^{1/p} \sup_{1 \leq k \leq n} |S_k - \bar{S}_k^n|^p, \quad (3)$$

and so we just have to prove that :

$$E(\sqrt{n} \sup_{1 \leq k \leq n} |S_k - \bar{S}_k^n|)^p \leq C(\log n)^p. \quad (4)$$

Recalling that for any positive random variable  $Z$ , and any  $p \geq 1$  :

$$EZ^p = \int_0^\infty pz^{p-1}P(Z > z)dz, \quad (5)$$

we deduce that

$$\begin{aligned} E(\sqrt{n} \sup_{1 \leq k \leq n} |S_k - \bar{S}_k^n|)^p &\leq \int_0^\infty pz^{p-1}P(\sqrt{n} \sup_{1 \leq k \leq n} |S_k - \bar{S}_k^n| > z)dz, \\ &\leq \int_0^{K \log n} pz^{p-1}dz + \int_{K \log n}^\infty pz^{p-1}P(\sqrt{n} \sup_{1 \leq k \leq n} |S_k - \bar{S}_k^n| > z)dz, \end{aligned} \quad (6)$$

where  $K$  is the constant given in Proposition 1. The first integral in the righthandside of (6) is clearly bounded by  $C(\log n)^p$ . For the second one, we have, using successively the change of variables  $z = x + K \log n$  and the Proposition 1

$$\int_{K \log n}^\infty pz^{p-1}P(\sqrt{n} \sup_{1 \leq k \leq n} |S_k - \bar{S}_k^n| > z)dz \leq C \int_0^\infty p(x + K \log n)^{p-1}e^{-\lambda x}dx \leq C(\log n)^p.$$

This gives (4) and Theorem 1 is proved. ◇

**Proposition 1** *One can construct on the same probability space a sequence of i.i.d. standard gaussian variables  $(Y_i)_{1 \leq i \leq n}$  and a sequence of i.i.d. variables  $(X_i^n)_{1 \leq i \leq n}$ , with  $X_i^n$  equal in law to  $\bar{Y}_i^n$ , such that for positive constants  $C$ ,  $K$  and  $\lambda$ , we have, for  $n$  large enough and for all  $x > 0$  :*

$$P(\sqrt{n} \sup_{1 \leq k \leq n} |S_k - \bar{S}_k^n| \geq K \log n + x) \leq C e^{-\lambda x},$$

where  $\bar{S}_k^n = \sum_{i=1}^k X_i^n$ , and  $S_k = \sum_{i=1}^k Y_i$ .

We improve the classical KMT result with a factor  $1/\sqrt{n}$ . This is not surprising since we are dealing with variables which are close to gaussian variables.

The proof of the Proposition 1 is postponed at the end of Section 4. It is obtained by using the KMT method developed in [6], [7]. The main tool for this construction is a gaussian coupling to the partial sums  $\bar{S}_k^n$ , which is based essentially on a large deviation expansion of  $\bar{p}_k^n(x)/\phi(x)$  where  $\bar{p}_k^n$  is the density function of  $\frac{1}{\sqrt{k}}\bar{S}_k^n$  and  $\phi$  the density function of the standard gaussian law. We state and prove this large deviation expansion and the associated coupling inequalities in Section 4.

### 3 Application to the Euler approximation of the geometric Brownian motion

In this section, we apply the preceding results to bound the pathwise Wasserstein distance between the geometric Brownian motion and its Euler approximation. Let  $(X_t)_{t \in [0,1]}$  be the geometric Brownian motion :  $X_t = e^{B_t}$ , where  $(B_t)$  is a standard Brownian motion and consider the continuous time Euler approximation of  $X_t$ , with step  $1/n$ , defined by :

$$\bar{X}_0^n = 1, \quad d\bar{X}_t^n = \frac{1}{2} \bar{X}_{\varphi_n(t)}^n dt + \bar{X}_{\varphi_n(t)}^n dB_t, \tag{7}$$

where  $\varphi_n(t) = \frac{[nt]}{n}$ , or analogously

$$\bar{X}_t^n = \prod_{k=1}^{[nt]} \left( 1 + (B_{\frac{k}{n}} - B_{\frac{k-1}{n}}) + \frac{1}{2n} \right) \left( 1 + (B_t - B_{\frac{[nt]}{n}}) + \frac{1}{2} \left( t - \frac{[nt]}{n} \right) \right). \tag{8}$$

Our main result is the following.

**Theorem 2** For  $p \geq 1$ , there exists a positive constant  $C$ , such that for  $n$  large enough :

$$\mathcal{W}_p((X_t)_{t \in [0,1]}, (\bar{X}_t^n)_{t \in [0,1]}) \leq C \frac{\log n}{n}.$$

To prove the result of Theorem 2 we proceed in several steps. First, we give a bound for the Wasserstein distance between the logarithm of the processes. We have  $(\log X_t)_{t \in [0,1]} = (B_t)_{t \in [0,1]}$  and since the logarithm of the Euler scheme  $(\bar{X}_t^n)_{t \in [0,1]}$  is not defined everywhere, we introduce an auxiliary process  $(L_t^n)_{t \in [0,1]}$ , defined in such a way that

$$\mathcal{W}_p((e^{L_t^n})_{t \in [0,1]}, (\bar{X}_t^n)_{t \in [0,1]}) \leq C/n.$$

We consider the process  $(L_t^n)$  defined by :

$$L_t^n = B_t - \int_0^t (B_s - B_{\varphi_n(s)}) dB_s, \quad t \in [0, 1]. \quad (9)$$

We can observe that the discrete process  $(L_{\frac{k}{n}}^n)$  satisfies :

$$L_{\frac{k}{n}}^n = B_{\frac{k}{n}} - \frac{1}{2} \sum_{i=1}^k ((B_{\frac{i}{n}} - B_{\frac{i-1}{n}})^2 - \frac{1}{n}), \quad 1 \leq k \leq n, \quad (10)$$

and consequently, the discrete time process  $(L_{\frac{k}{n}}^n)_{1 \leq k \leq n}$  is equal in law to  $\frac{1}{\sqrt{n}}(\bar{S}_k^n)_{1 \leq k \leq n}$ , where using the notations of Section 2,  $\bar{S}_k^n = \sum_{i=1}^k \bar{Y}_i^n$ , with  $\bar{Y}_i^n$  defined by (2). Similarly, we observe that  $(\log X_{\frac{k}{n}})_{1 \leq k \leq n}$  is equal in law to  $\frac{1}{\sqrt{n}}(S_k)_{1 \leq k \leq n}$ . This permits to derive immediately from Theorem 1 the following corollary.

**Corollary 1** For  $p \geq 1$ , there exists a positive constant  $C$ , such that for  $n$  large enough :

$$\mathcal{W}_p((\log X_{\frac{k}{n}})_{1 \leq k \leq n}, (L_{\frac{k}{n}}^n)_{1 \leq k \leq n}) \leq C \frac{\log n}{n}.$$

Next, we can extend this result to the continuous processes  $(\log X_t)$  and  $(L_t^n)$  using the strong approximation error on each interval with length  $1/n$ .

**Lemma 1** For  $p \geq 1$ , there exists a positive constant  $C$ , such that for  $n$  large enough :

$$\mathcal{W}_p((\log X_t)_{t \in [0,1]}, (L_t^n)_{t \in [0,1]}) \leq C \frac{\log n}{n}.$$

**Proof** We consider the process  $(L_t^n)$  defined by (9), driven by the Brownian motion  $(B_t)$ , and we introduce the process  $B_t^n$  :

$$B_t^n = L_{\frac{k-1}{n}}^n + B_t - B_{\frac{k-1}{n}}, \quad \text{for } \frac{k-1}{n} \leq t < \frac{k}{n}. \quad (11)$$

The process  $(B_t^n)$  is discontinuous and coincide with  $(L_t^n)$  at the discretization times  $(k/n)_{0 \leq k \leq n-1}$ .

First, we prove the following strong approximation result :

$$E \max_{1 \leq k \leq n} \sup_{t \in [\frac{k-1}{n}, \frac{k}{n}]} |B_t^n - L_t^n|^p \leq C \frac{(\log n)^p}{n^p}. \quad (12)$$

To prove (12), we will use (5) with  $Z = n \max_{1 \leq k \leq n} \sup_{t \in [\frac{k-1}{n}, \frac{k}{n}]} |B_t^n - L_t^n|$  and so we have to control  $P(Z > z)$ , for  $z > 0$ . We have :

$$P(Z > z) \leq n \max_k P(n \sup_{t \in [\frac{k-1}{n}, \frac{k}{n}]} |B_t^n - L_t^n| > z),$$

with  $n(B_t^n - L_t^n) = n \int_{\frac{k-1}{n}}^t (B_s - B_{\varphi_n(s)}) dB_s$ , for  $\frac{k-1}{n} \leq t < \frac{k}{n}$ . Observing that the processes  $(\int_{\frac{k-1}{n}}^t (B_s - B_{\varphi_n(s)}) dB_s)_{t \in [\frac{k-1}{n}, \frac{k}{n}]}$  and  $(\int_0^t B_s dB_s)_{t \in [0, \frac{1}{n}]}$  have the same law, we deduce :

$$P(Z > z) \leq n P(n \sup_{t \in [0, \frac{1}{n}]} |\int_0^t B_s dB_s| > z).$$

But from Ito's formula  $\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t)$ , consequently :

$$P(Z > z) \leq n P\left(n \sup_{t \in [0, \frac{1}{n}]} |B_t^2| > 2(z - \frac{1}{2})\right),$$

and by time rescaling :

$$P(Z > z) \leq n P\left(\sup_{t \in [0, 1]} |B_t^2| > 2(z - \frac{1}{2})\right).$$

Using the exponential inequality for the Brownian motion, we have  $P(\sup_{t \in [0, 1]} |B_t| > a) \leq 2e^{-a^2/2}$ , and this finally leads to :

$$P(Z > z) \leq C n e^{-(z - \frac{1}{2})}. \quad (13)$$

Turning back to (5), we have :

$$EZ^p \leq \int_0^{\log n} pz^{p-1} dz + \int_{z > \log n} pz^{p-1} P(Z > z) dz. \quad (14)$$

Reporting (13) in the second integral of (14) and using the change of variables  $x = z - \log n$ , we deduce :

$$EZ^p \leq C(\log n)^p.$$

This proves the strong approximation result (12).

We end the proof as in [1] (Proof of Theorem 3.2.). The Wasserstein distance in the left hand side of Corollary 1 is attained for a probability measure  $\pi$  on  $\mathbb{R}^n \times \mathbb{R}^n$  with marginal laws respectively



the law of a Brownian motion at times  $(k/n)_{0 \leq k \leq n}$  and the law of  $(L_{\frac{k}{n}})_k$ . We fix  $(L_{\frac{k}{n}})_k$  to be the discretization of the solution of (9) for a Brownian motion  $(B_t)$  and let  $(\overline{B}_{\frac{k}{n}})_k$  be distributed according to the first marginal of  $\pi$  given the second one equal to  $(L_{\frac{k}{n}})_k$ . The random variable  $((\overline{B}_{\frac{k}{n}})_k, L_{\frac{k}{n}})_k$  in  $\mathbb{R}^n \times \mathbb{R}^n$  is distributed according to  $\pi$  and attains the Wasserstein distance between  $(\log X_{\frac{k}{n}})_k$  and  $(L_{\frac{k}{n}})_k$ .

By the triangle inequality, we have:

$$\mathcal{W}_p((\log X_t)_{t \in [0,1]}, (L_t^n)_{t \in [0,1]}) \leq \mathcal{W}_p((\log X_t)_{t \in [0,1]}, (B_t^n)_{t \in [0,1]}) + \mathcal{W}_p((B_t^n)_{t \in [0,1]}, (L_t^n)_{t \in [0,1]}), \quad (15)$$

where  $(B_t^n)_{t \in [0,1]}$  is defined by (11). Let us note that the process  $(B_t^n)$  is not continuous and so the associated Wasserstein distance is defined in  $D([0,1])$ , the space of c-à-d-l-à-g functions, equipped with the supremum norm.

From the strong error approximation (12), the second right handside term in (15) is bound by  $C \log n/n$  and to end the proof it remains to estimate

$$\mathcal{W}_p((\log X_t)_{t \in [0,1]}, (B_t^n)_{t \in [0,1]}).$$

For this we consider a Brownian motion  $(W_t)_{t \in [0,1]}$  independent on  $((\overline{B}_{\frac{k}{n}})_k, (B_t)_t)$  and we construct the two Brownian Bridges driven by  $(W_t)$ :  $(W_t)_{t \in [\frac{k-1}{n}, \frac{k}{n}]}$  (starting from  $\overline{B}_{\frac{k-1}{n}}$  and ending at  $\overline{B}_{\frac{k}{n}}$ ), and  $(W_t)_{t \in [\frac{k-1}{n}, \frac{k}{n}]}$  (starting from  $B_{\frac{k-1}{n}}^n$  and ending at  $B_{\frac{k}{n}}^n$ , where  $B_{\frac{k}{n}}^n$  is the left hand limit at time  $\frac{k}{n}$  of  $(B_t^n)$ ). We set for  $t \in [\frac{k-1}{n}, \frac{k}{n}]$  and  $1 \leq k \leq n$ :

$$W_t^1 = W_t \frac{\overline{B}_{\frac{k-1}{n}}, \overline{B}_{\frac{k}{n}}}{n} = \overline{B}_{\frac{k-1}{n}} n \left( \frac{k}{n} - t \right) + \overline{B}_{\frac{k}{n}} n \left( t - \frac{k-1}{n} \right) + W_t - W_{\frac{k-1}{n}} - n \left( t - \frac{k-1}{n} \right) (W_{\frac{k}{n}} - W_{\frac{k-1}{n}}),$$

$$W_t^2 = W_t \frac{B_{\frac{k-1}{n}}^n, B_{\frac{k}{n}}^n}{n} = B_{\frac{k-1}{n}}^n n \left( \frac{k}{n} - t \right) + B_{\frac{k}{n}}^n n \left( t - \frac{k-1}{n} \right) + W_t - W_{\frac{k-1}{n}} - n \left( t - \frac{k-1}{n} \right) (W_{\frac{k}{n}} - W_{\frac{k-1}{n}}).$$

One can check that  $\mathcal{L}((W_t^1)_t) = \mathcal{L}((\log X_t)_t)$  and  $\mathcal{L}((W_t^2)_t) = \mathcal{L}((B_t^n)_t)$ , consequently,

$$\begin{aligned} \mathcal{W}_p((\log X_t)_{t \in [0,1]}, (B_t^n)_{t \in [0,1]}) &\leq E^{1/p} \sup_{t \in [0,1]} |W_t^1 - W_t^2|^p \\ &\leq E^{1/p} \max_k (|\overline{B}_{\frac{k-1}{n}} - B_{\frac{k-1}{n}}^n|^p \vee |\overline{B}_{\frac{k}{n}} - B_{\frac{k}{n}}^n|^p) \\ &\leq E^{1/p} \max_k |\overline{B}_{\frac{k}{n}} - L_{\frac{k}{n}}^n|^p + E^{1/p} \max_k |L_{\frac{k}{n}}^n - B_{\frac{k}{n}}^n|^p. \end{aligned}$$

We have

$$E^{1/p} \max_k |L_{\frac{k}{n}}^n - B_{\frac{k}{n}}^n|^p \leq E^{1/p} \max_k \sup_{t \in [\frac{k-1}{n}, \frac{k}{n}]} |B_t^n - L_t^n|^p,$$

and by construction of the process  $(\overline{B}_{\frac{k}{n}})_k$ ,

$$E^{1/p} \max_k |\overline{B}_{\frac{k}{n}} - L_{\frac{k}{n}}^n|^p = \mathcal{W}_p((\log X_{\frac{k}{n}})_k, (L_{\frac{k}{n}}^n)_k),$$

Consequently, using Corollary 1 and (12), we finally obtain :

$$\mathcal{W}_p((\log X_t)_{t \in [0,1]}, (B_t^n)_{t \in [0,1]}) \leq C \log n/n,$$

and Lemma 1 is proved. ◇

**Lemma 2** *For  $p \geq 1$ , there exists a positive constant  $C$ , such that for  $n$  large enough :*

$$\mathcal{W}_p((X_t)_{t \in [0,1]}, (e^{L_t^n})_{t \in [0,1]}) \leq C \frac{\log n}{n}.$$

**Proof** Let  $((\overline{B}_t)_{t \in [0,1]}, (\overline{L}_t^n)_{t \in [0,1]})$  be a random variable in  $\mathcal{C}([0,1]) \times \mathcal{C}([0,1])$  which attains the Wasserstein distance  $\mathcal{W}_{2p}$  in Lemma 1. We have :

$$\mathcal{W}_{2p}((\log X_t)_{t \in [0,1]}, (L_t^n)_{t \in [0,1]}) = E^{\frac{1}{2p}} \sup_{t \in [0,1]} |\overline{B}_t - \overline{L}_t^n|^{2p} \leq C \frac{\log n}{n}. \quad (16)$$

We deduce that :

$$\mathcal{W}_p((X_t)_{t \in [0,1]}, (e^{L_t^n})_{t \in [0,1]}) \leq E^{\frac{1}{p}} \sup_{t \in [0,1]} |e^{\overline{B}_t} - e^{\overline{L}_t^n}|^p.$$

But  $\sup_{t \in [0,1]} |e^{\overline{B}_t} - e^{\overline{L}_t^n}| \leq (\sup_{t \in [0,1]} e^{\overline{B}_t} + \sup_{t \in [0,1]} e^{\overline{L}_t^n}) \sup_{t \in [0,1]} |\overline{B}_t - \overline{L}_t^n|$ , and so from Cauchy-Schwarz inequality, this gives

$$\mathcal{W}_p((X_t)_{t \in [0,1]}, (e^{L_t^n})_{t \in [0,1]}) \leq (E^{\frac{1}{2p}}(\sup_{t \in [0,1]} e^{\overline{B}_t})^{2p} + E^{\frac{1}{2p}}(\sup_{t \in [0,1]} e^{\overline{L}_t^n})^{2p}) E^{\frac{1}{2p}} \sup_{t \in [0,1]} |\overline{B}_t - \overline{L}_t^n|^{2p}. \quad (17)$$

Using (16), to finish the proof of Lemma 2, we just have to verify that  $E(\sup_{t \in [0,1]} e^{\overline{B}_t})^{2p}$  and  $E(\sup_{t \in [0,1]} e^{\overline{L}_t^n})^{2p}$  are bounded.

Since  $(\overline{B}_t)_t$  is equal in law to a Brownian motion,  $E(\sup_{t \in [0,1]} e^{\overline{B}_t})^{2p}$  is bounded for any  $p \geq 1$ . Hence  $(\overline{L}_t^n)_t$  is equal in law to  $(L_t^n)$ , the process solution of (9), and so, it remains to show that  $E(\sup_{t \in [0,1]} e^{L_t^n})^p$  is bounded, for  $p \geq 1$ . From (9), the martingale  $(L_t^n)_{t \in [0,1]}$  can be written as :

$$L_t^n = \int_0^t (1 - (B_s - B_{\varphi_n(s)})) dB_s.$$

We first bound  $Ee^{a\langle L^n, L^n \rangle_t} = E(e^{a \int_0^1 (1 - (B_s - B_{\varphi_n(s)}))^2 ds})$ , for  $a > 0$ . We have

$$E(e^{a \int_0^1 (1 - (B_s - B_{\varphi_n(s)}))^2 ds}) \leq e^{2a} Ee^{2a \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (B_s - B_{\frac{k}{n}})^2 ds}.$$

Since the random variables  $(\int_{\frac{k}{n}}^{\frac{k+1}{n}} (B_s - B_{\frac{k}{n}})^2 ds)_{0 \leq k \leq n-1}$  are independent and identically distributed, we deduce

$$E(e^{a \int_0^1 (1 - (B_s - B_{\varphi_n(s)}))^2 ds}) \leq e^{2a} (Ee^{2a \int_0^{\frac{1}{n}} B_s^2 ds})^n \leq e^{2a} (Ee^{\frac{2a}{n} \sup_{s \leq 1/n} B_s^2})^n.$$

By time rescaling,  $\sup_{0 \leq s \leq 1/n} B_s^2$  is equal in law to  $\frac{1}{n} \sup_{0 \leq s \leq 1} B_s^2$  and consequently

$$E(e^{a \int_0^1 (1 - (B_s - B_{\varphi_n(s)}))^2 ds}) \leq e^{2a} (Ee^{\frac{2a}{n^2} \sup_{s \leq 1} B_s^2})^n.$$

From Hölder's inequality and Doob's maximal inequality applied to the positive submartingale  $(e^{\frac{2a}{n^2} B_t^2})_t$ , we have for  $q > 1$ ,

$$Ee^{\frac{2a}{n^2} \sup_{s \leq 1} B_s^2} \leq (E \sup_{s \in [0,1]} e^{\frac{2aq}{n^2} B_s^2})^{1/q} \leq \frac{q}{q-1} (Ee^{\frac{2aq}{n^2} B_1^2})^{1/q}.$$

Remarking that for  $\alpha < 1/2$ ,  $Ee^{\alpha B_1^2} = 1/\sqrt{1-2\alpha}$ , this gives, for  $n$  large enough and choosing  $q = n$ :

$$Ee^{\frac{2a}{n^2} \sup_{s \leq 1} B_s^2} \leq \frac{n}{n-1} (1 - \frac{4a}{n})^{-\frac{1}{2n}}.$$

This permits to obtain

$$Ee^{a\langle L^n, L^n \rangle_t} = E(e^{a \int_0^1 (1 - (B_s - B_{\varphi_n(s)}))^2 ds}) \leq e^{2a} \left( \frac{n}{n-1} \right)^n (1 - \frac{4a}{n})^{-\frac{1}{2}} \leq C_a, \quad (18)$$

where  $C_a$  is a constant depending on  $a$  but not on  $n$ . From Novikov's criterion, we deduce that for any  $a > 0$ ,  $(\mathcal{E}(aL_t^n))_{t \in [0,1]} = (e^{aL_t^n - \frac{a^2}{2} \langle L^n, L^n \rangle_t})_{t \in [0,1]}$  is a martingale. Observing that

$$e^{pL_t^n} = \mathcal{E}(2pL_t^n)^{1/2} e^{p^2 \langle L^n, L^n \rangle_t},$$

is a positive submartingale, and applying Doob's maximal inequality, we have :

$$E(\sup_{t \in [0,1]} e^{pL_t^n}) \leq C(Ee^{2pL_1^n})^{1/2} \leq C(Ee^{8p^2 \langle L^n, L^n \rangle_t})^{1/4},$$

and so from (18), this gives :

$$E(\sup_{t \in [0,1]} e^{pL_t^n}) \leq C_p. \quad (19)$$

This achieves the proof of Lemma 2. ◇

For the statement of the next Lemma, we recall that  $(\bar{X}_t^n)$  and  $(e^{L_t^n})$  are defined on the same probability space by (7) and (9).

**Lemma 3** For  $p \geq 1$ , there exists a positive constant  $C$ , such that for  $n$  large enough :

$$\left( E \sup_{t \in [0,1]} |\bar{X}_t^n - e^{L_t^n}|^p \right)^{1/p} \leq C/n.$$

**Proof** The process  $(e^{L_t^n})$  is solution of the equation

$$e^{L_0^n} = 1, \quad de^{L_t^n} = e^{L_t^n} (1 - (B_t - B_{\varphi_n(t)})) dB_t + \frac{1}{2} e^{L_t^n} (1 - (B_t - B_{\varphi_n(t)}))^2 dt. \quad (20)$$

To simplify the notation, we write  $\Delta B_t = B_t - B_{\varphi_n(t)}$  and  $\Delta_t = t - \varphi_n(t)$ . Considering the error process  $U_t = e^{L_t^n} - \bar{X}_t^n$ , for  $t \in [0, 1]$ , we first remark that for all  $p \geq 1$ ,  $E \sup_{t \in [0,1]} |\bar{X}_t^n|^p \leq C$ . Moreover from (19), we have  $E(\sup_{t \in [0,1]} e^{L_t^n})^p \leq C$ , and so  $E \sup_{t \in [0,1]} |U_t|^p \leq C$ , for some positive constant  $C$ .

Now, it is easy to verify from (7) and (20) that  $(U_t)$  satisfies the equation :

$$U_t = \int_0^t U_s (1 - \Delta B_s) dB_s + \frac{1}{2} \int_0^t U_s (1 - \Delta B_s)^2 ds + R_t^n, \quad (21)$$

with

$$dR_t^n = \bar{X}_t^n (1 - \Delta B_t) dB_t + \frac{1}{2} \bar{X}_t^n (1 - \Delta B_t)^2 dt - \bar{X}_{\varphi_n(t)}^n dB_t - \frac{1}{2} \bar{X}_{\varphi_n(t)}^n dt \quad (22)$$

Since  $\bar{X}_t^n - \bar{X}_{\varphi_n(t)}^n = \bar{X}_{\varphi_n(t)}^n \Delta B_t + \frac{1}{2} \bar{X}_{\varphi_n(t)}^n \Delta_t$ , we can rewrite  $R_t^n$  as :

$$\begin{aligned} dR_t^n &= (\bar{X}_t^n - \bar{X}_{\varphi_n(t)}^n) dB_t - \bar{X}_t^n \Delta B_t dB_t + \frac{1}{2} (\bar{X}_t^n - \bar{X}_{\varphi_n(t)}^n) dt + \frac{1}{2} \bar{X}_t^n ((\Delta B_t)^2 - 2\Delta B_t) dt, \\ &= -(\bar{X}_t^n - \bar{X}_{\varphi_n(t)}^n) \Delta B_t dB_t + \frac{1}{2} \bar{X}_{\varphi_n(t)}^n \Delta_t dB_t + \frac{1}{2} (\bar{X}_t^n - \bar{X}_{\varphi_n(t)}^n) dt + \frac{1}{2} \bar{X}_t^n ((\Delta B_t)^2 - 2\Delta B_t) dt. \end{aligned}$$

From Burkholder-Davis-Gundy inequality, we have

$$E \sup_{t \in [0,1]} \left| \int_0^t \bar{X}_{\varphi_n(s)}^n \Delta_s dB_s \right|^p \leq C/n^p$$

and since  $\int_0^t \bar{X}_{\varphi_n(s)}^n \Delta B_s ds = \int_0^t \bar{X}_{\varphi_n(s)}^n [(t \wedge (\varphi_n(s) + \frac{1}{n})) - s] dB_s$ , we obtain similarly

$$E \sup_{t \in [0,1]} \left| \int_0^t \bar{X}_{\varphi_n(s)}^n \Delta B_s ds \right|^p \leq C/n^p.$$

This permits to conclude, after a few computation, that :

$$E \sup_{t \in [0,1]} |R_t^n|^p \leq C_p/n^p. \quad (23)$$

Turning back to (21), we deduce using (23) and convexity and Burkholder-Davis-Gundy inequalities :

$$E \sup_{v \leq t} |U_v|^p \leq C \left( \int_0^t E \left( (1 + |\Delta B_s|)^{2p} \sup_{v \leq s} |U_v|^p \right) ds + \frac{1}{n^p} \right). \quad (24)$$

Moreover, introducing the truncation  $1_{|\Delta B_s| \leq C}$ , we have for any  $q \geq 1$  :

$$P(|\Delta B_s| > C) \leq C/n^q,$$

and so from Cauchy Schwarz inequality

$$E \left( (1 + |\Delta B_s|)^{2p} \sup_{v \leq s} |U_v|^p 1_{|\Delta B_u| > C} \right) \leq C/n^p,$$

this gives :

$$E \left( (1 + |\Delta B_s|)^{2p} \sup_{v \leq s} |U_v|^p \right) \leq C \left( E \sup_{v \leq s} |U_v|^p + \frac{1}{n^p} \right),$$

Reporting this in (24), we deduce :

$$E \sup_{v \leq t} |U_v|^p \leq C \left( \int_0^t E \sup_{v \leq s} |U_v|^p ds + \frac{1}{n^p} \right),$$

and the result of Lemma 3 follows from Gronwall's Lemma.

◇

## Proof of Theorem 2

The result of Theorem 2 is a straightforward consequence of Lemma 2 and Lemma 3, applying the triangle inequality and observing that  $\mathcal{W}_p((\bar{X}_t)_{t \in [0,1]}, (e^{L_t})_{t \in [0,1]}) \leq E^{1/p} \sup_{t \in [0,1]} |\bar{X}_t^n - e^{L_t^n}|^p$ .

◇

# 4 Quantile coupling inequalities and proof of the Proposition 1

## 4.1 Quantile coupling inequalities

We use the technique of the conjugated random variable (see [7], [4]). For technical reasons, essentially the non integrability of the characteristic function of the random variables  $\bar{Y}_k^n$ , we regularize them by adding independent normally distributed random variables. For that, we consider a sequence of independent identically distributed standard gaussian variables  $(\xi_k)_{k \geq 1}$ , independent of the sequence  $(Y_k)_{k \geq 1}$ , and we set :

$$\bar{Y}_k^{*,n} = \bar{Y}_k^n + \frac{1}{n} \xi_k, \tag{25}$$

$$\bar{S}_k^{*,n} = \sum_{i=1}^k \bar{Y}_i^{*,n}. \tag{26}$$

We have the following expansion.

**Lemma 4** Let  $\bar{p}_k^{*,n}$  be the density function of  $\frac{1}{\sqrt{k}}\bar{S}_k^{*,n}$ ,  $\phi$  and  $\Phi$  be respectively the density and the cumulative distribution function of the standard gaussian law. There exist some constants  $\varepsilon > 0$  and  $C > 0$  such that for all  $k \geq 1$  and  $n$  large enough, we have :

i) for  $|x| \leq \varepsilon\sqrt{kn}$  :

$$\bar{p}_k^{*,n}(x) = \phi(x)e^{\frac{1}{\sqrt{n}}T_k^1(x)},$$

ii) for  $0 \leq x \leq \varepsilon\sqrt{kn}$  :

$$P\left(\frac{1}{\sqrt{k}}\bar{S}_k^{*,n} > x\right) = (1 - \Phi(x))e^{\frac{1}{\sqrt{n}}T_k^2(x)},$$

$$P\left(\frac{1}{\sqrt{k}}\bar{S}_k^{*,n} \leq -x\right) = \Phi(-x)e^{\frac{1}{\sqrt{n}}T_k^3(x)},$$

where  $|T_k^j(x)| \leq C\frac{(1+|x|^3)}{\sqrt{k}}$ , for  $1 \leq j \leq 3$ .

**Remark 1** For  $k = n$ , the approximation of  $\bar{p}_k^{*,n}(x)$  by  $\phi(x)$  is of order  $1/n$ , for  $|x| \leq \varepsilon n$ . In the classical KMT result, the order of approximation is  $1/\sqrt{n}$  for  $|x| \leq \varepsilon\sqrt{n}$ . It is important to have a better approximation result which holds for larger values of  $x$  to improve the final bound in the KMT construction (compare for example to the refined quantile inequalities given in [9]).

An inspection of the proof below shows that without the regularization technique, the result of Lemma 4 holds with  $k \geq 3$ .

In the sequel,  $C$  denotes a constant which value does not depend on  $n$  and  $k$  and may change from one line to the other.

**Proof** We can prove i) and ii) by the technique of conjugated random variables. We only give the proof of i) in details, the proof of ii) being very similar (see for example [8]). We first compute the Laplace transform of the variables  $\bar{Y}_k^{*,n}$ . For  $t > -\sqrt{n}$ , let  $R(t) = Ee^{t\bar{Y}_k^{*,n}}$  and  $\Psi(t) = \log R(t)$ . A simple computation gives :

$$\Psi(t) = \frac{t}{2\sqrt{n}} + \frac{t^2}{2(1+t/\sqrt{n})} - \frac{1}{2}\log(1+t/\sqrt{n}) + \frac{t^2}{2n^2}. \quad (27)$$

In particular, we have  $\Psi(0) = 0$ ,  $\Psi^{(1)}(0) = 0$ ,  $\Psi^{(2)}(0) = 1 + \frac{1}{2n} + \frac{1}{n^2}$  and it is easy to verify from the computation of  $\Psi^{(2)}$  and  $\Psi^{(3)}$ , that, for  $0 \leq |t| \leq c\sqrt{n}$ ,  $0 < 1/C \leq \Psi^{(2)}(t) \leq C$  and  $|\Psi^{(3)}(t)| \leq C/\sqrt{n}$ , for some positive constants  $c$  and  $C$ . Fixing  $t$  such that  $|t| \leq c\sqrt{n}$ , we consider the sequence of independent random variables  $(Z_k^n)_{k \geq 1}$  such that,  $Z_k^n$  admits the density function  $\frac{e^{tx}}{R(t)}f_{\bar{Y}_k^{*,n}}(x)$ , where  $f_{\bar{Y}_k^{*,n}}$  denotes the density function of  $\bar{Y}_k^{*,n}$ . One can easily verify that  $E(Z_k^n) = \Psi^{(1)}(t)$  and  $V(Z_k^n) =$

$\Psi^{(2)}(t)$ . We denote by  $q_k^n$  the density function of the normalized sum  $\frac{1}{\sqrt{k\Psi^{(2)}(t)}} \sum_{i=1}^k (Z_i^n - \Psi^{(1)}(t))$ .

The following relation holds between  $\bar{p}_k^{*,n}$  and  $q_k^n$ :

$$\bar{p}_k^{*,n}(x) = \frac{e^{k\Psi(t)-tx\sqrt{k}}}{\sqrt{\Psi^{(2)}(t)}} q_k^n \left( \frac{x\sqrt{k} - k\Psi^{(1)}(t)}{\sqrt{k\Psi^{(2)}(t)}} \right). \quad (28)$$

The next step to obtain the result of Lemma 4 is to prove that that for  $|t| \leq c\sqrt{n}$

$$\sup_x |q_k^n(x) - \phi(x)| \leq C \frac{1}{\sqrt{kn}}. \quad (29)$$

Let  $\hat{q}_k^n(u)$  be the Fourier transform of  $q_k^n$ , we have  $\hat{q}_k^n(u) = \left( E e^{\frac{iu}{\sqrt{k\Psi^{(2)}(t)}} (Z_1^n - \Psi^{(1)}(t))} \right)^k$  and consequently

$$\hat{q}_k^n(u) = \left( \frac{R \left( t + \frac{iu}{\sqrt{k\Psi^{(2)}(t)}} \right)}{R(t)} \right)^k e^{-\frac{iu\Psi^{(1)}(t)}{\sqrt{\Psi^{(2)}(t)}} \sqrt{k}}. \quad (30)$$

Now, by the inversion formula,

$$\sup_x |q_k^n(x) - \phi(x)| \leq \frac{1}{2\pi} \int | \hat{q}_k^n(u) - e^{-\frac{u^2}{2}} | du \leq I_k^{n,0} + I_k^{n,1} + I_k^{n,2}, \quad (31)$$

where

$$I_k^{n,0} \leq C \int_{|u| \leq \alpha\sqrt{kn}} | \hat{q}_k^n(u) - e^{-\frac{u^2}{2}} | du, \quad (32)$$

$$I_k^{n,1} \leq C \int_{|u| > \alpha\sqrt{kn}} | \hat{q}_k^n(u) | du, \quad (33)$$

$$I_k^{n,2} \leq C \int_{|u| > \alpha\sqrt{kn}} e^{-\frac{u^2}{2}} du, \quad (34)$$

and  $\alpha$  is a positive constant which will be precised below.

Since for  $x > 0$ ,  $\int_x^\infty e^{-\frac{u^2}{2}} du \leq \frac{1}{x} e^{-\frac{x^2}{2}}$ , one can easily see that

$$I_k^{n,2} \leq \frac{C}{\sqrt{kn}}.$$

Turning to  $I_k^{n,1}$ , a tedious computation using (27) and (30) gives

$$| \hat{q}_k^n(u) | = \frac{e^{-\frac{u^2}{2n^2\Psi^{(2)}(t)}}}{\left( 1 + \frac{u^2}{kn\Psi^{(2)}(t)(1+t/\sqrt{n})^2} \right)^{k/4}} \times \frac{e^{\frac{kt^2(1+t/\sqrt{n})}{2((1+t/\sqrt{n})^2 + \frac{u^2}{kn\Psi^{(2)}(t)})}}}{e^{\frac{kt^2}{2(1+t/\sqrt{n})}}} g(u), \quad (35)$$

where

$$\log g(u) = \frac{-u^2(1-t/\sqrt{n})}{2\Psi^{(2)}(t) \left( (1+t/\sqrt{n})^2 + \frac{u^2}{kn\Psi^{(2)}(t)} \right)}. \quad (36)$$

We deduce then the bound

$$|\hat{q}_k^n(u)| \leq \frac{e^{-\frac{u^2}{2n^2\Psi^{(2)}(t)}}}{\left(1 + \frac{u^2}{kn\Psi^{(2)}(t)(1+t/\sqrt{n})^2}\right)^{k/4}} g(u). \quad (37)$$

We remark that the contribution of the regularization variables  $(\xi_k)$  in the numerator of equation (37) (the term  $e^{-\frac{u^2}{2n^2\Psi^{(2)}(t)}}$ ) ensures the integrability of  $|\hat{q}_k^n(u)|$  for the small values of  $k$  ( $k = 1$  and  $k = 2$ ).

Recalling that if  $|t| \leq c\sqrt{n}$ , we have  $0 < 1/C \leq \Psi^{(2)}(t) \leq C$ , we deduce that for  $|u| > \alpha\sqrt{kn}$ ,  $0 \leq g(u) \leq e^{-Ckn}$ . This finally yields :

$$I_k^{n,1} \leq C e^{-Ckn} \int e^{-C\frac{u^2}{2n^2}} du \leq C n e^{-Ckn} \leq \frac{C}{\sqrt{kn}}. \quad (38)$$

It remains to bound the main term  $I_k^{n,0}$ . With the previous notations, we rewrite (30) as :

$$\hat{q}_k^n(u) = e^{k\Psi\left(t + \frac{iu}{\sqrt{k\Psi^{(2)}(t)}}\right) - k\Psi(t) - \frac{iu\Psi^{(1)}(t)}{\sqrt{\Psi^{(2)}(t)}}\sqrt{k}}.$$

A Taylor expansion up to order three of  $v \mapsto \Psi\left(t + \frac{iv}{\sqrt{k\Psi^{(2)}(t)}}\right)$  on  $[0, u]$  gives :

$$\hat{q}_k^n(u) = e^{-\frac{u^2}{2} - \frac{iu^3}{6\Psi^{(2)}(t)\sqrt{k\Psi^{(2)}(t)}}\eta_u},$$

where  $|\eta_u| \leq \sup_{|y| \leq |u|} |\Psi^{(3)}(t + \frac{iy}{\sqrt{k\Psi^{(2)}(t)}})|$ . We deduce then that :

$$|\hat{q}_k^n(u) - e^{-\frac{u^2}{2}}| = e^{-\frac{u^2}{2}} \left| 1 - e^{-\frac{iu^3}{6\Psi^{(2)}(t)\sqrt{k\Psi^{(2)}(t)}}\eta_u} \right|.$$

Using the inequality  $|1 - e^z| \leq |z|e^{|z|}$  for any complex number  $z$ , we obtain by choosing  $\alpha$  such that  $\left| \frac{u}{\Psi^{(2)}(t)\sqrt{k\Psi^{(2)}(t)}}\eta_u \right| \leq 1$ , for  $|u| \leq \alpha\sqrt{kn}$ ,

$$|\hat{q}_k^n(u) - e^{-\frac{u^2}{2}}| \leq C \frac{|u|^3}{\sqrt{kn}} e^{-\frac{u^2}{2}} e^{\frac{u^2}{6}} = C \frac{|u|^3}{\sqrt{kn}} e^{-\frac{u^2}{3}}.$$

This gives

$$I_k^{n,0} \leq \frac{C}{\sqrt{kn}}. \quad (39)$$

This achieves the proof of (29).

We turn back to (28). We first recall that for  $|t| \leq c\sqrt{n}$ ,  $0 \leq 1/C \leq \Psi^{(2)}(t) \leq C$ . As a consequence,  $\Psi^{(1)}$  is increasing, for  $|t| \leq c\sqrt{n}$ , with values in  $[-C\sqrt{n}, C\sqrt{n}]$ , for some constant  $C$ . It follows that for  $|x| \leq \varepsilon\sqrt{kn}$ , the equation  $x = \Psi^{(1)}(t)\sqrt{k}$  admits a unique solution. In the sequel, we fix  $t$  to be the unique solution of :

$$x = \Psi^{(1)}(t)\sqrt{k}. \quad (40)$$



We have  $|t| \leq c\sqrt{n}$  and so combining (28) with (29), we obtain :

$$\bar{p}_k^{*,n}(x) = \frac{e^{k\Psi(t)-tx\sqrt{k}}}{\sqrt{\Psi^{(2)}(t)}} \frac{1}{\sqrt{2\pi}} \left( 1 + \frac{O(1)}{\sqrt{kn}} \right). \quad (41)$$

where  $O(1)$  is a bounded function. Next considering (40), by a Taylor expansion of  $[\Psi^{(1)}]^{-1}$  on  $[0, x/\sqrt{k}]$  up to order two, we obtain :

$$t = \frac{x}{\sqrt{k}\Psi^{(2)}(0)} + \frac{x^2}{k\sqrt{n}}O(1),$$

where we have used  $[\Psi^{(1)}]^{-1}(0) = 0$ ,  $([\Psi^{(1)}]^{-1})^{(1)}(0) = 1/\Psi^{(2)}(0)$ ,  $([\Psi^{(1)}]^{-1})^{(2)}(u) = \frac{\psi^{(3)}([\Psi^{(1)}]^{-1}(u))}{[\Psi^{(2)}([\Psi^{(1)}]^{-1}(u))]^2}$  and recalling that for  $|u| \leq c\sqrt{n}$ , we have  $0 \leq 1/C \leq \Psi^{(2)}(u) \leq C$  and  $|\Psi^{(3)}(u)| \leq C/\sqrt{n}$ .

Now, since  $\Psi(t) = \frac{t^2}{2}\Psi^{(2)}(0) + \frac{t^3}{6}\eta_t$ , with  $|\eta_t| \leq \sup_{|u| \leq |t|} |\Psi^{(3)}(u)|$ , we deduce the expansion :

$$k\Psi(t) = \frac{x^2}{2\Psi^{(2)}(0)} + \frac{x^3}{\sqrt{kn}}O(1),$$

where  $O(1)$  is a function which is bounded uniformly in  $k$  and  $n$ , for  $|x| \leq \varepsilon\sqrt{kn}$ . Using  $\Psi^{(2)}(0) = 1 + \frac{1}{2n} + \frac{1}{n^2}$ , this finally leads to :

$$k\Psi(t) - t\sqrt{k}x = -\frac{x^2}{2} + \frac{x^2}{n}O(1) + \frac{x^3}{\sqrt{kn}}O(1). \quad (42)$$

Reporting (42) in (41), it yields :

$$\bar{p}_k^{*,n}(x) = \phi(x) \frac{e^{\frac{x^2}{n}O(1) + \frac{x^3}{\sqrt{kn}}O(1)}}{\sqrt{\Psi^{(2)}(t)}} \left( 1 + \frac{O(1)}{\sqrt{kn}} \right).$$

We conclude, observing that  $\Psi^{(2)}(t) = \Psi^{(2)}(0) + \frac{x}{\sqrt{kn}}O(1)$ ,

$$\bar{p}_k^{*,n}(x) = \phi(x) e^{O(1)\frac{(1+x^3)}{\sqrt{kn}}},$$

and i) is proved. ◇

From Lemma 4, we can improve the KMT quantile coupling inequalities with a rate  $1/\sqrt{n}$ . Let  $F_k$  be the cumulative distribution function of  $\frac{1}{\sqrt{k}}\bar{S}_k^{*,n}$ . To simplify the notation we omit the dependence in  $n$ . We have  $\frac{1}{\sqrt{k}}\bar{S}_k^{*,n} =_d F_k^{-1}(\Phi(\frac{1}{\sqrt{k}}S_k))$  and in the next lemma, we assume  $\frac{1}{\sqrt{k}}\bar{S}_k^{*,n} = F_k^{-1}(\Phi(\frac{1}{\sqrt{k}}S_k))$ , for  $S_k = \sum_{i=1}^k Y_i$ , where  $(Y_i)_{i \geq 1}$  is a given sequence of i.i.d. standard gaussian variables. We have the following refined quantile inequalities.

**Lemma 5** *There exist  $\varepsilon > 0$  and  $C > 0$  such that, for all  $k \geq 1$  and  $n$  large enough :*

$$|\bar{S}_k^{*,n} - S_k| \leq \frac{C}{\sqrt{n}} \left( \frac{|\bar{S}_k^{*,n}|^2}{k} + 1 \right), \quad \text{if } |\bar{S}_k^{*,n}| \leq \varepsilon k \sqrt{n}.$$

**Proof** We just have to prove that for  $0 \leq |x| \leq \varepsilon \sqrt{kn}$ ,

$$\Phi(x - u(x)) \leq F_k(x) \leq \Phi(x + u(x)), \quad (43)$$

where  $u(x) = \frac{C}{\sqrt{n}} \left( \frac{x^2}{\sqrt{k}} + \frac{1}{\sqrt{k}} \right)$ . Observing that (43) is equivalent to  $x - u(x) \leq \Phi^{-1}(F_k(x)) \leq x + u(x)$ , the result of lemma 5 follows for  $x = \frac{1}{\sqrt{k}} \bar{S}_k^{*,n} = F_k^{-1}(\Phi(\frac{1}{\sqrt{k}} S_k))$ .

From Lemma (4), part ii), we have for  $0 \leq x \leq \varepsilon \sqrt{kn}$  :

$$(1 - \Phi(x)) e^{-C \frac{(1+x^3)}{\sqrt{kn}}} \leq 1 - F_k(x) \leq (1 - \Phi(x)) e^{C \frac{(1+x^3)}{\sqrt{kn}}}, \quad (44)$$

$$\Phi(-x) e^{-C \frac{(1+x^3)}{\sqrt{kn}}} \leq F_k(-x) \leq \Phi(-x) e^{C \frac{(1+x^3)}{\sqrt{kn}}}. \quad (45)$$

Now from Mason and Zhou [9], Lemma 3, we have for all  $A > 0$ ,  $k \geq 64A^2$  and  $0 \leq x \leq \sqrt{k}/(8A)$  :

$$\log \left( \frac{\Phi(-x+u)}{\Phi(-x)} \right) = \log \left( \frac{1 - \Phi(x-u)}{1 - \Phi(x)} \right) \geq A \left( \frac{1+x^3}{\sqrt{k}} \right), \quad (46)$$

$$\log \left( \frac{\Phi(-x-u)}{\Phi(-x)} \right) = \log \left( \frac{1 - \Phi(x+u)}{1 - \Phi(x)} \right) \leq -A \left( \frac{1+x^3}{\sqrt{k}} \right), \quad (47)$$

where  $u = 2A \frac{1+x^2}{\sqrt{k}}$ . Combining (46) and (47) with  $A = C/\sqrt{n}$  and  $n$  large enough, with (44) and (45), we deduce that  $\forall k \geq 1$  and  $0 \leq |x| \leq \varepsilon \sqrt{kn}$ , (43) holds. ◇

## 4.2 Conditional quantile inequalities

We complete this section with some conditional quantile inequalities, which will be used in the proof of the Proposition 1. We assume that  $k$  is even,  $k = 2p$ . We define  $\tilde{S}_k = 2S_p - S_k$  and  $\tilde{S}_k^{*,n} = 2\bar{S}_p^{*,n} - \bar{S}_k^{*,n}$  and we denote by  $\tilde{F}_k(\cdot|y)$  the conditional cumulative distribution function of  $\frac{1}{\sqrt{k}} \tilde{S}_k^{*,n}$  given  $\frac{1}{\sqrt{k}} \bar{S}_k^{*,n} = y$ . We assume moreover in the next Lemma that  $\frac{1}{\sqrt{k}} \tilde{S}_k^{*,n} = \tilde{F}_k^{-1}(\Phi(\frac{1}{\sqrt{k}} \tilde{S}_k) | \frac{1}{\sqrt{k}} \bar{S}_k^{*,n})$ . We have the following conditional quantile inequality :

**Lemma 6** *There exist  $\varepsilon > 0$  and  $C > 0$  such that, for all  $k \geq 1$  and  $n$  large enough :*

$$|\tilde{S}_k^{*,n} - \tilde{S}_k| \leq \frac{C}{\sqrt{n}} \left( \frac{|\tilde{S}_k^{*,n}|^2 + |\bar{S}_k^{*,n}|^2}{k} + 1 \right), \quad \text{if } \max(|\bar{S}_k^{*,n}|, |\tilde{S}_k^{*,n}|) \leq \varepsilon k \sqrt{n}.$$

Before proving the Lemma 6, we need the following three lemmas.

**Lemma 7** *Let  $\tilde{p}_k^{*,n}(\cdot | y)$  be the conditional density of  $\frac{1}{\sqrt{k}}\tilde{S}_k^{*,n}$  given  $\frac{1}{\sqrt{k}}\bar{S}_k^{*,n} = y$ . There exist some constants  $\varepsilon > 0$  and  $C > 0$  such that for all  $k \geq 1$  and  $n$  large enough, we have, for  $|x| \leq \varepsilon\sqrt{kn}$ ,  $|y| \leq \varepsilon\sqrt{kn}$  :*

$$\tilde{p}_k^{*,n}(x | y) = \phi(x) e^{\frac{1}{\sqrt{n}}\tilde{T}_k^1(x,y)}, \quad (48)$$

where  $\left| \tilde{T}_k^1(x, y) \right| \leq C \frac{1+|x|^3+|y|x^2+|y|}{\sqrt{k}}$ .

**Proof** First we show the following expansion for the density of  $\frac{1}{\sqrt{k}}\bar{S}_k^{*,n}$ ,

$$\exists \varepsilon > 0, \forall |x| \leq \varepsilon\sqrt{nk}, \quad \bar{p}_k^{*,n}(x) = \varphi(x) \exp \left( \frac{x^3}{\sqrt{nk}} r_n \left( \frac{x}{\sqrt{nk}} \right) + c_n x^2 + \frac{1+|x|}{\sqrt{nk}} B_{n,k}(x) \right), \quad (49)$$

where  $r_n$  is a sequence of smooth functions defined on some neighbourhood  $[-\varepsilon, \varepsilon]$  of 0 and whose derivatives up to order two are bounded independently of  $n$ ;  $c_n$  is a sequence such that  $c_n = O(1/n)$ ; and  $B_{n,k}(\cdot)$  is some measurable function bounded independently of  $k$  and  $n$ .

Recalling the representation (41), where  $t$  is the unique solution of (40),  $\Psi^{(2)}(t) = \Psi^{(2)}(0) + \frac{|x|}{\sqrt{kn}} O(1)$ , and  $\Psi^{(2)}(0) = 1 + \frac{1}{2n} + \frac{1}{n^2}$  yields to the representation

$$\bar{p}_k^{*,n}(x) = \frac{1}{\sqrt{2\pi}} \exp \left( k\Psi(t) - tx\sqrt{k} + \frac{1+|x|}{\sqrt{nk}} O(1) \right)$$

Let us denote  $\Phi_n(s) = \frac{1}{n}\Psi(\sqrt{ns})$  and by  $h_n$  the inverse of the function  $s \mapsto \frac{1}{\sqrt{n}}\Psi^{(1)}(\sqrt{ns})$ . Due to the expression (27), it is simple to check that both functions are well defined on some neighborhoods of 0 independent of  $n$ , and we can assume that  $h_n$  is well defined on the interval  $[-\varepsilon, \varepsilon]$ , up to reducing the value of  $\varepsilon$ . Using these notations, we get  $t = \sqrt{nh_n}(\frac{x}{\sqrt{nk}})$ , and in turn,  $\bar{p}_k^{*,n}(x) = \frac{1}{\sqrt{2\pi}} \exp \left( nk\Phi_n(h_n(\frac{x}{\sqrt{nk}})) - \sqrt{nk}h_n(\frac{x}{\sqrt{nk}})x + \frac{1+|x|}{\sqrt{nk}} O(1) \right)$ . Now, since  $\Phi_n(0) = \Phi_n^{(1)}(0) = 0$ ,  $\Phi_n^{(2)}(0) = 1 + \frac{1}{2n} + \frac{1}{n^2}$ , we can write  $\Phi_n(s) = \Phi_n^{(2)}(0)\frac{s^2}{2} + s^3\gamma_n(s) = (1/2 + O(1/n))s^2 + s^3\gamma_n(s)$ , where  $\gamma_n$  is some function. Using that  $\Phi_n$ , and all its derivatives, are bounded independently of  $n$  on  $[-\varepsilon, \varepsilon]$ , we deduce that the same property holds true for  $\gamma_n$ . Analogously, we can show that

$$h_n(s) = h_n'(0)s + s^2\beta_n(s) = (1 + O(1/n))s + s^2\beta_n(s) \quad (50)$$

where the function  $\beta_n$ , and its derivatives of any order, are bounded independently of  $n$  on  $[-\varepsilon, \varepsilon]$ .

With simple computation, we deduce that

$$\begin{aligned} nk\Phi_n(h_n(\frac{x}{\sqrt{nk}})) - \sqrt{nk}h_n(\frac{x}{\sqrt{nk}})x &= nk\Phi_n\left(\left(1 + O(1/n)\right)\frac{x}{\sqrt{nk}} + \frac{x^2}{nk}\beta_n(\frac{x}{\sqrt{nk}})\right) \\ &\quad - \sqrt{nk}x\left(\left(1 + O(1/n)\right)\frac{x}{\sqrt{nk}} + \frac{x^2}{nk}\beta_n(\frac{x}{\sqrt{nk}})\right) \\ &= -\frac{1}{2}x^2 + x^2O(1/n) + \frac{x^3}{\sqrt{nk}}r_n(\frac{x}{\sqrt{nk}}) \end{aligned}$$

where  $r_n$  is some bounded function, with bounded derivatives. This gives (49).

We now prove (48). Using the independence of the random variables  $\bar{S}_k^{*,n} + \tilde{S}_k^{*,n}$  and  $\bar{S}_k^{*,n} - \tilde{S}_k^{*,n}$ , we easily deduce

$$\forall x, y, \quad \tilde{p}_k^{*,n}(x | y) = \frac{\bar{p}_{k/2}^{*,n}(\frac{x+y}{\sqrt{2}})\bar{p}_{k/2}^{*,n}(\frac{-x+y}{\sqrt{2}})}{\bar{p}_k^{*,n}(y)}. \quad (51)$$

From (49), we readily get

$$\begin{aligned} \tilde{p}_k^{*,n}(x | y) &= \frac{\phi(\frac{x+y}{\sqrt{2}})\phi(\frac{-x+y}{\sqrt{2}})}{\phi(y)} \exp\left(\delta_n(x, y) + \frac{1 + |x| + |y|}{\sqrt{nk}}O(1) + c_n x^2\right), \\ &= \phi(x) \exp\left(\delta_n(x, y) + \frac{1 + |x| + |y|}{\sqrt{nk}}O(1) + c_n x^2\right), \end{aligned} \quad (52)$$

where  $\delta_n(x, y) = \frac{nk}{2} \left[ \left(\frac{x+y}{\sqrt{nk}}\right)^3 r_n\left(\frac{x+y}{\sqrt{nk}}\right) + \left(\frac{-x+y}{\sqrt{nk}}\right)^3 r_n\left(\frac{-x+y}{\sqrt{nk}}\right) - 2\left(\frac{y}{\sqrt{nk}}\right)^3 r_n\left(\frac{y}{\sqrt{nk}}\right) \right]$ . From a second order Taylor expansion of  $z \mapsto z^3 r_n(z)$  around  $\frac{y}{\sqrt{nk}}$  it can be shown that  $|\delta_n(x, y)| = x^2 O(\sum_{i=1}^3 \left|\frac{x}{\sqrt{nk}}\right|^i + \left|\frac{y}{\sqrt{nk}}\right|^i)$ . Using  $|x| \leq \varepsilon\sqrt{nk}$  and  $|y| \leq \varepsilon\sqrt{nk}$ , this yields to  $|\delta_n(x, y)| = \frac{1}{\sqrt{nk}}O(|x|^3 + x^2|y|)$ . Using the expansion (52) we deduce the lemma.  $\diamond$

**Lemma 8** *Let  $\varepsilon > 0$ , then there exist  $0 < \varepsilon'' < \varepsilon' < \varepsilon$ , and  $C > 0$ , such that for  $n$  large enough, and all  $k$ ,*

$$\forall |y| \leq \varepsilon''\sqrt{nk}, \forall |x| \geq \varepsilon'\sqrt{nk}, \text{ we have, } \tilde{p}_k^{*,n}(x | y) \leq C \exp\left(-\frac{\sqrt{nk}}{C} \left[|x| - \frac{\varepsilon'\sqrt{nk}}{2}\right]\right). \quad (53)$$

**Proof** For the proof, we focus on establishing (53) in the case  $x > 0$ , since the proof for  $x < 0$  is similar.

We first need to prove the following upper bound on the density of  $\bar{S}_k^{*,n}$ ,

$$\exists \varepsilon' > 0, \forall 0 \leq y \leq \varepsilon'\sqrt{nk}, \forall z \geq 0, \quad \bar{p}_k^{*,n}(y + z) \leq \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2} - \frac{yz}{2} + O(\frac{1+|y|^3}{\sqrt{nk}})}. \quad (54)$$

From (28) with  $x = y + z$  and for  $t$  given by  $t = \sqrt{n}h_n(\frac{y}{\sqrt{kn}})$  where  $h_n$  is defined in the proof of Lemma 7 we have

$$\bar{p}_k^{*,n}(y+z) = \frac{e^{k\Psi(t)-ty\sqrt{k}}}{\sqrt{\Psi^{(2)}(t)}} q_k^n \left( \frac{(y+z)\sqrt{k} - k\Psi^{(1)}(t)}{\sqrt{k\Psi^{(2)}(t)}} \right) e^{-tz\sqrt{k}}.$$

Remark that  $t$  is well defined for  $\frac{y}{\sqrt{nk}}$  in a neighbourhood  $[-\varepsilon', \varepsilon']$  of 0 and is solution to  $\Psi^{(1)}(t) = y/\sqrt{k}$ . Then, proceeding exactly as in the proof of Lemma 4 we can deduce that

$$\bar{p}_k^{*,n}(y+z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2} + O(\frac{1+|y|^3}{\sqrt{nk}})} e^{-tz\sqrt{k}}. \quad (55)$$

Now, from (50), we deduce  $-tz\sqrt{k} = -yz[1 + O(\frac{1}{n})] + O(\frac{|z|y^2}{\sqrt{kn}})$ . Using that  $\frac{|y|}{\sqrt{kn}} \leq \varepsilon' < 1/4$ , as soon as we choose the value of  $\varepsilon'$  small enough, we get  $-tz\sqrt{k} \leq -yz/2$ , for  $n$  large. Finally, (54) follows from (55).

We now prove (53). Up to a modification of the value of  $\varepsilon'$ , we can assume that the result of Lemma 4 i) holds true for  $x \leq \varepsilon'\sqrt{nk}$ . Thus, from (51), we get for  $x \in \mathbb{R}$ ,  $|y| \leq \varepsilon'\sqrt{nk}$

$$\tilde{p}_k^{*,n}(x|y) \leq \sqrt{2\pi} e^{\frac{y^2}{2} + C\frac{1+|y|^3}{\sqrt{nk}}} \bar{p}_{k/2}^{*,n}\left(\frac{x+y}{\sqrt{2}}\right) \bar{p}_{k/2}^{*,n}\left(\frac{-x+y}{\sqrt{2}}\right),$$

where  $C > 0$  is some constant. From (28)–(29), it is easily seen that  $\bar{p}^{*,n}$  is a bounded function. Hence, we deduce,

$$\tilde{p}_k^{*,n}(x|y) \leq C e^{\frac{y^2}{2} + C\frac{1+|y|^3}{\sqrt{nk}}} \bar{p}_{k/2}^{*,n}\left(\frac{x+y}{\sqrt{2}}\right), \quad (56)$$

where  $C > 0$  is some constant.

We assume, for the sequel, that  $x \geq \varepsilon'\sqrt{nk}$  and  $|y| \leq \varepsilon''\sqrt{nk}$  with  $\varepsilon'' = \varepsilon'/8$ . We write  $\bar{p}_{k/2}^{*,n}(\frac{x+y}{\sqrt{2}})$  as  $\bar{p}_{k/2}^{*,n}((\frac{\varepsilon'\sqrt{nk}}{2} + y)\frac{1}{\sqrt{2}} + (x - \frac{\varepsilon'\sqrt{nk}}{2})\frac{1}{\sqrt{2}})$  and use (54). Since  $\frac{\varepsilon'\sqrt{nk}}{2} + y \in [\frac{3}{8}\varepsilon'\sqrt{nk}, \frac{5}{8}\varepsilon'\sqrt{nk}] \subset [\frac{1}{4}\varepsilon'\sqrt{nk}, \frac{3}{4}\varepsilon'\sqrt{nk}]$ , we deduce

$$\bar{p}_{k/2}^{*,n}\left(\frac{x+y}{\sqrt{2}}\right) \leq C \exp\left(-\frac{1}{4^3}\varepsilon'^2 nk + \frac{C}{\sqrt{nk}} + C\varepsilon'^3 nk - \frac{\varepsilon'\sqrt{nk}}{16}(x - \varepsilon'\frac{\sqrt{nk}}{2})\right),$$

for some constant  $C > 0$ . From (56) and  $|y| \leq \varepsilon'\sqrt{nk}/8$ , we deduce,

$$\tilde{p}_k^{*,n}(x|y) \leq C \exp\left(-\frac{1}{2 \cdot 4^3}\varepsilon'^2 nk(1 - \varepsilon'C) + \frac{C}{\sqrt{nk}} - \frac{\varepsilon'\sqrt{nk}}{16}(x - \varepsilon'\frac{\sqrt{nk}}{2})\right),$$

where  $C$  is some positive constant. Up to a modification of  $\varepsilon'$ , we can assume that  $\varepsilon'C \leq 1$  and the latter equation gives (53).  $\diamond$

**Lemma 9** *There exist  $\varepsilon > 0$  and  $C > 0$ , such that for all  $0 < x \leq \varepsilon\sqrt{nk}$  and  $|y| \leq \varepsilon\sqrt{nk}$ ,*

$$1 - \tilde{F}_k(x | y) = (1 - \Phi(x))e^{\frac{1}{\sqrt{n}}\tilde{T}_k^2(x,y)}, \quad (57)$$

$$\tilde{F}_k(-x | y) = \Phi(-x)e^{\frac{1}{\sqrt{n}}\tilde{T}_k^3(x,y)}, \quad (58)$$

where  $|\tilde{T}_k^j(x, y)| \leq C \frac{1+|x|^3+|y|x^2+|y|}{\sqrt{k}}$ , for  $j = 2, 3$ .

**Proof** We only prove (57), since the proof of (58) is similar.

Using Lemma 7, let us consider  $\varepsilon_1 > 0$ , such that for all  $|x| \leq \varepsilon_1\sqrt{nk}$ ,  $|y| \leq \varepsilon_1\sqrt{nk}$ , we have

$$\tilde{p}_k^{*,n}(x | y) \leq \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{C_1 \frac{1+|x|^3+|y|x^2+|y|}{\sqrt{nk}}},$$

for some constant  $C_1 > 0$ . Hence, for  $0 \leq x < A \leq \varepsilon_1\sqrt{nk}$  and  $|y| \leq \varepsilon_1\sqrt{nk}$ , we can write

$$\tilde{F}_k(A | y) - \tilde{F}_k(x | y) \leq e^{C_1 \frac{1+|y|}{\sqrt{nk}}} \int_x^A e^{-\frac{u^2}{2}(1-2C_1 \frac{u+|y|}{\sqrt{nk}})} \frac{du}{\sqrt{2\pi}}. \quad (59)$$

Assume now on that  $\varepsilon_1 < \frac{1}{8C_1}$ , up to a modification of the value of  $\varepsilon_1$ . Then, the change of variable  $v = u \left(1 - 2C_1 \frac{u+|y|}{\sqrt{nk}}\right)^{1/2}$  is one to one as the variable  $u$  ranges in  $[x, A]$  and it is easy to see that  $|\frac{dv}{du}| \leq 1 + C_2 \frac{v+|y|}{\sqrt{nk}}$ , with some constant  $C_2 > 0$ . As a result, after a change of variable, we get

$$\begin{aligned} \int_x^A e^{-\frac{u^2}{2}(1-2C_1 \frac{u+|y|}{\sqrt{nk}})} \frac{du}{\sqrt{2\pi}} &\leq \int_{\hat{x}(y,n,k)}^\infty e^{-\frac{v^2}{2}} (1 + C_2 \frac{v+|y|}{\sqrt{nk}}) \frac{dv}{\sqrt{2\pi}} \\ &= [1 - \Phi(\hat{x}(y, n, k))] (1 + \frac{C_2 |y|}{\sqrt{nk}}) + \frac{C_2}{\sqrt{nk}} \phi(\hat{x}(y, n, k)) \end{aligned} \quad (60)$$

where we have noted  $\hat{x}(y, n, k) = x \left(1 - 2C_1 \frac{x+|y|}{\sqrt{nk}}\right)^{1/2}$ .

From the mean value theorem,

$$\log \left( \frac{1 - \Phi(\hat{x}(y, n, k))}{1 - \Phi(x)} \right) = (x - \hat{x}(y, n, k)) \frac{\phi(\xi)}{1 - \Phi(\xi)} \leq C_3 (x^2 + x|y|) \frac{\phi(\xi)}{1 - \Phi(\xi)},$$

where  $\xi \in [\hat{x}(y, n, k), x]$  and  $C_3 > 0$  is some constant. From Lemma 2 in [9] we know that  $z \mapsto \frac{\phi(z)}{1 - \Phi(z)}$  is increasing and Lemma 1 in [9] easily implies

$$\frac{\phi(z)}{1 - \Phi(z)} \leq C_4 (1 + z) \quad (61)$$

for any  $z \geq 0$  and  $C_4 > 0$  some constant. We deduce

$$\log \left( \frac{1 - \Phi(\hat{x}(y, n, k))}{1 - \Phi(x)} \right) \leq C_5 (x^2 + x|y|) (1 + x), \quad (62)$$

where  $C_5$  is some constant. Putting together (59), (60) and (62) we deduce,

$$\tilde{F}_k(A | y) - \tilde{F}_k(x | y) \leq [1 - \Phi(x)]e^{C_5 \frac{1+|x|^3+|y|x^2+|y|}{\sqrt{nk}}} (1 + C_2 \frac{1+|y|}{\sqrt{nk}}) + \frac{C_2}{\sqrt{nk}} \varphi(\hat{x}(y, n, k)).$$

Using (61) and (62), we have  $\phi(\hat{x}(y, n, k)) \leq C_4(1+|x|)(1-\Phi(x))e^{C_5(x^2+x|y|)(1+x)}$ . As a consequence, we easily deduce

$$\tilde{F}_k(A | y) - \tilde{F}_k(x | y) \leq [1 - \Phi(x)]e^{C_6 \frac{1+|x|^3+|y|x^2+|y|}{\sqrt{nk}}}, \quad \forall 0 \leq x < A \leq \varepsilon_1 \sqrt{nk}, |y| \leq \varepsilon_1 \sqrt{nk}, \quad (63)$$

and where  $C_6 > 0$  is some constant.

In order to prove (57), it remains to control  $1 - \tilde{F}_k(A | y) = \int_A^\infty \tilde{p}_k^{*,n}(u | y) du$ . From Lemma 8, there exists  $0 < \varepsilon_1'' < \varepsilon_1' < \varepsilon_1$  and  $C_7 > 0$  such that for all  $|y| \leq \varepsilon_1'' \sqrt{nk}$ ,  $u \geq \varepsilon_1' \sqrt{nk}$ ,  $\tilde{p}_k^{*,n}(u | y) \leq C_7 \exp\left(-\frac{\sqrt{nk}}{C_7} \left[|x| - \frac{\varepsilon_1' \sqrt{nk}}{2}\right]\right)$ . We choose  $A = \varepsilon_1' \sqrt{nk} \leq \varepsilon_1 \sqrt{nk}$ , and with easy computations deduce that  $\int_A^\infty \tilde{p}_k^{*,n}(u | y) du \leq \frac{C_7^2}{\sqrt{nk}} e^{-\frac{\varepsilon_1'}{4C_7} nk}$ . If we let  $\varepsilon_2 = \sqrt{\frac{\varepsilon_1'}{4C_7}} \wedge \varepsilon_1'$ , and if  $x \leq \varepsilon_2 \sqrt{nk}$ , we have,  $1 - \Phi(x) \geq \frac{\phi(x)}{C_4 + C_4 x} \geq \frac{e^{-\frac{\varepsilon_1' nk}{8C_7}}}{\sqrt{2\pi}(C_4 + C_4 \varepsilon_2 \sqrt{nk})}$ . This implies,

$$\forall 0 \leq x \leq \varepsilon_2 \sqrt{nk}, \quad \int_A^\infty \tilde{p}_k^{*,n}(u | y) du \leq C_8 [1 - \Phi(x)] \exp^{-\frac{nk}{C_8}}, \quad (64)$$

for some constant  $C_8 > 0$ . Joining (63) with (64) yields to the result (57).  $\diamond$

**Proof of Lemma 6.** We just have to prove that for  $0 \leq |x| \leq \varepsilon \sqrt{kn}$ , and  $0 \leq |y| \leq \varepsilon \sqrt{kn}$

$$1 - \Phi(x + u(x)) \leq 1 - \tilde{F}_k(x | y) \leq 1 - \Phi(x - u(x)), \quad (65)$$

where  $u(x) = \frac{C(1+x^2+y^2)}{\sqrt{nk}}$ . We focus on the case  $x \geq 0$ , as the proof is similar for  $x \leq 0$ . Using Lemma 9, there exist  $\varepsilon_1$  and  $C_1$  such that for  $0 \leq x \leq \varepsilon_1 \sqrt{nk}$ ,  $|y| \leq \varepsilon_1 \sqrt{nk}$ ,

$$(1 - \Phi(x))e^{-C_1 \frac{1+|x|^3+|y|x^2+|y|}{\sqrt{nk}}} \leq 1 - \tilde{F}_k(x | y) \leq (1 - \Phi(x))e^{C_1 \frac{1+|x|^3+|y|x^2+|y|}{\sqrt{nk}}}.$$

We set  $A = \frac{C_1}{\sqrt{n}} \frac{1+|x|^3+x^2|y|+|y|}{1+|x|^3}$ . Then it is simple to check that if  $\varepsilon_1$  is small enough and  $n$  sufficiently large, we have  $A^2 \leq \frac{2C_1^2}{n} + \frac{4C_1^2 y^2}{n} \leq \frac{k}{64}$ , for all  $k \geq 1$ . In the same way, we easily check that if  $\varepsilon_1$  is small enough, we have  $8Ax \leq \sqrt{k}$ . As a consequence, we can apply (46)–(47) to get

$$1 - \Phi(x + u) \leq [1 - \Phi(x)]e^{-A \frac{1+x^3}{\sqrt{k}}} \leq [1 - \Phi(x)]e^{A \frac{1+x^3}{\sqrt{k}}} \leq 1 - \Phi(x - u), \quad (66)$$

where  $u = 2A \frac{1+x^2}{\sqrt{k}} \leq C \frac{(1+x^2+y^2)}{\sqrt{nk}} = u(x)$ , for some constant  $C$ . Since  $A \frac{1+x^3}{\sqrt{k}} = C_1 \frac{1+|x|^3+|y|x^2+|y|}{\sqrt{nk}}$ , the equation (66) gives (65).  $\diamond$

### 4.3 Proof of the Proposition 1

The proof follows the dyadic construction scheme introduced by [6] (see also [4]) and we give it for the sake of completeness. We adopt the notation of [6]. We assume that  $\mathbf{n} = \mathbf{2}^N$ . Given a sequence of independent standard gaussian variables  $(Y_k)_{k \geq 1}$ , we will construct a sequence of independent identically distributed variables  $(X_k^n)_{1 \leq k \leq 2^N}$  such that  $X_k^n$  has the distribution of  $\bar{Y}_k^n$ , defined by (2). We first construct some independent variables  $(X_k^{*,n})_{1 \leq k \leq 2^N}$  equal in law to  $(\bar{Y}_k^{*,n})_{1 \leq k \leq 2^N}$ , defined by (25). As previously, for  $k$  equal 1 to  $2^N$ , we set :

$$\bar{S}_k^{*,n} = \sum_{i=1}^k \bar{Y}_i^{*,n}, \quad S_k = \sum_{i=1}^k Y_i.$$

We define :

$$\begin{aligned} U_{m,k} &= \bar{S}_{(k+1)2^m}^{*,n} - \bar{S}_{k2^m}^{*,n}, \quad , 0 \leq k < 2^{N-m}, \quad 0 \leq m \leq N, \\ \tilde{U}_{m,k} &= U_{m-1,2k} - U_{m-1,2k+1}, \quad , 0 \leq k < 2^{N-m}, \quad 1 \leq m \leq N. \end{aligned}$$

Similarly for the gaussian variables, we define :

$$\begin{aligned} V_{m,k} &= S_{(k+1)2^m} - S_{k2^m}, \quad , 0 \leq k < 2^{N-m}, \quad 0 \leq m \leq N, \\ \tilde{V}_{m,k} &= V_{m-1,2k} - V_{m-1,2k+1}, \quad , 0 \leq k < 2^{N-m}, \quad 1 \leq m \leq N. \end{aligned}$$

We have  $U_{m,0} = \bar{S}_{2^m}^{*,n}$  and  $V_{m,0} = S_{2^m}$ . We can also remark that the gaussian variables  $V_{m,k}$  and  $\tilde{V}_{m,k}$  are independent.

To complete the notation, for  $m$  equal 1 to  $N$ , we recall that  $F_{2^m}$  denotes the cumulative density function of  $\frac{1}{2^{m/2}}U_{m,0}$ , and we note  $\tilde{F}_{2^m}(\cdot|y)$  the cumulative density function of  $\frac{1}{2^{m/2}}\tilde{U}_{m,0}$  given  $\frac{1}{2^{m/2}}U_{m,0} = y$ .

Since the gaussian variables  $(Y_k)$  are given,  $(V_{m,k})_{m,k}$  and  $(\tilde{V}_{m,k})_{m,k}$  are also given. With these variables, we now proceed to the construction of some random variables, equal in law to the  $(U_{m,k})$ , that we will still denote by  $(U_{m,k})$  to simplify the notation. First, for  $m = N$ , we set :

$$U_{N,0} = 2^{N/2} F_{2^N}^{-1} \left( \Phi \left( \frac{V_{N,0}}{2^{N/2}} \right) \right), \quad \tilde{U}_{N,0} = 2^{N/2} \tilde{F}_{2^N}^{-1} \left( \Phi \left( \frac{\tilde{V}_{N,0}}{2^{N/2}} \right) \middle| \frac{1}{2^{N/2}} U_{N,0} \right).$$

We deduce then  $U_{N-1,0}$  and  $U_{N-1,1}$ , by the relations :

$$U_{N-1,0} = \frac{1}{2}(U_{N,0} + \tilde{U}_{N,0}), \quad U_{N-1,1} = \frac{1}{2}(U_{N,0} - \tilde{U}_{N,0}).$$



It is easy to verify that the random variables  $U_{N-1,0}$  and  $U_{N-1,1}$  are independent and distributed according to the expected law. We next continue the construction by induction on  $m$ . Assuming that  $U_{m,k}$  is constructed, for  $0 \leq k < 2^{N-m}$ , we set :

$$\tilde{U}_{m,k} = 2^{m/2} \tilde{F}_{2^m}^{-1} \left( \Phi \left( \frac{\tilde{V}_{m,k}}{2^{m/2}} \right) \middle| \frac{1}{2^{m/2}} U_{m,k} \right),$$

and we define

$$U_{m-1,2k} = \frac{1}{2}(U_{m,k} + \tilde{U}_{m,k}), \quad U_{m-1,2k+1} = \frac{1}{2}(U_{m,k} - \tilde{U}_{m,k}).$$

We can observe that the joint distribution of the pair  $(U_{m-1,2k}, U_{m-1,2k+1})$  is the prescribed one and so  $U_{m-1,2k}$  and  $U_{m-1,2k+1}$  are independent and distributed according to  $F_{2^{m-1}}$ . More generally, the constructed random variables  $U_{m,k}$ , for  $0 \leq k < 2^{N-m}$ , are independent.

At the final step,  $m = 0$ , this permits to construct a sequence of independent random variables  $(X_k^{*,n})_{1 \leq k \leq 2^N}$ , equal in law to  $(\bar{Y}_k^{*,n})_{1 \leq k \leq 2^N}$  by setting

$$X_k^{*,n} = U_{0,k-1}, \quad 1 \leq k \leq 2^N.$$

We end the construction with the definition of  $(X_k^n)_{1 \leq k \leq 2^N}$ , distributed according to  $(\bar{Y}_k^n)_{1 \leq k \leq 2^N}$ . Let  $F^*(\cdot|y)$  be the cumulative density function of  $\bar{Y}_k^n$  given  $\bar{Y}_k^{*,n} = y$ . It is clear by (2) and (25) that  $F^*$  does not depend on  $k$ . We set :

$$X_k^n = (F^*)^{-1}(\eta_k | X_k^{*,n}), \quad 1 \leq k \leq 2^N,$$

where  $(\eta_k)_{k \geq 1}$  is a sequence of independent random variables, uniformly distributed on  $[0, 1]$ , and independent of  $(Y_k)_{k \geq 1}$ . The pair  $(X_k^n, X_k^{*,n})$  has the prescribed distribution, so  $X_k^n$  has the law of  $\bar{Y}_k^n$ , and the difference  $X_k^n - X_k^{*,n}$  has the distribution of a centered gaussian variable with variance  $1/n^2$ .

In that follows, we set

$$\bar{S}_k^{*,n} = \sum_{i=1}^k X_i^{*,n}, \quad \bar{S}_k^n = \sum_{i=1}^k X_i^n, \quad 1 \leq k \leq 2^N,$$

where  $X_i^{*,n}$  and  $X_i^n$  are constructed above.

We first prove that for any positive constant  $K$  and  $\lambda$ , we have for  $n$  large enough and  $x > 0$  :

$$P(\sqrt{n} \sup_{1 \leq k \leq n} |\bar{S}_k^{*,n} - \bar{S}_k^n| \geq K \log n + x) \leq C e^{-\lambda x}. \quad (67)$$

By construction  $\sqrt{n} \sup_{1 \leq k \leq n} |\overline{S}_k^{*,n} - \overline{S}_k^n|$  is equal in law to  $\frac{1}{\sqrt{n}} \sup_{1 \leq k \leq n} |\sum_{i=1}^k \xi_i|$ , where  $(\xi_i)_i$  are i.i.d. standard gaussian variables. So we deduce that

$$P(\sqrt{n} \sup_{1 \leq k \leq n} |\overline{S}_k^{*,n} - \overline{S}_k^n| \geq K \log n + x) \leq 2e^{-\frac{1}{2}(K \log n + x)^2},$$

and (67) is proved. Consequently to prove Proposition 1, it is sufficient to prove that for some positive constant  $C$ ,  $K$  and  $\lambda$  (independent of  $n$  and  $x$ ), we have, for  $n$  large enough and for all  $x > 0$  :

$$P(\sqrt{n} \sup_{1 \leq k \leq n} |\overline{S}_k^{*,n} - S_k| \geq K \log n + x) \leq Ce^{-\lambda x},$$

or equivalently, there exist positive constants  $\gamma$ ,  $\alpha$  and  $\lambda$ , such that for all  $x$  and  $N$  :

$$P(\sup_{1 \leq k \leq n} |\overline{S}_k^{*,n} - S_k| \geq x) \leq \gamma e^{\alpha N - \lambda 2^{N/2} x}. \quad (68)$$

To prove (68), we distinguish between the cases  $x < 8\varepsilon 2^{N/2}$  and  $x \geq 8\varepsilon 2^{N/2}$ .

In that follows, we note

$$\Delta = \sup_{1 \leq k \leq n} |\overline{S}_k^{*,n} - S_k|. \quad (69)$$

First case :  $x < 8\varepsilon 2^{N/2}$

We set

$$A = \bigcap_{k=0}^{2^N - 1} \{|U_{0,k}| \leq \varepsilon 2^{N/2}\}, \quad (70)$$

where  $\varepsilon$  is smaller than the constants of Lemmas 5 and 6.

To prove (68), we use the decomposition :

$$P(\Delta \geq x) \leq P(\{\Delta \geq x\} \cap A) + P(A^c).$$

We first control  $P(A^c)$ . By (70), we have for  $t_0 > 0$  :

$$P(A^c) \leq 2^N P(|\overline{Y}_1^{*,n}| > \varepsilon 2^{N/2}) \leq 2^N (P(\overline{Y}_1^{*,n} > \varepsilon 2^{N/2}) + P(-\overline{Y}_1^{*,n} > \varepsilon 2^{N/2})) \leq 2^N ((E^{t_0 \overline{Y}_1^{*,n}} + E^{-t_0 \overline{Y}_1^{*,n}}) e^{-t_0 \varepsilon 2^{N/2}}).$$

But from (27), for  $t > -2^{N/2}/2$  :

$$E^{t \overline{Y}_1^{*,n}} \leq e^{\frac{3}{2}t^2 + \frac{t}{22^{N/2}}} \frac{1}{\sqrt{1 + t/2^{N/2}}},$$

so by choosing  $t_0 = \varepsilon 2^{N/2}/2$ , we obtain :

$$P(A^c) \leq C 2^N e^{-\varepsilon^2 2^N / 8},$$

where  $C$  is a positive constant depending on  $\varepsilon$ . Since  $x < 8\varepsilon 2^{N/2}$ , we deduce the expected bound :

$$P(A^c) \leq C 2^N e^{-\varepsilon x 2^{N/2}/64}. \quad (71)$$

Turning to the control of  $P(\{\Delta \geq x\} \cap A)$ , we first have :

$$P(\{\Delta \geq x\} \cap A) \leq 2^N \sup_{1 \leq k \leq 2^N} P(\{|\bar{S}_k^{*,n} - S_k| > x\} \cap A).$$

Now, from the dyadic construction, we observe that for  $1 \leq k \leq 2^N$  :

$$\bar{S}_k^{*,n} = \frac{k}{2^N} \bar{S}_{2^N}^{*,n} + \sum_{m=1}^N c_m \tilde{U}_{m,l(m,k)}, \quad (72)$$

where  $c_m \in (0, 1)$  and  $l(m, k)$  is defined by :

$$l(m, k) 2^m < k \leq (l(m, k) + 1) 2^m.$$

Using a similar decomposition for  $S_k$ , we deduce :

$$|\bar{S}_k^{*,n} - S_k| \leq |\bar{S}_{2^N}^{*,n} - S_{2^N}| + \sum_{m=1}^N |\tilde{U}_{m,l(m,k)} - \tilde{V}_{m,l(m,k)}|. \quad (73)$$

But on  $A$ , we have for all  $m$  and for all  $k$  (this is immediate from the definition of  $A$ ) :

$$|U_{m,k}| \leq \varepsilon 2^m 2^{N/2}, \quad |\tilde{U}_{m,k}| \leq \varepsilon 2^m 2^{N/2},$$

consequently, using the results of Lemmas 5 and 6, we obtain on  $A$  :

$$\begin{aligned} |U_{N,0} - V_{N,0}| &\leq \frac{C}{2^{N/2}} \left( \frac{|U_{N,0}|^2}{2^N} + 1 \right), \\ |\tilde{U}_{m,l(m,k)} - \tilde{V}_{m,l(m,k)}| &\leq \frac{C}{2^{N/2}} \left( \frac{|U_{m,l(m,k)}|^2 + |\tilde{U}_{m,l(m,k)}|^2}{2^m} + 1 \right). \end{aligned}$$

Plugging these bounds in (73), this gives on  $A$  :

$$|\bar{S}_k^{*,n} - S_k| \leq \frac{C}{2^{N/2}} \left( \frac{|U_{N,0}|^2}{2^N} + N + 1 + \delta_k \right). \quad (74)$$

where

$$\delta_k = \sum_{m=1}^N \frac{|U_{m,l(m,k)}|^2 + |\tilde{U}_{m,l(m,k)}|^2}{2^m}.$$

Observing that on  $A$  we have the equality in law :

$$\delta_k =_{\mathcal{L}} \delta_1 = \sum_{m=1}^N \frac{|U_{m,0}|^2 + |\tilde{U}_{m,0}|^2}{2^m},$$

we deduce :

$$P(\{\Delta \geq x\} \cap A) \leq 2^N P\left(\left\{\frac{C}{2^{N/2}}\left(\frac{|U_{N,0}|^2}{2^N} + N + 1 + \delta_1\right) \geq x\right\} \cap A\right).$$

Since  $U_{m,0} = U_{m-1,0} + U_{m-1,1}$  and  $\tilde{U}_{m,0} = U_{m-1,0} - U_{m-1,1}$ , we remark that :

$$\delta_1 = 2 \sum_{m=1}^N \frac{|U_{m-1,0}|^2 + |U_{m-1,1}|^2}{2^m} = \sum_{m=0}^{N-1} \frac{|U_{m,0}|^2 + |U_{m,1}|^2}{2^m}$$

and we obtain

$$P(\{\Delta \geq x\} \cap A) \leq 2^N P\left(\left\{\frac{C}{2^{N/2}}\left(\frac{|U_{N,0}|^2}{2^N} + N + 1 + \sum_{m=0}^{N-1} \frac{|U_{m,0}|^2 + |U_{m,1}|^2}{2^m}\right) \geq x\right\} \cap A\right).$$

Now, we get to bound  $\sum_{m=0}^N \frac{|U_{m,0}|^2}{2^m}$  by  $\sum_{m=0}^{N-1} \frac{|U_{m,1}|^2}{2^m}$ . First we have

$$|U_{m,0}|^2 = \left(U_{0,0} + \sum_{j=0}^{m-1} U_{j,1}\right)^2,$$

so by Cauchy-Schwarz inequality with  $q = 1/\sqrt{2}$ , we obtain :

$$|U_{m,0}|^2 \leq \left(\sum_{j \geq 0} q^j\right) \left(\frac{1}{q^m} U_{0,0}^2 + \sum_{j=0}^{m-1} \frac{1}{q^{m-j-1}} U_{j,1}^2\right) \leq C \left(\frac{1}{q^m} U_{0,0}^2 + \sum_{j=0}^{m-1} \frac{1}{q^{m-j-1}} U_{j,1}^2\right).$$

It yields after some calculus :

$$\sum_{m=0}^N \frac{|U_{m,0}|^2}{2^m} \leq C \left(U_{0,0}^2 + \sum_{m=0}^{N-1} \frac{U_{m,1}^2}{2^m}\right),$$

and finally :

$$P(\{\Delta \geq x\} \cap A) \leq 2^N P\left(\left\{\frac{C}{2^{N/2}}\left(U_{0,0}^2 + N + 1 + \sum_{m=0}^{N-1} \frac{U_{m,1}^2}{2^m}\right) \geq x\right\} \cap A\right). \quad (75)$$

To end the proof, we introduce the notation, for  $0 \leq m \leq N-1$ ,

$$\tau_m = \frac{U_{m,1}^2}{2^m} \mathbf{1}_{\{|U_{m,1}| \leq \varepsilon 2^m 2^{N/2}\}}, \quad (76)$$

and

$$\bar{\tau}_0 = U_{0,0}^2 \mathbf{1}_{\{|U_{0,0}| \leq \varepsilon 2^{N/2}\}}. \quad (77)$$

The random variables  $(\bar{\tau}_0, \tau_0, \tau_1, \dots, \tau_{N-1})$  are independent and this permits to deduce

$$\begin{aligned} P(\{\Delta \geq x\} \cap A) &\leq 2^N P\left(\frac{C}{2^{N/2}}(\bar{\tau}_0 + N + 1 + \sum_{m=0}^{N-1} \tau_m) \geq x\right) \\ &\leq 2^N e^{Ct(N+1)} E e^{Ct\bar{\tau}_0} \prod_{m=0}^{N-1} E e^{Ct\tau_m} e^{-t2^{N/2}x}, \end{aligned}$$

where the last inequality holds for any  $t > 0$ . Consequently to obtain the bound

$$P(\{\Delta \geq x\} \cap A) \leq \gamma e^{\alpha N - \lambda 2^{N/2} x},$$

for some positive constant  $\gamma$ ,  $\alpha$  and  $\lambda$  independent of  $N$  and  $x$ , it is sufficient to prove that

$$\exists t_0 > 0, \quad \exists C > 0, \quad \text{such that} \quad \forall 0 \leq m \leq N-1 \quad E e^{t_0 \tau_m} \leq C. \quad (78)$$

Integrating by parts, we first remark that :

$$E e^{t_0 \tau_m} = 1 + \int_0^{\varepsilon^2 2^m 2^N} t_0 e^{t_0 x} P(\tau_m > x) dx, \quad (79)$$

so to prove (78) we just have to prove that for  $0 \leq x \leq \varepsilon^2 2^m 2^N$  :

$$P(\tau_m > x) \leq C e^{-\eta x}, \quad (80)$$

for positive constant  $C$  and  $\eta$  ( independent of  $m$ ,  $x$ ,  $N$ ). We have

$$\begin{aligned} P(\tau_m > x) &\leq P(U_{m,1}^2 > 2^m x) \\ &\leq P(U_{m,1} > 2^{m/2} \sqrt{x}) + P(-U_{m,1} > 2^{m/2} \sqrt{x}), \\ &\leq E e^{t U_{m,1}} e^{-t 2^{m/2} \sqrt{x}} + E e^{-t U_{m,1}} e^{-t 2^{m/2} \sqrt{x}}, \end{aligned}$$

where the last inequality holds for any  $t > 0$ . We recall that  $U_{m,1}$  is equal in law to  $\sum_{i=1}^{2^m} \bar{Y}_i^{*,n}$ , this gives  $E e^{t U_{m,1}} = e^{2^m \Psi(t)}$ , where  $\Psi$  is defined by (27). Moreover, using the notation

$$\Lambda^*(u) = \inf_t (\Psi(t) - tu), \quad (81)$$

one has ( since  $\Psi^{(1)}(0) = 0$ ) : for  $u \geq 0$ ,  $\Lambda^*(u) = \inf_{t \geq 0} (\Psi(t) - tu)$  and  $\Lambda^*(-u) = \inf_{t \leq 0} (\Psi(t) + tu)$ .

This gives :

$$P(\tau_m > x) \leq e^{2^m \Lambda^*(2^{-m/2} \sqrt{x})} + e^{2^m \Lambda^*(-2^{-m/2} \sqrt{x})}.$$

From the estimation (42) (where  $t$  is defined by (40) with  $k = 2^m$ , changing  $x$  by  $\sqrt{x}$ ), we deduce, since  $\sqrt{x} \leq \varepsilon 2^{m/2} 2^{N/2}$ , that ( choosing  $\varepsilon$  small enough and  $N$  large enough),

$$2^m \Lambda^*(2^{-m/2} \sqrt{x}) \leq -\eta x \quad \text{and} \quad 2^m \Lambda^*(-2^{-m/2} \sqrt{x}) \leq -\eta x,$$

for some  $\eta > 0$  and (80) is proved. This achieves the proof of (68) in the first case.

Second case :  $x \geq 8\varepsilon 2^{N/2}$

Following [6] and [4], we first choose an integer  $M$  such that

$$\frac{x}{8\varepsilon} < 2^M 2^{N/2} \leq \frac{x}{4\varepsilon}.$$

Such an integer exists since  $x \geq 8\varepsilon 2^{N/2}$ . Without loss of generality, we may assume that  $M \leq N$  (if it is not the case the proof reduces to consider  $\Delta_1$  and  $\Delta_2$  below with  $M = N$ ). We set

$$B = \bigcap_{k=0}^{2^{N-M}-1} \{|U_{M,k}| \leq \varepsilon 2^M 2^{N/2}\}. \quad (82)$$

We remark that on  $B$ , we have for all  $m \geq M + 1$  and for all  $k$  :

$$|U_{m,k}| \leq \varepsilon 2^m 2^{N/2}, \quad |\tilde{U}_{m,k}| \leq \varepsilon 2^m 2^{N/2}.$$

Moreover we define :

$$\begin{aligned} \Delta_1 &= \max_{0 \leq k \leq 2^{N-M}} \max_{1 \leq l \leq 2^M} |\bar{S}_{k2^M+l}^{*,n} - \bar{S}_{k2^M}^{*,n}|, \\ \Delta_2 &= \max_{0 \leq k \leq 2^{N-M}} \max_{1 \leq l \leq 2^M} |S_{k2^M+l} - S_{k2^M}|, \\ \Delta_3 &= \max_{0 \leq k \leq 2^{N-M}} |\bar{S}_{k2^M}^{*,n} - S_{k2^M}|. \end{aligned}$$

We immediately see that

$$\Delta \leq \Delta_1 + \Delta_2 + \Delta_3,$$

where  $\Delta$  is defined by (69). Moreover, observing that  $B^c \subset \{\Delta_1 \geq \varepsilon 2^M 2^{N/2}\}$ , we have :

$$\{\Delta \geq x\} \subset \{\Delta_1 \geq \varepsilon 2^M 2^{N/2}\} \cup \{\Delta_2 \geq \varepsilon 2^M 2^{N/2}\} \cup (\{\Delta_3 \geq x/2\} \cap B),$$

and so

$$P(\Delta \geq x) \leq P(\Delta_1 \geq \varepsilon 2^M 2^{N/2}) + P(\Delta_2 \geq \varepsilon 2^M 2^{N/2}) + P(\Delta_3 \geq x/2 \cap B). \quad (83)$$

We first bound  $P(\Delta_3 \geq x/2 \cap B)$ . Starting with the decomposition (analogous to (72))

$$\bar{S}_{k2^M}^{*,n} = \frac{k}{2^N} \bar{S}_{2^N}^{*,n} + \sum_{m=M+1}^N c_m \tilde{U}_{m,l(m,k2^M)},$$

and proceeding as in the proof of the first case, one can show that

$$P(\Delta_3 \geq x/2 \cap B) \leq 2^N P\left(\frac{C}{2^N} (N+1 + \sum_{m=M}^{N-1} \tau_m) \geq x\right),$$

where the variables  $\tau_m$  are defined by (76). This permits to conclude, using the same arguments as previously, that

$$P(\Delta_3 \geq x/2 \cap B) \leq \gamma e^{\alpha N - \lambda 2^{N/2} x}.$$

Turning to  $P(\Delta_1 \geq \varepsilon 2^M 2^{N/2})$ , we have :

$$P(\Delta_1 \geq \varepsilon 2^M 2^{N/2}) \leq 2^N P\left(\max_{1 \leq k \leq 2^M} |\bar{S}_k^{*,n}| \geq \varepsilon 2^M 2^{N/2}\right)$$

Applying the Doob's maximal inequality to the positive submartingales  $(e^{t\bar{S}_k^{*,n}})$  and  $(e^{-t\bar{S}_k^{*,n}})$ , where  $t > 0$ , and optimizing on  $t$ , we deduce using the notation (81):

$$P(\Delta_1 \geq \varepsilon 2^M 2^{N/2}) \leq 2^N (e^{2^M \Lambda^*(\varepsilon 2^{N/2})} + e^{2^M \Lambda^*(-\varepsilon 2^{N/2})})$$

and from (42)

$$P(\Delta_1 \geq \varepsilon 2^M 2^{N/2}) \leq 2^{N+1} e^{-\eta \varepsilon^2 2^N 2^M} \leq 2^{N+1} e^{-\eta \varepsilon x 2^{N/2}/8},$$

for some  $\eta > 0$ . Similarly for the standard gaussian variables, we have :

$$P(\max_{1 \leq k \leq 2^M} |S_k| \geq \varepsilon 2^M 2^{N/2}) \leq 2e^{-\varepsilon^2 2^M 2^N/2},$$

and so

$$P(\Delta_2 \geq \varepsilon 2^M 2^{N/2}) \leq 2^{N+1} e^{-\varepsilon x 2^{N/2}/16}.$$

This achieves the proof of the Proposition 1.

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