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Vincent Koziarz, Julien Maubon

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MAXIMAL REPRESENTATIONS OF UNIFORM COMPLEX HYPERBOLIC LATTICES

VINCENT KOZIARZ AND JULIEN MAUBON

ABSTRACT. Let ρ be a maximal representation of a uniform lattice $\Gamma \subset \mathrm{SU}(n, 1)$, $n \geq 2$, in a classical Lie group of Hermitian type H . We prove that necessarily $H = \mathrm{SU}(p, q)$ with $p \geq qn$ and there exists a holomorphic or antiholomorphic ρ -equivariant map from complex hyperbolic space to the symmetric space associated to $\mathrm{SU}(p, q)$. This map is moreover a totally geodesic homothetic embedding. In particular, up to a representation in a compact subgroup of $\mathrm{SU}(p, q)$, the representation ρ extends to a representation of $\mathrm{SU}(n, 1)$ in $\mathrm{SU}(p, q)$.

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1. INTRODUCTION

Lattices in noncompact simple Lie groups can be regrouped in two broad classes: those which are superrigid and those which are not. A lattice Γ in a simple noncompact Lie group G is superrigid (over \mathbb{R} or \mathbb{C}) if for all simple noncompact Lie group H with trivial center, every homomorphism $\Gamma \rightarrow H$ with Zariski-dense image extends to a homomorphism $G \rightarrow H$. Lattices in simple Lie groups of real rank at least 2, such as $\mathrm{SL}(n, \mathbb{Z})$ in $\mathrm{SL}(n, \mathbb{R})$ for $n \geq 3$, as well as lattices in the real rank 1 Lie groups $\mathrm{Sp}(n, 1)$ and F_4^{-20} , are superrigid by [Mar91, Cor92, GS92]. This implies that these lattices are all arithmetic. On the other hand, lattices in the remaining simple Lie groups of real rank 1, $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$, are not superrigid in general. In particular, the study of their representations does not reduce to the study of the representations of the Lie group they live in. There are however important differences between real hyperbolic lattices, i.e. lattices in $\mathrm{SO}(n, 1)$, and complex hyperbolic lattices, i.e. lattices in $\mathrm{SU}(n, 1)$. Real hyperbolic objects are softer and more flexible than their complex counterparts. From the perspective of representations of lattices, for example, it is sometimes possible to deform non trivially lattices of $\mathrm{SO}(n, 1)$ in $\mathrm{SO}(m, 1)$, $m > n \geq 3$, see e.g. [JM87]. The analogous statement does not hold for lattices in $\mathrm{SU}(n, 1)$, $n \geq 2$: W. Goldman and J. Millson [GM87] proved that if $\Gamma \in \mathrm{SU}(n, 1)$, $n \geq 2$, is a uniform lattice and if $\rho : \Gamma \rightarrow \mathrm{SU}(m, 1)$, $m \geq n$, is the composition of the inclusion $\Gamma \hookrightarrow \mathrm{SU}(n, 1)$ with the natural embedding $\mathrm{SU}(n, 1) \hookrightarrow \mathrm{SU}(m, 1)$, then ρ , although not necessarily infinitesimally rigid, is locally rigid. From a maybe more subjective point of view, non arithmetic lattices in $\mathrm{SO}(n, 1)$ can be constructed for all n [GPS88] but there are no similar constructions in the complex case and examples of non arithmetic lattices in $\mathrm{SU}(n, 1)$ are very difficult to come by (and none are known for $n \geq 4$).

We will be interested here in global rigidity results for representations of complex hyperbolic lattices in semisimple Lie groups of Hermitian type with no compact factors which generalize the local rigidity we just mentioned. Recall that a Lie group H is of Hermitian type if its associated symmetric space is a Hermitian symmetric space. The classical noncompact groups of Hermitian type are $\mathrm{SU}(p, q)$ with $p \geq q \geq 1$, $\mathrm{SO}_0(p, 2)$ with $p \geq 3$, $\mathrm{Sp}(m, \mathbb{R})$ with $m \geq 2$ and $\mathrm{SO}^*(2m)$ with $m \geq 4$.

Let Γ be a lattice in $\mathrm{SU}(n, 1)$. The group Γ acts on complex hyperbolic n -space $\mathbb{H}_{\mathbb{C}}^n = \mathrm{SU}(n, 1)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$. The space $\mathbb{H}_{\mathbb{C}}^n$ is the rank 1 Hermitian symmetric space of non compact type and of complex dimension n . From a Riemannian point of view, it is up to isometry the unique complete simply connected Kähler manifold of constant negative holomorphic sectional curvature. The $\mathrm{SU}(n, 1)$ -invariant metric on $\mathbb{H}_{\mathbb{C}}^n$ will be normalized so that its holomorphic sectional curvature is -1 . As a bounded symmetric domain, $\mathbb{H}_{\mathbb{C}}^n$ is biholomorphic to the unit ball in \mathbb{C}^n .

For simplicity in this introduction, and because this is needed in our main result, the lattice Γ is assumed to be uniform (and torsion free) unless otherwise specified, so that the quotient $X := \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$ is a closed manifold.

Let also H be a semisimple Lie group of Hermitian type without compact factors, \mathcal{Y} the symmetric space associated to H and ρ a representation of Γ in H , i.e. a group homomorphism $\rho : \Gamma \rightarrow H$. There is a natural way to measure the “complex size” of the representation ρ by using the invariant Kähler forms of the involved symmetric spaces. The *Toledo invariant* of ρ is defined as follows.

$$\tau(\rho) = \frac{1}{n!} \int_X f^* \omega_{\mathcal{Y}} \wedge \omega^{n-1},$$

where $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow \mathcal{Y}$ is any ρ -equivariant map, ω is the Kähler form of X coming from the invariant Kähler form of $\mathbb{H}_{\mathbb{C}}^n$, $\omega_{\mathcal{Y}}$ is the H -invariant Kähler form of \mathcal{Y} normalized so that its holomorphic sectional curvatures are in $[-1, -1/\text{rk}\mathcal{Y}]$, and $f^*\omega_{\mathcal{Y}}$ is understood as a 2-form on X .

It should be noted that ρ -equivariant maps $\mathbb{H}_{\mathbb{C}}^n \rightarrow \mathcal{Y}$ always exist, because \mathcal{Y} is contractible, and that any two such maps are equivariantly homotopic, so that the Toledo invariant depends only on ρ , not on the choice of f . In fact, it depends only on the connected component of $\text{Hom}(\Gamma, H)$ containing ρ , because it can be seen as a characteristic class of the flat bundle on X associated to ρ . The definition of the Toledo invariant can be extended to non uniform lattices with a bit more work.

A fundamental fact about the Toledo invariant that was established in full generality by M. Burger and A. Iozzi in [BI07] is that it satisfies the following *Milnor-Wood type inequality*:

$$|\tau(\rho)| \leq \text{rk}(\mathcal{Y}) \text{vol}(X).$$

This allows to single out a special class of representations, namely those for which this inequality is an equality. These are the *maximal representations* we are interested in.

The Toledo invariant was first considered for representations of surface groups, i.e. when Γ is the fundamental group of a closed Riemann surface, which can be seen as a uniform lattice in $\text{SU}(1, 1)$. It appeared for the first time in D. Toledo's 1979 paper [Tol79] and more explicitly in [Tol89], where the Milnor-Wood inequality was proved for $n = 1$ and $\text{rk}\mathcal{Y} = 1$, namely when $H = \text{SU}(m, 1)$ for some $m \geq 1$. Toledo proved that maximal representations are faithful with discrete image, and stabilize a complex line in complex hyperbolic m -space, thus generalizing a theorem of Goldman for $H = \text{SL}(2, \mathbb{R})$ [Gol80, Gol88]. Analogous results in the non uniform case were proved in [BI07, KM08a]. L. Hernandez showed in [Her91] that maximal representations of surface groups in $H = \text{SU}(p, 2)$, $p \geq 2$, are also discrete and faithful and stabilize a symmetric subspace associated to the subgroup $\text{SU}(2, 2)$ in \mathcal{Y} . Maximal representations of surface groups are now known to be reductive, discrete and faithful, to stabilize a maximal tube type subdomain in \mathcal{Y} , and in general to carry interesting geometric structures, see e.g. [BIW10, GW12]. They are nevertheless quite flexible. They can for example always be deformed to representations that are Zariski-dense in the subgroup corresponding to the tube type subdomain they stabilize [BIW10].

On the other hand, as indicated by the local rigidity result of [GM87], maximal representations of higher dimensional complex hyperbolic lattices, that is, lattices in $\text{SU}(n, 1)$ for n greater than 1, are expected to be much more rigid.

This was confirmed for rank 1 targets by K. Corlette in [Cor88] (the statement was given for representations maximizing the so-called volume instead of the Toledo invariant but the proof for the Toledo invariant is essentially the same). Corlette proved that if ρ is a volume-maximal representation of a uniform lattice $\Gamma \subset \text{SU}(n, 1)$, $n \geq 2$, in $H = \text{SU}(m, 1)$, then there exists a ρ -equivariant holomorphic totally geodesic embedding $\mathbb{H}_{\mathbb{C}}^n \rightarrow \mathbb{H}_{\mathbb{C}}^m$. This answered a conjecture of Goldman and Millson and implies the local rigidity of [GM87]. This was later shown to hold also in the case of non uniform lattices [BI08, KM08a].

For $n \geq 2$ and higher rank targets, the situation was until now far from being well understood. The case of real rank 2 target Lie groups has been treated in [KM08b] (for uniform lattices), but the proof did not go through to higher ranks. Very recently, M. B. Pozzetti [Poz14] succeeded in generalizing the approach of [BI08] and proved that for $n \geq 2$ there are no Zariski dense maximal representations of a lattice $\Gamma \subset \text{SU}(n, 1)$ in $\text{SU}(p, q)$ if $p > q > 1$. There is no rank restriction in her result, and it is also valid for non uniform lattices, but as of now

it seems to depend strongly on having a non tube type target (this is the meaning of the assumption $p \neq q$).

In this paper, we prove the expected global rigidity for maximal representations of uniform lattices of $SU(n, 1)$, $n \geq 2$, in all classical Lie groups of Hermitian type:

Theorem 1.1. *Let Γ be a uniform (torsion free) lattice in $SU(n, 1)$, $n \geq 2$. Let ρ be a group homomorphism of Γ in a classical noncompact Lie group of Hermitian type H , i.e. H is either $SU(p, q)$ with $p \geq q \geq 1$, $SO_0(p, 2)$ with $p \geq 3$, $Sp(m, \mathbb{R})$ with $m \geq 2$, or $SO^*(2m)$ with $m \geq 4$.*

If ρ is maximal, then $H = SU(p, q)$ with $p \geq qn$, ρ is reductive and there exists a holomorphic or antiholomorphic ρ -equivariant map from $\mathbb{H}_{\mathbb{C}}^n$ to the symmetric space $\mathcal{Y}_{p,q}$ associated to $SU(p, q)$.

As a consequence, maximal representations can be described completely:

Corollary 1.2. *Let $n \geq 2$ and $p \geq qn$. Let $\rho : \Gamma \rightarrow SU(p, q)$ be a maximal representation of a uniform torsion free lattice $\Gamma \subset SU(n, 1)$. Then:*

- *the ρ -equivariant holomorphic or antiholomorphic map $\mathbb{H}_{\mathbb{C}}^n \rightarrow \mathcal{Y}_{p,q}$ whose existence is guaranteed by Theorem 1.1 is a totally geodesic homothetic embedding;*
- *the representation ρ is faithful, discrete, and $\rho(\Gamma)$ stabilizes (and acts cocompactly on) a totally geodesic copy of $\mathbb{H}_{\mathbb{C}}^n$ in $\mathcal{Y}_{p,q}$, of induced holomorphic sectional curvature $-\frac{1}{q}$;*
- *up to passing to a finite index subgroup and conjugacy, the representation ρ is a product $\rho_{\text{diag}} \times \rho_{\text{cpt}}$, where ρ_{diag} is the restriction to Γ of the standard diagonal embedding $SU(n, 1) \hookrightarrow SU(n, 1)^q \hookrightarrow SU(p, q)$, and ρ_{cpt} is a representation of Γ in the compact subgroup $S(U(1)^q \times U(p - qn))$ of $SU(p, q)$.*

Because as we said the Toledo invariant is constant on connected components of $\text{Hom}(\Gamma, H)$, this also implies the local rigidity of maximal representations and in particular we have:

Corollary 1.3. *Let $n \geq 2$ and $p \geq qn$. Then the restriction to a uniform lattice $\Gamma \subset SU(n, 1)$ of the standard diagonal embedding $SU(n, 1) \hookrightarrow SU(n, 1)^q \hookrightarrow SU(p, q)$ is locally rigid.*

This last corollary is in fact true without assuming the lattice Γ to be uniform [Poz14, Corollary 1.5]. It is also probably a special case of the main result of [Kli11], where B. Klingler gave a general algebraic condition for representations of uniform lattices in $SU(n, 1)$ induced by representations of $SU(n, 1)$ to be locally rigid.

To prove Theorem 1.1, we work with a reductive representation $\rho : \Gamma \rightarrow H$ (non reductive representations can be easily ruled out later) and we consider the *harmonic Higgs bundle* (E, θ) on the closed complex hyperbolic manifold $X = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$ associated to ρ by the work of K. Corlette [Cor88] and C. Simpson [Sim92]. This Higgs bundle is *polystable* and has a *real structure* which comes from the fact that it is constructed out of a representation in a Lie group of Hermitian type (and not merely in the general linear group). The Toledo invariant is interpreted in this setting as the degree of a vector bundle on X . See §3.1 and §4.1. These facts can be used in some situations to (re)prove the Milnor-Wood inequality and study maximal representations. This has been widely done for representations of surface groups, see e.g. [Xia00, MX02, BGPG03, BGPG06], and also, with limited success, for higher dimensional lattices [KM08b].

The main novelty here is the study of the interplay between the Higgs bundle point of view and the geometry and dynamics of the *tautological foliation* \mathcal{T} on the projectivized tangent bundle $\mathbb{P}T_X$ of the complex hyperbolic manifold X .

There is a well-defined notion of *complex geodesics* in complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$. This implies that $\mathbb{P}T_X$ carries a smooth complex 1-dimensional foliation \mathcal{T} by lifts of tangent spaces of (local) complex geodesics, see §2.1. The tangential subbundle L of the tangent bundle of $\mathbb{P}T_X$, i.e. the subbundle of tangent vectors tangent to the leaves of the foliation, identifies naturally with the tautological line bundle $\mathcal{O}_{\mathbb{P}T_X}(-1)$ on $\mathbb{P}T_X$. The tautological foliation is endowed with a homogeneous transverse structure, where the $SU(n, 1)$ -homogeneous space in question is the space \mathcal{G} of complex geodesics of $\mathbb{H}_{\mathbb{C}}^n$. This space supports an invariant indefinite but non degenerate Kähler metric, hence an invariant volume form which defines a transverse measure for the foliation \mathcal{T} , cf. §2.2. There is therefore a closed current of integration along the leaves of the foliation \mathcal{T} on $\mathbb{P}T_X$ and, as explained in §2.3, this current can be used to compute the Toledo invariant of the representation ρ .

The idea is then to pull-back the Higgs bundle (E, θ) to a Higgs bundle $(\tilde{E}, \tilde{\theta})$ over the projectivized tangent bundle $\mathbb{P}T_X$ and to study the behavior of this new Higgs bundle with respect to the tautological foliation \mathcal{T} . This is the content of §3.2. We define the *foliated degree* of a coherent sheaf on $\mathbb{P}T_X$ by integrating its first Chern class along the leaves of \mathcal{T} . We call a subsheaf of $\mathcal{O}(\tilde{E})$ a *leafwise Higgs subsheaf* of \tilde{E} if it is invariant by the Higgs field $\tilde{\theta}$ in the directions tangent to the leaves. With these definitions we introduce notions of *leafwise semistability* and *leafwise polystability* and we prove that they are satisfied by the Higgs bundle $(\tilde{E}, \tilde{\theta})$.

This allows to give in §4.2.1 a new proof of the Milnor-Wood inequality for reductive representations of uniform lattices and to gain a lot of information in the maximal case. To conclude one needs a dynamical argument to understand closures of projections to X of subsets of $\mathbb{P}T_X$ which are saturated under the tautological foliation. This is done using results of M. Ratner on unipotent flows, see §2.4 and §4.2.3.

The interpretation of the Toledo invariant as a “foliated Toledo number” is sketched by M. Burger and A. Iozzi in [BI08, p. 183], where it is attributed to F. Labourie. This point of view is indeed strongly related with their approach, and the one of M. B. Pozzetti, where one wants to prove that when a representation is maximal, there exists an equivariant measurable map between the Shilov boundaries that preserves a special incidence geometry. In the complex hyperbolic case, this incidence geometry is the geometry of *chains*, i.e. of boundaries at infinity of complex geodesics. Tautological foliations on the projectivized tangent bundle of manifolds carrying a holomorphic projective structure (in particular complex hyperbolic manifolds) are also discussed and used by N. Mok in [Mok05]. Some time ago, without at first grasping the foliated side of the story, the authors of the present paper made some quickly unsuccessful attempts at working with Higgs bundles on the projectivized tangent bundle. Reading F. Labourie’s suggestion in [BI08] and N. Mok’s article [Mok05] encouraged them to try again.

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2. THE TAUTOLOGICAL FOLIATION ON THE PROJECTIVIZED TANGENT BUNDLE OF COMPLEX HYPERBOLIC MANIFOLDS

Complex hyperbolic n -space $\mathbb{H}_{\mathbb{C}}^n$ is the set of negative lines in \mathbb{C}^{n+1} for a Hermitian form h of signature $(n, 1)$. It is an open set in the projective space $\mathbb{C}P^n$.

The Lie group $SU(n, 1) = SU(\mathbb{C}^{n+1}, h)$ is the subgroup of $SL(n+1, \mathbb{C})$ consisting of elements preserving the Hermitian form h . As a group of matrices, in a basis $(e_1, \dots, e_n, e_{n+1})$ of \mathbb{C}^{n+1} where the matrix of h is the diagonal matrix $I_{n,1} = \text{diag}(1, \dots, 1, -1)$,

$$SU(n, 1) = \{M \in SL(n+1, \mathbb{C}) \mid M^* I_{n,1} M = I_{n,1}\},$$

where M^* denotes the conjugate transpose of M .

The group $SU(n, 1)$ acts transitively on $\mathbb{H}_{\mathbb{C}}^n$. The stabilizer of a point is a maximal compact subgroup of $SU(n, 1)$ and is conjugated to $U(n) = S(U(n) \times U(1))$. This gives a realization of $\mathbb{H}_{\mathbb{C}}^n$ as the Hermitian symmetric space $SU(n, 1)/S(U(n) \times U(1))$.

Complex hyperbolic space also admits a realization as a bounded symmetric domain: it is biholomorphic to the unit ball in \mathbb{C}^n .

We equip the Lie algebra $\mathfrak{su}(n, 1)$ of $SU(n, 1)$ with the Killing form $b(A, B) = 2 \text{tr}(AB)$, normalized so that the holomorphic sectional curvature of the invariant Kähler metric ω it induces on $\mathbb{H}_{\mathbb{C}}^n$ is -1 .

An n -dimensional complex hyperbolic manifold X is the quotient of $\mathbb{H}_{\mathbb{C}}^n$ by a discrete torsion free subgroup Γ of $SU(n, 1)$.

2.1. Complex geodesics and the tautological foliation.

There are two kinds of real 2-dimensional totally geodesic subspaces of $\mathbb{H}_{\mathbb{C}}^n$: totally real copies of the Poincaré disc, of induced sectional curvature $-1/4$, and their complex counterparts, the *complex geodesics*, which are complex copies of the Poincaré disc, of induced sectional curvature -1 . The complex geodesics of $\mathbb{H}_{\mathbb{C}}^n \subset \mathbb{C}\mathbb{P}^n$ are just the intersections of $\mathbb{H}_{\mathbb{C}}^n$ with the complex lines $\mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^n$. Given a point in $\mathbb{H}_{\mathbb{C}}^n$ and a complex tangent line at this point, there is a unique complex geodesic through that point tangent to the complex line.

Let $T_{\mathbb{H}_{\mathbb{C}}^n} \rightarrow \mathbb{H}_{\mathbb{C}}^n$ be the holomorphic tangent bundle of $\mathbb{H}_{\mathbb{C}}^n$ and consider the *projectivized tangent bundle* $\pi : \mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n} \rightarrow \mathbb{H}_{\mathbb{C}}^n$ of $\mathbb{H}_{\mathbb{C}}^n$. It is a holomorphic bundle and the fiber over a point $x \in \mathbb{H}_{\mathbb{C}}^n$ is the projective space of lines in the tangent space $T_{\mathbb{H}_{\mathbb{C}}^n, x}$.

We can pull-back the tangent bundle $T_{\mathbb{H}_{\mathbb{C}}^n} \rightarrow \mathbb{H}_{\mathbb{C}}^n$ to $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ to obtain a vector bundle $\pi^*T_{\mathbb{H}_{\mathbb{C}}^n} \rightarrow \mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$. There is a natural line subbundle of $\pi^*T_{\mathbb{H}_{\mathbb{C}}^n}$, the tautological line bundle $\mathcal{O}_{\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}}(-1)$:

$$\mathcal{O}_{\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}}(-1)_{\xi} = \{u \in T_{\mathbb{H}_{\mathbb{C}}^n, \pi(\xi)} \mid u \in \xi\}.$$

The complex geodesics on $\mathbb{H}_{\mathbb{C}}^n$ define a foliation \mathcal{T} on $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ in the following way: if ξ is a point in $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$, there is a unique complex geodesic C in $\mathbb{H}_{\mathbb{C}}^n$ through $\pi(\xi)$ and tangent to the line $\xi \subset T_{\mathbb{H}_{\mathbb{C}}^n, \pi(\xi)}$. The holomorphic tangent bundle T_C of C defines a complex line in $T_{\mathbb{H}_{\mathbb{C}}^n, x}$ for each $x \in C$, hence a holomorphic curve \mathcal{L}_{ξ} in $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$. By definition, \mathcal{L}_{ξ} is the leaf through ξ of the foliation \mathcal{T} on $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$.

Of course this foliation defines a holomorphic line subbundle L of $T_{\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}}$: the fiber L_{ξ} of L at $\xi \in \mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ is the holomorphic tangent space at ξ of the leaf \mathcal{L}_{ξ} . By construction the differential π_* of π at ξ maps L_{ξ} to the line $\xi \subset T_{\mathbb{H}_{\mathbb{C}}^n, \pi(\xi)}$. This also means that when considered as a morphism from $T_{\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}}$ to $\pi^*T_{\mathbb{H}_{\mathbb{C}}^n}$, π_* realizes an isomorphism between the line subbundle L of $T_{\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}}$ and the tautological line subbundle $\mathcal{O}_{\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}}(-1)$ of $\pi^*T_{\mathbb{H}_{\mathbb{C}}^n}$. Therefore we will call \mathcal{T} the *tautological foliation*.

If Γ is a discrete torsion free subgroup in $SU(n, 1)$ and $X = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$ the corresponding complex hyperbolic manifold, all the objects that have just been defined descend to X , and will be denoted by the same letters: there is a tautological foliation \mathcal{T} on the projectivized tangent bundle $\pi : \mathbb{P}T_X \rightarrow X$ whose leaves are locally given by the tangent spaces to (local)

complex geodesics. This defines a line subbundle L of $T_{\mathbb{P}T_X}$ isomorphic to the tautological line subbundle $\mathcal{O}_{\mathbb{P}T_X}(-1)$ of $\pi^*T_X \rightarrow \mathbb{P}T_X$.

2.2. The transverse structure of the foliation.

Let us now consider the projectivized tangent bundle $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ of $\mathbb{H}_{\mathbb{C}}^n$ and its tautological foliation from the homogeneous point of view.

As we said, complex geodesics in $\mathbb{H}_{\mathbb{C}}^n$ are intersections of complex lines in $\mathbb{C}P^n$ with $\mathbb{H}_{\mathbb{C}}^n$, meaning that they are precisely the complex 2-planes in \mathbb{C}^{n+1} on which the Hermitian form h restricts to a Hermitian form of signature $(1, 1)$. Therefore, as a homogeneous space, the space \mathcal{G} of complex geodesics of $\mathbb{H}_{\mathbb{C}}^n$ is the homogeneous space $SU(n, 1)/S(U(n-1) \times U(1, 1))$. On the other hand, a point in the projectivized tangent bundle $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ of $\mathbb{H}_{\mathbb{C}}^n$ is given by two h -orthogonal complex lines in \mathbb{C}^{n+1} spanning a complex geodesic. Hence $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ is the quotient $SU(n, 1)/S(U(n-1) \times U(1) \times U(1))$. The central fiber of the holomorphic projection $\pi : \mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n} \rightarrow \mathbb{H}_{\mathbb{C}}^n$ is $U(n)/U(n-1) \times U(1) = \mathbb{C}P^{n-1}$ as it should.

Of course there is a map from $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ to \mathcal{G} associating to a point in the projectivized tangent bundle the complex geodesic it defines. This gives another holomorphic fibration $\pi' : \mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n} \rightarrow \mathcal{G}$. The central fiber $U(1, 1)/U(1) \times U(1)$ is isometric to the Poincaré disc and the fibers of the fibration π' are precisely the leaves of the tautological foliation on $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$.

For a general complex hyperbolic manifold X , the tautological foliation \mathcal{T} on $\mathbb{P}T_X$ is not given by a holomorphic fibration as in the case of $\mathbb{H}_{\mathbb{C}}^n$. But \mathcal{T} inherits a transverse structure of \mathcal{G} -homogeneous foliation, see [God91, Proposition 3.3, p. 165]. To understand this transverse structure, we need to describe the geometry of the space of complex geodesics \mathcal{G} . We will see that \mathcal{G} is equipped with a G -invariant indefinite Kähler form, which gives a G -invariant volume form, hence an invariant measure μ , which defines a transverse measure for the foliation \mathcal{T} on $\mathbb{P}T_X$, and hence a closed current of integration along the leaves of \mathcal{T} .

In order to lighten the notations, we set $G_{\mathbb{C}} = SL(n+1, \mathbb{C})$, $G = SU(n, 1)$, $K = S(U(n) \times U(1))$, $V = S(U(n-1) \times U(1) \times U(1))$, $V' = S(U(n-1) \times U(1, 1))$, and we denote their respective Lie algebras by $\mathfrak{g}_{\mathbb{C}}$, \mathfrak{g} , \mathfrak{k} , \mathfrak{v} , \mathfrak{v}' .

We also denote by Q , Q' , Q'' the parabolic subgroups of $G_{\mathbb{C}}$ stabilizing the flags $(\langle e_1, \dots, e_{n-1} \rangle \subset \langle e_1, \dots, e_n \rangle \subset \mathbb{C}^{n+1})$, $(\langle e_1, \dots, e_{n-1} \rangle \subset \mathbb{C}^{n+1})$ and $(\langle e_1, \dots, e_n \rangle \subset \mathbb{C}^{n+1})$ respectively. Here and in the rest of the paper, whenever v_1, \dots, v_k are elements of a vector space, $\langle v_1, \dots, v_k \rangle$ denotes their linear span. The Lie algebra of Q will be denoted by the corresponding fraktur letter. Then $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n} = G/V$ is the open G -orbit of eQ in $G_{\mathbb{C}}/Q$, $\mathcal{G} = G/V'$ is the open G -orbit of eQ' in $G_{\mathbb{C}}/Q'$ and $\mathbb{H}_{\mathbb{C}}^n = G/K$ is the open G -orbit of eQ'' in $G_{\mathbb{C}}/Q'' = \mathbb{C}P^n$.

We have the following matrix expressions:

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}, A \in \mathfrak{u}(n), a \in \mathbb{C}, a + \operatorname{tr} A = 0 \right\},$$

$$\mathfrak{v} = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, A \in \mathfrak{u}(n-1), a, b \in \mathbb{C}, a + b + \operatorname{tr} A = 0 \right\}.$$

$$\mathfrak{v}' = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, A \in \mathfrak{u}(n-1), B \in \mathfrak{u}(1, 1), \operatorname{tr} A + \operatorname{tr} B = 0 \right\}.$$

The real tangent space of $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ at eV is naturally identified with

$$\mathfrak{m} = \left\{ \xi = \begin{pmatrix} 0 & \xi_3 & \xi_2 \\ -\xi_3^* & 0 & \xi_1 \\ \xi_2^* & \xi_1^* & 0 \end{pmatrix}, \xi_1 \in \mathbb{C}, \xi_2, \xi_3 \in \mathbb{C}^{n-1} \right\}.$$

The holomorphic tangent space of $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ at eV is $\mathfrak{g}/\mathfrak{q}$ and the $(1,0)$ -part of the complexified tangent space is

$$\mathfrak{m}^{1,0} = \left\{ \begin{pmatrix} 0 & \xi_3 & \xi_2 \\ 0 & 0 & \xi_1 \\ 0 & 0 & 0 \end{pmatrix}, \xi_1 \in \mathbb{C}, \xi_2, \xi_3 \in \mathbb{C}^{n-1} \right\},$$

so that the invariant complex structure is given at eV by

$$J \begin{pmatrix} 0 & \xi_3 & \xi_2 \\ -\xi_3^* & 0 & \xi_1 \\ \xi_2^* & \xi_1^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-1}\xi_3 & \sqrt{-1}\xi_2 \\ \sqrt{-1}\xi_3^* & 0 & \sqrt{-1}\xi_1 \\ -\sqrt{-1}\xi_2^* & -\sqrt{-1}\xi_1^* & 0 \end{pmatrix}.$$

Define the subspaces

$$\mathfrak{m}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi_1 \\ 0 & \xi_1^* & 0 \end{pmatrix}, \xi_1 \in \mathbb{C} \right\},$$

$$\mathfrak{m}_2 = \left\{ \begin{pmatrix} 0 & 0 & \xi_2 \\ 0 & 0 & 0 \\ \xi_2^* & 0 & 0 \end{pmatrix}, \xi_2 \in \mathbb{C}^{n-1} \right\}$$

and

$$\mathfrak{m}_3 = \left\{ \begin{pmatrix} 0 & \xi_3 & 0 \\ -\xi_3^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \xi_3 \in \mathbb{C}^{n-1} \right\}$$

of \mathfrak{m} and the corresponding subspaces $\mathfrak{m}_1^{1,0}$, $\mathfrak{m}_2^{1,0}$, $\mathfrak{m}_3^{1,0}$ of $\mathfrak{m}^{1,0}$. It is plain that $\mathfrak{k} \oplus (\mathfrak{m}_1 \oplus \mathfrak{m}_2)$ is a Cartan decomposition of \mathfrak{g} . The subspaces \mathfrak{m}_1 , \mathfrak{m}_2 and \mathfrak{m}_3 are invariant under the adjoint action of V on \mathfrak{m} , and therefore define C^∞ subbundles of the real tangent bundle of $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$. The subspaces $\mathfrak{q} \oplus \mathfrak{m}_1^{1,0}$ and $\mathfrak{q} \oplus \mathfrak{m}_3^{1,0}$ are invariant under the adjoint action of Q and therefore define holomorphic subbundles of the holomorphic tangent bundle of $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$. These holomorphic subbundles correspond respectively to the holomorphic tangent bundle of the fibers of the fibrations $\pi' : \mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n} \rightarrow \mathcal{G}$ and $\pi : \mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n} \rightarrow \mathbb{H}_{\mathbb{C}}^n$.

Let $\omega_1, \omega_2, \omega_3$ be the skew-symmetric bilinear forms on \mathfrak{m} given by

$$\omega_j(\xi, \eta) = 2\sqrt{-1}(\eta_j^* \xi_j - \xi_j^* \eta_j),$$

for $\xi, \eta \in \mathfrak{m}$. These forms are invariant by V hence they define G -invariant 2-forms on $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n} = G/V$ which will be denoted by the same letters.

It is easily checked that the bilinear form $\omega_1 + \omega_2$ on $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ is invariant by K , hence that it defines a G -invariant 2-form ω on $\mathbb{H}_{\mathbb{C}}^n = G/K$. The form ω is closed (because it is an invariant 2-form on a symmetric space) and it is precisely the invariant Kähler form on $\mathbb{H}_{\mathbb{C}}^n$, normalized so as to have constant holomorphic sectional curvature -1 . It is also given by $\omega(\xi, \eta) = b(\zeta, [\xi, \eta]) = b(\text{ad}(\zeta)\xi, \eta)$ for $\xi, \eta \in \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where b is the Killing form on \mathfrak{g} and ζ is the element of the 1-dimensional center of \mathfrak{k} such that $\text{ad}(\zeta)$ gives the invariant complex structure of $\mathbb{H}_{\mathbb{C}}^n$:

$$\zeta = \begin{pmatrix} \frac{\sqrt{-1}}{n+1} 1_n & 0 \\ 0 & -\frac{n\sqrt{-1}}{n+1} \end{pmatrix}$$

Here and in the rest of the paper, if k is an integer, 1_k denotes the identity matrix of size k .

The homogeneous space \mathcal{G} of complex geodesics of $\mathbb{H}_{\mathbb{C}}^n$ is also a complex manifold and it admits a G -invariant non degenerate but indefinite Kähler form $\omega_{\mathcal{G}}$ (its signature is $(n-1, n-1)$). Indeed, one checks that the bilinear form $\omega_2 - \omega_3$ on $\mathfrak{m}_2 \oplus \mathfrak{m}_3$ is invariant by $V' = \mathrm{S}(\mathrm{U}(n-1) \times \mathrm{U}(1,1))$ and hence defines a G -invariant form $\omega_{\mathcal{G}}$ on $\mathcal{G} = G/V'$. Again, this form can be computed as $\omega_{\mathcal{G}}(\xi, \eta) = b(\zeta', [\xi, \eta]) = b(\mathrm{ad}(\zeta')\xi, \eta)$ for $\xi, \eta \in \mathfrak{m}_2 \oplus \mathfrak{m}_3$, where

$$\zeta' = \begin{pmatrix} \frac{2\sqrt{-1}}{n+1}1_{n-1} & 0 \\ 0 & -\frac{(n-1)\sqrt{-1}}{n+1}1_2 \end{pmatrix}$$

is the element of the 1-dimensional center of \mathfrak{v}' such that $\mathrm{ad}(\zeta')$ gives the invariant complex structure of \mathcal{G} . This expression of $\omega_{\mathcal{G}}$ implies in particular that it is closed. The indefinite Kähler form $\omega_{\mathcal{G}}$ defines a G -invariant volume form hence a G -invariant measure on \mathcal{G} . This measure is by construction a transverse measure for the tautological foliation on $\mathbb{P}T_X$ [God91, §3.2 i) p. 355].

The forms ω_j allow also to compute the curvature form of the dual $\mathcal{O}_{\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}}(1)$ of the tautological line bundle over $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ endowed with the natural metric induced from the one of $T_{\mathbb{H}_{\mathbb{C}}^n}$ (see e.g. [CW03, (3.7)]). This is a positive (1,1)-form which we take as the Kähler form $\omega_{\mathbb{P}T_X}$ on $\mathbb{P}T_X$. One finds

$$\omega_{\mathbb{P}T_X} = \sqrt{-1}\Theta(\mathcal{O}_{\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}}(1)) = \omega_1 + \frac{1}{2}(\omega_2 + \omega_3) = (\omega_1 + \omega_2) - \frac{1}{2}(\omega_2 - \omega_3) = \pi^*\omega - \frac{1}{2}\pi'^*\omega_{\mathcal{G}}.$$

(Note that we normalized the metric on $T_{\mathbb{H}_{\mathbb{C}}^n}$ in order to have constant holomorphic sectional curvature -1 and that ω_3 restricted to a fiber of π is $2\omega_{FS}$ in [CW03].)

If X is a complex hyperbolic manifold of the form $\Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$, the G -invariant 2-forms ω_1, ω_2 and ω_3 descend to forms on $\mathbb{P}T_X$ which will be denoted by the same letters. The G -invariant Kähler form ω on $\mathbb{H}_{\mathbb{C}}^n$ descends to a Kähler form on X , also denoted ω . Again, $\omega_1 + \omega_2$ is the pull-back $\pi^*\omega$ and $\omega_1 + \frac{1}{2}(\omega_2 + \omega_3)$ is the curvature form of $\mathcal{O}_{\mathbb{P}T_X}(1)$, which is isomorphic to the dual L^\vee of the tangent bundle L to the tautological foliation \mathcal{T} on $\mathbb{P}T_X$. The 2-form $\omega_2 - \omega_3$ is a closed form on $\mathbb{P}T_X$.

2.3. Integration along the leaves.

Assume from now on that Γ is uniform, so that $X = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$ is a closed complex hyperbolic manifold. Let

$$\Omega := \frac{(-1)^{n-1}}{\mathrm{vol}(\mathbb{C}\mathbb{P}^{n-1})} (\omega_2 - \omega_3)^{2n-2} = \frac{(-1)^{n-1}}{\mathrm{vol}(\mathbb{C}\mathbb{P}^{n-1})} \pi'^*\omega_{\mathcal{G}}^{2n-2}$$

be the form on $\mathbb{P}T_X$ defined by the pull-back to $\mathbb{P}T_{\mathbb{H}_{\mathbb{C}}^n}$ of the G -invariant volume form $\omega_{\mathcal{G}}^{2n-2}$ (suitably normalized) on the space of complex geodesics \mathcal{G} . It is a closed semi-positive $(2n-2, 2n-2)$ -form of rank $4n-4$ on $\mathbb{P}T_X$. It defines a closed semi-positive (1,1)-current on $\mathbb{P}T_X$ by the formula

$$\alpha \longmapsto \int_{\mathcal{T}} \alpha := \frac{1}{(2n-2)!} \int_{\mathbb{P}T_X} \alpha \wedge \Omega,$$

for α a (1,1)-form on $\mathbb{P}T_X$. This is the current of integration along the leaves of the foliation \mathcal{T} associated to the transverse measure μ (again suitably normalized) coming from the invariant volume form on \mathcal{G} . An alternate definition of this current, which explains its name, is as follows [God91, §3.5, p. 357]. Take a covering $(U_i)_{i \in I}$ of $\mathbb{P}T_X$ by regular open sets for the foliation \mathcal{T} , and a partition of unity $(\chi_i)_{i \in I}$ subordinated to it. Let T_i be the space of plaques of U_i and call again μ the measure on T_i given by the transverse measure. For a 2-form α on $\mathbb{P}T_X$, the forms $\chi_i \alpha$ are compactly supported in the open sets U_i and by integrating $\chi_i \alpha$ on

the plaques of U_i , we obtain a compactly supported function on the space T_i which we can then integrate against the measure μ to get

$$\int_{\mathcal{T}} \alpha = \sum_{i \in I} \int_{T_i} \left(\int_t \chi_i \alpha \right) d\mu(t).$$

The following easy but fundamental lemma will allow us to compute Toledo invariants of representations and degrees of vector bundles on X by integration along the leaves of \mathcal{T} .

Lemma 2.1. *Let β be a 2-form on X , then*

$$\frac{1}{n!} \int_X \beta \wedge \omega^{n-1} = \int_{\mathcal{T}} \pi^* \beta.$$

Proof. Let α be a 2-form on $\mathbb{P}T_X$. Then,

$$\begin{aligned} \int_{\mathbb{P}T_X} \alpha \wedge \omega_2^{n-1} \wedge \omega_3^{n-1} &= \frac{1}{2} \int_{\mathbb{P}T_X} \langle \alpha, \omega_1 \rangle \omega_1 \wedge \omega_2^{n-1} \wedge \omega_3^{n-1} \\ &= \frac{1}{2n} \int_{\mathbb{P}T_X} \langle \alpha, \omega_1 \rangle (\pi^* \omega)^n \wedge \omega_3^{n-1} \\ &= \frac{1}{2n} \int_X \left(\int_{\pi^{-1}(x)} \langle \alpha, \omega_1 \rangle \omega_3^{n-1} \right) \omega^n. \end{aligned}$$

If now β is a 2-form on X , one has

$$\int_{\pi^{-1}(x)} \langle \pi^* \beta, \omega_1 \rangle \frac{\omega_3^{n-1}}{(n-1)!} = \frac{\text{vol}(\mathbb{C}\mathbb{P}^{n-1})}{n} \langle \beta, \omega \rangle_x.$$

Hence

$$\begin{aligned} \int_{\mathbb{P}T_X} \pi^* \beta \wedge \omega_2^{n-1} \wedge \omega_3^{n-1} &= \frac{(n-1)! \text{vol}(\mathbb{C}\mathbb{P}^{n-1})}{2n^2} \int_X \langle \beta, \omega \rangle \omega^n \\ &= (n-1)!^2 \text{vol}(\mathbb{C}\mathbb{P}^{n-1}) \frac{1}{n!} \int_X \beta \wedge \omega^{n-1}, \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{n!} \int_X \beta \wedge \omega^{n-1} &= \frac{1}{\text{vol}(\mathbb{C}\mathbb{P}^{n-1})} \int_{\mathbb{P}T_X} \pi^* \beta \wedge \frac{\omega_2^{n-1}}{(n-1)!} \wedge \frac{\omega_3^{n-1}}{(n-1)!} \\ &= \frac{(-1)^{n-1}}{\text{vol}(\mathbb{C}\mathbb{P}^{n-1})} \int_{\mathbb{P}T_X} \pi^* \beta \wedge \frac{(\omega_2 - \omega_3)^{2n-2}}{(2n-2)!}. \end{aligned}$$

□

If ρ is a representation of our lattice $\Gamma \subset \text{SU}(n, 1)$ in the automorphism group of a Hermitian symmetric space \mathcal{Y} , recall from the introduction that the *Toledo invariant* of ρ is defined by

$$\tau(\rho) = \frac{1}{n!} \int_X f^* \omega_{\mathcal{Y}} \wedge \omega^{n-1},$$

where $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow \mathcal{Y}$ is any ρ -equivariant map, and $\omega_{\mathcal{Y}}$ is the invariant Kähler form of \mathcal{Y} , normalized so that the holomorphic sectional curvature of \mathcal{Y} lies in $[-1, -1/\text{rk}\mathcal{Y}]$. Here, using the ρ -equivariance of f and the invariance of $\omega_{\mathcal{Y}}$, $f^* \omega_{\mathcal{Y}}$ is seen as a form on $X = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$.

We therefore have

$$\tau(\rho) = \int_{\mathcal{T}} \pi^* f^* \omega_{\mathcal{Y}}.$$

2.4. A consequence of Ratner's theorem on orbit closures.

We state here a fundamental property of the leaves of the tautological foliation which follows from the resolution by M. Ratner of Raghunathan's conjecture on orbit closures.

Let \mathcal{L} be a leaf of the tautological foliation \mathcal{T} on $\mathbb{P}T_X = \Gamma \backslash G/V$. It is of the form $\Gamma \backslash \Gamma U_{\mathcal{L}} g_{\mathcal{L}} V/V$ for some $g_{\mathcal{L}} \in G$ and a group $U_{\mathcal{L}}$ locally isomorphic to $SU(1, 1)$. Because $U_{\mathcal{L}}$ is generated by unipotent elements, it follows from the work of Ratner [Rat91a] that the closure of the orbit $\Gamma e \cdot U_{\mathcal{L}}$ in $\Gamma \backslash G$ is homogeneous, namely that there exists a closed subgroup $S_{\mathcal{L}}$ of G such that $U_{\mathcal{L}} \subset S_{\mathcal{L}}$ and $\overline{\Gamma e \cdot U_{\mathcal{L}}} = \Gamma e \cdot S_{\mathcal{L}}$. This implies that $\Gamma \cap S_{\mathcal{L}}$ is a lattice in $S_{\mathcal{L}}$ [Rag, Theorem 1.13] and that $S_{\mathcal{L}}$ is a reductive group with compact center, for example because $\text{rk}_{\mathbb{R}} G = 1$ [Sha91].

By [Pay99], the fact that $\text{rk}_{\mathbb{R}} U_{\mathcal{L}} = \text{rk}_{\mathbb{R}} G$ implies that the Lie algebra of $S_{\mathcal{L}}$ is stable by the Cartan involution of G given by the point $g_{\mathcal{L}} K$ of $\mathbb{H}_{\mathbb{C}}^n = G/K$, so that the orbit $\tilde{Y}_{\mathcal{L}} := S_{\mathcal{L}} \cdot g_{\mathcal{L}} K$ of $g_{\mathcal{L}} K$ under $S_{\mathcal{L}}$ in $\mathbb{H}_{\mathbb{C}}^n$ is a totally geodesic submanifold and $Y_{\mathcal{L}} := \Gamma \backslash \Gamma \tilde{Y}_{\mathcal{L}}$ is a closed immersed totally geodesic submanifold of $X = \Gamma \backslash G/K$. This submanifold is the closure of the projection $\pi(\mathcal{L})$ of \mathcal{L} in X .

Summing up, we have:

Proposition 2.2. *Let \mathcal{L} be a leaf of the tautological foliation \mathcal{T} on $\mathbb{P}T_X$. The closure $\overline{\pi(\mathcal{L})}$ of the image of \mathcal{L} by the projection $\pi : \mathbb{P}T_X \rightarrow X$ is a closed immersed totally geodesic submanifold of X .*

3. HIGGS BUNDLES ON COMPLEX HYPERBOLIC MANIFOLDS

3.1. Harmonic Higgs bundles.

A representation ρ of the fundamental group Γ of a compact manifold X to in a real algebraic semisimple Lie group without compact factors $H \subset \text{SL}(N, \mathbb{C})$ is *reductive* if the Zariski closure of $\rho(\Gamma)$ in H is a reductive group.

K. Corlette [Cor88] proved that if $\rho : \pi_1(X) \rightarrow H$ is reductive, then there exists a ρ -equivariant *harmonic map* f from the universal cover \tilde{X} of X to the symmetric space \mathcal{Y} associated to H .

When the manifold X is Kähler, with Kähler form ω , it follows from a Bochner formula due to J. H. Sampson [Sam78] and Y.-T. Siu [Siu80], that the harmonic map f is moreover pluriharmonic, and that the image of the $(1, 0)$ -part $d^{1,0} f : T^{1,0} \mathbb{H}_{\mathbb{C}}^n \rightarrow T^{\mathbb{C}} \mathcal{Y}$ of its complexified differential is Abelian (as a subspace of the complexification of the Lie algebra of H). This has been shown by C. Simpson [Sim88, Sim92] to give a *harmonic Higgs bundle* (E, θ) on X .

Assume in this section that $H = \text{SL}(N, \mathbb{C})$. The bundle E , as a C^{∞} -bundle, is just the flat complex vector bundle of rank N associated to the representation ρ . The Higgs field θ is a $(1, 0)$ -form with values in $\text{End}(E)$, which can be seen as the $(1, 0)$ -part $d^{1,0} f$ of the complexified differential of the harmonic map f . It satisfies $[\theta, \theta] = 0$. The harmonic map can also be thought of as defining a Hermitian metric on E , called the *harmonic metric*, which has the following properties. If D is the flat connection on E and ∇ the component of D which preserves this metric, then $(\nabla'')^2 = 0$ and $\nabla'' \theta = 0$, so that ∇'' defines a holomorphic structure on E for which θ is holomorphic. Moreover D, ∇ and θ are related by

$$D = \nabla + \theta + \theta^*,$$

where θ^* is the adjoint of θ w.r.t. the harmonic metric. This, together with the Chern-Weil formula, implies that (E, θ) is a *polystable* Higgs bundle on X , see [Sim88] and the proof of Proposition 3.1 below. This means for one thing that (E, θ) is a *semistable* Higgs bundle,

namely that if $\mathcal{F} \subset \mathcal{O}_X(E)$ is a Higgs subsheaf of E , i.e. a subsheaf such that $\theta(\mathcal{F}) \subset \mathcal{F} \otimes \Omega_X^1$, then

$$\deg \mathcal{F} := \frac{1}{n!} \int_X c_1(\mathcal{F}) \wedge \omega^{n-1} \leq \deg E = 0.$$

(The last equality holds because E is flat). A semistable Higgs bundle (E, θ) is polystable if additionally whenever \mathcal{F} is a Higgs subsheaf of degree equal to $\deg E$, then \mathcal{F} is the sheaf of sections of a holomorphic vector subbundle F of E stable by θ and the orthogonal complement F^\perp of F w.r.t the harmonic metric is also a holomorphic subbundle of E stable by θ , so that we have a Higgs bundle orthogonal decomposition

$$(E, \theta) = (F, \theta|_F) \oplus (F^\perp, \theta|_{F^\perp}).$$

3.2. Foliated notions of stability on complex hyperbolic manifolds.

Let (E, θ) be a harmonic Higgs bundle on a closed complex hyperbolic manifold X . We may pull it back by $\pi : \mathbb{P}T_X \rightarrow X$ to get a Higgs bundle $(\tilde{E}, \tilde{\theta}) = (\pi^*E, \pi^*\theta)$ over the projectivized tangent bundle $\mathbb{P}T_X$. We lift everything (harmonic metric, flat connection, Chern connection, etc.) to $(\tilde{E}, \tilde{\theta})$ and we denote all the lifted objects with a “ \sim ”, e.g. we have

$$\tilde{D} = \tilde{\nabla} + \tilde{\theta} + \tilde{\theta}^*.$$

Our goal in this section is to understand the behaviour of the Higgs bundle $(\tilde{E}, \tilde{\theta})$ with respect to the tautological foliation \mathcal{T} on $\mathbb{P}T_X$. We begin by defining adapted notions of Higgs subsheaves, degree, and stability.

We restrict the Higgs field $\tilde{\theta} \in H^0(\mathbb{P}T_X, \text{End}(\tilde{E}) \otimes \Omega_{\mathbb{P}T_X}^1)$ to the tangent bundle L of the leaves of the foliation \mathcal{T} . This gives a new notion of invariance for subsheaves $\mathcal{F} \subset \mathcal{O}_{\mathbb{P}T_X}(\tilde{E})$: such a subsheaf \mathcal{F} is *invariant along the leaves* or is a *leafwise Higgs subsheaf* if $\tilde{\theta}$ maps $\mathcal{F} \otimes L$ to \mathcal{F} . (We use the same letter to denote the line bundle L and the invertible sheaf of its sections.)

The *foliated degree* $\deg_{\mathcal{T}} \mathcal{F}$ of a coherent sheaf \mathcal{F} on $\mathbb{P}T_X$ is defined by integration along the leaves as follows:

$$\deg_{\mathcal{T}} \mathcal{F} = \int_{\mathcal{T}} c_1(\mathcal{F}) = \frac{1}{(2n-2)!} \int_{\mathbb{P}T_X} c_1(\mathcal{F}) \wedge \Omega.$$

Lemma 2.1 readily implies that if \mathcal{F} is a coherent sheaf on X , then $\deg_{\mathcal{T}} \pi^* \mathcal{F} = \deg \mathcal{F}$.

The next result shows that the Higgs bundle $(\tilde{E}, \tilde{\theta})$ over $\mathbb{P}T_X$ enjoys some stability properties *along the leaves* of the tautological foliation \mathcal{T} . Before giving the statement, we recall a definition: a subset $A \subset \mathbb{P}T_X$ is *\mathcal{T} -saturated* if it is a union of leaves of the foliation \mathcal{T} , i.e. for all $\xi \in A$, the leaf \mathcal{L}_ξ of the foliation \mathcal{T} through ξ is included in A . Obviously if A is \mathcal{T} -saturated then so is $\mathbb{P}T_X \setminus A$.

Proposition 3.1. *Let $\mathcal{F} \subset \mathcal{O}_{\mathbb{P}T_X}(\tilde{E})$ be a leafwise Higgs subsheaf of $(\tilde{E}, \tilde{\theta})$ such that $\mathcal{O}_{\mathbb{P}T_X}(\tilde{E})/\mathcal{F}$ is torsion-free. Then*

- (1) (Semistability along the leaves) $\deg_{\mathcal{T}} \mathcal{F} \leq 0$;
- (2) (Polystability along the leaves) *if $\deg_{\mathcal{T}} \mathcal{F} = 0$, there exist a \mathcal{T} -saturated analytic subset \mathcal{S}_0 of $\mathbb{P}T_X$ of codimension at least 2 and a holomorphic subbundle F of \tilde{E} defined on $\mathbb{P}T_X \setminus \mathcal{S}_0$, such that \mathcal{F} is the sheaf of sections of F on $\mathbb{P}T_X \setminus \mathcal{S}_0$. Moreover if F^\perp is the orthogonal complement of F w.r.t. the pull-back on \tilde{E} of the harmonic metric, then $\tilde{\theta}(F^\perp \otimes L) \subset F^\perp$ and the C^∞ -decomposition $\tilde{E} = F \oplus F^\perp$ is holomorphic along the leaves of the foliation \mathcal{T} , i.e. for any leaf \mathcal{L} of \mathcal{T} such that $\mathcal{L} \subset \mathbb{P}T_X \setminus \mathcal{S}_0$, $\tilde{E}|_{\mathcal{L}} = F|_{\mathcal{L}} \oplus F^\perp|_{\mathcal{L}}$ is a holomorphic orthogonal direct sum on \mathcal{L} .*

Proof. We first prove the semistability along the leaves. It will follow from the Chern-Weil formula.

Because $\mathcal{O}_{\mathbb{P}T_X}(\tilde{E})/\mathcal{F}$ is torsion-free, there exist an analytic subset $\mathcal{S} \subset \mathbb{P}T_X$ of codimension at least 2, and a holomorphic subbundle F of \tilde{E} defined outside of \mathcal{S} , such that \mathcal{F} is the sheaf of sections of F on $\mathbb{P}T_X \setminus \mathcal{S}$. On $\mathbb{P}T_X \setminus \mathcal{S}$, we can decompose the flat connection \tilde{D} with respect to the orthogonal decomposition $\tilde{E} = F \oplus F^\perp$ where the background metric is the pull-back of the harmonic metric. Denoting by $\sigma \in C_{1,0}^\infty(\mathbb{P}T_X \setminus \mathcal{S}, \text{Hom}(F, F^\perp))$ the second fundamental form of F , we get:

$$\tilde{D} = \begin{pmatrix} \tilde{\nabla}_F & -\sigma^* \\ \sigma & \tilde{\nabla}_{F^\perp} \end{pmatrix} + \begin{pmatrix} \tilde{\theta}_1 & \tilde{\theta}_2 \\ \tilde{\theta}_3 & \tilde{\theta}_4 \end{pmatrix} + \begin{pmatrix} \tilde{\theta}_1^* & \tilde{\theta}_3^* \\ \tilde{\theta}_2^* & \tilde{\theta}_4^* \end{pmatrix}.$$

Since $\tilde{D}^2 = 0$, we have $(\tilde{\nabla}_F + \tilde{\theta}_1 + \tilde{\theta}_1^*)^2 = -(\tilde{\theta}_2 + \tilde{\theta}_3^* - \sigma^*) \wedge (\tilde{\theta}_3 + \tilde{\theta}_2^* + \sigma)$.

Now, the curvature Θ_F of the connection $\tilde{\nabla}_F + \tilde{\theta}_1 + \tilde{\theta}_1^*$ can be used to compute a representative of the first Chern class of \mathcal{F} on $\mathbb{P}T_X \setminus \mathcal{S}$, namely $c_1(F) = \frac{\sqrt{-1}}{2\pi} \text{tr} \Theta_F$, and we get

$$\begin{aligned} \text{tr} \Theta_F \wedge \Omega &= \text{tr}(-\tilde{\theta}_2 \wedge \tilde{\theta}_3^* + \sigma^* \wedge \sigma - \tilde{\theta}_3^* \wedge \tilde{\theta}_3 - \tilde{\theta}_3^* \wedge \sigma + \sigma^* \wedge \tilde{\theta}_3) \wedge \Omega \\ &= \text{tr}(-\tilde{\theta}_2 \wedge \tilde{\theta}_3^* + \sigma^* \wedge \sigma) \wedge \Omega. \end{aligned}$$

Indeed, if $[\eta] \in \mathbb{P}T_X$ and if $(\eta, \xi_1, \dots, \xi_{2n-2})$ is an orthonormal basis of the tangent space of $\mathbb{P}T_X$ at $[\eta]$ (with respect to $\omega_{\mathbb{P}T_X}$) and $(dz, dw_1, \dots, dw_{2n-2})$ is the dual basis, then $\tilde{\theta}_3(\eta) = 0$ since \mathcal{F} is invariant along the leaves. Hence $\tilde{\theta}_3$ can be written $\sum_{1 \leq j \leq 2n-2} \theta_3^j dw_j$. Since $\Omega = (2n-2)! \sqrt{-1}^{2n-2} dw_1 \wedge d\bar{w}_1 \wedge \dots \wedge dw_{2n-2} \wedge d\bar{w}_{2n-2}$, we get that $\text{tr}(\tilde{\theta}_3^* \wedge \tilde{\theta}_3) \wedge \Omega = \text{tr}(\tilde{\theta}_3^* \wedge \sigma) \wedge \Omega = \text{tr}(\sigma^* \wedge \tilde{\theta}_3) \wedge \Omega = 0$.

Hence $\frac{\sqrt{-1}}{2\pi} \text{tr} \Theta_F \wedge \Omega = a \omega_{\mathbb{P}T_X}^{2n-1}$ with $a : \mathbb{P}T_X \setminus \mathcal{S} \rightarrow \mathbb{R}$ a nonpositive function. This finishes the proof of (1), because integrating $\frac{\sqrt{-1}}{2\pi} \text{tr} \Theta_F \wedge \Omega$ on $\mathbb{P}T_X \setminus \mathcal{S}$ indeed computes the foliated degree of \mathcal{F} by the Chern-Weil Formula, see [Sim88, Lemma 3.2] and [Sib13, Theorem 2.23 & Lemma 4.6].

Now let us prove (2). We will follow the well-known proof that Einstein-Hermitian vector bundles are polystable in the usual sense, see [Kob87]. Assume that $\text{deg}_{\mathcal{T}}(\mathcal{F}) = 0$ for the subsheaf \mathcal{F} of the proof of Assertion (1). Then $\text{tr}(-\tilde{\theta}_2 \wedge \tilde{\theta}_3^* + \sigma^* \wedge \sigma) \wedge \Omega = 0$ on $\mathbb{P}T_X \setminus \mathcal{S}$ and this implies that for all $\eta \in L_{|\mathbb{P}T_X \setminus \mathcal{S}}$, $\tilde{\theta}_2(\eta) = 0$ and $\sigma(\eta) = 0$. This means on the first hand that $\tilde{\theta}(F^\perp \otimes L) \subset F^\perp$ and on the second hand that if \mathcal{L} is a leaf of \mathcal{T} , then on $\mathcal{L} \cap (\mathbb{P}T_X \setminus \mathcal{S})$, F is a parallel subbundle of \tilde{E} . As in the proof of [Kob87, Theorem 5.8.3], we deduce that the C^∞ -decomposition $\tilde{E} = F \oplus F^\perp$ is holomorphic when restricted to $\mathcal{L} \cap (\mathbb{P}T_X \setminus \mathcal{S})$. We will say that the decomposition, where it is defined, is holomorphic along the leaves of \mathcal{T} .

Let $\mathcal{S}_0 = \{\xi \in \mathcal{S} \text{ such that } \mathcal{L}_\xi \subset \mathcal{S}\}$. This set is clearly \mathcal{T} -saturated, and it is an analytic subset of codimension at least 2 in $\mathbb{P}T_X$. Indeed, on a regular open set $U \simeq \mathbb{C}^{2n-1}$ for the foliation \mathcal{T} , we may assume that the leaves of \mathcal{T} are the fibers of a linear projection $p : \mathbb{C}^{2n-1} \rightarrow \mathbb{C}^{2n-2}$. Then $\mathcal{S}_0 = \{\xi \in \mathcal{S} \text{ such that } \dim p^{-1}(p(\xi)) \geq 1\}$ and hence is analytic by [Fis76, p. 137].

We will prove that the holomorphic subbundle F which is defined outside \mathcal{S} can be extended to a holomorphic subbundle defined on $\mathbb{P}T_X \setminus \mathcal{S}_0$, and that the decomposition $\tilde{E} = F \oplus F^\perp$, which is C^∞ and holomorphic along the leaves outside \mathcal{S} , can also be extended to a decomposition on $\mathbb{P}T_X \setminus \mathcal{S}_0$, with the same regularity. This will be a consequence of the following variation on the second Riemann extension theorem:

Lemma 3.2. *Let V be a 1-dimensional linear subspace in \mathbb{C}^d . For $z \in \mathbb{C}^d$, let ℓ_z be the affine line $z + V$. Let Y be an analytic subset of \mathbb{C}^d , of codimension at least 2. Let $\varphi : \mathbb{C}^d \setminus Y \rightarrow \mathbb{C}$*

be a C^∞ map. Assume that φ is holomorphic in the V -direction, meaning that for every $z \in \mathbb{C}^d \setminus Y$, the restriction of φ to a neighborhood of z in ℓ_z is holomorphic. Let y be a point of Y which is an isolated point of $Y \cap \ell_y$. Then there exist a neighborhood \mathcal{U} of y in \mathbb{C}^d and a C^∞ map $\Phi : \mathcal{U} \rightarrow \mathbb{C}$, holomorphic in the V -direction, such that $\Phi = \varphi$ on $\mathcal{U} \setminus Y$.

(We postpone the proof of the lemma to the end of the present proof.)

Let ξ be a point of $\mathcal{S} \setminus \mathcal{S}_0$, i.e. ξ is an isolated point of $\mathcal{S} \cap \mathcal{L}_\xi$. We want to show that F and F^\perp can be extended in a neighborhood of ξ in $\mathbb{P}T_X$. Since this is a local problem, we may assume that we are on \mathbb{C}^{2n-1} , that the leaves of the tautological foliation \mathcal{T} are the affine lines of a given direction $V \subset \mathbb{C}^{2n-1}$ as in the lemma, and that \tilde{E} is a trivial bundle. Because of the regularity properties of F and F^\perp , the section ϕ of $\text{Hom}(\tilde{E}, \tilde{E})$ defined over $\mathbb{C}^{2n-1} \setminus \mathcal{S}$ and corresponding to the orthogonal projection on F^\perp is given by a matrix of functions (ϕ_{ij}) from $\mathbb{C}^{2n-1} \setminus \mathcal{S}$ to \mathbb{C} which are C^∞ and holomorphic in the V -direction. By the above lemma, ϕ extends to a section of $\text{Hom}(\tilde{E}, \tilde{E})$ defined in a neighborhood of ξ in \mathbb{C}^{2n-1} . By the lower semi-continuity of the rank, if $\xi \in \mathcal{S} \setminus \mathcal{S}_0$, $\text{rk } \phi(\xi) \leq \text{rk } \tilde{E} - \text{rk } F$. In the same way, $\text{id} - \phi$ can be extended to $\mathcal{S} \setminus \mathcal{S}_0$ and hence $\text{rk } \phi(\xi) = \text{rk } \tilde{E} - \text{rk } F$. Hence the subbundles F and F^\perp can be extended to $\mathbb{P}T_X \setminus \mathcal{S}_0$, as C^∞ -vector bundles holomorphic along the leaves of \mathcal{T} . Since F is holomorphic and orthogonal to F^\perp on $\mathbb{P}T_X \setminus \mathcal{S}$, this is also true on $\mathbb{P}T_X \setminus \mathcal{S}_0$. \square

Proof of Lemma 3.2. Choose coordinates (z_1, \dots, z_d) on \mathbb{C}^d such that $y = 0$ and $\ell_0 = V = \{z \mid z_1 = \dots = z_{d-1} = 0\}$. By assumption 0 is an isolated point of $\ell_0 \cap Y$, hence there exists $r > 0$ such that the circle $\{z \mid z_1 = \dots = z_{d-1} = 0, |z_d| = r\}$ does not meet Y . Let $\varepsilon > 0$ be such that $\{z \mid |z_i| < \varepsilon, \forall 1 \leq i \leq d-1, |z_d| \in (r - \varepsilon, r + \varepsilon)\} \cap Y = \emptyset$. Then the function Φ defined by

$$\Phi(z_1, \dots, z_d) = \frac{1}{2\pi\sqrt{-1}} \int_{|t|=r} \frac{\varphi(z_1, \dots, z_{d-1}, t)}{t - z_d} dt$$

is C^∞ on the polydisc $\mathcal{U} = \Delta(0, \varepsilon)^{d-1} \times \Delta(0, r)$. Moreover, for all $(z_1, \dots, z_{d-1}) \in \Delta(0, \varepsilon)^{d-1}$, the map $z \mapsto \Phi(z_1, \dots, z_{d-1}, z)$ is holomorphic on the disc $\Delta(0, r)$.

Let $\mathcal{U}' = \{z \in \mathcal{U} \mid \ell_z \cap Y \cap \mathcal{U} = \emptyset\}$. Because φ is holomorphic in the V -direction, for all z in \mathcal{U}' , the restriction of Φ to $\ell_z \cap \mathcal{U}$ equals φ by the Cauchy formula. Now \mathcal{U}' is dense in \mathcal{U} . Indeed, let p be the projection $\mathbb{C}^d \rightarrow \mathbb{C}^d/V$. Since $0 \in Y$ is an isolated point of $Y \cap p^{-1}(p(0))$, near $0 = p(0)$ the set $p(Y)$ is analytic of the same dimension as Y ([Fis76, p. 133]), hence it has codimension at least 1. Hence $\Phi = \varphi$ on $\mathcal{U} \setminus Y$. \square

4. MAXIMAL REPRESENTATIONS IN $\text{SU}(p, q)$, $p \geq q$

4.1. Preliminaries.

We recall here some necessary facts on the symmetric space $\mathcal{Y}_{p,q}$ associated to the group $\text{SU}(p, q)$, its holomorphic sectional curvature, and the consequences this has on special holomorphic maps $\mathbb{H}_{\mathbb{C}}^n \rightarrow \mathcal{Y}_{p,q}$. We refer to [KM08b, §3.1] for details.

Let \mathbb{E} be the vector space \mathbb{C}^{p+q} equipped with a Hermitian form $h_{p,q}$ of signature (p, q) , $p \geq q$. The group $H = \text{SU}(p, q) = \text{SU}(\mathbb{E}, h_{p,q})$ acts transitively on $\mathcal{Y}_{p,q}$, the open subset of the Grassmann manifold of q -dimensional subspaces of \mathbb{E} consisting of q -subspaces on which $h_{p,q}$ restricts to a negative definite Hermitian form. The stabilizer of such a q -subspace is a maximal compact subgroup of G and is isomorphic to $\text{S}(\text{U}(p) \times \text{U}(q))$. Hence $\mathcal{Y}_{p,q} = \text{SU}(p, q)/\text{S}(\text{U}(p) \times \text{U}(q))$. As a bounded symmetric domain, it is naturally identified with $\{Z \in M_{p,q}(\mathbb{C}) \mid 1_q - Z^*Z > 0\} \subset \mathbb{C}^{pq}$. We normalize the invariant metric so that, if $o \in \mathcal{Y}_{p,q}$, using the standard identification between the holomorphic tangent space of $\mathcal{Y}_{p,q}$ at o and

$M_{p,q}(\mathbb{C})$, the holomorphic sectional curvature for the complex line $\langle A \rangle$ generated by a nonzero $A \in M_{p,q}(\mathbb{C})$ is given by

$$\kappa(\langle A \rangle) = -\frac{\operatorname{tr}((A^*A)^2)}{(\operatorname{tr}(A^*A))^2}.$$

This formula shows that $\kappa(\langle A \rangle)$ is pinched between -1 and $-1/q$ and that $\kappa(\langle A \rangle) = -1/q$ if and only if the column vectors of A are pairwise orthogonal and have the same norm (for the standard Hermitian scalar product in \mathbb{C}^p).

The Ahlfors-Schwarz-Pick lemma [Roy80] therefore implies that if $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow \mathcal{Y}_{p,q}$ is holomorphic, then $f^*\omega_{\mathcal{Y}_{p,q}} \leq q\omega$. Moreover, this inequality is an equality only if the induced holomorphic sectional curvature on the image of f is everywhere maximal, i.e. equal to $-1/q$, and we proved the following lemma in [KM08b, §3.1]:

Lemma 4.1. *Let $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow \mathcal{Y}_{p,q}$ be a holomorphic map such that $f^*\omega_{\mathcal{Y}_{p,q}} = q\omega$. Then $p \geq qn$ and up to the composition of f by an isometry of $\mathcal{Y}_{p,q}$, f is equal to the following holomorphic totally geodesic embedding:*

$$f_{\max} : \mathbb{H}_{\mathbb{C}}^n \ni z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \mapsto Z = \begin{pmatrix} z_1 \mathbf{1}_q \\ z_2 \mathbf{1}_q \\ \vdots \\ z_n \mathbf{1}_q \\ 0_{p-nq,q} \end{pmatrix} \in \mathcal{Y}_{p,q}.$$

This lemma shows that Corollary 1.2 is indeed a consequence of Theorem 1.1, and that to prove Theorem 1.1 in the case of representations in $\mathrm{SU}(p, q)$, it suffices to prove that if ρ is maximal, then there exists a ρ -equivariant holomorphic or antiholomorphic map $\mathbb{H}_{\mathbb{C}}^n \rightarrow \mathcal{Y}_{p,q}$. For if f is holomorphic then the Ahlfors-Schwarz-Pick lemma and the maximality of ρ imply that $f^*\omega_{\mathcal{Y}_{p,q}} = q\omega$, hence by the lemma that $p \geq qn$ and essentially $f = f_{\max}$. If f is antiholomorphic then the maximality of ρ implies $f^*\omega_{\mathcal{Y}_{p,q}} = -q\omega$, so that again $p \geq qn$ and essentially $f = \bar{f}_{\max}$. The last assertion of Corollary 1.2 follows from the description of the stabilizer in $\mathrm{SU}(p, q)$ of the image of f_{\max} we gave at the end of [KM08b, §3.1].

4.2. Reductive representations.

Let $\rho : \Gamma \rightarrow \mathrm{SU}(p, q)$ be a reductive representation. We work with the associated Higgs bundle $(E, \theta) \rightarrow X$. We recall that Higgs bundles arising from reductive representations into real reductive subgroups H of $\mathrm{SL}(N, \mathbb{C})$ have an additional structure, that one could call a *real structure*, compared to those arising from representations in $\mathrm{SL}(N, \mathbb{C})$ without further restriction, see e.g. [Sim92, p. 90-91] or [Mau15, §3.6].

In particular, as explained for example in [KM08b, §2.4 & §3.2] or [Mau15, §3.6.2 & §3.6.3], in the case of $\mathrm{SU}(p, q) \subset \mathrm{SL}(p+q, \mathbb{C})$, the associated Higgs bundle E splits holomorphically as a sum $V \oplus W$, where V has rank p and W has rank q . This is because it is (as a smooth bundle) the bundle associated to the standard representation of $\mathrm{GL}(p+q, \mathbb{C})$ on $\mathbb{E} = \mathbb{C}^{p+q}$, but its structure group is the complexification $\mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))$ of the maximal compact subgroup $\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ of $\mathrm{SU}(p, q)$. Note that $\deg V + \deg W = \deg E = 0$. Moreover the Higgs field seen as a sheaf morphism $\theta : E \otimes T_X \rightarrow E$ is off-diagonal w.r.t. this decomposition: it has two components $\beta : W \otimes T_X \rightarrow V$ and $\gamma : V \otimes T_X \rightarrow W$.

If $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow \mathcal{Y}_{p,q}$ is the ρ -equivariant harmonic map used to construct (E, θ) , then as we said the Higgs field θ can be identified with the $(1, 0)$ -part $d^{1,0}f : T^{1,0}\mathbb{H}_{\mathbb{C}}^n \rightarrow T^{\mathbb{C}}\mathcal{Y}_{p,q}$ of the complexified differential of f . If $p^{1,0}$, resp. $p^{0,1}$, is the projection $T^{\mathbb{C}}\mathcal{Y}_{p,q} \rightarrow T^{1,0}\mathcal{Y}_{p,q}$, resp.

$T^{\mathbb{C}}\mathcal{Y}_{p,q} \rightarrow T^{0,1}\mathcal{Y}_{p,q}$, then β , resp. γ , can be identified with $p^{1,0} \circ d^{1,0}f$, resp. $p^{0,1} \circ d^{1,0}f$. Hence the harmonic map f is holomorphic, resp. antiholomorphic, if and only if $\gamma = 0$, resp. $\beta = 0$.

We lift the Higgs bundle $(V \oplus W, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}) \rightarrow X$ to the projectivized tangent bundle $\mathbb{P}T_X$. As before, all lifted objects are denoted with a “ \sim ”. In particular, we shall denote by

$$\begin{cases} \tilde{\beta} : \tilde{W} \otimes L \rightarrow \tilde{V} \\ \tilde{\gamma} : \tilde{V} \otimes L \rightarrow \tilde{W} \end{cases}$$

the two components of the lifted Higgs field restricted to the tangent bundle L of the tautological foliation \mathcal{T} on $\mathbb{P}T_X$. (We shall denote by the same letter a vector bundle defined on $\mathbb{P}T_X$, or X , and the sheaf of its sections.)

We also need to reformulate the Milnor-Wood inequality satisfied by the Toledo invariant of ρ in terms of foliated degrees of vector bundles on $\mathbb{P}T_X$.

If $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow \mathcal{Y}_{p,q}$ is the ρ -equivariant harmonic map, it is easily checked that $f^*T_{\mathcal{Y}_{p,q}} \simeq W^* \otimes V$, where $f^*T_{\mathcal{Y}_{p,q}}$ is here seen as a bundle on $X = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$. The symmetric space $\mathcal{Y}_{p,q}$ is a Kähler-Einstein manifold, and with our curvature normalization, $c_1(T_{\mathcal{Y}_{p,q}}) = -\frac{1}{2\pi} \frac{p+q}{2} \omega_{\mathcal{Y}_{p,q}}$. Hence the Toledo invariant $\tau(\rho)$ is given by

$$\tau(\rho) = -\frac{4\pi}{p+q} \deg f^*T_{\mathcal{Y}_{p,q}} = 4\pi \deg W = 4\pi \deg_{\mathcal{T}} \tilde{W},$$

where the last equality is given by Lemma 2.1.

At the end of §2.2 we saw that $c_1(L^{\vee}) = \frac{1}{2\pi}(\pi^*\omega - \frac{1}{2}(\omega_2 - \omega_3))$, so that

$$\deg_{\mathcal{T}} L^{\vee} = \frac{1}{2\pi(2n-2)!} \int_{\mathbb{P}T_X} \pi^*\omega \wedge \Omega = \frac{1}{2\pi n!} \int_X \omega^n = \frac{1}{2\pi} \text{vol}(X).$$

Therefore the Milnor-Wood inequality is equivalent to

$$|\deg_{\mathcal{T}} \tilde{W}| \leq \frac{q}{2} \deg_{\mathcal{T}} L^{\vee}$$

and reductive maximal representations are those for which this inequality is an equality.

The proof of Theorem 1.1 in the case of a reductive representation ρ into $\text{SU}(p, q)$ proceeds in four steps. Recall from the discussion above that all we need to prove is that one of the components β or γ of the Higgs field vanishes.

The first step is a new proof of the Milnor-Wood inequality, valid for reductive representations of torsion free uniform lattices of $\text{SU}(n, 1)$, $n \geq 1$, obtained by considering the lifted Higgs bundle $(\tilde{E}, \tilde{\theta})$ on $\mathbb{P}T_X$. The advantage of going up to the projectivized tangent bundle is that, thanks to the existence of the line bundle L tangent to the leaves of the tautological foliation \mathcal{T} , and to the semistability results of §3.2, the proof is exactly the same as the known one for surface groups, i.e. lattices in $\text{SU}(1, 1)$, see e.g. [Xia00, MX02, BGPG03].

When the representation is maximal, and say $\deg_{\mathcal{T}} \tilde{W} = \frac{q}{2} \deg_{\mathcal{T}} L^{\vee}$, our proof of the Milnor-Wood inequality shows that the subset \mathcal{S} of $\mathbb{P}T_X$ where $\tilde{\beta} : \tilde{W} \otimes L \rightarrow \tilde{V}$ is not injective, i.e.

$$\mathcal{S} = \{(x, [\xi]) \in \mathbb{P}T_X \mid \beta_x(\xi) : W_x \rightarrow V_x \text{ is not injective}\},$$

is small: it is a proper analytic subset of $\mathbb{P}T_X$. We want to prove that its projection $\pi(\mathcal{S})$ to X is also small.

This is achieved by first proving in the second step that \mathcal{S} is included in a proper analytic subset of $\mathbb{P}T_X$ which is saturated under the tautological foliation \mathcal{T} . This follows again

from our proof of the inequality, and from the polystability of \tilde{E} along the leaves of \mathcal{T} , see Proposition 3.1.

In the third step we show that this indeed implies that $\pi(\mathcal{S})$ is a proper analytic subset of X . This is where the dynamics of the tautological foliation play their crucial part via Proposition 2.2.

The fourth part is the conclusion where we prove that the injectivity of $\beta_x(\xi)$ for all $x \in X \setminus \pi(\mathcal{S})$ and $\xi \in T_{X,x} \setminus \{0\}$ forces γ to vanish, hence the ρ -equivariant harmonic map to be holomorphic. Here we use again the polystability along the leaves, the integrability condition $[\theta, \theta] = 0$, and our standing assumption that $n \geq 2$.

4.2.1. A new proof of the Milnor-Wood inequality.

As explained above, we need to show that

$$|\deg_{\mathcal{T}} \tilde{W}| \leq \frac{q}{2} \deg_{\mathcal{T}} L^{\vee}.$$

We first prove that $\deg_{\mathcal{T}} \tilde{W} \leq \frac{q}{2} \deg_{\mathcal{T}} L^{\vee}$.

If $\tilde{\beta} = 0$ then \tilde{W} is a leafwise Higgs subsheaf of \tilde{E} hence by Proposition 3.1 $\deg_{\mathcal{T}} \tilde{W} \leq 0$ and we are done.

Assume therefore that $\tilde{\beta} \neq 0$ and let $\mathcal{N} = \text{Ker}(\tilde{\beta} : \tilde{W} \otimes L \rightarrow \tilde{V}) \subset \tilde{W} \otimes L$ and \mathcal{I} be the saturation (as a sheaf) of $\text{Im}(\tilde{\beta} : \tilde{W} \otimes L \rightarrow \tilde{V}) \subset \tilde{V}$. By construction, $\mathcal{N} \otimes L^{\vee}$ and $\tilde{W} \oplus \mathcal{I}$ are leafwise Higgs subsheaves of \tilde{E} . Since $(\tilde{E}, \tilde{\theta})$ is semistable along the leaves of \mathcal{T} by Proposition 3.1,

$$\deg_{\mathcal{T}} \mathcal{N} \otimes L^{\vee} \leq 0 \quad \text{and} \quad \deg_{\mathcal{T}} \tilde{W} + \deg_{\mathcal{T}} \mathcal{I} \leq 0.$$

Moreover, we have $\deg_{\mathcal{T}}(\tilde{W} \otimes L) \leq \deg_{\mathcal{T}} \mathcal{N} + \deg_{\mathcal{T}} \mathcal{I}$ so that, if we denote by r the rank of \mathcal{I} (i.e. the generic rank of $\tilde{\beta}$),

$$\deg_{\mathcal{T}} \tilde{W} + q \deg_{\mathcal{T}} L \leq \deg_{\mathcal{T}} \mathcal{N} + \deg_{\mathcal{T}} \mathcal{I} \leq (q - r) \deg_{\mathcal{T}} L - \deg_{\mathcal{T}} \tilde{W}$$

and hence

$$\deg_{\mathcal{T}} \tilde{W} \leq \frac{r}{2} \deg_{\mathcal{T}} L^{\vee} \leq \frac{q}{2} \deg_{\mathcal{T}} L^{\vee}$$

since obviously $r \leq q$.

In the same manner, by considering $\tilde{\gamma}$, we obtain

$$\deg_{\mathcal{T}} \tilde{V} = -\deg_{\mathcal{T}} \tilde{W} \leq \frac{q}{2} \deg_{\mathcal{T}} L^{\vee}$$

and the Milnor-Wood inequality is proved.

We assume for the next three paragraphs that the representation is maximal, say that $\deg_{\mathcal{T}} \tilde{W} = \frac{q}{2} \deg_{\mathcal{T}} L^{\vee}$. Then we see from the proof above that $\tilde{\beta} : \tilde{W} \otimes L \rightarrow \tilde{V}$ has rank q , i.e. is injective as a sheaf morphism, hence generically injective as a vector bundle map. Let us call \mathcal{S} the singularity set of $\tilde{\beta}$, i.e. the analytic subspace of $\mathbb{P}T_X$ where $\tilde{\beta}$ is not injective as a vector bundle map.

4.2.2. \mathcal{T} -saturation.

Equality in the Milnor-Wood inequality implies moreover that the foliated degree of $\tilde{W} \oplus \mathcal{I}$, which is a leafwise Higgs subsheaf of \tilde{E} , is 0. By Proposition 3.1, there exists a codimension at least 2 \mathcal{T} -saturated subset \mathcal{S}_0 of $\mathbb{P}T_X$ and a holomorphic subbundle I of \tilde{V} , defined outside of \mathcal{S}_0 , such that on $\mathbb{P}T_X \setminus \mathcal{S}_0$, \mathcal{I} is the sheaf of sections of I . Note that the subbundle I has rank q where it is defined.

Since outside of \mathcal{S}_0 we have $\tilde{\beta} : \tilde{W} \otimes L \rightarrow I$ and $\text{rk } I = \text{rk } \tilde{W}$, the set of points of $\mathbb{P}T_X \setminus \mathcal{S}_0$ where $\tilde{\beta}$ is not injective is locally given by the vanishing of a single holomorphic function

and hence has codimension 1 if not empty. This means that the components of \mathcal{S} of higher codimension are included in \mathcal{S}_0 and hence that $\tilde{\beta} : \tilde{W} \otimes L \rightarrow \tilde{V}$ is injective, as a vector bundle map, outside $\mathcal{S}_0 \cup |\Delta|$, where $|\Delta|$ is the (possibly empty) divisorial part of \mathcal{S} , i.e. the union of its irreducible components Δ_j of codimension 1.

By an argument similar to [Kob87, Chap. V (8.5)], there is a line bundle $[\Delta]$ on $\mathbb{P}T_X$ corresponding to an effective divisor $\Delta = \sum_j a_j \Delta_j$ supported on $|\Delta|$ such that $\deg_{\mathcal{T}} \det \tilde{W} \leq \deg_{\mathcal{T}} (\det \tilde{W}) \otimes [\Delta] \leq \frac{q}{2} \deg_{\mathcal{T}} L^\vee$. Hence $\deg_{\mathcal{T}} [\Delta] = \sum_j a_j \int_{\Delta_j} \Omega = 0$. This means that for all j , and at each smooth point x of Δ_j , there is a tangent vector to Δ_j which is also tangent to the leaf of the foliation \mathcal{T} through x . Since the foliation is smooth, this leaf must stay in Δ_j . Now in Δ_j , the smooth points are dense and the set of points whose leaves stay in Δ_j is closed. Thus Δ_j is \mathcal{T} -saturated for all j .

The singularity set \mathcal{S} of $\tilde{\beta}$ is therefore included in a \mathcal{T} -saturated proper analytic subset of $\mathbb{P}T_X$.

4.2.3. Dynamics of the tautological foliation.

The fact that the leaves of the foliation \mathcal{T} come from orbits of groups generated by unipotent elements allowed to establish in Proposition 2.2 that the closure of the projection to X of any leaf of \mathcal{T} is a closed immersed totally geodesic submanifold of X . This in turn has the following consequence on projections to X of \mathcal{T} -saturated subsets of $\mathbb{P}T_X$:

Proposition 4.2. *Let Z be a closed \mathcal{T} -saturated proper subset of $\mathbb{P}T_X$. Then $\pi(Z)$ is a proper subset of X .*

Proof. The key point is that there is at most a countable number of closed immersed totally geodesic submanifolds in X which are of the form $\overline{\pi(\mathcal{L})}$ for \mathcal{L} a leaf of \mathcal{T} . This follows from [Rat91b, Theorem 1.1] but in our case a similar but simpler argument is available.

We use freely the notation of §2.4. We will show that, given a leaf $\mathcal{L} \subset \mathbb{P}T_X = \Gamma \backslash G/V$ of \mathcal{T} , the totally geodesic orbit $\tilde{Y}_{\mathcal{L}} = S_{\mathcal{L}} \cdot g_{\mathcal{L}}K$ of $S_{\mathcal{L}}$ in $\mathbb{H}_{\mathbb{C}}^n = G/K$ is entirely determined by the intersection $\Lambda_{\mathcal{L}}$ of Γ with the closed reductive subgroup $S_{\mathcal{L}}$ of G . Since on the one hand $\overline{\pi(\mathcal{L})} = \Gamma \backslash \Gamma \tilde{Y}_{\mathcal{L}}$ and on the other hand $\Lambda_{\mathcal{L}} = \Gamma \cap S_{\mathcal{L}}$ is finitely generated (it is a lattice in $S_{\mathcal{L}}$), and there are only countably many finite subsets in Γ , this will indeed prove the claim.

We note that there exists a simple (because $\text{rk}_{\mathbb{R}} G = 1$) noncompact subgroup $H_{\mathcal{L}}$ of $S_{\mathcal{L}}$ such that $\tilde{Y}_{\mathcal{L}} = H_{\mathcal{L}} \cdot g_{\mathcal{L}}K$. (In fact $H_{\mathcal{L}}$ is necessarily locally isomorphic to $\text{SU}(k, 1)$ for some $1 \leq k \leq n$.)

So let \mathcal{L}_1 and \mathcal{L}_2 be two leaves of \mathcal{T} . To lighten the notation, for $i \in \{1, 2\}$, we write S_i , H_i , \tilde{Y}_i and Λ_i for $S_{\mathcal{L}_i}$, $H_{\mathcal{L}_i}$, $\tilde{Y}_{\mathcal{L}_i}$ and $\Lambda_{\mathcal{L}_i}$ respectively. Assume that $\Lambda_1 = \Lambda_2$.

By a strengthening of the Borel density theorem, see e.g. [Dan80, Corollary 4.2], since $\Lambda_1 = \Gamma \cap S_1$ is a lattice in S_1 and H_1 is a simple noncompact subgroup of S_1 , the Zariski closure of Λ_1 in G contains H_1 . Now $\Lambda_1 = \Lambda_2 \subset S_2$ and hence Λ_1 normalizes S_2 , so that H_1 also normalizes S_2 . Therefore $(S_1 \cap S_2) \backslash \overline{(S_1 \cap S_2) H_1}$ is a compact group, since it is closed in $(S_1 \cap S_2) \backslash S_1$ which itself is a quotient of $\Lambda_1 \backslash S_1$ which is compact. Thus thus $\overline{S_1 \cap S_2}$ must contain every simple noncompact subgroup of every Levi subgroup of $\overline{(S_1 \cap S_2) H_1}$. We deduce that H_1 is a subgroup of S_2 . In the same way, H_2 is a subgroup of S_1 .

If d denotes the distance function on $\mathbb{H}_{\mathbb{C}}^n$, the function $x \mapsto d(x, \tilde{Y}_2)$ is convex on \tilde{Y}_1 because $\mathbb{H}_{\mathbb{C}}^n$ is a CAT(0)-space and \tilde{Y}_1 and \tilde{Y}_2 are totally geodesic (hence convex and complete). It is moreover bounded because \tilde{Y}_1 is an H_1 -orbit, \tilde{Y}_2 is an S_2 -orbit and $H_1 \subset S_2$. Hence it is constant. It must be identically zero, because if not, the convex hull of two distinct points in

\tilde{Y}_1 and their (distinct) projections in \tilde{Y}_2 is Euclidean by the flat quadrilateral theorem [BH99, p. 181], a contradiction since $\text{rk}_{\mathbb{R}}G = 1$. Hence $\tilde{Y}_1 \subset \tilde{Y}_2$. The same reasoning gives $\tilde{Y}_2 \subset \tilde{Y}_1$. This is what we wanted.

To conclude, let Z be a closed \mathcal{T} -saturated proper subset of $\mathbb{P}T_X$ and assume that $\pi(Z) = X$. Then because Z is a union of leaves, X is the union of the projections of the leaves of Z , hence of their closures. Since there are only countably many such objects, there must be a leaf $\mathcal{L} \subset Z$ such that $\pi(\overline{\mathcal{L}}) = X$. But this implies that $S_{\mathcal{L}} = G$, so that $\overline{\mathcal{L}} = \mathbb{P}T_X$. Hence $Z = \mathbb{P}T_X$, for Z is closed. A contradiction. \square

Therefore the projection to X of the set $\mathcal{S} \subset \mathbb{P}T_X$ where $\tilde{\beta}$ is not of maximal rank is a proper subset of X . In fact, since π is a proper map, it is a proper analytic subset of X .

4.2.4. Conclusion.

Let $x \in X$, $x \notin \pi(\mathcal{S})$. This means that for all $\xi \neq 0$ in $T_{X,x}$, $\beta_x(\xi) : W_x \rightarrow V_x$ is injective and we call $I_{\xi} \subset V_x$ its image.

We claim that $\bigcap_{\xi \neq 0} I_{\xi} = \{0\}$. Let indeed v be in $\bigcap_{\xi \neq 0} I_{\xi}$. For $\eta \neq 0$ in $T_{X,x}$, let p_{η} be the orthogonal projection from V_x to I_{η} . Then for all ξ in a sufficiently small neighborhood \mathcal{U} of η in $T_{X,x} \setminus \{0\}$, $p_{\eta}|_{I_{\xi}} : I_{\xi} \rightarrow I_{\eta}$ is an isomorphism, so that $p_{\eta} \circ \beta_x(\xi) : W_x \rightarrow I_{\eta}$ is also an isomorphism. Define $\varphi_{\eta} : \mathcal{U} \rightarrow W_x$ by $\varphi_{\eta}(\xi) = (p_{\eta} \circ \beta_x(\xi))^{-1}(v)$. Then clearly φ_{η} is holomorphic and $\beta_x(\xi)\varphi_{\eta}(\xi) = v$ for all $\xi \in \mathcal{U}$. By the injectivity of $\beta_x(\xi)$, the maps φ_{η} for $\eta \neq 0$ agree where they are simultaneously defined, so we get a holomorphic map $\varphi : T_{X,x} \setminus \{0\} \rightarrow W_x$ such that $\beta_x(\xi)\varphi(\xi) = v$ for all $\xi \neq 0$. Because $n \geq 2$, the map φ can be extended holomorphically to $0 \in T_{X,x}$ and necessarily $\beta_x(0)\varphi(0) = v$ so that $v = 0$ since $\beta_x : T_{X,x} \rightarrow \text{Hom}(W_x, V_x)$ is linear.

We may now conclude. The integrability property $[\theta, \theta] = 0$ of the Higgs field means that $\beta_x(\xi) \circ \gamma_x(\eta) = \beta_x(\eta) \circ \gamma_x(\xi)$ for all $\xi, \eta \in T_{X,x}$. Therefore, if $v \in \text{Ker } \gamma_x(\xi)$ for some $\xi \neq 0$, then for all η we have $\beta_x(\xi)(\gamma_x(\eta)v) = \beta_x(\eta)(\gamma_x(\xi)v) = 0$ and hence $\gamma_x(\eta)v = 0$ since $\beta_x(\xi)$ is injective. We know from Proposition 3.1 that $I_{\xi}^{\perp} \subset V_x$ is invariant by $\theta_x(\xi)$ and hence that $\gamma_x(\xi)$ vanishes on I_{ξ}^{\perp} , for $\theta_x(\xi)|_{V_x} = \gamma_x(\xi) : V_x \rightarrow W_x$. It follows that for all η and for all $\xi \neq 0$, $\gamma_x(\eta)$ is zero on I_{ξ}^{\perp} . Therefore $\gamma_x = 0$ since $\bigcap_{\xi \neq 0} I_{\xi} = \{0\}$ exactly means that the subspaces I_{ξ}^{\perp} for $\xi \neq 0$ generate V_x .

Since this holds outside a proper analytic subset, γ vanishes identically and the ρ -equivariant harmonic map f is holomorphic.

In the same manner, if $\text{deg}_{\mathcal{T}} \tilde{W} = -\frac{q}{2} \text{deg}_{\mathcal{T}} L^{\vee}$, β vanishes identically and the ρ -equivariant harmonic map f is antiholomorphic.

This ends the proof of Theorem 1.1 in the case of reductive representations in $\text{SU}(p, q)$.

4.3. Non reductive representations.

We first explain how to *semi-simplify* a non reductive representation in order to get a reductive representation with the same Toledo invariant. This is quite standard and follows from general results but we give some details in the $\text{SU}(p, q)$ case for completeness. The notation in this section may differ from those of the other sections.

Assume that the representation $\rho : \Gamma \rightarrow H = \text{SU}(\mathbb{C}^{p+q}, h_{p,q})$ is not reductive, i.e. that the Zariski closure $\overline{\rho(\Gamma)}^Z$ of $\rho(\Gamma)$ in H is not a reductive group, so that its unipotent radical U is not trivial. Then $\overline{\rho(\Gamma)}^Z$ stabilizes a flag of totally isotropic subspaces $0 = F_0 \subset F_1 \subset \dots \subset F_k$ in $\mathbb{E} = \mathbb{C}^{p+q}$ (for the Hermitian form $h_{p,q}$ defining H in $\text{SL}(p+q, \mathbb{C})$). Indeed, because U is

unipotent, $F := \{x \in \mathbb{E} \mid u(x) = x, \forall u \in U\} \neq \{0\}$. Then F^\perp is U -stable and therefore the totally isotropic subspace $F_1 := F \cap F^\perp$ is not reduced to $\{0\}$. Since U is normal in $\overline{\rho(\Gamma)}^Z$, the subspaces F , F^\perp and hence F_1 are $\overline{\rho(\Gamma)}^Z$ -stable. One completes the flag inductively by working on F_1^\perp/F_1 . This flag defines a parabolic subgroup of $\mathrm{SL}(p+q, \mathbb{C})$ and its intersection with H is a parabolic subgroup P of H which contains $\overline{\rho(\Gamma)}^Z$ and whose unipotent radical contains U .

If we let $r_i = \dim F_i - \dim F_{i-1}$, $1 \leq i \leq k$, we have that $r := r_1 + \dots + r_k \leq q$, and we may choose a basis of $\mathbb{C}^{p,q}$ in which the matrix C of $h_{p,q}$ is block anti-diagonal with anti-diagonal blocks given by $1_{r_1}, \dots, 1_{r_k}, 1_{q-r}, 1_{p-q}, 1_{q-r}, 1_{r_k}, \dots, 1_{r_1}$ (from the bottom left to the upper right). Conjugacy by this matrix defines a Cartan involution of H , hence a maximal compact subgroup L of H , hence a point o in the symmetric space $\mathcal{Y}_{p,q}$.

Let ξ be the endomorphism of \mathbb{E} whose matrix in our chosen basis is the block diagonal matrix $\mathrm{diag}(k 1_{r_1}, (k-1) 1_{r_2}, \dots, 1_{r_k}, 0_{q-r}, 0_{p-q}, 0_{q-r}, -1_{r_k}, \dots, -(k-1) 1_{r_2}, -k 1_{r_1})$. Then $\xi \in \mathfrak{q}$ where $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{q}$ is the Cartan decomposition of the Lie algebra \mathfrak{h} of H defined by C . Using the usual identification of $T_o\mathcal{Y}_{p,q}$ with \mathfrak{q} , the map $c : t \mapsto \exp_o(t\xi)$ is a geodesic through o and the parabolic subgroup P is precisely the stabilizer in H of the endpoint $c(+\infty) \in \partial_\infty\mathcal{Y}_{p,q}$ of c . We have the following description of P (see for example [Ebe96, §2.17]):

$$P = \{g \in H \mid \lim_{t \rightarrow +\infty} \exp(-t\xi) g \exp(t\xi) \text{ exists}\} = L_\xi.A_\xi.N_\xi$$

where $A_\xi = \exp(\{\zeta \in \mathfrak{q} \mid [\xi, \zeta] = 0\})$, $N_\xi = \{g \in P \mid \lim_{t \rightarrow +\infty} \exp(-t\xi) g \exp(t\xi) = 1\}$ is the unipotent radical of P , and $L_\xi = P \cap L$. The resulting decomposition of elements of P is unique. Moreover if $g \in L_\xi.A_\xi$ then $\lim_{t \rightarrow +\infty} \exp(-t\xi) g \exp(t\xi) = g$.

The orbit $L_\xi.A_\xi.o = A_\xi.o$ is the complete totally geodesic submanifold \mathcal{Y}_c of $\mathcal{Y}_{p,q}$ consisting of geodesics of $\mathcal{Y}_{p,q}$ that are parallel to c .

The *semi-simplification* ρ_{ss} of ρ is defined by $\rho_{ss}(\gamma) = \lim_{t \rightarrow +\infty} \exp(-t\xi) \rho(\gamma) \exp(t\xi) \in L_\xi.A_\xi$ for all $\gamma \in \Gamma$. The representation ρ_{ss} belongs to the connected component of ρ in the space $\mathrm{Hom}(\Gamma, H)$ and is reductive by construction. Thus we can apply the results of §4.2.1 to get the Milnor-Wood bound on $\tau(\rho) = \tau(\rho_{ss})$.

Assume moreover that ρ is maximal, say with $\tau(\rho) > 0$. Hence so is ρ_{ss} , and by §4.2 we know that $p \geq nq$ and that, up to composition with an isometry of $\mathcal{Y}_{p,q}$, the holomorphic map f_{\max} is ρ_{ss} -equivariant. Thus we may assume that $\rho_{ss}(\Gamma)$ stabilizes the totally geodesic copy $f_{\max}(\mathbb{H}_{\mathbb{C}}^n)$ of $\mathbb{H}_{\mathbb{C}}^n$ in $\mathcal{Y}_{p,q}$, and that it acts cocompactly on it.

On the other hand, as was mentioned before, $\rho_{ss}(\Gamma)$ stabilizes the totally geodesic submanifold $\mathcal{Y}_c = A_\xi.o \subset \mathcal{Y}_{p,q}$. As in the proof of Proposition 4.2, the function

$$\begin{aligned} f_{\max}(\mathbb{H}_{\mathbb{C}}^n) &\longrightarrow \mathbb{R} \\ y &\longmapsto d(y, \mathcal{Y}_c) \end{aligned}$$

is convex. Since it is bounded because $\rho_{ss}(\Gamma)$ acts cocompactly on $f_{\max}(\mathbb{H}_{\mathbb{C}}^n)$, it is constant, say equal to a .

If $a = 0$, $f_{\max}(\mathbb{H}_{\mathbb{C}}^n) \subset \mathcal{Y}_c$. Let x be a point in $\mathbb{H}_{\mathbb{C}}^n$, $y = f_{\max}(x)$, σ the geodesic parallel to c such that $\sigma(0) = y$. Let now $\mathfrak{l}_y \oplus \mathfrak{q}_y$ be the Cartan decomposition of \mathfrak{h} associated to the point y , so that \mathfrak{q}_y is identified with $T_y\mathcal{Y}_{p,q}$ and let $\eta \in \mathfrak{q}_y$ be the vector corresponding to $\dot{\sigma}(0)$. Then the tangent space of \mathcal{Y}_c at y identifies with $\{\zeta \in \mathfrak{q}_y \mid [\eta, \zeta] = 0\}$. However, due to the form of f_{\max} , it is easily checked that no nonzero vectors in \mathfrak{q}_y can commute to $df_{\max}(T_x\mathbb{H}_{\mathbb{C}}^n) = T_y f_{\max}(\mathbb{H}_{\mathbb{C}}^n) \subset T_y\mathcal{Y}_c$ as soon as $n \geq 2$. Hence $a > 0$. But in this case the convex hull in $\mathcal{Y}_{p,q}$ of $f_{\max}(\mathbb{H}_{\mathbb{C}}^n)$ and of its orthogonal projection to \mathcal{Y}_c is isometric to the

product $f_{\max}(\mathbb{H}_{\mathbb{C}}^n) \times [0, a]$ and this also implies that there is $\eta \in \mathfrak{q}_y$, $\eta \neq 0$, commuting with $T_y f_{\max}(\mathbb{H}_{\mathbb{C}}^n)$, a contradiction.

Therefore, non reductive representations in $SU(p, q)$ cannot be maximal.

5. REPRESENTATIONS IN CLASSICAL HERMITIAN LIE GROUPS OTHER THAN $SU(p, q)$

The remaining classical Hermitian Lie groups are $SO_0(p, 2)$ with $p \geq 3$, $Sp(m, \mathbb{R})$ with $m \geq 2$, and $SO^*(2m)$ with $m \geq 4$. Their associated symmetric spaces' ranks are 2, m and $\lfloor m/2 \rfloor$ respectively. We will prove that a reductive representation of a uniform complex hyperbolic lattice in one of these groups can never be maximal. Since it is always possible to semi-simplify a representation as in §4.3 to get a reductive representation with the same Toledo invariant, this will prove that there are no maximal representations in these groups. Hence in the following we assume that the representations under consideration are reductive.

5.1. Representations in $SO_0(p, 2)$, $p \geq 3$.

The case of representations in the groups $SO_0(p, 2)$ ($p \geq 3$) has already been treated in [KM08b] where it was shown that such representations satisfy the inequality $|\tau(\rho)| \leq \frac{n+1}{n} \text{vol}(X)$. This is stronger than the Milnor-Wood inequality since the rank of the symmetric space associated to $SO_0(p, 2)$ is 2 and $n \geq 2$. Hence there are no maximal representations in this case.

5.2. Representations in $Sp(m, \mathbb{R})$, $m \geq 2$.

This group may be described as

$$Sp(m, \mathbb{R}) = \{g \in SU(m, m) \mid g^{\top} J_{m,m} g = J_{m,m}\}$$

where g^{\top} is the transpose of the matrix g and $J_{m,m}$ is the $2m$ -by- $2m$ matrix

$$J_{m,m} = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}.$$

The associated symmetric space is totally geodesically embedded in the symmetric space $\mathcal{Y}_{m,m}$ associated to $SU(m, m)$ as

$$\mathcal{Y} = \{Z \in M_m(\mathbb{C}) \mid 1_m - Z^* Z > 0 \text{ and } Z^{\top} = Z\}$$

and the holomorphic sectional curvature of the metric on $\mathcal{Y}_{m,m}$ restricted to \mathcal{Y} is also pinched between -1 and $-1/m$, so that the Toledo invariant of a representation ρ with values in $Sp(m, \mathbb{R})$ is the same as the Toledo invariant of the induced representation into $SU(m, m)$.

The results of §4.2 give the desired bound $|\tau(\rho)| \leq m \text{vol}(X)$ and show that equality cannot be reached.

5.3. Representations in $SO^*(2m)$, $m \geq 4$.

The assumption $m \geq 4$ comes from the fact that for $m = 2$, $SO^*(4)$ is not simple (it is locally isomorphic to $SU(1, 1) \times SU(1, 1)$), whereas for $m = 3$, $SO^*(6)$ is locally isomorphic to $SU(3, 1)$ (and therefore in this case there are maximal representations of $SU(2, 1)$ and $SU(3, 1)$ in this group). Note that for $m = 4$, $SO^*(8)$ is locally isomorphic to $SO_0(6, 2)$.

The group $SO^*(2m)$ may be described as

$$SO^*(2m) = \{g \in SU(m, m) \mid g^{\top} J'_{m,m} g = J'_{m,m}\}$$

where $J'_{m,m}$ is the $2m$ -by- $2m$ matrix

$$J'_{m,m} = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix}.$$

The associated symmetric space is totally geodesically embedded in $\mathcal{Y}_{m,m}$ as

$$\mathcal{Y} = \{Z \in M_m(\mathbb{C}) \mid 1_m - Z^*Z > 0 \text{ and } Z^\top = -Z\}.$$

The holomorphic sectional curvature of the $\mathrm{SO}^*(2m)$ -invariant metric on \mathcal{Y} obtained by restriction of the $\mathrm{SU}(m,m)$ -invariant metric on $\mathcal{Y}_{m,m}$ is pinched between $-1/2$ and $-1/2\lfloor m/2 \rfloor$ according to the formula for the holomorphic sectional curvature of $\mathcal{Y}_{m,m}$ given in §4.1 and the first assertion of the following lemma (whose proof is postponed to the end of this paragraph).

Lemma 5.1. *Let $A \in M_m(\mathbb{C})$ be a non trivial skew-symmetric matrix. Then*

$$\frac{1}{2\lfloor m/2 \rfloor} \leq \frac{\mathrm{tr}((A^*A)^2)}{(\mathrm{tr}(A^*A))^2} \leq \frac{1}{2}$$

and if $m \geq 4$, the dimension of a maximal linear subspace of skew-symmetric matrices realizing the lower bound is 1.

As a consequence, with our normalization on the curvature, $\omega_{\mathcal{Y}} = 2\omega_{\mathcal{Y}_{m,m}|_{\mathcal{Y}}}$ and it is straightforward to see that the Milnor-Wood inequality for a representation ρ in $\mathrm{SO}^*(2m)$ translates to $|\tau(\rho)| \leq 2\lfloor m/2 \rfloor \mathrm{vol}(X)$ when the representation is seen as taking values in $\mathrm{SU}(m,m)$.

If m is even, a representation in $\mathrm{SO}^*(2m)$ is therefore maximal if and only if it is maximal when seen as a representation in $\mathrm{SU}(m,m)$. Section 4.2 hence show that there are no such representations.

We assume from now on that m is odd.

The equality case of the Ahlfors-Schwarz-Pick lemma [Roy80] and the second assertion of Lemma 5.1 show that if $n \geq 2$ and $m \geq 4$, there are no holomorphic map $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow \mathcal{Y} \subset \mathcal{Y}_{m,m}$ such that $f^*\omega_{\mathcal{Y}_{m,m}} = 2\lfloor m/2 \rfloor \omega$. Hence the following proposition implies that there are no maximal representations of a uniform lattice of $\mathrm{SU}(n,1)$, $n \geq 2$, in $\mathrm{SO}^*(2m)$, m odd, unless $m = 3$ (in which case as we said there are since $\mathrm{SO}^*(6)$ is locally isomorphic to $\mathrm{SU}(3,1)$).

Proposition 5.2. *Let ρ be a reductive maximal representation of a uniform lattice $\Gamma \subset \mathrm{SU}(n,1)$ in $\mathrm{SO}^*(2m)$. Assume that $n \geq 2$ and that $m \geq 3$ is odd. Then there exists a ρ -equivariant holomorphic or antiholomorphic map $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow \mathcal{Y}$.*

Proof. Here again we work with the Higgs bundle $(V \oplus W, \beta \oplus \gamma)$ associated to the reductive representation ρ seen as a representation in $\mathrm{SU}(m,m)$. However, because ρ takes its values in $\mathrm{SO}^*(2m) \subset \mathrm{SU}(m,m)$, we have an identification of V with W^* , and for all $\xi \in T_X$, $\beta(\xi) \in \mathrm{Hom}(W, W^*)$, resp. $\gamma(\xi) \in \mathrm{Hom}(W^*, W)$, satisfies $\beta(\xi)^\top = -\beta(\xi)$, resp. $\gamma(\xi)^\top = -\gamma(\xi)$.

Since m is odd and $\tilde{\beta}$ and $\tilde{\gamma}$ are skew-symmetric, their generic ranks are bounded above by $m - 1$. Thus the proof in §4.2.1 again gives the desired bound $|\tau(\rho)| \leq (m - 1) \mathrm{vol}(X)$.

If the representation is maximal, say with $\mathrm{deg}_{\mathcal{T}} \tilde{W} > 0$, then the generic rank of $\tilde{\beta} : \tilde{W} \otimes L \rightarrow \tilde{V}$ on $\mathbb{P}T_X$ is $m - 1$. Moreover, if \mathcal{N} and \mathcal{I} are respectively the kernel and image sheaf of $\tilde{\beta} : \tilde{W} \otimes L \rightarrow \tilde{V}$, by the polystability along the leaves of $(\tilde{E}, \tilde{\theta})$, there is a rank 1 subbundle N of \tilde{W} and a rank $m - 1$ subbundle I of V , defined outside a codimension at least 2 \mathcal{T} -saturated analytic subset of $\mathbb{P}T_X$, such that $\mathcal{N} \otimes L^\vee$ and \mathcal{I} are the sheaves of sections of N and I . We have that $\tilde{\beta}$ vanishes on $N \otimes L$, and is generically injective on $(\tilde{W}/N) \otimes L$, whereas $\tilde{\gamma}$ vanishes

on I^\perp , and maps $I \otimes L$ to N^\perp . Hence we can conclude as in §4.2.2 and §4.2.3 that if we now set

$$\mathcal{S} = \{(x, [\xi]) \in \mathbb{P}T_X \mid \beta_x(\xi) : W_x/N_{[\xi]} \longrightarrow I_{[\xi]} \text{ is not injective}\},$$

then $\pi(\mathcal{S})$ is a proper analytic subset of X .

We again work above a single point $x \in X$, $x \notin \pi(\mathcal{S})$. The letters ξ, η will denote tangent vectors at x . If $\xi \neq 0$, we write N_ξ for the 1-dimensional kernel of $\beta(\xi) : W_x \rightarrow V_x$, and I_ξ for its $(m-1)$ -dimensional image. We denote by N_ξ^\perp and I_ξ^\perp their orthogonal complements w.r.t. the harmonic metric. We recall that V_x is naturally identified with W_x^* and that if \mathcal{B} is a basis of W_x and \mathcal{B}^* the dual basis of V_x , then the matrices of $\beta(\xi)$ and $\gamma(\xi)$ in these bases are skew-symmetric.

We first prove that for all η , $\gamma(\eta)$ vanishes on $\langle I_\xi^\perp, \xi \neq 0 \rangle$ and hence maps V_x to $\cap_{\xi \neq 0} N_\xi^\perp$.

Let $\xi \neq 0$. We know that $\gamma(\xi)$ vanishes on I_ξ^\perp by polystability along the leaves. Hence for all η , $\beta(\xi) \circ \gamma(\eta) = \beta(\eta) \circ \gamma(\xi)$ vanishes on I_ξ^\perp so that $\gamma(\eta)$ maps I_ξ^\perp to N_ξ . But $\gamma(\eta)$ also maps V_x to N_η^\perp so that $\gamma(\eta)(I_\xi^\perp) \subset N_\xi \cap N_\eta^\perp$. Therefore for η close to ξ , and hence for all η , $\gamma(\eta)(I_\xi^\perp) = 0$. This implies by skew-symmetry that $\gamma(\eta)(V_x) \subset N_\xi^\perp$, hence our claim.

Since $n \geq 2$ we may choose two linearly independent tangent vectors ξ and η . The letter ζ will denote a tangent vector in $\langle \xi, \eta \rangle$. The set $\{\beta(\zeta), \zeta \in \langle \xi, \eta \rangle\}$ is a 2-dimensional linear subspace of $\text{Hom}(W_x, V_x)$, whose non zero elements are all of rank $(m-1)$. By [Wes72, Theorem 3.1], there exist $r \geq 1$, and decompositions $W_x = W_0 \oplus W_1 \oplus \cdots \oplus W_r$ and $V_x = V_0 \oplus V_1 \oplus \cdots \oplus V_r$ such that

- $\beta(\zeta)(W_0) = \{0\}$, for all ζ ;
- $\beta(\zeta)(W_i) \subset V_i$, for all ζ and all $1 \leq i \leq r$;
- $\dim V_i = \dim W_i \pm 1$ for all $1 \leq i \leq r$;
- $\text{rk } \beta(\zeta)|_{W_i} = \min\{\dim W_i, \dim V_i\}$, for all $\zeta \neq 0$ and all $1 \leq i \leq r$.

We remark that since ξ and η are linearly independent, the kernels N_ξ and N_η of $\beta(\xi)$ and $\beta(\eta)$ are distinct, and so are their images I_ξ and I_η . Indeed, the equality of the images is equivalent to the equality of the kernels by skew-symmetry. Therefore if they were equal, we would get that $\beta(\zeta)$ defines an isomorphism $N_\xi^\perp \rightarrow I_\xi$ for all $\zeta \neq 0$. This is impossible for example because since $\langle \xi, \eta \rangle$ is 2-dimensional, $\zeta \mapsto \det \beta(\zeta)$ cannot vanish only for $\zeta = 0$. Therefore $W_0 = \{0\}$. Also, $V_0 = \{0\}$ because if not then necessarily $\dim V_0 = 1$ and $I_\zeta = V_1 \oplus \cdots \oplus V_r$ for all $\zeta \neq 0$.

Moreover, since $\dim W_x = \dim V_x$, there must be at least one i such that $\dim V_i = \dim W_i + 1$. Say that $\dim V_i = \dim W_i + 1$ for $1 \leq i \leq s$ and $\dim V_i = \dim W_i - 1$ for $s+1 \leq i \leq r$. Then for $\zeta \neq 0$, $\text{rk } \beta(\zeta) = m - (r - s) = m - s$, so that $s = 1$ and $r = 2$. Hence we have $W_x = W_1 \oplus W_2$ and $V_x = V_1 \oplus V_2$ with

- $\beta(\zeta)(W_i) \subset V_i$, for all ζ and all $1 \leq i \leq 2$;
- $\dim V_1 = \dim W_1 + 1$ and $\dim V_2 = \dim W_2 - 1$;
- $\beta(\zeta)|_{W_1} : W_1 \rightarrow V_1$ is one-to-one for all $\zeta \neq 0$;
- $\beta(\zeta)|_{W_2} : W_2 \rightarrow V_2$ is onto for all $\zeta \neq 0$.

We deduce as in §4.2.4 that $\cap_{\zeta \neq 0} \beta(\zeta)W_1 = \{0\}$. By duality, the same reasoning also implies that $W_2 = \langle N_\zeta, \zeta \neq 0 \rangle$.

The first point implies that $\gamma(\eta)$ vanishes on V_2^\perp , since it vanishes on each I_ζ^\perp and

$$\langle I_\zeta^\perp, \zeta \neq 0 \rangle = \left(\bigcap_{\zeta \neq 0} I_\zeta \right)^\perp = \left(\bigcap_{\zeta \neq 0} (V_2 \oplus \beta(\zeta)W_1) \right)^\perp = \left(V_2 \oplus \left(\bigcap_{\zeta \neq 0} \beta(\zeta)W_1 \right) \right)^\perp = V_2^\perp.$$

The second point implies that $\gamma(\eta)(V_2) \subset W_2$, and hence that $\gamma(\eta)$ vanishes on V_2 . Indeed, since $\beta(\xi)|_{W_2} : W_2 \rightarrow V_2$ is surjective, V_2 is generated by vectors of the form $\beta(\xi)w$ with $w \in W_2$ such that $\beta(\zeta)w = 0$ for some $\zeta \neq 0$. Hence $\beta(\zeta) \circ \gamma(\eta) \circ \beta(\xi)w = \beta(\eta) \circ \gamma(\xi) \circ \beta(\zeta)w = 0$, so that $\gamma(\eta) \circ \beta(\xi)w \in N_\zeta \subset W_2$. Now we saw that $\gamma(\eta)(V_x) \subset \bigcap_{\zeta \neq 0} N_\zeta^\perp = W_2^\perp$. Hence $\gamma(\eta)(V_2) = \{0\}$.

We conclude that $\gamma = 0$ so that the ρ -equivariant harmonic map f is holomorphic.

In the same way, if the representation is maximal and $\deg W < 0$, we get that $\beta = 0$ so that the ρ -equivariant harmonic map f is antiholomorphic. \square

Proof of Lemma 5.1. Let A be a (non trivial) m -by- m skew-symmetric matrix. Then, by Youla's decomposition [You61], there exists a unitary matrix $U \in \mathbf{U}(m)$ such that $U^\top A U$ is a block diagonal matrix with $\lfloor m/2 \rfloor$ skew-symmetric 2-by-2 blocks $\begin{pmatrix} 0 & -\alpha_i \\ \alpha_i & 0 \end{pmatrix}$, where $\alpha_i \in \mathbb{R}$, and one additional zero on the diagonal if m is odd. Then, as $(U^\top A U)^\star (U^\top A U) = U^{-1} A^\star A U$, we get

$$\frac{1}{2\lfloor m/2 \rfloor} \leq \frac{\operatorname{tr}((A^\star A)^2)}{(\operatorname{tr}(A^\star A))^2} = \frac{2 \sum \alpha_i^4}{(2 \sum \alpha_i^2)^2} \leq \frac{1}{2}$$

where the lower bound is attained if and only if all the α_i are equal to some $\alpha \in \mathbb{R}^\star$ and in particular $A^\star A$ is unitary conjugate to $\alpha^2 1_m$ if m is even, or to $\alpha^2 \operatorname{diag}(1, \dots, 1, 0)$ if m is odd).

We will prove the last assertion of the lemma by contradiction. So let now B be another skew-symmetric matrix, assume that A and B are linearly independent and that each non trivial matrix in the two-dimensional vector space they generate realizes the lower bound. We normalize A and B such that $\operatorname{tr}(A^\star A) = \operatorname{tr}(B^\star B) = 2\lfloor m/2 \rfloor$. We can also suppose that A and B are orthogonal (i.e. $\operatorname{tr}(A^\star B) = 0$).

If m is even then clearly we have a contradiction. Indeed, for all $\lambda, \mu \in \mathbb{C}$, $(\lambda A + \mu B)^\star (\lambda A + \mu B)$ is a multiple of 1_m , hence $\bar{\lambda}\mu A^\star B + \lambda\bar{\mu} B^\star A$ is also a multiple of 1_m , but it is trace free so $A^\star B$ must be equal to zero, which is not possible as the column vectors of A (and B) generate \mathbb{C}^m .

From now on, we assume that m is odd. Then for any $\lambda, \mu \in \mathbb{C}$, $(\lambda, \mu) \neq (0, 0)$, the matrix

$$(\lambda A + \mu B)^\star (\lambda A + \mu B) - \frac{1}{m-1} \operatorname{tr}[(\lambda A + \mu B)^\star (\lambda A + \mu B)] 1_m$$

has rank 1 and since $\operatorname{tr}(A^\star B) = 0$, $\operatorname{tr}[(\lambda A + \mu B)^\star (\lambda A + \mu B)] = (m-1)(|\lambda|^2 + |\mu|^2)$. In other words $N_{\lambda, \mu} := M_{\lambda, \mu} - (|\lambda|^2 + |\mu|^2) 1_m$ has rank 1, where we denoted $(\lambda A + \mu B)^\star (\lambda A + \mu B)$ by $M_{\lambda, \mu}$.

There exists a hyperplane E (resp. F) in \mathbb{C}^m such that the endomorphism of \mathbb{C}^m whose matrix is $A^\star A$ (resp. $B^\star B$) is the identity when restricted to E (resp. F).

If $E = F$ then upon replacing A and B by $U^\top A U$ and $U^\top B U$ for some well chosen $U \in \mathbf{U}(m)$, we may assume that the m -th column vectors of A and B are trivial so that the m -th column and the m -th line of $M_{\lambda, \mu}$ both are trivial. As $N_{\lambda, \mu}$ has rank 1, this implies that for any $\lambda, \mu \in \mathbb{C}$, the upper left $(m-1)$ -by- $(m-1)$ block of $M_{\lambda, \mu}$ is equal to $(|\lambda|^2 + |\mu|^2) 1_{m-1}$. As in the case when m is even, the upper left $(m-1)$ -by- $(m-1)$ block of $A^\star B$ should be equal to zero and this is impossible because the column vectors of A and B generate hyperplanes which must intersect non trivially.

Assume now that $E \cap F$ has codimension 2. We will use the notation $\langle x, y \rangle = x^\star y$ for the standard Hermitian product on \mathbb{C}^m and write $|x|^2 = x^\star x$. Let us denote by v_1, \dots, v_m , resp.

w_1, \dots, w_m , the column vectors of A , resp. B . Again upon replacing A and B by $U^\top AU$ and $U^\top BU$ for some well chosen $U \in U(m)$, one can assume that $v_m = 0$ and that (v_1, \dots, v_{m-1}) and (w_1, \dots, w_{m-2}) are orthonormal families. Moreover, w_{m-1} and w_m are linearly dependent because the bottom right 2-by-2 block of B^*B must have determinant 0. Finally we also have $|w_{m-1}|^2 + |w_m|^2 = 1$.

The bottom right 2-by-2 block of $N_{\lambda, \mu}$ is

$$\begin{pmatrix} |\mu|^2(|w_{m-1}|^2 - 1) + \lambda\bar{\mu}\langle w_{m-1}, v_{m-1} \rangle + \bar{\lambda}\mu\langle v_{m-1}, w_{m-1} \rangle & |\mu|^2\langle w_{m-1}, w_m \rangle + \bar{\lambda}\mu\langle v_{m-1}, w_m \rangle \\ |\mu|^2\langle w_m, w_{m-1} \rangle + \lambda\bar{\mu}\langle w_m, v_{m-1} \rangle & -|\lambda|^2 + |\mu|^2(|w_m|^2 - 1) \end{pmatrix}$$

In the determinant of this block, the coefficient of $|\mu|^4$ is equal to 0 since

$$(1 - |w_{m-1}|^2)(1 - |w_m|^2) - |\langle w_m, w_{m-1} \rangle|^2 = |w_{m-1}|^2|w_m|^2 - |\langle w_m, w_{m-1} \rangle|^2$$

and w_m and w_{m-1} are linearly dependent.

The coefficient of $|\lambda|^2|\mu|^2$ is

$$1 - |w_{m-1}|^2 - |\langle w_m, v_{m-1} \rangle|^2$$

and by Schwarz inequality, it vanishes if and only if w_m and v_{m-1} are proportional since $|w_{m-1}|^2 + |\langle w_m, v_{m-1} \rangle|^2 \leq |w_{m-1}|^2 + |w_m|^2 = 1$. In this case, there exist complex numbers a and b such that $w_{m-1} = av_{m-1}$, $w_m = bv_{m-1}$ and $|a|^2 + |b|^2 = 1$. The above determinant is then equal to $-(|\lambda|^2 + |\mu|^2)(\lambda\bar{\mu}\bar{a} + \bar{\lambda}\mu a)$ and vanishes identically if and only if $a = 0$.

So $a = 0$ and $|b| = 1$. This immediately implies that $N_{\lambda, \mu}$ is block diagonal with a $(m-2)$ -by- $(m-2)$ upper left block and a 2-by-2 bottom right block because A^*A et B^*B have the same block decomposition, hence A^*B and B^*A too, since $w_{m-1} = v_m = 0$ and $w_m = bv_{m-1}$. The 2-by-2 bottom right block of $N_{\lambda, \mu}$ is equal to

$$\begin{pmatrix} -|\mu|^2 & b\bar{\lambda}\mu \\ \bar{b}\lambda\bar{\mu} & -|\lambda|^2 \end{pmatrix}$$

hence has rank 1 for each $(\lambda, \mu) \neq (0, 0)$. As the matrix $N_{\lambda, \mu}$ has rank 1 for all $(\lambda, \mu) \neq (0, 0)$, we must have $\langle v_i, w_j \rangle = 0$ for any $1 \leq i, j \leq m-2$. If $m \geq 5$, this is impossible since the two subspaces of codimension 2 generated respectively by the families $\{v_i\}_{1 \leq i \leq m-2}$ and $\{w_i\}_{1 \leq i \leq m-2}$ have a non trivial intersection. \square

5.4. Representations in Hermitian groups without exceptional factors.

It is easy to generalize the statement of Theorem 1.1 to the case where the lattice Γ is assumed uniform but not torsion free, and the target Lie group H is assumed to be a semisimple Lie group of Hermitian type without compact or exceptional factors. By this we mean that H is an almost-product of simple noncompact Lie groups of Hermitian type which are each isogeneous to one of the classical groups we have been considering.

In this case, there is a normal subgroup Γ' of finite index d in G such that Γ' is torsion free and the representation $\rho' = \rho|_{\Gamma'}$ is a product of k representations $\rho'_i : \Gamma' \rightarrow H_i$, where the H_i 's are classical Hermitian noncompact Lie groups. One defines the Toledo invariant of ρ to be $\frac{1}{d} \tau(\rho')$. Since $\text{vol}(\Gamma \backslash \mathbb{H}_{\mathbb{C}}^n) = \frac{1}{d} \text{vol}(\Gamma' \backslash \mathbb{H}_{\mathbb{C}}^n)$, the representation ρ is maximal if and only if the representation ρ' is. Since $\tau(\rho') = \sum_{i=1}^k \tau(\rho'_i)$ and $\text{rk}_{\mathbb{R}} H = \sum_{i=1}^k \text{rk}_{\mathbb{R}} H_i$, ρ' is maximal if and only if each ρ'_i is. Therefore in this case $H_i = \text{SU}(p_i, q_i)$ with $p_i \geq q_i n$ for all i and there is a ρ -equivariant holomorphic or antiholomorphic map from $\mathbb{H}_{\mathbb{C}}^n$ to the symmetric space $\mathcal{Y} = \prod_{i=1}^k \mathcal{Y}_{p_i, q_i}$ associated to H .

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(Vincent Koziarz) UNIV. BORDEAUX, IMB, UMR 5251, F-33400 TALENCE, FRANCE
E-mail address: vkoziarz@math.u-bordeaux1.fr

(Julien Maubon) IECL, UMR 7502, UNIVERSITÉ DE LORRAINE, B. P. 70239, F-54506 VANDŒUVRE-LÈS-NANCY CEDEX, FRANCE
E-mail address: julien.maubon@univ-lorraine.fr