CAN WE HEAR THE ECHOS OF CRACKS?

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ABSTRACT. The detection of cracks in mechanical engineering is mainly based on ultrasonic testing and Foucault currents. But even if they are efficient tools, this technology requires an important handling and is limited to the detection of cracks which are close to the source. Recently, several searchers have discussed the possibility of using waves as Lamb waves, for thin plates and shells, but also Love waves for bimaterials. In both cases the structure works as a wave guide and enables a long range propagation which is a promising possibility for detecting a crack quite far from the source. In this paper, we discuss the observability property of a crack inside an open set using Love waves. But there is a condition which connects the space representation of the excitation and the geometry of the structure containing the crack.

Introduction

The aim of this paper is to obtain a criterion of detection of a crack using local waves for bimaterial modelled by a transmission problem. The crack is located inside the material along the interface of transmission and the local waves that we consider are Love waves which are located in the low velocity side of the set occupied by the material. The detection of cracks in mechanical engineering is mainly based on ultrasonic testing and Foucault currents. But even if they are efficient tools, this technology requires an important handling and is limited to the detection of cracks which are close to the source. Recently, several searchers have discussed the possibility of using waves as Lamb waves, for thin plates and shells, but also Love waves for bimaterials (see [11], [13], [19] and [22]). In both cases, the structure works as a wave guide and enables a long range propagation which is a promising possibility for detecting a crack quite far from the source. In this paper, we discuss the observability property of a crack inside an open set using Love waves. In order to be able to detect cracks using Love waves, we state in the paper a condition which connects the space representation of the excitation and the geometry of the structure containing the crack.

Our plan is the following :

In section 1, we present the mathematical problem, its notations and the mathematical tools used in the paper such as the time-Fourier transform of solutions of the involved partial differential equations. We state some classical existence results.

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In section 2, we focus our study on the singularity of the time-Fourier transform of solutions. Let us notice that such results have been deeply studied by P. Grisvard in many cases but not for bimaterial and for stationary models (see the explanations below) and this is why we entirely detail it.

In section 3, we present different methods in order to compute the coefficients involved by the singularities due to cracks. The results presented here are based on works developed in [8].

In section 4, we introduce a criterion for detection of cracks which is based on an observability result. We state here our main theorem in non destructive testing which concerns an efficient method of detection of small cracks using Love waves and that can be explicitly computed. Let us notice that our main theorem requires an assumption (52) which is still an open problem. However, we think that it is generically true (and still work on it), and our feeling is illustrated by the next section.

Section 5 is concerned with numerical simulations of our results. We compute Love waves and we give illustrations of the previous work. Computations are performed with Matlab.

1. **Notations and preliminaries results.** Let us consider a rectangle -say R- as shown on figure 1. Let us assume that there is a crack parallel to the axis bearing the coordinate x_1. Its two extremities are at points A and B with the abscissa a and b and ordinate h. Both (up and down) sides of the closed segment AB are the crack lips and they are denoted by Γ_f. The line which bears the crack is denoted Γ_i.The wave equation that we consider is set on the open set $\Omega = \overset{\circ}{R} \backslash \Gamma_f$. We write $\Omega_+ = \Omega \cap (x_2 > h)$ and $\Omega_- = \Omega \cap (x_2 < h)$. The material can be different from both sides of Γ_i and thus the wave velocities are different. They are denoted respectively by c_+ in Ω_+ and by c_- in Ω_-. For any set $Q \subset \mathbb{R}^d$ $(d = 1, 2)$, we write $Q^+ = Q \cap \Omega_+$, $Q^- = Q \cap \Omega_-$ and 1_Q denotes the characteristic function of Q.

The boundary of Ω is $\Gamma = \Gamma_1 \cup \Gamma_f$ where Γ_1 denotes the boundary of the rectangle R. On Γ_1, the free edge condition is assumed. The unit normal outwards Ω and along its boundary is ν. Let $c = c_+ 1_{\Omega_+} + c_- 1_{\Omega_-}$ be the wave velocity which is piecewise constant for a bimaterial with $0 < c_- < c_+$.

The wave model that we consider is the following one, where u is the transverse displacement.

$$\left\{ \begin{array}{l} \text{Find } u \text{ such that:} \\ \\ \dfrac{\partial^2 u}{\partial t^2} - div(c^2 \nabla u) = f \ \text{ in } \ \Omega, \\ \\ \dfrac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \Gamma, \ \ u(x,0) = 0 \ \text{ and } \ \dfrac{\partial u}{\partial t}(x,0) = 0 \ \text{in } \ \Omega. \end{array} \right. \tag{1}$$

The right hand side f is the excitation and can be considered as a control variable. For sake of simplicity it is assumed that :

$$f(x,t) = z(t)q(x), \tag{2}$$

where z is for instance a wavelet function which enables one to generate an excitation on given frequencies and q is a smooth function which is the so-called space control. In fact, it has to be chosen in order to optimize the detection of the cracks, this is why we mention it as a control variable.

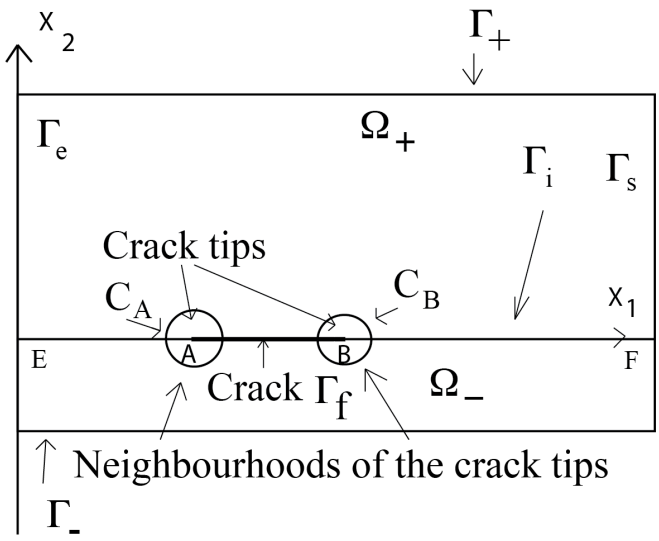

FIGURE 1. The open set used for the wave model

Let K be a compact set with $\{A, B\} \cap K = \emptyset$. This set K is fixed in all the paper. For sake of simplicity, we assume that $K = {}^cD_A \cap {}^cD_B \cap \bar{\Omega}$ where D_A and D_B are fixed open disk centered at A and B with same strictly positive radius. In the following it is assumed that the support of q is a subset of K. The existence and uniqueness of a solution are very classical results (see for instance [21]). The Helmoltz equation associated to this wave equation is obtained by taking the time Fourier transform. Setting (the initial conditions are homogeneous):

$$\hat{u}(x,\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} u(x,t)dt \quad \hat{z}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} z(t)dt, \tag{3}$$

one obtains:

$$\begin{cases} -\omega^2 \hat{u} - div(c^2 \nabla \hat{u}) = \hat{z}(\omega) q \ \text{ in } \ \Omega, \\ \dfrac{\partial \hat{u}}{\partial \nu} = 0 \ \text{ on } \Gamma. \end{cases} \tag{4}$$

Concerning existence and uniqueness of a solution one can derive the results from those known for (1) and from the Fourier transform. But, it is possible to prove it directly using Fredholm alternative : it enables one to precise for which values of ω the solution $\hat{u}$ is a unique element of the space $H^1(\Omega)$. First of all let us introduce the eigenvalue problem which is useful in this study.

Let us set:

$$\begin{cases} \text{find } (w, \lambda) \in H^1(\Omega) \times \mathbb{R} \text{ such that:} \\ -div(c^2 \nabla w) = \lambda w \text{ in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma, \quad \displaystyle\int_\Omega w^2(x)dx = 1. \end{cases} \tag{5}$$

The spectral theory (see for instance [12]) can be applied and enables one to state that there is a countable family of solutions $(w_n, \lambda_n) \in H^1(\Omega) \times \mathbb{R}^+$, $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots$, such that the family $\{w_n\}_{n\in\mathbb{N}}$, (respectively $\frac{1}{\sqrt{|\Omega|}} \oplus \{\frac{w_n}{\sqrt{\lambda_n}}\}_{n\in\mathbb{N}^*}$), is an Hilbert basis of the space $L^2(\Omega)$ (respectively of $H^1(\Omega)$). Furthermore, the sequence λ_n tends to infinity and the multiplicity of each eigenvalue is finite. The computation of these eigenmodes can be performed analytically when there is no crack, even and mainly, for a bimaterial as shown on figure 1. In this case, one obtains two families of eigenvectors. One contains the so-called Love stationary waves which are mainly localized in the open set Ω_- if $c_- < c_+$ with an exponential decay inside Ω_+ from the boundary Γ_i. They are computed explicitly at subsection 5.2. The second one contains global waves and their energy is sprayed in the whole domain Ω. Just for giving an idea on these two families of waves, we have plotted an element belonging to each of them on figure 2.

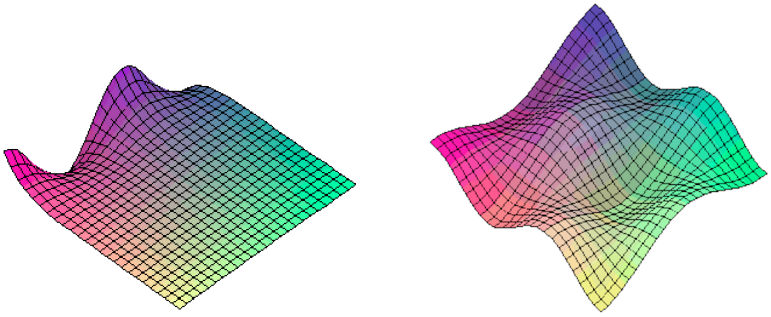

FIGURE 2. The two families of eigenvectors in case of a bimaterial (left is a Love wave and right a global one). The softest media is up side and the hardest is at the bottom.

The well-posedness of system (4) is resumed in the following theorem.

Theorem 1.1. *Let us introduce the set:*

$$\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}.$$

If $\omega^2 \notin \Lambda$, the system of equations (4) has a unique solution which is given by:

$$\hat{u}(x,\omega) = \hat{z}(\omega) \sum_{n\in\mathbb{N}} \frac{\displaystyle\int_\Omega q(x) w_n(x) dx}{\lambda_n - \omega^2} w_n(x).$$

Proof This is the classical Fredholm alternative as given in [12]-[21]. □

We now turn to section 2 where we describe the singularity due to the crack of the solution of (4).

2. **Singularity due to the crack.** We give a more precise description of the solution $\hat{u}$ in the neighborhood of the crack tip. In the case of a unique material, such characterizations are known since Kondratiev's work. A nice presentation is given in the book by Grisvard [14] in the following case : P. Grisvard analyses the case of a unique material (thus $c_+ = c_-$) in a polyhedral domain with general boundary conditions (Robin's type). One of the main tools is the use of polar coordinate and, in each angular sector, the use of the Hilbertian basis of eigenvector of second order differential equation in the polar coordinate satisfying the required boundary conditions. Let us notice that the case where the sector angle is 2π (which is the case of a crack) with homogeneous Neumann conditions at the two extremities is surprisingly not fully studied in the reference [14]. However, proving theorem below is inspired of P. Grisvard methods with a main change due to the non unique velocity which requires a suitable and not Hilbertian basis (for the polar coordinate).

Before stating our result, let us introduce some notations near the crack. The local polar coordinates are denoted respectively by (r_A, θ_A) for A and (r_B, θ_B) for B as shown on figure 3. The global cartesian coordinates are $x = (x_1, x_2)$.

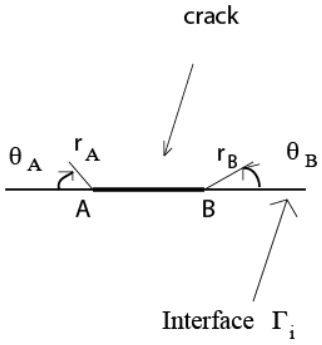

FIGURE 3. The neighborhood of the crack tips A and B

Let η_A (respectively η_B) be a smooth truncation function depending on the distance from point A (respectively B), equal to 1 in a close neighborhood D_{A_0} (respectively D_{B_0}) of A (respectively B) and null outside of a larger one denoted in the following by D_{A_1} (respectively D_{B_1}). We assume that their supports are subsets of the complementary of K.

We introduce the two local singular functions S_A and S_B defined by

$$\begin{cases} S_A(r_A, \theta_A) = \dfrac{\sqrt{r_A}}{c^2} \sin(\dfrac{\theta_A}{2}) \eta_A(x), \\ \\ S_B(r_B, \theta_B) = \dfrac{\sqrt{r_B}}{c^2} \sin(\dfrac{\theta_B}{2}) \eta_B(x). \end{cases} \tag{6}$$

We denote by V^R the space ($[\ ]_{\Gamma_i \setminus \Gamma_f}$ denotes the jump of a function across the boundary $\Gamma_i \setminus \Gamma_f$)

$$V^R = \{v \in H^1(\Omega),\ v_{|\Omega_\pm} \in H^2(\Omega_\pm),\ [c^2 \frac{\partial v}{\partial \nu}]_{\Gamma_i \setminus \Gamma_f} = 0,\ \ \frac{\partial v}{\partial \nu} = 0 \ on\ \partial\Omega\}. \tag{7}$$

We write for $v \in V^R$

$$||v||_{2,\Omega_+ \cup \Omega_-} = \sqrt{||v||^2_{2,\Omega_+} + ||v||^2_{2,\Omega_-}},$$

where $||v||_{p,X}$ denotes the $H^p(X)-$ norm of the function v. Let us finally underline that the norm in the space V^R is defined by:

$$v \in V^R \ \rightarrow \ ||v||_R = \sqrt{||v||^2_{1,\Omega} + ||v||^2_{2,\Omega_+} + ||v||^2_{2,\Omega_-}},$$

which takes into account the continuity of $v \in V_R$ across $\Gamma_i - \Gamma_f$.

Let us summarize the main features of the singularity analysis in the following theorem.

Theorem 2.1. *Let $q \in L^2(\Omega)$ with a compact support in K. There exist two complex numbers K_A, K_B (depending on ω) and a function $\hat{u}^R \in V^R$ such that the solution $\hat{u}$ of (4) satisfies on Ω*

$$\hat{u}(x,\omega) = \frac{K_A(\omega)}{c^2}\sqrt{r_A}\sin(\frac{\theta_A}{2})\eta_A(x) + \frac{K_B(\omega)}{c^2}\sqrt{r_B}\sin(\frac{\theta_B}{2})\eta_B(x) + \hat{u}^R(x,\omega),$$

The number K_A (respectively K_B) is called the stress intensity factor at point A (respectively B).

Furthermore, there exists a constant $c_0 > 0$ independent on the function q, but dependent on ω, such that:

$$|K_A| + |K_B| + ||\hat{u}^R||_{2,\Omega_+\cup\Omega_-} \leq c_0||q||_{0,\Omega}. \tag{8}$$

□

Remark 1. The conclusion of the theorem remains valid if one replaces the equation $-\omega^2\hat{u} - div(c^2\nabla\hat{u}) = \hat{z}(\omega)q$ by $-div(c^2\nabla\hat{u}) = f$ where $f \in L^2(\Omega)$ is null in a neighborhood of the interface between Ω_+ and Ω_- and satisfies the compatibility condition $\int_\Omega f = 0$ □

Proof

Even if the result is known for homogeneous materials and elliptic model (see [14]), we present the proof which takes into account the modifications necessary for multimaterials and stationary solutions (term $\omega^2\hat{u}$).

The function q is in the space $L^2(\Omega)$ but its support doesn't meet a close neighborhood of the crack tips . The only righthandside of the model is therefore the term $\omega^2\hat{u}$ which belongs to the space $H^1(\Omega)$. The proof is split into five steps for sake of clarity. In what follows, we focus at point B, similar results are valid of course at point A.

Step1: Localization. From classical Fredholm theorem [20], and for any $\omega^2 \notin \Lambda$ (see theorem 2.1), there exists a unique solution $\hat{u} \in H^1(\Omega)$ to the following model:

$$-\omega^2\hat{u} - div(c^2\nabla\hat{u}) = q \ in \ \Omega, \ \frac{\partial\hat{u}}{\partial\nu} = 0 \ on \ \Gamma.$$

Let us denote by $Q_B = [R_{B_0} < |x-B| < R_{B_1}]$ an open crown surrounded by two circles C_{B_0} and C_{B_1} of radius R_{B_0} and R_{B_1} ($R_{B_0} < R_{B_1}$) and both centered at the crack tip B. Let C_B be any circle centered at B inside the open set Q_B. One can easily prove (using a symmetrization around Γ_i) that $\hat{u}1_{Q_B^+} \in H^2(Q_B^+)$ and $\hat{u}1_{Q_B^-} \in H^2(Q_B^-)$. Hence, the restriction of $\hat{u}$ to C_B, C_B^+ or C_B^- satisfies $\hat{u}|_{C_B} \in H^{1/2}(C_B)$, $\hat{u}|_{C_B^+} \in H^{3/2}(C_{B^+})$ and $\hat{u}|_{C_B^-} \in H^{3/2}(C_{B^-})$ (trace theorem). Let $\rho = \rho(x) \in C_0^\infty(\mathbb{R}^2)$ with $\rho(x) = 1$ for $|x-B| \leq R$ and $\rho(x) = 0$ for $|x-B| > R_{B_1}$. We write $\hat{u} = \rho\hat{u} + (1-\rho)\hat{u}$. The function $(1-\rho)\hat{u}$ is null on $|x-B| \leq R$ therefore $\eta_B(1-\rho)\hat{u}1_{\Omega_+} \in H^2(\Omega_+)$ and $\eta_B(1-\rho)\hat{u}1_{\Omega_-} \in H^2(\Omega_-)$. In what follows, we study $\rho\hat{u}$ which is $\hat{u}$ on $|x-B| \leq R$.

Step 2: A Schauder basis for $L^2(]-\pi,\pi[)$ and $H^1(]-\pi,\pi[)$.

Let us consider the family of functions on C (which is the circle of radius R_B minus the point on Γ_f) :

$$p_n^s(\theta_B) = \frac{A_n}{c^2(\theta)} \sin(\frac{(2n+1)}{2}\theta_B) \quad \text{and} \quad p_n^c(\theta_B) = B_n \cos(n\theta_B)$$

where $c^2(\theta)$ is the velocity at the angle θ and where coefficients A_n and B_n are chosen such that

$$\int_{-\pi}^{\pi} p_n^s(\theta)^2 d\theta = \int_{-\pi}^{\pi} p_n^c(\theta)^2 d\theta = 1.$$

One gets

$$A_n = A_0 = \sqrt{\frac{2}{\pi}} \frac{c_+ c_-}{\sqrt{c_+^2 + c_-^2}} \quad \text{and} \quad B_n = B_0 = \frac{1}{\sqrt{\pi}}$$

Let us notice that these coefficients do not depend on n and we thus write them A_0 and B_0. If the functions p_n^c are C^∞ functions at the point $\theta = 0$, this is not the case of the functions p_n^s which are discontinuous at the point $\theta = 0$ (since the function c is discontinuous).

For a given function f defined on $]-\pi, \pi[$, we introduce the symmetrical and antisymmetrical parts of f :

$$f_a(\theta_B) = \frac{f(\theta_B) - f(-\theta_B)}{2}, \; f_s(\theta_B) = \frac{f(\theta_B) + f(\theta_B)}{2}.$$

The following Lemma will be useful for our study with the local polar coordinates near the point B (for instance) :

Lemma 2.2. *Let $g \in L^2(]-\pi, \pi[)$. There exists unique coefficients a_n and b_n such that*

$$g = \sum_{n\geq 0} a_n p_n^s + \sum_{n\geq 0} b_n p_n^c.$$

We have

$$\begin{cases} a_n = A_0 \displaystyle\int_{-\pi}^{\pi} c^2(\theta) g(\theta) p_n^s(\theta) d\theta, \\ b_n = B_0 \displaystyle\int_{-\pi}^{\pi} [g(\theta) - \sum_{n\geq 0} a_n (p_n^s)_s(\theta)] p_n^c(\theta) d\theta. \end{cases} \tag{9}$$

Furthermore,

$$\sum_{n\geq 0} [a_n^2 + b_n^2] < +\infty.$$

If $g \in H^1(]-\pi, \pi[)$, then its norm is equivalent to

$$||g||_{1,]-\pi,\pi[} \sim \sqrt{\sum_{n\geq 0} (1+n^2) a_n^2 + \sum_{n\geq 0} (1+n^2) b_n^2}.$$

□

Proof of Lemma 2.2

From classical results in Fourier analysis on $]0, \pi[$, one can write

$$g_a(\theta_B) = \sum_{n\geq 0} \frac{a_n}{2} (\frac{1}{c_+^2} + \frac{1}{c_-^2}) \sin((n+\frac{1}{2})\theta_B), \tag{10}$$

with

$$a_n = \frac{4c_+^2 c_-^2}{\pi(c_+^2 + c_-^2)} \int_0^\pi g_a(\theta_B) \sin((n+\frac{1}{2})\theta_B) d\theta_B.$$

The convergence of the previous series occurs in the space $L^2(]-\pi,\pi[)$ if $g \in L^2(]-\pi,\pi[)$ and in the space $H^1(]-\pi,\pi[)$ if $g \in H^1(]-\pi,\pi[)$.

Setting $g_a = g - g_s$ and the value of $A_0^2 = \dfrac{2c_+^2 c_-^2}{\pi(c_+^2 + c_-^2)}$, we get

$$\begin{aligned} a_n &= \frac{2c_+^2 c_-^2}{\pi(c_+^2 + c_-^2)} \int_{-\pi}^\pi g(\theta_B) \sin((n+\frac{1}{2})\theta_B) d\theta_B \\ &= \frac{2c_+^2 c_-^2}{\pi(c_+^2 + c_-^2)} \int_{-\pi}^\pi \frac{c^2(\theta_B)}{A} g(\theta_B) p_n^s(\theta_B) d\theta_B \\ &= A_0^2 \int_{-\pi}^\pi \frac{c^2(\theta_B)}{A_0} g(\theta_B) p_n^s(\theta_B) d\theta_B \\ &= A_0 \int_{-\pi}^\pi c^2(\theta_B) g(\theta_B) p_n^s(\theta_B) d\theta_B, \end{aligned} \tag{11}$$

which is the value given in Lemma 2.2.

The function p_n^s can be split into its symmetrical and antisymmetrical part and we obtain

$$(p_n^s)_a(\theta_B) = \frac{1}{2}(\frac{1}{c_+^2} + \frac{1}{c_-^2}) \sin((n+\frac{1}{2})\theta_B),$$

and

$$(p_n^s)_s(\theta_B) = \frac{1}{2}(\frac{1}{c_+^2} - \frac{1}{c_-^2}) \sin((n+\frac{1}{2})\theta_B),$$

therefore we proved that

$$g_a = \sum_{n\geq 0} a_n (p_n^s)_a = \sum_{n\geq 0} a_n p_n^s - \sum_{n\geq 0} a_n (p_n^s)_s.$$

Let

$$h(\theta_B) = \sum_{n\geq 0} a_n (p_n^s)_s = \sum_{n\geq 0} \frac{a_n}{2} (\frac{1}{c_+^2} - \frac{1}{c_-^2}) \sin((n+\frac{1}{2})\theta_B),$$

be the symmetrical part of the series $\displaystyle\sum_{n\geq 0} a_n p_n^s$.

The function $g_s - h$ is symmetrical and furthermore belongs to the space $H^1(]-\pi,\pi[)$ if $g \in H^1(]-\pi,\pi[)$. From Fourier theory, one can write:

$$g_s(\theta_B) - h(\theta_B) = \sum_{n\geq 0} b_n \cos(n\theta_B),$$

with

$$b_n = \frac{2}{\pi} \int_0^\pi [g_s(\theta_B) - h(\theta_B)] \cos(n\theta_B) d\theta_B. \tag{12}$$

The convergence occurs in the space $L^2(]-\pi,\pi[)$ and also in the space $H^1(]-\pi,\pi[)$ if $g \in H^1(]-\pi,\pi[)$.

We have :

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}[g_s(\theta_B) - h(\theta_B)]\cos(n\theta_B)d\theta_B$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}[g(\theta_B) - h(\theta_B)]\cos(n\theta_B)d\theta_B \tag{13}$$

$$= B_0\int_{-\pi}^{\pi}[g(\theta_B) - h(\theta_B)]p_n^c(\theta_B)d\theta_B.$$

Finally it has been proved that g can be written in a unique way as follows:

$$g(\theta_B) = g_a(\theta_B) + g_s(\theta_B) = \sum_{n\geq 0} a_n p_n^s(\theta_B) + \sum_{n\geq 0} b_n p_n^c(\theta_B). \tag{14}$$

The assertion $\sum_{n\geq 0}[a_n^2 + b_n^2] < +\infty$ is obvious from Fourier's theory.

Since $g \in H^1(]-\pi,\pi[) \Leftrightarrow (g_a, g_s) \in H^1(]-\pi,\pi[) \Leftrightarrow (h, g_s) \in H^1(]-\pi,\pi[)$, the $H^1(]-\pi,\pi[)-$ norm of g is equivalent to $||g_a||_{H^1(]-\pi,\pi[)} + ||g_s - h||_{H^1(]-\pi,\pi[)}$ which ends the proof of lemma 2.2. □

Remark 2. The family $\{p_n^s, p_n^c\}$ is a basis in $L^2(]-\pi,\pi[)$ and even in $H^1(]-\pi,\pi[)$. Let us point out that it is not an Hilbert basis as far as the functions are not two by two orthogonal (even if one can normalize them). The functions p_n^s belong to the space $H^1(]-\pi,\pi[)$, but not to $H^2(]-\pi,\pi[)$ because of the discontinuity of the first order derivatives at $\theta_B = 0$ when $c_+ \neq c_-$, which is the interesting case. But $c^2 p_n^s$ does belong to $H^2(]-\pi,\pi[)$. The functions p_n^c are polynomials in cartesian coordinates and thus they are $C^\infty([-\pi,\pi[)$. Moreover they are symmetrical on Q_B. Both functions p_n^s and p_n^c satisfy the homogeneous Neumann boundary conditions at $\theta_B = \pm\pi$ which correspond to the crack . □

Remark 3. It is also possible to consider an Hilbert basis of $L^2(]-\pi,\pi[)$ by computing the eigenfunctions solution of:

$$\begin{cases} \text{find } (\lambda, w) \in \mathbb{R}^+ \times v \in H^1(]-\pi,\pi[) \text{ such that:} \\ \\ \forall v \in H^1(]-\pi,\pi[),\ \displaystyle\int_{-\pi}^{\pi} c^2 \frac{dw}{d\theta}\frac{dv}{d\theta} = \lambda\int_{-\pi}^{\pi} wv, \\ \\ \displaystyle\int_{-\pi}^{\pi} w^2(\theta)d\theta = 1. \end{cases} \tag{15}$$

The analytical computation leads to the functions $p_n^c,\ (n \geq 0)$ on the one hand, and to the functions

$$p_n^{sc}(\theta) = \frac{D_n}{c^2}\sin(\frac{2n+1}{2}\theta)$$

on the other hand. The coefficient D_n is defined in order to satisfy the normalization condition:

$$D_n = \frac{2c_+^2 c_-^2}{\pi(c_+^2 + c_-^2)}.$$

The analysis is exactly the same as before if one choose to use this Hilbert basis of the space $L^2(]-\pi,\pi[)$. □

Let us introduce a family of Hilbert spaces denoted by D^s, and defined by:

$$D^s = \{g \in L^2(]-\pi,\pi[),\ g = \sum_{n\geq 0} a_n p_n^s + b_n p_n^c,\ \sum_{n\geq 0}(1+n^{2s})(a_n^2 + b_n^2) < \infty\} \tag{16}$$

The space D^s is endowed with its natural norm

$$||v||^2_{D^s} = \sum_{n\geq 0}(1+n^2)^s(a_n^2+b_n^2)$$

One has the following lemma which enables one to characterize few spaces D^s.

Lemma 2.3. *One has:*

$D^0 = L^2(]-\pi,\pi[)$,

$D^1 = H^1(]-\pi,\pi[)$,

$D^{3/2} \subset H^1(]-\pi,\pi[) \cap H^{3/2}(]-\pi,0[\cup]0,\pi[)$,

$$D^2 = \{v \in H^1(]-\pi,\pi[) \cap H^2(]-\pi,0[\cup]0,\pi[),\ \frac{dv}{d\theta_B}(\pm\pi) = 0,\ [c^2\frac{dv}{d\theta_B}]_{\Gamma_i-\Gamma_f}(0) = 0\}.$$

□

Proof

The two first equalities are a consequence of what has been done before. Therefore, we focus on $s = 2$ and $s = 3/2$. Let $g \in H^1(]-\pi,\pi[)$. From Lemma 2.2, one can write in $H^1(]-\pi,\pi[)$:

$$g = \sum_{n\geq 0}[a_n p_n^s + b_n p_n^c].$$

Let us now consider the second order derivative of the series (term by term), on $]0,\pi[$ for instance:

$$a_0\frac{d^2p_0^s}{d\theta_B^2} + \sum_{n\geq 1}[a_n\frac{d^2p_n^s}{d\theta_B^2} + b_n\frac{d^2p_n^c}{d\theta_B^2}]$$

$$= -\frac{1}{4}a_0\sin(\frac{\theta_B}{2}) - \sum_{n\geq 1}[(n+\frac{1}{2})^2 a_n\sin((n+\frac{1}{2})\theta_B) + n^2 b_n\cos(n\theta_B)].$$

The square of the $L^2(]0,\pi[)$-norm of the series is upper bounded in $L^2(]0,\pi[)$ by (c_0 is a constant independent on g):

$$c_0[a_0^2 + b_0^2 + \sum_{n\geq 1} n^4(a_n^2+b_n^2)].$$

Therefore the convergence of this series would ensure that $\frac{d^2g}{d\theta_B^2} \in L^2(]0,\pi[)$ and would prove the result. In fact if $g \in D^2$ this is satisfied; thus:

$$D^2 \subset \{v \in H^1(]-\pi,\pi[) \cap [H^2(]-\pi,0[\cup]0,\pi[),\ \frac{dv}{d\theta_B}(\pm\pi) = 0,\ \ [c^2\frac{dv}{d\theta_B}](0) = 0\}.$$

Conversely, if:

$$g \in \{v \in H^1(]-\pi,\pi[) \cap [H^2(]-\pi,0[\cup]0,\pi[),\ \frac{dv}{d\theta_B}(\pm\pi) = 0,\ [c^2\frac{dv}{d\theta_B}](0) = 0\},$$

from the definition of the coefficients a_n and b_n given earlier, the series $\sum_{n\geq 0} n^2 a_n p_n^s + n^2 b_n p_n^c$ converges for instance in $L^2(]0,\pi[)$ and finally $g \in D^2$. The interpolation between D^1 and D^2 (see [17]) leads to the result for $s = 3/2$ and Lemma 2.3 is proved. □

We now turn to the study of the explicit local solution $\hat{u}$.

Step 3 Analytical solution of the local model Let us look for analytical solutions of the local model and we still focus at point B. One can set on the circle C_B surrounding the neighborhood Q_B of B:

$$\hat{u}(r,\theta_B)=\sum_{n\geq 0} a_n(r)p_n^s(\theta_B)+b_n(r)p_n^c(\theta_B)$$

with

$$\forall\ s\in[0,3/2],\quad \sqrt{\sum_{n\geq 0}(1+n^{2s})(a_n(r)^2+b_n(r)^2)}\simeq ||\hat{u}(r)||_{s,]-\pi,\pi[}.$$

Remark 4. This convergence property is used as follows in the following (see step 3). Let us consider a series α_n satisfying:

$$\sum_{n\geq 3}\alpha_n^2 n^3<\infty. \tag{17}$$

Setting

$$k(r_B)=\sum_{n\geq 3}\alpha_n r_B^{\frac{n}{2}},\ |r|\leq 1,$$

and let us consider the second order derivative of this series with respect to r_B. In fact, the integration for the computing the H^2 norms are performed on the two dimensional open sets: $Q_B\cap\Omega_+$ and $Q_B\cap\Omega_-$. The surface element is $r_B dr_B\theta_B$. Therefore, in order to prove that the second order derivative of k is locally (in a neighborhood of B) in the space L^2, one has to check the integration with respect to r_B, of the following term:

$$\delta=\int_0^1\sum_{n\geq 3}\alpha_n^2(\frac{n^4}{16})r_B^{n-4}r_B dr_B\leq\sum_{n\geq 3}\alpha_n^2(\frac{n^4}{16})\int_0^1 r_B^{n-3}dr\leq\sum_{n\geq 3}\alpha_n^2\frac{n^4}{16(n-2)}.$$

And one can ensure from (17) that $\delta\in L^2(]0,1[)$ and therefore:

$$k\in H^2(]0,1[).$$

□

Let us now introduce the functions

$$H_n^s(r_B,\theta_B)=(\frac{r_B}{R_B})^{n+\frac{1}{2}}p_n^s(\theta_B)\text{ and }H_n^c(r_B,\theta_B)=(\frac{r_B}{R_B})^n p_n^c(\theta_B). \tag{18}$$

Since $(a_n,b_n)\in C([R/2,R])^2$, coefficients $A_n=a_n(R_B)$ and $B_n=b_n(R_B)$ are well defined. We have

$$\begin{cases} -div(c^2\nabla H_n^s)=-div(c^2\nabla H_n^c)=0\quad\text{in}\quad\Omega,\\ \\ \dfrac{\partial H_n^s}{\partial\nu}(r,\pm\pi)=\dfrac{\partial H_n^c}{\partial\nu}(r,\pm\pi)=0,\\ \\ \displaystyle\sum_{n\geq 0}A_nH_n^s(R_B,\theta_B)+B_nH_n^c(R_B,\theta_B)=\hat{u}(R_B,\theta_B).\end{cases}$$

Let us set on Q_B :

$$H(r_B,\theta_B)=\sum_{n\geq 0}A_nH_n^s(r_B,\theta_B)+B_nH_n^c(r_B,\theta_B). \tag{19}$$

The difference $d = \hat{u} - H \in H^1(Q_B)$ is the unique solution in the space $H^1(Q_B)$ of ($q = 0$ on Q_B):

$$\begin{cases} -div(c^2 \nabla d) = \omega^2 \hat{u} \; in \; Q_B, \\ \\ \forall r_B \in]0, R_B[: \; \dfrac{\partial d}{\partial \nu}(r_B, \pm\pi) = 0; \; \forall \theta_B \in]-\pi, \pi[: \; d(R_B, \theta_B) = 0. \end{cases} \tag{20}$$

Clearly, if $\omega = 0$, one would have $d = 0$. We assume $\omega \neq 0$. Lemma 2.2 leads to :

$$\omega^2 \hat{u} = \sum_{n \geq 0} [e_n^s(r_B) p_n^s(\theta_B) + e_n^c(r_B) p_n^c(\theta_B)],$$

where

$$\begin{cases} e_n^s(r_B) = A_0 \omega^2 \displaystyle\int_{-\pi}^{\pi} c^2 \hat{u}(r_B, \theta_B) p_n^s(\theta_B) d\theta_B, \\ \\ e_n^c(r_B) = B_0 \displaystyle\int_{-\pi}^{\pi} [\omega^2 \hat{u}(r_B, \theta_B) - \sum_j e_j^s(r_B) p_j^s(\theta_B)] p_n^c(\theta_B) d\theta_B. \end{cases}$$

We look for d as follows :

$$d(r_B, \theta_B) = d^s(r_B, \theta_B) + d^c(r_B, \theta_B),$$

where we have set

$$d^s(r_B, \theta_B) = \sum_{n \geq 0} \xi_n^s(r_B) p_n^s(\theta_B), \quad \text{and} \quad d^c(r_B, \theta_B) = \sum_{n \geq 0} \xi_n^c(r_B) p_n^c(\theta_B). \tag{21}$$

Let us compute ξ_n^s and ξ_n^c in $H^1(]0, R_B[)$, focusing on ξ_n^s (analogous computations are valid for ξ_n^c).

Since the functions p_n^s are orthogonal in $L^2(]0, \pi[)$, (and the same is true for the functions p_n^c) it is easy to prove that ξ_n^s is solution of (we set $\mu_n = n + \dfrac{1}{2}$)

$$\begin{cases} -r_B \dfrac{d}{dr_B}(r_B \dfrac{d\xi_n^s}{dr_B}) + \mu_n^2 \xi_n^s = r_B^2 e_n^s, \; 0 < r_B < R_B, \\ \\ \xi_n^s(R_B) = 0. \end{cases} \tag{22}$$

Let us notice that there is no boundary condition at $r_B = 0$, since it is replaced in this case by the fact that ξ_n^s and ξ_n^c must be in $H^1(]0, R_B[)$. By integrating this differential equation, we obtain

$$\xi_n^s(r_B) = r_B^{\mu_n} \int_{r_B}^{R_B} \frac{1}{s^{2\mu_n + 1}} \int_0^s \eta^{\mu_n + 1} e_n^s(\eta) d\eta ds. \tag{23}$$

Let us now estimate d^s (recalling that $\omega \hat{u} \in H^1(Q_B)$) with the following lemma

Lemma 2.4. *Let* $v(r, \theta_B) = \displaystyle\sum_{n \geq 0} a_n(r) p_n^s(\theta_B)$ *be in* $L^2(D_R)$. *Then*

1. $\dfrac{\partial v}{\partial r} \in L^2(Q_B) \quad \Leftrightarrow \quad \displaystyle\int_0^R r \sum_{n \geq 0} (\frac{da_n}{dr})^2 dr < \infty,$

2. $\dfrac{\partial v}{\partial \theta} \in L^2(Q_B) \quad \Leftrightarrow \quad \displaystyle\int_0^R r \sum_{n\geq 0}(1+n^2)a_n(r)^2 dr < \infty,$

3. $\dfrac{\partial^2 v}{\partial r^2} \in L^2(Q_B^+) \quad \Leftrightarrow \quad \displaystyle\int_0^R r \sum_{n\geq 0}(\frac{d^2 a_n}{dr^2})^2 dr < \infty,$

4. $\dfrac{\partial^2 v}{\partial r \partial \theta} \in L^2(Q_B^+) \quad \Leftrightarrow \quad \displaystyle\int_0^R r \sum_{n\geq 0}(1+n^2)(\frac{da_n}{dr})^2 dr < \infty,$

5. $\dfrac{\partial^2 v}{\partial \theta^2} \in L^2(Q_B^+) \quad \Leftrightarrow \quad \displaystyle\int_0^R r \sum_{n\geq 0}(1+n^4)a_n(r)^2 dr < \infty.$

□

The proof of this lemma is straightforwards using the orthogonality of the functions p_n^s on $H^s(]0,\pi[)$ $(s=0,1,2)$ and remark 4. □

The remaining proof concerning the estimate on ξ_n^s is rather technical and can be omitted in a first reading. Accepting it, the reader can jump to equation (30).

Let us now obtain the estimates of ξ_n^s using that $\displaystyle\sum_{n\geq 0} e_n^s p_n^s \in H^1(Q_B)$ and thus satisfies assertions 1 and 2 of Lemma 2.4 .

We write $A_n(s) = [\displaystyle\int_0^s \eta e_n^s(\eta)^2 d\eta]^{1/2}$ and, for sake of clarity during the proof, R instead of R_B.

First estimate :

We have for $n \geq 0$:

$$\begin{aligned}
|\xi_n^s(r)| &\leq r^{\mu_n} \int_r^R \frac{1}{s^{2\mu_n+1}} [\int_0^s \eta^{2\mu_n+1} d\eta]^{1/2} A_n(s) ds \\
&\leq r^{\mu_n} \int_r^R \frac{1}{s^{2\mu_n+1}} \frac{s^{\mu_n+1}}{\sqrt{2(\mu_n+1)}} A_n(s) ds \\
&\leq \frac{r^{\mu_n}}{\sqrt{2(\mu_n+1)}} A_n(r) [\frac{1}{\mu_n - 1}(\frac{1}{r^{\mu_n-1}} - \frac{1}{R^{\mu_n-1}})] \quad \text{if } n \geq 1 \\
&\leq c_0 \frac{r A_n(r)}{\mu_n \sqrt{\mu_n}} \quad \text{if } n \geq 1
\end{aligned}$$

and thus for $n \geq 1$:

$$|\xi_n^s(r)|^2 \leq c_0 \frac{r^2 A_n(r)^2}{\mu_n^3}. \tag{24}$$

The function d^s thus satisfies

$$\int_0^R \sum_{n\geq 1}(1+n^5) r |\xi_n^s(r)|^2 dr \leq c_0 \sum_{n\geq 0}(1+n^2) A_n(R)^2 < \infty.$$

With Lemma 2.4, $d^s - \xi_0^s p_0^s$ satisfies assertions 2 and 5 when $\omega \hat{u} \in L^2(Q_B)$.

Let us now turn to the study of the others assertions of Lemma 2.4 : they involve the computations of the derivatives with respect to r of the function ξ_n^s.

Second estimate :
We have:

$$\frac{d\xi_n^s}{dr}(r) = \frac{\mu_n}{r}\xi_n^s(r) - \frac{1}{r^{\mu_n+1}}\int_0^r \eta^{\mu_n+1} e_n^s(\eta)d\eta,$$

hence:

$$\begin{aligned}
|\frac{d\xi_n^s}{dr}(r)| &\leq \frac{\mu_n}{r}|\xi_n^s(r)| + \frac{1}{r^{\mu_n+1}}A_n(r)\frac{r^{\mu_n+1}}{\sqrt{2(\mu_n+1)}} \\
&\leq \frac{\mu_n}{r}|\xi_n^s(r)| + \frac{A_n(r)}{\sqrt{2(\mu_n+1)}} \\
&\leq c_0\frac{A_n(r)}{\sqrt{\mu}} \quad \text{with (24).}
\end{aligned}$$

We thus get (recall that $\mu_n = n + 1/2$):

$$\int_0^R \sum_{n\geq 0}(1+n^2)r|\frac{d\xi_n^s}{dr}(r)|^2 dr \leq c_0\sum_{n\geq 0}(1+n)A_n(R)^2 < \infty$$

which proves assertions 1 and 4 of Lemma 2.4.

Let us now turn to assertion 3 which involves $\dfrac{d^2\xi_n^s}{dr^2}$. Since

$$\frac{d^2\xi_n^s}{dr^2} = \frac{\mu_n^2}{r^2}\xi_n^s - e_n^s + \frac{1}{r}\frac{d\xi_n^s}{dr},$$

we study separately each term of the right hand side, starting with $\dfrac{\mu_n^2}{r^2}\xi_n^s$.

Integrating by parts, we get for $\mu_n \geq 1$ (which requires $n \geq 1$):

$$\begin{aligned}
\xi_n^s(r) &= r^{\mu_n}\int_r^R \frac{1}{s^{2\mu_n+1}}\Big([\frac{\eta^{\mu_n+2}}{\mu_n+2}e_n^s(\eta)]_0^s - \frac{1}{\mu_n+2}\int_0^s \eta^{\mu_n+2}\frac{de_n^s}{dr}(\eta)d\eta\Big)ds \\
&= \frac{r^{\mu_n}}{\mu_n+2}\int_r^R \frac{1}{s^{\mu_n-1}}e_n^s(s)ds - \frac{r^{\mu_n}}{\mu_n+2}\int_r^R \frac{1}{s^{2\mu_n+1}}\int_0^s \eta^{\mu_n+2}\frac{de_n^s}{dr}(\eta)d\eta\Big)ds \\
&\stackrel{def}{=} \xi_n^1(r) + \xi_n^2(r).
\end{aligned}$$

For $n \geq 2$, ξ_n^1 satisfies

$$\begin{aligned}
\xi_n^1(r) &= \frac{r^{\mu_n}}{\mu_n+2}\Big([\frac{s^{2-\mu_n}}{2-\mu_n}]_r^R - \int_r^R \frac{de_n^s}{dr}(s)\frac{s^{2-\mu_n}}{2-\mu_n}ds \\
&= \frac{r^{\mu_n}}{\mu_n^2-4}[\frac{e_n^s(r)}{r^{\mu_n-2}} - \frac{e_n^s(R)}{R^{\mu_n-2}}] - \frac{r^{\mu_n}}{\mu_n^2-4}\int_r^R \frac{1}{s^{\mu_n-2}}\frac{de_n^s}{dr}(s)ds,
\end{aligned}$$

thus (still for $n \geq 2$)

$$|\xi_n^1(r)| \leq \frac{1}{\mu_n^2-4}[r^2|e_n^s(r)| + (\frac{r}{R})^{\mu_n}R^2|e_n^s(R)| + r^{\mu_n}\int_r^R \frac{1}{s^{\mu_n-2}}|\frac{de_n^s}{dr}(s)|ds]$$

$$\leq \frac{1}{\mu_n^2-4}[[r^2|e_n^s(r)| + (\frac{r}{R})^{\mu_n}R^2|e_n^s(R)| + \frac{r^2}{\sqrt{2(\mu_n-2)}}(\int_0^R s|\frac{de_n^s}{dr}(s)|^2ds)^{1/2}]$$

Since $e_n^s \in H^1(]R/2,R[)$, the sequence $(e_n^s(R))_n \in l^\infty$ and for $n \geq 2$,

$$\frac{\mu_n^2}{r^2}|\xi_n^1(r)| \leq c_0[|e_n^s(r)| + r^{\mu_n-2}|e_n^s(R)| + \frac{1}{\sqrt{\mu_n}}(\int_0^R s|\frac{de_n^s}{dr}(s)|^2ds)^{1/2}]. \tag{25}$$

With simlar computations, one can prove that

$$|\xi_n^2(r)| \leq \frac{1}{\mu_n\sqrt{\mu_n}(\mu_n-2)}r^2(\int_0^R s|\frac{de_n^s}{dr}(s)|^2ds)^{1/2}$$

thus for $n \geq 2$,

$$\frac{\mu_n^2}{r^2}|\xi_n^2(r)| \leq \frac{1}{\sqrt{\mu_n}}r^2(\int_0^R s|\frac{de_n^s}{dr}(s)|^2ds)^{1/2}. \tag{26}$$

From (25) and (26), we deduce that

$$\sum_{n\geq 2}\int_0^R r\frac{\mu_n^2}{r^2}|\xi_n^s(r)|^2dr < \infty. \tag{27}$$

Let us now turn to the term involving $\frac{1}{r}\frac{d\xi_n^s}{dr}$. Writing

$$\frac{1}{r}\frac{d\xi_n^s}{dr} = \frac{\mu_n}{r^2}\xi_n^s - \frac{1}{r^{\mu_n+2}}\int_0^r \eta^{\mu_n+1}e_n^s(\eta)d\eta, \tag{28}$$

and integrating by parts, it is easy to check that

$$|\frac{1}{r^{\mu_n+2}}\int_0^r \eta^{\mu_n+1}e_n^s(\eta)d\eta| \leq c_0[\frac{1}{\mu_n}|e_n^s(r)| + \frac{1}{\mu_n\sqrt{\mu_n}}(\int_0^R s|\frac{de_n^s}{dr}(s)|^2ds)^{1/2}]. \tag{29}$$

With (27), (28) and (29), we deduce that

$$\sum_{n\geq 2}\int_0^R r(\frac{d^2\xi_n^s}{dr^2})^2dr \leq c_0[\sum_{n\geq 2}\int_0^R r(|e_n^s(r)|^2+|\frac{de_n^s}{dr}(r)|^2)dr + \sum_{n\geq 2}r^{2\mu_n-4}||e_n^s||^2_{C[R/2,R]}]$$

where the terms $||e_n^s||^2_{C[R/2,R]}$ are uniformly bounded in n therefore

$$d^s - \xi_0^s p_0^s - \xi_1^s p_1^s \in H^2(Q_B^+) \cap H^2(Q_B^-).$$

Since $\mu_1 = 3/2$, we get $\xi_1^s(r) = r^{3/2}\int_r^R \frac{1}{s^4}\int_0^s \eta^{5/2}e_1^s(\eta)d\eta ds$ and one can easily verifies that:

$$\xi_1^s(r)p_1^s(\theta_B) = \xi_1^s(r)\frac{A}{c^2}\sin(\frac{3\theta_B}{2}) \in H^2(Q_B^+) \cap H^2(Q_B^-).$$

Hence (with the notations used in theorem 1.1)

$$d^s - \xi_0^s p_0^s = d^s - a_0 S_B \in H^2(Q_B \cap [\Omega_+ \cup \Omega_-]).$$

Finally, it has been proved that on Q_B and with $\mu_n = n + \dfrac{1}{2}$:

$$d^s(r_B, \theta_B) = \sum_{n\geq 0} [A_n (\frac{r_B}{R_B})^{\mu_n} + \xi_n^s(r_B)] p_n^s(\theta_B). \tag{30}$$

In (30), the first series (with the coefficients A_n) takes into account the non homogeneous Dirichlet boundary condition on the circle C and the second one which satisfies an homogeneous Dirichlet boundary condition on C, takes into account the right handside of the equation (30) satisfied by ξ_n^s. The expression of $\hat{u}$ on Q_B is the sum of two series (d^s and d^c). It is worth noting that the functions $(\dfrac{r_B}{R_B})^n p_n^c$ which appear in the expression of d^c, are polynomials with respect to the cartesian coordinates (x_1, x_2) and therefore are C^∞. Furthermore the convergence of the corresponding series is uniform inside Q_B, $(r_B < R_B)$. But this is no more true for the terms $(\dfrac{r_B}{R_B})^{n+\frac{1}{2}} p_n^s$ which appear in the expression of d^s, because of the multiplicative term $\sqrt{r_B}$. In fact the first term which appears in the expression of d^s -say $\sqrt{\dfrac{r_B}{R_B}} p_0^s(\theta_B)$ is not smooth. It is named the singularity term (in the sense that it isn't in the space V_R see (7)) and denoted in the following by:

$$S_B(r_B, \theta_B) = \frac{1}{c^2} \sqrt{\frac{r_B}{R_B}} \sin(\frac{\theta_B}{2}).$$

It is in the space $H^1(Q_B)$ but not in $H^2(Q_B^+)$ or $H^2(Q_B^-)$.The second series which appears in the expression of d^s contains the term ξ_n^s which is explicit at (23). It has been noticed that the series converges to an element of the space $H^2(Q_B)$.

Step 4 : The *a priori* estimate (8).

Let us recall that K is a compact set with $\{A, B\} \cap K = \emptyset$. Let η_A and η_B be fixed such as in step 2 with compact supports in $\bar{K}$. Let $L^2_K = \{q \in L^2(\Omega), \quad \text{supp}(q) \subset K, \quad \int_\Omega q(x)dx = 0\}$. For $q \in L^2_K$, there exists a unique solution w of

$$\begin{cases} -\text{div}(c^2 \nabla w) = q \quad \text{in} \quad \Omega \\ \\ \dfrac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \end{cases}$$

and we proved in step 3 that there exists a unique $(k_A, k_B, v) \in \mathbb{R}^2 \times V^R$ such that $w = K_A S_A \eta_A + K_B S_B \eta_B + v^R$.

Applying the closed graph Theorem [16], il it easy to prove that the linear mapping

$$q \in L^2_K \to (K_A, K_B, v^R) \in \mathbb{R}^2 \times V^R$$

is continuous and estimate (8) is proved and so is Theorem 2.1. □

Let us now turn to the explicit computation of constants K_A and K_B.

3. **Computation of the stress intensity factors.** There are several possibilities for computing the coefficient K_A and K_B. Each of them has its own advantages and drawbacks depending of the goal one aims at. Let us sketch three of those methods recalling that q (see (3)) has a support far from the two crack tips A and B.

3.1. **The G^θ method.** Let us set $\theta = \{\theta_i\}$, $i = 1, 2$ where θ_i are at least $C^1(\overline{\Omega})$ functions with support included in a neighborhood $\tilde{V}_A$ of A. Let V_A be an open disk centered at A such that $V_A \subset\subset \tilde{V}_A$. We assume that $\theta = -e_1$ (a unit length growth of the crack which justifies the minus sign) on V_A. Using domain derivative tools, one gets the so-called G^θ expression [8] (the sign is changed at point B):

$$G^\theta = \frac{\pi K_A^2}{8}(\frac{1}{c_-^2} + \frac{1}{c_+^2}) = \frac{1}{2}\int_{\tilde{V}_A} [c^2|\nabla \hat{u}|^2 - \omega^2 \hat{u}^2] div(\theta) - \int_{\tilde{V}_A} c^2 (D\theta \nabla \hat{u}, \nabla \hat{u}). \quad (31)$$

Let us recall that this expression is obtained after multiplying the state equation of $\hat{u}$ by $\nabla u.\theta$ and by integrating on $\Omega_A = \Omega - \tilde{V}_A$ and then by evaluating the limit when the radius of $\tilde{V}_A$ tends to 0. This formula is independent on the choice of the vector field θ satisfying the previous requirements. This is a big advantage for the numerical computation of the coefficient K_A. Up to now this method introduced in 1979 is known as the most reliable and can be extended to any situation (3D, non linear elasticity, plates and shells...) et and gives good numerical results. This is due to the fact that it is energetically stable (with respect to energy norm of the solution $\hat{u}$ to the system (4)). This is an important feature for the inverse problem that will be defined in order to localize the crack in a forthcoming paper.

3.2. **The J-formula (Rice integral).** Let us denote by C_A a continuous curve (at least piecewise C^1 surrounding the crack tip A with the two extremities on the crack lips and which delimits an open set V_A including A (see figure 4). The unit normal to C_A and outwards V_A is denoted by ν and we set $(e_1, \nu) = \nu_1$. Using Stokes formula from the

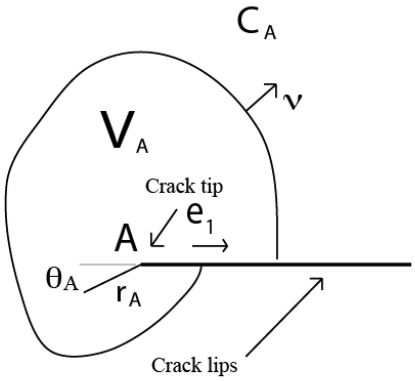

FIGURE 4. The neighborhood of the crack tip A

G^θ expression (31) and because of the equation satisfied by $\hat{u}$, one obtains the following expression for the stress intensity factor K_A where e_1 is the unit vector oriented from left to right on figure 4 (hence the sign is changed for point B) :

$$\frac{\pi K_A^2}{8}(\frac{1}{c_-^2} + \frac{1}{c_+^2}) = \frac{1}{2}\int_{C_A} [c^2|\nabla \hat{u}|^2 - \omega^2 \hat{u}^2]\nu_1 - \int_{C_A} c^2 \frac{\partial \hat{u}}{\partial \nu}(\nabla \hat{u}, e_1). \quad (32)$$

Unfortunately, this expression is not very convenient from the practical (ie. numerical) point of view, because the derivatives of $\hat{u}$ on a path inside Ω, are not stable quantities in a numerical approximation of the solution even if it is equal to (31) for the continuous model.

3.3. **The dual singular functions.** The idea of singular functions is a very old one and is based on the non uniqueness of a solution in $L^2(\Omega)$ to the homogeneous elasticity model as far as one looks for non smooth one. Let

$$L^2_K = \{q \in L^2(\Omega), \quad \text{supp}(q) \subset K, \quad \int_\Omega q(x)dx = 0\}.$$

Let us introduce

$$V_K = \{v \in H^1(\Omega) \cap L^2_0(\Omega), -div(c^2\nabla v) \in L^2_K(\Omega), \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\}. \tag{33}$$

and let us set

$$V^R_K = V^R \cap V_K. \tag{34}$$

The space V_K is a Banach space endowed with its natural norm

$$||v||_{V_K} = ||v||_{1,\Omega} + ||div(c^2\nabla v)||_{0,\Omega}.$$

If $f \in L^2_K(\Omega)$ is a given function, one can consider the following problem:

$$\text{find } z \in H^1(\Omega) \cap L^2_0(\Omega), \begin{cases} -div(c^2\nabla z) = f \text{ in } \Omega, \\ \dfrac{\partial z}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{35}$$

which has a unique solution and $z \in V_K$. The mapping T defined by

$$f \in L^2_K \ \rightarrow \ T(f) = z \in V_K$$

is linear and continuous. It is not difficult to see that T is an isomorphism from L^2_K onto V_K and

$$M = T^{-1} \in ISO(V_K, L^2_K).$$

From theorem 2.1, the solution z of (35) can be written

$$z = K_A S_A + K_B S_B + v^R, \text{ with } v^R \in V^R. \tag{36}$$

Let us recall that $K = ^c(D_A \cup D_B) \cap \bar{\Omega}$.

Let $\rho \in C^\infty_0(R)$ (R is the rectangle) with $\rho = 1$ on D_A and D_B. Since the singular parts of z at points A and B are uniquely defined on a neighborhood of A and B, one can also write:

$$z = K_A\sqrt{r_A}\sin(\frac{\theta_A}{2})\rho + K_B\sqrt{r_B}\sin(\frac{\theta_B}{2})\rho + z^R, \text{ with } z^R \in V^R. \tag{37}$$

Since the supports of any derivative of ρ is in K, the support of $-div(c^2\nabla z^R)$ is a subset of K and thus $z^R \in V^R_K$ (see 34).

Let us check that $M(V^R_K)$ is a closed subspace of L^2_K of codimension 2.

If :

$$\lim_{n\to\infty} ||f_n - f||_{0,\Omega} = 0,$$

the functions $v_n \in V^R_K$ such that $Mv_n = f_n$ satisfy from theorem 2.1:

$$||v_n||_{2,\Omega_+\cup\Omega_-} \leq c||f_n||_{0,\Omega}.$$

Therefore, the sequence $(v_n)_n$ is bounded in $H^2(\Omega_+ \cup \Omega_-)$ and (up to a subsequence) has a weak limit $v \in V^R_K$ such that $Mv = f$. Hence the range $M(V^R_K)$ is closed in the space L^2_K. We can write $L^2_K = M(V^R_K) \oplus M(V^R_K)^\perp$. Since the codimension of V^R_K in V_K is 2 (see (37)) and since M is an isomorphism from V_K onto L^2_K, we deduce that the dimension of $M(V^R_K)^\perp$ is 2. It is then spanned by two elements say S^*_A and S^*_B which can be chosen

as follows. First of all, they satisfy the orthogonality with the range of V_K^R by M in L_K^2. Since, the constants are in the kernel of M, we get

$$\forall v \in V_K^R \quad \int_K S_A^* div(c^2 \nabla v) = \int_K S_B^* div(c^2 \nabla v) = 0. \tag{38}$$

We now prove the following lemma:

Lemma 3.1. *There exists two functions S_A^* and S_B^* in $L^2(\Omega)$ such that for any compact set K with $\{A, B\} \cap K = \emptyset$, we have:*

$$L_K^2 = M(V_K^R) \oplus (\mathbb{R}S_A^* + \mathbb{R}S_B^*)1_K.$$

Furthermore, we have

$$\forall v \in V^R \quad \int_\Omega S_A^* div(c^2 \nabla v) = \int_\Omega S_B^* div(c^2 \nabla v) = 0. \tag{39}$$

S_A^* and S_B^* are called the dual singular functions. □

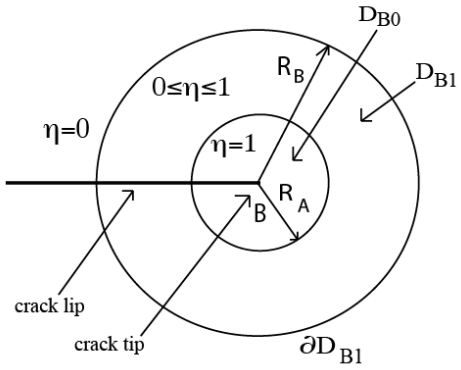

FIGURE 5. Two close circular neighborhoods of the crack tip B

Proof - Let us first remark that assertion (39) implies (38).

Let us consider for instance a circular neighborhood (see for example the figure 5) around the crack tip B denoted by D_{B_1}. The function S_B^* is locally solution of the following system deduced from (39) (see figure 5):

$$\begin{cases} -div(c^2 \nabla S_B^*) = 0 \text{ in } D_{B_1}, \\ \\ \dfrac{\partial S_B^*}{\partial \nu} = 0 \text{ on } \Gamma_i \cap \partial D_{B_1}, \quad S_B^* \in L^2(D_{B_1}) \quad (\Gamma_i \text{ are the crack's lips}). \end{cases} \tag{40}$$

The two functions S_A^* and S_B^* should be linearly independent in order to span the co-range of M. In order to localize S_B^* for instance, one can add the homogeneous Dirichlet condition on the external boundary of D_{B_1} $S_B^* = 0$ ($r = R_{B_1}$) and something similar can be done for S_A^*.

The local analytical solution using a local coordinate system (r_B, θ_B), is therefore, up to a multiplicative constant (classical computation on D_{B1}):

$$S_{Bloc}^*(r_B, \theta_B) = \frac{1}{c^2}\left(\frac{1}{\sqrt{r_B}} - \frac{\sqrt{r_B}}{R_{B_1}}\right) \sin\left(\frac{\theta_B}{2}\right) 1_{D_{B_1}}. \tag{41}$$

A similar expression is also true for the crack tip A. The extension (for instance in the case of S_B^*) to an arbitrary domain Ω can be done using a C^2 truncation function only dependent on the radius r_B, -say $\eta_(r_B)$- equal to one on a close neighborhood of the crack tip and to zero outside of a larger one. Then let us set (see figure 5 for the notations):

$$S_B^*(x) = \eta_B(r_B) S^*_{B_{loc}}(r_B, \theta_B) + S^*_{B_{ext}}(x)$$

$S^*_{B_{ext}}$ is zero on the close neighborhood of B where $\eta = 1$, for instance D_{B_0}.

The function $S^*_{B_{ext}}(x) \in H^1(\Omega)$ is the unique solution (thanks to (39)) defined up to a constant, of:

$$\begin{cases} -div(c^2 \nabla S^*_{B_{ext}}) = c^2 \nabla \eta_B . \nabla S^*_{B_{loc}} + div(c^2 S^*_{B_{loc}} \nabla \eta_B) \ \text{ in } \Omega \setminus D_{B_0}, \\ \\ \dfrac{\partial S^*_{B_{ext}}}{\partial \nu} = 0 \ \text{ on } \partial\Omega - \partial D_{B_0}, \\ \\ S^*_{B_{ext}} = 0 \text{ on } \partial D_{B_0}. \end{cases} \tag{42}$$

Let us underline that the two functions defined above (S_A^*, S_B^*) are linearly independent (because one is locally near one of the crack-tips) and they are in the space $L^2(\Omega)$ and not in $H^1(\Omega)$ because $S^*_{B_{loc}}$ is not in $H^1(D_{B_0})$. Lemma 3.1 is proved. □

The computation rules between singular functions and dual singular functions are mentioned in the following theorem.

Theorem 3.2. *Let S_A^* the dual singular function of crack tip A defined above and S_B the singular function associated to point B. One has:*

$$\int_\Omega S_A^* div(c^2 \nabla S_B) = 0.$$

Proof Let us notice that S_A^* is regular (H^1) on the support of S_B thus an integration by parts and the use of Stokes formula are both valid. The definitions of S_A^* and S_B enable one to write:

$$\begin{cases} \displaystyle\int_\Omega S_A^* div(c^2 \nabla S_B) = \int_{D_{B1}} S_A^* div(c^2 \nabla S_B) = \\ \\ \displaystyle\int_{\partial D_{B1}} c^2 S_A^* \frac{\partial S_B}{\partial \nu} - \int_{\partial D_{B1}} c^2 \frac{\partial S_A^*}{\partial \nu} S_B + \int_{D_{B1}} S_B div(c^2 \nabla S_A^*) = 0. \end{cases}$$

The Theorem 3.2 is proved. □

Let us now introduce the main result of this section. It is a formula which enables one to characterize the stress intensity factors. This is not necessarily the best one from a computational point of view, but it is very convenient for our purpose in this research work. Let us notice that, for numerical reasons, it is important that the dual singular functions do not depend on the compact K.

Theorem 3.3. *Let $\omega^2 \notin \Lambda$ (see Theorem 1.1 for the definition of Λ) and $f \in L^2(\Omega)$ a given function with support in K. Let z be the unique solution of the following stationary model $z \in H^1(\Omega)$ such that:*

$$\begin{cases} -\omega^2 z - div(c^2 \nabla z) = f \ \text{ in } \Omega, \\ \\ \dfrac{\partial z}{\partial \nu} = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Then one can write (see Theorem 2.1):

$$z = K_A S_A(r_A, \theta_A) + K_B S_B(r_B, \theta_B) + z_R(x) \;\; \textit{with } z_R \in V^R.$$

Furthermore, one can choose the dual singular functions associated to the crack tips A and B such that

$$-\int_\Omega S_A^* div(c^2 \nabla S_A) = -\int_\Omega S_B^* div(c^2 \nabla S_B) = 1.$$

Therefore one has:

$$\begin{cases} K_A(\omega) = \displaystyle\int_\Omega f S_A^* + \omega^2 \int_\Omega z S_A^*, \\ \\ K_B(\omega) = \displaystyle\int_\Omega f S_B^* + \omega^2 \int_\Omega z S_B^*. \end{cases}$$

□

Proof First of all, one can write (ν is hereafter the unit outwards normal to the boundary of the open crown $D_{B_1} \setminus D_{B_0}$ (see figure 5):

$$\begin{cases} -\displaystyle\int_\Omega div(c^2 \nabla z) S_A^* = \int_\Omega f S_A^* + \omega^2 \int_\Omega z S_A^*, \\ \\ -\displaystyle\int_\Omega div(c^2 \nabla z) S_B^* = \int_\Omega f S_B^* + \omega^2 \int_\Omega z S_B^*. \end{cases}$$

Hence from theorems 2.1 and (39), one deduces the relation given in Theorem 3.3.

Let $\rho \in C_0^\infty(D_{B_1})$ with $\rho = 1$ on D_{B_0}. Writing $S_B = \rho S_B + (1-\rho) S_B$ and since $(1-\rho) S_B \in V^R$, one obtains with Lemma 3.1

$$\int_\Omega S_B^* div(c^2 \nabla S_B) = \int_\Omega S_B^* div(c^2 \nabla(\rho S_B)).$$

The four arguments $div(c^2 \nabla S_B^*) = 0$ in Ω, $\rho = 1$ on ∂D_{B_0}, $\rho = 0$ on ∂D_{B_1} and $S_B^* = 0$ on ∂D_{B_1} lead to

$$\begin{aligned} \int_\Omega S_B^* div(c^2 \nabla S_B) &= \int_{\partial(D_{B_1} \setminus D_{B_0})} c^2 S_B^* \frac{\partial(\rho S_B)}{\partial \nu} - \int_{\partial(D_{B_1} \setminus D_{B_0})} c^2 \rho S_B \frac{\partial S_B^*}{\partial \nu} \\ &= \int_{\partial D_{B_0}} c^2 S_B^* \frac{\partial S_B}{\partial \nu} - \int_{\partial D_{B_0}} c^2 S_B \frac{\partial S_B^*}{\partial \nu} \\ &= -\int_{-\pi}^{\pi} \frac{1}{2c^2} \Big(\frac{1}{\sqrt{R_{B_0}}} - \frac{\sqrt{R_{B_0}}}{R_{B_1}} \Big) \sin^2\Big(\frac{\theta_B}{2}\Big) \sqrt{R}_{B_0} d\theta_B \\ &\qquad - \int_{-\pi}^{\pi} \frac{1}{2c^2} \Big(\frac{1}{R_{B_0}} + \frac{1}{R_{B_1}} \Big) \sin^2\Big(\frac{\theta_B}{2}\Big) R_{B_0} d\theta_B \\ &= -\frac{\pi}{4} \Big(\frac{1}{c_-^2} + \frac{1}{c_+^2} \Big) \Big(1 - \frac{R_{B_0}}{R_{B_1}} \Big) - \frac{\pi}{4} \Big(\frac{1}{c_-^2} + \frac{1}{c_+^2} \Big) \Big(1 + \frac{R_{B_0}}{R_{B_1}} \Big) \\ &= -\frac{\pi}{2} \Big(\frac{1}{c_-^2} + \frac{1}{c_+^2} \Big) < 0. \end{aligned}$$

The coefficient

$$\frac{\pi}{2}(\frac{1}{c_-^2}+\frac{1}{c_+^2})$$

can be used in the normalization of the dual singular functions in order to avoid a division in the expressions of the stress intensity factors K_A and K_B. The computation of the coefficients K_A and K_B can be easily deduced from the assertion $z = K_A S_A(r_A,\theta_A) + K_B S_B(r_B,\theta_B) + z_R(x)$ with $z_R \in V^R$ and the properties of the dual singular functions. Theorem 3.3 is proved. □

Remark 5. The result contained in Theorem 3.3 can be extended to a larger number of crack tips. □

We now turn to the main point of the paper : the detection of a crack.

4. **A criterion for a crack detection.** We first state a criterion. Let us denote by Ω^ε a sub-open set of Ω from which we have substracted two discs of radius ε centered respectively in A and B. The regularity of $\hat{u}$ on Ω^ε enables one to apply the Stokes formula. First of all, let us introduce an energetically invariant obtained by the domain derivative method. Multiplying the equation (4) by $\dfrac{\partial \hat{u}}{\partial x_1}$, one obtains:

$$-\int_\Omega \omega^2 \hat{u}\frac{\partial \hat{u}}{\partial x_1} - \int_\Omega div(c^2\nabla\hat{u})\frac{\partial \hat{u}}{\partial x_1} = \hat{z}(\omega)\int_\Omega q(x)\frac{\partial \hat{u}}{\partial x_1},$$

or else, after several integrations by parts on the open set Ω^ε:

$$-\int_{\Gamma_e\cup\Gamma_s\cup C_A^\varepsilon\cup C_B^\varepsilon}\frac{\omega^2}{2}\hat{u}^2\nu_1 - c^2\frac{\partial\hat{u}}{\partial\nu}\frac{\partial\hat{u}}{\partial x_1} + \int_\Omega c^2\frac{\partial}{\partial x_1}(\frac{|\nabla\hat{u}|^2}{2}) = 0,$$

and finally, using the results recalled in section 3.1:

$$\frac{1}{2}\int_{\Gamma_e\cup\Gamma_s}[-\omega^2\hat{u}^2 + c^2|\frac{\partial\hat{u}}{\partial x_2}|^2]\nu_1 - \hat{z}(\omega)\int_\Omega q(x)\frac{\partial\hat{u}}{\partial x_1} = \frac{\pi}{8}(\frac{1}{c_-^2}+\frac{1}{c_+^2})(K_B^2-K_A^2). \quad (43)$$

Hence one can consider the following quantity as a crack detector:

$$\omega \to Obs(\omega) = \frac{1}{2}|\int_{\Gamma_e\cup\Gamma_s}[-\omega^2\hat{u}^2 + c^2|\frac{\partial\hat{u}}{\partial x_2}|^2]\nu_1 - \hat{z}(\omega)\int_\Omega q(x)\frac{\partial\hat{u}}{\partial x_1}|^2.$$

The nonnegative quantity $Obs(\omega) = Obs(\omega, q)$ measures the difference between the input of a signal through Γ_e and the output through Γ_s, with the measure of the solution at the excitation. Let us notice that if it is different from zero, one can claim that there is a crack because necessarily $K_A^2 - K_B^2 \neq 0$ and thus K_A and K_B can't be both zero. In the contrary, if it is zero one has:

$$\forall\omega \in [\omega_1,\omega_2], K_A(\omega) = \pm K_B(\omega). \quad (44)$$

Let us now explain why this relation enables one to claim that there is no crack under suitable assumptions.

We will says that the function Obs is efficient for the detection of cracks if one can find a function q (independent of any crack) such that in presence of a crack, the function $Obs(\ .\ , q) > 0$ on an open interval in $\mathbb{R}$. We prove in this section that Obs is an efficient criterion of detection of small cracks and we explicit functions q that can be used for this purpose. We begin by a mathematical study of condition (44).

Lemma 4.1. *If there exists $q \in L^2(\Omega)$ such that $Obs(\omega, q) = 0$ for every ω in an open interval which contains a simple eigenvalue λ_{i_0} associated to the eigenvector w_{i_0}, then*

$$(q, w_{i_0})\ (S_A^* - S_B^*, w_{i_0})\ (S_A^* + S_B^*, w_{i_0}) = 0,$$

where $(\ ,\ )$ is the scalar product in $L^2(\Omega)$. □

Proof Let us consider the case where $(w_1 < w_2)$:

$$\forall \omega \in [\omega_1, \omega_2],\ K_A(\omega) = \pm K_B(\omega). \tag{45}$$

From the definitions of $K_A(\omega)$ and $K_B(\omega)$ (see Theorem 3.3), one has (assuming for sake of simplicity that the neighborhoods of A and B have the same radius):

$$\forall \omega \in [\omega_1, \omega_2],\ \int_\Omega [S_A^* \hat{z}(\omega) q + \omega^2 \hat{u} S_A^*] = \pm \hat{\int}_\Omega [S_B^* \hat{z}(\omega) q + \omega^2 \hat{u} S_B^*].$$

Let us introduce two new functions:

$$T_1 = S_A^* - S_B^* \ \text{ and } \ T_2 = S_A^* + S_B^*. \tag{46}$$

The relation between K_A and K_B leads to:

$$\forall \omega \in [\omega_1, \omega_2],\ \int_\Omega T_1(\omega^2 \hat{u} + \hat{z}(\omega) q) = 0 \text{ or } \int_\Omega T_2(\omega^2 \hat{u} + \hat{z}(\omega) q) = 0, \tag{47}$$

But, with Theorem 1.1, the solution $\hat{u}$ of (1.1) can be explicited in the eigenvectors basis $\{w_i\},\ i \geq 0$ defined at (5), by:

$$\hat{u}(x, \omega) = \hat{z}(\omega) \sum_{i \geq 0} \frac{(q, w_i)}{\lambda_i - \omega^2} w_i(x).$$

Introducing this expression in the previous relation, one obtains for $\omega \in [\omega_1, \omega_2]$ and $\hat{z}(\omega) \neq 0$:

$$\omega^2 [\sum_{i \geq 0} \frac{(q, w_i)(T_1, w_i)}{\lambda_i - \omega^2}] + \int_\Omega q T_1 = 0 \text{ or } \omega^2 [\sum_{i \geq 0} \frac{(q, w_i)(T_2, w_i)}{\lambda_i - \omega^2}] + \int_\Omega q T_2 = 0,$$

or else (writing $\int_\Omega q T_1 = \sum_i (q, w_i)(T_1, w_i)$):

$$\sum_{i \geq 0} [\frac{\lambda_i}{\lambda_i - \omega^2} \int_\Omega q w_i \int_\Omega T_1 w_i] = 0 \text{ or } \sum_{i \geq 0} [\frac{\lambda_i}{\lambda_i - \omega^2} \int_\Omega q w_i \int_\Omega T_2 w_i] = 0.$$

Let us introduce the function $h(\omega) = h(x, \omega)$ by:

$$h(x, \omega) = \chi_{[\omega_1, \omega_2]} \sum_{i \geq 0} \frac{\lambda_i (q, w_i)}{\lambda_i - \omega^2} w_i(x).$$

Therefore, the condition $K_A = \pm K_B$ implies:

$$\forall \omega \in [\omega_1, \omega_2],\ \int_\Omega h(x, \omega) T_1(x) dx = 0 \text{ or } \int_\Omega h(x, \omega) T_2(x) dx = 0. \tag{48}$$

Let $\delta_{i_0} > 0$ be

$$\delta_{i_0} = \min_{i \neq i_0} |\lambda_{i_0} - \lambda_i|.$$

and we observe that one can write:

$$h(x, \omega) = \frac{\lambda_{i_0}}{\lambda_{i_0} - \omega^2} w_{i_0}(x) + \sum_{i \neq i_0} \frac{\lambda_i (q, w_i)}{\lambda_i - \omega^2} w_i(x).$$

and therefore, setting: $\omega^2 = \lambda_{i_0}(1-\varepsilon)$ for ε small enough and choosing for instance the first relation (48):

$$\frac{(q, w_{i_0})}{\varepsilon}(w_{i_0}, T_1) + \sum_{i \neq i_0} \frac{\lambda_i (q, w_i)}{\lambda_i - \lambda_{i_0}(1-\omega^2)}(w_i, T_1) = 0.$$

Thus:

$$|(q, w_{i_0})(w_{i_0}, T_1)| \leq \varepsilon(1 + \frac{\lambda_{i_0}(1-\varepsilon)}{\delta_{i_0} + \lambda_{i_0}|\varepsilon|}) \sum_{i \neq i_0} |(q, w_i)| \ |(w_i, T_1)|.$$

Finally, from Schwarz inequality, one deduces that for every $\varepsilon > 0$ small enough:

$$|(q, w_{i_0})(w_{i_0}, T_1)| \leq \varepsilon(1 + \frac{\lambda_{i_0}}{\delta_{i_0}})||q||_{0,2,\Omega} \ \ ||T_1||_{0,2,\Omega},$$

or (in the case of T_2),

$$|(q, w_{i_0})(w_{i_0}, T_2)| \leq \varepsilon(1 + \frac{\lambda_{i_0}}{\delta_{i_0}})||q||_{0,2,\Omega} \ \ ||T_2||_{0,2,\Omega}.$$

Therefore :

$$(q, w_{i_0}) = 0 \ \ or \ \ (T_1, w_{i_0}) = 0 \ \ or \ \ (T_2, w_{i_0}) = 0, \tag{49}$$

and Lemma 4.1 is proved. □

The opposite of the first condition (48) traduces that the space excitation q shouldn't be orthogonal (in $L^2(\Omega)$) to the eigenvector w_{i_0} and the second relation (added to the one with T_2) is the contrary of the hearing capacity of the crack by the eigenvector w_{i_0}. Let us recall that the vectors w_i depends on the crack thus Lemma 4.1 is not efficient. But for a small crack (compared to the dimensions of the open set Ω), these eigenvectors are quite close to those without crack denoted by $\{w_i^0\}$. This is discussed in the next subsection. In order to explain how to use this property, let us assume that for instance that:

$$\int_\Omega q(x) w_{i_0}^0(x) dx \neq 0 \ \ and \ \ \int_\Omega T_1(x) w_{i_0}^0(x) dx \neq 0, \tag{50}$$

this will imply, for a small crack, that:

$$\int_\Omega q(x) w_{i_0}(x) dx \neq 0 \ \ and \ \ \int_\Omega T_1(x) w_{i_0}(x) dx \neq 0, \tag{51}$$

and therefore, from the condition (49), this implies that there is no crack. The advantage of condition (57) compared to (58) is that the eigenvectors $w_{i_0}^0$ don't depend on the crack length and furthermore they are known analytically.

A similar analysis can be performed for the proximity between the solution $\hat{u}$ with and without crack concerning the excitation term which appears in the criterion (the term is $-\hat{z}(\omega) \int_\Omega q \frac{\partial \hat{u}}{\partial x_1}$). We prove the following theorem that we will explicit and numerically illustrate in the next section.

Let us recall that the eigenfunction $(w_i^0)_{i \in \mathbb{N}}$ is a basis of $L^2(R)$ where R is the whole rectangle without crack. Let us introduce the following hypothesis :

$$\exists i \in \mathbb{N}, \ \ (S_A^*, w_i^0)^2 \neq (S_B^*, w_i^0)^2. \tag{52}$$

Theorem 4.2. *Assume that (52) holds. There exists $\alpha > 0$, there exists $q \in L^2(\Omega)$ such that the observability function $Obs(.\ , q)$ detects efficiently a crack among the cracks γ whose length $l(\gamma)$ satisfies $l(\gamma) \leq \alpha$.* □

Proof

Let us study the perturbation of the eigenvectors due to the crack γ. We denote by l the crack length $l(\gamma)$. Since the crack length varies in this proof, we denote by Ω^l and by an exponent l the object that depends on the crack. The eigenvalue $\lambda^l_{i_0}$ (as a function of l) for $i_0 \geq 1$, is characterized by (one can refer to [16] for details):

$$\lambda^l_{i_0} = \max_{\substack{E \subset H^1(\Omega^l), \\ codim\ (E) = i_0 - 1}} \min_{v \in E} \frac{\displaystyle\int_{\Omega^l} c^2 |\nabla v|^2}{\displaystyle\int_{\Omega^l} v^2}. \tag{53}$$

Because $H^1(\Omega) \subset H^1(\Omega^{l_1}) \subset H^1(\Omega^{l_2})$ for $l_1 \leq l_2$, one has:

$$\lambda^{l_2}_{i_0} \leq \lambda^{l_1}_{i_0} \leq \lambda^0_{i_0}. \tag{54}$$

Hence, the sequence $\lambda^l_{i_0}$ is increasing when l decreases. Furthermore, it is upper bounded and therefore it is convergent to a limit -say $\lambda^*_{i_0}$- such that $\lambda^*_{i_0} \leq \lambda^0_{i_0}$. The eigenvector $w^l_{i_0}$ is also a function of the crack length l and satisfies:

$$\begin{cases} ||w^l_{i_0}||_{0,\Omega^l} = 1, \\ \\ \forall v \in H^1(\Omega^l), \quad \lambda^l_{i_0} \displaystyle\int_{\Omega^l} w^l_{i_0} v = \int_{\Omega^l} c^2 \nabla w^l_{i_0} . \nabla v. \end{cases} \tag{55}$$

The sequence $w^l_{i_0}$ with respect to l, $\ l \leq l_0$, is bounded in the space $H^1(\Omega^{l_0})$ and converges weakly in $H^1(\Omega^{l_0})$ and strongly in $L^2(\Omega^{l_0})$ to a vector $w^*_{i_0} \in H^1(\Omega^{l_0})$. From the definition of w_{i_0}, one can state that $w^*_{i_0}$ is continuous across the line supporting the crack. Thus $w^*_{i_0} \in H^1(\Omega)$. This enables one to claim that $\lambda^*_{i_0} = \lambda^0_{i_0}$ and $w^*_{i_0} = w^0_{i_0}$. Hence for l small enough, one has:

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ such\ that\ for\ 0 \leq l \leq \delta,\ ||w^l_{i_0} - w^0_{i_0}||_{0,\Omega^l} \leq \varepsilon. \tag{56}$$

Let us choose an eigenfunction w_{i_0} which satisfies (52). We get

$$(T_1, w^0_{i_0})(T_2, w^0_{i_0}) = (S^*_A + S^*_B, w_{i_0})(S^*_A - S^*_B, w_{i_0}) = (S^*_A, w_{i_0})^2 - (S^*_B, w_{i_0})^2.$$

Considering a function q with

$$(q, w^0_{i_0})(T_1, w^0_{i_0})(T_2, w^0_{i_0}) \neq 0, \tag{57}$$

one gets for a small crack, that:

$$(q, w^l_{i_0})(T_1, w^l_{i_0})(T_2, w^l_{i_0}) \neq 0. \tag{58}$$

We get (from Lemma 4.1) that $Obs(w, q)$ can't be identically zero on an open interval containing λ^0_i and thus Theorem 4.2 is proved. □.

Remark 6. Assertion (52) is still an open problem. In the case where $c_+ = c_-$ and in the very special case where the crack is centered in the middle the interface, the assertion is not satisfied because $S^*_A + S^*_B$ is an even function with respect to x_1 and $S^*_A - S^*_B$ is an odd one, hence one has always in this very particular case

$$\forall i, \quad (T_1, w^l_i)\ (T_2, w^l_i) = 0.$$

But, this is a very particular case and there is no (good) reason for the crack to be perfectly centered in the middle interface. Furthermore, on one hand, if $c_+ = c_-$, there is no Love wave which are the waves used for numerical simulations hereafter and on the other hand,

if $c_+ \neq c_-$, the argument (odd and even function) is no more valid. We have some reasons to think that assertion (52) is generically true and it will be discuss and improve in future works. □

In the next section, we illustrate Theorem 4.2 with numerical examples. This numerical work suggests that hypothesis (52) seems to be valid with the choice of an interval for ω containing Love waves.

5. Few graphics representating the observability.

We begin this section by the computation of the eigenvectors without crack, and mainly with the Love waves.

5.1. Computation of the Love waves without crack.

The open set Ω is the whole rectangle $\Omega = R$. From a classical computation, one can derive the analytical expressions of the eigenvectors on the open set Ω. Due to the invariance of Ω with respect to the coordinate x_1, it is possible to look for solutions of (5) by setting: (L and H are the dimensions of the open set Ω):

$$w^0_{i_0}(x_1, x_2) = \sqrt{\frac{2}{L}} \cos(\frac{n\pi x_1}{L}) p_n(x_2), \tag{59}$$

with $n \in \mathbb{N}$. The functions p_n are therefore solutions of:

$$\begin{cases} \dfrac{d}{dx_2}(c^2 \dfrac{dp_n}{dx}) + (\lambda - \dfrac{n^2\pi^2 c^2}{L^2}) p_n = 0, \ 0 < x_2 < H, \\ \dfrac{dp_n}{dx}(0) = \dfrac{dp_n}{dx}(H) = 0, \\ \displaystyle\int_0^H p_n(x_2)^2 dx_2 = 1. \end{cases}$$

There are two families of solutions as far as $c_- < c_+$.

• *The interior waves* : they correspond to $\lambda > \dfrac{n^2\pi^2 c_+^2}{L^2}$. They are real trigonometric functions in the whole domain Ω (but with a derivative discontinuity at the interface between the two media along Γ_i). For each value of n, there is an infinite -but countable- number of solutions.

• *The Love waves* which concern our study : they correspond to

$$\frac{n^2\pi^2 c_-^2}{L^2} < \lambda < \frac{n^2\pi^2 c_+^2}{L^2}.$$

We prove hereafter that the eigenvectors are exponentially decreasing in Ω_+ and real trigonometric functions in Ω_- and thus they propagate in the softest media. Here again there are discontinuities on the normal derivatives across Γ_i. In this case, the number of solutions is finite for each value of n but this number is increasing with n.

The first step of our analysis is to characterize some of these localized waves. They can be computed for the structure without crack as follows. Let us recall that we set

$$w(x_1, x_2) = \cos(\frac{n\pi x_1}{L}) p_n(x_2), \tag{60}$$

where q_n is solution of (h is the ordinate of the boundary Γ_i):

$$\begin{cases} [\lambda - (\frac{cn\pi}{L})^2]p_n + \frac{d}{dx_2}(c^2\frac{dp_n}{dx_2}) = 0, \\ \\ \frac{dp_n}{dx_2}(0) = \frac{dp_n}{dx_2}(H) = 0, p_n(h^-) = p_n(h^+). \end{cases} \tag{61}$$

Setting for convenience:

$$\xi = \frac{\lambda L}{n\pi}, \quad K_+ = \frac{n\pi}{L}\sqrt{1 - (\frac{\xi}{c_+})^2}, \quad K_- = \frac{n\pi}{L}\sqrt{(\frac{\xi}{c_-})^2 - 1}, \tag{62}$$

for each value of n, the Love waves correspond to the solutions in $\xi \in]c_-, c_+[$ (see above) of the following equation:

$$(\frac{c_+}{c_-})\sqrt{\frac{c_+^2 - \xi^2}{\xi^2 - c_-^2}}\tanh(\frac{(H-h)n\pi}{Lc_+}\sqrt{c_+^2 - \xi^2}) = \tan(\frac{hn\pi}{Lc_-}\sqrt{\xi^2 - c_-^2}), \tag{63}$$

and the corresponding eigenvectors associated to p_n up to a multiplicative constant, are (the values of K_- and K_+ are given at (62)):

$$p_n(x_2) = \begin{cases} \frac{\cos(x_2 K_-)}{\cos(hK_-)} & \text{if } 0 \le x_2 \le h, \\ \\ \frac{\cosh((x_2 - H)K_+)}{\cosh((h-H)K_+)} & \text{if } h \le x_2 \le H. \end{cases} \tag{64}$$

We have solved the equation (63) for several values of n. A graphic representation of this equation is plotted on figure 6 ($n = 15$, $L = H = 1, h = .2$ and $1 = c_+ = 2c_-$).

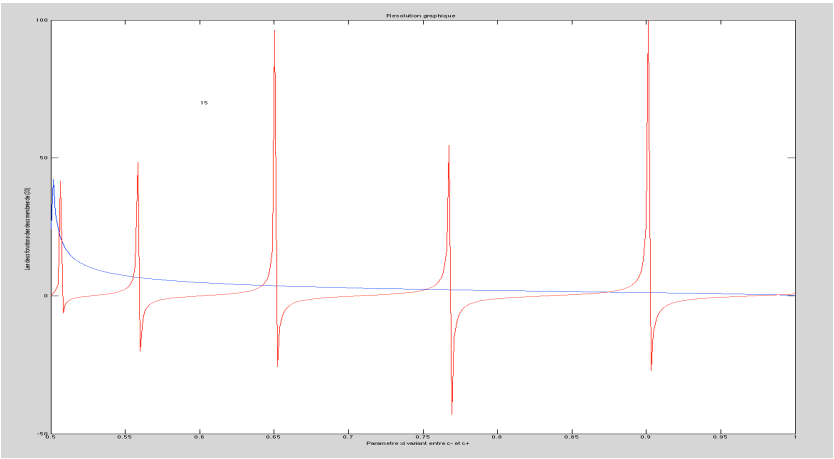

FIGURE 6. Graphic solution of (64) for $n = 15$

The solutions in ξ are for instance in this case:

For $n = 1$ there is one solution in ξ: .8657,
For $n = 2$ there is one solution in ξ: .7034,

For $n = 3$ there is one solution in ξ: .6152,
For $n = 4$ there are two solutions in ξ: .5731 .9238,
For $n = 5$ there are two solutions in ξ: .5502 .8311,
For $n = 6$ there are three solutions in ξ: .5361 .7615 .991,
For $n = 7$ there are three solutions in ξ: .5271 .7084 .9349,
For $n = 8$ there are three solutions in ξ: .521 .6673 .8768,
For $n = 9$ there are four solutions in ξ: .517 .6383 .8196 .992,
For $n = 10$ there are four solutions in ξ: .5134 .6154 .7742 .9465,
For $n = 11$ there are four solutions in ξ: .5110 .5972 .7395 .8928,
For $n = 12$ there are five solutions in ξ: .5106 .5832 .7064 .8507 .986,
For $n = 13$ there are five solutions in ξ: .5102 .5721 .6324 .8156 .9519,
For $n = 14$ there are five solutions in ξ: .5090 .5631 .6603 7815 9108,
For $n = 15$ there are five solutions in ξ: .5569 .6438 .7575 .8829 .9448,
For $n = 16$ there are six solutions in ξ: .5046 .5401 .6273 .7285 .8407 .9539,
For $n = 17$ there are six solutions in ξ: .504 .5441 .6152 .7074 .8116 .9188,
For $n = 18$ there are six solutions in ξ: .5032 .5391 .6042 .6834 .7866 .8888,
For $n = 19$ there are seven solutions in ξ: .5032 .5361 .5942 .6723 .7625 .8597 .9549,
For $n = 20$ there are seven solutions in ξ: .5031 .5321 .5862 .6573 .7425 .8337 .9279.

For $c_+ = 3200m/s$ and $c_- = 1600m/s$, there are ultrasonic waves as far as the frequencies are larger than 20 kHz. For instance, for $n = 10$ the frequency range of Love waves is between $\nu \simeq 30\ kHz$ and $\nu \simeq 60\ kHz$ and for $n = 20$ the frequency range is between $\nu = 60\ kHz$ and $\nu = 120\ kHz$.

We now turn to the numerical illustration of hypothesis (52).

5.2. **Visualization of the sensitivity factors for heterogeneous materials.** There are several parameters which can be discussed in the analysis of the observability of a crack. Let us enumerate them (see Figure 7):

1. a the position of point A on the crack line;
2. b the position of point B on the crack line;
3. h the position of the line supporting the crack measured from the lower boundary of the open set Ω;

Let us define

$$O_1(a,b,\alpha) = \int_\Omega w_{i_0}(x)T_1(x)dx, \quad O_2(a,b,\alpha) = \int_\Omega w_{i_0}(x)T_2(x)dx. \tag{65}$$

We ave represented on figures 8, 9 and 10, the two quantities O_1 and O_2 for different values of the frequencies $\nu_{n,m} = \dfrac{c\pi}{L}\sqrt{(\dfrac{n}{L})^2 + (\dfrac{m}{H})^2}$ parametrized by (n,m) which refer to the eigenmodes and for of $\alpha = 0$.

5.2.1. *Visualization of the sensitivity factors for homogeneous materials.* In a first step, the results are plotted for $c_+ = c_- = 1$. None of the terms computed are zero. It means that all the eigenmodes are able to detect the cracks, but the sensitivity is much larger for lower frequencies (c is about $3200m/s$ for steel and therefore the range of frequencies used is approximately between 3 kHz and 60 kHz). But w_{i_0} is replaced by the approximation w^0_{i0} and T_1 (respectively T_2) by the numerical approximation of $S^*_A + S^*_B$ (respectively $S^*_A - S^*_B$) using a finite element method using $Q1$-element (120×120). It clearly appears that the invisible area for the crack for the criterion which has been suggested, depends

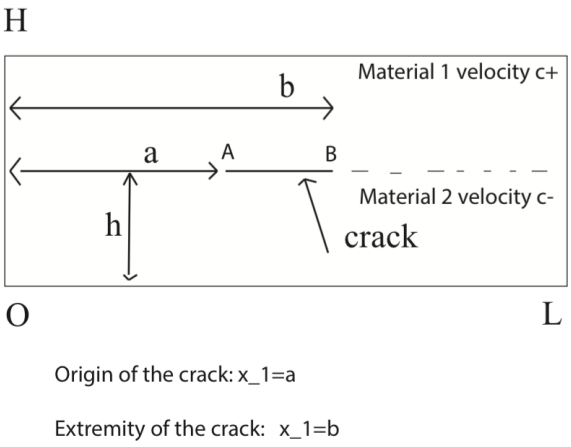

FIGURE 7. The parameters used in the discussion of the observability

on the frequencies used. When the frequency rises up, the impact of the singularities (due to the crack tips) is decreasing in this criterion. Nevertheless, the length of the crack tip should be small enough in order to justify the approximation.

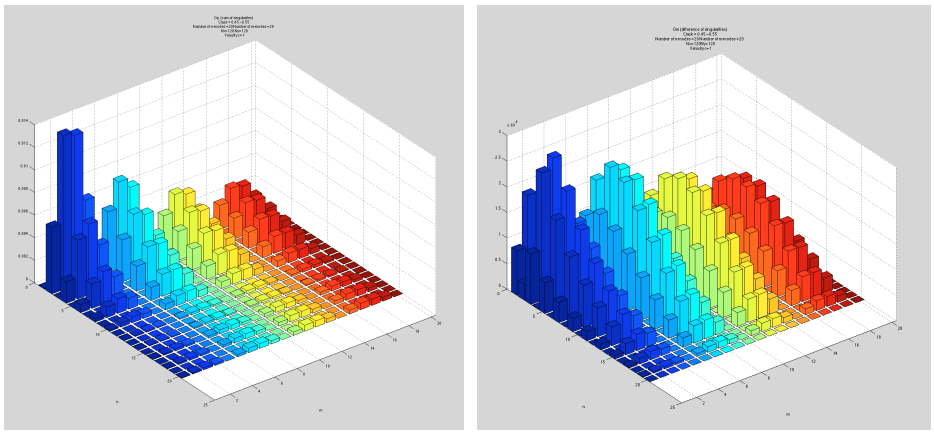

FIGURE 8. Homogeneous case. Long centered crack for which the computation are not very precise as far as we used the eigenvectors obtained without crack. One can see on this example that the smallest values of n and m are more efficient.

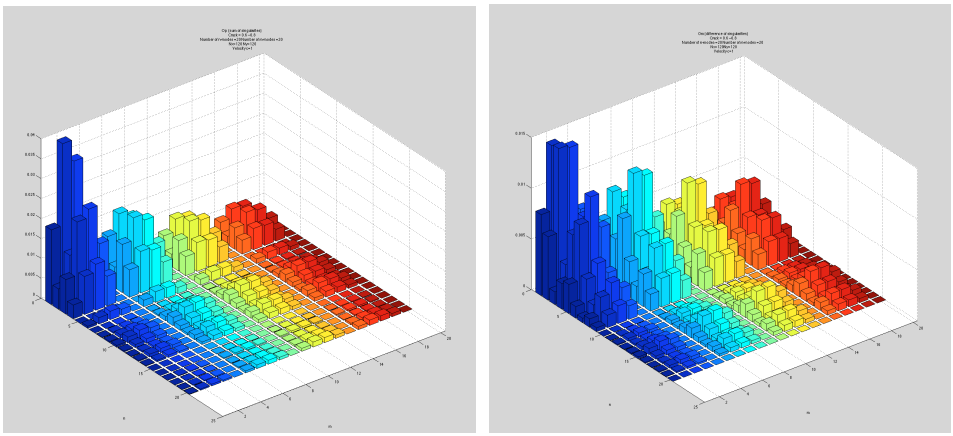

FIGURE 9. Homogeneous case. Small crack slightly moved to the right

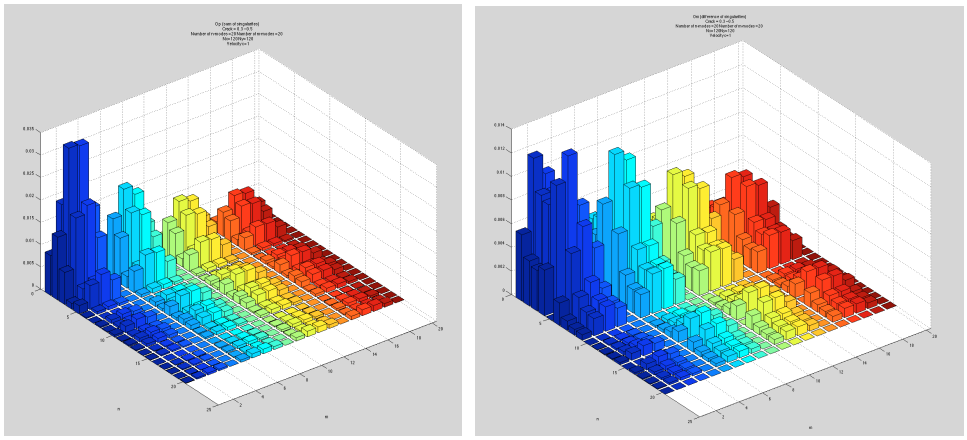

FIGURE 10. Homogeneous case. Small crack moved to the left

5.2.2. *Visualization of the sensitivity factors for bimaterials.* The terms O_1 (left) and O_2 (right) have been plotted on figure 11 and 12 for the same crack as on figure 9 and 10. The velocity are $c_+ = 2$ and $c_- = 0.5$. It appears that the sensitivity is much better than in the homogeneous case, due to the Love waves. It appears on this example that there are a lot of Love waves which can be used for detecting cracks (O_1 and O_2 shouldn't be zero for the same wave). But the sensitivity is clearly a function of the crack length and in this computation (100 elements are used) is not meaningful for very small cracks for instance ten times smaller that the one on figure 12).

6. **Conclusion.** Making use of an energetically invariant in the wave equation, it has been proved that the existence of a crack in a domain is strongly connected to quantities which can be estimated on a part of the boundary of the domain. The basic point has been to prove that for a given frequency windowing, the two crack tips don't send the same signal so that one can detect the presence of the crack. Furthermore, an inverse problem can be formulated in order to try to localize precisely the position of the crack inside the domain.

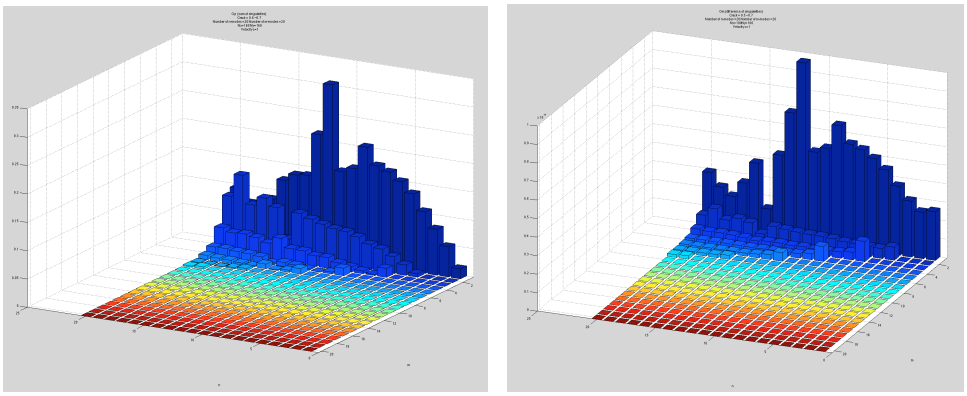

FIGURE 11. Heterogeneous case. The two sensitivity functions O_1 and O_2 for a bimaterial. Only the Love waves (eigenmodes) have been considered. The crack is set between $x_1 = .5$ and $x_1 = .7$ There is no internal waves used.

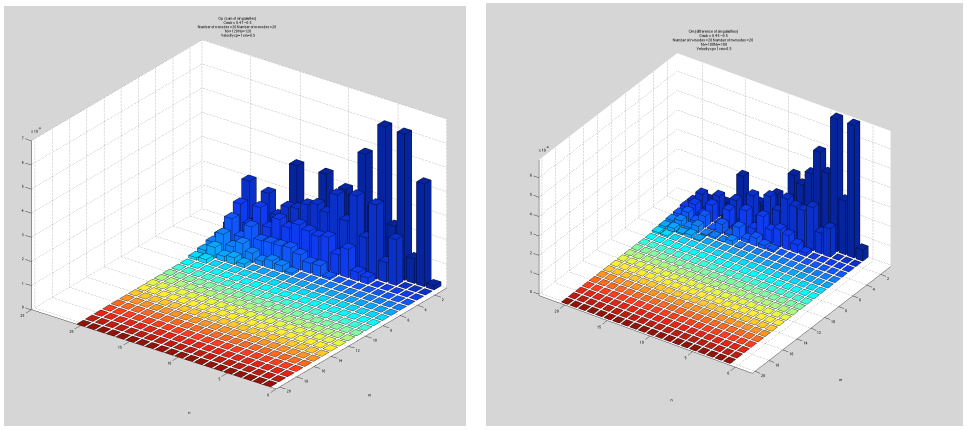

FIGURE 12. Heterogeneous case. The two sensitivity functions O_1 and O_2 for a bimaterial. Only the Love waves are considered. The crack is smaller and set between $x_1 = .47$ and $x_1 = .5$ There is no internal waves used.

The shape of the domain used (a rectangle) is a basic point as far as the invariant used implies the measure of mechanical quantities on the boundary which are not parallel to the coordinate x_1.

In the case of a bimaterial, as far as localized waves as Love waves can be used, the evaluation of the boundary terms for detecting the crack is easier to estimate because such waves are mainly localized in the softest part of the material. Furthermore, Love waves are well known for their properties to travel far from their origin and therefore, enable one to prospect a large range of the domain which is investigated for crack detection. Let us also point out that, in most cases, the crack if there is one, is at the interface between the two materials.

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