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A new bijection relating q -Eulerian polynomials

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Abstract

On the set of permutations of a finite set, we construct a bijection which maps the 3-vector of statistics $(\text{maj} - \text{exc}, \text{des}, \text{exc})$ to a 3-vector $(\text{maj}_2, \widetilde{\text{des}}_2, \text{inv}_2)$ associated with the q -Eulerian polynomials introduced by Shareshian and Wachs in *Chromatic quasisymmetric functions*, *arXiv:1405.4269(2014)*.

Keywords: q -Eulerian polynomials, descents, ascents, major index, exceedances, inversions.

Notations

For all pair of integers (n, m) such that $n < m$, the set $\{n, n + 1, \dots, m\}$ is indifferently denoted by $[n, m]$, $]n - 1, m]$, $[n, m + 1[$ or $]n - 1, m + 1[$.

The set of positive integers $\{1, 2, 3, \dots\}$ is denoted by $\mathbb{N}_{>0}$.

5 For all integer $n \in \mathbb{N}_{>0}$, we denote by $[n]$ the set $[1, n]$ and by \mathfrak{S}_n the set of the permutations of $[n]$. By abuse of notation, we assimilate every $\sigma \in \mathfrak{S}_n$ with the word $\sigma(1)\sigma(2)\dots\sigma(n)$.

If a set $S = \{n_1, n_2, \dots, n_k\}$ of integers is such that $n_1 < n_2 < \dots < n_k$, we sometimes use the notation $S = \{n_1 < n_2 < \dots < n_k\}$.

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10 **1. Introduction**

Let n be a positive integer and $\sigma \in \mathfrak{S}_n$. A *descent* (respectively *exceedance point*) of σ is an integer $i \in [n - 1]$ such that $\sigma(i) > \sigma(i + 1)$ (resp. $\sigma(i) > i$). The set of descents (resp. exceedance points) of σ is denoted by $\text{DES}(\sigma)$ (resp. $\text{EXC}(\sigma)$) and its cardinal by $\text{des}(\sigma)$ (resp. $\text{exc}(\sigma)$). The integers $\sigma(i)$ with
 15 $i \in \text{EXC}(\sigma)$ are called exceedance values of σ .

It is due to MacMahon [Mac15] and Riordan [Rio58] that

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)} = A_n(t)$$

where $A_n(t)$ is the n -th Eulerian polynomial [Eul55]. A statistic equidistributed with des or exc is said to be *Eulerian*. The statistic idcs defined by $\text{idcs}(\sigma) = \text{des}(\sigma^{-1})$ obviously is Eulerian.

The *major index* of a permutation $\sigma \in \mathfrak{S}_n$ is defined as

$$\text{maj}(\sigma) = \sum_{i \in \text{DES}(\sigma)} i.$$

It is also due to MacMahon that

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = \prod_{i=1}^n \frac{1 - q^i}{1 - q}.$$

A statistic equidistributed with maj is said to be *Mahonian*. Among Mahonian
 20 statistics is the statistic inv , defined by $\text{inv}(\sigma) = |\text{INV}(\sigma)|$ where $\text{INV}(\sigma)$ is the set of *inversions* of a permutation $\sigma \in \mathfrak{S}_n$, *i.e.* the pairs of integers $(i, j) \in [n]^2$ such that $i < j$ and $\sigma(i) > \sigma(j)$.

In [SW14], the authors consider analogous versions of the above statistics : let $\sigma \in \mathfrak{S}_n$, the set of *2-descents* (respectively *2-inversions*) of σ is defined as

$$\text{DES}_2(\sigma) = \{i \in [n - 1], \sigma(i) > \sigma(i + 1) + 1\}$$

(resp.

$$\text{INV}_2(\sigma) = \{1 \leq i < j \leq n, \sigma(i) = \sigma(j) + 1\}$$

and its cardinal is denoted by $\text{des}_2(\sigma)$ (resp. $\text{inv}_2(\sigma)$).

It is easy to see that $\text{inv}_2(\sigma) = \text{idcs}(\sigma)$. The *2-major index* of σ is defined as

$$\text{maj}_2(\sigma) = \sum_{i \in \text{DES}_2(\sigma)} i.$$

By using quasisymmetric function techniques, the authors of [SW14] proved the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\text{maj}_2(\sigma)} y^{\text{inv}_2(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{maj}(\sigma) - \text{exc}(\sigma)} y^{\text{exc}(\sigma)}. \quad (1)$$

Similarly, by using the same quasisymmetric function method as in [SW14], the authors of [HL12] proved the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\text{amaj}_2(\sigma)} y^{\widetilde{\text{asc}}_2(\sigma)} z^{\text{idcs}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{maj}(\sigma) - \text{exc}(\sigma)} y^{\text{des}(\sigma)} z^{\text{exc}(\sigma)} \quad (2)$$

where $\text{asc}_2(\sigma)$ is the number of *2-ascents* of a permutation $\sigma \in \mathfrak{S}_n$, i.e. the elements of the set $\text{ASC}_2(\sigma) = \{i \in [n-1], \sigma(i) < \sigma(i+1) + 1\}$, which rises the statistic amaj_2 defined by

$$\text{amaj}_2(\sigma) = \sum_{i \in \text{ASC}_2(\sigma)} i,$$

and where

$$\widetilde{\text{asc}}_2(\sigma) = \begin{cases} \text{asc}_2(\sigma) & \text{if } \sigma(1) = 1, \\ \text{asc}_2(\sigma) + 1 & \text{if } \sigma(1) \neq 1. \end{cases}$$

Definition 1.1. Let $\sigma \in \mathfrak{S}_n$. We consider the smallest 2-descent $d_2(\sigma)$ of σ such that $\sigma(i) = i$ for all $i \in [d_2(\sigma) - 1]$ (if there is no such 2-descent, we define $d_2(\sigma)$ as 0 and $\sigma(0)$ as $n+1$).

Now, let $d'_2(\sigma) > d_2(\sigma)$ be the smallest 2-descent of σ greater than $d_2(\sigma)$ (if there is no such 2-descent, we define $d'_2(\sigma)$ as n).

We define an inductive property $\mathcal{P}(d_2(\sigma))$ by :

1. $\sigma(d_2(\sigma)) < \sigma(i)$ for all $(i, j) \in \text{INV}_2(\sigma)$ such that $d_2(\sigma) < i < d'_2(\sigma)$;
2. if $(d'_2(\sigma), j) \in \text{INV}_2(\sigma)$ for some j , then either $\sigma(d_2(\sigma)) < \sigma(d'_2(\sigma))$, or $d'_2(\sigma)$ has the property $\mathcal{P}(d'_2(\sigma))$ (where the role of $d_2(\sigma)$ is played by $d'_2(\sigma)$ and that of $d'_2(\sigma)$ by $d''_2(\sigma)$ where $d''_2(\sigma) > d'_2(\sigma)$ is the smallest 2-descent of σ greater than $d'_2(\sigma)$, defined as n if there is no such 2-descent).

35 This property is well-defined because $(n, j) \notin \text{INV}_2(\sigma)$ for all $j \in [n]$.

Finally, we define a statistic $\widetilde{\text{des}}_2$ by :

$$\widetilde{\text{des}}_2(\sigma) = \begin{cases} \text{des}_2(\sigma) & \text{if the property } \mathcal{P}(d_2(\sigma)) \text{ is true,} \\ \text{des}_2(\sigma) + 1 & \text{otherwise.} \end{cases}$$

In the present paper, we prove the following theorem.

Theorem 1.2. *There exists a bijection $\varphi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ such that*

$$(\text{maj}_2(\sigma), \widetilde{\text{des}}_2(\sigma), \text{inv}_2(\sigma)) = (\text{maj}(\varphi(\sigma)) - \text{exc}(\varphi(\sigma)), \text{des}(\varphi(\sigma)), \text{exc}(\varphi(\sigma))).$$

As a straight corollary of Theorem 1.2, we obtain the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\text{maj}_2(\sigma)} y^{\widetilde{\text{des}}_2(\sigma)} z^{\text{inv}_2(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{maj}(\sigma) - \text{exc}(\sigma)} y^{\text{des}(\sigma)} z^{\text{exc}(\sigma)} \quad (3)$$

which implies Equality (1).

The rest of this paper is organised as follows.

40 In Section 2, we introduce two graphical representations of a given permutation so as to highlight either the statistic $(\text{maj} - \text{exc}, \text{des}, \text{exc})$ or $(\text{maj}_2, \widetilde{\text{des}}_2, \text{inv}_2)$. Practically speaking, the bijection φ of Theorem 1.2 will be defined by constructing one of the two graphical representations of $\varphi(\sigma)$ for a given permutation $\sigma \in \mathfrak{S}_n$.

45 We define φ in Section 3.

In Section 4, we prove that φ is bijective by constructing φ^{-1} .

2. Graphical representations

2.1. Linear graph

Let $\sigma \in \mathfrak{S}_n$. The linear graph of σ is a graph whose vertices are (from left to
50 right) the integers $\sigma(1), \sigma(2), \dots, \sigma(n)$ aligned in a row, where every $\sigma(k)$ (for $k \in \text{DES}_2(\sigma)$) is boxed, and where an arc of circle is drawn from $\sigma(i)$ to $\sigma(j)$ for every $(i, j) \in \text{INV}_2(\sigma)$.

For example, the permutation $\sigma = 34251 \in \mathfrak{S}_5$ (such that
($\text{maj}_2(\sigma), \widetilde{\text{des}}_2(\sigma), \text{inv}_2(\sigma)$) = $(6, 3, 2)$) has the linear graph depicted in Fig-
55 ure 1.

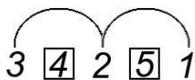


Figure 1: Linear graph of $\sigma = 34251 \in \mathfrak{S}_5$.

2.2. Planar graph

Let $\tau \in \mathfrak{S}_n$. The planar graph of τ is a graph whose vertices are the integers $1, 2, \dots, n$, organized in ascending and descending slopes (the height of each vertex doesn't matter) such that the i -th vertex (from left to right) is the i -th integer $\tau(i)$, and where every vertex $\tau(i)$ with $i \in \text{EXC}(\tau)$ is encircled.

For example, the permutation $\tau = 32541 \in \mathfrak{S}_5$ (such that $(\text{maj}(\tau) - \text{exc}(\tau), \text{des}(\tau), \text{exc}(\tau)) = (6, 3, 2)$) has the planar graph depicted in Figure 2.

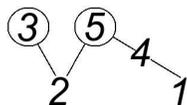


Figure 2: Planar graph of $\tau = 32541 \in \mathfrak{S}_5$.

3. Definition of the map φ of Theorem 1.2

Let $\sigma \in \mathfrak{S}_n$. We set $(r, s) = (\text{des}_2(\sigma), \text{inv}_2(\sigma))$, and

$$\text{DES}_2(\sigma) = \{d_2^k(\sigma), k \in [r]\},$$

$$\text{INV}_2(\sigma) = \{(i_l(\sigma), j_l(\sigma)), l \in [s]\}$$

with $d_2^k(\sigma) < d_2^{k+1}(\sigma)$ for all k and $i_l(\sigma) < i_{l+1}(\sigma)$ for all l .

We intend to define $\varphi(\sigma)$ by constructing its planar graph. To do so, we first construct (in Subsection 3.1) a graph $\mathcal{G}(\sigma)$ made of n circles or dots organized in ascending or descending slopes such that two consecutive vertices are necessarily in a same descending slope if the first vertex is a circle and the second vertex is a dot. Then, in Subsection 3.2, we label the vertices of this graph with the

integers $1, 2, \dots, n$ in such a way that, if y_i is the label of the i -th vertex $v_i(\sigma)$ (from left to right) of $\mathcal{G}(\sigma)$ for all $i \in [n]$, then :

1. $y_i < y_{i+1}$ if and only if v_i and v_{i+1} are in a same ascending slope;
2. $y_i > i$ if and only if v_i is a circle.

75 The permutation $\tau = \varphi(\sigma)$ will then be defined as $y_1 y_2 \dots y_n$, *i.e.* the permutation whose planar graph is the labelled graph $\mathcal{G}(\sigma)$.

With precision, we will obtain

$$\tau(\text{EXC}(\tau)) = \{j_k(\sigma), k \in [s]\}$$

(in particular $\text{exc}(\tau) = s = \text{inv}_2(\sigma)$), and

$$\text{DES}(\tau) = \begin{cases} \{d^k(\sigma), k \in [1, r]\} & \text{if } \widetilde{\text{des}}_2(\sigma) = r, \\ \{d^k(\sigma), k \in [0, r]\} & \text{if } \widetilde{\text{des}}_2(\sigma) = r + 1 \end{cases}$$

for integers $0 \leq d^0(\sigma) < d^1(\sigma) < \dots < d^r(\sigma) \leq n$ (with $d^0(\sigma) = 0 \Leftrightarrow \widetilde{\text{des}}_2(\sigma) = \text{des}_2(\sigma)$) defined by

$$d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma)$$

(with $d_2^0(\sigma) := 0$) where $(c_k(\sigma))_{k \in [0, r]}$ is a sequence defined in Subsection 3.1 such that $\sum_k c_k(\sigma) = \text{inv}_2(\sigma) = \text{exc}(\tau)$. Thus, we will obtain $\text{des}(\tau) = \widetilde{\text{des}}_2(\sigma)$ and $\text{maj}(\tau) = \text{maj}_2(\sigma) + \text{exc}(\tau)$.

80 *3.1. Construction of the unlabelled graph $\mathcal{G}(\sigma)$*

We set $(d_2^0(\sigma), \sigma(d_2^0(\sigma))) = (0, n + 1)$ and $(d_2^{r+1}(\sigma), \sigma(n + 1)) = (n, 0)$.

For all $k \in [r]$, we define the top $t_k(\sigma)$ of the 2-descent $d_2^k(\sigma)$ as

$$t_k(\sigma) = \min\{d_2^l(\sigma), 1 \leq l \leq k, d_2^l(\sigma) = d_2^k(\sigma) - (k - l)\}, \quad (4)$$

in other words $t_k(\sigma)$ is the smallest 2-descent $d_2^l(\sigma)$ such that the 2-descents $d_2^l(\sigma), d_2^{l+1}(\sigma), \dots, d_2^k(\sigma)$ are consecutive integers.

85 The following algorithm provides a sequence $(c_k^0(\sigma))_{k \in [0, r]}$ of nonnegative integers.

Algorithm 3.1. Let $I_r(\sigma) = \text{INV}_2(\sigma)$. For k from $r = \text{des}_2(\sigma)$ down to 0, we consider the set $S_k(\sigma)$ of sequences $(i_{k_1}(\sigma), i_{k_2}(\sigma), \dots, i_{k_m}(\sigma))$ such that :

1. $(i_{k_p}(\sigma), j_{k_p}(\sigma)) \in I_k(\sigma)$ for all $p \in [m]$;
- 90 2. $t_k(\sigma) \leq i_{k_1}(\sigma) < i_{k_2}(\sigma) < \dots < i_{k_m}(\sigma)$;
3. $\sigma(i_{k_1}(\sigma)) < \sigma(i_{k_2}(\sigma)) < \dots < \sigma(i_{k_m}(\sigma))$.

The *length* of such a sequence is defined as $l = \sum_{p=1}^m n_p$ where n_p is the number of *consecutive* 2-inversions whose beginning is i_{k_p} , *i.e.* the maximal number n_p of 2-inversions $(i_{k_p^1}(\sigma), j_{k_p^1}(\sigma)), (i_{k_p^2}(\sigma), j_{k_p^2}(\sigma)), \dots, (i_{k_p^{n_p}}(\sigma), j_{k_p^{n_p}}(\sigma))$ such that
95 $k_p^1 = k_p$ and $j_{k_p^i}(\sigma) = i_{k_p^{i+1}}(\sigma)$ for all i . If $I_k(\sigma) \neq \emptyset$, we consider the sequence $(i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \dots, i_{k_m^{max}}(\sigma)) \in I_k(\sigma)$ whose length $l^{max} = \sum_{p=1}^m n_p^{max}$ is maximal and whose elements $i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \dots, i_{k_m^{max}}(\sigma)$ also are maximal (as integers). Then,

- if $I_k(\sigma) \neq \emptyset$, we set $c_k^0(\sigma) = l^{max}$ and

$$I_{k-1}(\sigma) = I_k(\sigma) \setminus (\cup_{p=1}^m \{(i_{k_i^{max}}(\sigma), j_{k_i^{max}}(\sigma)), i \in [n_p^{max}]\});$$

- else we set $c_k^0(\sigma) = 0$ and $I_{k-1}(\sigma) = I_k(\sigma)$.

100 **Example 3.2.** Consider the permutation $\sigma = 549321867 \in \mathfrak{S}_9$, with $\text{DES}_2(\sigma) = \{3, 7\}$ and $I_2(\sigma) = \text{INV}_2(\sigma) = \{(1, 2), (2, 4), (3, 7), (4, 5), (5, 6), (7, 9)\}$. In Figure 3 are depicted the $\text{des}_2(\sigma) + 1 = 3$ steps $k \in \{2, 1, 0\}$ (at each step, the 2-inversions of the maximal sequence are drawn in red then erased at the following step) :

$k=2$	5 4 9 3 2 1 8 6 7	$c_2^0=1$
$k=1$	5 4 9 3 2 1 8 6 7	$c_1^0=2$
$k=0$	5 4 9 3 2 1 8 6 7	$c_0^0=3$

Figure 3: Computation of $(c_k^0(\sigma))_{k \in [0, \text{des}_2(\sigma)]}$ for $\sigma = 549321867 \in \mathfrak{S}_9$.

- 105 • $k = 2$: there is only one legit sequence $(i_{k_1}(\sigma)) = (7)$, whose length is $l = n_1 = 1$. We set $c_2^0(\sigma) = 1$ and $I_1(\sigma) = I_2(\sigma) \setminus \{(7, 9)\}$.
- $k = 1$: there are three legit sequences $(i_{k_1}(\sigma)) = (3)$ (whose length is $l = n_1 = 1$) then $(i_{k_1}(\sigma)) = (4)$ (whose length is $l = n_1 = 2$) and $(i_{k_1}(\sigma)) = (5)$ (whose length is $l = n_1 = 1$). The maximal sequence is the
110 second one, hence we set $c_1^0(\sigma) = 2$ and $I_0(\sigma) = I_1(\sigma) \setminus \{(4, 5), (5, 6)\}$.
- $k = 0$: there are three legit sequences $(i_{k_1}(\sigma), i_{k_2}(\sigma)) = (1, 3)$ (whose length is $l = n_1 + n_2 = 2 + 1 = 3$) then $(i_{k_1}(\sigma), i_{k_2}(\sigma)) = (2, 3)$ (whose length is $l = n_1 + n_2 = 1 + 1 = 2$) and $(i_{k_1}(\sigma)) = (3)$ (whose length is $l = n_1 = 1$). The maximal sequence is the first one, hence we set $c_0^0(\sigma) = 3$
115 and $I_{-1}(\sigma) = I_0(\sigma) \setminus \{(1, 2), (2, 4), (3, 7)\} = \emptyset$.

Lemma 3.3. *The sum $\sum_k c_k^0(\sigma)$ equals $inv_2(\sigma)$ (i.e. $I_{-1}(\sigma) = \emptyset$) and, for all $k \in [0, r] = [0, des_2(\sigma)]$, we have $c_k^0(\sigma) \leq d_2^{k+1}(\sigma) - d_2^k(\sigma)$ with equality only if $c_{k+1}^0(\sigma) > 0$ (where $c_{r+1}^0(\sigma)$ is defined as 0).*

Proof. With precision, we show by induction that, for all $k \in \{des_2(\sigma), \dots, 1, 0\}$,
120 the set $I_{k-1}(\sigma)$ contains no 2-inversion (i, j) such that $d_2^k(\sigma) < i$. For $k = 0$, it will mean $I_{-1}(\sigma) = \emptyset$ (recall that $d_2^0(\sigma)$ has been defined as 0).

★ If $k = des_2(\sigma) = r$, the goal is to prove that $c_r^0(\sigma) < n - d_2^r(\sigma)$. Suppose there exists a sequence $(i_{k_1}(\sigma), i_{k_2}(\sigma), \dots, i_{k_m}(\sigma))$ of length $c_r^0(\sigma) \geq n - d_2^r(\sigma)$ with $t_r(\sigma) \leq i_{k_1}(\sigma) < i_{k_2}(\sigma) < \dots < i_{k_m}(\sigma)$. In particular, there exist
125 $c_r^0(\sigma) \geq n - d_2^r(\sigma)$ 2-inversions (i, j) such that $d_2^r(\sigma) < j$, which forces $c_r^0(\sigma)$ to equal $n - d_2^r(\sigma)$ and every $j > d_2^r(\sigma)$ to be the arrival of a 2-inversion (i, j) such that $t_r(\sigma) \leq i$. In particular, this is true for $j = d_2^r(\sigma) + 1$, which is absurd because $\sigma(i) \geq \sigma(d_2^r(\sigma)) > \sigma(d_2^r(\sigma) + 1) + 1$ for all $i \in [t_r(\sigma), d_2^r(\sigma)]$. Therefore $c_r^0(\sigma) < n - d_2^r(\sigma)$. Also, it is easy to see that every $i > d_2^r(\sigma)$
130 that is the beginning of a 2-inversion (i, j) necessarily appears in the maximal sequence $(i_{k_1}^{max}(\sigma), i_{k_2}^{max}(\sigma), \dots, i_{k_m}^{max}(\sigma))$ whose length defines $c_r^0(\sigma)$, hence $(i, j) \notin I_{r-1}(\sigma)$.

★ Now, suppose that $c_k^0(\sigma) \leq d_2^{k+1}(\sigma) - d_2^k(\sigma)$ for some $k \in [des_2(\sigma)]$ with

equality only if $c_{k+1}^0(\sigma) > 0$, and that no 2-inversion (i, j) with $d_2^k(\sigma) < i$ belongs
135 to $I_{k-1}(\sigma)$.

If $t_{k-1}(\sigma) = t_k(\sigma)$ (i.e., if $d_2^{k-1}(\sigma) = d_2^k(\sigma) - 1$), since $I_{k-1}(\sigma)$ does not
contain any 2-inversion (i, j) with $d_2^k(\sigma) < i$, then $c_{k-1}^0(\sigma) \leq 1 = d_2^k(\sigma) -$
 $d_2^{k-1}(\sigma)$. Moreover, if $c_{k-1}^0(\sigma) = 1$, then there exists a 2-inversion $(i, j) \in$
 $I_{k-1}(\sigma) \subset I_k(\sigma)$ such that $i \in [t_{k-1}(\sigma), d_2^k(\sigma)]$. Consequently (i) was a legit
140 sequence for the computation of $c_k^0(\sigma)$ at the previous step (because $t_k(\sigma) =$
 $t_{k-1}(\sigma)$), which implies $c_k^0(\sigma)$ equals at least the length of (i) . In particular
 $c_k^0(\sigma) > 0$.

Else, consider a sequence $(i_{k_1}(\sigma), i_{k_2}(\sigma), \dots, i_{k_m}(\sigma))$ that fits the three con-
ditions of Algorithm 3.1 at the step $k - 1$. In particular $t_{k-1}(\sigma) \leq i_{k_1}(\sigma)$. Also
145 $i_{k_m}(\sigma) \leq d_2^k(\sigma)$ by hypothesis. Since $\sigma(i_{k_p}(\sigma)) < \sigma(i_{k_{p+1}}(\sigma))$ for all p , and since
 $\sigma(t_{k-1}(\sigma)) > \sigma(t_{k-1}(\sigma) + 1) > \dots > \sigma(d_2^{k-1}(\sigma)) > \sigma(d_2^{k-1}(\sigma) + 1)$, then only
one element of the set $[t_{k-1}(\sigma), d_2^{k-1}(\sigma) + 1]$ may equal $i_{k_p}(\sigma)$ for some $p \in [m]$.
Thus, the length l of the sequence verifies $l \leq d_2^k(\sigma) - d_2^{k-1}(\sigma)$, with equality
only if $i_{k_m}(\sigma) = d_2^k(\sigma)$ (which implies $c_k^0(\sigma) > 0$ as in the previous paragraph).
150 In particular, this is true for $l = c_{k-1}^0(\sigma)$.

Finally, as for $k = \text{des}_2(\sigma)$, every $i \in [d_2^{k-1}(\sigma) + 1, d_2^k(\sigma)]$ that is the
beginning of a 2-inversion (i, j) necessarily appears in the maximal sequence
 $(i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \dots, i_{k_m^{max}}(\sigma))$ whose length defines $c_{k-1}^0(\sigma)$, hence $(i, j) \notin$
 $I_{k-2}(\sigma)$.

155 So the lemma is true by induction. □

Definition 3.4. We define a graph $\mathcal{G}^0(\sigma)$ made of circles and dots organised in
ascending or descending slopes, by plotting :

- for all $k \in [0, r]$, an ascending slope of $c_k^0(\sigma)$ circles such that the first
circle has abscissa $d_2^k(\sigma) + 1$ and the last circle has abscissa $d_2^k(\sigma) + c_k^0(\sigma)$
(if $c_k^0(\sigma) = 0$, we plot nothing). All the abscissas are distinct because

$$d_2^0(\sigma) + c_0 < d_2^1(\sigma) + c_1 < \dots < d_2^r(\sigma) + c_r$$

in view of Lemma 3.3;

- dots at the remaining $n-s = n - \text{inv}_2(\sigma)$ abscissas from 1 to n , in ascending and descending slopes with respect to the descents and ascents of the word $\omega(\sigma)$ defined by

$$\omega(\sigma) = \sigma(u_1(\sigma))\sigma(u_2(\sigma)) \dots \sigma(u_{n-s}(\sigma)) \quad (5)$$

where

$$\{u_1(\sigma) < u_2(\sigma) < \dots < u_{n-s}(\sigma)\} := \mathfrak{S}_n \setminus \{i_1(\sigma) < i_2(\sigma) < \dots < i_s(\sigma)\}.$$

Example 3.5. The permutation $\sigma_0 = 425736981 \in \mathfrak{S}_9$ (with $\text{DES}_2(\sigma_0) = \{1, 4, 8\}$ and $\text{INV}_2(\sigma_0) = \{(1, 5), (2, 9), (4, 6), (7, 8)\}$), which yields the sequence $(c_k^0(\sigma_0))_{k \in [0,3]} = (1, 1, 2, 0)$ (see Figure 4 where all the 2-inversions involved in the computation of a same $c_k^0(\sigma_0)$ are drawn in a same color) and the word $\omega(\sigma_0) = 53681$, provides the unlabelled graph $\mathcal{G}^0(\sigma_0)$ depicted in Figure 5.

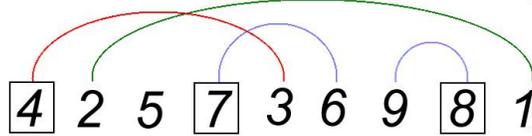


Figure 4: $(c_k^0(\sigma_0))_{k \in [0,3]} = (1, 1, 2, 0)$.

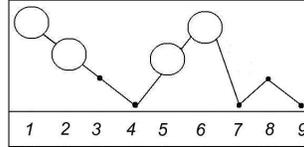


Figure 5: Graph $\mathcal{G}^0(\sigma_0)$.

The following lemma is easy.

Lemma 3.6. For all $i \in [n]$, if the i -th vertex (from left to right) $v_i^0(\sigma)$ of $\mathcal{G}^0(\sigma)$ is a dot and if i is a descent of $\mathcal{G}^0(\sigma)$ (i.e., if $v_i^0(\sigma)$ and $v_{i+1}^0(\sigma)$ are two dots in a same descending slope) whereas $i \notin \text{DES}_2(\sigma)$, let k_i such that

$$d_2^{k_i}(\sigma) + c_{k_i}^0(\sigma) < i < d_2^{k_i+1}(\sigma)$$

165 and let $p \in [n - s]$ such that $v_i^0(\sigma)$ is the p -th dot (from left to right) of $\mathcal{G}^0(\sigma)$.

Then :

1. $u_p(\sigma)$ is the greatest integer $u < d_2^{k_i+1}(\sigma)$ that is not the beginning of a 2-inversion of σ ;
2. $u_{p+1}(\sigma)$ is the smallest integer $u > d_2^{k_i+1}(\sigma)$ that is not a 2-descent or the
170 beginning of a 2-inversion of σ ;
3. $c_k^0(\sigma) > 0$ for all k such that $d_2^{k_i+1}(\sigma) \leq d_2^k(\sigma) \leq u_{p+1}(\sigma)$.

In particular $c_{k_i+1}^0(\sigma) > 0$.

Lemma 3.6 motivates the following definition.

Definition 3.7. For i from 1 to $n - 1$, let $k_i \in [0, r]$ such that

$$d_2^{k_i}(\sigma) + c_{k_i}^0(\sigma) < i < d_2^{k_i+1}(\sigma).$$

If i fits the conditions of Lemma 3.6, then we define a sequence $(c_k^i(\sigma))_{k \in [0, r]}$ by

$$\begin{aligned} c_{k_i}^i(\sigma) &= c_{k_i}^{i-1}(\sigma) + 1, \\ c_{k_i+1}^i(\sigma) &= c_{k_i+1}^{i-1}(\sigma) - 1, \\ c_k^i(\sigma) &= c_k^{i-1}(\sigma) \text{ for all } k \notin \{k_i, k_i + 1\}. \end{aligned}$$

Else, we define $(c_k^i(\sigma))_{k \in [0, r]}$ as $(c_k^{i-1}(\sigma))_{k \in [0, r]}$.

The final sequence $(c_k^n(\sigma))_{k \in [0, r]}$ is denoted by

$$(c_k(\sigma))_{k \in [0, r]}.$$

175 By construction, and from Lemma 3.3, the sequence $(c_k(\sigma))_{k \in [0, r]}$ has the same properties as $(c_k^0(\sigma))_{k \in [0, r]}$ detailed in Lemma 3.3.

Consequently, we may define an unlabelled graph

$$\mathcal{G}(\sigma)$$

by replacing $(c_k^0(\sigma))_{k \in [0, r]}$ with $(c_k(\sigma))_{k \in [0, r]}$ in Definition 3.4.

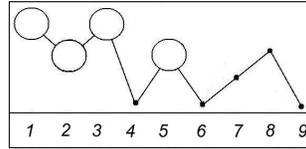


Figure 6: Graph $\mathcal{G}(\sigma_0)$.

Example 3.8. In the graph $\mathcal{G}^0(\sigma)$ depicted in Figure 5 where $\sigma_0 = 425736981 \in \mathfrak{S}_9$, we can see that the dot $v_3^0(\sigma_0)$ is a descent whereas $3 \notin \text{DES}_2(\sigma_0)$, hence, from the sequence $(c_k^0(\sigma_0))_{k \in [0,3]} = (1, 1, 2, 0)$, we compute $(c_k(\sigma_0))_{k \in [0,3]} = (1, 2, 1, 0)$ and we obtain the graph $\mathcal{G}(\sigma_0)$ depicted in Figure 6.

Let $v_1(\sigma), v_2(\sigma), \dots, v_n(\sigma)$ be the n vertices of $\mathcal{G}(\sigma)$ from left to right.

By construction, the descents of the unlabelled graph $\mathcal{G}(\sigma)$ (*i.e.*, the integers $i \in [n-1]$ such that $v_i(\sigma)$ and $v_{i+1}(\sigma)$ are in a same descending slope) are the integers

$$d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma)$$

for all $k \in [0, r]$.

3.2. Labelling of the graph $\mathcal{G}(\sigma)$

3.2.1. Labelling of the circles

We intend to label the circles of $\mathcal{G}(\sigma)$ with the integers

$$j_1(\sigma), j_2(\sigma), \dots, j_s(\sigma).$$

Algorithm 3.9. For all $i \in [n]$, if the vertex $v_i(\sigma)$ is a circle (hence $i < n$), we label it first with the set

$$[i+1, n] \cap \{j_1(\sigma), j_2(\sigma), \dots, j_s(\sigma)\}.$$

Afterwards, if a circle $v_i(\sigma)$ is found in a descending slope such that there exists a quantity of a circles above $v_i(\sigma)$, and in an ascending slope such that there exists a quantity of b circles above $v_i(\sigma)$, then we remove the $a+b$ greatest integers from the current label of $v_i(\sigma)$ (this set necessarily had at least $a+b+1$

190 elements) and the smallest integer from every of the $a + b$ labels of the $a + b$ circles above $v_i(\sigma)$ in the two related slopes. At the end of this step, if an integer $j_k(\sigma)$ appears in only one label of a circle $v_i(\sigma)$, then we replace the label of $v_i(\sigma)$ with $j_k(\sigma)$.

Finally, we replace every label that is still a set by the unique integer it may contain with respect to the order of the elements in the sequence

$$(j_1(\sigma), j_2(\sigma), \dots, j_s(\sigma))$$

(from left to right).

195 **Example 3.10.** For $\sigma_0 = 425736981$ (see Figure 4) whose graph $\mathcal{G}(\sigma_0)$ is depicted in Figure 6, we have $s = \text{inv}_2(\sigma) = 4$ and $\{j_1(\sigma_0), j_2(\sigma_0), j_3(\sigma_0), j_4(\sigma_0)\} = \{5, 6, 8, 9\}$, which provides first the graph labelled by sets depicted in Figure 7. Afterwards, since the circle $v_2(\sigma_0)$ is in a descending slope with $a = 1$ circle above it (the vertex $v_1(\sigma_0)$) and in an ascending slope with also $b = 1$ circle above it (the vertex $v_3(\sigma_0)$), then we remove the $a + b = 2$ integers 8 and 9 from its label, which becomes $\{5, 6\}$, and we remove 5 from the labels of $v_1(\sigma_0)$ and $v_3(\sigma_0)$. Also, since the label of $v_2(\sigma_0)$ is the only set that contains 5, then we label $v_2(\sigma_0)$ with 5 (see Figure 8). Finally, the sequence $(j_1(\sigma_0), j_2(\sigma_0), j_3(\sigma_0), j_4(\sigma_0)) = (5, 9, 6, 8)$ gives the order (from left to right) of
 200
 205 apparition of the remaining integers 6, 8, 9 (see Figure 9).

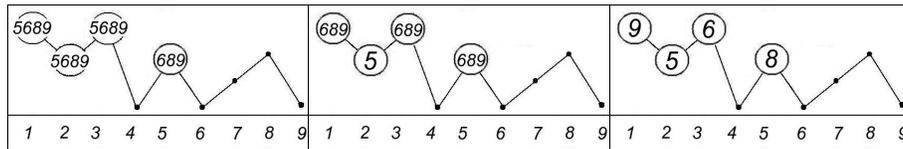


Figure 7

Figure 8

Figure 9

3.2.2. Labelling of the dots

Let

$$\{p_1(\sigma) < p_2(\sigma) < \dots < p_{n-s}(\sigma)\} = [n] \setminus \bigsqcup_{k=0}^r d_2^k(\sigma), d^k(\sigma).$$

We intend to label the dots $\{v_{p_i(\sigma)}(\sigma), i \in [n-s]\}$ of $\mathcal{G}(\sigma)$ with the elements of

$$\{1 = e_1(\sigma) < e_2(\sigma) < \dots < e_{n-s}(\sigma)\} = [n] \setminus \{j_1(\sigma), j_2(\sigma), \dots, j_s(\sigma)\}.$$

Algorithm 3.11. 1. For all $k \in [n-s]$, we label first the dot $v_{p_k(\sigma)}(\sigma)$ with the set

$$[\min(p_k(\sigma), u_k(\sigma))] \cap ([n] \setminus \{j_1(\sigma), j_2(\sigma), \dots, j_s(\sigma)\})$$

where $u_1(\sigma), u_2(\sigma), \dots, u_{n-s}(\sigma)$ are the integers introduced in (5).

2. Afterwards, similarly as for the labelling of the circles, if a dot $v_i(\sigma)$ is found in a descending slope such that a dots are above $v_i(\sigma)$, and in an ascending slope such that b dots are above $v_i(\sigma)$, then we remove the $a+b$ greatest integers from the current label of $v_i(\sigma)$ and the smallest integer from every of the $a+b$ labels of the dots above $v_i(\sigma)$ in the two related slopes. At the end of this step, if an integer l appears in only one label of a dot $v_i(\sigma)$, then we replace the label of $v_i(\sigma)$ with l .
3. Finally, for k from 1 to $n-s$, let

$$w_1^k(\sigma) < w_2^k(\sigma) < \dots < w_{q_k(\sigma)}^k(\sigma) \quad (6)$$

such that

$$\{p_{w_i^k(\sigma)}(\sigma), i\} = \{p_i(\sigma), e_k(\sigma) \text{ appear in the label of } p_i(\sigma)\},$$

and let $i(k) \in [q_k(\sigma)]$ such that

$$\sigma(u_{w_{i(k)}^k(\sigma)}(\sigma)) = \min\{\sigma(u_{w_i^k(\sigma)}(\sigma)), i \in [q_k(\sigma)]\}.$$

- Then, we replace the label of the dot $p_{w_{i(k)}^k(\sigma)}(\sigma)$ with the integer $e_k(\sigma)$ and we erase $e_k(\sigma)$ from any other label (and if an integer l appears in only one label of a dot $v_i(\sigma)$, then we replace the label of $v_i(\sigma)$ with l).

Example 3.12. For $\sigma_0 = 425736981$ whose graph $\mathcal{G}(\sigma_0)$ has its circles labelled in Figure 9, the sequence $(u_1(\sigma_0), u_2(\sigma_0), u_3(\sigma_0), u_4(\sigma_0), u_5(\sigma_0)) = (3, 5, 6, 8, 9)$ provides first the graph labelled by sets depicted in Figure 10. The rest of the algorithm goes from $k = 1$ to $n-s = 9-4 = 5$.

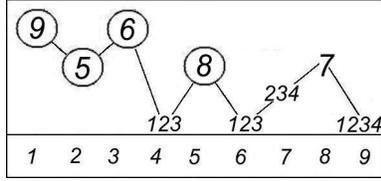


Figure 10

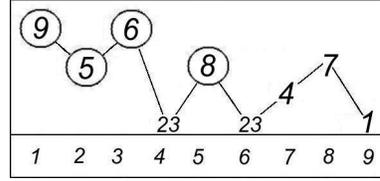


Figure 11

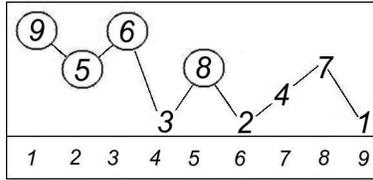


Figure 12: Labelled graph $\mathcal{G}(\sigma_0)$.

- $k = 1$: in Figure 10, the integer $e_1(\sigma_0) = 1$ appears in the labels of the dots $v_{p_1(\sigma_0)}(\sigma_0) = v_4(\sigma_0)$, $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$ and $v_{p_5(\sigma_0)}(\sigma_0) = v_9(\sigma_0)$, so, from

$$(\sigma_0(u_1(\sigma_0)), \sigma_0(u_2(\sigma_0)), \sigma_0(u_5(\sigma_0))) = (5, 3, 1),$$

we label the dot $v_{p_5(\sigma_0)}(\sigma_0) = v_9(\sigma_0)$ with the integer $e_1(\sigma_0) = 1$ and we erase 1 from any other label, and since the integer 4 now only appears in the label of the dot $v_7(\sigma_0)$, then we label $v_7(\sigma_0)$ with 4 (see Figure 11).

- $k = 2$: in Figure 11, the integer $e_2(\sigma_0) = 2$ appears in the labels of the dots $v_{p_1(\sigma_0)}(\sigma_0) = v_4(\sigma_0)$ and $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$ so, from

$$(\sigma_0(u_1(\sigma_0)), \sigma_0(u_2(\sigma_0))) = (5, 3),$$

225 we label the dot $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$ with the integer $e_2(\sigma_0) = 2$ and we erase 2 from any other label, which provides the graph labelled by integers depicted in Figure 12.

- The three steps $k = 3, 4, 5$ change nothing because every dot of $\mathcal{G}(\sigma_0)$ is already labelled by an integer at the end of the previous step.

230 So the final version of the labelled graph $\mathcal{G}(\sigma_0)$ is the one depicted in Figure 12.

3.3. Definition of $\varphi(\sigma)$

By construction of the labelled graph $\mathcal{G}(\sigma)$, the word $y_1 y_2 \dots y_n$ (where the integer y_i is the label of the vertex $v_i(\sigma)$ for all i) obviously is a permutation of the set $[n]$, whose planar graph is $\mathcal{G}(\sigma)$.

235 We define $\varphi(\sigma) \in \mathfrak{S}_n$ as this permutation.

For the example $\sigma_0 = 425736981 \in \mathfrak{S}_9$ whose labelled graph $\mathcal{G}(\sigma_0)$ is depicted in Figure 12, we obtain $\varphi(\sigma_0) = 956382471 \in \mathfrak{S}_9$.

In general, by construction of $\tau = \varphi(\sigma) \in \mathfrak{S}_n$, we have

$$\tau(\text{EXC}(\tau)) = \{j_k(\sigma), k \in [\text{inv}_2(\sigma)]\} \quad (7)$$

and

$$\text{DES}(\tau) = \begin{cases} \{d^k(\sigma), k \in [1, \text{des}_2(\sigma)]\} & \text{if } c_0(\sigma) = 0 (\Leftrightarrow d^0(\sigma) = 0), \\ \{d^k(\sigma), k \in [0, \text{des}_2(\sigma)]\} & \text{otherwise.} \end{cases} \quad (8)$$

Equality (7) provides

$$\text{exc}(\tau) = \text{inv}_2(\sigma).$$

By $d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma)$ for all k , Equality (8) provides

$$\text{maj}(\tau) = \text{maj}_2(\sigma) + \sum_{k \geq 0} c_k(\sigma),$$

and by definition of $(c_k(\sigma))_k$ and Lemma 3.3 we have $\sum_{k \geq 0} c_k(\sigma) = \sum_{k \geq 0} c_k^0(\sigma) = \text{inv}_2(\sigma) = \text{exc}(\tau)$ hence

$$\text{maj}(\tau) - \text{exc}(\tau) = \text{maj}_2(\sigma).$$

Finally, it is easy to see that $\widetilde{\text{des}}_2(\sigma) = \text{des}_2(\sigma)$ if and only if $c_0(\sigma) = 0$, so Equality (8) also provides

$$\text{des}(\tau) = \widetilde{\text{des}}_2(\sigma).$$

As a conclusion, we obtain

$$(\text{maj}(\tau) - \text{exc}(\tau), \text{des}(\tau), \text{exc}(\tau)) = (\text{maj}_2(\sigma), \widetilde{\text{des}}_2(\sigma), \text{inv}_2(\sigma))$$

as required by Theorem 1.2.

4. Construction of φ^{-1}

To end the proof of Theorem 1.2, it remains to show that $\varphi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ is surjective. Let $\tau \in \mathfrak{S}_n$. We introduce integers $r \geq 0$, $s = \text{exc}(\tau)$, and

$$0 \leq d^{0,\tau} < d^{1,\tau} < \dots < d^{r,\tau} < n$$

such that

$$\begin{aligned} \text{DES}(\tau) &= \{d^{k,\tau}, k \in [0, r]\} \cap \mathbb{N}_{>0}, \\ d^{0,\tau} &= 0 \Leftrightarrow \tau(1) = 1. \end{aligned}$$

240 In particular $\text{des}(\tau) = \begin{cases} r & \text{if } \tau(1) = 1, \\ r + 1 & \text{otherwise.} \end{cases}$

For all $k \in [0, r]$, we define

$$\begin{aligned} c_k^\tau &= \text{EXC}(\tau) \cap]d^{k-1,\tau}, d^{k,\tau}] \text{ (with } d^{-1,\tau} := 0), \\ d_2^{k,\tau} &= d^{k,\tau} - c_k^\tau. \end{aligned}$$

We have

$$0 = d_2^{0,\tau} < d_2^{1,\tau} < \dots < d_2^{r,\tau} < n$$

and similarly as Formula 4, we define

$$t_k^\tau = \min\{d_2^{l,\tau}, 1 \leq l \leq k, d_2^{l,\tau} = d_2^{k,\tau} - (k - l)\} \quad (9)$$

for all $k \in [r]$.

We intend to construct a graph $\mathcal{H}(\tau)$ which is the linear graph of permutation $\sigma \in \mathfrak{S}_n$ such that $\varphi(\sigma) = \tau$.

4.1. Skeleton of the graph $\mathcal{H}(\tau)$

245 We consider a graph $\mathcal{H}(\tau)$ whose vertices $v_1^\tau, v_2^\tau, \dots, v_n^\tau$ (from left to right) are n dots, aligned in a row, among which we box the $d_2^{k,\tau}$ -th vertex $v_{d_2^{k,\tau}}^\tau$ for all $k \in [r]$. We also draw the end of an arc of circle above every vertex v_j^τ such that $j = \tau(i)$ for some $i \in \text{EXC}(\tau)$.

For the example $\tau_0 = 956382471 \in \mathfrak{S}_9$ (whose planar graph is depicted in Figure 12), we have $r = \text{des}(\tau_0) - 1 = 3$ and

$$\begin{aligned} (c_k^{\tau_0})_{k \in [0,3]} &= (1, 2, 1, 0), \\ (d_2^{k, \tau_0})_{k \in [0,3]} &= (1 - 1, 3 - 2, 5 - 1, 8 - 0) = (0, 1, 4, 8), \\ \tau_0(\text{EXC}(\tau_0)) &= \{5, 6, 8, 9\}, \end{aligned}$$

and we obtain the graph $\mathcal{H}(\tau_0)$ depicted in Figure 13.

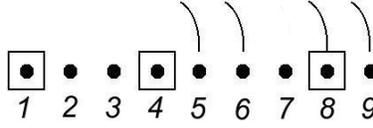


Figure 13: Incomplete graph $\mathcal{H}(\tau_0)$.

250 In general, by definition of $\varphi(\sigma)$ for all $\sigma \in \mathfrak{S}_n$, if $\varphi(\sigma) = \tau$, then $r = \text{des}_2(\sigma)$ and $d_2^k(\sigma)$ (respectively $c_k(\sigma), d^k(\sigma), t_k(\sigma)$) equals $d_2^{k, \tau}$ (resp. $c_k^\tau, d^{k, \tau}, t_k^\tau$) for all $k \in [0, r]$ and $\{j_l(\sigma), l \in [\text{inv}_2(\sigma)]\} = \tau(\text{EXC}(\tau))$. Consequently, the linear graph of σ necessarily have the same skeleton as that of $\mathcal{H}(\tau)$.

The following lemma is easy.

255 **Lemma 4.1.** *If $\tau = \varphi(\sigma)$ for some $\sigma \in \mathfrak{S}_n$, then :*

1. *If $j = \tau(l)$ with $l \in \text{EXC}(\tau)$ such that $l \in]d_2^{k, \tau}, d^{k, \tau}]$, and if $(i, j) \in \text{INV}_2(\sigma)$, then $t_k^\tau \leq i$.*
2. *A pair $(i, i + 1)$ cannot be a 2-inversion of σ if $i \in \text{DES}_2(\sigma)$ (\Leftrightarrow if the vertex v_i^τ of $\mathcal{H}(\tau)$ is boxed).*
- 260 3. *For all pair $(l, l') \in \text{EXC}(\tau)^2$, if the labels of the two circles $v_l(\sigma)$ and $v_{l'}(\sigma)$ can be exchanged without modifying the skeleton of $\mathcal{G}(\sigma)$, let i and i' such that $(i, l) \in \text{INV}_2(\sigma)$ and $(i', l') \in \text{INV}_2(\sigma)$, then $i < i' \Leftrightarrow l < l'$.*

Consequently, in order to construct the linear graph of a permutation $\sigma \in \mathfrak{S}_n$ such that $\tau = \varphi(\sigma)$ from $\mathcal{H}(\tau)$, it is necessary to extend the arcs of circles of $\mathcal{H}(\tau)$ to reflect the three facts of Lemma 4.1. When a vertex is necessarily the beginning of an arc of circle, we draw the beginning of an arc of circle above it.

When there is only one vertex v_i^τ that can be the beginning of an arc of circle, we complete the latter by making it start from v_i^τ .

Example 4.2. For $\tau_0 = 956382471 \in \mathfrak{S}_9$, the graph $\mathcal{H}(\tau_0)$ becomes as depicted in Figure 14. Note that the arc of circle ending at $v_6^{\tau_0}$ cannot begin at $v_5^{\tau_0}$ because

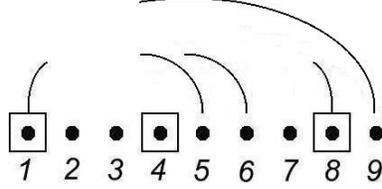


Figure 14: Incomplete graph $\mathcal{H}(\tau_0)$.

270

otherwise, from the third point of Lemma 4.1, and since $(6, 8) = (\tau_0(l), \tau_0(l'))$ with $3 = l < l' = 5$, it would force the arc of circle ending at $v_8^{\tau_0}$ to begin at $v_{i'}^{\tau_0}$ with $6 \leq i'$, which is absurd because a permutation $\sigma \in \mathfrak{S}_9$ whose linear graph would be of the kind $\mathcal{H}(\tau_0)$ would have $c_2(\sigma) = 2 \neq 1 = c_2^{\tau_0}$. Also, still
 275 in view of the third point of Lemma 4.1, and since $\tau_0^{-1}(9) < \tau_0^{-1}(6)$, the arc of circle ending at $v_9^{\tau_0}$ must start before the arc of circle ending at $v_6^{\tau_0}$, hence the configuration of $\mathcal{H}(\tau_0)$ in Figure 14.

The following two facts are obvious.

Facts 4.3. If $\tau = \varphi(\sigma)$ for some $\sigma \in \mathfrak{S}_n$, then :

280

1. A vertex v_i^τ of $\mathcal{H}(\tau)$ is boxed if and only if $i \in \text{DES}_2(\sigma)$. In that case, in particular i is a descent of σ .
2. If a pair $(i, i + 1)$ is not a 2-descent of σ and if v_i^τ is not boxed, then i is an ascent of σ , *i.e.* $\sigma(i) < \sigma(i + 1)$.

To reflect Facts 4.3, we draw an ascending arrow (respectively a descending
 285 arrow) between the vertices v_i^τ and v_{i+1}^τ of $\mathcal{H}(\tau)$ whenever it is known that $\sigma(i) < \sigma(i + 1)$ (resp. $\sigma(i) > \sigma(i + 1)$) for all $\sigma \in \mathfrak{S}_n$ such that $\varphi(\sigma) = \tau$.

For the example $\tau_0 = 956382471 \in \mathfrak{S}_9$, the graph $\mathcal{H}(\tau_0)$ becomes as depicted in Figure 15. Note that it is not known yet if there is an ascending or descending arrow between $v_7^{\tau_0}$ and $v_8^{\tau_0}$.

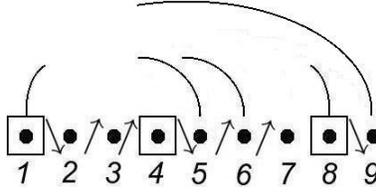


Figure 15: Incomplete graph $\mathcal{H}(\tau_0)$.

290 4.2. Completion and labelling of $\mathcal{H}(\tau)$

The following lemma is analogous to the third point of Lemma 4.1 for the dots instead of the circles and follows straightly from the definition of $\varphi(\sigma)$ for all $\sigma \in \mathfrak{S}_n$.

Lemma 4.4. *Let $\sigma \in \mathfrak{S}_n$ such that $\varphi(\sigma) = \tau$. For all pair $(l, l') \in ([n] \setminus EXC(\tau))^2$,
 295 if the labels of the two dots $v_l(\sigma)$ and $v_{l'}(\sigma)$ can be exchanged without modifying the skeleton of $\mathcal{G}(\sigma)$, let k and k' such that $l = p_k(\sigma)$ and $l' = p_{k'}(\sigma)$, then $\tau(l) < \tau(l') \Leftrightarrow \sigma(u_k(\sigma)) < \sigma(u_{k'}(\sigma))$.*

Now, the ascending and descending arrows between the vertices of $\mathcal{H}(\tau)$ introduced earlier, and Lemma 4.4, induce a partial order on the set $\{v_i^\tau, i \in [n]\}$:

300 **Definition 4.5.** We define a partial order \succ on $\{v_i^\tau, i \in [n]\}$ by :

- $v_i^\tau \prec v_{i+1}^\tau$ (resp. $v_i^\tau \succ v_{i+1}^\tau$) if there exists an ascending (resp. descending) arrow between v_i^τ and v_{i+1}^τ ;
- $v_i^\tau \succ v_j^\tau$ (with $i < j$) if there exists an arc of circle from v_i^τ to v_j^τ ;
- if two vertices v_i^τ and v_j^τ are known to be respectively the k -th and k' -th
 305 vertices of $\mathcal{H}(\tau)$ that cannot be the beginning of a complete arc of circle, let l and l' be respectively the k -th and k' -th non-exceedance point of τ (from left to right), if (l, l') fits the conditions of Lemma 4.4, then we set $v_i^\tau \prec v_j^\tau$ (resp. $v_i^\tau \succ v_j^\tau$) if $\tau(l) < \tau(l')$ (resp. $\tau(l) > \tau(l')$).

Example 4.6. For the example $\tau_0 = 956382471$, according to the first point of Definition 4.5, the arrows of Figure 15 provide

$$v_1^{\tau_0} \succ v_2^{\tau_0} \prec v_3^{\tau_0} \prec v_4^{\tau_0} \succ v_5^{\tau_0} \prec v_6^{\tau_0} \prec v_7^{\tau_0}$$

and

$$v_8^{\tau_0} \succ v_9^{\tau_0}.$$

Definition 4.7. A vertex v_i^τ of $\mathcal{H}(\tau)$ is said to be *minimal* on a subset $S \subset [n]$
 310 if $v_i^\tau \not\prec v_j^\tau$ for all $j \in S$.

Let

$$1 = e_1^\tau < e_2^\tau < \dots < e_{n-s}^\tau$$

be the non-exceedance values of τ (*i.e.*, the labels of the dots of the planar graph of τ).

Algorithm 4.8. Let $S = [n]$ and $l = 1$. While the vertices $\{v_i^\tau, i \in [n]\}$ have not all been labelled with the elements of $[n]$, apply the following algorithm.

- 315 1. If there exists a unique minimal vertex v_i^τ of τ on S , we label it with l , then we set $l := l + 1$ and $S := S \setminus \{v_i^\tau\}$. Afterwards,
- (a) If v_i^τ is the ending of an arc of circle starting from a vertex v_j^τ , then we label v_j^τ with the integer l and we set $l := l + 1$ and $S := S \setminus \{v_j^\tau\}$.
 - (b) If v_i^τ is the arrival of an incomplete arc of circle (in particular $i = \tau(l)$
 320 for some $l \in \text{EXC}(\tau)$), we intend to complete the arc by making it start from a vertex v_j^τ for some integer $j \in [t_k^\tau, j[$ (where $l \in]d_2^{k,\tau}, d^{k,\tau}]$) in view of the first point of Lemma 4.1. We choose v_j^τ as the rightest minimal vertex on $[t_k^\tau, j[\cap S$ from which it may start in view of the third point of Lemma 4.1, and we label this vertex v_j^τ
 325 with the integer l . Then we set $l := l + 1$ and $S := S \setminus \{v_j^\tau\}$.

Now, if there exists an arc of circle from v_j^τ (for some j) to v_i^τ , we apply steps (a),(b) and (c) to the vertex v_j^τ in place of v_i^τ .

2. Otherwise, let $k \geq 0$ be the number of vertices v_i^τ that have already been labelled and that are not the beginning of arcs of circles. Let

$$l_1 < l_2 < \dots < l_q$$

be the integers $l \in [n]$ such that $l \geq \tau(l) \geq e_{k+1}^\tau$ and such that we can exchange the labels of dots $\tau(l)$ and e_{k+1}^τ in the planar graph of τ without

330 modifying the skeleton of the graph. It is easy to see that q is precisely the number of minimal vertices of τ on S . Let $l_{i_{k+1}} = \tau^{-1}(e_{k+1}^\tau)$ and let v_j^τ be the i_{k+1} -th minimal vertex (from left to right) on S . We label v_j^τ with l , then we set $l := l + 1$ and $S := S \setminus \{v_j^\tau\}$, and we apply steps 1.(a), (b) and (c) to v_j^τ instead of v_i^τ .

By construction, the labelled graph $\mathcal{H}(\tau)$ is the linear graph of a permutation $\sigma \in \mathfrak{S}_n$ such that

$$\text{DES}_2(\sigma) = \{d_2^{k,\tau}, k \in [r]\}$$

and

$$\{j_l(\sigma), l \in [\text{inv}_2(\sigma)]\} = \tau(\text{EXC}(\tau)).$$

335 **Example 4.9.** Consider $\tau_0 = 956382471 \in \mathfrak{S}_9$ whose unlabelled and incomplete graph $\mathcal{H}(\tau_0)$ is depicted in Figure 15.

- As stated in Example 4.6, the minimal vertices of τ_0 on $S = [9]$ are $(v_2^{\tau_0}, v_5^{\tau_0}, v_9^{\tau_0})$. Following step 2 of Algorithm 4.8, $k = 0$ and the integers $l \in [9]$ such that $\tau_0(l) \geq e_{k+1}^{\tau_0} = 1$ and such that the labels of dots $\tau_0(l)$ can be exchanged with 1 in the planar graph of τ_0 (see Figure 12) are $(l_1, l_2, l_3) = (4, 6, 9)$. By $\tau_0^{-1}(1) = 9 = l_3$, we label the third minimal vertex on $[9]$, *i.e.* the vertex $v_9^{\tau_0}$, with the integer $l = 1$.

340 Afterwards, following step 1.(b), since $v_9^{\tau_0}$ is the arrival of an incomplete arc of circle starting from a vertex $v_j^{\tau_0}$ with $1 = t_1^{\tau_0} \leq j$, and with $j < 5$ because that arc of circle must begin before the arc of circle ending at $v_6^{\tau_0}$ in view of Fact 3 of Lemma 4.1, we complete that arc of circle by making it start from the unique minimal vertex $v_j^{\tau_0}$ on $[1, 5[$, *i.e.* $j = 2$, and we label $v_2^{\tau_0}$ with the integer $l = 2$ (see Figure 16). Note that as from now we know that the arc of circle ending at $v_5^{\tau_0}$ necessarily begins at $v_1^{\tau_0}$, because otherwise $v_1^{\tau_0}$, being the beginning of an arc of circle, would be the beginning of the arc of circle ending at $v_6^{\tau_0}$, which is absurd in view of Fact 3 of Lemma 4.1 because $\tau_0^{-1}(9) < \tau^{-1}(6)$, so we complete that arc of circle by making it start from $v_1^{\tau_0}$, which has been depicted in Figure 16.

350

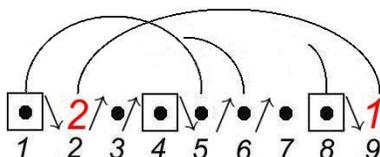


Figure 16: Beginning of the labelling of $\mathcal{H}(\tau_0)$.

We now have $S = [9] \setminus \{2, 9\}$ and $l = 3$.

- 355 • From Figure 16, the minimal vertices on $S = [9] \setminus \{2, 9\}$ are $(v_3^{\tau_0}, v_5^{\tau_0})$. Following step 2 of Algorithm 4.8, $k = 1$ and the integers $l \in [9]$ such that $l \geq \tau_0(l) \geq e_{k+1}^{\tau_0} = 2$ and such that the labels of dots $\tau_0(l)$ can be exchanged with 2 in the planar graph of τ_0 (see Figure 12) are $(l_1, l_2) = (4, 6)$. By $\tau_0^{-1}(2) = 6 = l_2$, we label the second minimal vertex on S , *i.e.*
- 360 the vertex $v_5^{\tau_0}$, with the integer $l = 3$.

Afterwards, following step 1.(a), since $v_5^{\tau_0}$ is the arrival of the arc of circle starting from the vertex $v_1^{\tau_0}$, we label $v_1^{\tau_0}$ with the integer $l = 4$ (see Figure 17).

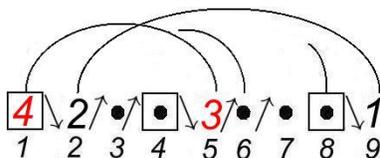


Figure 17: Beginning of the labelling of $\mathcal{H}(\tau_0)$.

We now have $S = [9] \setminus \{1, 2, 5, 9\}$ and $l = 5$.

- 365 • From Figure 17, the minimal vertices on $S = \{3, 4, 6, 7, 8\}$ are $(v_3^{\tau_0}, v_6^{\tau_0})$. Following step 2 of Algorithm 4.8, $k = 2$ and the integers $l \in [9]$ such that $l \geq \tau_0(l) \geq e_{k+1}^{\tau_0} = 3$ and such that the labels of dots $\tau_0(l)$ can be exchanged with 3 in the planar graph of τ_0 (see Figure 12) are $(l_1, l_2) = (4, 7)$. By $\tau_0^{-1}(3) = 4 = l_1$, we label the first minimal vertex on S , *i.e.*
- 370 the vertex $v_3^{\tau_0}$, with the integer $l = 5$ (see Figure 18). Note that as from now we know that the arc of circle ending at $v_6^{\tau_0}$ necessarily begins at $v_4^{\tau_0}$

since it is the only vertex left it may start from. Consequently, the arc of circle ending at $v_8^{\tau_0}$ necessarily starts from $v_7^{\tau_0}$ (otherwise it would start from $v_6^{\tau_0}$, which is prevented by Definition 4.5 because we cannot have $v_8^{\tau_0} \prec v_6^{\tau_0} \prec v_7^{\tau_0} \prec v_8^{\tau_0}$). The two latter remarks are taken into account in Figure 18.

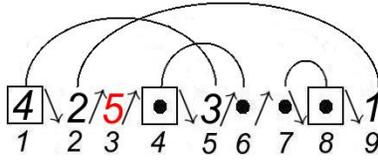


Figure 18: Beginning of the labelling of $\mathcal{H}(\tau_0)$.

We now have $S = \{4, 6, 7, 8\}$ and $l = 6$.

- From Figure 18, there is only one minimal vertex on $S = \{4, 6, 7, 8\}$, *i.e.* the vertex $v_6^{\tau_0}$. Following step 1 of Algorithm 4.8, we label $v_6^{\tau_0}$ with $l = 6$. Afterwards, following step 1.(a), since $v_6^{\tau_0}$ is the arrival of the arc of circle starting from the vertex $v_4^{\tau_0}$, we label $v_4^{\tau_0}$ with the integer $l = 7$ (see Figure 19).

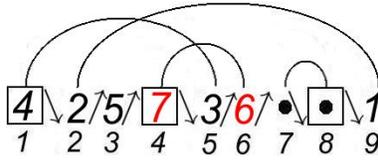


Figure 19: Beginning of the labelling of $\mathcal{H}(\tau_0)$.

We now have $S = \{7, 8\}$ and $l = 8$.

- From Figure 19, there is only one minimal vertex on $S = \{7, 8\}$, *i.e.* the vertex $v_8^{\tau_0}$. Following step 1 of Algorithm 4.8, we label $v_8^{\tau_0}$ with $l = 8$. Afterwards, following step 1.(a), since $v_8^{\tau_0}$ is the arrival of the arc of circle starting from the vertex $v_7^{\tau_0}$, we label $v_7^{\tau_0}$ with the integer $l = 9$ (see Figure 20).

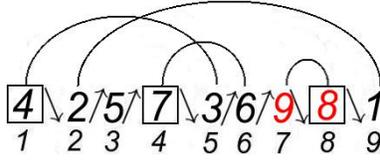


Figure 20: Labelled graph $\mathcal{H}(\tau_0)$.

As a conclusion, the graph $\mathcal{H}(\tau_0)$ is the linear graph of the permutation
390 $\sigma_0 = 425736981 \in \mathfrak{S}_9$, which is mapped to τ_0 by φ .

Proposition 4.10. *We have $\varphi(\sigma) = \tau$, hence φ is bijective.*

Proof. By construction, for all $k \in [0, \text{des}_2(\sigma)] = [0, r]$,

$$d_2^k(\sigma) = d^{k,\tau} - c_k^\tau,$$

$$c_k(\sigma) = c_k^\tau,$$

$$d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma) = d_2^{k,\tau} + c_k^\tau = d^{k,\tau},$$

so $\mathcal{G}(\sigma)$ has the same skeleton as the planar graph of τ , *i.e.* $\text{DES}(\varphi(\sigma)) = \text{DES}(\tau)$ and $\text{EXC}(\varphi(\sigma)) = \text{EXC}(\tau)$.

The labels of the circles of $\mathcal{G}(\sigma)$ are the elements of

$$\{j_l(\sigma), l \in [s]\} = \tau(\text{EXC}(\tau)),$$

and by construction of σ , every pair $(l, l') \in \text{EXC}(\tau)^2$ such that we can exchange the labels $\tau(l)$ and $\tau(l')$ in the planar graph of τ is such that

$$i < i' \Leftrightarrow l < l'$$

where $(i, \tau(l))$ and $(i, \tau(l'))$ are the two corresponding 2-inversions of σ . Consequently, by definition of $\varphi(\sigma)$, the labels of the circles of $\mathcal{G}(\sigma)$ appear in the same
395 order as in the planar graph of τ (*i.e.* $\varphi(\sigma)(i) = \tau(i)$ for all $i \in \text{EXC}(\varphi(\sigma)) = \text{EXC}(\tau)$).

As a consequence, the dots of $\mathcal{G}(\sigma)$ and the planar graph of τ are labelled by the elements

$$1 = e_1(\sigma) = e_1^\tau < e_2(\sigma) = e_2^\tau < \dots < e_{n-s}(\sigma) = e_{n-s}^\tau.$$

As for the labels of the circles, to show that the above integers appear in the same order among the labels of $\mathcal{G}(\sigma)$ and the planar graph of τ , it suffices to prove that

$$\varphi(\sigma)^{-1}(e_i^{\tau}) < \varphi(\sigma)^{-1}(e_j^{\tau}) \Leftrightarrow \tau^{-1}(e_i^{\tau}) < \tau^{-1}(e_j^{\tau})$$

for all pair (i, j) such that we can exchange the labels e_i^{τ} and e_j^{τ} in the planar graph of τ (hence in $\mathcal{G}(\sigma)$ since the two graphs have the same skeleton). This is
400 guaranteed by Definition 4.5 because the vertices v_i^{τ} that are not the beginning of an arc of circle correspond with the labels of the dots of the planar graph of τ .

As a conclusion, the planar graph of τ is in fact $\mathcal{G}(\sigma)$, *i.e.* $\tau = \varphi(\sigma)$. \square

5. Open problem

405 In view of Formula (2) and Theorem 1.2, it is natural to look for a bijection $\mathfrak{S}_n \rightarrow \mathfrak{S}_n$ that maps $(\text{maj}_2, \widetilde{\text{des}}_2, \text{inv}_2)$ to $(\text{amaj}_2, \widetilde{\text{asc}}_2, \text{ides})$.

Recall that $\text{ides} = \text{des}_2$ and that for a permutation $\sigma \in \mathfrak{S}_n$, the equality $\widetilde{\text{des}}_2(\sigma) = \text{des}_2(\sigma)$ is equivalent to $\varphi(\sigma)(1) = 1$, which is similar to the equivalence $\widetilde{\text{asc}}_2(\tau) = \text{asc}_2(\tau) \Leftrightarrow \tau(1) = 1$ for all $\tau \in \mathfrak{S}_n$.

Note that if $\text{DES}_2(\sigma) = \bigsqcup_{p=1}^r [i_p, j_p]$ with $j_p + 1 < i_{p+1}$ for all p , the permutation $\pi = \rho_1 \circ \rho_2 \circ \dots \circ \rho_r \circ \sigma$, where ρ_p is the $(j_p - i_p + 2)$ -cycle

$$\begin{pmatrix} i_p & i_p + 1 & i_p + 2 & \dots & j_p & j_p + 1 \\ \sigma(j_p + 1) & \sigma(j_p) & \sigma(j_p - 1) & \dots & \sigma(i_p + 1) & \sigma(i_p) \end{pmatrix}$$

410 for all p , is such that $\text{DES}_2(\sigma) \subset \text{ASC}_2(\pi)$ and $\text{INV}_2(\sigma) = \text{INV}_2(\pi)$. One can try to get rid of the eventual unwanted 2-ascents $i \in \text{ASC}_2(\pi) \setminus \text{DES}_2(\sigma)$ by composing π with adequate permutations.

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