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# A new bijection relating $q$ -Eulerian polynomials

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## Abstract

On the set of permutations of a finite set, we construct a bijection which maps the 3-vector of statistics  $(\text{maj} - \text{exc}, \text{des}, \text{exc})$  to a 3-vector  $(\text{maj}_2, \widetilde{\text{des}}_2, \text{inv}_2)$  associated with the  $q$ -Eulerian polynomials introduced by Shareshian and Wachs in *Chromatic quasisymmetric functions*, *arXiv:1405.4269(2014)*.

*Keywords:*  $q$ -Eulerian polynomials, descents, ascents, major index, exceedances, inversions.

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## Notations

For all pair of integers  $(n, m)$  such that  $n < m$ , the set  $\{n, n + 1, \dots, m\}$  is indifferently denoted by  $[n, m]$ ,  $]n - 1, m]$ ,  $[n, m + 1[$  or  $]n - 1, m + 1[$ .

The set of positive integers  $\{1, 2, 3, \dots\}$  is denoted by  $\mathbb{N}_{>0}$ .

5 For all integer  $n \in \mathbb{N}_{>0}$ , we denote by  $[n]$  the set  $[1, n]$  and by  $\mathfrak{S}_n$  the set of the permutations of  $[n]$ . By abuse of notation, we assimilate every  $\sigma \in \mathfrak{S}_n$  with the word  $\sigma(1)\sigma(2)\dots\sigma(n)$ .

If a set  $S = \{n_1, n_2, \dots, n_k\}$  of integers is such that  $n_1 < n_2 < \dots < n_k$ , we sometimes use the notation  $S = \{n_1 < n_2 < \dots < n_k\}$ .

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10 **1. Introduction**

Let  $n$  be a positive integer and  $\sigma \in \mathfrak{S}_n$ . A *descent* (respectively *exceedance point*) of  $\sigma$  is an integer  $i \in [n - 1]$  such that  $\sigma(i) > \sigma(i + 1)$  (resp.  $\sigma(i) > i$ ). The set of descents (resp. exceedance points) of  $\sigma$  is denoted by  $\text{DES}(\sigma)$  (resp.  $\text{EXC}(\sigma)$ ) and its cardinal by  $\text{des}(\sigma)$  (resp.  $\text{exc}(\sigma)$ ). The integers  $\sigma(i)$  with  
 15  $i \in \text{EXC}(\sigma)$  are called exceedance values of  $\sigma$ .

It is due to MacMahon [Mac15] and Riordan [Rio58] that

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)} = A_n(t)$$

where  $A_n(t)$  is the  $n$ -th Eulerian polynomial [Eul55]. A statistic equidistributed with  $\text{des}$  or  $\text{exc}$  is said to be *Eulerian*. The statistic  $\text{idcs}$  defined by  $\text{idcs}(\sigma) = \text{des}(\sigma^{-1})$  obviously is Eulerian.

The *major index* of a permutation  $\sigma \in \mathfrak{S}_n$  is defined as

$$\text{maj}(\sigma) = \sum_{i \in \text{DES}(\sigma)} i.$$

It is also due to MacMahon that

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = \prod_{i=1}^n \frac{1 - q^i}{1 - q}.$$

A statistic equidistributed with  $\text{maj}$  is said to be *Mahonian*. Among Mahonian  
 20 statistics is the statistic  $\text{inv}$ , defined by  $\text{inv}(\sigma) = |\text{INV}(\sigma)|$  where  $\text{INV}(\sigma)$  is the set of *inversions* of a permutation  $\sigma \in \mathfrak{S}_n$ , *i.e.* the pairs of integers  $(i, j) \in [n]^2$  such that  $i < j$  and  $\sigma(i) > \sigma(j)$ .

In [SW14], the authors consider analogous versions of the above statistics : let  $\sigma \in \mathfrak{S}_n$ , the set of *2-descents* (respectively *2-inversions*) of  $\sigma$  is defined as

$$\text{DES}_2(\sigma) = \{i \in [n - 1], \sigma(i) > \sigma(i + 1) + 1\}$$

(resp.

$$\text{INV}_2(\sigma) = \{1 \leq i < j \leq n, \sigma(i) = \sigma(j) + 1\}$$

and its cardinal is denoted by  $\text{des}_2(\sigma)$  (resp.  $\text{inv}_2(\sigma)$ ).

It is easy to see that  $\text{inv}_2(\sigma) = \text{idcs}(\sigma)$ . The *2-major index* of  $\sigma$  is defined as

$$\text{maj}_2(\sigma) = \sum_{i \in \text{DES}_2(\sigma)} i.$$

By using quasisymmetric function techniques, the authors of [SW14] proved the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\text{maj}_2(\sigma)} y^{\text{inv}_2(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{maj}(\sigma) - \text{exc}(\sigma)} y^{\text{exc}(\sigma)}. \quad (1)$$

Similarly, by using the same quasisymmetric function method as in [SW14], the authors of [HL12] proved the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\text{amaj}_2(\sigma)} y^{\widetilde{\text{asc}}_2(\sigma)} z^{\text{idcs}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{maj}(\sigma) - \text{exc}(\sigma)} y^{\text{des}(\sigma)} z^{\text{exc}(\sigma)} \quad (2)$$

where  $\text{asc}_2(\sigma)$  is the number of *2-ascents* of a permutation  $\sigma \in \mathfrak{S}_n$ , i.e. the elements of the set  $\text{ASC}_2(\sigma) = \{i \in [n-1], \sigma(i) < \sigma(i+1) + 1\}$ , which rises the statistic  $\text{amaj}_2$  defined by

$$\text{amaj}_2(\sigma) = \sum_{i \in \text{ASC}_2(\sigma)} i,$$

and where

$$\widetilde{\text{asc}}_2(\sigma) = \begin{cases} \text{asc}_2(\sigma) & \text{if } \sigma(1) = 1, \\ \text{asc}_2(\sigma) + 1 & \text{if } \sigma(1) \neq 1. \end{cases}$$

**Definition 1.1.** Let  $\sigma \in \mathfrak{S}_n$ . We consider the smallest 2-descent  $d_2(\sigma)$  of  $\sigma$  such that  $\sigma(i) = i$  for all  $i \in [d_2(\sigma) - 1]$  (if there is no such 2-descent, we define  $d_2(\sigma)$  as 0 and  $\sigma(0)$  as  $n+1$ ).

Now, let  $d'_2(\sigma) > d_2(\sigma)$  be the smallest 2-descent of  $\sigma$  greater than  $d_2(\sigma)$  (if there is no such 2-descent, we define  $d'_2(\sigma)$  as  $n$ ).

We define an inductive property  $\mathcal{P}(d_2(\sigma))$  by :

1.  $\sigma(d_2(\sigma)) < \sigma(i)$  for all  $(i, j) \in \text{INV}_2(\sigma)$  such that  $d_2(\sigma) < i < d'_2(\sigma)$ ;
2. if  $(d'_2(\sigma), j) \in \text{INV}_2(\sigma)$  for some  $j$ , then either  $\sigma(d_2(\sigma)) < \sigma(d'_2(\sigma))$ , or  $d'_2(\sigma)$  has the property  $\mathcal{P}(d'_2(\sigma))$  (where the role of  $d_2(\sigma)$  is played by  $d'_2(\sigma)$  and that of  $d'_2(\sigma)$  by  $d''_2(\sigma)$  where  $d''_2(\sigma) > d'_2(\sigma)$  is the smallest 2-descent of  $\sigma$  greater than  $d'_2(\sigma)$ , defined as  $n$  if there is no such 2-descent).

35 This property is well-defined because  $(n, j) \notin \text{INV}_2(\sigma)$  for all  $j \in [n]$ .

Finally, we define a statistic  $\widetilde{\text{des}}_2$  by :

$$\widetilde{\text{des}}_2(\sigma) = \begin{cases} \text{des}_2(\sigma) & \text{if the property } \mathcal{P}(d_2(\sigma)) \text{ is true,} \\ \text{des}_2(\sigma) + 1 & \text{otherwise.} \end{cases}$$

In the present paper, we prove the following theorem.

**Theorem 1.2.** *There exists a bijection  $\varphi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$  such that*

$$(\text{maj}_2(\sigma), \widetilde{\text{des}}_2(\sigma), \text{inv}_2(\sigma)) = (\text{maj}(\varphi(\sigma)) - \text{exc}(\varphi(\sigma)), \text{des}(\varphi(\sigma)), \text{exc}(\varphi(\sigma))).$$

As a straight corollary of Theorem 1.2, we obtain the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\text{maj}_2(\sigma)} y^{\widetilde{\text{des}}_2(\sigma)} z^{\text{inv}_2(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{maj}(\sigma) - \text{exc}(\sigma)} y^{\text{des}(\sigma)} z^{\text{exc}(\sigma)} \quad (3)$$

which implies Equality (1).

The rest of this paper is organised as follows.

40 In Section 2, we introduce two graphical representations of a given permutation so as to highlight either the statistic  $(\text{maj} - \text{exc}, \text{des}, \text{exc})$  or  $(\text{maj}_2, \widetilde{\text{des}}_2, \text{inv}_2)$ . Practically speaking, the bijection  $\varphi$  of Theorem 1.2 will be defined by constructing one of the two graphical representations of  $\varphi(\sigma)$  for a given permutation  $\sigma \in \mathfrak{S}_n$ .

45 We define  $\varphi$  in Section 3.

In Section 4, we prove that  $\varphi$  is bijective by constructing  $\varphi^{-1}$ .

## 2. Graphical representations

### 2.1. Linear graph

Let  $\sigma \in \mathfrak{S}_n$ . The linear graph of  $\sigma$  is a graph whose vertices are (from left to  
50 right) the integers  $\sigma(1), \sigma(2), \dots, \sigma(n)$  aligned in a row, where every  $\sigma(k)$  (for  $k \in \text{DES}_2(\sigma)$ ) is boxed, and where an arc of circle is drawn from  $\sigma(i)$  to  $\sigma(j)$  for every  $(i, j) \in \text{INV}_2(\sigma)$ .

For example, the permutation  $\sigma = 34251 \in \mathfrak{S}_5$  (such that  
( $\text{maj}_2(\sigma), \widetilde{\text{des}}_2(\sigma), \text{inv}_2(\sigma)$ ) =  $(6, 3, 2)$ ) has the linear graph depicted in Fig-  
55 ure 1.

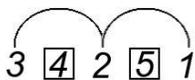


Figure 1: Linear graph of  $\sigma = 34251 \in \mathfrak{S}_5$ .

### 2.2. Planar graph

Let  $\tau \in \mathfrak{S}_n$ . The planar graph of  $\tau$  is a graph whose vertices are the integers  $1, 2, \dots, n$ , organized in ascending and descending slopes (the height of each vertex doesn't matter) such that the  $i$ -th vertex (from left to right) is the  $i$ -th integer  $\tau(i)$ , and where every vertex  $\tau(i)$  with  $i \in \text{EXC}(\tau)$  is encircled.

For example, the permutation  $\tau = 32541 \in \mathfrak{S}_5$  (such that  $(\text{maj}(\tau) - \text{exc}(\tau), \text{des}(\tau), \text{exc}(\tau)) = (6, 3, 2)$ ) has the planar graph depicted in Figure 2.

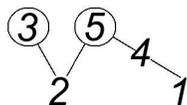


Figure 2: Planar graph of  $\tau = 32541 \in \mathfrak{S}_5$ .

### 3. Definition of the map $\varphi$ of Theorem 1.2

Let  $\sigma \in \mathfrak{S}_n$ . We set  $(r, s) = (\text{des}_2(\sigma), \text{inv}_2(\sigma))$ , and

$$\text{DES}_2(\sigma) = \{d_2^k(\sigma), k \in [r]\},$$

$$\text{INV}_2(\sigma) = \{(i_l(\sigma), j_l(\sigma)), l \in [s]\}$$

with  $d_2^k(\sigma) < d_2^{k+1}(\sigma)$  for all  $k$  and  $i_l(\sigma) < i_{l+1}(\sigma)$  for all  $l$ .

We intend to define  $\varphi(\sigma)$  by constructing its planar graph. To do so, we first construct (in Subsection 3.1) a graph  $\mathcal{G}(\sigma)$  made of  $n$  circles or dots organized in ascending or descending slopes such that two consecutive vertices are necessarily in a same descending slope if the first vertex is a circle and the second vertex is a dot. Then, in Subsection 3.2, we label the vertices of this graph with the

integers  $1, 2, \dots, n$  in such a way that, if  $y_i$  is the label of the  $i$ -th vertex  $v_i(\sigma)$  (from left to right) of  $\mathcal{G}(\sigma)$  for all  $i \in [n]$ , then :

1.  $y_i < y_{i+1}$  if and only if  $v_i$  and  $v_{i+1}$  are in a same ascending slope;
2.  $y_i > i$  if and only if  $v_i$  is a circle.

75 The permutation  $\tau = \varphi(\sigma)$  will then be defined as  $y_1 y_2 \dots y_n$ , *i.e.* the permutation whose planar graph is the labelled graph  $\mathcal{G}(\sigma)$ .

With precision, we will obtain

$$\tau(\text{EXC}(\tau)) = \{j_k(\sigma), k \in [s]\}$$

(in particular  $\text{exc}(\tau) = s = \text{inv}_2(\sigma)$ ), and

$$\text{DES}(\tau) = \begin{cases} \{d^k(\sigma), k \in [1, r]\} & \text{if } \widetilde{\text{des}}_2(\sigma) = r, \\ \{d^k(\sigma), k \in [0, r]\} & \text{if } \widetilde{\text{des}}_2(\sigma) = r + 1 \end{cases}$$

for integers  $0 \leq d^0(\sigma) < d^1(\sigma) < \dots < d^r(\sigma) \leq n$  (with  $d^0(\sigma) = 0 \Leftrightarrow \widetilde{\text{des}}_2(\sigma) = \text{des}_2(\sigma)$ ) defined by

$$d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma)$$

(with  $d_2^0(\sigma) := 0$ ) where  $(c_k(\sigma))_{k \in [0, r]}$  is a sequence defined in Subsection 3.1 such that  $\sum_k c_k(\sigma) = \text{inv}_2(\sigma) = \text{exc}(\tau)$ . Thus, we will obtain  $\text{des}(\tau) = \widetilde{\text{des}}_2(\sigma)$  and  $\text{maj}(\tau) = \text{maj}_2(\sigma) + \text{exc}(\tau)$ .

80 *3.1. Construction of the unlabelled graph  $\mathcal{G}(\sigma)$*

We set  $(d_2^0(\sigma), \sigma(d_2^0(\sigma))) = (0, n + 1)$  and  $(d_2^{r+1}(\sigma), \sigma(n + 1)) = (n, 0)$ .

For all  $k \in [r]$ , we define the top  $t_k(\sigma)$  of the 2-descent  $d_2^k(\sigma)$  as

$$t_k(\sigma) = \min\{d_2^l(\sigma), 1 \leq l \leq k, d_2^l(\sigma) = d_2^k(\sigma) - (k - l)\}, \quad (4)$$

in other words  $t_k(\sigma)$  is the smallest 2-descent  $d_2^l(\sigma)$  such that the 2-descents  $d_2^l(\sigma), d_2^{l+1}(\sigma), \dots, d_2^k(\sigma)$  are consecutive integers.

85 The following algorithm provides a sequence  $(c_k^0(\sigma))_{k \in [0, r]}$  of nonnegative integers.

**Algorithm 3.1.** Let  $I_r(\sigma) = \text{INV}_2(\sigma)$ . For  $k$  from  $r = \text{des}_2(\sigma)$  down to 0, we consider the set  $S_k(\sigma)$  of sequences  $(i_{k_1}(\sigma), i_{k_2}(\sigma), \dots, i_{k_m}(\sigma))$  such that :

1.  $(i_{k_p}(\sigma), j_{k_p}(\sigma)) \in I_k(\sigma)$  for all  $p \in [m]$ ;
- 90 2.  $t_k(\sigma) \leq i_{k_1}(\sigma) < i_{k_2}(\sigma) < \dots < i_{k_m}(\sigma)$ ;
3.  $\sigma(i_{k_1}(\sigma)) < \sigma(i_{k_2}(\sigma)) < \dots < \sigma(i_{k_m}(\sigma))$ .

The *length* of such a sequence is defined as  $l = \sum_{p=1}^m n_p$  where  $n_p$  is the number of *consecutive* 2-inversions whose beginning is  $i_{k_p}$ , *i.e.* the maximal number  $n_p$  of 2-inversions  $(i_{k_p^1}(\sigma), j_{k_p^1}(\sigma)), (i_{k_p^2}(\sigma), j_{k_p^2}(\sigma)), \dots, (i_{k_p^{n_p}}(\sigma), j_{k_p^{n_p}}(\sigma))$  such that  
95  $k_p^1 = k_p$  and  $j_{k_p^i}(\sigma) = i_{k_p^{i+1}}(\sigma)$  for all  $i$ . If  $I_k(\sigma) \neq \emptyset$ , we consider the sequence  $(i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \dots, i_{k_m^{max}}(\sigma)) \in I_k(\sigma)$  whose length  $l^{max} = \sum_{p=1}^m n_p^{max}$  is maximal and whose elements  $i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \dots, i_{k_m^{max}}(\sigma)$  also are maximal (as integers). Then,

- if  $I_k(\sigma) \neq \emptyset$ , we set  $c_k^0(\sigma) = l^{max}$  and

$$I_{k-1}(\sigma) = I_k(\sigma) \setminus (\cup_{p=1}^m \{(i_{k_i^{max}}(\sigma), j_{k_i^{max}}(\sigma)), i \in [n_p^{max}]\});$$

- else we set  $c_k^0(\sigma) = 0$  and  $I_{k-1}(\sigma) = I_k(\sigma)$ .

100 **Example 3.2.** Consider the permutation  $\sigma = 549321867 \in \mathfrak{S}_9$ , with  $\text{DES}_2(\sigma) = \{3, 7\}$  and  $I_2(\sigma) = \text{INV}_2(\sigma) = \{(1, 2), (2, 4), (3, 7), (4, 5), (5, 6), (7, 9)\}$ . In Figure 3 are depicted the  $\text{des}_2(\sigma) + 1 = 3$  steps  $k \in \{2, 1, 0\}$  (at each step, the 2-inversions of the maximal sequence are drawn in red then erased at the following step) :

$k=2$	5 4 <span style="border: 1px solid black; padding: 2px;">9</span> 3 2 1 <span style="border: 1px solid black; padding: 2px;">8</span> 6 7	$c_2^0=1$
$k=1$	5 4 <span style="border: 1px solid black; padding: 2px;">9</span> 3 2 1 <span style="border: 1px solid black; padding: 2px;">8</span> 6 7	$c_1^0=2$
$k=0$	5 4 <span style="border: 1px solid black; padding: 2px;">9</span> 3 2 1 <span style="border: 1px solid black; padding: 2px;">8</span> 6 7	$c_0^0=3$

Figure 3: Computation of  $(c_k^0(\sigma))_{k \in [0, \text{des}_2(\sigma)]}$  for  $\sigma = 549321867 \in \mathfrak{S}_9$ .

- 105 •  $k = 2$  : there is only one legit sequence  $(i_{k_1}(\sigma)) = (7)$ , whose length is  $l = n_1 = 1$ . We set  $c_2^0(\sigma) = 1$  and  $I_1(\sigma) = I_2(\sigma) \setminus \{(7, 9)\}$ .
- $k = 1$  : there are three legit sequences  $(i_{k_1}(\sigma)) = (3)$  (whose length is  $l = n_1 = 1$ ) then  $(i_{k_1}(\sigma)) = (4)$  (whose length is  $l = n_1 = 2$ ) and  $(i_{k_1}(\sigma)) = (5)$  (whose length is  $l = n_1 = 1$ ). The maximal sequence is the  
110 second one, hence we set  $c_1^0(\sigma) = 2$  and  $I_0(\sigma) = I_1(\sigma) \setminus \{(4, 5), (5, 6)\}$ .
- $k = 0$  : there are three legit sequences  $(i_{k_1}(\sigma), i_{k_2}(\sigma)) = (1, 3)$  (whose length is  $l = n_1 + n_2 = 2 + 1 = 3$ ) then  $(i_{k_1}(\sigma), i_{k_2}(\sigma)) = (2, 3)$  (whose length is  $l = n_1 + n_2 = 1 + 1 = 2$ ) and  $(i_{k_1}(\sigma)) = (3)$  (whose length is  $l = n_1 = 1$ ). The maximal sequence is the first one, hence we set  $c_0^0(\sigma) = 3$   
115 and  $I_{-1}(\sigma) = I_0(\sigma) \setminus \{(1, 2), (2, 4), (3, 7)\} = \emptyset$ .

**Lemma 3.3.** *The sum  $\sum_k c_k^0(\sigma)$  equals  $inv_2(\sigma)$  (i.e.  $I_{-1}(\sigma) = \emptyset$ ) and, for all  $k \in [0, r] = [0, des_2(\sigma)]$ , we have  $c_k^0(\sigma) \leq d_2^{k+1}(\sigma) - d_2^k(\sigma)$  with equality only if  $c_{k+1}^0(\sigma) > 0$  (where  $c_{r+1}^0(\sigma)$  is defined as 0).*

**Proof.** With precision, we show by induction that, for all  $k \in \{des_2(\sigma), \dots, 1, 0\}$ ,  
120 the set  $I_{k-1}(\sigma)$  contains no 2-inversion  $(i, j)$  such that  $d_2^k(\sigma) < i$ . For  $k = 0$ , it will mean  $I_{-1}(\sigma) = \emptyset$  (recall that  $d_2^0(\sigma)$  has been defined as 0).

★ If  $k = des_2(\sigma) = r$ , the goal is to prove that  $c_r^0(\sigma) < n - d_2^r(\sigma)$ . Suppose there exists a sequence  $(i_{k_1}(\sigma), i_{k_2}(\sigma), \dots, i_{k_m}(\sigma))$  of length  $c_r^0(\sigma) \geq n - d_2^r(\sigma)$  with  $t_r(\sigma) \leq i_{k_1}(\sigma) < i_{k_2}(\sigma) < \dots < i_{k_m}(\sigma)$ . In particular, there exist  
125  $c_r^0(\sigma) \geq n - d_2^r(\sigma)$  2-inversions  $(i, j)$  such that  $d_2^r(\sigma) < j$ , which forces  $c_r^0(\sigma)$  to equal  $n - d_2^r(\sigma)$  and every  $j > d_2^r(\sigma)$  to be the arrival of a 2-inversion  $(i, j)$  such that  $t_r(\sigma) \leq i$ . In particular, this is true for  $j = d_2^r(\sigma) + 1$ , which is absurd because  $\sigma(i) \geq \sigma(d_2^r(\sigma)) > \sigma(d_2^r(\sigma) + 1) + 1$  for all  $i \in [t_r(\sigma), d_2^r(\sigma)]$ . Therefore  $c_r^0(\sigma) < n - d_2^r(\sigma)$ . Also, it is easy to see that every  $i > d_2^r(\sigma)$   
130 that is the beginning of a 2-inversion  $(i, j)$  necessarily appears in the maximal sequence  $(i_{k_1}^{max}(\sigma), i_{k_2}^{max}(\sigma), \dots, i_{k_m}^{max}(\sigma))$  whose length defines  $c_r^0(\sigma)$ , hence  $(i, j) \notin I_{r-1}(\sigma)$ .

★ Now, suppose that  $c_k^0(\sigma) \leq d_2^{k+1}(\sigma) - d_2^k(\sigma)$  for some  $k \in [des_2(\sigma)]$  with

equality only if  $c_{k+1}^0(\sigma) > 0$ , and that no 2-inversion  $(i, j)$  with  $d_2^k(\sigma) < i$  belongs  
135 to  $I_{k-1}(\sigma)$ .

If  $t_{k-1}(\sigma) = t_k(\sigma)$  (i.e., if  $d_2^{k-1}(\sigma) = d_2^k(\sigma) - 1$ ), since  $I_{k-1}(\sigma)$  does not  
contain any 2-inversion  $(i, j)$  with  $d_2^k(\sigma) < i$ , then  $c_{k-1}^0(\sigma) \leq 1 = d_2^k(\sigma) -$   
 $d_2^{k-1}(\sigma)$ . Moreover, if  $c_{k-1}^0(\sigma) = 1$ , then there exists a 2-inversion  $(i, j) \in$   
 $I_{k-1}(\sigma) \subset I_k(\sigma)$  such that  $i \in [t_{k-1}(\sigma), d_2^k(\sigma)]$ . Consequently  $(i)$  was a legit  
140 sequence for the computation of  $c_k^0(\sigma)$  at the previous step (because  $t_k(\sigma) =$   
 $t_{k-1}(\sigma)$ ), which implies  $c_k^0(\sigma)$  equals at least the length of  $(i)$ . In particular  
 $c_k^0(\sigma) > 0$ .

Else, consider a sequence  $(i_{k_1}(\sigma), i_{k_2}(\sigma), \dots, i_{k_m}(\sigma))$  that fits the three con-  
ditions of Algorithm 3.1 at the step  $k - 1$ . In particular  $t_{k-1}(\sigma) \leq i_{k_1}(\sigma)$ . Also  
145  $i_{k_m}(\sigma) \leq d_2^k(\sigma)$  by hypothesis. Since  $\sigma(i_{k_p}(\sigma)) < \sigma(i_{k_{p+1}}(\sigma))$  for all  $p$ , and since  
 $\sigma(t_{k-1}(\sigma)) > \sigma(t_{k-1}(\sigma) + 1) > \dots > \sigma(d_2^{k-1}(\sigma)) > \sigma(d_2^{k-1}(\sigma) + 1)$ , then only  
one element of the set  $[t_{k-1}(\sigma), d_2^{k-1}(\sigma) + 1]$  may equal  $i_{k_p}(\sigma)$  for some  $p \in [m]$ .  
Thus, the length  $l$  of the sequence verifies  $l \leq d_2^k(\sigma) - d_2^{k-1}(\sigma)$ , with equality  
only if  $i_{k_m}(\sigma) = d_2^k(\sigma)$  (which implies  $c_k^0(\sigma) > 0$  as in the previous paragraph).  
150 In particular, this is true for  $l = c_{k-1}^0(\sigma)$ .

Finally, as for  $k = \text{des}_2(\sigma)$ , every  $i \in [d_2^{k-1}(\sigma) + 1, d_2^k(\sigma)]$  that is the  
beginning of a 2-inversion  $(i, j)$  necessarily appears in the maximal sequence  
 $(i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \dots, i_{k_m^{max}}(\sigma))$  whose length defines  $c_{k-1}^0(\sigma)$ , hence  $(i, j) \notin$   
 $I_{k-2}(\sigma)$ .

155 So the lemma is true by induction. □

**Definition 3.4.** We define a graph  $\mathcal{G}^0(\sigma)$  made of circles and dots organised in  
ascending or descending slopes, by plotting :

- for all  $k \in [0, r]$ , an ascending slope of  $c_k^0(\sigma)$  circles such that the first  
circle has abscissa  $d_2^k(\sigma) + 1$  and the last circle has abscissa  $d_2^k(\sigma) + c_k^0(\sigma)$   
(if  $c_k^0(\sigma) = 0$ , we plot nothing). All the abscissas are distinct because

$$d_2^0(\sigma) + c_0 < d_2^1(\sigma) + c_1 < \dots < d_2^r(\sigma) + c_r$$

in view of Lemma 3.3;

- dots at the remaining  $n-s = n - \text{inv}_2(\sigma)$  abscissas from 1 to  $n$ , in ascending and descending slopes with respect to the descents and ascents of the word  $\omega(\sigma)$  defined by

$$\omega(\sigma) = \sigma(u_1(\sigma))\sigma(u_2(\sigma)) \dots \sigma(u_{n-s}(\sigma)) \quad (5)$$

where

$$\{u_1(\sigma) < u_2(\sigma) < \dots < u_{n-s}(\sigma)\} := \mathfrak{S}_n \setminus \{i_1(\sigma) < i_2(\sigma) < \dots < i_s(\sigma)\}.$$

**Example 3.5.** The permutation  $\sigma_0 = 425736981 \in \mathfrak{S}_9$  (with  $\text{DES}_2(\sigma_0) = \{1, 4, 8\}$  and  $\text{INV}_2(\sigma_0) = \{(1, 5), (2, 9), (4, 6), (7, 8)\}$ ), which yields the sequence  $(c_k^0(\sigma_0))_{k \in [0,3]} = (1, 1, 2, 0)$  (see Figure 4 where all the 2-inversions involved in the computation of a same  $c_k^0(\sigma_0)$  are drawn in a same color) and the word  $\omega(\sigma_0) = 53681$ , provides the unlabelled graph  $\mathcal{G}^0(\sigma_0)$  depicted in Figure 5.

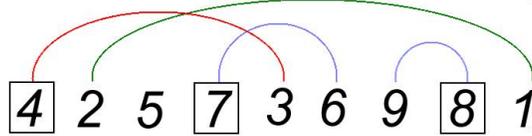


Figure 4:  $(c_k^0(\sigma_0))_{k \in [0,3]} = (1, 1, 2, 0)$ .

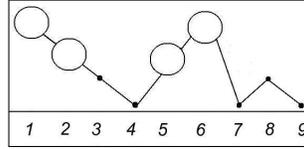


Figure 5: Graph  $\mathcal{G}^0(\sigma_0)$ .

The following lemma is easy.

**Lemma 3.6.** For all  $i \in [n]$ , if the  $i$ -th vertex (from left to right)  $v_i^0(\sigma)$  of  $\mathcal{G}^0(\sigma)$  is a dot and if  $i$  is a descent of  $\mathcal{G}^0(\sigma)$  (i.e., if  $v_i^0(\sigma)$  and  $v_{i+1}^0(\sigma)$  are two dots in a same descending slope) whereas  $i \notin \text{DES}_2(\sigma)$ , let  $k_i$  such that

$$d_2^{k_i}(\sigma) + c_{k_i}^0(\sigma) < i < d_2^{k_i+1}(\sigma)$$

165 and let  $p \in [n - s]$  such that  $v_i^0(\sigma)$  is the  $p$ -th dot (from left to right) of  $\mathcal{G}^0(\sigma)$ .

Then :

1.  $u_p(\sigma)$  is the greatest integer  $u < d_2^{k_i+1}(\sigma)$  that is not the beginning of a 2-inversion of  $\sigma$ ;
2.  $u_{p+1}(\sigma)$  is the smallest integer  $u > d_2^{k_i+1}(\sigma)$  that is not a 2-descent or the  
170 beginning of a 2-inversion of  $\sigma$ ;
3.  $c_k^0(\sigma) > 0$  for all  $k$  such that  $d_2^{k_i+1}(\sigma) \leq d_2^k(\sigma) \leq u_{p+1}(\sigma)$ .

In particular  $c_{k_i+1}^0(\sigma) > 0$ .

Lemma 3.6 motivates the following definition.

**Definition 3.7.** For  $i$  from 1 to  $n - 1$ , let  $k_i \in [0, r]$  such that

$$d_2^{k_i}(\sigma) + c_{k_i}^0(\sigma) < i < d_2^{k_i+1}(\sigma).$$

If  $i$  fits the conditions of Lemma 3.6, then we define a sequence  $(c_k^i(\sigma))_{k \in [0, r]}$  by

$$\begin{aligned} c_{k_i}^i(\sigma) &= c_{k_i}^{i-1}(\sigma) + 1, \\ c_{k_i+1}^i(\sigma) &= c_{k_i+1}^{i-1}(\sigma) - 1, \\ c_k^i(\sigma) &= c_k^{i-1}(\sigma) \text{ for all } k \notin \{k_i, k_i + 1\}. \end{aligned}$$

Else, we define  $(c_k^i(\sigma))_{k \in [0, r]}$  as  $(c_k^{i-1}(\sigma))_{k \in [0, r]}$ .

The final sequence  $(c_k^n(\sigma))_{k \in [0, r]}$  is denoted by

$$(c_k(\sigma))_{k \in [0, r]}.$$

175 By construction, and from Lemma 3.3, the sequence  $(c_k(\sigma))_{k \in [0, r]}$  has the same properties as  $(c_k^0(\sigma))_{k \in [0, r]}$  detailed in Lemma 3.3.

Consequently, we may define an unlabelled graph

$$\mathcal{G}(\sigma)$$

by replacing  $(c_k^0(\sigma))_{k \in [0, r]}$  with  $(c_k(\sigma))_{k \in [0, r]}$  in Definition 3.4.

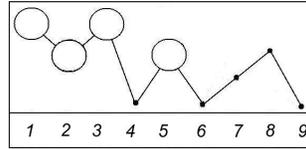


Figure 6: Graph  $\mathcal{G}(\sigma_0)$ .

**Example 3.8.** In the graph  $\mathcal{G}^0(\sigma)$  depicted in Figure 5 where  $\sigma_0 = 425736981 \in \mathfrak{S}_9$ , we can see that the dot  $v_3^0(\sigma_0)$  is a descent whereas  $3 \notin \text{DES}_2(\sigma_0)$ , hence, from the sequence  $(c_k^0(\sigma_0))_{k \in [0,3]} = (1, 1, 2, 0)$ , we compute  $(c_k(\sigma_0))_{k \in [0,3]} = (1, 2, 1, 0)$  and we obtain the graph  $\mathcal{G}(\sigma_0)$  depicted in Figure 6.

Let  $v_1(\sigma), v_2(\sigma), \dots, v_n(\sigma)$  be the  $n$  vertices of  $\mathcal{G}(\sigma)$  from left to right.

By construction, the descents of the unlabelled graph  $\mathcal{G}(\sigma)$  (*i.e.*, the integers  $i \in [n-1]$  such that  $v_i(\sigma)$  and  $v_{i+1}(\sigma)$  are in a same descending slope) are the integers

$$d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma)$$

for all  $k \in [0, r]$ .

### 3.2. Labelling of the graph $\mathcal{G}(\sigma)$

#### 3.2.1. Labelling of the circles

We intend to label the circles of  $\mathcal{G}(\sigma)$  with the integers

$$j_1(\sigma), j_2(\sigma), \dots, j_s(\sigma).$$

**Algorithm 3.9.** For all  $i \in [n]$ , if the vertex  $v_i(\sigma)$  is a circle (hence  $i < n$ ), we label it first with the set

$$[i+1, n] \cap \{j_1(\sigma), j_2(\sigma), \dots, j_s(\sigma)\}.$$

Afterwards, if a circle  $v_i(\sigma)$  is found in a descending slope such that there exists a quantity of  $a$  circles above  $v_i(\sigma)$ , and in an ascending slope such that there exists a quantity of  $b$  circles above  $v_i(\sigma)$ , then we remove the  $a+b$  greatest integers from the current label of  $v_i(\sigma)$  (this set necessarily had at least  $a+b+1$

190 elements) and the smallest integer from every of the  $a + b$  labels of the  $a + b$  circles above  $v_i(\sigma)$  in the two related slopes. At the end of this step, if an integer  $j_k(\sigma)$  appears in only one label of a circle  $v_i(\sigma)$ , then we replace the label of  $v_i(\sigma)$  with  $j_k(\sigma)$ .

Finally, we replace every label that is still a set by the unique integer it may contain with respect to the order of the elements in the sequence

$$(j_1(\sigma), j_2(\sigma), \dots, j_s(\sigma))$$

(from left to right).

195 **Example 3.10.** For  $\sigma_0 = 425736981$  (see Figure 4) whose graph  $\mathcal{G}(\sigma_0)$  is depicted in Figure 6, we have  $s = \text{inv}_2(\sigma) = 4$  and  $\{j_1(\sigma_0), j_2(\sigma_0), j_3(\sigma_0), j_4(\sigma_0)\} = \{5, 6, 8, 9\}$ , which provides first the graph labelled by sets depicted in Figure 7. Afterwards, since the circle  $v_2(\sigma_0)$  is in a descending slope with  $a = 1$  circle above it (the vertex  $v_1(\sigma_0)$ ) and in an ascending slope with also  $b = 1$  circle above it (the vertex  $v_3(\sigma_0)$ ), then we remove the  $a + b = 2$  integers 8 and 9 from its label, which becomes  $\{5, 6\}$ , and we remove 5 from the labels of  $v_1(\sigma_0)$  and  $v_3(\sigma_0)$ . Also, since the label of  $v_2(\sigma_0)$  is the only set that contains 5, then we label  $v_2(\sigma_0)$  with 5 (see Figure 8). Finally, the sequence  $(j_1(\sigma_0), j_2(\sigma_0), j_3(\sigma_0), j_4(\sigma_0)) = (5, 9, 6, 8)$  gives the order (from left to right) of  
 200  
 205 apparition of the remaining integers 6, 8, 9 (see Figure 9).

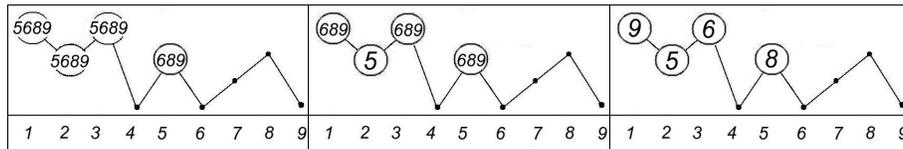


Figure 7

Figure 8

Figure 9

### 3.2.2. Labelling of the dots

Let

$$\{p_1(\sigma) < p_2(\sigma) < \dots < p_{n-s}(\sigma)\} = [n] \setminus \bigsqcup_{k=0}^r d_2^k(\sigma), d^k(\sigma).$$

We intend to label the dots  $\{v_{p_i(\sigma)}(\sigma), i \in [n-s]\}$  of  $\mathcal{G}(\sigma)$  with the elements of

$$\{1 = e_1(\sigma) < e_2(\sigma) < \dots < e_{n-s}(\sigma)\} = [n] \setminus \{j_1(\sigma), j_2(\sigma), \dots, j_s(\sigma)\}.$$

**Algorithm 3.11.** 1. For all  $k \in [n-s]$ , we label first the dot  $v_{p_k(\sigma)}(\sigma)$  with the set

$$[\min(p_k(\sigma), u_k(\sigma))] \cap ([n] \setminus \{j_1(\sigma), j_2(\sigma), \dots, j_s(\sigma)\})$$

where  $u_1(\sigma), u_2(\sigma), \dots, u_{n-s}(\sigma)$  are the integers introduced in (5).

2. Afterwards, similarly as for the labelling of the circles, if a dot  $v_i(\sigma)$  is found in a descending slope such that  $a$  dots are above  $v_i(\sigma)$ , and in an ascending slope such that  $b$  dots are above  $v_i(\sigma)$ , then we remove the  $a+b$  greatest integers from the current label of  $v_i(\sigma)$  and the smallest integer from every of the  $a+b$  labels of the dots above  $v_i(\sigma)$  in the two related slopes. At the end of this step, if an integer  $l$  appears in only one label of a dot  $v_i(\sigma)$ , then we replace the label of  $v_i(\sigma)$  with  $l$ .
3. Finally, for  $k$  from 1 to  $n-s$ , let

$$w_1^k(\sigma) < w_2^k(\sigma) < \dots < w_{q_k(\sigma)}^k(\sigma) \quad (6)$$

such that

$$\{p_{w_i^k(\sigma)}(\sigma), i\} = \{p_i(\sigma), e_k(\sigma) \text{ appear in the label of } p_i(\sigma)\},$$

and let  $i(k) \in [q_k(\sigma)]$  such that

$$\sigma(u_{w_{i(k)}^k(\sigma)}(\sigma)) = \min\{\sigma(u_{w_i^k(\sigma)}(\sigma)), i \in [q_k(\sigma)]\}.$$

- Then, we replace the label of the dot  $p_{w_{i(k)}^k(\sigma)}(\sigma)$  with the integer  $e_k(\sigma)$  and we erase  $e_k(\sigma)$  from any other label (and if an integer  $l$  appears in only one label of a dot  $v_i(\sigma)$ , then we replace the label of  $v_i(\sigma)$  with  $l$ ).

**Example 3.12.** For  $\sigma_0 = 425736981$  whose graph  $\mathcal{G}(\sigma_0)$  has its circles labelled in Figure 9, the sequence  $(u_1(\sigma_0), u_2(\sigma_0), u_3(\sigma_0), u_4(\sigma_0), u_5(\sigma_0)) = (3, 5, 6, 8, 9)$  provides first the graph labelled by sets depicted in Figure 10. The rest of the algorithm goes from  $k = 1$  to  $n-s = 9-4 = 5$ .

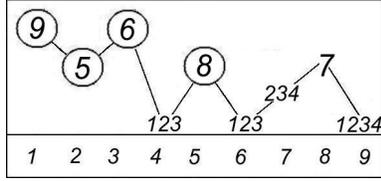


Figure 10

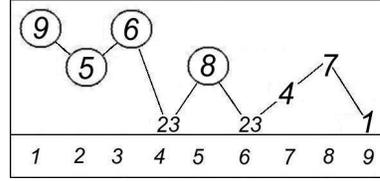


Figure 11

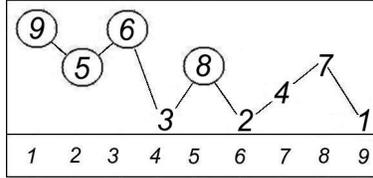


Figure 12: Labelled graph  $\mathcal{G}(\sigma_0)$ .

- $k = 1$  : in Figure 10, the integer  $e_1(\sigma_0) = 1$  appears in the labels of the dots  $v_{p_1(\sigma_0)}(\sigma_0) = v_4(\sigma_0)$ ,  $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$  and  $v_{p_5(\sigma_0)}(\sigma_0) = v_9(\sigma_0)$ , so, from

$$(\sigma_0(u_1(\sigma_0)), \sigma_0(u_2(\sigma_0)), \sigma_0(u_5(\sigma_0))) = (5, 3, 1),$$

we label the dot  $v_{p_5(\sigma_0)}(\sigma_0) = v_9(\sigma_0)$  with the integer  $e_1(\sigma_0) = 1$  and we erase 1 from any other label, and since the integer 4 now only appears in the label of the dot  $v_7(\sigma_0)$ , then we label  $v_7(\sigma_0)$  with 4 (see Figure 11).

- $k = 2$ : in Figure 11, the integer  $e_2(\sigma_0) = 2$  appears in the labels of the dots  $v_{p_1(\sigma_0)}(\sigma_0) = v_4(\sigma_0)$  and  $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$  so, from

$$(\sigma_0(u_1(\sigma_0)), \sigma_0(u_2(\sigma_0))) = (5, 3),$$

225

we label the dot  $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$  with the integer  $e_2(\sigma_0) = 2$  and we erase 2 from any other label, which provides the graph labelled by integers depicted in Figure 12.

- The three steps  $k = 3, 4, 5$  change nothing because every dot of  $\mathcal{G}(\sigma_0)$  is already labelled by an integer at the end of the previous step.

230

So the final version of the labelled graph  $\mathcal{G}(\sigma_0)$  is the one depicted in Figure 12.

### 3.3. Definition of $\varphi(\sigma)$

By construction of the labelled graph  $\mathcal{G}(\sigma)$ , the word  $y_1 y_2 \dots y_n$  (where the integer  $y_i$  is the label of the vertex  $v_i(\sigma)$  for all  $i$ ) obviously is a permutation of the set  $[n]$ , whose planar graph is  $\mathcal{G}(\sigma)$ .

235 We define  $\varphi(\sigma) \in \mathfrak{S}_n$  as this permutation.

For the example  $\sigma_0 = 425736981 \in \mathfrak{S}_9$  whose labelled graph  $\mathcal{G}(\sigma_0)$  is depicted in Figure 12, we obtain  $\varphi(\sigma_0) = 956382471 \in \mathfrak{S}_9$ .

In general, by construction of  $\tau = \varphi(\sigma) \in \mathfrak{S}_n$ , we have

$$\tau(\text{EXC}(\tau)) = \{j_k(\sigma), k \in [\text{inv}_2(\sigma)]\} \quad (7)$$

and

$$\text{DES}(\tau) = \begin{cases} \{d^k(\sigma), k \in [1, \text{des}_2(\sigma)]\} & \text{if } c_0(\sigma) = 0 (\Leftrightarrow d^0(\sigma) = 0), \\ \{d^k(\sigma), k \in [0, \text{des}_2(\sigma)]\} & \text{otherwise.} \end{cases} \quad (8)$$

Equality (7) provides

$$\text{exc}(\tau) = \text{inv}_2(\sigma).$$

By  $d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma)$  for all  $k$ , Equality (8) provides

$$\text{maj}(\tau) = \text{maj}_2(\sigma) + \sum_{k \geq 0} c_k(\sigma),$$

and by definition of  $(c_k(\sigma))_k$  and Lemma 3.3 we have  $\sum_{k \geq 0} c_k(\sigma) = \sum_{k \geq 0} c_k^0(\sigma) = \text{inv}_2(\sigma) = \text{exc}(\tau)$  hence

$$\text{maj}(\tau) - \text{exc}(\tau) = \text{maj}_2(\sigma).$$

Finally, it is easy to see that  $\widetilde{\text{des}}_2(\sigma) = \text{des}_2(\sigma)$  if and only if  $c_0(\sigma) = 0$ , so Equality (8) also provides

$$\text{des}(\tau) = \widetilde{\text{des}}_2(\sigma).$$

As a conclusion, we obtain

$$(\text{maj}(\tau) - \text{exc}(\tau), \text{des}(\tau), \text{exc}(\tau)) = (\text{maj}_2(\sigma), \widetilde{\text{des}}_2(\sigma), \text{inv}_2(\sigma))$$

as required by Theorem 1.2.

#### 4. Construction of $\varphi^{-1}$

To end the proof of Theorem 1.2, it remains to show that  $\varphi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$  is surjective. Let  $\tau \in \mathfrak{S}_n$ . We introduce integers  $r \geq 0$ ,  $s = \text{exc}(\tau)$ , and

$$0 \leq d^{0,\tau} < d^{1,\tau} < \dots < d^{r,\tau} < n$$

such that

$$\begin{aligned} \text{DES}(\tau) &= \{d^{k,\tau}, k \in [0, r]\} \cap \mathbb{N}_{>0}, \\ d^{0,\tau} &= 0 \Leftrightarrow \tau(1) = 1. \end{aligned}$$

240 In particular  $\text{des}(\tau) = \begin{cases} r & \text{if } \tau(1) = 1, \\ r + 1 & \text{otherwise.} \end{cases}$

For all  $k \in [0, r]$ , we define

$$\begin{aligned} c_k^\tau &= \text{EXC}(\tau) \cap ]d^{k-1,\tau}, d^{k,\tau}] \text{ (with } d^{-1,\tau} := 0), \\ d_2^{k,\tau} &= d^{k,\tau} - c_k^\tau. \end{aligned}$$

We have

$$0 = d_2^{0,\tau} < d_2^{1,\tau} < \dots < d_2^{r,\tau} < n$$

and similarly as Formula 4, we define

$$t_k^\tau = \min\{d_2^{l,\tau}, 1 \leq l \leq k, d_2^{k,\tau} = d_2^{l,\tau} - (k - l)\} \quad (9)$$

for all  $k \in [r]$ .

We intend to construct a graph  $\mathcal{H}(\tau)$  which is the linear graph of permutation  $\sigma \in \mathfrak{S}_n$  such that  $\varphi(\sigma) = \tau$ .

##### 4.1. Skeleton of the graph $\mathcal{H}(\tau)$

245 We consider a graph  $\mathcal{H}(\tau)$  whose vertices  $v_1^\tau, v_2^\tau, \dots, v_n^\tau$  (from left to right) are  $n$  dots, aligned in a row, among which we box the  $d_2^{k,\tau}$ -th vertex  $v_{d_2^{k,\tau}}^\tau$  for all  $k \in [r]$ . We also draw the end of an arc of circle above every vertex  $v_j^\tau$  such that  $j = \tau(i)$  for some  $i \in \text{EXC}(\tau)$ .

For the example  $\tau_0 = 956382471 \in \mathfrak{S}_9$  (whose planar graph is depicted in Figure 12), we have  $r = \text{des}(\tau_0) - 1 = 3$  and

$$\begin{aligned} (c_k^{\tau_0})_{k \in [0,3]} &= (1, 2, 1, 0), \\ (d_2^{k, \tau_0})_{k \in [0,3]} &= (1 - 1, 3 - 2, 5 - 1, 8 - 0) = (0, 1, 4, 8), \\ \tau_0(\text{EXC}(\tau_0)) &= \{5, 6, 8, 9\}, \end{aligned}$$

and we obtain the graph  $\mathcal{H}(\tau_0)$  depicted in Figure 13.

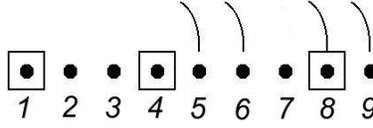


Figure 13: Incomplete graph  $\mathcal{H}(\tau_0)$ .

250 In general, by definition of  $\varphi(\sigma)$  for all  $\sigma \in \mathfrak{S}_n$ , if  $\varphi(\sigma) = \tau$ , then  $r = \text{des}_2(\sigma)$  and  $d_2^k(\sigma)$  (respectively  $c_k(\sigma), d^k(\sigma), t_k(\sigma)$ ) equals  $d_2^{k, \tau}$  (resp.  $c_k^\tau, d^{k, \tau}, t_k^\tau$ ) for all  $k \in [0, r]$  and  $\{j_l(\sigma), l \in [\text{inv}_2(\sigma)]\} = \tau(\text{EXC}(\tau))$ . Consequently, the linear graph of  $\sigma$  necessarily have the same skeleton as that of  $\mathcal{H}(\tau)$ .

The following lemma is easy.

255 **Lemma 4.1.** *If  $\tau = \varphi(\sigma)$  for some  $\sigma \in \mathfrak{S}_n$ , then :*

1. *If  $j = \tau(l)$  with  $l \in \text{EXC}(\tau)$  such that  $l \in ]d_2^{k, \tau}, d^{k, \tau}]$ , and if  $(i, j) \in \text{INV}_2(\sigma)$ , then  $t_k^\tau \leq i$ .*
2. *A pair  $(i, i + 1)$  cannot be a 2-inversion of  $\sigma$  if  $i \in \text{DES}_2(\sigma)$  ( $\Leftrightarrow$  if the vertex  $v_i^\tau$  of  $\mathcal{H}(\tau)$  is boxed).*
- 260 3. *For all pair  $(l, l') \in \text{EXC}(\tau)^2$ , if the labels of the two circles  $v_l(\sigma)$  and  $v_{l'}(\sigma)$  can be exchanged without modifying the skeleton of  $\mathcal{G}(\sigma)$ , let  $i$  and  $i'$  such that  $(i, l) \in \text{INV}_2(\sigma)$  and  $(i', l') \in \text{INV}_2(\sigma)$ , then  $i < i' \Leftrightarrow l < l'$ .*

Consequently, in order to construct the linear graph of a permutation  $\sigma \in \mathfrak{S}_n$  such that  $\tau = \varphi(\sigma)$  from  $\mathcal{H}(\tau)$ , it is necessary to extend the arcs of circles of  $\mathcal{H}(\tau)$  to reflect the three facts of Lemma 4.1. When a vertex is necessarily the beginning of an arc of circle, we draw the beginning of an arc of circle above it.

When there is only one vertex  $v_i^\tau$  that can be the beginning of an arc of circle, we complete the latter by making it start from  $v_i^\tau$ .

**Example 4.2.** For  $\tau_0 = 956382471 \in \mathfrak{S}_9$ , the graph  $\mathcal{H}(\tau_0)$  becomes as depicted in Figure 14. Note that the arc of circle ending at  $v_6^{\tau_0}$  cannot begin at  $v_5^{\tau_0}$  because

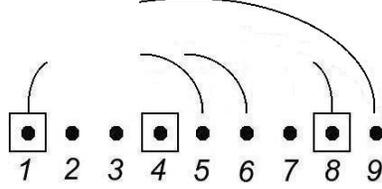


Figure 14: Incomplete graph  $\mathcal{H}(\tau_0)$ .

270

otherwise, from the third point of Lemma 4.1, and since  $(6, 8) = (\tau_0(l), \tau_0(l'))$  with  $3 = l < l' = 5$ , it would force the arc of circle ending at  $v_8^{\tau_0}$  to begin at  $v_{i'}^{\tau_0}$  with  $6 \leq i'$ , which is absurd because a permutation  $\sigma \in \mathfrak{S}_9$  whose linear graph would be of the kind  $\mathcal{H}(\tau_0)$  would have  $c_2(\sigma) = 2 \neq 1 = c_2^{\tau_0}$ . Also, still  
 275 in view of the third point of Lemma 4.1, and since  $\tau_0^{-1}(9) < \tau_0^{-1}(6)$ , the arc of circle ending at  $v_9^{\tau_0}$  must start before the arc of circle ending at  $v_6^{\tau_0}$ , hence the configuration of  $\mathcal{H}(\tau_0)$  in Figure 14.

The following two facts are obvious.

**Facts 4.3.** If  $\tau = \varphi(\sigma)$  for some  $\sigma \in \mathfrak{S}_n$ , then :

280

1. A vertex  $v_i^\tau$  of  $\mathcal{H}(\tau)$  is boxed if and only if  $i \in \text{DES}_2(\sigma)$ . In that case, in particular  $i$  is a descent of  $\sigma$ .
2. If a pair  $(i, i + 1)$  is not a 2-descent of  $\sigma$  and if  $v_i^\tau$  is not boxed, then  $i$  is an ascent of  $\sigma$ , *i.e.*  $\sigma(i) < \sigma(i + 1)$ .

To reflect Facts 4.3, we draw an ascending arrow (respectively a descending  
 285 arrow) between the vertices  $v_i^\tau$  and  $v_{i+1}^\tau$  of  $\mathcal{H}(\tau)$  whenever it is known that  $\sigma(i) < \sigma(i + 1)$  (resp.  $\sigma(i) > \sigma(i + 1)$ ) for all  $\sigma \in \mathfrak{S}_n$  such that  $\varphi(\sigma) = \tau$ .

For the example  $\tau_0 = 956382471 \in \mathfrak{S}_9$ , the graph  $\mathcal{H}(\tau_0)$  becomes as depicted in Figure 15. Note that it is not known yet if there is an ascending or descending arrow between  $v_7^{\tau_0}$  and  $v_8^{\tau_0}$ .

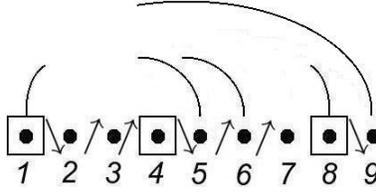


Figure 15: Incomplete graph  $\mathcal{H}(\tau_0)$ .

290 4.2. Completion and labelling of  $\mathcal{H}(\tau)$

The following lemma is analogous to the third point of Lemma 4.1 for the dots instead of the circles and follows straightly from the definition of  $\varphi(\sigma)$  for all  $\sigma \in \mathfrak{S}_n$ .

**Lemma 4.4.** *Let  $\sigma \in \mathfrak{S}_n$  such that  $\varphi(\sigma) = \tau$ . For all pair  $(l, l') \in ([n] \setminus EXC(\tau))^2$ ,  
 295 if the labels of the two dots  $v_l(\sigma)$  and  $v_{l'}(\sigma)$  can be exchanged without modifying the skeleton of  $\mathcal{G}(\sigma)$ , let  $k$  and  $k'$  such that  $l = p_k(\sigma)$  and  $l' = p_{k'}(\sigma)$ , then  $\tau(l) < \tau(l') \Leftrightarrow \sigma(u_k(\sigma)) < \sigma(u_{k'}(\sigma))$ .*

Now, the ascending and descending arrows between the vertices of  $\mathcal{H}(\tau)$  introduced earlier, and Lemma 4.4, induce a partial order on the set  $\{v_i^\tau, i \in [n]\}$ :

300 **Definition 4.5.** We define a partial order  $\succ$  on  $\{v_i^\tau, i \in [n]\}$  by :

- $v_i^\tau \prec v_{i+1}^\tau$  (resp.  $v_i^\tau \succ v_{i+1}^\tau$ ) if there exists an ascending (resp. descending) arrow between  $v_i^\tau$  and  $v_{i+1}^\tau$ ;
- $v_i^\tau \succ v_j^\tau$  (with  $i < j$ ) if there exists an arc of circle from  $v_i^\tau$  to  $v_j^\tau$ ;
- if two vertices  $v_i^\tau$  and  $v_j^\tau$  are known to be respectively the  $k$ -th and  $k'$ -th  
 305 vertices of  $\mathcal{H}(\tau)$  that cannot be the beginning of a complete arc of circle, let  $l$  and  $l'$  be respectively the  $k$ -th and  $k'$ -th non-exceedance point of  $\tau$  (from left to right), if  $(l, l')$  fits the conditions of Lemma 4.4, then we set  $v_i^\tau \prec v_j^\tau$  (resp.  $v_i^\tau \succ v_j^\tau$ ) if  $\tau(l) < \tau(l')$  (resp.  $\tau(l) > \tau(l')$ ).

**Example 4.6.** For the example  $\tau_0 = 956382471$ , according to the first point of Definition 4.5, the arrows of Figure 15 provide

$$v_1^{\tau_0} \succ v_2^{\tau_0} \prec v_3^{\tau_0} \prec v_4^{\tau_0} \succ v_5^{\tau_0} \prec v_6^{\tau_0} \prec v_7^{\tau_0}$$

and

$$v_8^{\tau_0} \succ v_9^{\tau_0}.$$

**Definition 4.7.** A vertex  $v_i^\tau$  of  $\mathcal{H}(\tau)$  is said to be *minimal* on a subset  $S \subset [n]$  if  $v_i^\tau \not\prec v_j^\tau$  for all  $j \in S$ .

Let

$$1 = e_1^\tau < e_2^\tau < \dots < e_{n-s}^\tau$$

be the non-exceedance values of  $\tau$  (i.e., the labels of the dots of the planar graph of  $\tau$ ).

**Algorithm 4.8.** Let  $S = [n]$  and  $l = 1$ . While the vertices  $\{v_i^\tau, i \in [n]\}$  have not all been labelled with the elements of  $[n]$ , apply the following algorithm.

1. If there exists a unique minimal vertex  $v_i^\tau$  of  $\tau$  on  $S$ , we label it with  $l$ , then we set  $l := l + 1$  and  $S := S \setminus \{v_i^\tau\}$ . Afterwards,
  - (a) If  $v_i^\tau$  is the ending of an arc of circle starting from a vertex  $v_j^\tau$ , then we label  $v_j^\tau$  with the integer  $l$  and we set  $l := l + 1$  and  $S := S \setminus \{v_j^\tau\}$ .
  - (b) If  $v_i^\tau$  is the arrival of an incomplete arc of circle (in particular  $i = \tau(l)$  for some  $l \in \text{EXC}(\tau)$ ), we intend to complete the arc by making it start from a vertex  $v_j^\tau$  for some integer  $j \in [t_k^\tau, j[$  (where  $l \in ]d_2^{k,\tau}, d^{k,\tau}]$ ) in view of the first point of Lemma 4.1. We choose  $v_j^\tau$  as the rightest minimal vertex on  $[t_k^\tau, j[ \cap S$  from which it may start in view of the third point of Lemma 4.1, and we label this vertex  $v_j^\tau$  with the integer  $l$ . Then we set  $l := l + 1$  and  $S := S \setminus \{v_j^\tau\}$ .

Now, if there exists an arc of circle from  $v_j^\tau$  (for some  $j$ ) to  $v_i^\tau$ , we apply steps (a),(b) and (c) to the vertex  $v_j^\tau$  in place of  $v_i^\tau$ .

2. Otherwise, let  $k \geq 0$  be the number of vertices  $v_i^\tau$  that have already been labelled and that are not the beginning of an arc of circles. Let

$$l_1 < l_2 < \dots < l_q$$

be the integers  $l \in [n]$  such that  $l \geq \tau(j) \geq e_{k+1}^\tau$  and such that we can exchange the labels of dots  $\tau(l)$  and  $e_{k+1}^\tau$  in the planar graph of  $\tau$  without

330 modifying the skeleton of the graph. It is easy to see that  $q$  is precisely the number of minimal vertices of  $\tau$  on  $S$ . Let  $l_{i_{k+1}} = \tau^{-1}(e_{k+1}^\tau)$  and let  $v_j^\tau$  be the  $i_{k+1}$ -th minimal vertex (from left to right) on  $S$ . We label  $v_j^\tau$  with  $l$ , then we set  $l := l + 1$  and  $S := S \setminus \{v_j^\tau\}$ , and we apply steps 1.(a), (b) and (c) to  $v_j^\tau$  instead of  $v_i^\tau$ .

By construction, the labelled graph  $\mathcal{H}(\tau)$  is the linear graph of a permutation  $\sigma \in \mathfrak{S}_n$  such that

$$\text{DES}_2(\sigma) = \{d_2^{k,\tau}, k \in [r]\}$$

and

$$\{j_l(\sigma), l \in [\text{inv}_2(\sigma)]\} = \tau(\text{EXC}(\tau)).$$

335 **Example 4.9.** Consider  $\tau_0 = 956382471 \in \mathfrak{S}_9$  whose unlabelled and incomplete graph  $\mathcal{H}(\tau_0)$  is depicted in Figure 15.

- As stated in Example 4.6, the minimal vertices of  $\tau_0$  on  $S = [9]$  are  $(v_2^{\tau_0}, v_5^{\tau_0}, v_9^{\tau_0})$ . Following step 2 of Algorithm 4.8,  $k = 0$  and the integers  $l \in [9]$  such that  $\tau_0(l) \geq e_{k+1}^{\tau_0} = 1$  and such that the labels of dots  $\tau_0(l)$  can be exchanged with 1 in the planar graph of  $\tau_0$  (see Figure 12) are  $(l_1, l_2, l_3) = (4, 6, 9)$ . By  $\tau_0^{-1}(1) = 9 = l_3$ , we label the third minimal vertex on  $[9]$ , *i.e.* the vertex  $v_9^{\tau_0}$ , with the integer  $l = 1$ .

340 Afterwards, following step 1.(b), since  $v_9^{\tau_0}$  is the arrival of an incomplete arc of circle starting from a vertex  $v_j^{\tau_0}$  with  $1 = t_1^{\tau_0} \leq j$ , and with  $j < 5$  because that arc of circle must begin before the arc of circle ending at  $v_6^{\tau_0}$  in view of Fact 3 of Lemma 4.1, we complete that arc of circle by making it start from the unique minimal vertex  $v_j^{\tau_0}$  on  $[1, 5[$ , *i.e.*  $j = 2$ , and we label  $v_2^{\tau_0}$  with the integer  $l = 2$  (see Figure 16). Note that as from now we know that the arc of circle ending at  $v_5^{\tau_0}$  necessarily begins at  $v_1^{\tau_0}$ , because otherwise  $v_1^{\tau_0}$ , being the beginning of an arc of circle, would be the beginning of the arc of circle ending at  $v_6^{\tau_0}$ , which is absurd in view of Fact 3 of Lemma 4.1 because  $\tau_0^{-1}(9) < \tau^{-1}(6)$ , so we complete that arc of circle by making it start from  $v_1^{\tau_0}$ , which has been depicted in Figure 16.

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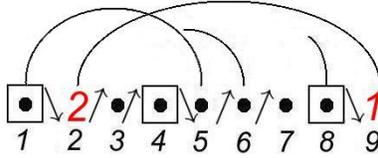


Figure 16: Beginning of the labelling of  $\mathcal{H}(\tau_0)$ .

We now have  $S = [9] \setminus \{2, 9\}$  and  $l = 3$ .

- 355 • From Figure 16, the minimal vertices on  $S = [9] \setminus \{2, 9\}$  are  $(v_3^{\tau_0}, v_5^{\tau_0})$ . Following step 2 of Algorithm 4.8,  $k = 1$  and the integers  $l \in [9]$  such that  $l\tau_0(l) \geq e_{k+1}^{\tau_0} = 2$  and such that the labels of dots  $\tau_0(l)$  can be exchanged with 2 in the planar graph of  $\tau_0$  (see Figure 12) are  $(l_1, l_2) = (4, 6)$ . By  $\tau_0^{-1}(2) = 6 = l_2$ , we label the second minimal vertex on  $S$ , *i.e.* the vertex
   
360  $v_5^{\tau_0}$ , with the integer  $l = 3$ .

Afterwards, following step 1.(a), since  $v_5^{\tau_0}$  is the arrival of the arc of circle starting from the vertex  $v_1^{\tau_0}$ , we label  $v_1^{\tau_0}$  with the integer  $l = 4$  (see Figure 17).

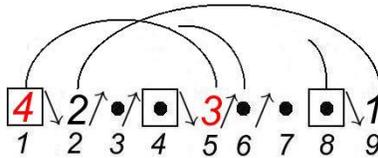


Figure 17: Beginning of the labelling of  $\mathcal{H}(\tau_0)$ .

We now have  $S = [9] \setminus \{1, 2, 5, 9\}$  and  $l = 5$ .

- 365 • From Figure 17, the minimal vertices on  $S = \{3, 4, 6, 7, 8\}$  are  $(v_3^{\tau_0}, v_6^{\tau_0})$ . Following step 2 of Algorithm 4.8,  $k = 2$  and the integers  $l \in [9]$  such that  $l\tau_0(l) \geq e_{k+1}^{\tau_0} = 3$  and such that the labels of dots  $\tau_0(l)$  can be exchanged with 3 in the planar graph of  $\tau_0$  (see Figure 12) are  $(l_1, l_2) = (4, 7)$ . By  $\tau_0^{-1}(3) = 4 = l_1$ , we label the first minimal vertex on  $S$ , *i.e.* the vertex
   
370  $v_3^{\tau_0}$ , with the integer  $l = 5$  (see Figure 18). Note that as from now we know that the arc of circle ending at  $v_6^{\tau_0}$  necessarily begins at  $v_4^{\tau_0}$  since

it it is the only vertex left it may start from. Consequently, the arc of circle ending at  $v_8^{\tau_0}$  necessarily starts from  $v_7^{\tau_0}$  (otherwise it would start from  $v_6^{\tau_0}$ , which is prevented by Definition 4.5 because we cannot have  $v_8^{\tau_0} \prec v_6^{\tau_0} \prec v_7^{\tau_0} \prec v_8^{\tau_0}$ ). The two latter remarks are taken into account in Figure 18.

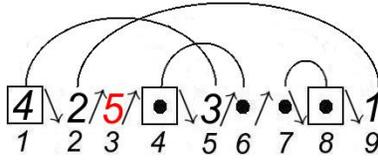


Figure 18: Beginning of the labelling of  $\mathcal{H}(\tau_0)$ .

We now have  $S = \{4, 6, 7, 8\}$  and  $l = 6$ .

- From Figure 18, there is only one minimal vertex on  $S = \{4, 6, 7, 8\}$ , *i.e.* the vertex  $v_6^{\tau_0}$ . Following step 1 of Algorithm 4.8, we label  $v_6^{\tau_0}$  with  $l = 6$ . Afterwards, following step 1.(a), since  $v_6^{\tau_0}$  is the arrival of the arc of circle starting from the vertex  $v_4^{\tau_0}$ , we label  $v_4^{\tau_0}$  with the integer  $l = 7$  (see Figure 19).

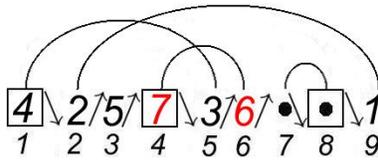


Figure 19: Beginning of the labelling of  $\mathcal{H}(\tau_0)$ .

We now have  $S = \{7, 8\}$  and  $l = 8$ .

- From Figure 19, there is only one minimal vertex on  $S = \{7, 8\}$ , *i.e.* the vertex  $v_8^{\tau_0}$ . Following step 1 of Algorithm 4.8, we label  $v_8^{\tau_0}$  with  $l = 8$ . Afterwards, following step 1.(a), since  $v_8^{\tau_0}$  is the arrival of the arc of circle starting from the vertex  $v_7^{\tau_0}$ , we label  $v_7^{\tau_0}$  with the integer  $l = 9$  (see Figure 20).

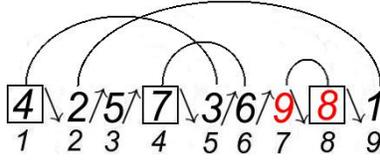


Figure 20: Labelled graph  $\mathcal{H}(\tau_0)$ .

As a conclusion, the graph  $\mathcal{H}(\tau_0)$  is the linear graph of the permutation  
390  $\sigma_0 = 425736981 \in \mathfrak{S}_9$ , which is mapped to  $\tau_0$  by  $\varphi$ .

**Proposition 4.10.** *We have  $\varphi(\sigma) = \tau$ , hence  $\varphi$  is bijective.*

**Proof.** By construction, for all  $k \in [0, \text{des}_2(\sigma)] = [0, r]$ ,

$$d_2^k(\sigma) = d^{k,\tau} - c_k^\tau,$$

$$c_k(\sigma) = c_k^\tau,$$

$$d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma) = d_2^{k,\tau} + c_k^\tau = d^{k,\tau},$$

so  $\mathcal{G}(\sigma)$  has the same skeleton as the planar graph of  $\tau$ , *i.e.*  $\text{DES}(\varphi(\sigma)) = \text{DES}(\tau)$  and  $\text{EXC}(\varphi(\sigma)) = \text{EXC}(\tau)$ .

The labels of the circles of  $\mathcal{G}(\sigma)$  are the elements of

$$\{j_l(\sigma), l \in [s]\} = \tau(\text{EXC}(\tau)),$$

and by construction of  $\sigma$ , every pair  $(l, l') \in \text{EXC}(\tau)^2$  such that we can exchange the labels  $\tau(l)$  and  $\tau(l')$  in the planar graph of  $\tau$  is such that

$$i < i' \Leftrightarrow l < l'$$

where  $(i, \tau(l))$  and  $(i, \tau(l'))$  are the two corresponding 2-inversions of  $\sigma$ . Consequently, by definition of  $\varphi(\sigma)$ , the labels of the circles of  $\mathcal{G}(\sigma)$  appear in the same  
395 order as in the planar graph of  $\tau$  (*i.e.*  $\varphi(\sigma)(i) = \tau(i)$  for all  $i \in \text{EXC}(\varphi(\sigma)) = \text{EXC}(\tau)$ ).

As a consequence, the dots of  $\mathcal{G}(\sigma)$  and the planar graph of  $\tau$  are labelled by the elements

$$1 = e_1(\sigma) = e_1^\tau < e_2(\sigma) = e_2^\tau < \dots < e_{n-s}(\sigma) = e_{n-s}^\tau.$$

As for the labels of the circles, to show that the above integers appear in the same order among the labels of  $\mathcal{G}(\sigma)$  and the planar graph of  $\tau$ , it suffices to prove that

$$\varphi(\sigma)^{-1}(e_i^{\tau}) < \varphi(\sigma)^{-1}(e_j^{\tau}) \Leftrightarrow \tau^{-1}(e_i^{\tau}) < \tau^{-1}(e_j^{\tau})$$

for all pair  $(i, j)$  such that we can exchange the labels  $e_i^{\tau}$  and  $e_j^{\tau}$  in the planar graph of  $\tau$  (hence in  $\mathcal{G}(\sigma)$  since the two graphs have the same skeleton). This is  
 400 guaranteed by Definition 4.5 because the vertices  $v_i^{\tau}$  that are not the beginning of an arc of circle correspond with the labels of the dots of the planar graph of  $\tau$ .

As a conclusion, the planar graph of  $\tau$  is in fact  $\mathcal{G}(\sigma)$ , *i.e.*  $\tau = \varphi(\sigma)$ .  $\square$

## 5. Open problem

405 In view of Formula (2) and Theorem 1.2, it is natural to look for a bijection  $\mathfrak{S}_n \rightarrow \mathfrak{S}_n$  that maps  $(\text{maj}_2, \widetilde{\text{des}}_2, \text{inv}_2)$  to  $(\text{amaj}_2, \widetilde{\text{asc}}_2, \text{ides})$ .

Recall that  $\text{ides} = \text{des}_2$  and that for a permutation  $\sigma \in \mathfrak{S}_n$ , the equality  $\widetilde{\text{des}}_2(\sigma) = \text{des}_2(\sigma)$  is equivalent to  $\varphi(\sigma)(1) = 1$ , which is similar to the equivalence  $\widetilde{\text{asc}}_2(\tau) = \text{asc}_2(\tau) \Leftrightarrow \tau(1) = 1$  for all  $\tau \in \mathfrak{S}_n$ .

Note that if  $\text{DES}_2(\sigma) = \bigsqcup_{p=1}^r [i_p, j_p]$  with  $j_p + 1 < i_{p+1}$  for all  $p$ , the permutation  $\pi = \rho_1 \circ \rho_2 \circ \dots \circ \rho_r \circ \sigma$ , where  $\rho_p$  is the  $(j_p - i_p + 2)$ -cycle

$$\begin{pmatrix} i_p & i_p + 1 & i_p + 2 & \dots & j_p & j_p + 1 \\ \sigma(j_p + 1) & \sigma(j_p) & \sigma(j_p - 1) & \dots & \sigma(i_p + 1) & \sigma(i_p) \end{pmatrix}$$

410 for all  $p$ , is such that  $\text{DES}_2(\sigma) \subset \text{ASC}_2(\pi)$  and  $\text{INV}_2(\sigma) = \text{INV}_2(\pi)$ . One can try to get rid of the eventual unwanted 2-ascents  $i \in \text{ASC}_2(\pi) \setminus \text{DES}_2(\sigma)$  by composing  $\pi$  with adequate permutations.

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