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# A transmission problem across a fractal self-similar interface

Yves Achdou<sup>\*</sup>, Thibaut Deheuvels<sup>†</sup>

## Abstract

We consider a transmission problem where the interior domain has infinitely ramified structures. Transmission between the interior and exterior domains only takes place at the fractal component of the boundary of the interior domain. We also consider transmission problems in which the interior domain is obtained by stopping the self-similar construction after a finite number of steps; the transmission condition is then posed on a prefractal approximation of the fractal interface. We prove the convergence in the sense of Mosco of the energy forms associated with these problems to the energy form of the limit problem. In particular, this implies the convergence of the solutions of the approximated problems to the solution of the problem with fractal interface. The proof relies in particular on an extension property of the interior domain.

Emphasis is put on the geometry of the ramified domain. The convergence result is proved when the fractal interface has no self-contact, and for a particular geometry with self-contacts for which an extension result is proved.

## 1 Introduction

Transmission problems naturally arise in various fields of physics and have been extensively studied. An introduction to this class of problems can be found in [18] and several applications are detailed in [12]. Such problems were more recently studied in the case when the interface is irregular, Lipschitz continuous or even fractal. These problems find many applications, such as the study of rough electrodes in electrochemistry or diffusion across irregular membranes in physiological processes, etc. (see [38, 17]). Several transmission problems with fractal interfaces have been studied in the case of the Koch flake or the Sierpiński gasket in 2D and 3D (see *e.g.* [26, 35, 36, 27, 10]).

This paper deals with transmission problems between two domains  $\Omega_{\text{int}}$  and  $\Omega_{\text{ext}}$  where  $\Omega_{\text{int}}$  is a ramified bounded domain as defined in Section 2.2 (see Figure 2). The domain  $\Omega_{\text{int}}$  presents infinitely many ramifications, and its boundary contains a fractal self-similar set  $\Gamma$  which plays the role of the interface.

The domain  $\Omega_{\text{int}}$  can be seen as a bidimensional idealization of the bronchial tree, for example. Since the exchanges between the lungs and the circulatory system take place only in the last generations of the bronchial tree (the smallest structures), it is reasonable to consider transmission problems with specific transmission conditions across the fractal boundary  $\Gamma$  of the tree. We will however limit ourselves to simple transmission conditions.

The fractal boundary  $\Gamma$  belongs to a family of self-similar sets introduced by Mandelbrot *et al.* in [28]. Elliptic boundary value problems in the domain  $\Omega_{\text{int}}$  have been studied in [3], and traces and extension results for these domains have been proved in [1, 14].

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The considered problem can be formally stated as

$$\left\{ \begin{array}{ll} -\Delta u = f & \text{in } \Omega_{\text{int}} \cup \Omega_{\text{ext}}, \\ [u] = 0 & \text{on } \Gamma, \\ [\partial_n u] = \alpha u & \text{on } \Gamma, \\ \partial_n u_{\text{int}} = \partial_n u_{\text{ext}} = 0 & \text{on } \partial\Omega \setminus \Gamma, \\ \partial_n u_{\text{ext}} = 0, \quad u_{\text{int}} = u_0 & \text{on } \Gamma^0, \\ \partial_n u_{\text{ext}} = 0 & \text{on } \partial D, \end{array} \right. \quad (P)$$

where  $\alpha > 0$ ,  $D$  is a regular bounded open domain in the plane with  $\overline{D} = \overline{\Omega_{\text{int}}} \cup \overline{\Omega_{\text{ext}}}$  such that  $\Gamma \Subset D$ , and  $\Gamma^0$  is a line segment included in the boundary of  $\Omega_{\text{int}}$ . The sets  $\Omega_{\text{int}}$  and  $\Omega_{\text{ext}}$  are disjoint subdomains of  $D$ , and  $[u]$  (*resp.*  $[\partial_n u]$ ) denotes the jump of  $u$  (*resp.* of the “normal derivative”  $\partial_n u$  of  $u$ ) across the fractal set  $\Gamma$ . Since the interface  $\Gamma$  is fractal, the normal derivative on  $\Gamma$  has to be understood in a suitable weak sense, which will be made precise later.

The goal is to study approximations  $(P_n)$  of problem  $(P)$ , obtained by stopping the construction of the ramified domain  $\Omega_{\text{int}}$  at step  $n$ . The interfaces in the problems  $(P_n)$  are called prefractal approximations of the fractal set  $\Gamma$ . They consist of disjoint finite unions of line segments, which makes problems  $(P_n)$  much simpler than problem  $(P)$ . A natural question is to understand the asymptotic behavior of the problems  $(P_n)$  as  $n \rightarrow \infty$ , and in particular to investigate the convergence of the solutions  $u_n$  of the problems  $(P_n)$  to the solution  $u$  of problem  $(P)$ .

**Remark 1.** *In contrast with the references [26, 35, 36, 27, 10], the boundary value problem does not only involve transmission conditions at the interface between  $\Omega_{\text{int}}$  and  $\Omega_{\text{ext}}$ : there are also homogeneous Neumann conditions on the polygonal part of  $\partial\Omega_{\text{int}}$ ; as a consequence, the traces of  $u_{\text{int}}$  and  $u_{\text{ext}}$  on this set do not match a priori. Coping with these discontinuities will be a difficulty in studying the convergence of the solutions  $u_n$  of the problems  $(P_n)$  to the solution  $u$  of problem  $(P)$ .*

**Remark 2.** *We have chosen that the source term in  $(P)$  appear both in the Dirichlet boundary condition on  $\Gamma_0$  for  $u_{\text{int}}$  and in the Poisson equations in  $\Omega_{\text{int}}$  and  $\Omega_{\text{ext}}$ ; this is of course completely arbitrary.*

A crucial step in the study of the asymptotic behavior of  $(P_n)$  is the question of extending functions defined in the domain  $\Omega_{\text{int}}$ . More precisely, it is of particular importance that  $\Omega_{\text{int}}$  should satisfy a  $W^{1,p}$ -extension property for some  $p \in [1, \infty]$ , *i.e.* there exists a bounded linear operator

$$\mathcal{E} : W^{1,p}(\Omega_{\text{int}}) \rightarrow W^{1,p}(\mathbb{R}^2)$$

such that  $\mathcal{E}(u)|_{\Omega_{\text{int}}} = u$  for all  $u \in W^{1,p}(\Omega_{\text{int}})$ . The domains satisfying this property for all  $p \in [1, \infty]$  will often be referred to as *Sobolev extension domains*.

It is well known that every Lipschitz domain in  $\mathbb{R}^n$ , that is every domain whose boundary is locally the graph of a Lipschitz function, is a Sobolev extension domain. Calderón proved the  $W^{1,p}$ -extension property for  $p \in (1, \infty)$  (see [9]), and Stein extended the result to the cases  $p = 1$  and  $p = \infty$  (see [39]). In [20], Jones generalized this result to the class of  $(\varepsilon, \delta)$ -domains, also referred to as Jones domain, or locally uniform domains (see [29]). In dimension two, the definition of  $(\varepsilon, \delta)$ -domains is equivalent to that of quasi-disks, see [30]. This extension result is almost optimal in the plane, in the sense that every plane finitely connected Sobolev extension domain is an  $(\varepsilon, \delta)$ -domain, see [20, 30]. The case of an unbounded domain in  $\mathbb{R}^2$  has been studied in [41]. Extension properties for domains which do not have the  $(\varepsilon, \delta)$ -property have been studied *e.g.* in [31], where the authors examine in particular the case of domains with cusps.

When the fractal boundary  $\Gamma$  of the domains  $\Omega_{\text{int}}$  studied in this paper has no self-contact,  $\Omega_{\text{int}}$  can be proved to be an  $(\varepsilon, \delta)$ -domain, and is therefore a Sobolev extension domain. However, in the case when the boundary self-intersects,  $\Omega_{\text{int}}$  does not have the  $W^{1,p}$ -extension property for all  $p \geq 1$ . In particular, the extension property does not hold for  $p = 2$ , which is the relevant case since the variational formulations naturally involve the spaces  $H^1(\Omega_{\text{int}})$ . In the particular geometry considered in Section 6, it will be proved that the transmission condition imposed on  $\Gamma$  yields a better regularity of the trace on  $\Gamma$ , for functions belonging to the function space arising in the variational formulation. It is possible to deduce from this trace result that there exists a continuous extension operator from  $H^1(\Omega_{\text{int}})$  to  $H^1(\mathbb{R}^2)$

in this case (see Theorem 9).

The main question investigated in this paper is the question of the convergence in the sense of Mosco of the energy forms associated with problem  $(P_n)$  to the energy form of the problem  $(P)$ . The notion of Mosco-convergence, or  $M$ -convergence, was introduced in [33], see also [34]. It is a stronger convergence in the space of forms than  $\Gamma$ -convergence. In particular, it also implies the convergence of minimizers of the energy forms to the minimizer of the limit form. The  $M$ -convergence of forms is equivalent to the convergence of the resolvent operators associated with the relaxed forms in the strong operator topology (see [34]).

The main results of this paper are Theorems 6 and 9. Theorem 6 is about the convergence of the energy forms associated with  $(P_n)$  in the sense of Mosco to the energy form associated with  $(P)$ , in the case when the fractal interface has no self-contact. The proof uses the extension operator from  $H^1(\Omega_{\text{int}})$  to  $H^1(\mathbb{R}^2)$  as a main ingredient. The existence of a continuous extension operator from  $H^1(\Omega_{\text{int}})$  to  $H^1(\mathbb{R}^2)$ , for a particular geometry where the fractal interface self-intersects, is stated in Theorem 9. As a consequence, the proof of Theorem 6 can be reproduced in this case, to show the  $M$ -convergence of the energy forms.

The paper is organized as follows: the geometry of the interior and exterior domains is detailed in Section 2, as well as the prefractal geometry. Section 3 is devoted to the study of the function spaces involved in the paper, and emphasis is put on trace and extension results for the domains under study. The considered transmission problem is described in Section 4. Section 5 is devoted to the  $M$ -convergence of the energy forms associated with the problem with prefractal interface to the energy form of the problem with fractal interface, in the case when the boundary of  $\Omega_{\text{int}}$  has no self-contact. In Section 6, a particular geometry in which the fractal part of the boundary of  $\Omega_{\text{int}}$  self-intersects is considered; an extension result is proved in this particular case and the  $M$ -convergence of the energy forms follows.

## 2 The geometry

### 2.1 The fractal interface

#### 2.1.1 Definitions

Consider four real numbers  $r, \beta_1, \beta_2, \theta$  such that  $0 < r < 1/\sqrt{2}$ ,  $\beta_1 > 0$ ,  $\beta_2 > 0$  and  $0 \leq \theta < \pi/2$ . Let  $f_i$ ,  $i = 1, 2$  be the two similitudes in  $\mathbb{R}^2$  given by

$$f_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\beta_1 \\ \beta_2 \end{pmatrix} + r \begin{pmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \end{pmatrix},$$

$$f_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + r \begin{pmatrix} x_1 \cos \theta + x_2 \sin \theta \\ -x_1 \sin \theta + x_2 \cos \theta \end{pmatrix}.$$

The two similitudes have the same dilation ratio  $r$  and opposite angles  $\pm\theta$ . One can obtain  $f_2$  by composing  $f_1$  with the symmetry with respect to the vertical axis  $\{x_1 = 0\}$ .

Let  $\Gamma$  denote the self-similar set associated with the similitudes  $f_1$  and  $f_2$ , *i.e.*  $\Gamma$  is the unique compact subset of  $\mathbb{R}^2$  such that

$$\Gamma = f_1(\Gamma) \cup f_2(\Gamma).$$

It was stated in [28] (see [13] for a complete proof) that for any  $\theta$ ,  $0 \leq \theta < \pi/2$ , there exists a unique positive number  $r_\theta^* < 1/\sqrt{2}$  which only depends on the angle  $\theta$  such that

- ◇ if  $0 < r < r_\theta^*$ , then  $\Gamma$  is totally disconnected,
- ◇ if  $r = r_\theta^*$ , then  $\Gamma$  is connected.

In the following, we will often write  $r^*$  instead of  $r_\theta^*$ .

**Notations** For every integer  $n > 0$ , we note  $\mathcal{A}_n = \{1, 2\}^n$ . For  $\sigma \in \mathcal{A}_n$ , we note  $f_\sigma$  the similitude  $f_{\sigma_1} \circ \dots \circ f_{\sigma_n}$ . We agree to extend the notation to the case  $n = 0$ :  $f_\sigma = \text{Id}$  if  $\sigma \in \mathcal{A}_0$ . We also introduce the notation  $\mathcal{A} := \bigcup_{n \geq 0} \mathcal{A}_n$ .

For  $\sigma \in \mathcal{A}$ , we note  $\Gamma^\sigma = f_\sigma(\Gamma^0)$ , and for every integer  $n \geq 0$ ,  $\Gamma^n = \bigcup_{\sigma \in \mathcal{A}_n} f_\sigma(\Gamma^0)$ .

### 2.1.2 Hausdorff dimension of $\Gamma$

If  $r \leq r^*$ , then it can be seen that the open set condition (or Moran condition) holds, see [32] or [23] for a definition. The open set condition implies that the Hausdorff dimension of  $\Gamma$  is

$$\dim_H \Gamma = -\frac{\log 2}{\log r}$$

see [32, 23]. If  $0 < \theta < \pi/2$ , then  $0 < r < r^* < 1/\sqrt{2}$  and thus  $\dim_H \Gamma < 2$ .

In the case when  $r = r^*$ , let  $\Xi$  be the set  $\Xi = f_1(\Gamma) \cap f_2(\Gamma)$ . In this case, the fractal set  $\Gamma$  self-intersects, and the images of  $\Xi$  under the similitudes  $f_{\sigma_1} \circ \dots \circ f_{\sigma_n}$ ,  $\sigma_1, \dots, \sigma_n \in \{1, 2\}^n$  can be seen as the multiple points of  $\Gamma$ .

Two situations can occur, depending on the angle  $\theta$  (see [28]):

- if  $\theta$  is not of the form  $\frac{\pi}{2k}$  for any integer  $k > 0$ , then  $\Xi$  is reduced to a single point, and  $\Gamma$  has countably many multiple points,
- if  $\theta$  is of the form  $\frac{\pi}{2k}$  where  $k > 0$  is an integer, then  $\Xi$  is a Cantor set, and the Hausdorff dimension of  $\Xi$ , noted  $\dim_H \Xi$  is  $\frac{\dim_H \Gamma}{2}$ .

### 2.1.3 The self-similar measure $\mu$

Recall the classical result on self-similar measures, see [15, 19] and [23] page 26.

**Theorem 1.** *There exists a unique Borel regular probability measure  $\mu$  on  $\Gamma$  such that for any Borel set  $A \subset \Gamma$ ,*

$$\mu(A) = \frac{1}{2}\mu(f_1^{-1}(A)) + \frac{1}{2}\mu(f_2^{-1}(A)). \quad (1)$$

The measure  $\mu$  is called the *self-similar measure defined in the self-similar triplet*  $(\Gamma, f_1, f_2)$ .

Let  $L^p(\Gamma)$ ,  $p \in [1, +\infty)$  be the space of the measurable functions  $v$  on  $\Gamma$  such that  $\int_\Gamma |v|^p d\mu < \infty$ , endowed with the norm  $\|v\|_{L^p(\Gamma)} = (\int_\Gamma |v|^p d\mu)^{1/p}$ . A Hilbert basis of  $L^2_\mu(\Gamma)$  can be constructed with *e.g.* Haar wavelets.

The space  $W^{s,p}(\Gamma)$  for  $s \in (0, 1)$  and  $p \in [1, \infty)$  is defined as the space of functions  $v \in L^p(\Gamma)$  such that  $|v|_{W^{s,p}(\Gamma)} < \infty$ , where

$$|v|_{W^{s,p}(\Gamma)} = \left( \int_\Gamma \int_\Gamma \frac{|v(x) - v(y)|^p}{|x - y|^{d+ps}} d\mu(x) d\mu(y) \right)^{\frac{1}{p}}.$$

Endowed with the norm  $\|v\|_{W^{s,p}(\Gamma)} = \|v\|_{L^p(\Gamma)} + |v|_{W^{s,p}(\Gamma)}$ , the spaces  $W^{s,p}(\Gamma)$  are Banach spaces. In the special case  $p = 2$ , the space  $W^{s,p}(\Gamma)$  is a Hilbert space, and is noted  $H^s(\Gamma)$ .

**Remark 3.** *It can be proved that if  $s \in (0, 1)$ , then an equivalent norm of the space  $W^{s,p}(\Gamma)$  is given by*

$$\|v\|_{\text{Lip}_s^{p,p}(\Gamma)}^p := \int_\Gamma |v|^p d\mu + \sum_{k \geq 0} 2^{skp} \sum_{\sigma \in \mathcal{A}_k} \int_{\tilde{\Gamma}^\sigma} |v - \langle v \rangle_{\tilde{\Gamma}^\sigma}|^p dx \quad (2)$$

where  $\tilde{\Gamma}^\sigma$  is the intersection with  $\Gamma$  of the segment obtained by expanding the line segment  $\Gamma^\sigma$  with a factor 2 around its center (see [21, 22]). As in the rest of the paper, if  $v$  is a measurable function in a measured space  $(X, m)$ , the notation  $\langle v \rangle_X$  refers to the mean value  $\frac{1}{m(X)} \int_X v dm$ .

## 2.2 The domains $\Omega_{\text{int}}$ and $\Omega_{\text{ext}}$

Call  $P_1 = (-1, 0)$  and  $P_2 = (1, 0)$  and  $\Gamma^0$  the line segment  $\Gamma^0 = [P_1 P_2]$ . Let us assume that  $f_2(P_1)$  and  $f_2(P_2)$  have positive coordinates, *i.e.* that

$$r \cos \theta < \beta_1 \quad \text{and} \quad r \sin \theta < \beta_2. \quad (3)$$

Let us also assume that the open domain  $Y^0$  inside the closed polygonal line joining the points  $P_1, P_2, f_2(P_2), f_2(P_1), f_1(P_2), f_1(P_1), P_1$  in this order must be convex and hexagonal, except if  $\theta = 0$ , in which case it is trapezoidal. With (3), this is true if and only if

$$(\beta_1 - 1) \sin \theta + \beta_2 \cos \theta > 0. \quad (4)$$

Under assumptions (3) and (4), the domain  $Y^0$  is contained in the half-plane  $x_2 > 0$  and symmetric with respect to the vertical axis  $x_1 = 0$ .

Call  $K^0 = \overline{Y^0}$ . It is possible to glue together  $K^0, f_1(K^0)$  and  $f_2(K^0)$  and obtain a new polygonal domain. The assumptions (3) and (4) imply that  $Y^0 \cap f_1(Y^0) = \emptyset$  and  $Y^0 \cap f_2(Y^0) = \emptyset$ .

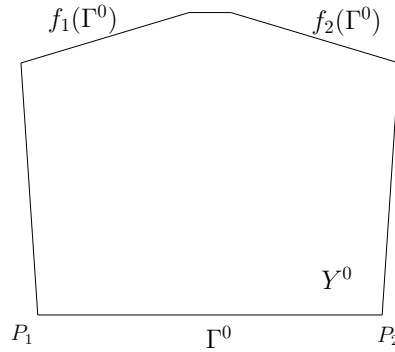


Figure 1: The construction of the first cell  $Y^0$

Let the open domain  $\Omega_{\text{int}}$  (see Figure 2) be defined as follows:

$$\Omega_{\text{int}} = \text{Interior} \left( \bigcup_{\sigma \in \mathcal{A}} f_{\sigma}(K^0) \right), \quad (5)$$

with the notations of §2.1.1.

For a given  $\theta$ , with  $r^*$  defined as above, the following assumption on  $(\alpha, \beta)$  will be made:

**Assumption 1.** For  $0 \leq \theta < \pi/2$ , the parameters  $\alpha$  and  $\beta$  satisfy (3) and (4) for  $r = r^*$ , and are such that

$$\begin{cases} i. & \text{for all } r, 0 < r \leq r^*, \text{ the sets } Y^0, f_{\sigma}(Y^0), \sigma \in \mathcal{A}, \text{ are pairwise disjoint,} \\ ii. & \text{for all } r, 0 < r < r^*, f_1(\overline{\Omega}) \cap f_2(\overline{\Omega}) = \emptyset, \\ iii. & \text{for } r = r^*, f_1(\overline{\Omega}) \cap f_2(\overline{\Omega}) \neq \emptyset. \end{cases}$$

**Remark 4.** Assumption 1 implies that if  $r = r^*$ , then  $f_1(\Omega) \cap f_2(\Omega) = \emptyset$ .

The following theorem, proved in [5], asserts that for all  $\theta, 0 < \theta < \pi/2$ , there exists  $(\alpha, \beta)$  satisfying Assumption 1.

**Theorem 2.** If  $\theta \in [0, \pi/2)$ , then for every  $\beta_1 > r_{\theta}^* \cos \theta$ , there exists  $\beta > 0$  such that for all  $\beta_2 \geq \beta$ ,  $(\alpha, \beta)$  satisfies Assumption 1.

Let  $D$  be an open bounded domain with a Lipschitz boundary, containing the closure of  $\Omega_{\text{int}}$ . The exterior domain  $\Omega_{\text{ext}}$  is defined by

$$\Omega_{\text{ext}} := \text{Interior}(D \setminus \Omega_{\text{int}}). \quad (6)$$

**Remark 5.** The assumption that  $\Omega_{\text{int}} \Subset D$  may be relaxed: in fact, it would be enough to assume that  $\Gamma \Subset D$ .

Displayed on Figure 2 are examples of the domains  $\Omega_{\text{int}}$  and  $\Omega_{\text{ext}}$ , for the parameters  $\theta = \pi/5$  in the left-hand side and  $\theta = \pi/4$  in the right-hand side. Note that these cases correspond to the two different kinds of geometries discussed in §2.1 in the subcritical case  $r < r_\theta^*$ .

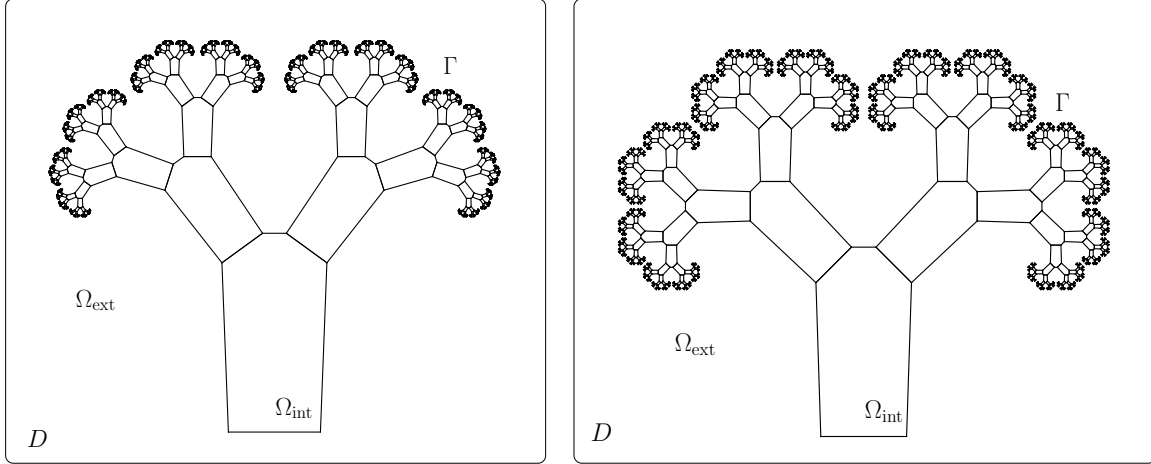


Figure 2: The ramified domain  $\Omega$  for  $\theta = \pi/5$  (left) and  $\theta = \pi/4$  (right) when  $r < r^*$ ,  $\beta_1 = 0.7$ ,  $\beta_2 = 4$ .

### 2.3 The truncated domain $\Omega_{\text{int}}^n$ and the prefractal interface

For every integer  $n \geq 0$ , the truncated domain  $\Omega_{\text{int}}^n$  is defined by

$$\Omega_{\text{int}}^n = \text{Interior} \left( \bigcup_{0 \leq k \leq n} \bigcup_{\sigma \in \mathcal{A}_k} f_\sigma(K^0) \right), \quad (7)$$

with the notations of §2.1.1. As above, the exterior domain associated to  $\Omega_{\text{int}}^n$  is

$$\Omega_{\text{ext}}^n = \text{Interior}(D \setminus \Omega_{\text{int}}^n). \quad (8)$$

Note that the set  $\Gamma^n$  defined in §2.1.1 is a part of the boundary of  $\Omega_{\text{int}}^n$ . The sets  $\Gamma^n$ ,  $n \geq 0$ , will be referred to as *prefractal approximations* of the fractal set  $\Gamma$ .

## 3 Function spaces

Hereafter, we consider a domain  $\Omega_{\text{int}}$  as defined in 2.2, with  $\theta$  in  $[0, \pi/2)$  and  $r \leq r_\theta^*$ , and we assume that the parameters  $\alpha, \beta$  are such that Assumption 1 is satisfied.

We define  $W^{1,p}(\Omega)$ ,  $p = 1, \dots, \infty$ ,  $\Omega = \Omega_{\text{int}}$  or  $\Omega = \Omega_{\text{ext}}$ , to be the space of functions in  $L^p(\Omega)$  with first order partial derivatives in  $L^p(\Omega)$ .

The space  $W^{1,p}(\Omega)$  is a Banach space with the norm  $\left( \|u\|_{L^p(\Omega)}^p + \|\frac{\partial u}{\partial x_1}\|_{L^p(\Omega)}^p + \|\frac{\partial u}{\partial x_2}\|_{L^p(\Omega)}^p \right)^{1/p}$ , see for example [6]. Elementary calculus shows that  $\|u\|_{W^{1,p}(\Omega)} := \left( \|u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega)}^p \right)^{1/p}$  is an equivalent norm, with  $\|\nabla u\|_{L^p(\Omega)}^p := \int_\Omega |\nabla u|^p$  and  $|\nabla u| = \sqrt{|\frac{\partial u}{\partial x_1}|^2 + |\frac{\partial u}{\partial x_2}|^2}$ .

In the special case  $p = 2$ , the space  $W^{1,p}(\Omega)$  is a Hilbert space, and is noted  $H^1(\Omega)$ .

The spaces  $W^{1,p}(\Omega_{\text{int}})$  as well as elliptic boundary value problems in  $\Omega_{\text{int}}$  have been studied in [3], with, in particular Poincaré inequalities and a Rellich compactness theorem. The same results in a similar but different geometry were proved by Berger [8] with other methods.

### 3.1 Trace results

#### 3.1.1 The classical definition of traces

We recall the classical definition of a trace operator on  $\partial\omega$  when  $\omega$  is an open subset of  $\mathbb{R}^2$  (see for instance [22] p. 206).

**Definition 1.** Consider an open set  $\omega \subset \mathbb{R}^2$ . The function  $u \in L^1_{\text{loc}}(\omega)$  can be strictly defined at  $x \in \overline{\Omega}$  if the limit

$$\overline{u}(x) = \lim_{r \rightarrow 0} \frac{1}{|B(x,r) \cap \omega|} \int_{B(x,r) \cap \omega} u(y) \, dy \quad (9)$$

exists, where  $|B(x,r) \cap \omega|$  is the 2-dimensional Lebesgue measure of the set  $B(x,r) \cap \omega$ . In this case,  $x$  is said to be a Lebesgue point of  $u$ .

The trace  $u|_{\partial\Omega}$  is defined to be the function given by  $u|_{\partial\Omega}(x) = \overline{u}(x)$  at every point  $x \in \partial\omega$  such that the limit  $\overline{u}(x)$  exists.

**Remark 6.** Recall that for any  $p > 1$ , a function which belongs to  $W^{1,p}(\mathbb{R}^n)$  can be strictly defined except on a set with zero  $p$ -capacity, see for example [16] and [24].

#### 3.1.2 A trace theorem on $\Gamma$

It has been shown in [2] (see Theorem 11) that every function in  $W^{1,p}(\Omega_{\text{int}})$  can be strictly defined on  $\Gamma$   $\mathcal{H}^1$ -almost everywhere, where  $\mathcal{H}^1$  is the one-dimensional Hausdorff measure. Moreover, the following trace result holds.

**Theorem 3.** (see [1])

- If  $r < r^*$ , then for all  $u \in W^{1,p}(\Omega_{\text{int}})$ ,  $u|_{\Gamma} \in W^{1-\frac{2-d}{p},p}(\Gamma)$ , and there exists a constant  $C > 0$  independent of  $u$  such that

$$\|u|_{\Gamma}\|_{W^{1-\frac{2-d}{p},p}(\Gamma)} \leq C \|u|_{\Gamma}\|_{W^{1,p}(\Omega_{\text{int}})}.$$

- If  $r = r^*$ , then for all  $u \in W^{1,p}(\Omega_{\text{int}})$ ,  $u|_{\Gamma} \in W^{s,p}(\Gamma)$  for every  $s < \frac{1}{p}(d - \dim_H \Xi)$ , and there exists a constant  $C' > 0$  independent of  $u$  such that

$$\|u|_{\Gamma}\|_{W^{s,p}(\Gamma)} \leq C' \|u|_{\Gamma}\|_{W^{1,p}(\Omega_{\text{int}})},$$

for every  $s < \frac{1}{p}(d - \dim_H \Xi)$ . Moreover, if  $s > \frac{1}{p}(d - \dim_H \Xi)$ , then  $W^{1,p}(\Omega_{\text{int}})|_{\Gamma} \not\subset W^{s,p}(\Gamma)$ .

**Remark 7.** It can be proved that if  $r \leq r^*$ ,  $\mu(f_1(\Gamma) \cap f_2(\Gamma)) = 0$ . Introduce the set  $\mathcal{N} = \bigcup_{\sigma \in \mathcal{A}} f_{\sigma}(f_1(\Gamma) \cap f_2(\Gamma))$ . For every  $x \in \Gamma \setminus \mathcal{N}$ , there exists a unique  $\sigma_x \in \mathcal{A}_{\infty}$  such that  $x = \lim_{n \rightarrow \infty} f_{\sigma_x(1)} \circ \dots \circ f_{\sigma_x(n)}(O)$  where  $O$  is the origin. The mapping  $\Phi : \Gamma \setminus \mathcal{N} \rightarrow \Gamma^0$  defined by

$$\Phi(x) = \left( -1 + 2 \sum_{i \geq 0} \frac{\sigma_x(i) - 1}{2^i}, 0 \right) \quad (10)$$

is injective, and for every  $v \in L^1(\Gamma^0)$ ,

$$\frac{1}{|\Gamma^0|} \int_{\Gamma^0} v \, dx = \int_{\Gamma} v \circ \Phi \, d\mu,$$

(see [4]).



Similarly, for every integer  $n$ , introduce the mapping  $\Phi_n : \Gamma \setminus \mathcal{N} \rightarrow \Gamma^n$  defined by  $\Phi_n(x) = f_\sigma \circ \Phi \circ f_\sigma^{-1}(x)$  if  $x \in f_\sigma(\Gamma) \setminus \mathcal{N}$ ,  $\sigma \in \mathcal{A}_n$ . Then, for every  $v \in L^1(\Gamma^n)$ ,

$$\frac{1}{|\Gamma^n|} \int_{\Gamma^n} v \, dx = \int_{\Gamma} v \circ \Phi_n \, d\mu. \quad (11)$$

**Proposition 1.** For every  $u \in H^1(\Omega_{\text{int}})$ ,

$$\frac{1}{|\Gamma^n|} \int_{\Gamma^n} u|_{\Gamma^n}{}^2 \, dx \xrightarrow{n \rightarrow \infty} \int_{\Gamma} u|_{\Gamma}{}^2 \, d\mu.$$

*Proof.* Remark 7 implies that it is enough to show that  $u \circ \Phi_n \rightarrow u$  in  $L^2_\mu(\Gamma)$ . We first suppose that  $u \in C^1(\overline{\Omega}_{\text{int}})$ . Hence

$$\int_{\Gamma} (u \circ \Phi_n - u)^2 \, d\mu \leq \|\nabla u\|_\infty r^n \text{Diam } \Omega,$$

since for every  $x \in \Gamma$ ,  $x$  and  $\Phi_n(x)$  both lie in  $f_\sigma(\Omega_{\text{int}})$  for some  $\sigma \in \mathcal{A}_n$ , which yields the result.

In the general case, take  $\varepsilon > 0$  and  $v \in C^1(\overline{\Omega}_{\text{int}})$  such that  $\|u - v\|_{H^1(\Omega_{\text{int}})} \leq \varepsilon$ . There exists a constant  $c > 0$  such that

$$\int_{\Gamma} (u \circ \Phi_n - u)^2 \, d\mu \leq c \left( \frac{1}{|\Gamma^n|} \int_{\Gamma^n} (u - v)^2 \, dx + \int_{\Gamma} (v \circ \Phi_n - v)^2 \, d\mu + \int_{\Gamma} (u - v)^2 \, d\mu \right). \quad (12)$$

By the previous point,  $\int_{\Gamma} (v \circ \Phi_n - v)^2 \, d\mu \leq \varepsilon$  for  $n$  large enough. A trace inequality yields  $\int_{\Gamma} (u - v)^2 \, d\mu \leq C\|u - v\|_{H^1(\Omega_{\text{int}})}$  for some constant  $C > 0$  (see [3]). We are left with dealing with the first term of the right hand side of (12).

$$\begin{aligned} \frac{1}{|\Gamma^n|} \int_{\Gamma^n} (u - v)^2 \, dx &= \frac{1}{2^n} \sum_{\sigma \in \mathcal{A}_n} \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} (u - v)^2 \, dx \\ &\leq \frac{C'}{(2r^2)^n} \sum_{\sigma \in \mathcal{A}_n} \|u - v\|_{H^1(f_\sigma(\Omega_{\text{int}}))}^2 \end{aligned}$$

for some constant  $C' > 0$ . Therefore,  $\frac{1}{|\Gamma^n|} \int_{\Gamma^n} (u - v)^2 \, dx \leq C'\|u - v\|_{H^1(\Omega_{\text{int}})}^2$ , since  $2r^2 < 1$ , which implies the desired result.  $\square$

**Remark 8.** The space of the traces on  $\Gamma$  of functions in  $W^{1,p}(\Omega_{\text{int}})$ ,  $1 < p < \infty$  was characterized in [5], whether  $r < r^*$  or  $r = r^*$ , as the space  $\text{JLip}(1 - \frac{2-d}{p}, p, p; 0; \Gamma)$ , which was first introduced in [21]. Of course, if  $r < r^*$ , then  $\text{JLip}(1 - \frac{2-d}{p}, p, p; 0; \Gamma)$  coincides with  $W^{1-\frac{2-d}{p}, p}(\Gamma)$ . An easy consequence of this characterization is that space of the traces on  $\Gamma$  of functions in  $W^{1,p}(\Omega_{\text{int}})$  is relatively compact in  $L^p_\mu(\Gamma)$ .

We also recall the following refined trace inequality, we refer to [5] for the proof.

**Theorem 4.** [see [5], Th. 11] Assume that  $r \geq 1/2$ , then for all real number  $\kappa \in (2r^2, 1)$ , there exists a constant  $C$  such that for all  $v \in H^1(\Omega_{\text{int}})$  with  $v|_{\Gamma^0} = 0$ ,

$$\|v|_{\Gamma}\|_{L^2_\mu(\Gamma)}^2 \leq C \sum_{m \geq 0} \kappa^m \sum_{\tau \in \mathcal{A}_n} \|\nabla v\|_{L^2(f_\tau(Y^0))}^2. \quad (13)$$

## 3.2 Extension results

### 3.2.1 The subcritical case $r < r^*$

As seen in § 1, it was proved in [4] that if  $r < r^*$ , then  $\Omega_{\text{int}}$  is an  $(\varepsilon, \delta)$ -domain (see [20] for a definition), or in an equivalent manner, a quasi-disk (see [30]). Hence, the extension result of Jones and Vodop'janov *et al.* applies, and  $\Omega_{\text{int}}$  is a Sobolev extension domain (see [20]), *i.e.*  $\Omega_{\text{int}}$  has the  $W^{1,p}$ -extension property for every  $p \in [1, \infty]$ : there exists a continuous linear operator  $\mathcal{E}$  from  $W^{1,p}(\Omega_{\text{int}})$  to  $W^{1,p}(\mathbb{R}^2)$  such that

$$\mathcal{E}(u)|_{\Omega_{\text{int}}} = u, \quad \forall u \in W^{1,p}(\Omega_{\text{int}}). \quad (14)$$

### 3.2.2 The critical case $r = r^*$

When  $r = r^*$ , it is easily seen that  $\Omega_{\text{int}}$  is not an  $(\varepsilon, \delta)$ -domain, and the extension results of Jones and Vodop'janov *et al.* do not apply. In fact, if  $p \in (1, \infty)$ , it is easy to construct a function  $u \in W^{1,p}(\Omega_{\text{int}})$  such that  $u \equiv 1$  in  $f_1(\Omega_{\text{int}})$  and  $u \equiv -1$  in  $f_2(\Omega_{\text{int}})$ . If  $p > 2$ , because of the Sobolev imbedding  $W^{1,p}(\mathbb{R}^2) \rightarrow C(\mathbb{R}^2)$ ,  $u$  cannot be extended to a function in  $W^{1,p}(\mathbb{R}^2)$ .

In the case when  $r = r^*$ , the situation depends in fact on the Hausdorff dimension of the set  $\Xi = f_1(\Gamma) \cap f_2(\Gamma)$ . The following extension theorem holds.

**Theorem 5.** *see [14, 2] Set  $p^* = 2 - \dim_H \Xi$ .*

1. *If  $p \in (1, p^*)$ , then  $\Omega_{\text{int}}$  has the  $W^{1,p}$ -extension property.*
2. *If  $p > p^*$ , then  $\Omega_{\text{int}}$  does not have the  $W^{1,p}$ -extension property.*

Point 1 in Theorem 5 was obtained in [14]. Point 2 is a consequence of [1] and [2]: by Theorem 3, if  $p > p^*$ , then  $W^{1,p}(\Omega_{\text{int}})|_{\Gamma} \not\subset W^{1-\frac{2-d}{p}, p}(\Gamma) = W^{1,p}(\mathbb{R}^2)|_{\Gamma}$ . This is in contradiction with the existence of a continuous extension operator from  $W^{1,p}(\Omega_{\text{int}})$  to  $W^{1,p}(\mathbb{R}^2)$  (see [1] for the proof that the notions of traces coincide  $\mu$ -almost everywhere on  $\Gamma$ ).

**Remark 9.** *As it was seen in §2.1.2, only two situations can occur, depending on the geometry of  $\Omega_{\text{int}}$ :*

- *if  $\theta$  is not of the form  $\pi/(2k)$  for any integer  $k$ , then  $\dim_H \Xi = 0$  and  $p^* = 2$ ,*
- *if  $\theta$  is of the form  $\pi/(2k)$  for an integer  $k$ , then  $\dim_H \Xi = (\dim_H \Gamma)/2$ , and  $p^* = 2 - (\dim_H \Gamma)/2$ .*

**Remark 10.** *The special case  $p = p^*$  is not dealt with in Theorem 5. The latter is of particular importance in case 1 of Remark 9 above, since the case  $p = p^* = 2$  corresponds to the question of the  $H^1$ -extension property.*

*In fact, it was proved by Koskela in [25] that if a domain in  $\mathbb{R}^n$  has the  $W^{1,n}$ -extension property, then it must have the  $W^{1,p}$ -extension property for every  $p \geq n$ . Hence, a consequence of Theorem 5 is that  $\Omega_{\text{int}}$  cannot have the  $W^{1,p}$ -extension property when  $p = 2$ . In particular, the domains we will consider in Section 6 fail to satisfy the  $H^1$ -extension property.*

*To the best of our knowledge, the question of the extension property for  $p = p^*$  in case 2 of Remark 9 seems to be open.*

## 4 The transmission problem

### 4.1 The transmission problem with fractal interface

The transmission problem can be formally stated as

$$\left\{ \begin{array}{ll} -\Delta u = f & \text{in } \Omega_{\text{int}} \cup \Omega_{\text{ext}}, \\ [u] = 0 & \text{on } \Gamma, \\ [\partial_n u] = \alpha u & \text{on } \Gamma, \\ \partial_n u_{\text{int}} = \partial_n u_{\text{ext}} = 0 & \text{on } \Sigma, \\ \partial_n u_{\text{ext}} = 0, \quad u_{\text{int}} = u_0 & \text{on } \Gamma^0, \\ \partial_n u_{\text{ext}} = 0 & \text{on } \partial D, \end{array} \right. \quad (15)$$

where  $\alpha > 0$ ,  $\Sigma = \partial\Omega_{\text{int}} \setminus (\Gamma \cup \Gamma^0)$ , and  $[u]$  (resp.  $[\partial_n u]$ ) denotes the jump of  $u$  (resp. of the normal derivative of  $u$ ) across  $\Gamma$ , and  $f \in L^2(D)$ . We also use the notations  $u_{\text{int}} := u|_{\Omega_{\text{int}}}$  and  $u_{\text{ext}} := u|_{\Omega_{\text{ext}}}$ .

The transmission condition  $[\partial_n u] = \alpha u$  on  $\Gamma$  has no real meaning, since the normal is not defined on  $\Gamma$ . The rigorous meaning of (15) is the following variational formulation:

$$\begin{aligned} & \text{find } u \in V \text{ such that for all } v \in V_0, \\ & a(u, v) = \int_D f v \, dx, \end{aligned} \quad (P)$$

where  $V$  is the affine space defined by

$$V = \{u \in L^2(D), u_{\text{int}} \in H^1(\Omega_{\text{int}}), u_{\text{ext}} \in H^1(\Omega_{\text{ext}}), u_{\text{int}}|_{\Gamma^0} = u_0, u_{\text{int}}|_{\Gamma} = u_{\text{ext}}|_{\Gamma}\}. \quad (16)$$

The vector space  $V_0$  is defined as  $V$ , except that the condition  $u_{\text{int}}|_{\Gamma^0} = u_0$  is replaced by  $u_{\text{int}}|_{\Gamma^0} = 0$ . Finally,

$$a(u, v) = \int_{\Omega_{\text{int}}} \nabla u_{\text{int}} \cdot \nabla v_{\text{int}} \, dx + \int_{\Omega_{\text{ext}}} \nabla u_{\text{ext}} \cdot \nabla v_{\text{ext}} \, dx + \alpha \int_{\Gamma} u|_{\Gamma} v|_{\Gamma} \, d\mu. \quad (17)$$

**Remark 11.** *The traces in the condition  $u_{\text{int}}|_{\Gamma} = u_{\text{ext}}|_{\Gamma}$  in (16) are meant in the sense of Definition 1. The above definition of the space  $V$  is suitable when  $r < r^*$  and in the special case when  $r = r^*$  and  $\theta = 0$  discussed in Section 6. In the other cases, the transmission condition has to be considerably changed, see Remark 13.*

Note that the space  $V_0$ , equipped with the norm  $a(u, u)^{1/2}$  is a Hilbert space. From the Lax-Milgram theorem, we see that for every function  $f$  given in  $L^2(D)$ , there exists a unique weak solution  $u \in V$  to (P). Moreover,  $u$  minimizes the quadratic functional

$$v \in V \mapsto a(v, v) - 2 \int_D f v \, dx. \quad (18)$$

## 4.2 The transmission problem with prefractal interface

For any positive integer  $n$ , let us consider the similar transmission problem in which the interior domain has been truncated by stopping the construction at step  $n$ . This class of problems is much more standard since the interface  $\Gamma^n$  consists of  $2^n$  pairwise disjoint line segments. The boundary value problem reads:

$$\begin{cases} -\Delta u = f & \text{in } \Omega_{\text{int}}^n \cup \Omega_{\text{ext}}^n, \\ [u] = 0 & \text{on } \Gamma^n, \\ [\partial_n u] = \frac{\alpha}{|\Gamma^n|} u & \text{on } \Gamma^n, \\ \partial_n u_{\text{int}}^n = \partial_n u_{\text{ext}}^n = 0 & \text{on } \Sigma^n, \\ \partial_n u_{\text{ext}}^n = 0, \quad u_{\text{int}}^n = u_0 & \text{on } \Gamma^0, \\ \partial_n u_{\text{ext}}^n = 0 & \text{on } \partial D, \end{cases} \quad (19)$$

where  $\Sigma^n = \partial\Omega_{\text{int}}^n \setminus (\Gamma^n \cup \Gamma^0)$ , and  $[u]$  (resp.  $[\partial_n u]$ ) denotes the jump of  $u$  (resp. of the normal derivative of  $u$ ) across  $\Gamma^n$ . We also use the notations  $u_{\text{int}}^n = u|_{\Omega_{\text{int}}^n}$  and  $u_{\text{ext}}^n = u|_{\Omega_{\text{ext}}^n}$ .

The variational formulation of problem (19) can be stated as follows:

$$\begin{aligned} & \text{find } u \in V^n \text{ such that for all } v \in V_0^n, \\ & a_n(u, v) = \int_D f v \, dx, \end{aligned} \quad (P_n)$$

where  $V^n$  is the affine space defined by

$$V^n = \{u \in L^2(D), u_{\text{int}}^n \in H^1(\Omega_{\text{int}}^n), u_{\text{ext}}^n \in H^1(\Omega_{\text{ext}}^n), u_{\text{int}}^n|_{\Gamma^0} = u_0, u_{\text{int}}^n|_{\Gamma^n} = u_{\text{ext}}^n|_{\Gamma^n}\}, \quad (20)$$

and here also,  $V_0^n$  is defined as  $V^n$ , except that the condition  $u_{\text{int}}^n|_{\Gamma^0} = u_0$  is replaced by  $u_{\text{int}}^n|_{\Gamma^0} = 0$ , and  $a_n$  is defined by

$$a_n(u, v) = \int_{\Omega_{\text{int}}^n} \nabla u_{\text{int}}^n \cdot \nabla v_{\text{int}}^n \, dx + \int_{\Omega_{\text{ext}}^n} \nabla u_{\text{ext}}^n \cdot \nabla v_{\text{ext}}^n \, dx + \frac{\alpha}{|\Gamma^n|} \int_{\Gamma^n} u|_{\Gamma^n} v|_{\Gamma^n} \, dx. \quad (21)$$

The space  $V_0^n$ , equipped with the norm  $a_n(u, u)^{1/2}$  is a Hilbert space. We also remark that  $V^n \subset V$  with a continuous imbedding. Again, the Lax-Milgram theorem implies that for every function  $f$  given in  $L^2(D)$ , there exists a unique weak solution  $u_n \in V^n$  to that problem, and  $u_n$  minimizes the quadratic functional

$$v \in V^n \mapsto a_n(v, v) - 2 \int_D f v \, dx. \quad (22)$$

Therefore, the sequence  $u_n$  is bounded in  $V$ .

## 5 Convergence of the energy forms in the case $r < r^*$

### 5.1 M-convergence of the energy forms

We start by extending the definition of the forms  $a$  and  $a_n$  to the whole space  $L^2(D)$  by setting

$$a(u, u) = \infty \quad \text{if } u \in L^2(D) \setminus V, \quad (23)$$

$$a_n(u, u) = \infty \quad \text{if } u \in L^2(D) \setminus V^n. \quad (24)$$

We are interested in proving the convergence of the forms  $a_n$  to  $a$  in the following sense, introduced by Mosco (see [34]).

**Definition 2.** A sequence of forms  $(a_n)_n$  is said to M-converge to a form  $a$  in  $L^2(D)$  if

*i.* for every sequence  $(u_n)_n$  weakly converging to a function  $u$  in  $L^2(D)$ ,

$$\underline{\lim} a_n(u_n, u_n) \geq a(u, u) \quad \text{as } n \rightarrow \infty, \quad (25)$$

*ii.* for every  $u \in L^2(D)$ , there exists a sequence  $(u_n)_n$  converging strongly in  $L^2(D)$  such that

$$\overline{\lim} a_n(u_n, u_n) \leq a(u, u) \quad \text{as } n \rightarrow \infty. \quad (26)$$

**Theorem 6.** Assume that  $r < r^*$ , then the energy forms  $a_n$  M-converge in  $L^2(D)$  to the form  $a$ .

**Remark 12.** The M-convergence of forms differs from the  $\Gamma$ -convergence only in that the sequence  $(u_n)$  in point *i.* of Definition 2 is assumed to converge weakly instead of strongly. In the following, only the  $\Gamma$ -convergence of the energy forms  $a_n$  will be needed.

*Proof.* We will prove separately points *i.* and *ii.* in Definition 2.

*Proof of point i.* Suppose that  $(u_n)$  weakly converges to  $u$  in  $L^2(D)$ . We may assume that there exists a subsequence, still called  $(u_n)$ , such that  $a_n(u_n, u_n)$  converges to some real number as  $n \rightarrow \infty$ ; as a consequence, there exists a constant  $c$  independent of  $n$  such that

$$a_n(u_n, u_n) \leq c. \quad (27)$$

In particular, for all  $n$ ,  $u_n \in V^n$ , which implies that  $u_n \in V$ . Then, (27) implies that  $(u_n|_{\Omega_{\text{int}}})$  is bounded in  $H^1(\Omega_{\text{int}})$ , and  $(u_n|_{\Omega_{\text{ext}}})$  is bounded in  $H^1(\Omega_{\text{ext}})$ . Therefore, there is a subsequence that we still note  $(u_n)$  such that  $(u_n|_{\Omega_{\text{int}}})$  converges weakly in  $H^1(\Omega_{\text{int}})$ , and strongly in  $L^2(\Omega_{\text{int}})$ . Since  $u_n \rightharpoonup u$  in  $L^2(D)$ , we see that  $u_n|_{\Omega_{\text{int}}} \rightharpoonup u_{\text{int}}$  in  $H^1(\Omega_{\text{int}})$ . Similarly, up to a further extraction of a subsequence,  $u_n|_{\Omega_{\text{ext}}} \rightharpoonup u_{\text{ext}}$  in  $H^1(\Omega_{\text{ext}})$ . Consequently,

$$\underline{\lim} \left( \int_{\Omega_{\text{int}}} |\nabla u_n|^2 dx + \int_{\Omega_{\text{ext}}} |\nabla u_n|^2 dx \right) \geq \int_{\Omega_{\text{int}}} |\nabla u|^2 dx + \int_{\Omega_{\text{ext}}} |\nabla u|^2 dx.$$

We will now prove that  $\frac{1}{|\Gamma^n|} \int_{\Gamma^n} u_n^2 dx \rightarrow \int_{\Gamma} u^2 d\mu$  as  $n \rightarrow \infty$ , which will yield point *i.*

The following inequality was proved in [3]: for every  $v \in H^1(\Omega_{\text{int}})$ ,  $\|v - \langle v \rangle_{\Gamma^0}\|_{L^2_{\mu}(\Gamma)} \leq C \|\nabla v\|_{L^2(\Omega_{\text{int}})}$ .

This implies that

$$|\langle v \rangle_{\Gamma^0}| \leq \|v\|_{L^2_{\mu}(\Gamma)} + C \|\nabla v\|_{L^2(\Omega_{\text{int}})}.$$

Similarly,  $\|v - \langle v \rangle_{\Gamma^0}\|_{L^2(\Gamma^0)} \leq C \|\nabla v\|_{L^2(\Omega_{\text{int}})}$  implies that

$$\frac{1}{\sqrt{|\Gamma^0|}} \|v\|_{L^2(\Gamma^0)} \leq |\langle v \rangle_{\Gamma^0}| + C \|\nabla v\|_{L^2(\Omega_{\text{int}})}.$$

for some constant independent of  $v$  that we still note  $C$ . Combining these two inequalities, we obtain that

$$\frac{1}{\sqrt{|\Gamma^0|}} \|v\|_{L^2(\Gamma^0)} \leq \|v\|_{L^2_{\mu}(\Gamma)} + C \|\nabla v\|_{L^2(\Omega_{\text{int}})}.$$

Hence, for every  $\sigma \in \mathcal{A}_n$ ,

$$\begin{aligned} \frac{1}{\sqrt{|\Gamma^\sigma|}} \|v\|_{L^2(\Gamma^\sigma)} &\leq \|v \circ f_\sigma\|_{L_\mu^2(\Gamma)} + C \|\nabla(v \circ f_\sigma)\|_{L^2(\Omega_{\text{int}})} \\ &= 2^{\frac{n}{2}} \|v\|_{L_\mu^2(f_\sigma(\Gamma))} + C \|\nabla v\|_{L^2(f_\sigma(\Omega_{\text{int}}))}. \end{aligned}$$

This yields that

$$\frac{1}{|\Gamma^\sigma|} \|v\|_{L^2(\Gamma^\sigma)}^2 \leq 2^n \|v\|_{L_\mu^2(f_\sigma(\Gamma))}^2 + 2C^2 2^{\frac{n}{2}} \|v\|_{L_\mu^2(f_\sigma(\Gamma))} \|\nabla v\|_{L^2(f_\sigma(\Omega_{\text{int}}))} + C^2 \|\nabla v\|_{L^2(f_\sigma(\Omega_{\text{int}}))}^2.$$

Therefore

$$\begin{aligned} \frac{1}{|\Gamma^n|} \int_{\Gamma^n} u_n^2 \, dx &= \frac{1}{2^n} \sum_{\sigma \in \mathcal{A}_n} \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} u_n^2 \, dx \\ &\leq \|u_n\|_{L_\mu^2(\Gamma)}^2 + 2C^2 2^{-\frac{n}{2}} \sum_{\sigma \in \mathcal{A}_n} \|u_n\|_{L_\mu^2(f_\sigma(\Gamma))} \|\nabla u_n\|_{L^2(f_\sigma(\Omega_{\text{int}}))} + \frac{C^2}{2^n} \sum_{\sigma \in \mathcal{A}_n} \|\nabla u_n\|_{L^2(f_\sigma(\Omega_{\text{int}}))}^2 \\ &\leq \|u_n\|_{L_\mu^2(\Gamma)}^2 + 2C^2 2^{-\frac{n}{2}} \|u_n\|_{L_\mu^2(\Gamma)} \left( \sum_{\sigma \in \mathcal{A}_n} \|\nabla u_n\|_{L^2(f_\sigma(\Omega_{\text{int}}))}^2 \right)^{\frac{1}{2}} + \frac{C^2}{2^n} \sum_{\sigma \in \mathcal{A}_n} \|\nabla u_n\|_{L^2(f_\sigma(\Omega_{\text{int}}))}^2. \end{aligned}$$

Since  $u_n|_{\Omega_{\text{int}}}$  is a bounded sequence in  $H^1(\Omega_{\text{int}})$ , there exists a constant  $M$  such that

$$\frac{1}{|\Gamma^n|} \int_{\Gamma^n} u_n^2 \, dx \leq \|u_n\|_{L_\mu^2(\Gamma)}^2 + 2CM^2 2^{-\frac{n}{2}} \|u_n\|_{L_\mu^2(\Gamma)} + \frac{C^2 M^2}{2^n}.$$

Moreover, since  $u_n$  weakly converges to  $u$  in  $H^1(\Omega_{\text{int}})$ , then up to the extraction of a subsequence,  $u_n|_\Gamma$  strongly converges to  $u|_\Gamma$  in  $L_\mu^2(\Gamma)$ , from Remark 8. Hence, we obtain that

$$\overline{\lim} \frac{1}{|\Gamma^n|} \int_{\Gamma^n} u_n^2 \, dx \leq \int_\Gamma u^2 \, d\mu.$$

Similarly, the following inequality holds for every  $v \in H^1(\Omega_{\text{int}})$ :

$$\|v\|_{L_\mu^2(\Gamma)} \leq \frac{1}{\sqrt{|\Gamma^0|}} \|v\|_{L^2(\Gamma^0)} + C' \|\nabla v\|_{L^2(\Omega_{\text{int}})} \quad (28)$$

for some constant  $C'$  independent of  $v$ . As above, we deduce that

$$\underline{\lim} \frac{1}{|\Gamma^n|} \int_{\Gamma^n} u_n^2 \, dx \geq \int_\Gamma u^2 \, d\mu,$$

and we obtain the desired result.

*Proof of point ii.* Take  $u \in L^2(D)$ . By (23), we may assume that  $u \in V$ . Consider a neighborhood  $\omega \subset D$  of  $\Omega_{\text{int}}$  such that  $\Omega_{\text{int}} \Subset \omega$ . We introduce the notations  $\omega^\sigma = f_\sigma(\omega)$  for  $\sigma \in \mathcal{A}$  and  $\omega_n = \bigcup_{\sigma \in \mathcal{A}_n} \omega^\sigma$  for all integer  $n$ .

For every  $n$ , introduce the cut-off function  $\chi_n$  in  $D$  defined by

$$\chi_n(x) = (1 - r^{-n} d(x, \omega_n))^+, \quad (29)$$

where  $d(x, \omega_n) = \inf_{y \in \omega_n} |x - y|$  and  $\alpha^+$  stands for the positive part of a real number  $\alpha$ . Hence,  $\chi_n \equiv 1$  in  $\omega_n$  and  $\chi_n \equiv 0$  outside  $\tilde{\omega}_n := \{x \in D, d(x, \omega_n) < r^n\}$ . Note that if we set  $\tilde{\omega} := \{x \in D, d(x, \omega) < 1\}$  and  $\tilde{\omega}^\sigma := f_\sigma(\tilde{\omega}) = \{x \in D, d(x, \omega^\sigma) < r^n\}$  for  $\sigma \in \mathcal{A}_n$ , then  $\tilde{\omega}_n = \bigcup_{\sigma \in \mathcal{A}_n} \tilde{\omega}^\sigma$ .

We now define a sequence of functions  $u_n$  by

$$u_n = (1 - \chi_n)u + \chi_n \mathcal{E}(u_{\text{int}}), \quad (30)$$

where  $\mathcal{E}$  is the extension operator introduced in (14) and as above,  $u_{\text{int}} = u|_{\Omega_{\text{int}}}$ . Obviously,  $u_n$  belongs to the space  $V^n$  and the sequence  $(u_n)$  strongly converges to  $u$  in  $L^2(D)$ . We will prove that  $\lim a_n(u_n, u_n) = a(u, u)$  as  $n \rightarrow \infty$ . We start by showing that

$$I_n := \int_{\Omega_{\text{int}}} |\nabla u|^2 dx + \int_{\Omega_{\text{ext}}} |\nabla u|^2 dx - \left( \int_{\Omega_{\text{int}}^n} |\nabla u_n|^2 dx + \int_{\Omega_{\text{ext}}^n} |\nabla u_n|^2 dx \right) \longrightarrow 0 \quad (31)$$

as  $n \rightarrow \infty$ . First observe that

$$I_n = \int_{\tilde{\omega}_n \setminus \Omega_{\text{int}}} |\nabla u|^2 dx - \int_{\tilde{\omega}_n \setminus \Omega_{\text{int}}} |\nabla u_n|^2 dx.$$

Hence, it is enough to show that  $\int_{\tilde{\omega}_n \setminus \Omega_{\text{int}}} |\nabla(u - u_n)|^2 dx \rightarrow 0$  as  $n \rightarrow \infty$ . Note that

$$\int_{\tilde{\omega}_n \setminus \Omega_{\text{int}}} |\nabla(u - u_n)|^2 dx = \int_{\tilde{\omega}_n \setminus \Omega_{\text{int}}} |\nabla(\chi_n(\mathcal{E}(u_{\text{int}}) - u))|^2 dx \leq 2(I_n^1 + I_n^2) \quad (32)$$

where  $I_n^1 = \int_{\tilde{\omega}_n \setminus \Omega_{\text{int}}} |\nabla(\mathcal{E}(u_{\text{int}}) - u)|^2 dx$  and  $I_n^2 = \int_{\tilde{\omega}_n \setminus \Omega_{\text{int}}} |(\nabla \chi_n)(\mathcal{E}(u_{\text{int}}) - u)|^2 dx$ .

First observe that  $I_n^1 \rightarrow 0$  as  $n \rightarrow \infty$  since  $\nabla(\mathcal{E}(u_{\text{int}}) - u) \in L^2(\Omega_{\text{ext}})$ . We are left with dealing with  $I_n^2$ . One has

$$\begin{aligned} I_n^2 &\leq c r^{-2n} \int_{\tilde{\omega}_n \setminus \Omega_{\text{int}}} |\mathcal{E}(u_{\text{int}}) - u|^2 dx \\ &\leq c r^{-2n} \sum_{\sigma \in \mathcal{A}_n} \int_{\tilde{\omega}^\sigma \setminus \Omega_{\text{int}}} |\mathcal{E}(u_{\text{int}}) - u|^2 dx. \end{aligned}$$

where  $c > 0$  is a constant independent of  $n$ . We have the following Poincaré inequality: there exists a constant  $C$  such that for every  $v \in H^1(\tilde{\omega} \setminus \Omega_{\text{int}})$  such that  $v|_\Gamma = 0$ ,

$$\int_{\tilde{\omega} \setminus \Omega_{\text{int}}} |v|^2 dx \leq C \int_{\tilde{\omega} \setminus \Omega_{\text{int}}} |\nabla v|^2 dx. \quad (33)$$

Applying a rescaled version of (33) to the function  $\mathcal{E}(u_{\text{int}}) - u$ , we get

$$I_n^2 \leq c' \sum_{\sigma \in \mathcal{A}_n} \int_{\tilde{\omega}^\sigma \setminus \Omega_{\text{int}}} |\nabla(\mathcal{E}(u_{\text{int}}) - u)|^2 dx.$$

where  $c' > 0$  is independent of  $n$ . By Lemma 1 in [14], for every  $\sigma \in \mathcal{A}_n$ , the number of strings  $\tau \in \mathcal{A}_n$  such that  $\tilde{\omega}^\sigma \cap \tilde{\omega}^\tau \neq \emptyset$  is bounded independently of  $n$ . Therefore,

$$I_n^2 \leq c'' \int_{\tilde{\omega}_n \setminus \Omega_{\text{int}}} |\nabla(\mathcal{E}(u_{\text{int}}) - u)|^2 dx$$

where  $c'' > 0$  does not depend on  $n$ . We deduce that  $I_n^2 \rightarrow 0$  as  $n \rightarrow \infty$ , since  $\mathcal{E}(u_{\text{int}}) - u \in H^1(\Omega_{\text{ext}})$ , which yields (31).

We will now prove that

$$\frac{1}{|\Gamma^n|} \int_{\Gamma^n} u_n^2 dx \longrightarrow \int_\Gamma u^2 d\mu \quad (34)$$

as  $n \rightarrow \infty$ , which will conclude the proof of point *ii.*. Observe that for every integer  $n$ ,  $\mathcal{E}(u_{\text{int}})|_{\Gamma^n} = u$ , which implies that  $u_n|_{\Gamma^n} = u$ . We are left with proving that

$$\frac{1}{|\Gamma^n|} \int_{\Gamma^n} u^2 dx \longrightarrow \int_\Gamma u^2 dx \quad (35)$$

as  $n \rightarrow \infty$ , which holds by Proposition 1.  $\square$

## 5.2 Convergence of the minimizers

A consequence of Theorem 6 (in fact a consequence of the  $\Gamma$ -convergence of the forms  $a_n$ ) is the convergence of the solutions of problems  $(P_n)$  to the solution of problem  $(P)$  in  $L^2(D)$  (see *e.g.* [11]). The following theorem shows the convergence of the minimizers in the space  $V$ .

**Theorem 7.** *Take  $f \in L^2(D)$ , and note  $u_n$  (resp.  $u$ ) the solution of problem  $(P_n)$  (resp.  $(P)$ ). The sequence  $(u_n)$  converges to  $u$  in the space  $V$ .*

*Proof.* We know that  $u_n$  is bounded in  $V$ . Therefore, there exists a subsequence of  $u_n$ , still noted  $u_n$ , which converges to some function  $u^*$  weakly in  $V$ , and strongly in  $L^2(D)$ . We will start by showing that  $u^* = u$  and  $a(u_n, u_n) \rightarrow a(u, u)$  as  $n \rightarrow \infty$ .

By Theorem 6, point *i.* in Definition 2 implies that  $a(u^*, u^*) \leq \underline{\lim} a_n(u_n, u_n)$ . Hence,

$$\underline{\lim} \left( a_n(u_n, u_n) - 2 \int_D f u_n \, dx \right) \geq a(u^*, u^*) - 2 \int_D f u^* \, dx. \quad (36)$$

We deduce from point *ii.* in Definition 2 that there exists a sequence  $(v_n)$  in  $L^2(D)$  such that  $v_n \rightarrow u$  in  $L^2(D)$  and  $\overline{\lim} a_n(v_n, v_n) \leq a(u, u)$ . Therefore,

$$a(u, u) - 2 \int_D f u \, dx \geq \overline{\lim} \left( a_n(v_n, v_n) - 2 \int_D f v_n \, dx \right) \quad (37)$$

$$\geq \overline{\lim} \left( a_n(u_n, u_n) - 2 \int_D f u_n \, dx \right) \quad (38)$$

by (22).

We deduce from (36) that

$$a(u, u) - 2 \int_D f u \, dx \geq a(u^*, u^*) - 2 \int_D f u^* \, dx. \quad (39)$$

Consequently, by (18) and by uniqueness of the solution of  $(P)$ , we obtain that  $u = u^*$ , and  $a(u, u) \leq \underline{\lim} a_n(u_n, u_n)$ . On the other hand, since  $u_n$  converges to  $u$  in  $L^2(D)$ , (26) yields  $\overline{\lim} a_n(u_n, u_n) \leq a(u, u)$ , and we obtain that  $a_n(u_n, u_n) \rightarrow a(u, u)$  as  $n \rightarrow \infty$ .

Since  $u_n \rightarrow u$  in  $H^1(\Omega_{\text{int}})$ , we deduce as in the proof of Theorem 6 point *i.*, that  $\frac{1}{|\Gamma^n|} \int_{\Gamma^n} u_n^2 \, dx \rightarrow \int_{\Gamma} u^2 \, d\mu$  as  $n \rightarrow \infty$ . Since, by Remark 8,  $u_n|_{\Gamma} \rightarrow u|_{\Gamma}$  in  $L^2_{\mu}(\Gamma)$ , we deduce that

$$a(u_n, u_n) \xrightarrow{n \rightarrow \infty} a(u, u). \quad (40)$$

We now prove that  $u_n \rightarrow u$  in  $V$ . Since  $u_n|_{\Gamma} \rightarrow u|_{\Gamma}$  in  $L^2_{\mu}(\Gamma)$ , (40) implies that

$$\int_{\Omega_{\text{int}}} |\nabla u_n|^2 \, dx + \int_{\Omega_{\text{ext}}} |\nabla u_n|^2 \, dx \rightarrow \int_{\Omega_{\text{int}}} |\nabla u|^2 \, dx + \int_{\Omega_{\text{ext}}} |\nabla u|^2 \, dx.$$

By well known weak lower semi-continuity properties, this can happen if and only if  $\int_{\Omega_{\text{int}}} |\nabla u_n|^2 \, dx \rightarrow \int_{\Omega_{\text{int}}} |\nabla u|^2 \, dx$  and  $\int_{\Omega_{\text{ext}}} |\nabla u_n|^2 \, dx \rightarrow \int_{\Omega_{\text{ext}}} |\nabla u|^2 \, dx$  as  $n \rightarrow \infty$ .

This implies that  $\int_{\Omega_{\text{int}}} |\nabla(u - u_n)|^2 \, dx \rightarrow 0$  since  $u_n \rightarrow u$  in  $H^1(\Omega_{\text{int}})$ . Similarly,  $\int_{\Omega_{\text{ext}}} |\nabla(u - u_n)|^2 \, dx \rightarrow 0$ . The claim is proved.

Since  $(u_n)$  is bounded in  $V$  and has  $u$  for unique cluster point, it converges to  $u$  in  $V$ , which concludes the proof.  $\square$

## 6 A particular geometry with $r = r^*$

As seen before, the proof of Theorem 6 is based on the extension result of §3.2.1. In the case  $r = r^*$ , the  $H^1$ -extension property is no longer true for the domain  $\Omega_{\text{int}}$  (see Remark 10). In what follows, we focus

on the special case when  $\theta = 0$ . We will see that in this case, the transmission condition imposed on  $\Gamma$  yields an extension result (see Theorem 9) which is the main ingredient for proving the  $M$ -convergence of the energy forms.

In the case  $\theta = 0$ , it can be seen that  $r^* = \frac{1}{2}$ , and the ramified domain described in §2.2 is for example as in Figure 3. In this particular case, the set  $f_1(\Gamma) \cap f_2(\Gamma)$  is reduced to a single point that we call  $A$ . Observe that the self-similar part  $\Gamma$  of the boundary is a line segment, and the self-similar measure  $\mu$  associated with  $\Gamma$  is the normalized one-dimensional Hausdorff measure.

Since  $r = r^*$ , the domain  $\Omega_{\text{ext}}$  has infinitely many connected components. Call  $U$  the outer connected component of  $\Omega_{\text{ext}}$ , ie the only connected component which has a nonempty intersection with  $\partial D$  (see Figure 3). Observe that  $\Gamma$  is a subset of  $\partial U$ , and that the intersection of  $\Gamma$  with the boundary of every other connected component of  $\Omega_{\text{ext}}$  is reduced to a single point.

We consider the transmission problem  $(P)$ , where the space  $V$  is still given by (16). The trace  $u_{\text{ext}|_{\Gamma}}$  in the transmission condition

$$u_{\text{int}|_{\Gamma}} = u_{\text{ext}|_{\Gamma}} \quad (41)$$

is meant as the trace of the function  $u|_U$  on the set  $\Gamma$ .

**Remark 13.** When  $\theta > 0$  and  $r = r^*$ , the situation is quite different:  $\Gamma$  is not entirely contained in the boundary of any connected component of  $\Omega_{\text{ext}}$ . It can be shown that there exists  $\delta \in (1, d)$  such that the intersections of  $\Gamma$  with the boundary of the connected components of  $\Omega_{\text{ext}}$  have Hausdorff dimension  $\delta$ . These sets are called the canopies of the domain  $\Omega_{\text{int}}$ . In this case, the transmission condition has to be described more carefully. This is the topic of a work in progress.

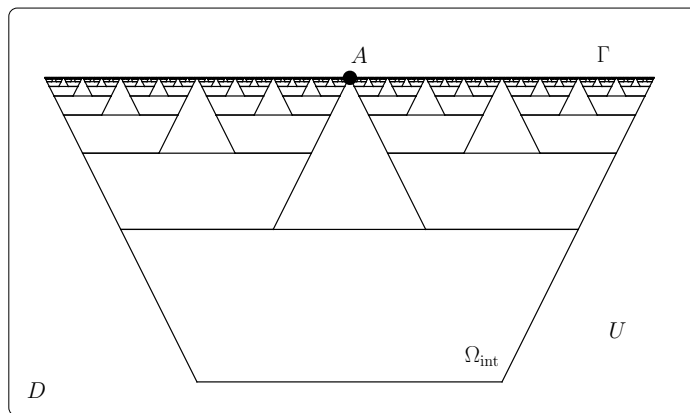


Figure 3: An exemple of the domains  $\Omega_{\text{int}}$  and  $\Omega_{\text{ext}}$  in the case  $\theta = 0$

**Theorem 8.** Assume that  $\theta = 0$  and  $r = r^* = \frac{1}{2}$ . Then the energy forms  $a_n$  defined in (21)  $M$ -converge in  $L^2(D)$  to the form  $a$  defined in (17).

Since  $\partial U$  is Lipschitz-continuous, a standard trace result yields that for every  $u \in V$ ,  $u_{\text{ext}|_{\Gamma}} \in H^{1/2}(\Gamma)$ . Hence, the transmission condition (41) implies that

$$u_{\text{int}|_{\Gamma}} \in H^{1/2}(\Gamma). \quad (42)$$

Note that (42) is not only a consequence of the fact that  $u_{\text{int}} \in H^1(\Omega_{\text{int}})$ , because the latter property only implies that  $u_{\text{int}|_{\Gamma}} \in H^s(\Gamma)$  for all  $s < \frac{1}{2}$  (see Theorem 3).

We start by proving the following extension result.

**Theorem 9.** If  $v \in H^1(\Omega_{\text{int}})$  and  $v|_{\Gamma} \in H^{1/2}(\Gamma)$ , then there exists an extension  $\tilde{v} \in H^1(D)$  of  $v$  such that

$$\|\tilde{v}\|_{H^1(D)}^2 \leq C \left( \|v\|_{H^1(\Omega_{\text{int}})}^2 + \|v|_{\Gamma}\|_{H^{1/2}(\Gamma)}^2 \right), \quad (43)$$

where the constant  $C > 0$  is independent of  $v$ .



If  $u \in V$ , Theorem 9 can be applied to  $u_{\text{int}}$ . Hence, the proof of Theorem 6 can be entirely reproduced, using the extension given by Theorem 9 instead of the extension operator  $\mathcal{E}$  introduced in (14). This achieves the proof of Theorem 8.

Theorems 10 and 11 below will play an important role in the proof of Theorem 9. We start by recalling an extension result for multiple cones from [7].

**Theorem 10.** [see [7], Th. 5.1] *Call  $C$  the double cone in  $\mathbb{R}^2$  defined by  $|x_1| < |x_2|$ . Write  $\rho(x) = \|x\|$  (the notation  $\|\cdot\|$  stands for the euclidean norm), and, for every  $v \in H^1(C)$ , introduce the antiradial part  $v_a$  of  $v$  in the cone  $C$ , defined by*

$$v_a(x) = v(x) - \langle v \rangle_{S_{\rho(x)}},$$

where for any  $R > 0$ ,  $S_R = \{x \in C, \|x\| = R\}$ , and  $\langle v \rangle_{S_R}$  is the mean value of  $v$  along the arc  $S_R$ . There exists a linear extension operator

$$\Lambda : \left\{ v \in H^1(C), \frac{v_a}{\rho} \in L^2(C) \right\} \rightarrow H^1(\mathbb{R}^2), \quad (44)$$

such that for every  $v \in \{v \in H^1(C), v_a/\rho \in L^2(C)\}$ ,

$$\|\Lambda v\|_{H^1(\mathbb{R}^2)} \leq c \left( \|u\|_{H^1(C)} + \left\| \frac{v_a}{\rho} \right\|_{L^2(C)} \right)$$

where  $c > 0$  is a constant independent of  $v$ .

As mentioned in [7], Theorem 10 can be immediately extended in  $\mathbb{R}^n$  to the case of a union of two half-cones sharing the same vertex, separated by a hyperplane passing through the vertex and not containing any direction of the boundaries.

**Theorem 11 : Peetre-Tartar.** [see [37, 40]] *Let  $E, E_1, E_2, F$  be Banach spaces, and let  $A_i, i = 1, 2$ , be continuous linear operators from  $E$  to  $E_i$ , and suppose  $A_1$  is compact. Further assume that there exists a constant  $c_0 > 0$  such that for any  $v \in E$ ,*

$$\|v\|_E \leq c_0 (\|A_1 v\|_{E_1} + \|A_2 v\|_{E_2}). \quad (45)$$

*If  $L$  is a continuous linear operator from  $E$  to  $F$  such that  $L|_{\ker A_2} \equiv 0$ , then there exists a constant  $c_1 > 0$  such that for any  $v \in E$ ,*

$$\|Lv\|_F \leq c_1 \|A_2 v\|_{E_2}. \quad (46)$$

**Notations** We start by introducing notations for the proof of Theorem 9.

We first introduce a domain  $C$  which is the union of two truncated half-cones included in  $\Omega_{\text{int}}$ , whose common vertex is the point  $A$ . Call  $\tilde{\Omega}_{\text{int}}$  the convex hull of the domain  $\Omega_{\text{int}}$ , and introduce the set

$$T = \tilde{\Omega}_{\text{int}} \setminus \overline{Y_0 \cup f_1(\tilde{\Omega}_{\text{int}}) \cup f_2(\tilde{\Omega}_{\text{int}})}. \quad (47)$$

Note that  $T$  is the main hole of domain  $\Omega_{\text{int}}$  (see Figure 4). Call  $\varphi_0 \in (0, \frac{\pi}{2})$  the upper half-angle of the triangular domain  $T$  (see Figure 4), and take  $\varphi_1 > \varphi_0$ . Call  $\mathcal{C}$  the half-cone whose boundary is made of the two half-lines through  $A$  with respective angles  $\varphi_0$  and  $\varphi_1$  with the vertical axis (see Figure 4). Call  $C_2 = \mathcal{C} \cap (\tilde{\Omega}_{\text{int}} \setminus \overline{Y^0})$ . We can assume that  $\varphi_1 > \varphi_0$  is small enough so that  $C_2 \subset \Omega_{\text{int}}$ , in other words  $C_2$  does not intersect any of the holes of  $\Omega_{\text{int}}$ . We define  $C_1$  to be the symmetric of  $C_2$  with respect to the vertical axis  $x_2 = 0$ , and we write  $C = C_1 \cup C_2$ .

We also introduce the sets  $Y^{k,1} := f_1 \circ f_2^{k-1}(Y^0)$  and  $Y^{k,2} := f_2 \circ f_1^{k-1}(Y^0)$  for every  $k \geq 1$  (see Figure 4), and we write  $Y^k := \overline{Y^{k,1} \cup Y^{k,2}}$ . We also note  $\gamma := \overline{f_1^2(\Gamma) \cup f_2^2(\Gamma)}$  (see Figure 5).

Call  $\Omega^{1,1} := f_1(\Omega_{\text{int}} \setminus \overline{f_2(\Omega_{\text{int}})})$  and  $\Omega^{1,2} := f_2(\Omega_{\text{int}} \setminus \overline{f_1(\Omega_{\text{int}})})$  (see Figure 5). We introduce the sets  $\Omega^{k,i}$  defined by  $\Omega^{k,i} := g^{k-1}(\Omega^i)$ ,  $k \geq 1$ ,  $i = 1, 2$ , where  $g$  is the homothety centered at  $A$  with ratio  $1/2$ , see Figure 5.

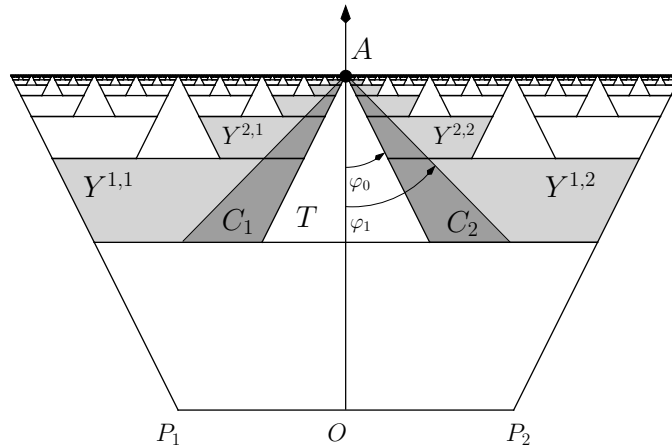


Figure 4: The region  $C = C_1 \cup C_2$  and the domains  $Y^{k,i}$ ,  $k, i = 1, 2$ .

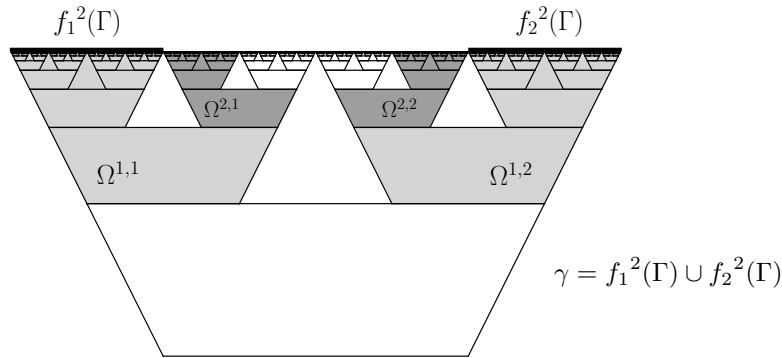


Figure 5: The domains  $\Omega^{k,i}$ ,  $k, i = 1, 2$  (in light grey:  $\Omega^{1,1}$  and  $\Omega^{1,2}$ , in dark grey:  $\Omega^{2,1}$  and  $\Omega^{2,2}$ ) and the sets  $f_i^2(\Gamma)$ ,  $i = 1, 2$ .

For every integer  $k \geq 1$ , we introduce  $\Sigma^k := \{\tau \in \mathcal{A}, f_\tau(Y^0) \subset \Omega^{k,1} \cup \Omega^{k,2}\}$ , with the notations of §2.1.1. Take  $\kappa \in (1/2, 1)$ . We introduce the space  $G = \{v \in L_{\text{loc}}^1(\Omega^{1,1} \cup \Omega^{1,2}), \|v\|_G < \infty\}$ , where

$$\|v\|_G^2 = \sum_{m \geq 1} \kappa^m \sum_{\tau \in \Sigma^1 \cap \mathcal{A}_m} \int_{f_\tau(Y^0)} |v|^2 dx. \quad (48)$$

Endowed with the norm  $\|\cdot\|_G$ , the space  $G$  is a Banach space.

We also introduce the space  $H = \{v \in L^2(\Omega^{1,1} \cup \Omega^{1,2}), \nabla v \in G\}$ , which is a Hilbert space with the norm

$$\left( \|\nabla v\|_G^2 + \|v\|_{L^2(\Omega^{1,1} \cup \Omega^{1,2})}^2 \right)^{\frac{1}{2}}.$$

Moreover, from Theorem 4, we see that  $v \in H \mapsto v|_\gamma$  is a continuous operator from  $H$  to  $L_\mu^2(\gamma)$ . Arguing by contradiction, we can show that

$$\left( \|\nabla v\|_G^2 + \|v|_\gamma\|_{L_\mu^2(\gamma)}^2 \right)^{\frac{1}{2}}$$

is an equivalent norm on  $H$ .

We first state and prove two lemmas which will be useful in the proof of Theorem 9.

**Lemma 1.** *There exists a constant  $c > 0$  such that for every  $v \in H$ ,*

$$\int_{Y^1} \left( |v(x) - \langle v \rangle_{Y^{1,1}}|^2 + |v(x) - \langle v \rangle_{Y^{1,2}}|^2 \right) dx \leq c \left( \int_{\gamma} |v - \langle v \rangle_{\gamma}|^2 d\mu + \|\nabla v\|_G^2 \right). \quad (49)$$

*Proof.* We introduce the Hilbert space  $E = \{(v, w) \in H \times L_{\mu}^2(\gamma), v|_{\gamma} = w\}$ , endowed with the norm  $\|\cdot\|_E$  given by  $\|(v, w)\|_E^2 = \|\nabla v\|_G^2 + \|w\|_{L_{\mu}^2(\gamma)}^2$ .

We now introduce the operators

$$\begin{aligned} A_1 &: (v, w) \in E \mapsto \langle v \rangle_{\gamma}, \\ A_2 &: (v, w) \in E \mapsto (\nabla v, w - \langle w \rangle_{\gamma}), \\ L &: (v, w) \in E \mapsto (v - \langle v \rangle_{Y^{1,1}}, v - \langle v \rangle_{Y^{1,2}}). \end{aligned}$$

With the notations of Theorem 11,  $E_1 = \mathbb{R}$ ,  $E_2 = G \times L_{\mu}^2(\gamma)$  and  $F = L^2(Y^1)^2$ . It is easily seen that  $A_1$ ,  $A_2$  and  $L$  are continuous linear operators, and  $A_1$  is compact. Moreover, (45) is clearly satisfied. Observe that  $(v, w) \in E$  lies in  $\ker A_2$  if and only if  $v$  is constant in  $\Omega^{1,1}$  and in  $\Omega^{1,2}$ . Hence, it is obvious that  $L|_{\ker A_2} \equiv 0$ . From this, we deduce by Theorem 11 that there exists a constant  $c > 0$  such that  $\|L(v, w)\|_F \leq c\|A_2(v, w)\|_{E_2}$  for all  $(v, w) \in E$ , which yields (49).  $\square$

**Lemma 2.** *Assume that  $v \in H^1(\Omega_{\text{int}})$  and  $v|_{\Gamma} \in H^{1/2}(\Gamma)$ , then  $\frac{v_a}{\rho} \in L^2(C)$ , and*

$$\int_C \left| \frac{v_a}{\rho} \right|^2 dx \leq c \left( \sum_{k \geq 1} 2^k \int_{\gamma_k} |v - \langle v \rangle_{\gamma_k}|^2 d\mu + \sum_{k \geq 1} \sum_{p \geq k} \kappa^{p-k+1} \sum_{\tau \in \Sigma^k \cap \mathcal{A}_p} \int_{f_{\tau}(Y^0)} |\nabla v|^2 dx \right)$$

for some constant  $c > 0$  independent of  $v$ , where  $\gamma^k = g^{k-1}(\gamma)$  (recall that  $g$  is the homothety centered at  $A$ , with ratio  $1/2$ ).

*Proof.* We first observe that, by self-similarity, there exists a constant  $c_1 > 0$  such that for all  $x \in Y^k$ ,  $\rho(x) \geq \frac{c_1}{2^k}$ . Therefore,

$$\int_C \left| \frac{v_a}{\rho} \right|^2 dx \leq c_1 \sum_{k \geq 1} 2^{2k} \int_{Y^k} |v_a|^2 dx,$$

since  $C \subset \bigcup_{k \geq 1} Y^k$  by construction. Hence, there is a constant  $c_2 > 0$  such that  $\int_C \left| \frac{v_a}{\rho} \right|^2 dx \leq c_2(I_1 + I_2)$ , where

$$I_1 = \sum_{k \geq 1} 2^{2k} \int_{Y^k} |v(x) - \langle v \rangle_{Y^k}|^2 dx, \quad (50)$$

$$I_2 = \sum_{k \geq 1} 2^{2k} \int_{Y^k} |\langle v \rangle_{Y^k} - \langle v \rangle_{S_{\rho(x)}}|^2 dx. \quad (51)$$

We start by dealing with  $I_1$ . We note that

$$I_1 \leq \frac{1}{2} \sum_{k \geq 1} 2^{2k} \int_{Y^k} \left( |v - \langle v \rangle_{Y^{k,1}}|^2 + |v - \langle v \rangle_{Y^{k,2}}|^2 \right) dx. \quad (52)$$

For every  $k \geq 1$ , we can apply Lemma 1 to the function  $v \circ g^{k-1}$ . Since  $\gamma^k = g^{k-1}(\gamma)$ , we obtain

$$\begin{aligned} I_1 &\leq c \left( \sum_{k \geq 1} 2^k \int_{\gamma_k} |v - \langle v \rangle_{\gamma_k}|^2 d\mu + \sum_{k \geq 1} \sum_{m \geq 1} \kappa^m \sum_{\tau \in \Sigma^1 \cap \mathcal{A}_m} \int_{g^{k-1}(f_{\tau}(Y^0))} |\nabla v|^2 dx \right) \\ &= c \left( \sum_{k \geq 1} 2^k \int_{\gamma_k} |v - \langle v \rangle_{\gamma_k}|^2 d\mu + \sum_{k \geq 1} \sum_{p \geq k} \kappa^{p-k+1} \sum_{\tau \in \Sigma^k \cap \mathcal{A}_p} \int_{f_{\tau}(Y^0)} |\nabla v|^2 dx \right), \end{aligned} \quad (53)$$

for some constant  $c > 0$  independent of  $v$ , since  $\{g^{k-1} \circ f_\tau, \tau \in \Sigma^1 \cap \mathcal{A}_m\} = \{f_\tau, \tau \in \Sigma^k \cap \mathcal{A}_{m+k-1}\}$ . Let us now deal with  $I_2$ . For every  $R > 0$  and  $i = 1, 2$ , call  $S_R^i = S_R \cap C_i$  and  $\langle v \rangle_{S_R^i}$  the mean value of  $v$  on the set  $S_R^i$ . We note that

$$I_2 \leq \frac{1}{2} \sum_{i=1,2} \sum_{k \geq 1} 2^{2k} \left( \int_{Y^k} |\langle v \rangle_{Y^{k,i}} - \langle v \rangle_{S_{\rho(x)}^i}|^2 \right). \quad (54)$$

Take  $i \in \{1, 2\}$  and  $x_0 \in f_i(Y^0)$ , and, for every integer  $k \geq 1$ ,  $\rho_k = \rho(x_0)/2^{k-1}$ .

We observe that  $\sum_k 2^{2k} \int_{Y^k} |\langle v \rangle_{Y^{k,i}} - \langle v \rangle_{S_{\rho(x)}^i}|^2 \leq 2(J_1 + J_2)$ , where

$$J_1 = \sum_{k \geq 1} 2^{2k+1} |Y^{k,i}| (\langle v \rangle_{Y^{k,i}} - \langle v \rangle_{S_{\rho_k}^i})^2, \quad (55)$$

$$J_2 = \sum_{k \geq 1} 2^{2k} \int_{Y^k} |\langle v \rangle_{S_{\rho_k}^i} - \langle v \rangle_{S_{\rho(x)}^i}|^2 dx. \quad (56)$$

Let us first examine  $J_1$ . The following Poincaré inequality holds in  $Y^{1,i} = f_i(Y^0)$ : for every  $v \in H^1(Y^{1,i})$ ,

$$\int_{Y^{1,i}} |v(x) - \langle v \rangle_{S_{\rho_i}^i}|^2 dx \leq M \int_{Y^{1,i}} |\nabla v|^2 dx, \quad (57)$$

where the constant  $M > 0$  is independent of  $v$ .

Observe that for every integer  $k \geq 1$ ,  $Y^{k,i} = g^{k-1}(Y^{1,i})$ , and  $S_{\rho_k}^i = g^{k-1}(S_{\rho_1}^i)$ . Then

$$\begin{aligned} |\langle v \rangle_{Y^{k,i}} - \langle v \rangle_{S_{\rho_k}^i}|^2 &= \frac{1}{|Y^{k,i}|^2} \left| \int_{Y^{k,i}} (v(x) - \langle v \rangle_{S_{\rho_k}^i}) dx \right|^2 \\ &\leq \frac{1}{|Y^{k,i}|} \int_{Y^{k,i}} |v(x) - \langle v \rangle_{S_{\rho_k}^i}|^2 dx \\ &\leq \frac{1}{2^{2(k-1)} |Y^{k,i}|} \int_{Y^{1,i}} |v \circ g^{k-1}(x) - \langle v \circ g^{k-1} \rangle_{S_{\rho_1}^i}|^2 dx \\ &\leq \frac{M}{2^{2(k-1)} |Y^{k,i}|} \int_{Y^{1,i}} |\nabla(v \circ g^{k-1})|^2 dx \\ &= \frac{M}{2^{2(k-1)} |Y^{k,i}|} \int_{Y^{k,i}} |\nabla v|^2 dx, \end{aligned}$$

where we used (57). From this and (55), we deduce that

$$J_1 \leq 8M \sum_{k \geq 1} \int_{Y^{k,i}} |\nabla v|^2. \quad (58)$$

To deal with  $J_2$ , we use polar coordinates  $(\rho, \varphi)$  centered at  $A$ . Introduce the positive constants  $R_0, R_1$  such that for all  $(\rho, \varphi) \in Y^{1,i}$ ,  $R_0 \leq \rho \leq R_1$ . Observe that if  $x \in Y^k$ ,

$$\begin{aligned} |\langle v \rangle_{S_{\rho(x)}^i} - \langle v \rangle_{S_{\rho_k}^i}|^2 &= \left| \frac{1}{\varphi_1 - \varphi_0} \int_{\varphi_0}^{\varphi_1} (v(\rho(x), \varphi) - v(\rho_k, \varphi)) d\varphi \right|^2 \\ &= \left| \frac{1}{\varphi_1 - \varphi_0} \int_{\varphi_0}^{\varphi_1} \int_{\rho_k}^{\rho(x)} \frac{\partial v}{\partial \rho}(s, \varphi) ds d\varphi \right|^2 \\ &\leq \frac{1}{(\varphi_1 - \varphi_0)^2} \left| \int_{\varphi_0}^{\varphi_1} \int_{\rho_k}^{\rho(x)} \left| \frac{\partial v}{\partial \rho}(s, \varphi) \right|^2 s ds d\varphi \right| \times \left| \int_{\varphi_0}^{\varphi_1} \int_{\rho_k}^{\rho(x)} \frac{ds}{s} d\varphi \right| \\ &\leq \frac{\log \frac{R_1}{R_0}}{\varphi_1 - \varphi_0} \int_{\varphi_0}^{\varphi_1} \int_{R_0/2^{k-1}}^{R_1/2^{k-1}} \left| \frac{\partial v}{\partial \rho}(s, \varphi) \right|^2 s ds d\varphi \\ &= \frac{\log \frac{R_1}{R_0}}{\varphi_1 - \varphi_0} \int_{C(k)} |\nabla v(y)|^2 dy, \end{aligned}$$

where  $C(k) = \{(\rho, \varphi), \frac{R_0}{2^{k-1}} < \rho < \frac{R_1}{2^{k-1}}, \varphi_1 < |\varphi| < \varphi_0\}$ . Hence,

$$J_2 \leq \frac{\log \frac{R_1}{R_0}}{\varphi_1 - \varphi_0} \sum_{k \geq 1} 2^{2k} \int_{Y^k} \int_{C(k)} |\nabla v(y)|^2 dy dx,$$

and there is a constant  $c_3 > 0$  independent of  $v$  such that

$$J_2 \leq c_3 \sum_{k \geq 1} \int_{C(k)} |\nabla v(y)|^2 dy.$$

Note that every point  $(\rho, \varphi)$  in  $C(k)$  lies in at most  $\log_2 \frac{R_1}{R_0}$  sets  $C(l)$ ,  $l \geq 1$ . Therefore,

$$J_2 \leq c_3 \log_2 \frac{R_1}{R_0} \int_C |\nabla v(y)|^2 dy \leq c_3 \log_2 \frac{R_1}{R_0} \sum_{k \geq 1} \int_{Y^k} |\nabla v|^2 dx. \quad (59)$$

By the inequalities (58) and (59),  $I_2 \leq c_4 \sum_{k \geq 1} \int_{Y^k} |\nabla v|^2 dx$  for some constant  $c_4 > 0$  independent of  $v$ . Hence,

$$I_2 \leq \frac{c_4}{\kappa} \sum_{k \geq 1} \sum_{p \geq k} \kappa^{p-k+1} \sum_{\tau \in \Sigma^k \cap \mathcal{A}_p} \int_{f_\tau(Y^0)} |\nabla v|^2 dx. \quad (60)$$

Indeed, it can be easily seen that for all  $k \geq 1$ , one has  $Y^k = \bigcup \{f_\tau(Y^0), \tau \in \Sigma^k \cap \mathcal{A}_k\}$ .

Therefore, (53) and (60) yield, which (50), which achieves the proof.  $\square$

*Proof of Theorem 9.* By Lemma 2 and Theorem 10, for every  $v \in H^1(\Omega_{\text{int}})$  such that  $v|_\Gamma \in H^{1/2}(\Gamma)$ , there exists  $\Lambda v \in H^1(\mathbb{R}^2)$  such that

$$\|\Lambda v\|_{H^1(\mathbb{R}^2)}^2 \leq c \left( \|v\|_{H^1(C)}^2 + \left\| \frac{v_a}{\rho} \right\|_{L^2(C)}^2 \right) \quad (61)$$

where the constant  $c > 0$  is independent of  $v$ .

We will now define the extension  $\tilde{v}$  of a function  $v$  as in Theorem 9. Note that  $\{f_\sigma(T), \sigma \in \mathcal{A}\}$  is the collection of the holes of the domain  $\Omega_{\text{int}}$ , where the main hole  $T$  was defined in (47). Introduce the function  $\tilde{v}$  defined in  $\tilde{\Omega}_{\text{int}}$  by

$$\begin{cases} \tilde{v} := v & \text{in } \Omega_{\text{int}}, \\ \tilde{v} := \Lambda(v \circ f_\sigma) \circ f_\sigma^{-1} & \text{in } f_\sigma(T), \sigma \in \mathcal{A}. \end{cases}$$

We will prove that  $\tilde{v} \in H^1(\tilde{\Omega}_{\text{int}})$ . By (61) and by Lemma 2, we get the estimate

$$\begin{aligned} & \int_T |\nabla(\Lambda v)|^2 dx \\ & \leq C \left( \sum_{k \geq 1} 2^k \int_{\gamma_k} |v - \langle v \rangle_{\gamma_k}|^2 d\mu + \sum_{k \geq 1} \sum_{p \geq k} \kappa^{p-k+1} \sum_{\tau \in \Sigma^k \cap \mathcal{A}_p} \int_{f_\tau(Y^0)} |\nabla v|^2 dx \right). \end{aligned} \quad (62)$$

Observe that for every integer  $k \geq 1$ ,  $\gamma^k \subset \tilde{\Gamma}^{\sigma^k}$  where  $\sigma^k = (1, 2, \dots, 2) \in \mathcal{A}_{k-2}$ , (recall that the sets  $\tilde{\Gamma}^\sigma$  have been introduced in Remark 3). Therefore, there exists a constant  $c > 0$  independent of  $k$  such that

$$\int_{\gamma_k} |v - \langle v \rangle_{\gamma_k}|^2 d\mu \leq c \int_{\tilde{\Gamma}^{\sigma^k}} |v - \langle v \rangle_{\tilde{\Gamma}^{\sigma^k}}|^2 dx, \quad (63)$$

for  $i = 1, 2$ . Take  $\sigma \in \mathcal{A}_n$ , one has

$$\begin{aligned} \int_{f_\sigma(T)} |\nabla \tilde{v}|^2 dx &= \int_T |\nabla(\Lambda(v \circ f_\sigma))|^2 dx \\ &\leq C' \left( \sum_{k \geq 1} 2^{k+n} \int_{f_\sigma(\tilde{\Gamma}^{\sigma^k})} |v - \langle v \rangle_{f_\sigma(\tilde{\Gamma}^{\sigma^k})}|^2 d\mu \right. \\ &\quad \left. + \sum_{k \geq 1} \sum_{p \geq k} \kappa^{p-k+1} \sum_{\tau \in \Sigma^k \cap \mathcal{A}_p} \int_{f_{\sigma\tau}(Y^0)} |\nabla v|^2 dx \right), \end{aligned}$$

for a constant  $C' > 0$  independent of  $v$ , where we applied (62) to the function  $v \circ f_\sigma$ , and we used (63) and the notation  $\sigma\tau$  for  $\tau \in \mathcal{A}_k$ , which stands for  $(\sigma(1), \dots, \sigma(n), \tau(1), \dots, \tau(k)) \in \mathcal{A}_{n+k}$ .

We can write

$$\begin{aligned} \int_{\tilde{\Omega}_{\text{int}}} |\nabla \tilde{v}|^2 dx &= \int_{\Omega_{\text{int}}} |\nabla \tilde{v}|^2 dx + \sum_{\sigma \in \mathcal{A}} \int_{f_\sigma(T)} |\nabla \tilde{v}|^2 dx \\ &\leq \int_{\Omega_{\text{int}}} |\nabla \tilde{v}|^2 dx + C'(S_1 + S_2), \end{aligned}$$

where

$$\begin{aligned} S_1 &= \sum_{n \geq 0} \sum_{\sigma \in \mathcal{A}_n} \sum_{k \geq 1} 2^{k+n} \int_{f_\sigma(\tilde{\Gamma}^{\sigma^k})} |v - \langle v \rangle_{f_\sigma(\tilde{\Gamma}^{\sigma^k})}|^2 d\mu, \\ S_2 &= \sum_{\sigma \in \mathcal{A}} \sum_{k \geq 1} \sum_{p \geq k} \kappa^{p-k+1} \sum_{\tau \in \Sigma^k \cap \mathcal{A}_p} \int_{f_{\sigma\tau}(Y^0)} |\nabla v|^2 dx. \end{aligned}$$

We first deal with  $S_1$ . Take  $\sigma, \tau \in \mathcal{A}$  and  $k, l \geq 1$ . Note that if  $\sigma\sigma^k = \tau\tau^l$ , then  $k = l$  and  $\sigma = \tau$ . Therefore,

$$S_1 \leq \sum_{k \geq 0} 2^k \sum_{\sigma \in \mathcal{A}_k} \int_{\tilde{\Gamma}^\sigma} |v - \langle v \rangle_{\tilde{\Gamma}^\sigma}|^2 d\mu \leq c'' \|v|_\Gamma\|_{H^{1/2}(\Gamma)}^2 \quad (64)$$

by (2).

We are left with dealing with  $S_2$ . Assume that  $\eta \in \mathcal{A}_N$  and  $\eta = \sigma\tau$  with  $\tau \in \Sigma^k \cap \mathcal{A}_p$ , then  $p \leq N$ . Since the sets  $\Sigma^k$ ,  $k \geq 1$ , are pairwise disjoint, this means that the term  $\int_{f_\eta(Y^0)} |\nabla v|^2 dx$  appears at most  $N$  times in the sum  $S_2$ .

Moreover, we observe that  $p - k + 1 \in [1, N]$ . It can be seen that there is at most one quadruplet  $(\sigma', \tau', l, q)$  with  $\sigma' \in \mathcal{A}$ ,  $\tau' \in \Sigma^l \cap \mathcal{A}_q$ ,  $l \geq 1$  and  $q \geq l$  distinct from  $(\sigma, \tau, k, p)$  such that  $\eta = \sigma'\tau'$  and  $p - k + 1 = q - l + 1$ . As a consequence,

$$\begin{aligned} S_2 &\leq 2 \sum_{N \geq 0} \sum_{\eta \in \mathcal{A}_N} \sum_{m=1}^N \kappa^m \int_{f_\eta(Y^0)} |\nabla v|^2 dx \\ &\leq \frac{2}{1 - \kappa} \sum_{\eta \in \mathcal{A}} \int_{f_\eta(Y^0)} |\nabla v|^2 dx = \frac{2}{1 - \kappa} \int_{\Omega_{\text{int}}} |\nabla v|^2 dx. \end{aligned} \quad (65)$$

Therefore, (64) and (65) give the existence of a constant  $C > 0$  independent of  $v$  such that

$$\|\tilde{v}\|_{H^1(\tilde{\Omega}_{\text{int}})}^2 \leq C \left( \|v\|_{H^1(\Omega_{\text{int}})}^2 + \|v|_\Gamma\|_{H^{1/2}(\Gamma)}^2 \right).$$

Since  $\tilde{\Omega}_{\text{int}}$  is a polygonal domain, we can further extend  $\tilde{v}$  into a function in  $H^1(D)$  satisfying (43).  $\square$

## References

- [1] Y. Achdou, T. Deheuvels, and N. Tchou. *JLip* versus Sobolev spaces on a class of self-similar fractal foliages. *Journal de Mathématiques Pures et Appliquées*, 97:142–172, 2012.
- [2] Y. Achdou, T. Deheuvels, and N. Tchou. Comparison of different definitions of traces for a class of ramified domains with self-similar fractal boundaries. *Potential Anal.*, 40(4):345–362, 2014.
- [3] Y. Achdou and N. Tchou. Neumann conditions on fractal boundaries. *Asymptotic Analysis*, 53(1-2):61–82, 2007.
- [4] Y. Achdou and N. Tchou. Trace results on domains with self-similar fractal boundaries. *Journal de Mathématiques Pures et Appliquées*, 89:596–623, 2008.
- [5] Y. Achdou and N. Tchou. Trace theorems for a class of ramified domains with self-similar fractal boundaries. *SIAM j. of Math. Anal.*, 42(4):828–860 (electronic), 2010.
- [6] R.A. Adams and J.J.F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.
- [7] P. Auscher and N. Badr. Sobolev spaces on multiple cones. *Ann. Fac. Sci. Toulouse Math. (6)*, 19(3-4):707–733, 2010.
- [8] G. Berger. Eigenvalue distribution of elliptic operators of second order with Neumann boundary conditions in a snowflake domain. *Math. Nachr.*, 220:11–32, 2000.
- [9] A.-P. Calderón. Lebesgue spaces of differentiable functions and distributions. In *Proc. Sympos. Pure Math., Vol. IV*, pages 33–49. American Mathematical Society, Providence, R.I., 1961.
- [10] R. Capitanelli and M. A. Vivaldi. On the Laplacean transfer across fractal mixtures. *Asymptot. Anal.*, 83(1-2):1–33, 2013.
- [11] G. Dal Maso. *An introduction to  $\Gamma$ -convergence*. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston, Inc., Boston, MA, 1993.
- [12] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 2*. Springer-Verlag, Berlin, 1988. Functional and variational methods, With the collaboration of Michel Artola, Marc Authier, Philippe Bénilan, Michel Cessenat, Jean Michel Combes, Hélène Lanchon, Bertrand Mercier, Claude Wild and Claude Zuily, Translated from the French by Ian N. Sneddon.
- [13] T. Deheuvels. Contributions à l'étude d'espaces de fonctions et d'EDP pour une classe de domaines frontière fractale auto-similaire. PhD dissertation.
- [14] T. Deheuvels. Sobolev extension property for tree-shaped domains with self-contacting fractal boundary. To appear in *Annali della Scuola Normale Superiore di Pisa*.
- [15] K.J. Falconer. *Techniques in fractal geometry*. John Wiley & Sons Ltd., Chichester, 1997.
- [16] H. Federer and W.P. Ziemer. The Lebesgue set of a function whose distribution derivatives are  $p$ -th power summable. *Indiana Univ. Math. J.*, 22:139–158, 1972/73.
- [17] M. Filoche and B. Sapoval. Transfer across random versus deterministic fractal interfaces. *Phys. Rev. Lett.*, 84:5776–5779, Jun 2000.
- [18] P.H. Hung and E. Sánchez-Palencia. Phénomènes de transmission à travers des couches minces de conductivité élevée. *J. Math. Anal. Appl.*, 47:284–309, 1974.
- [19] J.E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.
- [20] P.W. Jones. Quasiconformal mappings and extendability of functions in Sobolev spaces. *Acta Math.*, 147(1-2):71–88, 1981.
- [21] A. Jonsson. Haar wavelets of higher order on fractals and regularity of functions. *J. Math. Anal. Appl.*, 290(1):86–104, 2004.
- [22] A. Jonsson and H. Wallin. Function spaces on subsets of  $\mathbf{R}^n$ . *Math. Rep.*, 2(1):xiv+221, 1984.
- [23] J. Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.
- [24] J. Kinnunen and O. Martio. The Sobolev capacity on metric spaces. *Ann. Acad. Sci. Fenn. Math.*, 21(2):367–382, 1996.
- [25] P. Koskela. Extensions and imbeddings. *J. Funct. Anal.*, 159(2):369–383, 1998.
- [26] M.R. Lancia. A transmission problem with a fractal interface. *Z. Anal. Anwendungen*, 21(1):113–133, 2002.

- 
- [27] M.R. Lancia and P. Vernole. Irregular heat flow problems. *SIAM J. Math. Anal.*, 42(4):1539–1567, 2010.
- [28] B.B. Mandelbrot and M. Frame. The canopy and shortest path in a self-contacting fractal tree. *Math. Intelligencer*, 21(2):18–27, 1999.
- [29] O. Martio and J. Sarvas. Injectivity theorems in plane and space. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 4(2):383–401, 1979.
- [30] V.G. Maz’ja. *Sobolev spaces*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.
- [31] V.G. Maz’ya and S.V. Poborchi. *Differentiable functions on bad domains*. World Scientific Publishing Co., Inc., River Edge, NJ, 1997.
- [32] P.A.P. Moran. Additive functions of intervals and Hausdorff measure. *Proc. Cambridge Philos. Soc.*, 42:15–23, 1946.
- [33] U. Mosco. Convergence of convex sets and of solutions of variational inequalities. *Advances in Math.*, 3:510–585, 1969.
- [34] U. Mosco. Composite media and asymptotic Dirichlet forms. *J. Funct. Anal.*, 123(2):368–421, 1994.
- [35] U. Mosco and M.A. Vivaldi. Variational problems with fractal layers. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)*, 27:237–251, 2003.
- [36] U. Mosco and M.A. Vivaldi. An example of fractal singular homogenization. *Georgian Math. J.*, 14(1):169–193, 2007.
- [37] J. Peetre. Espaces d’interpolation et théorème de Soboleff. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1):279–317, 1966.
- [38] B. Sapoval. General formulation of laplacian transfer across irregular surfaces. *Phys. Rev. Lett.*, 73:3314–3316, Dec 1994.
- [39] E.M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
- [40] L. Tartar. Nonlinear partial differential equations using compactness methods. *Mathematics Research Center, Univ. of Wisconsin, Madison*, Report 1584, 1975.
- [41] S. K. Vodop’janov, V. M. Gol’dšteĭn, and T. G. Latfullin. A criterion for the extension of functions of the class  $L^1_2$  from unbounded plane domains. *Sibirsk. Mat. Zh.*, 20(2):416–419, 464, 1979.