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Microstructural effects in non linear stratified composites

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ABSTRACT

In this paper, we analyze the microstructural effects on non linear elastic and periodic composites within the framework of asymptotic homogenization. We assume that the constitutive laws of the individual constituents derive from strain potentials. The microstructural effects are incorporated by considering the higher order terms, which come from the asymptotic series expansion. The complete solution at any order requires the resolution of a chain of cell problems in which the source terms depend on the solution at the lower order. The influence of these terms on the macroscopic response of the non linear composite is evaluated in the particular case of a stratified microstructure. The analytic solutions of the cell problems at the first and second order are provided for arbitrary local strain-stress laws which derive from potentials. As classically, the non-linear dependence on the applied macroscopic strain is retrieved for the solution at the first order. It is proved that the second order term in the expansion series also exhibits a non linear dependence with the macroscopic strain but linearly depends on the gradient of macroscopic strain. As a consequence, the macroscopic potential obtained by homogenization is a quadratic function of the macroscopic strain gradient when the expansion series is truncated at the second order. This model generalizes the well known first strain gradient elasticity theory to the case of non linear elastic material. The influence of the non local correctors on the macroscopic potential is investigated in the case of power law elasticity under macroscopic plane strain or antiplane conditions.

Keywords:

Homogenization
Non linear material
Strain gradient
Composite material
Asymptotic series

1. Introduction

The analysis of the microstructural effects on the macroscopic response of non linear solid materials is of paramount importance for many practical problems in mechanics such as for example the localization in shear band, the fracture or the microindentation. Conventional theories of elasticity and plasticity are not adapted to treat adequately the problems which are dominated by the microstructural effects. This has motivated various strain gradient theories. There is an abundant literature on this subject among which [Toupin \(1962\)](#), [Mindlin \(1964\)](#), [Green and Rivlin \(1964\)](#), [Kröner \(1967\)](#), [Eringen and Edelen \(1972\)](#) for the case of elasticity and [Aifantis \(1984\)](#), [Fleck and Hutchinson \(1993, 1997\)](#), [Nix and Gao \(1998\)](#), [Gao et al. \(1999a,b\)](#) and [Gurtin and Anand \(2005\)](#) for plasticity. For instance [Fleck and Hutchinson \(1997\)](#) have proposed an extension of the first strain gradient elasticity theory of [Toupin \(1962\)](#) and [Mindlin \(1964\)](#) to the case of plasticity in which the plastic dissipation is written with a new definition for the

equivalent strain rate that depends on first strain rate gradient tensor and also on an internal lengthscale.

These strain gradient plasticity theories have been applied to many problems where strain gradient effects are expected to be important, including for example, the analysis of crack tip fields in [Xia and Hutchinson \(1996\)](#) and [Huang et al. \(1997\)](#). Another example is the localization in thin bands that standard plasticity theories cannot reproduce adequately with the mesh refinement when these models are implemented numerically. Indeed, it is well admitted that the loss of homogeneity in the macroscopic response of the structure is due to the presence of a microstructure in the material. When the localization in shear bands occurs, the microstructure is involved in the deformation of the solid. Consequently, standard continuum models are not adapted to describe such kind of phenomena. By using plasticity strain gradient theories, the problem of localization has been addressed in [Aifantis \(1984\)](#), [Zbib and Aifantis \(1988\)](#) and [Chambon et al. \(2001\)](#), including the mesh dependency investigation, the band width and the distributions of field variables within and outside the band.

However, all the above cited strain gradient plasticity models are not constructed in a rigorous homogenization approach able to provide a mathematical meaning of the strain gradient dependence at the macroscopic scale. The formulation of strain gradient

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models in the framework of homogenization methods has been the subject of intense research. For instance, Diener et al. (1981, 1982) and later Drugan and Willis (1996), and Drugan (2000, 2003) have extended the Hashin–Shtrikman variational approach (Hashin and Shtrikman, 1962, 1963; Willis, 1977) in order to derive a non local macroscopic constitutive law of linear elastic heterogeneous materials. This approach corrects the standard homogenization procedure by accounting for the dependence of ensemble average stress on ensemble average strain by addition of terms which depend on the strain gradients in the ensemble average strain. However, such a method is restricted to the case of linear constituents. Other methods are based on the resolution of unit cells of random media with prescribed strain gradient at the boundary (Gologanu et al., 1997; Forest, 1998; Bouyge et al., 2001, 2002; Kouznetsova et al., 2002, 2004; Yuan et al., 2008). They use a generalization of the Hill–Mandel lemma for deriving the higher-order constitutive relations and they can deal with non-linear materials (the reader could refer to Forest (2006) for a review on these multiscale methods). In particular, Gologanu et al. (1997) have extended the Gurson model (1977) by accounting for strain gradient at the boundary of the hollow sphere and provide an analytic expression for the macroscopic yield criterion which depends on macroscopic stress and hyperstress. As already mentioned in Fleck and Hutchinson (1997), the results are however somewhat surprising: the dominant strain gradient effect is not influenced by the void volume fraction or void spacing and this effect persists even when the void volume fraction tends to zero. This last result is however surprising, because the strain gradient effect is only due to the presence of the microstructure. Therefore, when the volume fraction tends to zero, the solid is homogeneous at the microscopic scale and the microstructural effects cannot be observed at the macroscopic scale. Note that a similar conclusion has been formulated by Yuan et al. (2008) about the use of this kind of generalized boundary conditions. This suggests that the resulting macroscopic model overevaluates the macroscopic internal energy of gradient elastic media and gives rise to some unphysical results such as the persistence of strain gradient effect when the unit cell is homogeneous.

The asymptotic homogenization is a powerful tool which has shown to be efficient to incorporate the microstructural effects on the effective behavior of periodic elastic composites (Gambin and Kroner, 1989; Boutin, 1996; Smyshlyaev and Cherednichenko, 2000; Peerlings and Fleck, 2004; Tran et al., 2012). The general solution of a linear elastic composite subjected to an external macroscopic loading is expanded in series with respect to a small parameter, the scale factor ϵ , which is defined as the ratio between the microstructure lengthscale and a macroscopic lengthscale. This last one is characteristic of the macroscopic structure dimensions or of the applied loading. When the scale factor is small but not completely negligible or when strong values of macroscopic strain gradient occur, it is reasonable to incorporate higher order terms which come from asymptotic series. For instance, in Tran et al. (2012), these higher order terms are kept in the homogenization procedure to evaluate the correction in the macroscopic elastic energy density. More particularly, it has been found that the resulting model coincides with the phenomenological strain gradient theory of Green and Rivlin (1964) (which accounts for the derivatives of macroscopic strain at any order) and also contains the second gradient theory of Toupin (1962) and Mindlin (1964) when it is truncated. Moreover, the microstructural effects do not persist when the cell is homogeneous. The reason for this feature is that the source terms in the higher order cell problems vanish when the cell is homogeneous. In the same way, the higher order terms of the expansions series, which introduce the non local correctors, also vanish in the macroscopic energy density. Mention must be made of the studies of

Seppelcher and Pideri (1997a,b) and Soubestre and Boutin (2011) who show, still in the framework of asymptotic homogenization, important effects of the strain gradient on the effective properties of elastic composites reinforced by highly rigid slender inclusions. When the elastic coefficients of the inclusions have an appropriate order of magnitude with respect to the scale factor, the effective material exhibits a strain gradient dependence.

However, these results are restricted to the case of linear elastic constituents while the microstructure effect are expected to be prominent for non linear materials. Note that the case of composites made up of materials displaying a rate dependent behavior at finite strain has been addressed by Triantafyllidis and Bardenhagen (1996) to investigate the relation between macro and micro instabilities. In the present paper, we extend the recent work of Tran et al. (2012) by constructing the strain gradient dependent effective potential of non linear composites materials. Due to the high complexity of the problem, the non linear effects are investigated in the case of a stratified composite with non linear elastic constituents. The paper is organized as follows. In Section 2, we apply the asymptotic homogenization procedure for non linear solid and for an arbitrary microstructure. In Section 3, we focus our study on stratified composites for which analytic solutions can be derived. The last part of the paper is devoted to some applications in the case of power law elasticity under antiplane or plane strain conditions.

2. Application of the asymptotic series method to non linear composites

2.1. Description of the problem

We consider a periodic, non linear composite whose constituents have the following constitutive equation:

$$\boldsymbol{\sigma}(\underline{X}) = \frac{\partial \psi}{\partial \boldsymbol{\epsilon}}(\underline{X}, \boldsymbol{\epsilon}(\underline{X})) \quad (1)$$

where \underline{X} denotes the vector position and ψ is defined by:

$$\psi(\underline{X}, \boldsymbol{\epsilon}(\underline{X})) = \sum_r \chi_r(\underline{X}) \psi_r(\boldsymbol{\epsilon}(\underline{X})) \quad (2)$$

in which $\chi_r(\underline{X})$ are the characteristic function of phase “r” and $\psi_r(\underline{X})$ for $r = 1, 2, \dots$ are non linear functions of the strain. Functions $\psi_r(\boldsymbol{\epsilon}(\underline{X}))$ are assumed to be strictly convex and C^∞ with respect to the variable $\boldsymbol{\epsilon}(\underline{X})$. The strain and stress tensors comply with compatibility and equilibrium equations:

$$\text{div}_X(\boldsymbol{\sigma}(\underline{X})) = 0, \quad \boldsymbol{\epsilon}(\underline{X}) = \text{sym}(\nabla_X \underline{u}(\underline{X})) \quad (3)$$

where $\underline{u}(\underline{X})$ is the displacement. In the above relations, index “X” in div_X and ∇_X means that the derivatives are made with respect to the vector position \underline{X} . This distinction is needed since two space variables, namely a slow and a rapid position vector, will be introduced thereafter when using the asymptotic expansion. For simplicity, the body force has been neglected in the balance equation. Additionally, the displacement and traction are continuous across the interfaces of the individual constituents:

$$[\underline{u}(\underline{X})]_\Gamma = 0, \quad [\boldsymbol{\sigma}(\underline{X}) \cdot \underline{n}]_\Gamma = 0 \quad (4)$$

The notation $[\bullet]_\Gamma$ represents the jump of \bullet across the interface Γ defined as follows: $[\bullet]_\Gamma = \bullet^+ - \bullet^-$, where \bullet^+ and \bullet^- are the values of \bullet calculated on both sides of Γ .

2.2. Cell problems issued from the asymptotic expansion

In the following, we use the asymptotic expansion method initiated by Sanchez-Palencia (1974, 1980) and Bensoussan et al. (1978) in order to derive a set of cell problems at the microscopic

scale. To this aim, we introduce the two non-dimensional space variables $\underline{x} = \underline{X}/L$ and $\underline{y} = \underline{X}/h$ where h and L are respectively a characteristic lengthscale of the microstructure (the size of the unit cell, of the inclusions, the distance between two neighboring inclusions. . .) and of the (macro) structure. Variable \underline{x} is "slow" while \underline{y} is said rapid. All the fields, i.e. displacement, strain and stress, are assumed to depend on these two variables. The dependence in \underline{x} is attributed to the variations induced by the macroscopic loading while the one in \underline{y} is due to the local heterogeneity. Due to the periodicity of the microstructure, the characteristic functions $\chi_r(\underline{y})$ only depend on the rapid variable \underline{y} . The local strain-stress relation (1) reads:

$$\boldsymbol{\sigma}(\underline{x}, \underline{y}) = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\underline{y}, \boldsymbol{\varepsilon}(\underline{x}, \underline{y})) \quad \text{with} \quad \boldsymbol{\varepsilon}(\underline{x}, \underline{y}) = \sum_r \chi_r(\underline{y}) \boldsymbol{\psi}_r(\boldsymbol{\varepsilon}(\underline{x}, \underline{y})) \quad (5)$$

The scale factor, $\epsilon = h/L$, is assumed to be small compared to 1 but not negligible. In presence of this small parameter, the solution can be expanded as a power series in ϵ :

$$\underline{u}(\underline{x}, \underline{y}) = L \sum_{n=0}^{+\infty} \epsilon^n \underline{u}^{(n)}(\underline{x}, \underline{y}) \quad (6)$$

where the displacements $\underline{u}^{(n)}(\underline{x}, \underline{y})$ are \underline{y} -periodic and non dimensional due to the presence of the lengthscale L before the sum. In the compatibility equation, the gradient operator is decomposed into two parts: $\nabla \bullet = \frac{1}{L} (\nabla_x \bullet + \frac{1}{\epsilon} \nabla_y \bullet)$, where the indices x and y indicate that the derivatives are taken with respect to \underline{x} and \underline{y} respectively. The strain tensor takes the form:

$$\boldsymbol{\varepsilon}(\underline{x}, \underline{y}) = \epsilon^{-1} \boldsymbol{\varepsilon}^{(-1)}(\underline{x}, \underline{y}) + \sum_{n=0}^{n=+\infty} \epsilon^n \boldsymbol{\varepsilon}^{(n)}(\underline{x}, \underline{y}) \quad (7)$$

where $\boldsymbol{\varepsilon}^{(-1)}(\underline{x}, \underline{y})$ and $\boldsymbol{\varepsilon}^{(n)}(\underline{x}, \underline{y})$ for $n \geq 0$ read:

$$\begin{aligned} \boldsymbol{\varepsilon}^{(-1)}(\underline{x}, \underline{y}) &= \boldsymbol{\varepsilon}_y(\underline{u}^{(0)}(\underline{x}, \underline{y})) \\ \boldsymbol{\varepsilon}^{(n)}(\underline{x}, \underline{y}) &= \boldsymbol{\varepsilon}_x(\underline{u}^{(n)}(\underline{x}, \underline{y})) + \boldsymbol{\varepsilon}_y(\underline{u}^{(n+1)}(\underline{x}, \underline{y})) \quad \text{for} \quad n \geq 0 \end{aligned} \quad (8)$$

The strain must remain finite when the scale factor tends to zero. Consequently, one has $\boldsymbol{\varepsilon}^{(-1)}(\underline{x}, \underline{y}) = 0$ and then the displacement $\underline{u}^{(0)}(\underline{x}, \underline{y})$ is only a function of the slow space variable, \underline{x} . It is thereafter denoted $\underline{U}(\underline{x})$ and is in fact the macroscopic displacement.¹ It follows that the strain $\boldsymbol{\varepsilon}^{(0)}(\underline{x}, \underline{y})$ reads:

$$\boldsymbol{\varepsilon}^{(0)}(\underline{x}, \underline{y}) = \mathbf{E}(\underline{x}) + \boldsymbol{\varepsilon}_y(\underline{u}^{(1)}(\underline{x}, \underline{y})) \quad \text{with} \quad \mathbf{E}(\underline{x}) = \boldsymbol{\varepsilon}_x(\underline{U}(\underline{x})) \quad (9)$$

in which $\mathbf{E}(\underline{x})$ is the macroscopic strain and where $\boldsymbol{\varepsilon}_x$ and $\boldsymbol{\varepsilon}_y$ are the strain tensors calculated with respect to \underline{x} and \underline{y} . The microscopic stress is:

$$\boldsymbol{\sigma}(\underline{x}, \underline{y}) = \sum_{n=0}^{n=+\infty} \epsilon^n \boldsymbol{\sigma}^{(n)}(\underline{x}, \underline{y}) \quad (10)$$

with:

$$\begin{aligned} \boldsymbol{\sigma}^{(0)}(\underline{x}, \underline{y}) &= \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}^{(0)}(\underline{x}, \underline{y})) \\ \boldsymbol{\sigma}^{(1)}(\underline{x}, \underline{y}) &= \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}^{(0)}(\underline{x}, \underline{y})) : \boldsymbol{\varepsilon}^{(1)}(\underline{x}, \underline{y}) \\ \boldsymbol{\sigma}^{(2)}(\underline{x}, \underline{y}) &= \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}^{(0)}(\underline{x}, \underline{y})) : \boldsymbol{\varepsilon}^{(2)}(\underline{x}, \underline{y}) \\ &\quad + \frac{1}{2} \frac{\partial^3 \psi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}^{(0)}(\underline{x}, \underline{y})) :: (\boldsymbol{\varepsilon}^{(1)}(\underline{x}, \underline{y}) \otimes \boldsymbol{\varepsilon}^{(1)}(\underline{x}, \underline{y})) \\ &\quad \dots \end{aligned} \quad (11)$$

¹ $\underline{U}(\underline{x})$ is the average of the microscopic displacement and physically represents the displacement of the centroid of the unit cell.

In the equilibrium equation (the first equation in (3)), the divergence operator is also decomposed into two parts "div_x" and "div_y". This leads to:

$$\frac{1}{\epsilon} \text{div}_y(\boldsymbol{\sigma}^{(0)}(\underline{x}, \underline{y})) + \sum_{n=0}^{n=+\infty} \epsilon^n [\text{div}_y(\boldsymbol{\sigma}^{(n+1)}(\underline{x}, \underline{y})) + \text{div}_x(\boldsymbol{\sigma}^{(n)}(\underline{x}, \underline{y}))] = 0 \quad (12)$$

By collecting all the terms having the same power in ϵ we obtain:

$$\begin{aligned} \text{div}_y(\boldsymbol{\sigma}^{(0)}(\underline{x}, \underline{y})) &= 0 \\ \text{div}_x(\boldsymbol{\sigma}^{(n)}(\underline{x}, \underline{y})) + \text{div}_y(\boldsymbol{\sigma}^{(n+1)}(\underline{x}, \underline{y})) &= 0 \quad \text{for} \quad n \geq 0 \end{aligned} \quad (13)$$

By taking the average of each term in (13) over the volume of the cell (denoted V) we obtain:

$$\begin{aligned} \int_{\partial V} \boldsymbol{\sigma}^{(0)}(\underline{x}, \underline{y}) \cdot \underline{n} dS &= 0 \\ \text{div}_x(\langle \boldsymbol{\sigma}^{(n)}(\underline{x}, \underline{y}) \rangle_V) + \int_{\partial V} \boldsymbol{\sigma}^{(n+1)}(\underline{x}, \underline{y}) \cdot \underline{n} dS &= 0 \quad \text{for} \quad n \geq 0 \end{aligned} \quad (14)$$

in which the divergence theorem has been used in order to replace the volume integrals into integrals over the boundary ∂V of the cell. In the relation above, $\langle \bullet \rangle_V$ denotes the average of \bullet over the volume of the cell (here the integration is performed with respect to variable \underline{y}). Each surface integral over ∂V is null due to the equilibrium between two neighboring cells. It remains:

$$\text{div}_x(\langle \boldsymbol{\sigma}^{(n)}(\underline{x}, \underline{y}) \rangle_V) = 0 \quad \text{for} \quad n \geq 0 \quad (15)$$

It is possible to eliminate the mean part $\langle \boldsymbol{\sigma}^{(n)}(\underline{x}, \underline{y}) \rangle_V$ in the second relation in (13):

$$\text{div}_x^*(\boldsymbol{\sigma}^{(n)}(\underline{x}, \underline{y})) + \text{div}_y(\boldsymbol{\sigma}^{(n+1)}(\underline{x}, \underline{y})) = 0 \quad \text{for} \quad n \geq 0 \quad (16)$$

in which we have introduced div_x^* , defined, for any function $\mathbf{f}(\underline{x}, \underline{y})$, by:

$$\text{div}_x^*[\mathbf{f}(\underline{x}, \underline{y})] = \text{div}_x[\mathbf{f}(\underline{x}, \underline{y}) - \langle \mathbf{f}(\underline{x}, \underline{y}) \rangle_V] \quad (17)$$

Finally, by using (8) and (11) for modifying (16), we obtain a hierarchy of equations for the unknowns fields $\underline{u}^{(1)}(\underline{x}, \underline{y})$, $\underline{u}^{(2)}(\underline{x}, \underline{y})$, $\underline{u}^{(3)}(\underline{x}, \underline{y})$, etc. The problem at the first order reads:

$$\begin{cases} \text{div}_y[\boldsymbol{\sigma}^{(0)}(\underline{x}, \underline{y})] = 0, & \boldsymbol{\sigma}^{(0)}(\underline{x}, \underline{y}) = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\mathbf{E}(\underline{x}) + \boldsymbol{\varepsilon}_y(\underline{u}^{(1)}(\underline{x}, \underline{y}))) \\ [\underline{u}^{(1)}(\underline{x}, \underline{y})]_\Gamma = 0, & [\boldsymbol{\sigma}^{(0)}(\underline{x}, \underline{y}) \cdot \underline{n}]_\Gamma = 0, \\ \underline{u}^{(1)}(\underline{x}, \underline{y}) \underline{y}\text{-periodic}, & \boldsymbol{\sigma}^{(0)}(\underline{x}, \underline{y}) \cdot \underline{n} \underline{y}\text{-antiperiodic} \end{cases} \quad (18)$$

In the above equations we search the displacement $\underline{u}^{(1)}(\underline{x}, \underline{y})$ for a given value of the applied macroscopic strain $\mathbf{E}(\underline{x})$. The problem at the second order is:

$$\begin{cases} \text{div}_y[\boldsymbol{\sigma}^{(1)}(\underline{x}, \underline{y})] = -\text{div}_x^*[\boldsymbol{\sigma}^{(0)}(\underline{x}, \underline{y})], \\ \boldsymbol{\sigma}^{(1)}(\underline{x}, \underline{y}) = \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}^{(0)}(\underline{x}, \underline{y})) : [\boldsymbol{\varepsilon}_y(\underline{u}^{(2)}(\underline{x}, \underline{y})) + \boldsymbol{\varepsilon}_x(\underline{u}^{(1)}(\underline{x}, \underline{y}))] \\ [\underline{u}^{(2)}(\underline{x}, \underline{y})]_\Gamma = 0, & [\boldsymbol{\sigma}^{(1)}(\underline{x}, \underline{y}) \cdot \underline{n}]_\Gamma = 0, \\ \underline{u}^{(2)}(\underline{x}, \underline{y}) \underline{y}\text{-periodic}, & \boldsymbol{\sigma}^{(1)}(\underline{x}, \underline{y}) \cdot \underline{n} \underline{y}\text{-antiperiodic} \end{cases} \quad (19)$$

In this problem, $\underline{u}^{(1)}(\underline{x}, \underline{y})$ is known since it has been computed at the first order. Also, $\boldsymbol{\varepsilon}^{(0)}(\underline{x}, \underline{y})$ is known because it depends on the displacement $\underline{u}^{(1)}(\underline{x}, \underline{y})$ (see Eq. (9)). In the system of Eq. (19), the unknown variable is $\underline{u}^{(2)}(\underline{x}, \underline{y})$.

The problem at the third order takes the form:

$$\begin{cases} \operatorname{div}_y [\boldsymbol{\sigma}^{(2)}(\underline{x}, y)] = -\operatorname{div}_x^* [\boldsymbol{\sigma}^{(1)}(\underline{x}, y)], \\ \boldsymbol{\sigma}^{(2)}(\underline{x}, y) = \frac{\partial^2 \psi}{\partial \varepsilon_i \partial \varepsilon_j}(\boldsymbol{\varepsilon}^{(0)}(\underline{x}, y)) : [\boldsymbol{e}_y(\underline{u}^{(2)}(\underline{x}, y)) + \boldsymbol{e}_x(\underline{u}^{(2)}(\underline{x}, y))] \\ + \frac{1}{2} \frac{\partial^2 \psi}{\partial \varepsilon_i \partial \varepsilon_j}(\boldsymbol{\varepsilon}^{(1)}(\underline{x}, y)) : (\boldsymbol{\varepsilon}^{(1)}(\underline{x}, y) \otimes \boldsymbol{\varepsilon}^{(1)}(\underline{x}, y)) \\ \left[\underline{u}^{(3)}(\underline{x}, y) \right]_{\Gamma} = 0, \quad \left[\boldsymbol{\sigma}^{(2)}(\underline{x}, y) \cdot \underline{n} \right]_{\Gamma} = 0, \\ \underline{u}^{(3)}(\underline{x}, y) \text{ } y\text{-periodic, } \boldsymbol{\sigma}^{(2)}(\underline{x}, y) \cdot \underline{n} \text{ } y\text{-antiperiodic} \end{cases} \quad (20)$$

In these equations, the fields $\underline{u}^{(1)}(\underline{x}, y)$, $\underline{u}^{(2)}(\underline{x}, y)$, $\boldsymbol{\varepsilon}^{(0)}(\underline{x}, y)$ and $\boldsymbol{\varepsilon}^{(1)}(\underline{x}, y)$ are known, being computed at lower orders. The unknown is the displacement $\underline{u}^{(3)}(\underline{x}, y)$.

To summarize, all terms of the series (6) are computed iteratively. The first cell problem (18) is non linear for the displacement $\underline{u}^{(1)}(\underline{x}, y)$ and depends on the macroscopic variable \underline{x} through $\boldsymbol{E}(\underline{x})$. Interestingly, it is observed that higher order cell problems are linear for $\underline{u}^{(2)}(\underline{x}, y)$, $\underline{u}^{(3)}(\underline{x}, y)$, etc. and can be interpreted as linear elasticity problems with source terms which depend on the solution at lower orders and involve the tangent modulus computed with the strain obtained at the first order. Since the local potentials ψ are assumed to be strictly convex, the tangent modulus is strictly positive definite. This ensures the uniqueness of the higher order cell problems. In the next section we propose to derive some analytic expressions for the displacements $\underline{u}^{(1)}(\underline{x}, y)$ and $\underline{u}^{(2)}(\underline{x}, y)$ in the particular case of a stratified composite.

3. Solutions for a stratified composite

In this section, we propose to derive the general solution of the first and second order cell problems for the stratified composite. This solution is provided for arbitrary microscopic strain potentials. The macroscopic potential which includes the strain gradient effect at the second order is also provided and compared with the phenomenological model of Fleck and Hutchinson (1997).

3.1. The unit cell

Let us consider the case of a three dimensional layered material, whose layers have the respective thicknesses f_a and f_b (see Fig. 1). The dimensions of the cells are equal to 1; consequently f_a and f_b represent the volume fractions of the phases. The layers being periodically distributed along the axis Ox_1 , the displacement locally oscillates with the non dimensional coordinate y_1 which varies from $-f_b$ to f_a in a period (y_1 is denoted y thereafter for the sake of simplicity). The composite is subjected to a multiaxial macro-

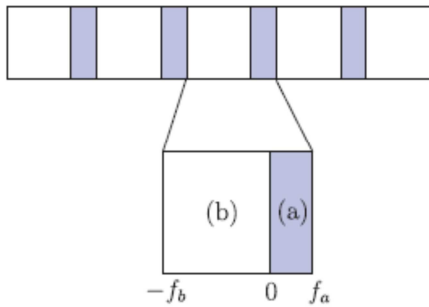


Fig. 1. The unit cell of the periodic stratified composite.

scopic strain, involving the components E_{ij} for $i, j = 1, 2, 3$. Also, the dependence of the macroscopic variables with the coordinates x_1, x_2, x_3 is omitted for clarity.

3.2. Solution of the first order cell problem

Owing to the one-dimensional geometry of the unit cell (periodic arrangement of thin layers in the direction Oy_1) the local fields only fluctuate with the coordinate $y \equiv y_1$. As a consequence, all the derivatives with respect to y_2 and y_3 in the system of Eq. (18) vanish. The problem becomes:

$$\begin{cases} \frac{d^2 \underline{u}^{(1)}}{dy^2}(y) = 0, \\ \sigma_{ij}^{(0)}(y) = \frac{\partial \psi}{\partial \varepsilon_{ij}}(\boldsymbol{\varepsilon}^{(0)}(y)), \\ \varepsilon_{ij}^{(0)}(y) = E_{ij} + \frac{1}{2} \left[\frac{du_i^{(1)}}{dy}(y) \delta_{j1} + \frac{du_j^{(1)}}{dy}(y) \delta_{i1} \right], \\ \left[u_i^{(1)}(y=0) \right] = 0, \quad \left[\sigma_{i1}^{(0)}(y=0) \right] = 0, \\ u_i^{(1)}(y=f_a) = u_i^{(1)}(y=-f_b), \quad \sigma_{i1}^{(0)}(y=f_a) = \sigma_{i1}^{(0)}(y=-f_b) \end{cases} \quad (21)$$

The equilibrium equation combined with the continuity of $\sigma_{i1}^{(0)}(y)$ at the interface in $y=0$ implies that the components $\sigma_{i1}^{(0)}(y)$ for $i = 1, 2, 3$ are constant within the cell. As a consequence, the components of the microscopic strain tensor $\boldsymbol{\varepsilon}^{(0)}(y)$ are piecewise constant functions. The components of $\underline{u}^{(1)}(y)$ are then piecewise linear functions, which are continuous at $y=0$ and periodic. Such functions have the following form:

$$u_i^{(1)}(y) = \alpha_i(\boldsymbol{E}) F(y) \quad (22)$$

where $\alpha_i(\boldsymbol{E})$ for $i = 1, 2, 3$ are three functions of the macroscopic strain \boldsymbol{E} and $F(y)$ is given by:

$$F(y) = \begin{cases} F_a(y) = \frac{1}{2} - \frac{y}{f_a} & \text{for } y \in (a) \\ F_b(y) = \frac{1}{2} + \frac{y}{f_b} & \text{for } y \in (b) \end{cases} \quad (23)$$

The microscopic strain $\boldsymbol{\varepsilon}^{(0)}(y)$ reads:

$$\begin{cases} \varepsilon_{ij}^{(0)} = E_{ij} - \frac{1}{2f_a} [\alpha_i(\boldsymbol{E}) \delta_{j1} + \alpha_j(\boldsymbol{E}) \delta_{i1}], \\ \varepsilon_{ij}^{(0)} = E_{ij} + \frac{1}{2f_b} [\alpha_i(\boldsymbol{E}) \delta_{j1} + \alpha_j(\boldsymbol{E}) \delta_{i1}] \end{cases} \quad (24)$$

where $\varepsilon_{ij}^{(0)}(y)$ and $\varepsilon_{ij}^{(0)}(y)$ are the values of local strains in phase (a) and (b) respectively. The stress components read in both layers:

$$\begin{cases} \sigma_{ij}^{(0)} = \frac{\partial \psi_a}{\partial \varepsilon_{ij}}(\boldsymbol{\varepsilon}_a^{(0)}), \\ \sigma_{ij}^{(0)} = \frac{\partial \psi_b}{\partial \varepsilon_{ij}}(\boldsymbol{\varepsilon}_b^{(0)}) \end{cases} \quad (25)$$

where ψ_a and ψ_b are the expressions of ψ taken in layer (a) and (b) respectively. The continuity of $\sigma_{i1}^{(0)}(y)$ at $y=0$ provides three non linear algebraic equations defining the function $\alpha_i(\boldsymbol{E})$ for $i = 1, 2, 3$:

$$\frac{\partial \psi_a}{\partial \varepsilon_{i1}}(\boldsymbol{\varepsilon}_a^{(0)}) = \frac{\partial \psi_b}{\partial \varepsilon_{i1}}(\boldsymbol{\varepsilon}_b^{(0)}) \quad (26)$$

Obviously, the derivation $\alpha_i(\boldsymbol{E})$ for $i = 1, 2, 3$ could be effected by giving the expression of the potential $\psi(\boldsymbol{\varepsilon})$.

3.3. Solution at the second order

We derive now the solution of the second order cell problem (19) which introduces the microstructural effects and the dependence on the macroscopic strain gradient. Again, in (19), all the derivatives with respect to y_2 and y_3 vanish. Moreover, the stress $\boldsymbol{\sigma}^{(0)}(y)$ and the displacement $\underline{u}^{(1)}(y)$, given by (25) and (22), are introduced as source terms. The second order problem reads:

$$\begin{cases} \frac{d\sigma_{ij}^{(1)}}{dy}(y) + g_i F'(y) = 0, \\ \sigma_{ij}^{(1)}(y) = C_{ijk1}(y) \frac{du_i^{(2)}}{dy}(y) + C_{ijk2}(y) e_{kl} F(y) \\ \left[u_i^{(2)}(y=0) = 0, \quad \left[\sigma_{ij}^{(1)}(y=0) = 0 \right. \right. \\ \left. \left. u_i^{(2)}(y=f_a) = u_i^{(2)}(y=-f_b), \quad \sigma_{ij}^{(1)}(y=f_a) = \sigma_{ij}^{(1)}(y=-f_b) \right. \right. \end{cases} \quad (27)$$

where g_i and e_{ij} are constant while the tangent modulus $C_{ijk}(y)$ is a piecewise constant function. Their expressions are:

$$C_{ijk}(y) = \frac{\partial^2 \psi}{\partial \epsilon_{ij} \partial \epsilon_{kl}}(\epsilon^{(0)}(y))$$

$$g_i = f_a f_b \left[C_{ijk}^b - C_{ijk}^a \right] E_{pqj} + f_a f_b \left[f_b C_{ijk}^a + f_a C_{ijk}^b \right] D_{kl} \left[C_{1kpq}^a - C_{1kpq}^b \right] E_{pqj} \quad (28)$$

$$e_{ij} = \frac{f_a f_b}{2} \left[D_{ik} \delta_{jr} + D_{jk} \delta_{ir} \right] \left[C_{1kpq}^a - C_{1kpq}^b \right] E_{pqr}$$

In the relation above, E_{ijk} are the components of the macroscopic gradient of strain $\nabla_x \mathbf{E}$ while C_{ijk}^a and C_{ijk}^b are the values taken by $C_{ijk}(y)$ in layer (a) and (b) respectively. The components D_{ij} are defined as:

$$(\mathbf{D}^{-1})_{ij} = f_b C_{ijj1}^a + f_a C_{ijj1}^b \quad (29)$$

The details about the derivations of g_i and e_{ij} can be found in [Appendix A](#).

The solution of (27) is searched with the form:

$$\frac{du_i^{(2)}}{dy}(y) = q_i(y) F(y) \quad (30)$$

where $q_i(y)$ is constant in each layer. Introducing this expression into (27) leads to:

$$q_i(y) = -Q_{ij}(y) [g_j + C_{j1k}(y) e_{kl}] \quad (31)$$

where $Q_{ij}(y)$ is defined by:

$$\left[\mathbf{Q}^{-1} \right]_{ij} = C_{ijj1}(y) \quad (32)$$

The components of the strain tensor $\epsilon^{(1)}(y)$ are:

$$\epsilon_{ij}^{(1)}(y) = \left[e_{ij} + \frac{q_i(y) \delta_{j1} + q_j(y) \delta_{i1}}{2} \right] F(y) \quad (33)$$

The microscopic displacement $u^{(2)}(y)$ can be obtained by the integration of (30) with the periodic conditions at the boundary of the cell and the continuity at the interface in $y=0$. The analytic expression of $u^{(2)}(y)$ is not provided here since the computation of the macroscopic elastic energy involves only the microscopic strains.

3.4. The macroscopic potential

The macroscopic potential, denoted by $\Psi(\underline{x})$, is obtained from the standard average rule:

$$\Psi = \langle \psi(\epsilon(y)) \rangle_V \quad (34)$$

This macroscopic potential is evaluated by keeping in $\epsilon(\underline{x}, y)$ the first two terms of the series:

$$\epsilon(y) = \epsilon^{(0)}(y) + \epsilon \epsilon^{(1)}(y) + o(\epsilon) \quad (35)$$

Let us recall that $\epsilon^{(0)}(y)$ and $\epsilon^{(1)}(y)$ are non linear functions of the macroscopic strain and $\epsilon^1(y)$ also depends on the gradient of macroscopic strain. Higher order terms which introduce a dependence on higher order derivatives of macroscopic strain (double gradient of macroscopic strain, triple gradient of macroscopic strain etc.) are not considered here. It follows that the macroscopic potential

(34) with (35) only depends on the macroscopic strain and its gradient. The resulting macroscopic model then provides a non linear version of the second gradient model of [Mindlin \(1964\)](#) for the stratified composite.

Due to the presence of the small parameter ϵ , the macroscopic potential is also expanded along a power series in ϵ :

$$\Psi = \Psi_0 + \epsilon \Psi_1 + \epsilon^2 \Psi_2 + o(\epsilon^2) \quad (36)$$

with:

$$\begin{aligned} \Psi_0 &= \langle \psi(\underline{\epsilon}^{(0)}(y)) \rangle_V \\ \Psi_1 &= \langle \psi_{,\alpha}(\epsilon^{(0)}(y)) : \epsilon^{(1)}(y) \rangle_V \\ \Psi_2 &= \frac{1}{2} \langle \epsilon^{(1)}(y) : \psi_{,\alpha\alpha}(\epsilon^{(0)}(y)) : \epsilon^{(1)}(y) \rangle_V \end{aligned} \quad (37)$$

It can be noticed that the expansion of the macroscopic potential at the first order in ϵ is licit since only the first two terms in the expansion of the microscopic strain have been kept (see Eq. (35)). The strain $\epsilon^{(2)}(y)$ could be also considered for the computation of Ψ_2 . However the term Ψ_1 is null due to the geometry of the elementary cell and a truncation at the first order for the stratified composite is then not able to capture the microstructural effects. These effects are accounted by the second order term in the expansion of Ψ , but in which $\epsilon^{(2)}(y)$ is not considered since only the effect of the first gradient of macroscopic strain is taken into account.

Owing to expression (33), we obtain for Ψ_0 , Ψ_1 and Ψ_2 :

$$\begin{aligned} \Psi_0 &= f_a \psi_a(\underline{\epsilon}_a^{(0)}) + f_b \psi_b(\underline{\epsilon}_b^{(0)}) \\ \Psi_1 &= 0 \\ \Psi_2 &= \frac{1}{24} \left[f_a l_{ijpq}^a + f_b l_{ijpq}^b \right] e_{ij} e_{pq} + \frac{1}{24} \left[f_a Q_{ij}^a + f_b Q_{ij}^b \right] g_i g_j \end{aligned} \quad (38)$$

with:

$$l_{ijk}^a = C_{ijpq}^a - C_{ijk}^a Q_{kl}^a C_{1kpq}^a, \quad \beta = a, b \quad (39)$$

In (38), Ψ_0 has a non linear dependence on $\underline{\epsilon}^{(0)}(y)$ and then on the macroscopic strain \mathbf{E} . The term Ψ_2 has also a non linear dependence on \mathbf{E} (through C_{ijk}^a and C_{ijk}^b) and is quadratic with respect to the macroscopic gradient of strain $\nabla_x \mathbf{E}$ (see the definitions of g_i and e_{ij} in (28) which show that these terms linearly depend on $\nabla_x \mathbf{E}$). These solutions generalize those obtained by [Boutin \(1996\)](#) which can be retrieved by putting:

$$C_{ijk}^a = \lambda_a \delta_{ij} \delta_{kl} + 2\mu_a l_{ijk}, \quad C_{ijk}^b = \lambda_b \delta_{ij} \delta_{kl} + 2\mu_b l_{ijk} \quad (40)$$

Still in the case of linear elasticity, Ψ_0 is quadratic with the macroscopic strain while Ψ_2 is independent of \mathbf{E} and is quadratic with respect to the gradient of strain. One recovers the first strain gradient model of [Mindlin \(1964\)](#). The components of strain gradient elasticity can be found in [Tran et al. \(2012\)](#).

Various extensions of the second gradient theory have been provided in the literature among which the Fleck and Hutchinson model ([Fleck and Hutchinson, 1993, 1997](#)). The authors postulated the following expression for the macroscopic power law type elastic energy density:

$$\Psi = \frac{\Sigma_0 E_0}{1+m} \left[\frac{\mathcal{E}}{E_0} \right]^{(1+m)/2} \quad (41)$$

where Σ_0 and E_0 are material constants, m is the power law exponent defined such that $m \geq 0$ and \mathcal{E} is the normalized² elastic energy density of the first strain gradient theory of [Mindlin \(1964\)](#) for an incompressible material. Particularly, power law elasticity is

² The elastic energy density of the second gradient for an incompressible material is divided by μ (the shear modulus) and multiplied by 2/3 in order to recover the square of the von Mises equivalent strain for \mathcal{E} when the strain gradient effects vanish.

retrieved when the energy density reduces to $\mathcal{E} = E_{eq}^2 = 2/3 \text{tr}(\bar{\mathbf{E}} \cdot \bar{\mathbf{E}})$, $\bar{\mathbf{E}}$ being the deviatoric part of the macroscopic strain. The Fleck and Hutchinson strain gradient model is not quadratic with the gradient of strain (except for the linear case corresponding to $m = 1$) that constitutes the main difference with the macroscopic potential derived from asymptotic homogenization.

4. Analysis for power law elasticity

4.1. Power law elasticity

The potential $\psi(\boldsymbol{\varepsilon})$ is chosen in the form:

$$\psi(\boldsymbol{\varepsilon}) = \begin{cases} \frac{k_a}{2} \text{tr}(\boldsymbol{\varepsilon})^2 + \frac{3\mu_a}{m+1} (e_{eq})^{m+1} & \text{in (a)} \\ \frac{k_b}{2} \text{tr}(\boldsymbol{\varepsilon})^2 + \frac{3\mu_b}{m+1} (e_{eq})^{m+1} & \text{in (b)} \end{cases} \quad (42)$$

where k_a and μ_a (resp. k_b and μ_b) are the compressibility and the shear moduli of layer (a) (resp. layer (b)) and e_{eq} is the equivalent strain (within von Mises meaning):

$$e_{eq} = \sqrt{\frac{2}{3}} \bar{\mathbf{E}} : \bar{\mathbf{E}}, \quad \bar{\mathbf{E}} = \boldsymbol{\varepsilon} - \frac{1}{3} \text{tr}(\boldsymbol{\varepsilon}) \mathbf{I} \quad (43)$$

where \mathbf{I} is the identity for second order tensors. The case of linear elasticity is retrieved for $m = 1$ while the case of a rigid plastic material corresponds to $m = 0$ and simultaneously k_a and k_b tend to infinity. The stress-strain relation is:

$$\boldsymbol{\sigma} = k_\beta \text{tr}(\boldsymbol{\varepsilon}) \mathbf{I} + 2\mu_\beta (e_{eq})^{m-1} \bar{\mathbf{E}} \quad (44)$$

with $\beta = a$ in layer (a) and $\beta = b$ in layer (b). The tangent modulus (needed for the second order homogenization problem) is given by:

$$\mathbb{C}^\beta = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}) = 3k_\beta \mathbf{J} + 2\mu_\beta (e_{eq})^{m-1} \mathbb{K} + (m-1) \frac{4\mu_\beta}{3} (e_{eq})^{m-3} \bar{\mathbf{E}} \otimes \bar{\mathbf{E}} \quad (45)$$

The fourth order tensors \mathbf{J} and \mathbb{K} are given by:

$$\mathbf{J} = \frac{1}{3} \mathbf{I} \otimes \mathbf{I}, \quad \mathbb{K} = \mathbf{I} - \mathbf{J}, \quad I_{ijkl} = \frac{1}{2} [\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}] \quad (46)$$

The case of power law incompressible materials is retrieved when $k_a = k_b \rightarrow +\infty$. When the hypothesis of incompressibility is considered, the power law corresponds to rigid viscoplastic materials in the range of power law exponent $0 < m < 1$ and corresponds to

the case of rigid plastic materials for $m = 0$ while the strain is replaced by the strain rate tensor. In such a case, the macroscopic elastic energy must be interpreted as the effective dissipation.

4.2. The antiplane solution

The derivation of explicit solutions in the non linear case requires the resolution of the non linear Eq. (26) giving the function $\alpha_i(\mathbf{E})$ as a functions of the macroscopic strain \mathbf{E} . It is possible to provide the analytic solutions under antiplane conditions, corresponding to a macroscopic displacement $U_1(x_2, x_3)$ and $U_2 = U_3 = 0$. The resulting non null components of the macroscopic strain are E_{12} and E_{13} while the components of macroscopic strain gradient are $E_{12,2}$, $E_{13,3}$ and $E_{12,3} = E_{13,2}$. The details about the derivation of functions $\alpha_i(\mathbf{E})$ and macroscopic potentials can be found in Appendix B. Additionally we assume that both constituents are incompressible. Note however that the case of compressible materials can also be investigated by using the present approach and the results remain quite similar. The macroscopic potentials have the following expressions:

$$\begin{aligned} \Psi_0 &= \frac{3\mu^{hom}}{1+m} E_{eq}^{m+1} \\ \Psi_2 &= \eta^{hom} E_{eq}^{m-1} \left[E_{12,\beta} E_{12,\beta} + \frac{1}{3} (E_{12,\alpha})^2 \right] \end{aligned} \quad (47)$$

in which μ^{hom} and η^{hom} are given by:

$$\begin{aligned} \mu^{hom} &= \frac{\mu_a \mu_b}{(f_a \mu_a^n + f_b \mu_b^n)^m} \\ \eta^{hom} &= \frac{2}{3} f_a^2 f_b^2 \mu_a \mu_b \frac{(\mu_a^n - \mu_b^n)^2}{(f_b \mu_a^n + f_a \mu_b^n)^{m+1}} \left[\frac{f_a}{\mu_a^n} + \frac{f_b}{\mu_b^n} \right] \end{aligned} \quad (48)$$

where $n = 1/m$ and E_{eq} is the macroscopic equivalent strain under antiplane sollicitation:

$$E_{eq} = \sqrt{\frac{4}{3}} (E_{12}^2 + E_{13}^2) \quad (49)$$

A first result is that the macroscopic potential Ψ_0 is a function of the equivalent strain E_{eq} of degree $m+1$ while Ψ_2 is a function of the equivalent strain E_{eq} of degree $m-1$. For a null strain of state, corresponding to $E_{eq} = 0$, the potential Ψ_0 is null and Ψ_2 is also null when $m \geq 1$. In the range $m < 1$, Ψ_2 is singular. This singularity is inherent to the use of the elastic power law at the microscopic scale. Indeed, higher order homogenization problems use the tangent elastic tensor that is singular when the strain is null. From a physical point of view, the strain energy of a solid due to an external loading must remain finite. It suggests that, when the strain tends to zero, the gradient of strain must also tend to zero so that the total strain energy of the solid remains finite.

Expressions in (47) introduce two coefficients, the effective shear modulus μ^{hom} and the coefficient η^{hom} which depend on the power law exponent, the volume fraction of the phase and the local shear moduli. On Figs. 2 and 3, we represent the variations of these two effective coefficients with the volume fraction f_a for a contrast $\mu_b/\mu_a = 2$ and for various values of the power law exponent in the range $[0.5, 1]$. When the volume fraction f_a is equal to 0 or 1 the unit cell is made up of a homogeneous material of shear modulus μ_b or μ_a respectively. In these two particular cases, we retrieve for the effective shear modulus μ^{hom} the values μ_b and μ_a . It is observed on Fig. 3 that the effective modulus η^{hom} which enters into the expression of Ψ_2 is null in both cases corresponding to $f_a = 0$ and $f_a = 1$. Since the strain gradient effects at the macroscopic scale are due to the existence of a microstructure at the lower scale, it is coherent that these effects vanish in the special cases for which the unit cell becomes homogeneous. The higher

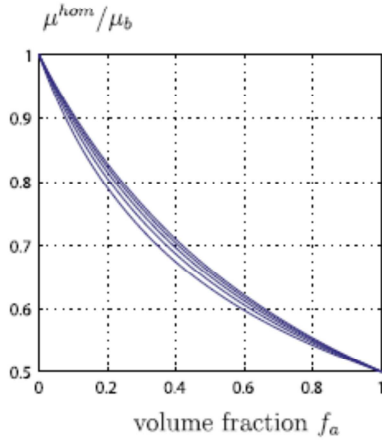


Fig. 2. Variations of μ^{hom}/μ_b with the volume fraction f_a for various values of the power law exponent. From the top: $m = 0.9$, $m = 0.8$, $m = 0.7$, $m = 0.6$, $m = 0.5$.

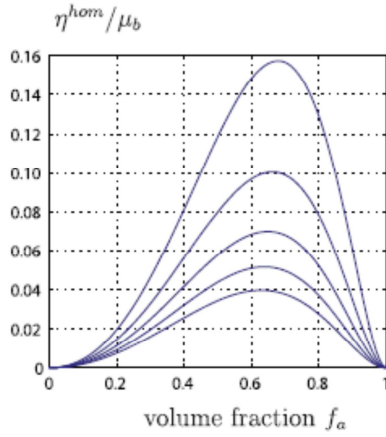


Fig. 3. Variations of η^{hom}/μ_b with the volume fraction f_α for various values of the power law exponent. From the top: $m = 0.5$, $m = 0.6$, $m = 0.7$, $m = 0.8$, $m = 0.9$.

values of η^{hom} are then attained at intermediate values of the volume fraction.

Another particular case for which the unit cell becomes homogeneous corresponds to $\mu_a = \mu_b$. It will be easily verified that in this last case the effective modulus η^{hom} is also null. Consequently, it would be expected to obtain more prominent microstructural effects by increasing the contrast. This is confirmed on Figs. 4 and 5 in which we represent the effective moduli μ^{hom} and η^{hom} as functions of the power law exponent in the range $[0.1, 100]$ and for different values of the contrast. On Fig. 4 we observe that the influence of the power law exponents on μ^{hom} is relatively small. Conversely, the power law exponent has a great influence on η^{hom} when m is small and this, particularly for high values of the contrast. On another hand, it is also observed that for large values of m the microstructural effects vanish.

Another particular case of great importance is $m = 0$ corresponding to a rigid ideally plastic material. However, some precaution must be applied when taking the limit $m \rightarrow 0$. This is explained in details in the following.

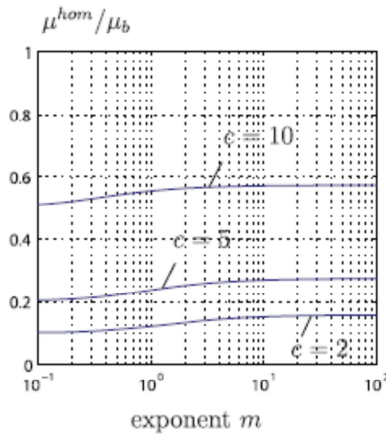


Fig. 4. Variations of μ^{hom}/μ_b as a function of the power law exponent m for three values of contrast $c = 2$, $c = 5$, $c = 10$.

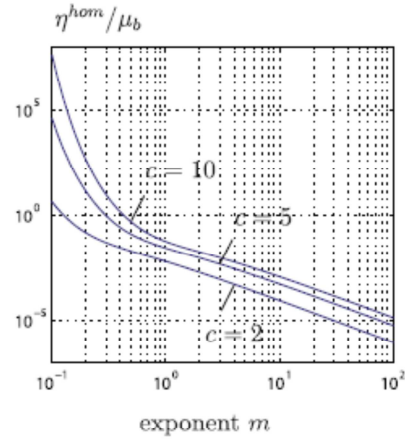


Fig. 5. Variations of η^{hom}/μ_b as a function of the power law exponent m for three values of contrast $c = 2$, $c = 5$, $c = 10$.

When the limit $m \rightarrow 0$ is taken in the expression of η^{hom} given by (48), two expressions are obtained, depending on the choice of the contrast $c = \mu_b/\mu_a$. Indeed, for $c \neq 1$ the limit is $\eta^{hom} = +\infty$ but for $c = 1$ the limit is $\eta^{hom} = 0$. In fact, as shown below, the case $c \neq 1$ and $m = 0$ is not physically admissible.

By accounting for relations (24) together with (B.3) and (B.4), the strain at the first order reads:

$$\epsilon_{1\alpha}^{(0)} = \begin{cases} \frac{c^\alpha E_{1\alpha}}{b^\alpha \mu_a^\alpha} & \text{in phase (a)} \\ \frac{E_{1\alpha}}{b^\alpha \mu_b^\alpha} & \text{in phase (b)} \end{cases} \quad (50)$$

where index α takes the values 2 or 3. There are three limits for the components of the strain when $m \rightarrow 0$ (or equivalently $n \rightarrow \infty$) depending on the value of c :

$$\begin{aligned} \bullet \quad c > 1 : \epsilon_{1\alpha}^{(0)} &= \begin{cases} \frac{E_{1\alpha}}{b^\alpha} & \text{in phase (a)} \\ 0 & \text{in phase (b)} \end{cases} \\ \bullet \quad c < 1 : \epsilon_{1\alpha}^{(0)} &= \begin{cases} 0 & \text{in phase (a)} \\ \frac{E_{1\alpha}}{b^\alpha} & \text{in phase (b)} \end{cases} \\ \bullet \quad c = 1 : \epsilon_{1\alpha}^{(0)} &= \begin{cases} E_{1\alpha} & \text{in phase (a)} \\ E_{1\alpha} & \text{in phase (b)} \end{cases} \end{aligned} \quad (51)$$

On another hand, the equilibrium at the interface between the phases implies that $\sigma_{1\alpha}^{(0)}$ is constant in the cell. When the limit $m = +\infty$ is taken, the stress attains the saturation values $3\mu_a$ and $3\mu_b$ respectively in phases (a) and (b) as illustrated on Fig. 6. However the stress strain curve is not bijective when $m = +\infty$. For a stress $\sigma_{1\alpha}^0$ equal to $3\mu_a$ in layer a (resp. $3\mu_b$ in layer b), the strain admits an infinite number of possible values comprised in the range $[0, +\infty[$. Particularly, it can be null whenever the stress is equal to $3\mu_a$, that is the case when $c > 1$ in (51). Similarly, the strain is null in phase b whenever the stress is equal to $3\mu_b$ for $c < 1$ in (51). The only physical situation corresponding to a rigid plastic material is $c = 1$ for which the strain is constant and equal to the macroscopic strain $E_{1\alpha}$ in both layers. In other words, it is not possible, under antiplane loading, to have the plastification of both layers having two different yield stresses.

It is then not possible to detect the microstructural effects in the plastic regime for an antiplane macroscopic loading. Such effects will be investigated by considering another particular macroscopic

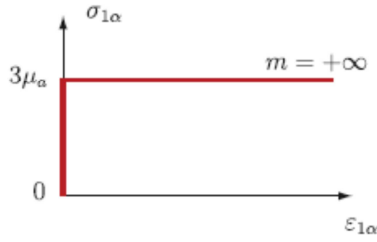


Fig. 6. Stress strain curve in the case $m = +\infty$.

loading. This is effected in the next section by considering the case of plane strain conditions.

4.3. The solution under plane strain conditions

A second particular case for which explicit expressions of the macroscopic potentials can be derived corresponds to a macroscopic loading with plane strain conditions. We consider a macroscopic displacement on the form: $U_1 = 0$, $U_2 = U_2(x_2, x_3)$ and $U_3 = U_3(x_2, x_3)$. The remaining non null components of the macroscopic strain are $E_{\alpha\beta}$ where the Greek letters indices take the values 2 or 3. Also, the hypothesis of incompressible materials is used ($k_a \rightarrow +\infty$ and $k_b \rightarrow +\infty$). This implies for the macroscopic strain that $E_{22} = -E_{33}$. The components of the macroscopic gradient of strain are then $E_{\alpha\beta,\gamma}$ with $E_{22,\gamma} = -E_{33,\gamma}$ due to the incompressibility. Again, the details leading to the analytic expressions of the macroscopic potentials can be found in [Appendix C](#).

Under plane strain conditions and for incompressible materials, the macroscopic potential Ψ_0 has the following expression:

$$\Phi_0 = \frac{3\mu^{hom}}{1+m} E_{\alpha\beta}^{m+1} \quad (52)$$

with:

$$\mu^{hom} = f_a \mu_a + f_b \mu_b \quad (53)$$

and:

$$E_{\alpha\beta} = \sqrt{\frac{4}{3} (E_{22}^2 + E_{23}^2)} \quad (54)$$

The macroscopic potential Ψ_2 can be put into the form:

$$\Psi_2 = \frac{1}{2} \begin{bmatrix} E_{22,2} \\ E_{22,3} \\ E_{33,2} \\ E_{33,3} \end{bmatrix}^T \begin{bmatrix} A & 0 & C & D \\ 0 & A & -D & C \\ C & -D & B & 0 \\ D & C & 0 & B \end{bmatrix} \begin{bmatrix} E_{22,2} \\ E_{22,3} \\ E_{33,2} \\ E_{33,3} \end{bmatrix} \quad (55)$$

In which A, B, C and D depend on the macroscopic strain and are given by:

$$A = \frac{4R}{3} E_{\alpha\beta}^{m-3} [m^2 E_{22}^2 + E_{23}^2], \quad B = \frac{4R}{3} E_{\alpha\beta}^{m-3} [E_{22}^2 + m^2 E_{23}^2] \quad (56)$$

$$C = \frac{4R}{3} E_{\alpha\beta}^{m-3} (m^2 - 1) E_{22} E_{23}, \quad D = R E_{\alpha\beta}^{m-1} m$$

with:

$$R = \frac{1}{3} \frac{f_a^2 f_b^2}{f_a + f_b} (\mu_a - \mu_b)^2 \left[\frac{f_a}{\mu_a} + \frac{f_b}{\mu_b} \right] \quad (57)$$

All functions A, B, C, D are null when $\mu_a = \mu_b$ or $f_a = 0$ or $f_b = 0$: that corresponds to a homogeneous material at the local scale. Ψ_0 is a function of the macroscopic strain of degree $m+1$ while Ψ_2 is a function of the macroscopic strain of degree $m-1$. The macroscopic potential Ψ_2 is then singular when $E_{\alpha\beta} = 0$ and $m < 1$. It must be noticed that the limit case $m \rightarrow 0$ can be applied here without any ambiguity. Indeed, for the plane strain solution, the longitudinal components of the local stress are null as it can be shown from the results of [Appendix C](#). The equilibrium at the interface between the layers is therefore verified implicitly while, for the antiplane solution, the equilibrium is verified only if $\mu_a = \mu_b$.

The variations of A, B, C and D with the macroscopic equivalent strain are represented from [Figs. 7-11](#) for the following values of the power law exponent: $m=0$ (plastic case),

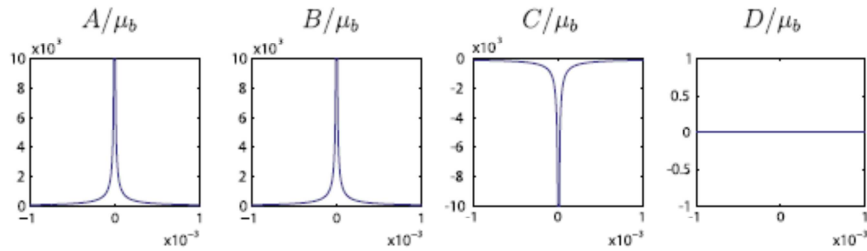


Fig. 7. Variations of A, B, C and D with the macroscopic equivalent strain for $f_a = f_b = 0.5$, $\mu_b/\mu_a = 10$, $E_{22} = E_{23}$ and for a power law exponent $m = 0$.

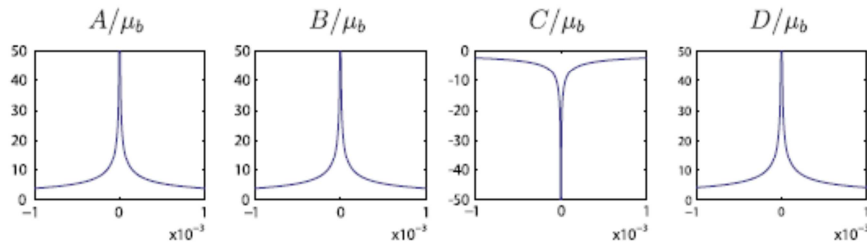


Fig. 8. Variations of A, B, C and D with the macroscopic equivalent strain for $f_a = f_b = 0.5$, $\mu_b/\mu_a = 10$, $E_{22} = E_{23}$ and for a power law exponent $m = 0.5$.

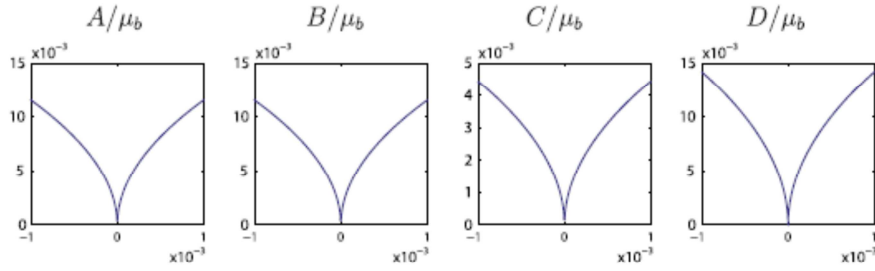


Fig. 9. Variations of A , B , C and D with the macroscopic equivalent strain for $f_a = f_b = 0.5$, $\mu_b/\mu_a = 10$, $E_{22} = E_{23}$ and for a power law exponent $m = 1.5$.

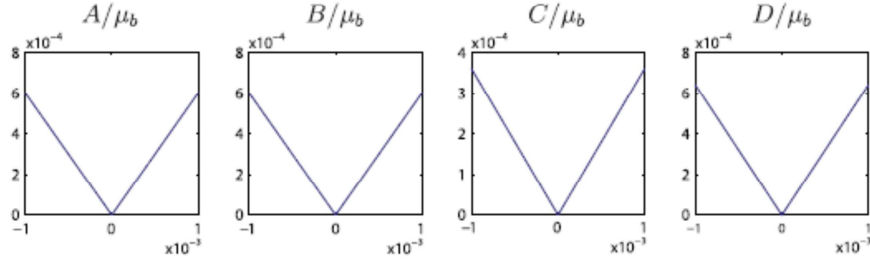


Fig. 10. Variations of A , B , C and D with the macroscopic equivalent strain for $f_a = f_b = 0.5$, $\mu_b/\mu_a = 10$, $E_{22} = E_{23}$ and for a power law exponent $m = 2$.

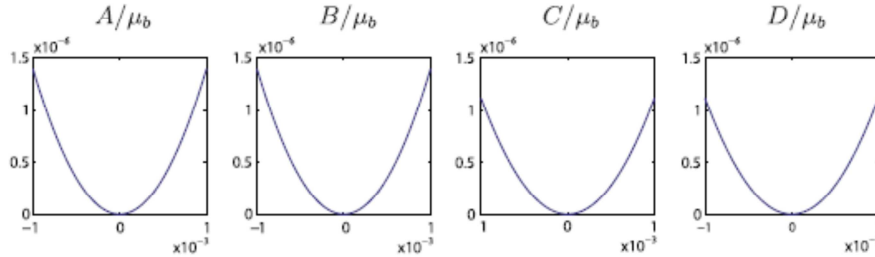


Fig. 11. Variations of A , B , C and D with the macroscopic equivalent strain for $f_a = f_b = 0.5$, $\mu_b/\mu_a = 10$, $E_{22} = E_{23}$ and for a power law exponent $m = 3$.

$m = 0.5$, $m = 1.5$, $m = 2$ and $m = 3$. In these illustrations, we consider the macroscopic loading $E_{22} = E_{23}$. The volume fraction of the phases are $f_a = f_b = 0.5$ and we consider, for the contrast, $\mu_b/\mu_a = 10$. In cases $m = 0$ and $m = 0.5$ the macroscopic potential is singular at $E_{eq} = 0$. The variations of A , B , C and D in the case $m = 1$ are not represented since they are independent of the macroscopic equivalent strain. For the values of power law exponent $m = 1.5$, $m = 2$ and $m = 3$, the macroscopic potential does not exhibit a singularity at $E_{eq} = 0$. It is also observed that the values of A , B , C and D decrease when increasing the value of the power law exponent. It suggests that the microstructural effects vanish for large values of the power law exponent but become more predominant for small values of m . This result has been already observed in the case of antiplane macroscopic loading.

5. Conclusion

In this paper we have investigated the microstructural effects in non linear composites. This analysis has been conducted in the framework of periodic homogenization combined with the method based on matched asymptotic series. The microstructural effects are introduced by keeping the second order term of the series which explicitly induces the effect of the macroscopic strain

gradient. All the calculations have been performed in the case of a stratified composite and by choosing stress strain relations for the phases which derive from microscopic strain potentials.

The consideration of stratified composites allows to produce various analytical expressions for the macroscopic potential. By including the higher order term which comes from the series expansion, it is proved that the macroscopic potential is of Mindlin (1964) type since it is quadratic with respect to the macroscopic gradient of strain. However, due to the non linearity in the local response of the phase, the macroscopic potential has a non linear dependence with the applied macroscopic strain. This non linearity introduces some couplings between the macroscopic strain and the gradient of macroscopic strain which do not exist in the second gradient theory of Mindlin (1964).

In the second part of the paper we analyze these microstructural effects in the case of non linear constituents characterized by power law constitutive equations. The explicit expressions of the macroscopic potential are provided under macroscopic antiplane and plane strain conditions. It has been shown that the microstructure effects become very significant for high values of the contrast and for small values of the power law exponent m (in the range $0 < m < 1$). Conversely, for large values of the power law exponent (in the range $m > 1$) the microstructural effects

vanish. The analytic results have also shown an unusual behavior for the macroscopic potential which becomes singular when the macroscopic strain is null. This is attributed to the use of power law which gives a tangent tensor which is singular when the strain is null. The analysis of microstructural effects can also be performed by considering a more realistic non linear local behavior such as elastoplasticity.

The extension of the present work to other geometries such as composites with fibers must be developed in a numerical approach. When dealing with standard non linear homogenization, the first order cell problem can be solved at each increment of the applied macroscopic strain. Such an approach has been often considered in the literature (Feyel, 1999; Ghosh et al., 2001). Considering now the microstructural effects, we may solve the second order cell problem enforced with the gradient of first order microscopic fields, but the gradient is taken with respect to the macroscopic space variable. The resolution of this cell problem is possible only if the fundamental solution (obtained at the first order) is computed but also the derivative of this solution with respect to the applied macroscopic strain, which is not usually produced by standard numerical methods. Therefore, new numerical strategies must be developed to compute this derivative, this will be the subject of a future work.

Appendix A. Detailed calculation of g_i and e_y

When the derivatives with respect to y_2 and y_3 vanish in (19), the equilibrium equation becomes:

$$\frac{d\sigma_{11}^{(1)}}{dy}(y) = - \left[\frac{\partial \sigma_{ij}^{(0)}}{\partial x_j} - \left\langle \frac{\partial \sigma_{ij}^{(0)}}{\partial x_j} \right\rangle_V \right] \quad (\text{A.1})$$

Note that $\sigma_{ij}^{(0)}(y)$ takes the constant values $\sigma_{ij_a}^{(0)}$ and $\sigma_{ij_b}^{(0)}$ in each layer which are given by (25). Consequently, we have:

$$\left\langle \frac{\partial \sigma_{ij}^{(0)}}{\partial x_j} \right\rangle_V = f_a \frac{\partial \sigma_{ij_a}^{(0)}}{\partial x_j} + f_b \frac{\partial \sigma_{ij_b}^{(0)}}{\partial x_j} \quad (\text{A.2})$$

It follows that the equilibrium can be put into the form:

$$\frac{d\sigma_{11}^{(1)}}{dy}(y) + g_i F'(y) = 0 \quad (\text{A.3})$$

where $F(y)$ is given by (23) and g_i reads:

$$g_i = f_a f_b \frac{\partial}{\partial x_i} \left[\sigma_{ij_b}^{(0)} - \sigma_{ij_a}^{(0)} \right] \quad (\text{A.4})$$

Accounting for the definition (25) and expression (9) for $\sigma^{(0)}(y)$, it follows that:

$$\frac{\partial \sigma_{ij}^{(0)}}{\partial x_j}(y) = C_{ijpq}(y) \frac{\partial \varepsilon_{kl}^{(0)}}{\partial x_k}(y) = C_{ijpq}(y) E_{pqj} + C_{ijk1}(y) \frac{\partial \alpha_k}{\partial x_j} F'(y) \quad (\text{A.5})$$

It leads to:

$$g_i = f_a f_b \left[C_{ijpq}^b - C_{ijpq}^a \right] E_{pqj} + \left[f_a C_{ijk1}^b + f_b C_{ijk1}^a \right] \frac{\partial \alpha_k}{\partial x_j} \quad (\text{A.6})$$

On another hand, since $\sigma_{11}^{(0)}$ is constant in the cell, we have:

$$C_{i1pq}^a - \frac{C_{i1ik}^a}{f_a} \frac{\partial \alpha_k}{\partial E_{pq}} = C_{i1pq}^b + \frac{C_{i1ik}^b}{f_b} \frac{\partial \alpha_k}{\partial E_{pq}} \quad (\text{A.7})$$

which provides the expression for the derivative of α_k with respect to E_{pq} :

$$\frac{\partial \alpha_k}{\partial E_{pq}} = f_a f_b D_{kr} \left[C_{1rpq}^a - C_{1rpq}^b \right] \quad (\text{A.8})$$

where D_{ij} is given by (29). Accounting for relations (A.6) and (A.8), we obtain expression g_i in (28). On another hand, e_y is given by:

$$e_y = \frac{1}{2} \left[\frac{\partial \alpha_i}{\partial x_j} + \frac{\partial \alpha_j}{\partial x_i} \right] \quad (\text{A.9})$$

Accounting for (A.8), we retrieve the expression of e_y given in (28).

Appendix B. Antiplane solution

The non linear Eq. (26) reads for $i = 1, 2, 3$:

$$\begin{aligned} i = 1: & \quad -k_a \frac{\alpha_1}{f_a} - \frac{4\mu_a}{3} (\varepsilon_{eq,a}^{(0)})^{m-1} \frac{\alpha_1}{f_a} = k_b \frac{\alpha_1}{f_b} + \frac{4\mu_b}{3} (\varepsilon_{eq,b}^{(0)})^{m-1} \frac{\alpha_1}{f_b} \\ i = 2: & \quad 2\mu_a (\varepsilon_{eq,a}^{(0)})^{m-1} \left(E_{12} - \frac{\alpha_2}{2f_a} \right) = 2\mu_b (\varepsilon_{eq,b}^{(0)})^{m-1} \left(E_{12} + \frac{\alpha_2}{2f_b} \right) \\ i = 3: & \quad 2\mu_a (\varepsilon_{eq,a}^{(0)})^{m-1} \left(E_{13} - \frac{\alpha_3}{2f_a} \right) = 2\mu_b (\varepsilon_{eq,b}^{(0)})^{m-1} \left(E_{13} + \frac{\alpha_3}{2f_b} \right) \end{aligned} \quad (\text{B.1})$$

in which $\varepsilon_{eq,a}^{(0)}$ and $\varepsilon_{eq,b}^{(0)}$ are given by:

$$\begin{aligned} \varepsilon_{eq,a}^{(0)} &= \sqrt{\frac{4}{9} \frac{\alpha_1^2}{f_a^2} + \frac{4}{3} \left(E_{12} - \frac{\alpha_2}{2f_a} \right)^2 + \frac{4}{3} \left(E_{13} - \frac{\alpha_3}{2f_a} \right)^2} \\ \varepsilon_{eq,b}^{(0)} &= \sqrt{\frac{4}{9} \frac{\alpha_1^2}{f_b^2} + \frac{4}{3} \left(E_{12} + \frac{\alpha_2}{2f_b} \right)^2 + \frac{4}{3} \left(E_{13} + \frac{\alpha_3}{2f_b} \right)^2} \end{aligned} \quad (\text{B.2})$$

The solutions to these equations are:

$$\alpha_1 = 0, \quad \alpha_2 = \alpha E_{12}, \quad \alpha_3 = \alpha E_{13} \quad (\text{B.3})$$

where the constant α is given by:

$$\alpha = 2f_a f_b \frac{\mu_a^n - \mu_b^n}{f_a \mu_a^n + f_b \mu_b^n} \quad (\text{B.4})$$

with $n = 1/m$. The computation of Φ_0 is trivial and leads to:

$$\Psi_0 = \frac{3\mu^{hom}}{1+m} E_{eq}^{m+1} \quad (\text{B.5})$$

in which μ^{hom} is given by:

$$\mu^{hom} = \frac{\mu_a \mu_b}{(f_a \mu_a^n + f_b \mu_b^n)^m} \quad (\text{B.6})$$

For the computation of Ψ_2 , we first have to calculate g_i and e_y . To this end, it is easier to use expressions (A.6) and (A.9) together with (B.3). It leads to:

$$\begin{aligned} e_y &= \alpha E_{1ij} \\ g_i &= f_a f_b \left[C_{ij1k}^b - C_{ij1k}^a \right] E_{1kj} + \alpha \left[f_a C_{ijk1}^b + f_b C_{ijk1}^a \right] E_{1kj} \end{aligned} \quad (\text{B.7})$$

Let us recall that g_i is null in any case. Since E_{1kj} is non null only for $j, k = 2, 3$, the components g_α are given by using indices 2, 3:

$$g_\alpha = f_a f_b \left[C_{\alpha p\gamma 1}^b - C_{\alpha p\gamma 1}^a \right] E_{1\gamma p} + \alpha \left[f_a C_{\alpha p\gamma 1}^b + f_b C_{\alpha p\gamma 1}^a \right] E_{1\gamma p} \quad (\text{B.8})$$

The tangent modulus C_{ijkl}^c reads:

$$C_{ijkl}^c = 3k_a J_{ijkl} + 2\mu_a E_{eq}^{m-1} \left(1 - \frac{\alpha}{2f_a} \right)^{m-1} \times \left[K_{ijkl} + \frac{2(m-1)}{3E_{eq}^2} E_{ij} E_{kl} \right] \quad (\text{B.9})$$

where the non null components E_{ij} are E_{12} and E_{13} . It can be easily verified that the components $C_{\alpha p\gamma 1}^c$ are null. Similarly, all components $C_{\alpha p\gamma 1}^b$ are also null. Then g_i is null. It follows that Ψ_2 reads:

$$\Psi_2 = \frac{\alpha^2}{24} \left[f_a L_{\alpha p\gamma \delta}^a + f_b L_{\alpha p\gamma \delta}^b \right] E_{1\alpha p} E_{1\gamma \delta} \quad (\text{B.10})$$

The components $L_{\alpha\beta\gamma\delta}^a$ are given by:

$$L_{\alpha\beta\gamma\delta}^a = C_{\alpha\beta\gamma\delta}^a - C_{\alpha\beta k l}^a Q_{k l}^a C_{l i j \delta}^a \quad (\text{B.11})$$

The only non null components of $C_{\alpha\beta k l}^a$ are $C_{\alpha\beta 11}^a$ which read:

$$C_{\alpha\beta 11}^a = \frac{1}{3} \left[3k_a - 2\mu_a E_{eq}^{m-1} \left(1 - \frac{\alpha}{2f_a} \right)^{m-1} \right] \delta_{\alpha\beta} \quad (\text{B.12})$$

On another hand:

$$C_{i i j j}^a = k_a \delta_{i j} \delta_{j i} + 2\mu_a E_{eq}^{m-1} \left(1 - \frac{\alpha}{2f_a} \right)^{m-1} \times \left[K_{i i j j} + \frac{2(m-1)}{3E_{eq}^2} E_{i i} E_{j j} \right] \quad (\text{B.13})$$

It appears that $Q_{11}^a = 1/k_a$ since the term proportional to μ_a is null for $i = 1$ and $j = 2, 3$. The components $C_{\alpha\beta\gamma\delta}^a$ read:

$$C_{\alpha\beta\gamma\delta}^a = 3k_a J_{\alpha\beta\gamma\delta} + 2\mu_a E_{eq}^{m-1} \left(1 - \frac{\alpha}{2f_a} \right)^{m-1} K_{\alpha\beta\gamma\delta} \quad (\text{B.14})$$

Accounting for (B.10)–(B.12) and (B.14), and taking the limit $k_a \rightarrow +\infty$ one has:

$$L_{\alpha\beta\gamma\delta}^a = 2\mu_a E_{eq}^{m-1} \left(1 - \frac{\alpha}{2f_a} \right)^{m-1} (I_{\alpha\beta\gamma\delta} + J_{\alpha\beta\gamma\delta}) \quad (\text{B.15})$$

Similarly, we have for $L_{\alpha\beta\gamma\delta}^b$:

$$L_{\alpha\beta\gamma\delta}^b = 2\mu_b E_{eq}^{m-1} \left(1 + \frac{\alpha}{2f_b} \right)^{m-1} (I_{\alpha\beta\gamma\delta} + J_{\alpha\beta\gamma\delta}) \quad (\text{B.16})$$

It leads to the following expression for the macroscopic potential:

$$\Psi_2 = \frac{\alpha^2}{12} \left[f_a \mu_a \left(1 - \frac{\alpha}{2f_a} \right) + f_b \mu_b \left(1 + \frac{\alpha}{2f_b} \right) \right] \times \left[E_{1\alpha\beta} E_{1\alpha\beta} + \frac{1}{3} (E_{1\alpha\alpha})^2 \right] \quad (\text{B.17})$$

Finally, α can be replaced by its expression (B.4) into the potential.

Appendix C. Solution under plane strain conditions

The non linear Eq. (26) reads for $i = 1, 2, 3$:

$$\begin{aligned} i = 1: & -k_a \frac{\alpha_1}{f_a} - \frac{4\mu_a}{3} \left(\epsilon_{\alpha\beta a}^{(0)} \right)^{m-1} \frac{\alpha_1}{f_a} = k_b \frac{\alpha_1}{f_b} + \frac{4\mu_b}{3} \left(\epsilon_{\alpha\beta b}^{(0)} \right)^{m-1} \frac{\alpha_1}{f_b} \\ i = 2: & -2\mu_a \left(\epsilon_{\alpha\beta a}^{(0)} \right)^{m-1} \frac{\alpha_2}{2f_a} = 2\mu_b \left(\epsilon_{\alpha\beta b}^{(0)} \right)^{m-1} \frac{\alpha_2}{2f_b} \\ i = 3: & -2\mu_a \left(\epsilon_{\alpha\beta a}^{(0)} \right)^{m-1} \frac{\alpha_3}{2f_a} = 2\mu_b \left(\epsilon_{\alpha\beta b}^{(0)} \right)^{m-1} \frac{\alpha_3}{2f_b} \end{aligned} \quad (\text{C.1})$$

These equations admit the trivial solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Consequently, $\epsilon_a = \mathbf{E}$ and $\epsilon_b = \mathbf{E}$. The macroscopic potential Φ_0 reads then:

$$\Phi_0 = \left[\frac{3f_a \mu_a}{1+m} + \frac{3f_b \mu_b}{1+m} \right] E_{eq}^{m+1} \quad (\text{C.2})$$

with:

$$E_{eq} = \sqrt{\frac{4}{3} (E_{33}^2 + E_{23}^2)} \quad (\text{C.3})$$

From expressions (A.6) and (A.9), we have:

$$g_1 = 0, \quad g_\alpha = f_a b \left[C_{\alpha\beta\gamma\delta}^b - C_{\alpha\beta\gamma\delta}^a \right] E_{\beta\gamma\delta}, \quad e_{ij} = 0 \quad (\text{C.4})$$

Under plane strain conditions, the tangent modulus C_{ijkl}^a reads:

$$C_{ijkl}^a = 3k_a J_{ijkl} + 2\mu_a E_{eq}^{m-1} \left[K_{ijkl} + \frac{2(m-1)}{3E_{eq}^2} E_{ij} E_{kl} \right] \quad (\text{C.5})$$

The components of C_{ijkl}^b are:

$$C_{ijkl}^b = 3k_b J_{ijkl} + 2\mu_b E_{eq}^{m-1} \left[K_{ijkl} + \frac{2(m-1)}{3E_{eq}^2} E_{ij} E_{kl} \right] \quad (\text{C.6})$$

It follows that, by assuming the same compressibility in each layer (the case of incompressibility will be after considered by taking the limit $k = k_a = k_b \rightarrow +\infty$), g_α reads:

$$g_\alpha = 2f_a f_b (\mu_a - \mu_b) E_{eq}^{m-1} \left[E_{\alpha\beta\gamma} + \frac{2(m-1)}{3E_{eq}^2} E_{\alpha\beta} E_{\gamma\delta} E_{\gamma\delta\beta} \right] \quad (\text{C.7})$$

The macroscopic potential Ψ_2 is:

$$\Psi_2 = \frac{1}{24} \left[f_a Q_{\alpha\beta}^a + f_b Q_{\alpha\beta}^b \right] g_\alpha g_\beta \quad (\text{C.8})$$

The components of C_{ij1j}^a are:

$$C_{i i k l}^a = \left[k_a - \frac{2\mu_a E_{eq}^{m-1}}{3} \right] \delta_{i j} \delta_{j i} + \mu_a E_{eq}^{m-1} \delta_{ij} \quad (\text{C.9})$$

It follows that:

$$Q_{\alpha\beta}^a = \frac{1}{\mu_a E_{eq}^{m-1}} \delta_{\alpha\beta} \quad (\text{C.10})$$

and also:

$$Q_{\alpha\beta}^b = \frac{1}{\mu_b E_{eq}^{m-1}} \delta_{\alpha\beta} \quad (\text{C.11})$$

The macroscopic potential reads then:

$$\Psi_2 = \frac{1}{24 E_{eq}^{m-1}} \left[\frac{f_a}{\mu_a} + \frac{f_b}{\mu_b} \right] g_\alpha g_\alpha \quad (\text{C.12})$$

where g_α is given by (C.7). Replacing g_α by its expression (C.7) leads to (55).

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