A PLATONIC (EUCLIDEAN-PROJECTIVE) MODEL FOR THE SPECIAL THEORY OF RELATIVITY

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UN MODÈLE PLATONICIEN (EUCLIDIEN-PROJECTIF) POUR LA THÉORIE DE LA RELATIVITÉ RESTREINTE

Alain JEGAT

Abstract

The usual framework for Einstein's special theory of relativity is the pseudo-Euclidean space-time proposed by Hermann Minkowski.
This article aims at proposing a ***different model***.
The framework is a four-dimensional Euclidean space in which ***all the objects move regularly*** (i. e. ***the distances covered are equal*** regardless of the trajectories), and the events are considered ***in projection*** according to a privileged direction, as we are going to explain.
The rather surprising remark which is the object of this article is that we thus find the Lorentz transformation and all the equations of the special relativity.

Résumé

Le cadre géométrique usuel de la Relativité Restreinte est l'espace-temps pseudo-euclidien de Minkowski.
Cet article a pour but d'exposer un modèle différent.
Le cadre en est un espace euclidien de dimension quatre, dans lequel les objets se déplacent tous de façon uniforme (c'est-à-dire qu'entre deux observations, quelle que soit leur trajectoire, ils parcourent tous la même distance), mais où les événements sont vus en projection selon une direction privilégiée, comme nous allons l'expliquer.
La remarque, assez surprenante, qui fait l'objet de cet article est qu'on retrouve ainsi les transformations de Lorentz et toutes les formules de la Relativité Restreinte.

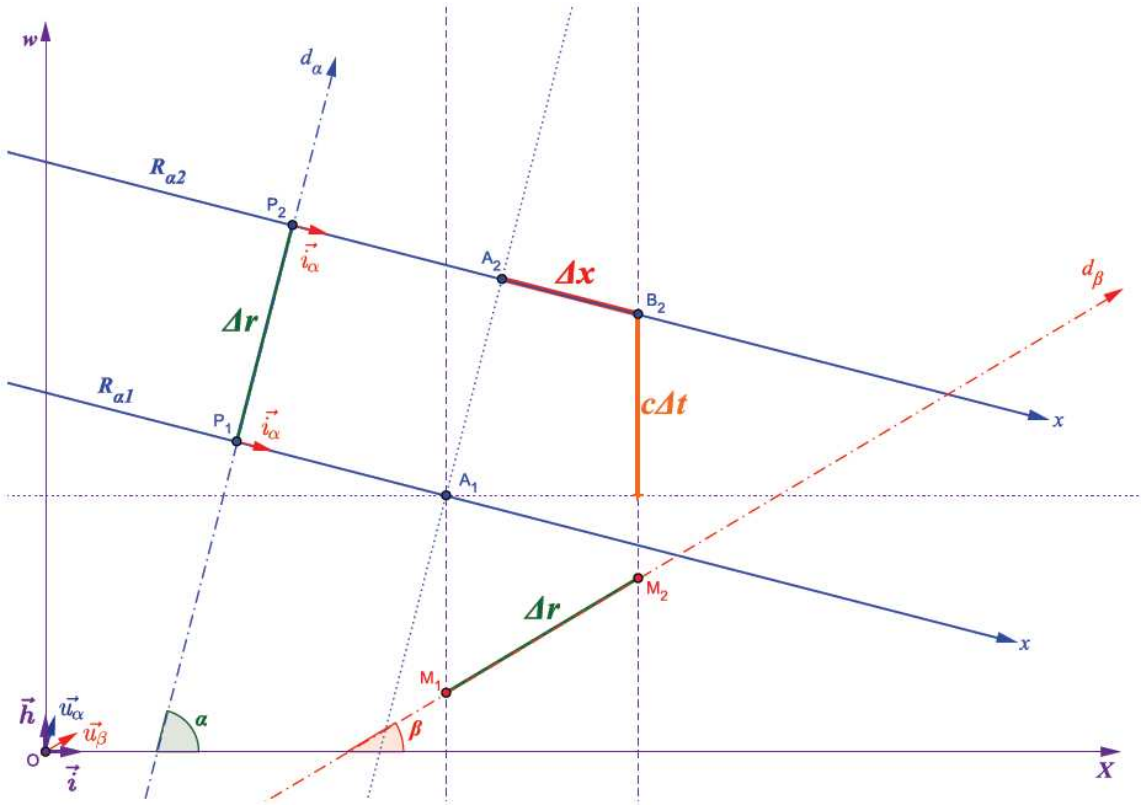
1. Coordinate system in this space, uniform motion, speed measurement.

$\left(O,\vec{i},\vec{j},\vec{k},\vec{h}\right)$ is a frame for the four-dimensional Euclidean space whose axes are denoted $\left(OX\right)$, $\left(OY\right)$, $\left(OZ\right)$, $\left(Ow\right)$; the direction of the projection is that of the vector $\vec{h}$.

We consider a punctual mobile *P* in rectilinear uniform motion and associate to this mobile the hyperplane H_P which contains it and is orthogonal to the direction of its motion. The points of this hyperplane, which thus have parallel trajectories to that of *P*, constitute what we will refer to as the "observers" of the reference frame related to *P*.

To simplify the following diagrams, we shall consider only mobiles whose motion is located in the plan $\left(XOw\right)$, as well as the "observers" connected to *P* located in this plan.

We will name d_α the straight line traveled by *P*, $\overrightarrow{u_\alpha}$ its guiding vector (oriented in the direction of the motion), R_α the reference frame related to *P*, α the angle $\left(\vec{i},\overrightarrow{u_\alpha}\right)$.



The diagram above represents the observation by the reference frame R_α of a second punctual mobile *M* whose rectilinear trajectory d_β is guided by the vector $\overrightarrow{u_\beta}$, with $\left(\vec{i},\overrightarrow{u_\beta}\right)=\beta$.

M is observed by the reference frame R_α when it is located in M_1 and in M_2.

We will suppose here that α is not a multiple of π.

The event "*M is in* M_1" occurs while the mobile *P* is in P_1: in projection, this event is perceived by the observer A_1.

The event "*M is in* M_2" occurs while the mobile *P* is in P_2: in projection, this event is perceived by the observer B_2, while the observer A_1 is now in A_2.

Between the two observations, both mobiles traveled the same distance $\Delta r = P_1P_2 = M_1M_2$, by hypothesis (cf. the abstract).

In the reference frame R_α we will define:

1.1 ***The algebraic distance*** Δx separating both events "*M is in M₁*" and "*M is in M₂*" from the Euclidian distance A_2B_2, affected by a positive or negative sign according to the situations (we will use the relation $\overrightarrow{A_2B_2} = \Delta x \vec{i}_\alpha$ where $\vec{i}_\alpha = \sin\alpha\,\vec{i} - \cos\alpha\,\vec{h}$).

1.2 ***The time interval*** Δt separating these two events in R_α by the relation $c\Delta t = w(B_2) - w(A_1)$, where c is a strictly positive constant (we will see that c corresponds to the light speed in the space).

Lemma. *The speed* $v = \dfrac{\Delta x}{\Delta t}$ *of M measured by* R_α *is:* $v = c\dfrac{\cos\beta - \cos\alpha}{1 - \cos\alpha\cos\beta}$. (1)

Proof. By identifying vectors with their columns of coordinates in the basis $\left(\vec{i}, \vec{h}\right)$, we have:

$$\overrightarrow{M_1M_2} = \begin{pmatrix} \Delta r\cos\beta \\ \Delta r\sin\beta \end{pmatrix},\ \overrightarrow{A_1A_2} = \begin{pmatrix} \Delta r\cos\alpha \\ \Delta r\sin\alpha \end{pmatrix},$$

$$\text{and}\ \ \overrightarrow{A_2B_2} = \Delta x\vec{i}_\alpha = \begin{pmatrix} \Delta x\sin\alpha \\ -\Delta x\cos\alpha \end{pmatrix},\ \text{hence}\ \overrightarrow{A_1B_2} = \begin{pmatrix} \Delta r\cos\alpha + \Delta x\sin\alpha \\ \Delta r\sin\alpha - \Delta x\cos\alpha \end{pmatrix}.$$

However, $X\left(M_2\right) - X\left(M_1\right) = X\left(B_2\right) - X\left(A_1\right)$, so $\Delta r\cos\alpha + \Delta x\sin\alpha = \Delta r\cos\beta$.

It can be deduced that: $\Delta x = \dfrac{\cos\beta - \cos\alpha}{\sin\alpha}\Delta r$, (2)

and the relation $c\Delta t = w(B_2) - w(A_1) = \Delta r\sin\alpha - \Delta x\cos\alpha$ becomes:

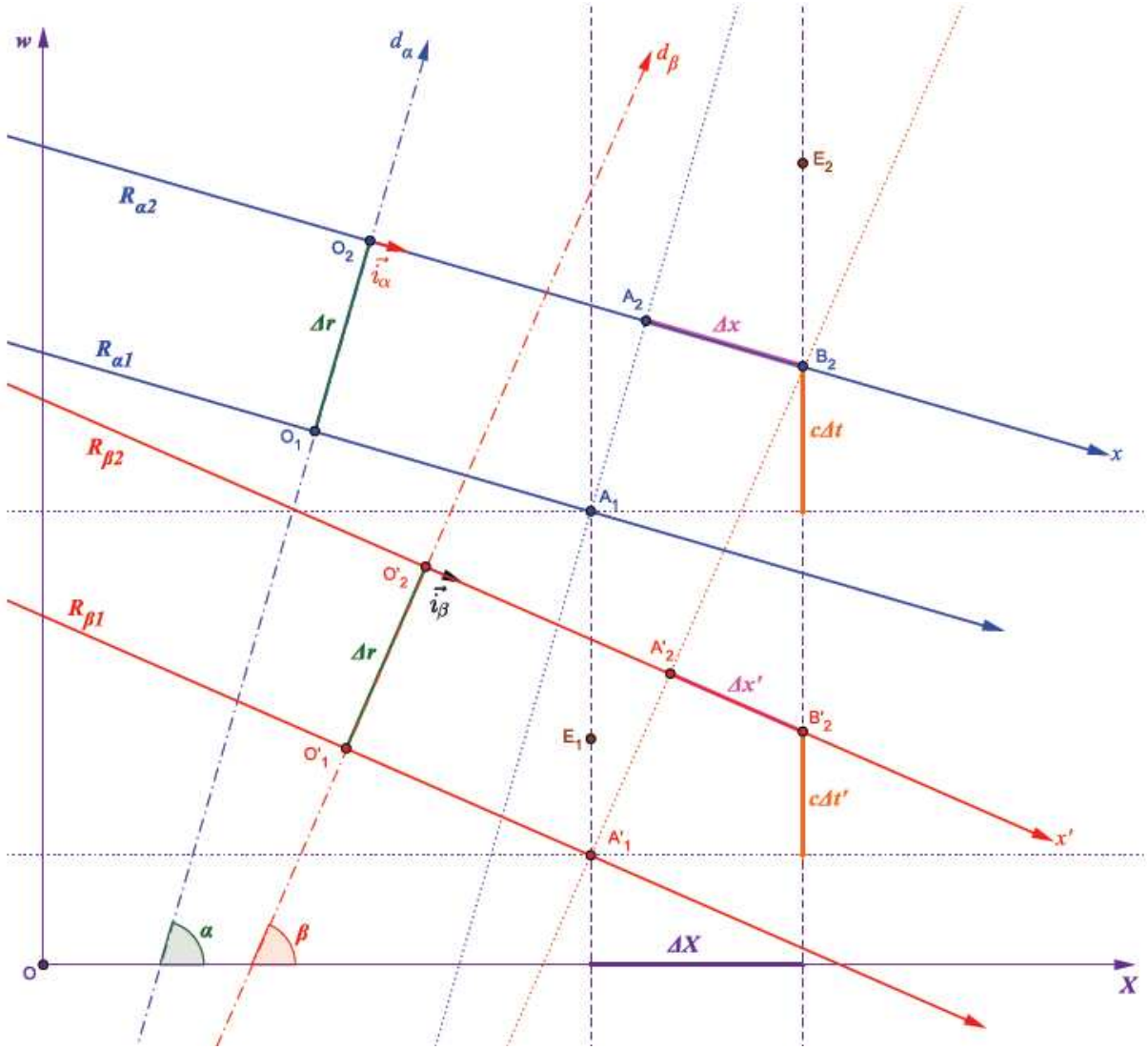
$$c\Delta t = \frac{1 - \cos\alpha\cos\beta}{\sin\alpha}\Delta r\,. \tag{3}$$

The lemma results from (2) and (3). ■

Remarks. The lemma implies that we always find $|v| \le c$. Regardless of the reference frame of observation R_α, we obtain $v = c$ for $\beta = 0$ and $v = -c$ for $\beta = \pm\pi$ (mod. 2π); and in the cases where $\alpha = m\pi$ (m is an integer), regardless of the value of β (not a multiple of π), the values of Δx and $c\Delta t$ aim towards the infinity but their quotient leads to $v = -c$ if $\alpha = 0$ or $v = c$ if $\alpha = \pm\pi$ (mod. 2π).

2. Lorentz transformation

Two reference frames R_α and R_β in uniform translation (i.e. with rectilinear trajectories) are considered here, and two events E_1 and E_2 are observed.
For the convenience of calculations, we will refer to the notations indicated on the diagram below and we will put $\Delta X = X(E_2) - X(E_1)$.



Theorem (Lorentz transformation). Supposing that $\sin\alpha \sin\beta > 0$, then:

$$\Delta x' = \frac{\Delta x - v\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad c\Delta t' = \frac{c\Delta t - \frac{v}{c}\Delta x}{\sqrt{1-\frac{v^2}{c^2}}} \tag{4}$$

Proof. We have $\overrightarrow{A_1B_2} = \begin{pmatrix} \Delta r\cos\alpha + \Delta x\sin\alpha \\ \Delta r\sin\alpha - \Delta x\cos\alpha \end{pmatrix} = \begin{pmatrix} \Delta X \\ c\Delta t \end{pmatrix}$,

from which we find $\Delta x = \dfrac{\Delta X - \Delta r\cos\alpha}{\sin\alpha}$ and $c\Delta t = \dfrac{\Delta r - \Delta X\cos\alpha}{\sin\alpha}$. (5)

Similarly, we find $\Delta x' = \dfrac{\Delta X - \Delta r\cos\beta}{\sin\beta}$ and $c\Delta t' = \dfrac{\Delta r - \Delta X\cos\beta}{\sin\beta}$ (6)

(1) and (5) result in: $\Delta x - \dfrac{v}{c}c\Delta t = \dfrac{(\Delta X - \Delta r\cos\beta)\sin\alpha}{1-\cos\alpha\cos\beta}$

As (1) and $\sin\alpha\sin\beta > 0$ result in $\sqrt{1-\dfrac{v^2}{c^2}} = \dfrac{\sin\alpha\sin\beta}{1-\cos\alpha\cos\beta}$, then:

$$\frac{\Delta x - \dfrac{v}{c}c\Delta t}{\sqrt{1-\dfrac{v^2}{c^2}}} = \frac{\Delta X - \Delta r\cos\beta}{\sin\beta} = \Delta x'.$$

Similarly, we obtain $c\Delta t - \dfrac{v}{c}\Delta x = \dfrac{(\Delta r - \Delta X\cos\beta)\sin\alpha}{1-\cos\alpha\cos\beta}$, then finally:

$$\frac{c\Delta t - \dfrac{v}{c}\Delta x}{\sqrt{1-\dfrac{v^2}{c^2}}} = \frac{\Delta r - \Delta X\cos\beta}{\sin\beta} = c\Delta t'. \qquad \blacksquare$$

Remarks:

1. The formulae of Lorentz (4) may also be written as follows:

$$\Delta x' = \frac{(1-\cos\alpha\cos\beta)\Delta x - (\cos\beta - \cos\alpha)c\Delta t}{\sin\alpha\sin\beta};$$

$$c\Delta t' = \frac{(1-\cos\alpha\cos\beta)c\Delta t - (\cos\beta - \cos\alpha)\Delta x}{\sin\alpha\sin\beta}. \qquad (7)$$

(Contrary to what formulae (7) suggest, the relations (6) clearly show that the quantities $\Delta x'$ and $c\Delta t'$ measuring the space and the time separating the events E_1 and E_2 in the reference frame R_β are independent from the reference frame R_α.)

2. We thus find all the well-known formulae of the special theory of relativity and in particular the formula of ***speed addition which*** results from the following identity:

$$\frac{\frac{\cos\beta-\cos\alpha}{1-\cos\alpha\cos\beta}+\frac{\cos\gamma-\cos\beta}{1-\cos\beta\cos\gamma}}{1+\frac{\cos\beta-\cos\alpha}{1-\cos\alpha\cos\beta}.\frac{\cos\gamma-\cos\beta}{1-\cos\beta\cos\gamma}}=\frac{\cos\gamma-\cos\alpha}{1-\cos\alpha\cos\gamma}.$$

3. If a punctual mobile *M* in the four-dimensional Euclidian space describes a curved trajectory (i.e. the angle β depends on *r*), at identical linear speed with the other mobiles, it will be seen in a reference frame R_α in uniform translation as a mobile in ***accelerated motion***.
Its acceleration a_α measured by the reference frame R_α will be given (see (3) and (1))

by: $$\frac{a_\alpha}{c^2}=\frac{d\left(v/c\right)}{cdt}=\frac{-d\beta}{dr}.\frac{\sin^3\alpha\sin\beta(r)}{\left(1-\cos\alpha\cos\beta(r)\right)^3}$$

i.e. $$\frac{a_\alpha}{c^2}=\frac{-d\beta}{dr}\frac{1}{\sin^2\beta(r)}\left(1-\frac{v^2}{c^2}\right)^{3/2}$$

where v is the speed of *M* measured in R_α;
resulting in, for two such reference frames R_α and R_γ:

$$a_\gamma=a_\alpha\frac{\sin^3\gamma}{\sin^3\alpha}\frac{\left(1-\cos\alpha\cos\beta(r)\right)^3}{\left(1-\cos\gamma\cos\beta(r)\right)^3},\quad\text{i.e.}\quad a_\gamma=a_\alpha\left(1-\frac{v_\gamma^2}{c^2}\right)^{3/2}\left(1-\frac{v_\alpha^2}{c^2}\right)^{-3/2}$$

(where v_γ and v_α are the speeds of *M* measured in R_γ and R_α).

3. Conclusion

This model offers a geometrical formulation of the special theory of Relativity that differs from Minkowski's one. However, it raises two major issues from a physical point of view:

3.1 The first one concerns the motion of objects (observed mobiles, reference frames): we impose them to travel the same distance between two observations, regardless of their trajectories.
This constraint evokes undulatory phenomena where wave speed is defined by the propagation medium.

3.2 The second, of course, concerns projection: in this model, everything takes place as if an event *E* was considered in the same way by a given reference frame regardless of its coordinate w (cf. the second diagram, where both events observed are seen in A_1 and B_2 regardless of their coordinates w).