A two-dimensional oblique extension of Bessel processes

Dominique Lepingle

A TWO-DIMENSIONAL OBLIQUE EXTENSION OF BESSEL PROCESSES

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Abstract. We consider the problem of strong existence and uniqueness of a Brownian motion forced to stay in the nonnegative quadrant by an electrostatic repulsion from the sides that works obliquely. We obtain a kind of two-dimensional Bessel process with oblique drift. Because of the obliqueness the components are not independent each other. To construct the solution, we consider the case of a normal drift and use as main tool a comparison lemma. A few of the properties of one-dimensional Bessel processes are preserved in this setting. The results are reminiscent of the study of a Brownian motion with oblique reflection in a wedge. Actually, the same skew symmetry condition is involved when looking for a stationary distribution in product form. The terms of the product are now gamma distributions in place of exponential ones.

1. Introduction

In the late seventies the study of heavy traffic limits in open multi-station queueing networks has put the question of existence and properties of the Brownian motion obliquely reflected on the sides of a wedge and more generally on the faces of a polyhedron. In the following decade there was an extensive literary output on that topic, among which we mention the works of Harrison, Reiman, Williams and co-authtors ([7],[8],[19],[21],[9],[3], to cite a few of them). But there is another way than normal or oblique reflection to prevent Brownian motion from overstepping a linear barrier. We may add as drift term the gradient of a concave function that explodes in the neighborhood of the faces of the polyhedron. To be more specific, let $\mathbf{n_1}, \ldots, \mathbf{n_k}$ be unit vectors in $\mathbb{R}^d$ and $b_1, \ldots, b_k$ be real numbers. The state space S is defined by

$$S := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{n}_r.\mathbf{x} \geq b_r, r = 1, \ldots, k\}\,.$$

Let $\phi_1, \ldots, \phi_k$ be k convex C^1 functions on $(0, \infty)$ with $\phi_r(0+) = +\infty$ for any $r = 1, \ldots, k$. The potential function Φ on S is defined by

$$\Phi(\mathbf{x}) := \sum_{r=1}^{k} \phi_r(\mathbf{n}_r.\mathbf{x} - b_r)\,.$$

From the general existence and uniqueness theorem on multivalued stochastic differential systems established in [1], completed with identification of the drift term as in Lemma 3.4 in [2], we know there exists a unique strong solution living in S to the equation

$$\begin{array}{rcl} d\mathbf{X}_t & = & d\mathbf{B}_t - \nabla\Phi(\mathbf{X}_t)dt \\ \mathbf{X}_0 & \in & S \end{array} \tag{1}$$

where $\mathbf{B}$ is a Brownian motion in $\mathbb{R}^d$. As was proved in Proposition 4.1 of [14], the hypotheses $\phi_r(0+) = +\infty$ for any $r = 1, \ldots, k$ entail that there is no additional boundary process of

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local time type in the r.h.s. of (1) since the repulsion forces are sufficiently strong. As

$$-\nabla\Phi(\mathbf{x}) = -\sum_{r=1}^{k} \phi_r'(\mathbf{n}_r.\mathbf{x} - b_r)\mathbf{n}_r \,,$$

the repulsion from the faces $F_r = \{\mathbf{x} \in S : \mathbf{n}_r.\mathbf{x} = b_r\}$ points in a normal direction into the interior of S. We now introduce vectors $\mathbf{q}_1, \ldots, \mathbf{q}_k$ with $\mathbf{q}_r.\mathbf{n}_r = 0$ for $r = 1, \ldots, k$ and consider a new equation

$$\begin{array}{rcl} d\mathbf{X}_t & = & d\mathbf{B}_t - \sum_{r=1}^{k} \phi_r'(\mathbf{n}_r.\mathbf{x} - b_r)(\mathbf{n}_r + \mathbf{q}_r)dt \\ \mathbf{X}_0 & \in & S \end{array} \quad . \tag{2}$$

This is a singular drift and because of the obliqueness induced by $\mathbf{q}_1, \ldots, \mathbf{q}_k$ convex analysis cannot be used as in [1] to get strong existence and uniqueness of the associated stochastic differential system. A similar problem has been recently tackled by Gassous, Răşcanu and Rotenstein in [6]. They assume the domain of the subdifferential operator to be closed, which is not compatible with our hypotheses $\phi_r(0+) = +\infty$.

In this paper we shall concentrate on the nonnegative quadrant in $\mathbb{R}^2$ as state space and a drift term that derives from electrostatic repulsive forces. In dimension one the associated process is a Bessel process. Thus in this new case we may call the solution to (2) a two-dimensional Bessel process with oblique drift, or more briefly an oblique two-dimensional Bessel process (O2BP for short). When the drift acts normally ($\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{0}$), we recognize the usual two-dimensional Bessel process which is made of two independent Bessel processes.

We obtain strong existence and uniqueness for a large set of parameters and initial conditions, but not for all possible values. Our results are reminiscent of the thorough study in [19] of the oblique reflection in a wedge. A key tool in this work was an appropriate harmonic function that made a weak approach possible and fruitful. Thus full results were obtained, while our strong approach merely provides a partial answer to the crucial question of hitting the corner. A few from all the nice properties of one-dimensional Bessel processes are preserved in our setting: same scaling property, abolute continuity between the laws of processes with different parameters, a martingale property. Our methods of proof are elementary. We make an essential use of the comparison of solutions of stochastic differential equations and of the asymptotic path behavior of local martingales.

For theoretical as well as practical reasons, a great deal of interest was taken in the question of existence and computation of the invariant measure of the Brownian motion with a constant drift vector and oblique reflection ([8],[21],[9]). Under the assumption that the directions of reflection satisfy a skew symmetry condition, it was proved that the invariant measure has exponential product form density. Motivated by the so-called Atlas model of equity markets presented in [5], some authors ([16],[10],[12]) have recently studied Brownian motions on the line with rank dependent local characteristics. This model is strongly related to Brownian motion reflected in polyhedral domains. The invariant probability density has an explicit exponential product form when the volatility is constant [16] and a sum of products of exponentials form when the volatility coefficients depend on the rank ([12],[11]). This last kind of density was previously obtained in [4] for a Brownian motion in a wedge with oblique reflection.

A neighboring way has been recently explored by O'Connell and Ortmann in [15]. Here the process is a Brownian motion with a drift term that is continuous and depends obliquely, via a regular potential function, on the position of the process relative to a polyhedral domain. Under the same skew symmetry condition as in [9], the invariant density has an explicit product form again. In Section 9, we consider a Brownian motion with a constant drift living in the nonnegative quadrant and an oblique electrostatic repulsion from the sides. Under the

skew symmetry condition, there is still an invariant measure in product form. Now the terms of the product are two gamma distributions with explicit parameters.

An earlier version of the present paper was first posted in 2012 on arXiv (1208.6399) under the name *Brownian motion in the quadrant with oblique repulsion from the sides*. After the death of Marc Yor in January 2014, I decided in memory to him to add three new sections stressing the relationship with the theory of Bessel processes where his contribution has been so significant. In the same spirit I also changed the title of the paper.

2. Some properties of Bessel processes

A Bessel process of dimension $d > 1$, starting at $r \geq 0$, is the unique solution of the stochastic differential equation

$$R_t = r + W_t + \frac{d-1}{2}\int_0^t \frac{1}{R_s} ds. \tag{3}$$

where W is a real-valued Brownian motion starting at 0.

We know that:

- $d \geq 2$: the point 0 is a polar set ([17], Proposition V.2.7);
- $d = 2$: $\limsup_{t\to\infty} R_t = +\infty$, $\liminf_{t\to\infty} R_t = 0$ ([17], Theorem V.2.8);
- $1 < d < 2$: the point 0 is instantaneously reflecting ([17], Proposition XI.1.5).

More precisely ([13], p.337 and p.339),

- **Dimension** $d > 2$:
$$\begin{array}{l}\mathbb{P}(R_t > 0, \forall t > 0) = 1,\\ \mathbb{P}(R_t \to \infty, t \to \infty) = 1,\\ R^{2-d} \text{ is a local martingale.}\end{array}$$
- **Dimension** $d = 2$:
$$\begin{array}{l}\mathbb{P}(R_t > 0, \forall t > 0) = 1,\\ \mathbb{P}(\sup_{t>a} R_t = \infty, \inf_{t>a} R_t = 0) = 1 \text{ for any a} > 0,\\ \ln R \text{ is a local martingale.}\end{array}$$
- **Dimension** $1 < d < 2$:
$$P(R_t > 0, \forall t > a) = 0 \text{ for any a} > 0.$$

Corollary 1. *When $d \geq 2$, for any $a > 0$,*

$$\int_a^\infty \frac{ds}{R_s^2} = \infty \ a.s.$$

Proof. From Itô formula

$$\ln R_t \ = \ \ln R_a + \int_a^t \frac{dW_s}{R_s} + \frac{d-2}{2}\int_a^t \frac{ds}{R_s^2}.$$

From ([17], Proposition IV.1.26) we deduce that on the set $\{\int_a^\infty \frac{ds}{R_s^2} < \infty\}$, the continuous local martingale $\int_0^t \frac{dW_s}{R_s}$ converges as $t \to \infty$. But we have seen that $\ln R_t$ does not converges a.s. and the conclusion follows. ■

We shall also use an absolute continuity result ([17], Exercise XI.1.22 or [13], Proposition 6.1.5.1). On the canonical space $\Omega = C(\mathbb{R}_+, \mathbb{R}_+)$, we denote by R the canonical map $R_t(\omega) = \omega(t)$, by $\mathcal{R}_t = \sigma(R_s, s \leq t)$ the canonical filtration and by $\mathbb{P}_r^d$ the law of the Bessel process of dimension $d \geq 2$ starting at $r > 0$. Then,

$$\mathbb{P}_r^d \mid_{\mathcal{R}_t} = \left(\frac{R_t}{r}\right)^{\frac{d-2}{2}} \exp\left(-\frac{(d-2)^2}{8}\int_0^t \frac{ds}{R_s^2}\right) . \mathbb{P}_r^2 \mid_{\mathcal{R}_t} . \tag{4}$$

3. Oblique Bessel process in the quadrant

The general state space is the nonnegative quadrant $S = \mathbb{R}_+ \times \mathbb{R}_+$. The corner $\mathbf{0} = (0,0)$ will play a crucial role and in some cases it will be necessary to restrict the state space to the punctured nonnegative quadrant $S^{\mathbf{0}} = S \setminus \{\mathbf{0}\}$.

Definition 2. *Let (B_t, C_t) be a Brownian motion in the plane starting from $\mathbf{0}$, adapted to a filtration $\mathcal{F} = (\mathcal{F}_t)$ with usual conditions. Let $\alpha, \beta, \gamma, \delta$ be four real constants with $\alpha > 0, \delta > 0$. We say that an $\mathcal{F}$-adapted continuous process (X, Y) with values in S is an Oblique Two-dimensional Bessel Process (O2BP) if for any $t \geq 0$*

$$\begin{array}{rcl} X_t &=& X_0 + B_t + \alpha \int_0^t \frac{ds}{X_s} + \beta \int_0^t \frac{ds}{Y_s} \quad \geq \quad 0 \\ Y_t &=& Y_0 + C_t + \gamma \int_0^t \frac{ds}{X_s} + \delta \int_0^t \frac{ds}{Y_s} \quad \geq \quad 0 \end{array} \tag{5}$$

where X_0 and Y_0 are non-negative $\mathcal{F}_0$-measurable random variables, and

$$\begin{array}{ll} \int_0^t \mathbf{1}_{\{X_s=0\}} ds = 0 & \int_0^t \mathbf{1}_{\{Y_s=0\}} ds = 0 \\ \int_0^t \mathbf{1}_{\{X_s>0\}} \frac{ds}{X_s} < \infty & \int_0^t \mathbf{1}_{\{Y_s>0\}} \frac{ds}{Y_s} < \infty \,. \end{array}$$

The drift in (5) is of type (2) with $d = k = 2$ and

$$\begin{array}{ll} \phi_1(x) = -\alpha \ln(x) & \phi_2(y) = -\delta \ln(y) \\ b_1 = 0 & b_2 = 0 \\ \mathbf{n}_1 = (1,0) & \mathbf{n}_2 = (0,1) \\ \mathbf{q}_1 = (0, \frac{\gamma}{\alpha}) & \mathbf{q}_2 = (\frac{\beta}{\delta}, 0) \end{array}$$

The case with $\beta = \gamma = 0$ is a particular case of (1). In the sequel, we will note (U, V) the solution of the system

$$\begin{array}{rcl} U_t &=& X_0 + B_t + \alpha \int_0^t \frac{ds}{U_s} \quad \geq \quad 0 \\ V_t &=& Y_0 + C_t + \delta \int_0^t \frac{ds}{V_s} \quad \geq \quad 0 \,. \end{array} \tag{6}$$

The processes U and V are independent Bessel processes (if X_0 and Y_0 are independent variables). Actually, U is a Bessel process of dimension $2\alpha + 1$, and the point 0 is intanstaneously reflecting for U if $\alpha < \frac{1}{2}$ and polar if $\alpha \geq \frac{1}{2}$. Moreover, $U^2 + V^2$ is the square of a Bessel process of dimension $2\alpha + 2\delta + 2$ [18], and so the corner $\mathbf{0}$ is polar for (U, V) in any case.

Comparison between X and U, Y and V will play a key role in the construction of the solution (X, Y) and the study of its behavior close to the sides of the quadrant. The following simple lemma will be of constant use.

Lemma 3. *For $T > 0$, $\alpha > 0$, let x_1 and x_2 be nonnegative continuous solutions on $[0, T]$ of the equations*

$$\begin{array}{rcl} x_1(t) &=& v_1(t) + \alpha \int_0^t \frac{ds}{x_1(s)} \\ x_2(t) &=& v_2(t) + \alpha \int_0^t \frac{ds}{x_2(s)} \end{array}$$

where v_1, v_2 are continuous functions such that $0 \leq v_1(0) \leq v_2(0)$, and $v_2 - v_1$ is nondecreasing. Then $x_1(t) \leq x_2(t)$ on $[0, T]$.

Proof. Assume there exists $t \in (0, T]$ such that $x_2(t) < x_1(t)$. Set

$$\tau := \max\{s \leq t : x_1(s) \leq x_2(s)\} \,.$$

Then,

$$\begin{array}{rcl} x_2(t) - x_1(t) &=& x_2(\tau) - x_1(\tau) + (v_2(t) - v_1(t)) - (v_2(\tau) - v_1(\tau)) + \alpha \int_\tau^t (\frac{1}{x_2(s)} - \frac{1}{x_1(s)}) ds \\ &\geq& 0 \,, \end{array}$$

a contradiction. ■

We will also need the following consequence of the results in [1] on multivalued stochastic differential systems, completed with the method used in [2] and developed in [14] to check the lack of additional boundary process.

Proposition 4. *Let $\alpha > 0$, $\delta \geq 0$, $\sigma = (\sigma_j^i; i,j = 1,2)$ a 2×2-matrix, (B_1, B_2) a Brownian motion in the plane, b_1 and b_2 two Lipschitz functions on $\mathbb{R}^2$, Z_0^1 and Z_0^2 two $\mathcal{F}_0$-measurable nonnegative random variables. There exists a unique strong solution (Z^1, Z^2) to the system*

$$\begin{array}{rcl} Z_t^1 & = & Z_0^1 + \sigma_1^1 B_t^1 + \sigma_2^1 B_t^2 + \alpha \int_0^t \frac{ds}{Z_s^1} + \int_0^t b_1(Z_s^1, Z_s^2) ds \\ Z_t^2 & = & Z_0^2 + \sigma_1^2 B_t^1 + \sigma_2^2 B_t^2 + \delta \int_0^t \frac{ds}{Z_s^2} + \int_0^t b_2(Z_s^1, Z_s^2) ds \end{array} \tag{7}$$

with the conditions $Z_t^1 \geq 0$ if $\delta = 0$ and $Z_t^1 \geq 0, Z_t^2 \geq 0$ if $\delta > 0$.

It is worth noticing that the solutions to (5) enjoy the Brownian scaling property. It means that if (X, Y) is a solution to (5) starting from (X_0, Y_0) with driving Brownian motion (B_t, C_t), then for any $c > 0$ the process $(X'_t := c^{-1} X_{c^2 t}, Y'_t := c^{-1} Y_{c^2 t}; t \geq 0)$ is a solution to (5) starting from $(c^{-1}X_0, c^{-1}Y_0)$ with driving Brownian motion $(c^{-1}B_{c^2 t}, c^{-1}C_{c^2 t})$.

4. Avoiding the corner

We shall see in Section 6 that existence and uniqueness of the solution to (5) are easily obtained as soon as the solution process keeps away from the corner. Thus the question of attaining the corner in finite time is of great interest.

Theorem 5. *Let (X, Y) be a solution to (5). We set*

$$\tau^{\mathbf{0}} := \inf\{t > 0 : (X_t, Y_t) = \mathbf{0}\}$$

with the usual convention $\inf \emptyset = \infty$. Then $\mathbb{P}(\tau^{\mathbf{0}} < \infty) = 0$ if one of the following conditions is satisfied:

1. *$C_1 : \beta \geq 0$ and $\gamma \geq 0$*
2. *$C_{2a} : \alpha \geq \frac{1}{2}$ and $\beta \geq 0$*
3. *$C_{2b} : \delta \geq \frac{1}{2}$ and $\gamma \geq 0$*
4. *C_3 : There exist $\lambda > 0$ and $\mu > 0$ such that*
   - *$\lambda\alpha + \mu\gamma \geq 0$*
   - *$\lambda\beta + \mu\delta \geq 0$*
   - *$(\sqrt{\lambda(\lambda\alpha + \mu\gamma)} + \sqrt{\mu(\lambda\beta + \mu\delta)})^2 \geq \frac{1}{2}(\lambda^2 + \mu^2)$.*

Proof.
Condition C_1. From Lemma 1 we get $X_t \geq U_t$, $Y_t \geq V_t$, where (U, V) is the solution to (6), and we know that $\mathbf{0}$ is polar for (U, V).
Condition C_{2a} (resp. *C_{2b}*). From Lemma 1 we get $X_t \geq U_t$ (resp. $Y_t \geq V_t$) and in this case 0 is polar for U (resp. V), so $U_t > 0$ (resp. $V_t > 0$) for $t > 0$.
Condition C_3. For $\epsilon > 0$ let

$$\begin{array}{rcl} \sigma^\epsilon & = & \mathbf{1}_{\{(X_0,Y_0)=\mathbf{0}\}} \inf\{t > 0 : X_t + Y_t \geq \epsilon\} \\ \tau^{\mathbf{0},\epsilon} & = & \inf\{t > \sigma^\epsilon : (X_t, Y_t) = \mathbf{0}\}. \end{array}$$

As $\epsilon \downarrow 0$, $\sigma^\epsilon \downarrow 0$ and $\tau^{0,\epsilon} \downarrow \tau^{\mathbf{0}}$. We set $S_t = \lambda X_t + \mu Y_t$ for $t \geq 0$, $\lambda > 0$ and $\mu > 0$. From Itô formula we get for $t \in [\sigma^\epsilon, \tau^{\mathbf{0},\epsilon})$

$$\begin{array}{rcl} & & \ln S_t \\ & = & \ln S_{\sigma^\epsilon} + \int_{\sigma^\epsilon}^t \frac{\lambda dB_s + \mu dC_s}{S_s} + (\lambda\alpha + \mu\gamma) \int_{\sigma^\epsilon}^t \frac{ds}{X_s S_s} + (\lambda\beta + \mu\delta) \int_{\sigma^\epsilon}^t \frac{ds}{Y_s S_s} - \frac{1}{2}(\lambda^2 + \mu^2) \int_{\sigma^\epsilon}^t \frac{ds}{S_s^2} \\ & = & \ln S_{\sigma^\epsilon} + M_t + \int_{\sigma^\epsilon}^t \frac{P(X_s, Y_s)}{X_s Y_s S_s^2} ds \end{array}$$

where M is a continuous local martingale and $P(x,y)$ is the second degree homogeneous polynomial

$$P(x,y) = \lambda(\lambda\beta+\mu\delta)x^2 + \mu(\lambda\alpha+\mu\gamma)y^2 + (\lambda(\lambda\alpha+\mu\gamma)+\mu(\lambda\beta+\mu\delta) - \frac{1}{2}(\lambda^2+\mu^2))xy\,.$$

Condition C_3 is exactly the condition for P being nonnegative on S. Therefore

$$0 \le \int_{\sigma^\epsilon}^t \frac{P(X_s,Y_s)}{X_sY_sS_s^2}ds < \infty$$

and so

$$0 \le \int_{\sigma^\epsilon}^{\tau^{\mathbf{0},\epsilon}} \frac{P(X_s,Y_s)}{X_sY_sS_s^2}ds \le \infty\,.$$

As $t \to \tau^{\mathbf{0},\epsilon}$, the continuous local martingale M either converges to a finite limit or oscillates between $+\infty$ and $-\infty$. It cannot converge to $-\infty$ and thus $S_{\tau^{\mathbf{0},\epsilon}} > 0$ on $\{\tau^{\mathbf{0},\epsilon} < \infty\}$, proving that $\mathbb{P}(\tau^{\mathbf{0},\epsilon} < \infty) = 0$ and finally $\mathbb{P}(\tau^{\mathbf{0}} < \infty) = 0$. ■

Example. When $\alpha = \delta$ and $|\beta| = |\gamma|$, condition C_3 is satisfied (with $\lambda = \mu$) if

$$\begin{array}{ll} \bullet & \beta^2 \le \alpha - \frac{1}{4} \quad \text{when} \quad \beta = -\gamma \\ \bullet & -\beta \le \alpha - \frac{1}{4} \quad \text{when} \quad \beta = \gamma < 0\,. \end{array} \tag{8}$$

5. Avoiding the sides

We may also be interested in hitting a single side. Then we set

$$\begin{array}{rcl} \tau_X^0 & := & \inf\{t>0 : X_t = 0\} \\ \tau_Y^0 & := & \inf\{t>0 : Y_t = 0\}\,. \end{array} \tag{9}$$

We already know that $\mathbb{P}(\tau_X^0 < \infty) = 0$ if $\alpha \ge \frac{1}{2}$ and $\beta \ge 0$. Conversely we can prove that $\mathbb{P}(\tau_X^0 < \infty) = 1$ if $\alpha < \frac{1}{2}$ and $\beta \le 0$. If we know that the corner is not hit and $\alpha \ge \frac{1}{2}$, we can get rid of the nonnegativity assumption on β.

Proposition 6. *Assume $\mathbb{P}(\tau^{\mathbf{0}} < \infty) = 0$. If $\alpha \ge \frac{1}{2}$, then $\mathbb{P}(\tau_X^0 < \infty) = 0$.*

Proof. For $\eta > 0$ let

$$\begin{array}{rcl} \theta_X^\eta & = & \mathbf{1}_{\{X_0=0\}} \inf\{t>0 : X_t \ge \eta\} \\ \tau_X^{0,\eta} & = & \inf\{t > \theta_X^\eta : X_t = 0\}\,. \end{array}$$

As $\eta \downarrow 0$, $\theta_X^\eta \downarrow 0$ and $\tau_X^{0,\eta} \downarrow \tau_X^0$. For $t \in [\theta_X^\eta, \tau_X^{0,\eta})$,

$$\ln X_t \;=\; \ln X_{\theta_X^\eta} + \int_{\theta_X^\eta}^t \frac{dB_s}{X_s} + (\alpha - \tfrac{1}{2})\int_{\theta_X^\eta}^t \frac{ds}{X_s^2} + \beta\int_{\theta_X^\eta}^t \frac{ds}{X_sY_s}\,. \tag{10}$$

Since $\mathbb{P}(\tau^{\mathbf{0}} < \infty) = 0$, we have $Y_{\tau_X^{0,\eta}} > 0$ on $\{\tau_X^{0,\eta} < \infty\}$. On this set,

$$\int_{\theta_X^\eta}^{\tau_X^{0,\eta}} \frac{ds}{X_s} < \infty\,, \qquad \int_{\theta_X^\eta}^{\tau_X^{0,\eta}} \frac{ds}{Y_s} < \infty$$

and $X_s > 0$ on $[\theta_X^\eta, \tau_X^{0,\eta})$, which proves that

$$\beta\int_{\theta_X^\eta}^{\tau_X^{0,\eta}} \frac{ds}{X_sY_s} > -\infty\,.$$

As $t \to \tau_X^{0,\eta}$, the local martingale in the r.h.s. of (10) cannot converge to $-\infty$. This entails that $\mathbb{P}(\tau_X^{0,\eta} < \infty) = 0$ and therefore $\mathbb{P}(\tau_X^0 < \infty) = 0$. ■

We may use again the method in Theorem 5 and Proposition 6 to learn more about hitting the sides of the quadrant.

Proposition 7. *Assume $\alpha \geq \frac{1}{2}$ and $\delta \geq \frac{1}{2}$. Then $\mathbb{P}(\tau_X^0 < \infty) = \mathbb{P}(\tau_Y^0 < \infty) = 0$ if one of the following conditions is satisfied:*

(1) $\beta > 0$
(2) $\gamma > 0$
(3) $\beta\gamma \leq (\alpha - \frac{1}{2})(\delta - \frac{1}{2})$.

Proof. Statements (1) and (2) are direct consequences of Theorem 5 and Proposition 6. For $\epsilon > 0$ let

$$\begin{array}{rcl} \rho^\epsilon & = & \mathbf{1}_{\{X_0Y_0=0\}} \inf\{t > 0 : X_tY_t \geq \epsilon\} \\ \rho^{0,\epsilon} & = & \inf\{t > \rho^\epsilon : X_tY_t = 0\} \, . \end{array}$$

For $\lambda > 0$ and $\mu > 0$ we set

$$R_t = \lambda \ln X_t + \mu \ln Y_t.$$

From Itô formula we get for $t \in [\rho^\epsilon, \rho^{0,\epsilon})$

$$\begin{array}{rcl} R_t & = & R_{\rho^\epsilon} + \int_{\rho^\epsilon}^t (\frac{\lambda}{X_s} dB_s + \frac{\mu}{Y_s} dC_s) + \int_{\rho^\epsilon}^t [\frac{\lambda(\alpha - \frac{1}{2})}{X_s^2} + \frac{\mu(\delta - \frac{1}{2})}{Y_s^2} + \frac{(\lambda\beta + \mu\gamma)}{X_sY_s}] ds \\ & = & R_{\rho^\epsilon} + N_t + \int_{\rho^\epsilon}^t \frac{Q(X_s, Y_s)}{X_s^2 Y_s^2} ds \end{array}$$

where N is a continuous local martingale and $Q(x, y)$ is the second degree homogeneous polynomial

$$Q(x, y) = \mu(\delta - \frac{1}{2})x^2 + \lambda(\alpha - \frac{1}{2})y^2 + (\lambda\beta + \mu\gamma)xy.$$

If the condition (3) is satisfied, then we may find $\lambda > 0$ and $\mu > 0$ such that Q is nonnegative on S and the proof terminates as in Theorem 5 under condition C_3. ■

6. Existence and uniqueness

We now proceed to the question of existence and uniqueness of a global solution to (5). We consider separately the three cases: $\beta \geq 0$ and $\gamma \geq 0$, then $\beta > 0$ and $\gamma < 0$, then $\beta \leq 0$ and $\gamma < 0$.

6.1. Case $\beta \geq 0$ and $\gamma \geq 0$. This is exactly condition C_1.

Theorem 8. *Assume $\beta \geq 0$ and $\gamma \geq 0$.*

(1) *There is a unique solution to (5) in $S^{\mathbf{0}}$.*
(2) *There is a solution to (5) in S starting from $\mathbf{0}$.*
(3) *If $\alpha\delta \geq \beta\gamma$, there is a unique solution to (5) in S.*

Proof. 1. Let $a > 0$, $\epsilon > 0$ and define for $(x, z) \in \mathbb{R}_+ \times \mathbb{R}$

$$\psi_\epsilon(x, z) := \frac{1}{\max(\gamma x + z, \alpha\epsilon)} \, .$$

This is a Lipschitz function. From Proposition 4 we know that the system

$$\begin{array}{rcl} X_t^\epsilon & = & X_0 + B_t + \alpha \int_0^t \frac{ds}{X_s^\epsilon} + \alpha\beta \int_0^t \psi_\epsilon(X_s^\epsilon, Z_s^\epsilon) ds \geq 0 \\ Z_t^\epsilon & = & -\gamma X_0 + \alpha(Y_0 + \mathbf{1}_{\{Y_0=0\}} a) - \gamma B_t + \alpha C_t + \alpha(\alpha\delta - \beta\gamma) \int_0^t \psi_\epsilon(X_s^\epsilon, Z_s^\epsilon) ds \end{array} \tag{11}$$

has a unique solution. Let

$$\tau_Y^\epsilon := \inf\{t > 0 : \gamma X_t^\epsilon + Z_t^\epsilon < \alpha\epsilon\} \, .$$

If $0 < \eta < \epsilon < a$ we deduce from the uniqueness that (X^ϵ, Z^ϵ) and (X^η, Z^η) are identical on $[0, \tau_Y^\epsilon]$. Patching together we can set

$$\begin{array}{rcl} X_t & := & \lim_{\epsilon\to 0} X_t^\epsilon \\ Y_t & := & \lim_{\epsilon\to 0} \frac{1}{\alpha}(\gamma X_t^\epsilon + Z_t^\epsilon) \end{array}$$

on $\{Y_0 > 0\} \times [0, \tau_Y^0)$, where

$$\tau_Y^0 := \lim_{\epsilon\to 0} \tau_Y^\epsilon \,.$$

On this set, (X, Y) is the unique solution to (5). As we noted in the proof of Theorem 3 with condition C_1, we have $X_t \geq U_t$ and $Y_t \geq V_t$. Therefore, on $\{Y_0 > 0\} \cap \{\tau_Y^0 < \infty\}$,

$$\int_0^{\tau_Y^0} \tfrac{ds}{X_s} \leq \int_0^{\tau_Y^0} \tfrac{ds}{U_s} < \infty \quad \text{and} \quad \int_0^{\tau_Y^0} \tfrac{ds}{Y_s} \leq \int_0^{\tau_Y^0} \tfrac{ds}{V_s} < \infty$$

and we can define

$$\begin{array}{rcl} X_{\tau_Y^0} := \lim_{t\to\tau_Y^0} X_t & = & X_0 + B_{\tau_Y^0} + \alpha \int_0^{\tau_Y^0} \tfrac{ds}{X_s} + \beta \int_0^{\tau_Y^0} \tfrac{ds}{Y_s} \\ Y_{\tau_Y^0} := \lim_{t\to\tau_Y^0} Y_t & = & Y_0 + C_{\tau_Y^0} + \gamma \int_0^{\tau_Y^0} \tfrac{ds}{X_s} + \delta \int_0^{\tau_Y^0} \tfrac{ds}{Y_s} \,. \end{array} \tag{12}$$

We have $Y_{\tau_Y^0} = 0$ and as $\mathbf{0}$ is polar for (U, V), then $X_{\tau_Y^0} > 0$. In exactly the same way we can construct a solution on $\{Y_0 > 0\}$ in the interval $[T_1, T_2]$, where $T_1 = \tau_Y^0$, $T_2 = \inf\{t > T_1 : X_t = 0\}$. Iterating, we get a solution on $\{Y_0 > 0\} \times [0, \lim_{n\to\infty} T_n)$ where

$$\begin{array}{rcl} T_{2p} & := & \inf\{t > T_{2p-1} : X_t = 0\} \\ T_{2p+1} & := & \inf\{t > T_{2p} : Y_t = 0\} \,. \end{array}$$

On $\{Y_0 > 0\} \cap \{\lim_{n\to\infty} T_n < \infty\}$ we set $X_{\lim_{n\to\infty} T_n} := \lim_{p\to\infty} X_{T_{2p}} = 0$ and $Y_{\lim_{n\to\infty} T_n} := \lim_{p\to\infty} Y_{T_{2p+1}} = 0$. The polarity of $\mathbf{0}$ entails this is not possible in finite time and thus $\lim_{n\to\infty} T_n = \infty$. So we have obtained a unique global solution on $\{Y_0 > 0\}$. In the same way we obtain a unique global solution on $\{X_0 > 0\}$ and as $\mathbb{P}((X_0, Y_0) = \mathbf{0}) = 0$ the proof is complete.

2. Assume now $X_0 = Y_0 = 0$. Let $(y_n)_{n\geq 1}$ be a sequence of real numbers (strictly) decreasing to 0. From the above paragraph it follows there exists for any $n \geq 1$ a unique solution (X^n, Y^n) with values in $S^{\mathbf{0}}$ to the system

$$\begin{array}{rcl} X_t^n & = & B_t + \alpha \int_0^t \tfrac{ds}{X_s^n} + \beta \int_0^t \tfrac{ds}{Y_s^n} \\ Y_t^n & = & y_n + C_t + \gamma \int_0^t \tfrac{ds}{X_s^n} + \delta \int_0^t \tfrac{ds}{Y_s^n} \,. \end{array}$$

Let

$$\tau := \inf\{t > 0 : X_t^{n+1} < X_t^n\} \,.$$

Using Lemma 3 we obtain $Y_t^{n+1} \leq Y_t^n$ on $[0, \tau]$. We note that $(X_\tau^n, Y_\tau^n) \in S^{\mathbf{0}}$ on $\{\tau < \infty\}$. On $\{Y_\tau^{n+1} = Y_\tau^n\} \cap \{\tau < \infty\}$, since $X_\tau^{n+1} = X_\tau^n$ and the solution starting at time τ is unique, it follows that $X_t^{n+1} = X_t^n$ and $Y_t^{n+1} = Y_t^n$ on $[\tau, \infty)$. On $\{Y_\tau^{n+1} < Y_\tau^n\} \cap \{\tau < \infty\}$, the continuity of solutions at time τ entails there exists $\rho > 0$ such that $Y_t^{n+1} \leq Y_t^n$ on $[\tau, \tau + \rho]$. A second application of Lemma 1 proves that $X_t^{n+1} \geq X_t^n$ on $[\tau, \tau + \rho]$, a contradiction to the definition of τ. Therefore $\mathbb{P}(\tau = \infty) = 1$. It follows that $X_t^{n+1} \geq X_t^n$ and $Y_t^{n+1} \leq Y_t^n$ for any $t \in [0, \infty)$, and we may define

$$X_t := \lim_{n\to\infty} \uparrow X_t^n \qquad Y_t := \lim_{n\to\infty} \downarrow Y_t^n \,.$$

As $Y_t^n \geq V_t$ where (U, V) is the solution to (6) with $X_0 = Y_0 = 0$, we have

$$\begin{array}{rcl} X_t & = & B_t + \alpha \lim_{n\to\infty} \int_0^t \tfrac{ds}{X_s^n} + \beta \lim_{n\to\infty} \int_0^t \tfrac{ds}{Y_s^n} \\ & = & B_t + \alpha \int_0^t \tfrac{ds}{X_s} + \beta \int_0^t \tfrac{ds}{Y_s} \\ & < & \infty \end{array}$$

and also

$$\begin{aligned} Y_t &= \lim_{n\to\infty} y_n + C_t + \gamma \lim_{n\to\infty} \int_0^t \tfrac{ds}{X_s^n} + \delta \lim_{n\to\infty} \int_0^t \tfrac{ds}{Y_s^n} \\ &= C_t + \gamma \int_0^t \tfrac{ds}{X_s} + \delta \int_0^t \tfrac{ds}{Y_s} \\ &< \infty\,. \end{aligned}$$

3. Assume finally $\alpha\delta - \beta\gamma \geq 0$. As the conclusion holds true if $\beta = \gamma = 0$, we may also assume $\beta > 0$. Let (X, Y) be the solution to (5) with $X_0 = Y_0 = 0$ obtained in the previous paragraph and let (X', Y') be another solution. Considering (X^n, Y^n) again and replacing (X^{n+1}, Y^{n+1}) with (X', Y'), the previous proof works and we finally obtain $X'_t \geq X_t$ and $Y'_t \leq Y_t$. Then,

$$0 \leq \delta(X'_t - X_t) - \beta(Y'_t - Y_t) = \int_0^t (\alpha\delta - \beta\gamma)(\frac{1}{X'_s} - \frac{1}{X_s})ds \leq 0. \tag{13}$$

Thus $X'_t = X_t$ and $Y'_t = Y_t$, proving uniqueness. Replacing (δ, β) with (γ, α) in equation (13) we obtain the same conclusion if $\gamma > 0$. ■

6.2. **Case $\beta > 0$ and $\gamma < 0$.**

Theorem 9. *Assume $\beta > 0$, $\gamma < 0$ and one of the conditions C_{2a} or C_3 is satisfied. Then, there exists a unique solution to (5) in $S^{\mathbf{0}}$.*

Proof. The proof is similar to the proof of 1 in Theorem 8. The only change is that now $Y_t \leq V_t$. Therefore, on $\{Y_0 > 0\} \cap \{\tau_Y^0 < \infty\}$,

$$\delta \int_0^{\tau_Y^0} \frac{ds}{Y_s} \leq V_{\tau_Y^0} - Y_0 - C_{\tau_Y^0} - \gamma \int_0^{\tau_Y^0} \frac{ds}{U_s} < \infty$$

and we can define $X_{\tau_Y^0}$ and $Y_{\tau_Y^0}$ as previously done. The application of Theorem 5 to the process on the time interval $[0, \tau_Y^0]$ shows that $X_{\tau_Y^0} > 0$. ■

There is obviously an analnous statement if $\beta < 0$, $\gamma > 0$.

6.3. **Case $\beta \leq 0$ and $\gamma < 0$.** In this case we can give a full answer to the question of existence and uniqueness. Our condition of existence is exactly the condition found in [20] for the reflected Brownian in a wedge being a semimartingale, i.e. there is a convex combination of the directions of reflection that points into the wedge from the corner.

Theorem 10. *Assume $\beta \leq 0$ and $\gamma < 0$.*

1. *If $\alpha\delta > \beta\gamma$, there exists a unique solution to (5) in S.*
2. *If $\alpha\delta \leq \beta\gamma$, there does not exist any solution.*

Proof. 1. Assume first $\alpha\delta > \beta\gamma$.
a) *Existence.* Let $(h_n, n \geq 1)$ be a (strictly) increasing sequence of bounded positive nonincreasing Lipschitz functions converging to $1/x$ on $(0, \infty)$ and to $+\infty$ on $(-\infty, 0]$. For instance we can take

$$\begin{aligned} h_n(x) &= (1 - \tfrac{1}{n})\tfrac{1}{x} \qquad \text{on} \quad [\tfrac{1}{n}, \infty) \\ &= n - 1 \qquad\qquad \text{on} \quad (-\infty, \tfrac{1}{n}]\,. \end{aligned}$$

We consider for each $n \geq 1$ the system

$$\begin{aligned} X_t^n &= X_0 + B_t + \alpha \int_0^t \tfrac{ds}{X_s^n} + \beta \int_0^t h_n(Y_s^n)ds \\ Y_t^n &= Y_0 + C_t + \gamma \int_0^t h_n(X_s^n)ds + \delta \int_0^t \tfrac{ds}{Y_s^n}\,. \end{aligned} \tag{14}$$

From Proposition 4 it follows there exists a unique solution to this system. We set

$$\tau := \inf\{s > 0 : X_s^{n+1} > X_s^n\}\,.$$

We have $h_{n+1}(X_t^{n+1}) \geq h_n(X_t^n)$ on $[0, \tau]$. A first application of Lemma 3 shows that $Y_t^{n+1} \leq Y_t^n$ on $[0, \tau]$. Since $h_{n+1}(Y_\tau^{n+1}) > h_n(Y_\tau^n)$ on $\{\tau < \infty\}$, we deduce from the continuity of

solutions that there exists $\rho > 0$ such that $h_{n+1}(Y_t^{n+1}) \geq h_n(Y_t^n)$ on $[\tau, \tau+\rho]$. A second application of Lemma 3 shows that $X_t^{n+1} \leq X_t^n$ on $[\tau, \tau+\rho]$, a contradiction to the definition of τ. Thus $\mathbb{P}(\tau = \infty) = 1$ proving that on the whole $[0,\infty)$ we have $X_t^{n+1} \leq X_t^n$ and $Y_t^{n+1} \leq Y_t^n$. Then we can set for any $t \in [0,\infty)$

$$X_t := \lim_{n\to\infty} X_t^n \quad \text{and} \quad Y_t := \lim_{n\to\infty} Y_t^n \,.$$

If $\alpha\delta > \beta\gamma$, there is a convex combination of the directions of repulsion that points into the positive quadrant, i.e. there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda\alpha + \mu\gamma > 0$ and $\mu\delta + \lambda\beta > 0$. For $n \geq 1$ and $t \geq 0$,

$$\begin{array}{rcl} \lambda U_t + \mu V_t & \geq & \lambda X_t^n + \mu Y_t^n \\ & \geq & \lambda X_0 + \mu Y_0 + \lambda B_t + \mu C_t + (\lambda\alpha + \mu\gamma)\int_0^t \frac{ds}{X_s^n} + (\mu\delta + \lambda\beta)\int_0^t \frac{ds}{Y_s^n} \,. \end{array} \tag{15}$$

Letting $n \to \infty$ in (15) we obtain

$$\int_0^t \frac{ds}{X_s} < \infty \quad \text{and} \quad \int_0^t \frac{ds}{Y_s} < \infty \,.$$

Then we may let n go to ∞ in (14) proving that (X, Y) is a solution to (5).
b) *Uniqueness.* Let (X', Y') be another solution to (5). Replacing (X^{n+1}, Y^{n+1}) with (X', Y') we follow the above proof to obtain for $t \in [0,\infty)$ and $n \geq 1$

$$X_t' \leq X_t^n \quad \text{and} \quad Y_t' \leq Y_t^n$$

Letting $n \to \infty$ we conclude

$$X_t' \leq X_t \quad \text{and} \quad Y_t' \leq Y_t \,.$$

With the same $\lambda > 0$ and $\mu > 0$ as above,

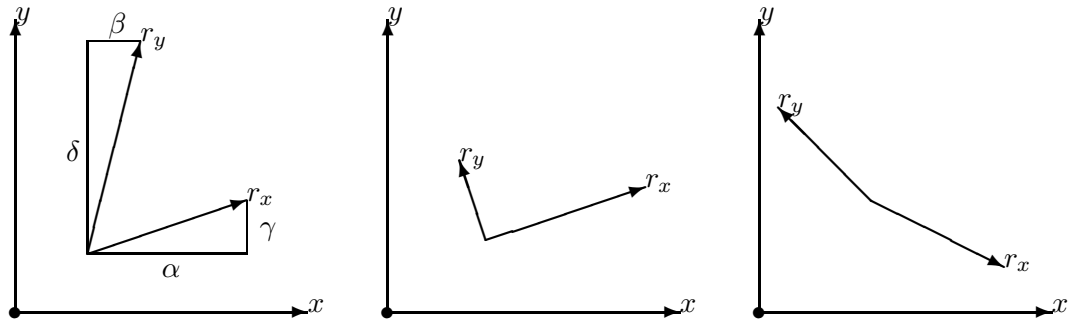
$$0 \leq \lambda(X_t - X_t') + \mu(Y_t - Y_t') = \int_0^t [(\lambda\alpha + \mu\gamma)(\frac{1}{X_s} - \frac{1}{X_s'}) + (\mu\delta + \lambda\beta)(\frac{1}{Y_s} - \frac{1}{Y_s'})]ds \leq 0$$

and therefore $X_t' = X_t$, $Y_t' = Y_t$.
2. If $\alpha\delta \leq \beta\gamma$ there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda\alpha + \mu\gamma \leq 0$ and $\mu\delta + \lambda\beta \leq 0$. Thus, if (X, Y) is a solution to (5),

$$0 \leq \lambda X_t + \mu Y_t \leq \lambda X_0 + \mu Y_0 + \lambda B_t + \mu C_t \,.$$

This is not possible since the paths of the Brownian motion $(\lambda^2 + \mu^2)^{-1/2}(\lambda B_t + \mu C_t)$ are not bounded below. So there is no global solution. ■

In the following pictures, we display the x-repulsion direction vector $r_x = (\alpha, \gamma)$ and the y-repulsion direction vector $r_y = (\beta, \delta)$ in three illustrative instances.



$\beta > 0,\ \gamma > 0,\ \alpha\delta > \beta\gamma$ $\qquad$ $\beta < 0, \gamma > 0, \alpha\beta + \gamma\delta = 0$ $\qquad$ $\beta < 0,\ \gamma < 0,\ \alpha\delta > \beta\gamma$

7. Absolute continuity properties

In some cases we easily obtain an absolute continuity property between the laws of O2BPs with various parameters. When there exists a unique solution to (5), we denote by $\mathbb{P}^{\alpha,\beta,\gamma,\delta}_{x,y}$ the law on $C(\mathbb{R}_+, S) = C(\mathbb{R}_+, \mathbb{R}_+) \times C(\mathbb{R}_+, \mathbb{R}_+)$ of the solution starting at $(x, y) \in S$. We denote by (U, V) the canonical map $(U_t(u, v), V_t(u, v)) = (u(t), v(t))$ and by $\mathcal{U}_t = \sigma((U_s, V_s), s \le t)$ the canonical filtration. Recall that we denote by $\mathbb{P}^d_r$ the law of one-dimensional Bessel process of dimension d starting at $r \ge 0$.

Proposition 11. *Assume $\delta \ge \frac{1}{2}$ and $y > 0$. Then*

$$\mathbb{P}^{\alpha,\beta,0,\delta}_{x,y} \mid_{\mathcal{U}_t} = \exp\left\{\beta \int_0^t \frac{dU_s}{V_s} - \alpha\beta \int_0^t \frac{ds}{U_s V_s} - \frac{\beta^2}{2} \int_0^t \frac{ds}{V_s^2}\right\} . \mathbb{P}^{2\alpha+1}_x \otimes \mathbb{P}^{2\delta+1}_y \mid_{\mathcal{U}_t} . \tag{16}$$

Proof. Under $\mathbb{P}^{2\alpha+1}_x \otimes \mathbb{P}^{2\delta+1}_y$ the process

$$B_t := U_t - x - \alpha \int_0^t \frac{ds}{U_s}$$

is a one-dimensional Brownian motion. The assumptions on δ and on y imply that $V_t > 0$ for any $t \ge 0$ and thus

$$\int_0^t \frac{ds}{v^2(s)} < \infty \qquad \mathbb{P}^{2\delta+1}_y - a.s.$$

Therefore, for $\mathbb{P}^{2\delta+1}_y$-almost every v, $\int_0^t \frac{dB_s}{v(s)}$ is a $\mathbb{P}^{2\alpha+1}_x$-centered Gaussian variable with variance $\int_0^t \frac{ds}{v^2(s)}$ and

$$\int \left(\int \exp\left\{\beta \int_0^t \frac{dB_s}{v(s)} - \frac{\beta^2}{2} \int_0^t \frac{ds}{v(s)^2}\right\} d\mathbb{P}^{2\alpha+1}_x(u) \right) d\mathbb{P}^{2\delta+1}_y(v) = 1.$$

We see that

$$Z_t := \exp\left\{\beta \int_0^t \frac{dB_s}{V_s} - \frac{\beta^2}{2} \int_0^t \frac{ds}{V_s^2}\right\}$$

is a $\mathbb{P}^{2\alpha+1}_x \otimes \mathbb{P}^{2\delta+1}_y$-positive martingale with expectation 1. Setting for any $T > 0$

$$\mathbb{Q}_T := Z_T \, . \mathbb{P}^{2\alpha+1}_x \otimes \mathbb{P}^{2\delta+1}_y \mid_{\mathcal{U}_T}$$

we check that under $\mathbb{Q}_T$

$$U_t - x - \alpha \int_0^t \frac{ds}{U_s} - \beta \int_0^t \frac{ds}{V_s}, \qquad 0 \le t \le T$$

is a real Brownian motion independent of the Brownian motion $\{V_t - y - \delta \int_0^t \frac{ds}{V_s}, 0 \le t \le T\}$. Thus,

$$\mathbb{Q}_T = \mathbb{P}^{\alpha,\beta,0,\delta}_{x,y} \mid_{\mathcal{U}_T} . \qquad \blacksquare$$

A second set of conditions is obtained with the help of Novikov criterion.

Proposition 12. *Assume the following set of conditions is satisfied:*

$$\begin{array}{cc} |\beta| \le \delta - 1/2 & |\gamma| \le \alpha - 1/2 \\ x > 0 & y > 0. \end{array} \tag{17}$$

The process

$$Z_t := \exp\left\{\beta \int_0^t \frac{dU_s}{V_s} - \gamma \int_0^t \frac{dV_s}{U_s} - (\alpha\beta + \gamma\delta) \int_0^t \frac{ds}{U_s V_s} - \frac{\beta^2}{2} \int_0^t \frac{ds}{V_s^2} - \frac{\gamma^2}{2} \int_0^t \frac{ds}{U_s^2}\right\},$$

is a $\mathbb{P}_x^{2\alpha+1} \otimes \mathbb{P}_y^{2\delta+1}$*-positive martingale with expectation* 1 *and*

$$\mathbb{P}_{x,y}^{\alpha,\beta,\gamma,\delta} \mid_{\mathcal{U}_t} = Z_t \,.\, \mathbb{P}_x^{2\alpha+1} \otimes \mathbb{P}_y^{2\delta+1} \mid_{\mathcal{U}_t} . \tag{18}$$

Proof. Under $\mathbb{P}_x^{2\alpha+1} \otimes \mathbb{P}_y^{2\delta+1}$ the processes

$$B_t := U_t - x - \alpha \int_0^t \frac{ds}{U_s}$$

and

$$C_t := V_t - y - \delta \int_0^t \frac{ds}{V_s}$$

are independent Brownian motions. Then Z_t may be written

$$Z_t = \exp\left\{ \beta \int_0^t \frac{dB_s}{V_s} + \gamma \int_0^t \frac{dC_s}{U_s} - \frac{\beta^2}{2} \int_0^t \frac{ds}{V_s^2} - \frac{\gamma^2}{2} \int_0^t \frac{ds}{U_s^2} \right\},$$

which shows that Z_t is a positive local martingale. Using (4) we compute

$$\begin{array}{rcl} \int\int \exp\{\frac{\gamma^2}{2} \int_0^t \frac{ds}{U_s^2}\} \, d\mathbb{P}_x^{2\alpha+1} \otimes d\mathbb{P}_y^{2\delta+1} & = & \int \exp\{\frac{\gamma^2}{2} \int_0^t \frac{ds}{R_s^2}\} \, d\mathbb{P}_x^{2\alpha+1} \\ & = & \int (\frac{R_t}{x})^{\alpha-1/2} \exp\{-\frac{(\alpha-1/2)^2-\gamma^2}{2} \int_0^t \frac{ds}{R_s^2}\} \, d\mathbb{P}_x^2 \\ & < & \infty \end{array}$$

since a Bessel process of dimension two has finite moments of any order. Finally,

$$\begin{array}{rcl} & & \int\int \exp\{\frac{\beta^2}{2} \int_0^t \frac{ds}{V_s^2}\} \exp\{\frac{\gamma^2}{2} \int_0^t \frac{ds}{U_s^2}\} \, d\mathbb{P}_x^{2\alpha+1} \otimes d\mathbb{P}_y^{2\delta+1} \\ & = & (\int \exp\{\frac{\beta^2}{2} \int_0^t \frac{ds}{R_s^2}\} \, d\mathbb{P}_x^{2\delta+1})(\int \exp\{\frac{\gamma^2}{2} \int_0^t \frac{ds}{R_s^2}\} \, d\mathbb{P}_x^{2\alpha+1}) \\ & < & \infty \end{array}$$

and Novikov's criterion ([17], Proposition VIII.1.15) proves that Z_t has expectation 1 with respect to $\mathbb{P}_x^{2\alpha+1} \otimes \mathbb{P}_y^{2\delta+1}$. We easily see that $\{B_t - \beta \int_0^t \frac{ds}{V_s}, 0 \le t \le T\}$ and $\{C_t - \gamma \int_0^t \frac{ds}{U_s}, 0 \le t \le T\}$ are independent Brownian motions under the probability with density Z_T and this proves that the new probability is $\mathbb{P}_{x,y}^{\alpha,\beta,\gamma,\delta} \mid_{\mathcal{U}_T}$. ■

8. Associated local martingales

Scale functions play a major role in the study of linear continuous Markov Processes ([17] Section VII.3). There is no equivalent functions on the plane. However, in some particular cases we can find simple functions of O2BPs that are local martingales.

Proposition 13. *Assume the following set of conditions:*

$$\begin{array}{l} \alpha > 1/2 \\ \delta > 1/2 \\ (2\alpha-1)\beta + (2\delta-1)\gamma = 0 \\ (X_0, Y_0) \neq (0,0). \end{array} \tag{19}$$

Then

$$M_t := X_t^{1-2\alpha} Y_t^{1-2\delta}$$

is for any $a > 0$ *a positive local martingale on* $[a, \infty)$ *which tends to* 0 *as* $t \to \infty$.

Proof. Using Proposition 7 we first remark that $X_t > 0$ and $Y_t > 0$ for any $t > 0$. From Itô formula we get

$$M_t = M_a + \int_0^t [(1-2\alpha) X_s^{-2\alpha} Y_s^{1-2\delta} dB_s + (1-2\delta) X_s^{1-2\alpha} Y_s^{-2\delta} dC_s].$$

Thus M is a positive local martingale which tends to a limit $H \geq 0$ as $t \to \infty$ ([17], Corollary II.2.11 and Exercise IV.1.46). Moreover, the quadratic variation $< M >$ satisfies

$$\begin{aligned} < M >_t &= < M >_a + \int_a^t X_s^{-4\alpha} Y_s^{-4\delta} [(1-2\alpha)^2 Y_s^2 + (1-2\delta)^2 X_s^2] ds \\ &= < M >_a + \int_a^t M_s^2 [\tfrac{(1-2\alpha)^2}{X_s^2} + \tfrac{(1-2\delta)^2}{Y_s^2}] ds. \end{aligned}$$

We may assume $\beta \geq 0$ and $\gamma \leq 0$. Then $Y \leq V$ where V is the process in (6). From Corollary 1 we deduce that

$$\int_a^\infty \frac{ds}{Y_s^2} \geq \int_a^\infty \frac{ds}{V_s^2} = \infty.$$

On the set $\{H > 0\}$ we should have $< M >_\infty = \infty$ and then $\limsup_t M_t = -\liminf_t M_t = \infty$, a contradiction with $M_t \to H$. Thus $H = 0$ a.s. ■

There is another case of interest.

Proposition 14. *Assume $\alpha = \delta = 1/2$ and one of the following set of conditions is satisfied:*

- *$\beta > 0$ and $\gamma > 0$*
- *$\beta\gamma < 0$ and $(X_0, Y_0) \neq (0,0)$*
- *$\beta < 0$, $\gamma < 0$ and C_3.*

Then

$$M_t := \gamma \ln X_t - \beta \ln Y_t$$

is for any $a > 0$ a continuous local martingale on $[a, \infty)$ and

$$\limsup_{t\to\infty} M_t = -\liminf_{t\to\infty} M_t = \infty$$

.

Proof. We deduce from Theorem 5 and Proposition 6 that $X_t > 0$ and $Y_t > 0$ for any $t > 0$. Now Itô formula gives

$$M_t = M_a + \int_a^t \left(\gamma \frac{dB_s}{X_s} - \beta \frac{dC_s}{Y_s} \right)$$

and

$$< M >_t = < M >_a + \int_a^t \left(\frac{\gamma^2}{X_s^2} + \frac{\beta^2}{Y_s^2} \right) ds.$$

If $\beta < 0$, then $X_t \leq U_t$ where U is the Bessel process of dimension two in (6). From Corollary 1 we get

$$< M >_\infty \geq \int_a^\infty \gamma^2 \frac{ds}{X_s^2} \geq \gamma^2 \int_a^\infty \frac{ds}{U_s^2} = \infty$$

and the conclusion follows. If $\beta > 0$ we know that

$$\limsup_{t\to\infty} (\ln X_t) \geq \limsup_{t\to\infty} (\ln U_t) = \infty$$

where

$$\ln X_t = \ln X_a + \int_a^t \frac{dB_s}{X_s} + \beta \int_a^t \frac{ds}{X_s Y_s}.$$

But $\ln X_t$ would converge a.s. as $t \to \infty$ if

$$\int_a^\infty \frac{ds}{X_s^2} < \infty \quad \text{and} \quad \int_a^\infty \frac{\mathrm{ds}}{\mathrm{Y_s^2}} < \infty.$$

Therefore $< M >_\infty = \infty$ again. ■

9. Product form stationary distribution

We introduce an additional constant drift $(-\mu, -\nu)$ in order to make the solution a recurrent process in the nonnegative quadrant. We consider the system

$$\begin{array}{rcl} X_t &=& X_0 + B_t + \alpha \int_0^t \frac{ds}{X_s} + \beta \int_0^t \frac{ds}{Y_s} - \mu t \\ Y_t &=& Y_0 + C_t + \gamma \int_0^t \frac{ds}{X_s} + \delta \int_0^t \frac{ds}{Y_s} - \nu t \end{array} \tag{20}$$

with the conditions $X_t \geq 0$, $Y_t \geq 0$. If now

$$\begin{array}{rcl} U_t &=& X_0 + B_t + \alpha \int_0^t \frac{ds}{U_s} - \mu t \\ V_t &=& Y_0 + C_t + \delta \int_0^t \frac{ds}{V_s} - \nu t \end{array} \tag{21}$$

with $U_t \geq 0$, $V_t \geq 0$, we can check that $\mathbf{0}$ is still polar for (U, V) (as well, 0 is polar for U if $\alpha \geq \frac{1}{2}$ and for V if $\delta \geq \frac{1}{2}$). Therefore the results in Sections 4, 5 and 6 are still valid for the solution to (20). We are now looking for conditions on the set of parameters in order to obtain a stationary distribution for the Markov process (X, Y) in the remarkable form of a product of two gamma distributions. Our proof is just an adaptation of the proof in [15].

Theorem 15. *Assume there exists a unique solution to (20) in $S^{\mathbf{0}}$ or in S. This process has an invariant distribution in the form $\Gamma(a, c) \otimes \Gamma(b, d)$ if and only if*

- $\alpha\beta + \gamma\delta = 0$ *(skew symmetry)*
- $a = 2\alpha + 1$, $b = 2\delta + 1$
- $c = 2\delta \frac{\mu\alpha + \nu\gamma}{\alpha\delta - \beta\gamma}$, $d = 2\alpha \frac{\mu\beta + \nu\delta}{\alpha\delta - \beta\gamma}$
- $\mu\alpha + \nu\gamma > 0$, $\mu\beta + \nu\delta > 0$.

Proof. Let

$$\rho(x, y) = x^{a-1} e^{-cx} y^{b-1} e^{-dy} \qquad \text{for } x \geq 0, y \geq 0\,.$$

The infinitesimal generator of the diffusion (20) is given by

$$L = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \left(\frac{\alpha}{x} + \frac{\beta}{y} - \mu\right)\frac{\partial}{\partial x} + \left(\frac{\gamma}{x} + \frac{\delta}{y} - \nu\right)\frac{\partial}{\partial y}\,.$$

By a density argument, to prove that ρ is an invariant density, it is enough to check that

$$\int_0^\infty \int_0^\infty Lf(x, y)\rho(x, y)\, dxdy \; = 0$$

for any $f(x, y) = g(x)h(y)$ with $g, h \in C_c^2((0, \infty))$ (compactly supported twice continuously differentiable functions on $(0, \infty)$). Integrating by parts, we get

$$\int_0^\infty \int_0^\infty L(gh)(x, y)\rho(x, y)\, dxdy \; = \int_0^\infty \int_0^\infty g(x)h(y)J(x, y)\, dxdy$$

where

$$J(x, y) = \rho(x, y)\left[A + Bx^{-1} + Cx^{-2} + Dy^{-1} + Ey^{-2} + Fx^{-1}y^{-1}\right]$$

with

$$\begin{array}{rcl} A &=& \frac{1}{2}c^2 + \frac{1}{2}d^2 - \mu c - \nu d \\ B &=& -(a-1)c + \mu(a-1) + \alpha c + \gamma d \\ C &=& \frac{1}{2}(a-1)(a-2) - \alpha(a-2) \\ D &=& -(b-1)d + \nu(b-1) + \beta c + \delta d \\ E &=& \frac{1}{2}(b-1)(b-2) - \delta(b-2) \\ F &=& \beta(a-1) + \gamma(b-1)\,. \end{array}$$

Letting $A = B = C = D = E = F = 0$ we obtain the specified values for a, b, c, d and the skew symmetry condition $\alpha\beta + \gamma\delta = 0$, which means that r_x and r_y are orthogonal. The last condition in the statement of the theorem is written out so that the invariant density ρ is

integrable on S. It is satisfied if (μ, ν) points into the interior of the quadrant designed by r_x and r_y. ■

Remark. With the same proof, we may check that under the skew symmetry condition, when $\mu = \nu = 0$, the function $\rho(x, y) = x^{2\alpha} y^{2\delta}$ is a non-integrable invariant density that does not depend on the obliqueness parameter β.

A simple change of variables (beta-gamma algebra) provides the following result, where $\beta(a, b)$ denotes the usual Beta distribution with parameters a and b.

Corollary 16. *With the conditions and notations of Theorem 15, the two-dimensional process*

$$\begin{array}{rcl} W_t & := & \frac{cX_t}{cX_t + dY_t} \\ Z_t & := & cX_t + dY_t \end{array} \tag{22}$$

has $\beta(a, b) \otimes \gamma(a + b, 1)$ *for invariant distribution in* $(0, 1) \times (0, \infty)$.

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