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Re-projection without Reconstruction

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Abstract

Discrete tomography focuses on image representation by its discrete projections, and the related inversion algorithms (or image reconstruction). Our study is based on redundant representations (considering more than just few projections). We propose a new approach to compute further redundancy (i.e. new projections) from a set of existing projections. While this technique relies on the geometric properties of ghosts, which are elements of the 2D image that sum to zero following some projection directions, we show an equivalent method using 1D convolutions, thus avoiding the explicit image reconstruction. This technique has interesting applications in distributed storage systems, where the use of redundancy data is key for system reliability.

Keywords: discrete tomography, redundancy, ghost, convolution, storage.

1. Introduction

The inverse problem in 2D computed tomography (CT) is to reconstruct an image from a set of projections recorded at different angles. A sufficient amount of projections is required in order to inverse the operation, guaranteeing the existence and uniqueness of the solution [Kat78]. The forward operation computes a set of projections of a given 2D image by the Radon transform [Dea07]. The Mojette transform [GN05, Gué09] is a discrete and exact Radon transform. It consists in computing a set of projections from a lattice of pixels along different discrete directions. The values of projection elements are the addition of the pixel values along these directions. While other discrete versions of the Radon transform (e.g. the Finite Radon Transform [MF93]) can compute limited amount of projections, the Mojette transform can be used to compute redundant projections (i.e. beyond the sufficient amount required for the reconstruction process).

Redundant representations are useful in different applications such as reliable transmission [NGPB96] or storage systems [GPN01]. Redundancy is the key to provide reliability in systems that face breakdowns such as packet losses, disk failures or server unavailability. The loss of some projections is counteracted by the redundant information contained in the extra projections computed during the forward transform.

Once projections are computed, we consider that the image is only available by reconstruction. Indeed, in distributed storage systems, the image is only used to compute projections which are afterwards distributed over several storage nodes. When projections are permanently lost, it is necessary to rebuild it to restore the system fault-tolerance. This issue, well-known in the storage community as the *repair problem*, deals with the node repairing after a permanent failure. Furthermore, it could be necessary to improve the system redundancy during its lifetime, by computing further extra projections (i.e. after the forward transform process).

A simple way to compute new projections consists in reconstructing the image from a sufficient amount of projections followed by projecting the pixel values along the desired directions. While simple to design, this technique has several cons. First, the image should not be accessible by anyone but the owner (i.e. not fully reconstructed by a tier). Furthermore, the computation workload should be distributed and not dedicated to the reconstructing tier to avoid computation congestion. Finally, the number of projections transmitted to the reconstructing tier should be minimized. Indeed, the amount of data in large storage clusters continuously increases and the network load for such a task might induce network congestions.

In this paper, we propose a new technique to generate redundant projections from an existing set of projections. After recalling the Mojette transform basics in Section 2, we show that the linear Mojette recon-

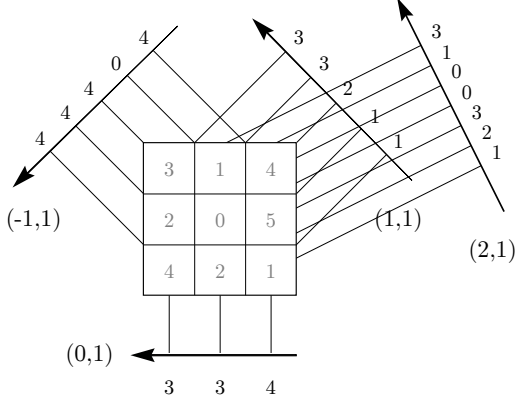


Figure 1: Mojette transform of a 3×3 image for directions (p, q) in the set $\{(-1, 1), (0, 1), (1, 1), (2, 1)\}$. Each projection bin is the sum of the pixels centered on a discrete line of equation $b = -kq + lp$. Addition is done modulo 6.

struction can be decomposed into partial reconstructions in Section 3. Particularly, this section shows how each projection contributes independently to the re-projection process and how the explicit reconstruction of the image is avoided. Then, we discuss the storage applications related to this technique in Section 4 before the conclusion.

2. Mojette Transform

The Mojette transform [GN05, Gué09] is a discrete and exact version of the Radon transform. This section presents the Mojette transform, the uniqueness conditions of the reconstruction solution, and the inverse algorithm.

2.1. Forward Mojette Transform

The forward Mojette transform computes a linear set of 1D projections at different angles, from a discrete image $f : (k, l) \mapsto \mathbb{N}$, composed of $P \times Q$ pixels. A projection direction is defined by a couple of co-prime integers (p, q) . Projections are vectors of variable sizes whose elements are called *bins*. A bin in the Mojette transform of f is characterised by its position b in the projection which corresponds to a discrete line of equation $b = -kq + lp$. Its value is the sum of the centered pixels along the line:

$$(M_{(p,q)}f)(b) = \sum_{k=0}^{P-1} \sum_{l=0}^{Q-1} f(k, l) [b = -kq + lp], \quad (1)$$

where, $[\cdot]$ is the Iverson bracket ($[P] = 1$ whenever P is true, 0 otherwise). The number of bins B of a projection depends on the projection direction (p, q) and the lattice size:

$$B(p, q, P, Q) = |p|(Q - 1) + |q|(P - 1) + 1. \quad (2)$$

Figure 1 gives an example of the forward Mojette

transform for a 3×3 integer image. The process transforms the 2D image into a linear set of four projections along the directions of the following set: $\{(-1, 1), (0, 1), (1, 1), (2, 1)\}$ based on modulo-6 additions. The next section concerns the inverse problem and the reconstruction algorithm.

2.2. Inverse Mojette Transform

The following part describes the existence of a reconstruction solution, its uniqueness, and the inversion algorithm to retrieve the image given a projection set.

2.2.1. Ghosts and Reconstruction Criteria

Katz has shown that for a $P \times Q$ lattice, the reconstruction is possible given a projection set S_I if one of the following criteria is verified [Kat78]:

$$P \leq \sum_{i=0}^{I-1} |p_i| \text{ or } Q \leq \sum_{i=0}^{I-1} |q_i|, \quad (3)$$

where I is the number of projections. For instance, the image in Fig. 1 is reconstructible since $\sum_{i=0}^2 |p_i| = 3$ for the given set, which is not less than the image height. We can also notice that each subset of three projections $\{(p_{j_0}, q_{j_0}), \dots, (p_{j_2}, q_{j_2})\}$ is such that $\sum_{i=0}^2 |p_{j_i}| = 3$. Thus, Fig. 1 depicts a redundant representation of the image, where any projection among four can be omitted for reconstruction.

The criterion for an arbitrary shape is given by the geometric properties of ghosts. Ghosts are elements of the image made of positive and negative values that sum to zero along a set of projection directions. While ghosts influence the image itself, they are invisible in different projection directions. Each projection defined by (p, q) is related to a simple ghost as shown in Fig. 2 (a):

$$G_{\{(p,q)\}} : p \mapsto \begin{cases} 1 & \text{if } p = (0, 0) \\ -1 & \text{if } p = (p, q) \\ 0 & \text{otherwise} \end{cases}.$$

Any image convolved with $G_{\{(p,q)\}}$ has null projection values in the direction (p, q) . As a consequence, a composed ghost $G_{\{(p_i, q_i)\}}$, for the set of projections $\{(p_i, q_i)\}$, is obtained by convolving the simple ghosts G_{p_i, q_i} . Figure 2 gives some examples of simple and composed ghosts. Part (a) shows the simple ghost for the projection direction $(0, 1)$. Parts (b-d) show respectively the composed ghosts:

$$\begin{aligned} G_{\{(0,1),(1,1)\}} &= G_{\{(0,1)\}} * G_{\{(1,1)\}}, \\ G_{\{(-1,1),(0,1),(1,1)\}} &= G_{\{(-1,1)\}} * G_{\{(0,1),(1,1)\}}, \\ G_{\{(2,1),(-1,1),(0,1),(1,1)\}} &= G_{\{(2,1)\}} * G_{\{(-1,1),(0,1),(1,1)\}}. \end{aligned}$$

Normand and Guédon proved that any rectangular domain that does not contain the composed ghost for a given projection set, has necessarily a unique inverse solution [NG98]. This proposal generalises Katz' criterion for images of arbitrary shape.

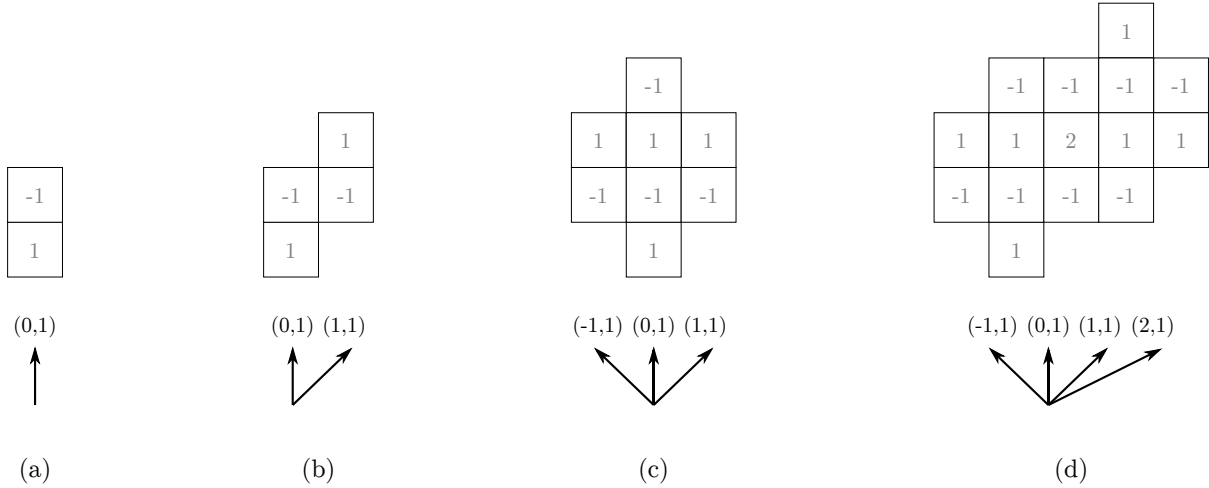


Figure 2: (a) Simple ghost $G_{(0,1)}$ with null projection values in direction $(0,1)$. The following constructs the composed ghost $G_{\{(-1,1),(0,1),(1,1),(2,1)\}}$ by the iterative 2D convolutions with $G_{\{(p,q)\}}$ for directions (p,q) in the set $\{(-1,1), (1,1), (2,1)\}$.

2.2.2. Inverse Mojette Algorithm

The reconstruction algorithm aims at finding a reconstructible bin and at writing its value in the image by back-projection. Bins are reconstructible when they result from a unique pixel of the image. Once a bin is reconstructed, its contribution is subtracted from all the projections involved in the reconstruction. While a bin is reconstructed, the related pixel is removed from the problem, paving the way to reconstruct further bins. Figure 3 shows the first step of the reconstruction of a 3×3 image from a projection set whose directions (p,q) are in the set $\{(1,1), (0,1), (-1,1)\}$. The image is reconstructible since the composed ghost is not con-

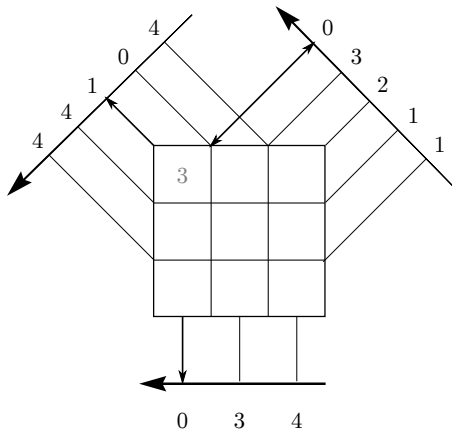


Figure 3: First step of the Mojette inverse of a 3×3 image by three projections whose directions (p,q) are in the set $\{(0,1), (-1,1), (1,1)\}$. The first bin of the projection $(1,1)$ is a reconstructible bin as it fully defined the first pixel. Its value is back-projected in the image and is subtracted from all the projections.

tained in the image domain (this ghost is depicted in Fig. 2 (c)). The first bin of the projection $(1,1)$ is reconstructible as it sums only the pixel on the top left corner. After the back-projection, the value of the related bins in the three projections are subtracted by the value of the constructible bin.

Observing that the reconstruction propagation from the image corners to its center, Normand et al. showed that given an image domain and a projection set, a dependency graph between the image pixels can be found [NKÉ06]. Within this graph, considering that a single projection is dedicated to the reconstruction of one line of the image, a reconstruction path can be pre-determined. Figure 4 illustrates such a path inside the dependency graph of a 6×4 image with the following projection set $S = \{(2,1), (1,1), (0,1), (-1,1)\}$. We consider here that the reconstruction propagation progresses from left to right.

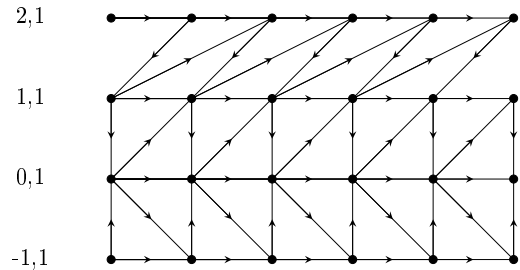


Figure 4: Dependency graph for a 6×4 image given the four projection directions (p,q) in the set $\{(2,1), (1,1), (0,1), (-1,1)\}$. The reconstruction is from left to right. Vertices represent image pixels while directed edges correspond to dependencies between pixels.

3. Mojette Re-projection

In this section, we present a new approach based on the linearity of the Mojette transform to decompose its inverse operation into partial reconstructions. Particularly, we show how each projection contributes independently to the re-projection process. The second part highlights an equivalent method based on 1D convolutions, that avoids the explicit reconstruction of the image.

3.1. Partial Reconstructions

The linear property of the Mojette inverse enables the decomposition of the reconstruction process into partial reconstructions that can be individually computed from each projection and re-projected along any arbitrary direction.

Let S be a set of Q projection directions in the form $(p_i, q_i = 1)$. Then $\sum q_i = Q$ and, according to Eq. (3), any $P \times Q$ domain is uniquely reconstructible by the set of projections with directions in S . Let R be a subset of S , a partial reconstruction is the process that reconstructs an image f_S^R from the set of projections R (i.e. disregarding Eq. (3) if $R \subsetneq S$), completed by null-valued projections in the missing directions $S \setminus R$. If subsets R_i form a partition of S then, by linearity, $f = \sum_i f_S^{R_i}$. Figure 5 shows an example of partial reconstructions (a-c) with projections in the set $S = \{(-1, 1), (0, 1), (1, 1)\}$. Figure 5 (a) to (c) respectively depict the partial reconstructions from the subsets $\{(0, 1)\}$, $\{(1, 1)\}$ and $\{(-1, 1)\}$. The sum of the three partially reconstructed images $f_S^{\{(0,1)\}}$, $f_S^{\{(1,1)\}}$ and $f_S^{\{(-1,1)\}}$ yields the reconstructed image f as shown in Fig. 5 (d) since $\{(0, 1)\}$, $\{(1, 1)\}$ and $\{(-1, 1)\}$ form a partition of S .

Similarly to the inverse Mojette transform, the forward transform is linear. Hence the sum of the projections along direction (p, q) computed from partial reconstructions yields the projection $M_{(p,q)}f$ of the original image along this direction (p, q) :

$$M_{(p,q)}f = \sum_{R \in P} M_{(p,q)}f_S^R, \quad (4)$$

where S is a sufficient set of projection directions for image reconstruction, and f_S^R is the partial reconstruction from the subset R in the partition P of S .

An example of the re-projection process is given in Fig. 5. The image used in this example is the same 3×3 integer image as the one depicted in Fig. 1. The Mojette transform gives a set of the following projection set: $S = \{(2, 1), (1, 1), (0, 1), (-1, 1)\}$ where addition is realised modulo 6. In what follows, we consider the computation of the projection $(2, 1)$ from the three other projections. Figure 5 gives successively the partial reconstructions from $\{(0, 1)\}$, $\{(1, 1)\}$, and $\{(-1, 1)\}$ with their relative re-projection along the projection direction $(2, 1)$.

While the partial reconstructions have infinite

length, the result can be truncated since the part outside the original image is zero-valued. In the next section, we investigate a technique to avoid the 2D reconstructions of images.

3.2. Reconstruction by 1D Convolutions

The partial reconstruction $f_S^{\{(p_i, q_i)\}}$ from the single projection $M_{(p_i, q_i)}f$ is such that, by definition, the other projections of S are zero-valued. It is equivalent to say that the image $f_S^{\{(p_i, q_i)\}}$ is a ghost for those directions. We consider the composed ghost $G_{S \setminus \{(p_i, q_i)\}}$. Then, the image resulting from a partial reconstruction is composed of this ghost kernel. More particularly, this image can be expressed as the convolution of some sequence h and $G_{S \setminus \{(p_i, q_i)\}}$:

$$f_S^{\{(p_i, q_i)\}} = h * G_{S \setminus \{(p_i, q_i)\}}$$

Then for any direction $(p_k, q_k = 1)$,

$$M_{(p_k, q_k)}f_S^{\{(p_i, q_i)\}} = h * (M_{(p_k, q_k)}G_{S \setminus \{(p_i, q_i)\}}). \quad (5)$$

By definition of $f_S^{\{(p_i, q_i)\}}$,

$$M_{(p_i, q_i)}f_S^{\{(p_i, q_i)\}} = M_{(p_i, q_i)}f$$

therefore

$$h = (M_{(p_i, q_i)}f) *^{-1} (M_{(p_i, q_i)}G_{S \setminus \{(p_i, q_i)\}}). \quad (6)$$

Using Eq. (5) and h from Eq. (6), we are able to compute the projection of the partial reconstruction $f_S^{\{(p_i, q_i)\}}$ along any direction (p_k, q_k) with 1D operations (provided that $M_{(p_k, q_k)}G_{S \setminus \{(p_i, q_i)\}}$ and $M_{(p_i, q_i)}G_{S \setminus \{(p_i, q_i)\}}$ are precomputed).

Furthermore, since $G_{S \setminus \{(p_i, q_i)\}}$ is composed by convolutions:

$$\begin{aligned} M_{(p_k, q_k)}f_S^{\{(p_i, q_i)\}} &= (M_{(p_i, q_i)}f) \\ &\quad *^{-1}_{(p_j, q_j) \in S \setminus \{(p_i, q_i)\}} (M_{(p_i, q_i)}G_{\{(p_j, q_j)\}}) \\ &\quad *_{(p_j, q_j) \in S \setminus \{(p_k, q_k)\}} (M_{(p_i, q_i)}G_{\{(p_j, q_j)\}}) \end{aligned} \quad (7)$$

In Eq. (7), each $M_{(p_i, q_i)}G_{(p_j, q_j)}$ corresponds to the projection along direction (p_i, q_i) of the simple ghost for direction (p_j, q_j) and is equal to the sequence:

$$t \mapsto \begin{cases} 1 & \text{if } t = 0 \\ -1 & \text{if } t = j - i \\ 0 & \text{otherwise} \end{cases}$$

Then, the re-projection of the partial reconstruction can be computed from a single projection and the related ghost kernel by Eq. (7) where the projection of the ghost is expressed as 1D deconvolutions and 1D convolutions with basic kernels.

4. Applications to Distributed Storage Systems

In distributed storage systems, redundancy is key to provide reliability and face inevitable software and

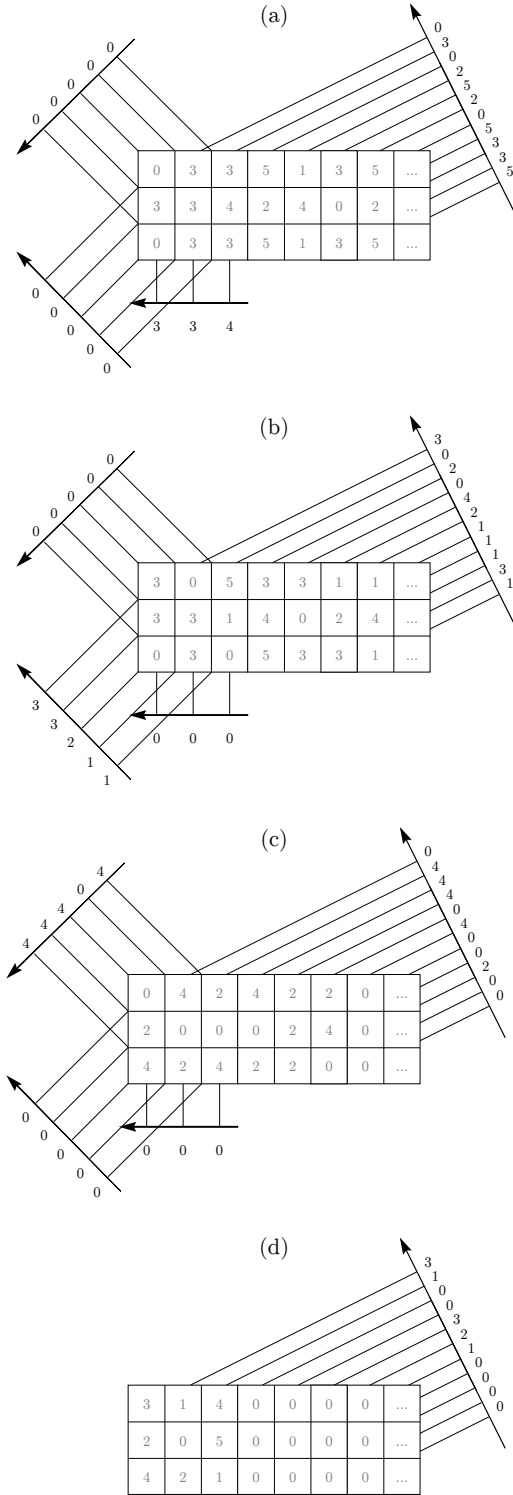


Figure 5: Re-projection of three partial reconstructions. The figures depict the partial reconstructions respectively from the projections $(0, 1)$, $(1, 1)$ and $(-1, 1)$. The re-projection is done following the direction $(2, 1)$. The last figure corresponds to the sum of the three partial reconstructions which equals the original image.

hardware failures. Plain replication is the standard technique that distributes several copies of an image on remote storage nodes to handle data unavailability. While simple, this technique induces high storage overhead defined by the factor of replication.

Erasure coding has become a crucial alternative that provides similar fault-tolerance and reduces significantly this storage overhead. An (n, k) erasure code divides an image into k blocks that are used to compute $r = n - k$ redundant blocks. A Maximal Distance Separable (MDS) code (e.g. Reed Solomon [PD03]) depicts a code that is able to retrieve the image from any set of k blocks among the n . These n blocks are then distributed to different storage nodes. While the system is able to face r failures, its storage overhead is $1 + \frac{r}{k}$.

4.1. Mojette Erasure Coding

The Mojette transform can be used as an erasure code [NGPB96]. Since the number of projections is not limited, redundancy is generated when it is beyond the sufficient number of projections required to satisfy reconstruction criteria (as depicted in Section 2.2.1). In this case, the reconstruction is possible even when some projections are erased.

We consider the following constraint over projections: $q_i = 1$. According to Katz' criterion, and considering a rectangular grid, it is possible to reconstruct the grid using any subset of Q projections. Thus, the (n, k) Mojette erasure code is defined such as k is the image height, and n corresponds to the number of projections. Then, the image is reconstructible even if $n - k$ projections are lost. While the Mojette erasure code is able to retrieve the image from any subset of k projections among n , the size of projections is variable and the code is $(1 + \epsilon)$ MDS, i.e. asymptotically optimal in terms of code size.

4.2. Node Repair and Redundancy Rate

Node repairing and fault-tolerance adaptation are two applications of the re-projection process in distributed storage systems. The repair problem deals with the reconstruction of the whole information contained in a storage node after suffering from permanent failure. A naive repairing process consists in fetching k projections for each file concerned by the projections contained in the failed node. Then, the auxiliary node fully reconstructs the related files, and generates the desired redundant information. Many problems arise from this method. The first concerns the computation centralization at the auxiliary node. The re-projection technique in this paper enables a distributed computation since each node involved in the reconstruction process transmits its contribution as partial reconstruction projection along the desired direction. The auxiliary node needs only to sum the received projections.

Node repairing is a growing concern as the volume

of data increases in storage systems. Thus, fetching k projections for each file contained in a disk can induce tera-bytes of data to transfer. Major cloud services as Facebook [RSG*13] significantly suffer from it. Wang et al. [WDB10] showed that it is possible to send a amount of information lower than k to rebuild a node. In our case, network coding appears as a solution to reduce the network load since any intermediate node between the contributors and the reconstructing node is able to merge the contributions from different nodes.

When the re-projection direction corresponds to a new direction, this technique corresponds to generating new redundant projections. Thus, it appears as an efficient way to increase the fault-tolerance of a system by adapting the redundancy rate after the transform.

5. Conclusion

In this paper, we rely on the linearity of the Mojette transform to decompose the reconstruction process into partial reconstructions. We show that the projection of these reconstructions following new directions can be aggregated in order to generate redundant projections. This novel approach induces distributed load during re-projection. Furthermore, we show that this process is equivalent to a 1D convolution and deconvolution, avoiding to fully reconstruct images. This technique has significant advantages in reliable storage systems as a way to both repair erased projections and increase data reliability. Then, our future work will focus on the capability to decrease the network load.

References

- [Dea07] DEANS S. R.: *The Radon transform and some of its applications*. Courier Dover Publications, 2007.
- [GN05] GUÉDON J., NORMAND N.: The Mojette transform: The first ten years. In *Discrete Geometry for Computer Imagery*, Andres E., Damiand G., Lienhardt P., (Eds.), vol. 3429 de *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2005, pp. 79–91.
- [GPN01] GUÉDON J. P., PARREIN B., NORMAND N.: Internet distributed image information system. *Integrate Computer-Aided Engineering*. Vol. 8, Num. 3 (August 2001), 205–214.
- [Gué09] GUÉDON J.: *The Mojette transform. Theory and applications*. ISTE-WILEY, January 2009.
- [Kat78] KATZ M.: *Questions of uniqueness and resolution in reconstruction from projections*. Springer-Verlag Berlin ; New York, 1978.
- [MF93] MATUŠ F., FLUSSER J.: Image representation via a finite radon transform. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*. Vol. 15, Num. 10 (October 1993), 996–1006.
- [NG98] NORMAND N., GUÉDON J.-P.: La transformée Mojette: une représentation redondante pour l'image. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*. Vol. 326, Num. 1 (1998), 123 – 126.
- [NGPB96] NORMAND N., GUEDON J. V., PHILIPPE O., BARBA D.: Controlled redundancy for image coding and high-speed transmission. vol. 2727, pp. 1070–1081.
- [NKÉ06] NORMAND N., KINGSTON A., ÉVENOU P.: A geometry driven reconstruction algorithm for the Mojette transform. In *DGCI*, vol. 4245 de *LNCS*. Springer Berlin Heidelberg, 2006, pp. 122–133.
- [PD03] PLANK J. S., DING Y.: *Note: Correction to the 1997 Tutorial on Reed-Solomon Coding*. Tech. Rep. CS-03-504, University of Tennessee, April 2003.
- [RSG*13] RASHMI K. V., SHAH N. B., GU D., KUANG H., BORTHAKUR D., RAMCHANDRAN K.: A solution to the network challenges of data recovery in erasure-coded distributed storage systems: A study on the facebook warehouse cluster. In *Proceedings of the 5th USENIX Conference on Hot Topics in Storage and File Systems* (Berkeley, CA, USA, 2013), HotStorage'13, USENIX Association, pp. 8–8.
- [WDB10] WANG Z., DIMAKIS A. G., BRUCK J.: Rebuilding for array codes in distributed storage systems. In *GLOBECOM Workshops (GC Wkshps), 2010 IEEE* (2010), IEEE, pp. 1905–1909.