

# Carleman estimates for some non smooth anisotropic media

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June 3, 2015

## Abstract

Let  $B$  be a  $n \times n$  block diagonal matrix in which the first block  $C_\tau$  is an hermitian matrix of order  $(n - 1)$  and the second block  $c$  is a positive function. Both are piecewise smooth in  $\bar{\Omega}$ , a bounded domain of  $\mathbb{R}^n$ . If  $S$  denotes the set where discontinuities of  $C_\tau$  and  $c$  can occur, we suppose that  $\Omega$  is stratified in a neighborhood of  $S$  in the sense that locally it takes the form  $\Omega' \times (-\delta, \delta)$  with  $\Omega' \subset \mathbb{R}^{n-1}$ ,  $\delta > 0$  and  $S = \Omega' \times \{0\}$ . We prove a Carleman estimate for the elliptic operator  $A = -\nabla \cdot (B\nabla)$  with an arbitrary observation region. This Carleman estimate is obtained through the introduction of a suitable mesh of the neighborhood of  $S$  and an associated approximation of  $c$  involving the Carleman large parameters.

*AMS 2010 subject classification:* 35B37, 35J15, 35J60.

**Keywords:** anisotropic elliptic operators; approximation; non-smooth coefficients; stratified media; Carleman estimate; observation location.

## Addendum

The supplementary hypothesis  $C_\tau(x) = a(x_n)C'_\tau(x')$  is missing in Assumption 1.1 to validate the used approach. For the more general case  $C_\tau(x) = a(x)C'_\tau(x')$  we must slightly modify in (2.2) and (2.4), the terms  $c^l$ ,  $f_{j,k}$ ,  $g_{j,k}$  and  $h_{j,k}$  without any consequence for the method and the result.

## 1 Introduction, notation and main results

Carleman estimates [10] have originally been introduced for uniqueness results for partial differential operators and later generalized (see e.g. [14, Chapter 8], [15, Chapter 28], [25]). They have been successfully used for inverse problems [9] and for the null controllability of linear parabolic equations [21] and the null controllability of classes of semilinear parabolic equations [3, 12, 13].

For a second-order elliptic operator, say  $A = -\Delta_x$ , acting in a bounded open set  $\Omega \subset \mathbb{R}^n$ , (local) Carleman estimates take the form

$$s^3 \lambda^4 \|\varphi^{\frac{3}{2}} e^{s\varphi} u\|_{L^2(\Omega)}^2 + s \lambda^2 \|\varphi^{\frac{1}{2}} e^{s\varphi} \nabla_x u\|_{L^2(\Omega)}^2 \leq C \|e^{s\varphi} A u\|_{L^2(\Omega)}^2, \quad u \in \mathcal{C}_c^\infty(\Omega), \quad s \geq s_0, \quad \lambda \geq \lambda_0, \quad (1.1)$$

for a properly chosen weight function  $\beta$  such that  $|\beta'| \neq 0$ ,  $\varphi(x) = e^{\lambda\beta(x)}$  and  $s_0, \lambda_0, C$  sufficiently large (see [13]). Difficulties arise if one attempts to derive Carleman estimates in the case of non-smooth coefficients in the principal part of the operator, by example for a regularity lower than Lipschitz. In fact, Carleman

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L. Thevenet was partially supported by l'Agence Nationale de la Recherche under grant ANR-07-JCJC-0139-01.

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estimates imply the unique continuation property which does not hold in general for a  $\mathcal{C}^{0,\alpha}$  Hölder regularity of the coefficients with  $0 < \alpha < 1$  [22, 23].

Here we are interested in coefficients that are non continuous across an interface  $S$ . When the observation takes place in the region where the diffusion coefficient  $c$  is the 'lowest', this question was solved in [11] for a parabolic operator  $P = \partial_t - \nabla_x \cdot (c(x)\nabla_x)$  (see [24] for additional improvements). In the one dimensional case, and without assumption on the localization of the observation, the question was solved for general piecewise  $C^1$  coefficients [5, 6] and for coefficients with bounded variations [16]. The work [7] generalizes [5, 6] to some stratified media with dimension  $n \geq 1$ . *Without* Carleman estimate, the controllability for a one dimensional parabolic operator was proved in [2] for  $c \in L^\infty$  but this approach does not authorize semilinear operators.

Recently, Carleman estimates for an arbitrary dimension without any condition on the localization of the observation were obtained in [4, 19], in the elliptic case, and in the parabolic case in [7, 20, 24], but the methods used in [4, 18, 17, 19, 20] require strong regularity for the coefficients and for the interface. Moreover, they fall short if the interface crosses the boundary whereas this configuration is typical in bounded stratified media, examples falling into the framework considered here and in [7]. In [7] the authors assumed that the diffusion coefficients have a 'stratified' structure. More precisely, they have considered operators of the form  $A = -\nabla \cdot (B(\cdot)\nabla)$  in which the diffusion matrix  $B(x)$  has the following block diagonal form

$$B(x) = B(x', x_n) = \begin{pmatrix} c_1(x_n)C_\tau(x') & 0 \\ 0 & c_2(x_n) \end{pmatrix}$$

where  $\Omega = \Omega' \times (-H, H)$ ,  $x = (x', x_n)$ ,  $C_\tau$  is a smooth hermitian matrix and the coefficients  $c_1, c_2$  have a possible jump at  $x_n = 0$ . The object of the present work is to obtain a Carleman estimate for more general diffusion coefficients without a stratified structure that separates variables. We shall consider a diffusion matrix of the form

$$B(x) = \begin{pmatrix} C_\tau(x) & 0 \\ 0 & c(x) \end{pmatrix}$$

with  $C_\tau$  and  $c$  having possible jump at  $x_n = 0$ .

Here, to understand the difficulties that we face, the reader can observe that attempting to prove the Carleman estimate by extending the proof as is done in the one dimensional case, leads in fact to tangential terms at the interface  $S$  that cannot be controlled. These terms existed also in [11] where  $B$  is a scalar function  $c$  which led the authors to add conditions on the jump of diffusion coefficient  $c$  at the interface. Not to mention our approach, the main contribution of our paper is to derive an estimate of these tangential terms allowing to conclude the proof of the Carleman estimate.

In [7], these tangential terms at the interface are controlled by using Fourier series in the tangential direction. By a suitable choice of the weight function, the low frequencies lead to a positive quadratic form. The treatment of the high frequencies needs more computations. It uses the ideas developed in [4, 19, 20] where the normal part of the elliptic operator can be inverted. In [7] this argument uses the assumption of the separation of the tangential and normal variables in the diffusion matrix  $B$ .

In the case we consider here, the diffusion coefficients depend on  $x = (x', x_n)$  and, contrary to [7], one cannot decompose the operator  $A$  as  $\partial_{x_n} c_2(x_n) \partial_{x_n} + A_\tau$  with  $A_\tau$  a tangential elliptic operator on  $\Omega'$ . Our method consists in the introduction of a suitable decomposition,  $(\Omega_{j,\delta})$ , of a neighborhood of the interface and, on each  $\Omega_{j,\delta}$ , an approximation of the diffusion coefficient  $c$  by a function depending only on the normal variable,  $x_n$ , for which the result of [7] can be used. As these approximations depend on the Carleman large parameters  $s$  and  $\lambda$  (see (1.1)), we shall need a refined estimate of the tangential derivative (more precisely, for the high frequencies, see Lemma 3.1).

The question of the derivation of Carleman estimates in the case where the diffusion coefficients are *totally* anisotropic in the neighborhood of a point where the interface  $S$  meets the boundary  $\partial\Omega$  is left open and its solution is not a mere technical point. Solve this question we will allow to solve the case of an interface  $S$  (place where the discontinuities of  $C_\tau$  and  $c$  can occur) transverse but non orthogonal to the

boundary  $\partial\Omega$ . The difficulties seem equivalent and come from the changes of coordinates (far from the boundary we could consider anisotropic coefficients by introduction of normal geodesic coordinates as in [19]). More generally, extensions to manifolds will follow. Note also that deriving Carleman estimates for the parabolic operator associated to the elliptic operator we consider here, is also an open question. In fact, if we follow the same idea as for the elliptic case we present here, and if we use singular weight functions as introduced in [13], we then have to consider approximations of order  $\frac{1}{\sqrt{t(T-t)s\lambda\varphi_S}}$  (connected to the Carleman parameters). These approximations blow up near  $t = 0$  and  $t = T$ .

For each pair  $(s, \lambda)$  of Carleman parameters, we introduce several meshes that seem to indicate a connection to numerical methods. We believe that this connection should be further investigated.

## 1.1 Setting and notation

Let  $\Omega$  be an open subset in  $\mathbb{R}^n$ , with  $\Omega = \Omega' \times (-H, H)^1$ , where  $\Omega'$  is a nonempty bounded open subset of  $\mathbb{R}^{n-1}$  with  $\mathcal{C}^2$  boundary<sup>2</sup>. We shall use the notation  $x = (x', x_n) \in \Omega' \times (-H, H)$ . We set  $S = \Omega' \times \{0\}$ , that will be understood as an interface where coefficients and functions may jump. For a function  $f = f(x)$  we define by  $[f]_S$  its jump at  $S$ , i.e.,  $[f]_S(x') = f(x)|_{x_n=0^+} - f(x)|_{x_n=0^-}$ . For a function  $u$  defined on both sides of  $S$ , we set  $u|_{S^\pm} = (u|_{\Omega^\pm})|_S$ , with  $\Omega^+ = \Omega' \times (0, H)$  and  $\Omega^- = \Omega' \times (-H, 0)$ .

Let  $B(x)$ ,  $x \in \Omega$ , be with values in  $M_n(\mathbb{R})$ , the space of square matrices with real coefficients of order  $n$ . We make the following assumption.

**Assumption 1.1.** *The diffusion matrix  $B(x', x_n)$  has the following block diagonal form*

$$B(x) = \begin{pmatrix} C_\tau(x) & 0 \\ 0 & c(x) \end{pmatrix}$$

where

1. the functions  $C_\tau, c$  are  $\mathcal{C}^1(\overline{\Omega^\pm})$  with a possible jump at  $x_n = 0$ ,
2. the two restrictions to the interface  $S$  of the function  $c : x' \rightarrow c(x', 0^\pm)$  are  $\mathcal{C}^2$ ,
3.  $C_\tau(x)$  is an hermitian matrix of order  $n - 1$ .

We further assume uniform ellipticity

$$\begin{aligned} 0 < c_{\min} \leq c(x) \leq c_{\max} < \infty, \quad x \in \Omega, \\ 0 < c_{\min} \text{Id}_{n-1} \leq C_\tau(x) \leq c_{\max} \text{Id}_{n-1}, \quad x \in \Omega. \end{aligned}$$

We consider the symmetric bilinear  $H_0^1$ -coercive form

$$a(u, v) = \int_{\Omega} (B(\cdot)\nabla u) \cdot \nabla v dx,$$

with domain  $D(a) = H_0^1(\Omega)$ . It defines a selfadjoint operator  $A = -\nabla \cdot (B(\cdot)\nabla)$  in  $L^2(\Omega)$  with compact resolvent and its domain is  $D(A) = \{u \in H_0^1(\Omega); \nabla \cdot (B(\cdot)\nabla u) \in L^2(\Omega)\}$ . We shall denote by  $\|\cdot\|_{L^2(\Omega)}$  the  $L^2$  norm over  $\Omega$ , by  $|\cdot|_{L^2(S)}$  the  $L^2$  norm over the interface  $S$  of codimension 1 and by  $|\cdot|_{\mathbb{R}^p}$  the euclidean norm in  $\mathbb{R}^p$ .

In this article, when the constant  $C$  is used, it refers to a constant that is independent of all the parameters. Its value may however change from one line to another. If we want to keep track of the value of a constant we shall use another letter or add a subscript.

<sup>1</sup>As a matter of fact, we only ask that  $\Omega$  is a cylinder in a neighborhood of the interface  $S$ . See the end of section 2.

<sup>2</sup>For some particular geometries we can suppose that  $\Omega'$  is piecewise smooth. Nevertheless the technics used for building our approximation in a neighborhood of the interface seems to require better than  $\mathcal{C}^1$ . We shall take  $\mathcal{C}^2$  to ease the readability.

## 1.2 Statements of the main results

We consider  $\omega$ , a nonempty open subset of  $\Omega$ . For a function  $\beta$  in  $\mathcal{C}^0(\overline{\Omega})$  we set

$$\varphi(x) = e^{\lambda\beta(x)}, \quad \lambda > 0,$$

to be used as a weight function. A proper choice of the function  $\beta$ , with respect to the operator  $A$ ,  $\omega$  and  $\Omega$  (see Assumption 2.4 and Assumption 4.1), yields the following Carleman estimate for the elliptic operator  $A$ . In particular,  $\beta$  will depend only on  $x_n$ .

**Theorem 1.2.** *There exist  $C > 0$ ,  $\lambda_0$  and  $s_0 > 0$  such that*

$$\begin{aligned} s\lambda^2 \|e^{s\varphi} \varphi^{\frac{1}{2}} \nabla u\|_{L^2(\Omega)}^2 + s^3 \lambda^4 \|e^{s\varphi} \varphi^{\frac{3}{2}} u\|_{L^2(\Omega)}^2 + s\lambda \left( |e^{s\varphi} \varphi^{\frac{1}{2}} \nabla_\tau u|_{L^2(S)}^2 + |e^{s\varphi} \varphi^{\frac{1}{2}} \partial_{x_n} u|_{L^2(S)}^2 \right) \\ + s^3 \lambda^3 |e^{s\varphi} \varphi^{\frac{3}{2}} u|_{L^2(S)}^2 \leq C \left( \|e^{s\varphi} Au\|_{L^2(\Omega)}^2 + s^3 \lambda^4 \|e^{s\varphi} \varphi^{\frac{3}{2}} u\|_{L^2(\omega)}^2 \right), \end{aligned}$$

for all  $u \in D(A)$ ,  $\lambda \geq \lambda_0$ , and  $s \geq s_0$ .

Here,  $\nabla_\tau$  is the tangential gradient, i.e. parallel to the interface  $S$ . Note that the condition  $u \in D(A)$  implies some constraints on the function  $u$  at the interface  $S$ , namely  $u \in H_0^1(\Omega)$  and  $B\nabla_x u \in H(\text{div}, \Omega) := \{v \in L^2(\Omega)^n; \text{div } v \in L^2(\Omega)\}$ . We shall first prove the result for piecewise  $C^2$  functions satisfying

$$u|_{S^-} = u|_{S^+}, \quad (c\partial_{x_n} u)|_{S^-} = (c\partial_{x_n} u)|_{S^+},$$

and then use their density in  $D(A)$  (see Appendix C).

## 1.3 Outline

Choosing  $0 < \delta < H$ , our starting point is the following local Carleman estimate in the open set  $\Omega_\delta := \Omega' \times (-\delta, \delta)$ , neighborhood of the interface  $S$ .

*There exist a weight function  $\beta$  and  $C, C' > 0$ ,  $\lambda_0 > 0$ ,  $s_0 > 0$  such that*

$$\begin{aligned} C \left( s\lambda^2 \|\varphi^{\frac{1}{2}} e^{s\varphi} \nabla u\|_{L^2(\Omega_\delta)}^2 + s^3 \lambda^4 \|\varphi^{\frac{3}{2}} e^{s\varphi} u\|_{L^2(\Omega_\delta)}^2 \right) + s\lambda \varphi|_S \left( \int_S [c^2 \beta' |e^{s\varphi} \partial_{x_n} u|^2]_S d\sigma \right. \\ \left. + \int_S |s\lambda \varphi e^{s\varphi} u|_S|^2 [c^2 \beta'^3]_S d\sigma \right) \leq C' \|e^{s\varphi} Au\|_{L^2(\Omega_\delta)}^2 + s\lambda \varphi|_S \int_S |e^{s\varphi} \nabla_\tau u|^2 \|\beta' c C_\tau\|_S d\sigma \quad (1.2) \end{aligned}$$

for all  $u \in D(A)$ ,  $\lambda \geq \lambda_0$ ,  $s \geq s_0$  and  $\text{supp } u \subset \overline{\Omega'} \times (-\delta, \delta)$ .

We have to understand  $[\beta' c C_\tau]_S$  as the matrix of jumps of each term of the matrix and  $\|\beta' c C_\tau\|_S$  is its norm that we can take in  $L^\infty(S)$ . As a matter of fact, (1.2) is obtained in two steps. Firstly, by adapting the derivations in [11] for instance and a suitable choice of the weight function  $\beta$ , the last term on the r.h.s. of (1.2) is  $\int_S \langle [\beta' c C_\tau]_S e^{s\varphi} \nabla_\tau u, e^{s\varphi} \nabla_\tau u \rangle d\sigma$  where  $\langle \cdot, \cdot \rangle$  is the scalar product in  $\mathbb{R}^{n-1}$ . This integral arises a problem since we cannot exclude to have a positive quantity. For the second step, we consider the worst possibility, namely (1.2). In other words, the main difficulty is to estimate the tangential derivative of  $u$  at the interface  $S$ .

In Section 2, we introduce a covering  $(\Omega'_j)_j$  of a neighborhood of  $\overline{\Omega'}$  related to the Carleman's parameters  $s, \lambda$  and coefficients  $(c^j)_j$ , approximations of the diffusion coefficient  $c$ , then we associate to each  $\Omega'_j$  the open set  $\Omega_{j,\delta} := \Omega'_j \times (-\delta, \delta)$  which gives a covering of  $\Omega_\delta$ . We build an adapted partition of unity  $(\chi_j)_j$  subordinated to  $(\Omega'_j)_j$  and we define, for each  $x_n \in (-\delta, \delta)$  and each  $j$ , the tangential part of  $A$ , i.e.  $A_\tau(x_n) = -\nabla_\tau \cdot C_\tau(\cdot, x_n) \nabla_\tau$  with  $D_j(A_\tau(x_n)) = \{u \in H_0^1(\Omega'_j); \nabla_\tau \cdot (C_\tau(\cdot, x_n) \nabla_\tau u) \in L^2(\Omega'_j)\}$ . So, for  $u \in D(A)$  that solves  $Au = f$  with  $f \in L^2(\Omega)$ ,  $u_j := \chi_j u$  solves

$$\begin{cases} A_\tau(x_n)u_j - c^j(x_n)\partial_{x_n}^2 u_j = f_j + g_j + h_j & \text{on } \Omega_{j,\delta}^\pm = \{x = (x', x_n); x' \in \Omega'_j, 0 < \pm x_n < \delta\}, \\ u_j = 0 & \text{on } \partial\Omega_{j,\delta}, \\ [u_j]_{S_j} = 0 \quad \text{and} \quad [c^j \partial_{x_n} u_j]_{S_j} = [(c^j - c)\partial_{x_n} u_j]_{S_j} =: \theta_j \end{cases} \quad (1.3)$$

with

$$S_j = \Omega_{j,\delta} \cap S, \quad f_j = \chi_j f, \quad g_j = (c - c^j)\partial_{x_n}^2 u_j \quad \text{and} \quad h_j = [A_\tau, \chi_j]u + (\partial_{x_n} c)(\partial_{x_n} u_j), \quad (1.4)$$

where  $[A_\tau, \chi_j]$  denotes the commutator of  $A_\tau$  and  $\chi_j$ .

Our approach shall show that it will be sufficient to estimate the tangential derivative of  $u_j$  defined below to prove Theorem 1.2. Here, we cannot directly apply the results of [7] for three main reasons:

1. the dependence on  $x_n$  of  $A_\tau$ ,
2. the presence of  $\theta_j$  and  $g_j$  involving the normal derivative of  $u$  on  $S_j$  and the second derivative of  $u$  on  $\Omega_j$ ,
3. the presence of  $h_j$  that in fact depends on both parameters  $s, \lambda$ .

To take into account the first constraint, we consider  $(\mu_{j,k}^2(x_n))_{k \geq 1}$ , the family of eigenvalues of  $(A_\tau(x_n), D_j(A_\tau(x_n)))$ , and denote by  $u_{j,k}, f_{j,k}, g_{j,k}, h_{j,k}$  and  $\theta_{j,k}$  the respective Fourier coefficients of  $u_j, f_j, g_j, h_j$  and  $\theta_j$  in an orthonormal basis associated to the previous eigenvalues.

To overcome the two other constraints, we prove in Section 3 a refined estimate of the Fourier coefficients of the tangential derivatives of  $u_j$  corresponding to the high frequencies (i.e. coefficients associated to the large eigenvalues in the decomposition on the eigenfunctions): there exist a constant  $C$  independent of  $s, \lambda, \mu_{j,k}$ , a constant  $\mu_0 := \mu_0(s, \lambda) > 0$  such that, for all  $\mu_{j,k}(0^+) \geq \mu_0$ , one has

$$s\lambda\varphi_s |\mu_{j,k}(0^+) e^{s\varphi_s} u_{j,k}|_{S_j}^2 \leq C \left( |e^{s\varphi} f_{j,k}|_{L^2(-\delta,\delta)}^2 + \varphi_s |e^{s\varphi} \varphi^{-1/2} g_{j,k}|_{L^2(-\delta,\delta)}^2 + \varphi_s^{-1} |e^{s\varphi} \varphi^{1/2} h_{j,k}|_{L^2(-\delta,\delta)}^2 + s^{3/2} \lambda \varphi_s e^{2s\varphi_s} |\theta_{j,k}|^2 \right) \quad (1.5)$$

for  $s, \lambda$  sufficiently large (as we allow  $C_\tau(x)$  to be discontinuous through  $S$ ,  $\mu_{j,k}(0^+)$  denotes  $\lim_{x_n \downarrow 0^+} \mu_{j,k}(x_n)$ ). Note the peculiar weights in the right hand side (in brief r.h.s.):  $\varphi_s^{1/2} \varphi^{-1/2}$  and  $\varphi_s^{-1/2} \varphi^{1/2}$ .

The coefficients corresponding to the low frequencies are treated as in [7]. We conclude, still as in [7], by verifying that there exists a weight function  $\beta$  such that the estimates obtained for low and high frequencies are simultaneously correct and that no frequency of  $(A_\tau(0^+), D_j(A_\tau(0^+)))$  has been forgotten (see Section 4). It remains to eliminate the three last terms of the r.h.s. of (1.5) (the additional terms with respect to [7]). These properties of the functions  $\chi_j$  and the definition of  $c^j$ , occurring in the two sides of (1.3), shall be used in this step ( $\|\cdot\|_{L^\infty}$  is denoted by  $\|\cdot\|_\infty$ ):

$$\|\nabla \chi_j\|_\infty \leq C \sqrt{s\lambda\varphi_s}, \quad \|\nabla \cdot (B\nabla \chi_j)\|_\infty \leq Cs\lambda\varphi_s, \quad \|c^j - c\|_\infty \leq C \frac{1}{\sqrt{s\lambda\varphi_s}}.$$

Collecting all the previous results will yield a proof of Theorem 1.2 and this will conclude Section 4.

In order to point out the main ideas of this work, we have put almost all technical results in the appendices.

## 2 Preparation of data

### 2.1 The partition of unity

**Theorem 2.1.** *For each pair  $(s, \lambda)$ ,  $s > 0, \lambda > 0$ , there exist a finite family  $(\Omega'_j)_{j \in J}$  of open sets such that  $\overline{\Omega'} \subset \cup_j \Omega'_j$  and a partition of unity  $(\chi_j)_{j \in J}$  subordinated to this open covering with*

$$|x' - y'|_{\mathbb{R}^{n-1}} \leq \frac{C}{\sqrt{s\lambda\varphi_{|s}}}, \forall x', y' \in \Omega'_j, \quad \|\nabla_\tau \chi_j\|_\infty \leq C \sqrt{s\lambda\varphi_{|s}} \quad \text{and} \quad \|\nabla_\tau \cdot (C_\tau \nabla_\tau \chi_j)\|_\infty \leq Cs\lambda\varphi_{|s}, \quad (2.1)$$

where the constant  $C$  is independant on  $s, \lambda$  and  $j$ . Moreover, at most  $N$  functions  $\chi_j$  are non equal to 0 in each point of  $\overline{\Omega'}$  with  $N$  only depending on  $\Omega'$ .

As a matter of fact, the proof is tricky when  $\Omega'$  is not a cube. In this case, we begin by defining open sets  $\Omega'_j := \Omega'_j(s, \lambda)$  such that  $\overline{\Omega'_j} \subset \Omega'$ . They are cubes with edges of length  $h = h(s, \lambda)$ . Next, we define additional open sets  $\Omega'_j$  that intersect  $\partial\Omega'$ . These are not cubes. The complete proof of Theorem 2.1 is given in Appendix A.

We recover  $\overline{\Omega_\delta}$  by the family of cylindrical subdomains

$$\Omega_{j,\delta}(s, \lambda) = \{(x', x_n) \in \mathbb{R}^n; x' \in \Omega'_j(s, \lambda), -\delta < x_n < \delta\}.$$

In the sequel, we will denote  $\Omega_{j,\delta} := \Omega_{j,\delta}(s, \lambda) = \Omega'_j \times (-\delta, \delta)$  and we recall that  $\Omega_{j,\delta}^\pm = \{(x', x_n); x' \in \Omega'_j, 0 < \pm x_n < \delta\}$  (as already introduced in (1.3)).

### 2.2 Partition and transverse operators

On each subdomain  $\Omega_{j,\delta}$  we define the following approximation of the diffusion coefficient  $c(x', x_n)$ :

$$c^j(x_n) = \begin{cases} c_+^j(x_n) = \frac{1}{|\Omega'_j|} \int_{\Omega'_j} c(x', x_n) dx', & \forall x_n \in (0, \delta), \\ c_-^j(x_n) = \frac{1}{|\Omega'_j|} \int_{\Omega'_j} c(x', x_n) dx', & \forall x_n \in (-\delta, 0). \end{cases} \quad (2.2)$$

So, for each  $x_n \in (-\delta, 0) \cup (0, \delta)$ , we have given sense to (1.3) where the operators  $A_\tau(x_n)$  act in a section of  $\Omega_{j,\delta}^\pm$  parallel to the interface  $S_j$ . We also define  $c^j(0^\pm) := \lim_{\pm x_n \downarrow 0} c^j(x_n)$  since  $c \in \mathcal{C}^1(\overline{\Omega^\pm})$  that we use in Section 3. If  $\Omega'$  is a cube, we can go straight to Lemma 2.2 taking into account the construction of the partition of unity (see Step 2 in Appendix A). Otherwise, an additional work needs to be carried out with the open cylinders  $\Omega_{j,\delta}(s, \lambda)$  intersecting  $\partial\Omega$ , which needs modifications in a neighborhood of  $\partial\Omega'$ . The idea is to extend the coefficients  $c(x', x_n)$ , outside  $\Omega$  and independently of  $(s, \lambda)$ , in such a way that we control the behavior of the extended solutions  $u_j$  that are associated to these cylinders. The reader may refer to Appendix A for more explanations.

Now we mention the following result which will be useful in the next section.

**Lemma 2.2.** *With Assumption 1.1, we have*

$$\frac{c_{\min}}{c_{\max}} \leq \frac{\mu_{j,k}^2(x_n)}{\mu_{j,k}^2(0^+)} \leq \frac{c_{\max}}{c_{\min}}. \quad (2.3)$$

**Proof.** We shall easily deduce these inequalities from the variational presentation of the Min-Max Principle since all the symmetric bilinear  $H_0^1$ -coercive forms  $a_{\tau, x_n, \Omega'_j}$  of the operators  $A_\tau(x_n)$  have same domain up to a translation of variables, i.e.  $H_0^1(\Omega'_j)$ . If  $V_k$  denotes the generic  $k$ -dimensional linear space of  $L^2(\Omega'_j)$  and if

$V_k^\perp$  is its orthogonal space in  $L^2(\Omega'_j)$  for the scalar product  $(u, v) = \int uv dx'$  (specific notation to this Lemma, as well as  $\|u\|^2 = (u, u)$ ), we know that

$$\mu_{j,k}^2(x_n) = \max_{V_{k-1} \subset L^2(\Omega'_j)} \left( \min_{u \in V_k^\perp \cap H_0^1(\Omega'_j), \|u\|=1} a_{\tau, x_n, \Omega'_j}(u, u) \right),$$

which implies, by Assumption 1.1,

$$c_{\min} \max_{V_{k-1} \subset L^2(\Omega'_j)} \left( \min_{u \in V_k^\perp \cap H_0^1(\Omega'_j), \|u\|=1} \|\nabla u\|^2 \right) \leq \mu_{j,k}^2(x_n) \leq c_{\max} \max_{V_{k-1} \subset L^2(\Omega'_j)} \left( \min_{u \in V_k^\perp \cap H_0^1(\Omega'_j), \|u\|=1} \|\nabla u\|^2 \right),$$

from which one may conclude. ■

**Remark 2.3.** When  $\Omega'_j \cap \partial\Omega' \neq \emptyset$ , we only have to modify, without repercussions, the values of  $c_{\min}$  and  $c_{\max}$  that appear in (2.3). We shall find again this situation throughout the proofs of this work.

In order to evaluate the awkward term  $\int_S |e^{s\varphi} \nabla_\tau u|^2 \|[\beta' c C_\tau]\|_S d\sigma$  that occurs in (1.2), we have to estimate  $\nabla_\tau u$  on the interface  $S$ . In fact, we need this estimate for  $u_j := \chi_j u$ . We write  $u_j(x) = \sum_k u_{j,k}(x_n) \varphi_k(x', x_n)$  where the family  $(\varphi_k(\cdot, x_n))_{k \geq 1}$  is an orthonormal basis associated to the eigenvalues of  $A_\tau(x_n)$ . So, the first line of (1.3) becomes

$$\mu_{j,k}^2(x_n) u_{j,k} - c^j(x_n) \partial_{x_n}^2 u_{j,k} = f_{j,k} + g_{j,k} + h_{j,k}, \quad 0 < |x_n| < \delta. \quad (2.4)$$

For  $x_n = 0$ , the same relation is valid if one distinguishes the cases  $x_n = 0^+$  and  $x_n = 0^-$  for the coefficients  $\mu_{j,k}^2$  and  $c^j$ . Finally, reasoning as in [7], section 2, we find

$$(c_{\max})^{-1} \sum_{k=1}^{\infty} \mu_{j,k}^2(x_n) |u_{j,k}(x_n)|^2 \leq \|\nabla_\tau u_j(\cdot, x_n)\|_{L^2(\Omega'_j)}^2 \leq (c_{\min})^{-1} \sum_{k=1}^{\infty} \mu_{j,k}^2(x_n) |u_{j,k}(x_n)|^2. \quad (2.5)$$

### 2.3 The weight function $\beta$

The open set  $\omega$  having been fixed in section 1.2, we choose a weight function  $\beta$  that satisfies the following properties.

**Assumption 2.4.** *The function  $\beta \in \mathcal{C}^0(\Omega)$ , and  $\beta_{|\Omega^\pm} \in \mathcal{C}^2(\overline{\Omega^\pm})$  and*

$$\begin{aligned} \beta &\geq C > 0, \quad |\nabla_x \beta| \geq C > 0 \text{ in } \Omega \setminus \omega, \\ \beta &= \text{Cst on } \Omega' \times \{-H\} \quad \text{and} \quad \beta = \text{Cst on } \Omega' \times \{H\}, \\ \nabla_{x'} \beta &= 0 \text{ on } \partial\Omega' \times (-H, H), \\ \partial_{x_n} \beta &> 0 \text{ on } \Omega' \times \{-H\}, \quad \text{and} \quad \partial_{x_n} \beta < 0 \text{ on } \Omega' \times \{H\}. \end{aligned}$$

*There exists a neighborhood  $V$  of  $S$  in  $\Omega$  of the form  $V = \Omega' \times (-\delta, \delta)$  in which  $\beta$  solely depends on  $x_n$  and is a piecewise affine function of  $x_n$ .*

We draw the reader's attention on two points: firstly, the trace  $\beta|_S$  is constant on the interface  $S$  and, secondly, we can assume that  $\omega \cap (\Omega' \times (-\delta, \delta)) = \emptyset$ . Such a weight function  $\beta$  can be obtained by first designing a function that satisfies the proper properties at the boundaries and at the interface and then construct  $\beta$  by means of Morse functions following the method introduced in [13].

In the remainder of this paper we assume that  $\partial_{x_n} \beta = \beta' > 0$  on  $S^+$  and  $S^-$ , which means that the observation region  $\omega$  is chosen in  $\Omega' \times (0, H)$ , i.e., where  $x_n \geq 0$ . This is done without any lose of generality as we can change  $x_n$  into  $-x_n$  to treat the case of an observation  $\omega \subset \Omega' \times (-H, 0)$ .

Note that Assumption 2.4 will be completed below by Assumption 4.1 that will make the value of the jump of  $\partial_{x_n} \beta$  precise at the interface  $S$ .

### 3 A refined estimation for the high frequencies of the tangential derivative

This section is a first step to achieve inequality (1.5). Taking into account (2.4) we fix  $j \in J, k \in \mathbb{N}^*$ , for the moment, and consider  $w$  solution of

$$\begin{cases} \mu_{j,k}^2(x_n)w - c^j(x_n)\partial_{x_n}^2 w = F & \text{on } (-\delta, 0) \cup (0, \delta), \\ w(\pm\delta) = 0 \\ [w]_S = 0 \quad \text{and} \quad [c^j \partial_{x_n} w]_S = \theta, \end{cases}$$

with, here,  $S = \{0\}$ ,  $F \in L^2(-\delta, \delta)$ ,  $\theta \in \mathbb{R}$ . One has

**Lemma 3.1.** *Let  $F$  belong to  $L^2(-\delta, \delta)$ . There exist a constant  $C$  independent of  $s, \lambda, j, k$ , a constant  $\mu_0 := \mu_0(s, \lambda) > 0$  such that for all  $\mu_{j,k}(0^+) \geq \mu_0$ , the following estimates are satisfied for  $s, \lambda$  sufficiently large and  $w$  solution of (3)*

$$s\lambda\varphi_{|s}\mu_{j,k}(0^+)^2|e^{s\varphi}w|_S^2 \leq C\left(|e^{s\varphi}F|_{L^2(-\delta,\delta)}^2 + s^{3/2}\lambda\varphi_{|s}e^{2s\varphi_{|s}}|\theta|^2\right), \quad (3.1)$$

$$s\lambda\varphi_{|s}\mu_{j,k}(0^+)^2|e^{s\varphi}w|_S^2 \leq C\left(\varphi_{|s}|e^{s\varphi}\varphi^{-1/2}F|_{L^2(-\delta,\delta)}^2 + s^{3/2}\lambda\varphi_{|s}e^{2s\varphi_{|s}}|\theta|^2\right), \quad (3.2)$$

$$s\lambda\varphi_{|s}\mu_{j,k}(0^+)^2|e^{s\varphi}w|_S^2 \leq C\left(\varphi_{|s}^{-1}|e^{s\varphi}\varphi^{1/2}F|_{L^2(-\delta,\delta)}^2 + s^{3/2}\lambda\varphi_{|s}e^{2s\varphi_{|s}}|\theta|^2\right). \quad (3.3)$$

**Remark 3.2.** Even if Lemma 3.1 seems similar to Proposition 3.5 of [7] (for elliptic operators), there are two important differences: the weights for the sources  $F$  and the presence of  $\theta$  which is zero in that proposition.

The difference among these three inequalities is the weight associated with the source term  $F$ . It should be noted that there is no comparison relation between them. The source terms resulting from the approximation of the coefficient  $c$  yields three terms ( $f, g$  and  $h$  in (1.3)-(1.4)). The first one is just the localization of the initial source term, the second one is the difference between the elliptic operator and its approximation and the third one comes from the action of the cut-off function  $\chi_j$  on the elliptic operator. As we shall see later, they should be treated differently to be absorbed by the source  $\|e^{s\varphi}Au\|_{L^2(\Omega_s)}^2$  and the l.h.s. of the Carleman estimate (1.2).

*Proof.* We begin by setting

$$\sigma^2 := \inf_{j \in J, k \geq 1} \inf_{x_n \in (-\delta, \delta)} \frac{\mu_{j,k}^2(x_n)}{c^j(x_n)\mu_{j,k}^2(0^+)} \quad \text{and} \quad \mu_0 := \mu_0(s, \lambda) = \frac{2s\lambda\varphi_{|s}\beta'_{|s^-} + \lambda\beta'_{|s^-}}{\sigma}, \quad (3.4)$$

and next we introduce

$$W(x_n) = \frac{1}{2}s\lambda\varphi_{|s}e^{2s\varphi_{|s}}|w(x_n)|^2.$$

On the one hand it follows from (2.3) that we have  $\sigma > 0$ , on the other hand we observe that  $W \geq 0$  and it verifies

$$\begin{cases} \partial_{x_n}^2 W - \sigma^2\mu_{j,k}(0^+)^2 W = \ell - \left(\sigma^2\mu_{j,k}^2(0^+) - \frac{\mu_{j,k}^2(x_n)}{c^j(x_n)}(2-\gamma)\right)W(x_n) =: -d, & 0 < |x_n| < \delta, \\ W(-\delta) = W(\delta) = 0, \quad W(0^-) = W(0^+), \quad c_+^j W'(0^+) = c_-^j W'(0^-) + \theta w_{|s} s\lambda\varphi_{|s} e^{2s\varphi_{|s}}, \end{cases}$$

with

$$\ell = \frac{1}{c^j} \left( -s\lambda\varphi_{|s}e^{2s\varphi_{|s}}Fw + s\lambda\varphi_{|s}e^{2s\varphi_{|s}}c^j(\partial_{x_n}w)^2 + \gamma\mu_{j,k}^2(x_n)W \right), \quad (3.5)$$



where the real number  $\gamma$  will be made precise later and where we omitted the subscript  $j$  at places since the function  $\beta$ , and therefore  $\varphi$ , depend only on  $x_n$  if  $-\delta < x_n < \delta$ . Applying Lemma B.2, one gets, with  $c_{\pm}^j := c^j(0^{\pm})$  to lighten notation,

$$s\lambda\varphi_{\text{ls}} |\mu_{j,k}(0^+) e^{s\varphi_{\text{ls}}} w(0)|^2 = \frac{2\mu_{j,k}(0^+)}{(c_+^j + c_-^j)} \int_0^{\delta} \frac{\sinh(\sigma\mu_{j,k}(0^+)(\delta - x_n))}{\sigma \cosh(\sigma\mu_{j,k}(0^+)\delta)} (c_+^j d(x_n) + c_-^j d(-x_n)) dx_n - \frac{2\mu_{j,k}(0^+) \tanh(\sigma\mu_{j,k}(0^+)\delta) \theta w_{\text{ls}}}{\sigma(c_+^j + c_-^j)} s\lambda\varphi_{\text{ls}} e^{2s\varphi_{\text{ls}}} \quad (3.6)$$

which is exactly the left-hand side of the estimates of Lemma 3.1. Setting  $r(x_n) := \sigma^2 \mu_{j,k}^2(0^+) - \frac{\mu_{j,k}^2(x_n)}{c^j(x_n)} (2 - \gamma)$ , we have  $d(x_n) = r(x_n)W(x_n) - \ell(x_n)$  and we note that the definition of  $\sigma$  implies that  $r(x_n) \leq 0$  if  $\gamma \leq 1$ . As a consequence the following non positive contribution can be eliminated if we choose  $\gamma \leq 1$

$$\int_0^{\delta} \frac{\sinh(\sigma\mu_{j,k}(0^+)(\delta - x_n))}{\sigma \cosh(\sigma\mu_{j,k}(0^+)\delta)} (c_+^j r(x_n)W(x_n) + c_-^j r(-x_n)W(-x_n)) dx_n \leq 0. \quad (3.7)$$

Now, we consider the contribution coming from  $-\ell$  and, similarly to the previous observation, the second term of (3.5) yields a non positive contribution:

$$- \int_0^{\delta} \frac{\sinh(\sigma\mu_{j,k}(0^+)(\delta - x_n))}{\cosh(\sigma\mu_{j,k}(0^+)\delta)} s\lambda\varphi_{\text{ls}} e^{2s\varphi_{\text{ls}}} (c_+^j (\partial_{x_n} w)^2(x_n) + c_-^j (\partial_{x_n} w)^2(-x_n)) dx_n \leq 0. \quad (3.8)$$

The estimation of the other terms of (3.6) requires more computations. Temporarily we omit the coefficient  $\frac{2}{\sigma(c_+^j + c_-^j)}$  in front of the first term in the r.h.s. of (3.6). Let us begin by

$$I_{\pm} := -\mu_{j,k}(0^+) \int_0^{\delta} \frac{\sinh(\sigma\mu_{j,k}(0^+)(\delta - x_n))}{\cosh(\sigma\mu_{j,k}(0^+)\delta)} \frac{c_{\pm}^j}{c^j} (-s\lambda\varphi_{\text{ls}} e^{2s\varphi_{\text{ls}}} F(\pm x_n)w(\pm x_n) + \gamma\mu_{j,k}^2(\pm x_n)W(\pm x_n)) dx_n.$$

On the one hand, applying the Young inequality we have, for any  $\alpha > 0$ ,

$$\mu_{j,k}(0^+) |s\lambda\varphi_{\text{ls}} e^{2s\varphi_{\text{ls}}} F(\pm x_n)w(\pm x_n)| \leq \frac{1}{2\gamma\alpha} e^{2s\varphi_{\text{ls}}} |F(\pm x_n)|^2 + \frac{\gamma}{2} \alpha s^2 \lambda^2 \varphi_{\text{ls}}^2 e^{2s\varphi_{\text{ls}}} \mu_{j,k}^2(0^+) |w(\pm x_n)|^2,$$

and, on the other hand, observing that

$$\frac{\sinh(\sigma\mu_{j,k}(0^+)(\delta - x_n))}{\cosh(\sigma\mu_{j,k}(0^+)\delta)} = \frac{e^{\sigma\mu_{j,k}(0^+)(\delta - x_n)} - e^{-\sigma\mu_{j,k}(0^+)(\delta - x_n)}}{e^{\sigma\mu_{j,k}(0^+)\delta} + e^{-\sigma\mu_{j,k}(0^+)\delta}} \leq e^{-\sigma\mu_{j,k}(0^+)x_n},$$

we obtain

$$\begin{aligned} I_{\pm} &= \mu_{j,k}(0^+) \int_0^{\delta} \frac{\sinh(\sigma\mu_{j,k}(0^+)(\delta - x_n))}{\cosh(\sigma\mu_{j,k}(0^+)\delta)} \frac{c_{\pm}^j}{c^j} (s\lambda\varphi_{\text{ls}} e^{2s\varphi_{\text{ls}}} F(\pm x_n)w(\pm x_n) - \gamma\mu_{j,k}^2 W)(\pm x_n) dx_n \\ &\leq \int_0^{\delta} \frac{e^{-\sigma\mu_{j,k}(0^+)x_n + 2s\varphi_{\text{ls}}}}{2\gamma\alpha} \frac{c_{\pm}^j}{c^j} |F(\pm x_n)|^2 dx_n \\ &\quad + \int_0^{\delta} \frac{\sinh(\sigma\mu_{j,k}(0^+)(\delta - x_n))}{\cosh(\sigma\mu_{j,k}(0^+)\delta)} \frac{c_{\pm}^j}{c^j} \frac{\gamma}{2} \mu_{j,k}^2(0^+) s\lambda\varphi_{\text{ls}} |e^{s\varphi_{\text{ls}}} w(\pm x_n)|^2 \left( \alpha s\lambda\varphi_{\text{ls}} - \frac{\mu_{j,k}^2(\pm x_n)}{\mu_{j,k}(0^+)} \right) dx_n. \end{aligned}$$

Since we assume that  $\mu_{j,k}(0^+) \geq \mu_0$ , the definition of  $\mu_0$  implies that  $2\frac{\beta'_{\text{ls}}}{\sigma} \leq \frac{\mu_{j,k}(0^+)}{s\lambda\varphi_{\text{ls}}}$ . Taking  $\alpha \leq 2\frac{\beta'_{\text{ls}}}{\sigma} \frac{c_{\text{min}}}{c_{\text{max}}}$ ,

we have  $\alpha s\lambda\varphi_{\text{ls}} \leq \mu_{j,k}(0^+) \frac{c_{\text{min}}}{c_{\text{max}}}$  that we write  $\alpha s\lambda\varphi_{\text{ls}} \leq \frac{\mu_{j,k}^2(0^+)}{\mu_{j,k}^2(x_n)} \frac{\mu_{j,k}^2(x_n)}{\mu_{j,k}(0^+)} \frac{c_{\text{min}}}{c_{\text{max}}}$ . We deduce from (2.3) that  $(\alpha s\lambda\varphi_{\text{ls}} - \frac{\mu_{j,k}^2(\pm x_n)}{\mu_{j,k}(0^+)}) \leq 0$ . That allows us to omit the corresponding term and gives  $I_{\pm} \leq \int_0^{\delta} \frac{e^{-\sigma\mu_{j,k}(0^+)x_n + 2s\varphi_{\text{ls}}}}{2\gamma\alpha} \frac{c_{\pm}^j}{c^j} |F(\pm x_n)|^2 dx_n$ . This term will be estimated by three different and non-comparable ways that will lead to the three estimates of Lemma 3.1.

**Case 1.** We follow exactly the way described in [7] (point 2(a) of section 4) with the necessary adaptations since the time is not present here. We have to evaluate  $\int_0^\delta \frac{e^{-\sigma\mu_{j,k}(0^+)x_n+2s\varphi|_S}}{2\gamma\alpha} \frac{c_\pm^j}{c^j(x_n)} |F(\pm x_n)|^2 dx_n$  but it will be sufficient to look at  $\int_0^\delta e^{-\sigma\mu_{j,k}(0^+)x_n+2s\varphi|_S} |F(\pm x_n)|^2 dx_n$  (we will put back  $2\gamma\alpha$  at the end). We ask that

$$-\sigma\mu_{j,k}(0^+)x_n + 2s\varphi|_S \leq 2s\varphi(\pm x_n), 0 < x_n < \delta.$$

The function  $\varphi$  being increasing we have already  $-\sigma\mu_{j,k}(0^+)x_n + 2s\varphi|_S \leq 2s\varphi(x_n)$ . We still have to prove

$$-\sigma\mu_{j,k}(0^+)x_n + 2s\varphi|_S \leq 2s\varphi(-x_n) \text{ if } 0 < x_n < \delta. \quad (3.9)$$

We chose an affine piecewise function  $\beta$  which gives  $\varphi(-x_n) - \varphi|_S \geq -x_n\lambda(\beta'_-\varphi)|_S$  where  $\beta_-$  corresponds to the negative values of  $x_n$ . The inequality (3.9) will be true if  $\sigma\mu_{j,k}(0^+) \geq 2s\lambda(\beta'_-\varphi)|_S$ . It is the case since  $\mu_{j,k}(0^+) \geq \mu_0$ . With (3.10) below this leads to (3.1), the first estimate of Lemma 3.1.

**Case 2.** Since we suppose  $\mu_{j,k}(0^+) \geq \mu_0$  with  $\mu_0$  defined in (3.4), estimate (B.3) of Lemma B.1 leads to

$$I_\pm \leq \varphi|_S \frac{1}{2\gamma\alpha_0} \int_0^\delta e^{2s\varphi(\pm x_n)} \frac{c_\pm^j}{c^j(x_n)} |\varphi^{-1/2}(\pm x_n) F(\pm x_n)|^2 dx_n.$$

With (3.10) below this will give (3.2), the second estimate of Lemma 3.1.

**Case 3.** With the assumption made on  $\mu_{j,k}(0^+)$ , estimate (B.2) of Lemma B.1 leads to

$$I_\pm \leq \varphi|_S^{-1} \frac{1}{2\gamma\alpha_0} \int_0^\delta e^{2s\varphi(\pm x_n)} \frac{c_\pm^j}{c^j(x_n)} |\varphi^{1/2}(\pm x_n) F(\pm x_n)|^2 dx_n.$$

With (3.10) below this will give (3.3), the third estimate of Lemma 3.1.

Finally the last term in (3.6) can be estimated as follows

$$\frac{2\mu_{j,k}(0^+) \tanh(\sigma\mu_{j,k}(0^+)\delta) \theta w|_S}{\sigma(c_+^j + c_-^j)} s\lambda\varphi|_S e^{2s\varphi|_S} \leq C \left( s^{3/2} \lambda\varphi|_S e^{2s\varphi|_S} |\theta|^2 + s^{1/2} \lambda\varphi|_S |\mu_{j,k}(0^+) e^{s\varphi|_S} w|_S|^2 \right), \quad (3.10)$$

which permits to conclude for  $s$  chosen large enough if we collect these results with (3.7) and (3.8).  $\blacksquare$

## 4 Proof of the Theorem 1.2

We start off from inequality (1.2) where sign of the interface term (second parenthesis) in the l.h.s. must be determined. We introduce the following quadratic form

$$\mathcal{B}(u) = s\lambda\varphi|_S e^{2s\varphi} \left( [c^2\beta'|\partial_{x_n}u|^2]_S + |s\lambda\varphi u|_S|^2 [c^2\beta'^3]_S \right)$$

(recall that the function  $e^{2s\varphi}$  is continuous across  $S$ ) and

$$c_\pm(x') = c(x', 0^\pm), \quad L = \frac{\beta'_{|S^+}}{\beta'_{|S^-}}, \quad K_c(x') = \frac{c_-(x')}{c_+(x')}, \quad K_\tau = \|C_\tau(x', 0^-)(C_\tau(x', 0^+))^{-1}\|_{L^\infty(S)}$$

$$\underline{K}_c = \inf_{x' \in \Omega'} K_c(x'), \quad \overline{K}_c = \sup_{x' \in \Omega'} K_c(x').$$

Finally, we set

$$D = D(L) = \beta'_{|S^-} \sup_{x' \in \Omega'} \left( c_+(x') \|C_\tau(x', 0^+)\| \right) (L + \overline{K}_c K_\tau)$$

and we make the following assumption on the weight function in addition to Assumption 2.4.

**Assumption 4.1.** The weight function  $\beta$  is chosen such that  $L \geq \underline{L} = \max\{\bar{K}_c, 2\}$  and

$$\frac{(\underline{K}_c + L^3(L - \underline{L}))}{(L + \bar{K}_c K_\tau)(L - 1)} \geq \frac{36N \sup_{x' \in \Omega'} (c_+(x') \|C_\tau(x', 0^+)\|)}{\sigma^2 \inf_{x' \in \Omega'} c_+^2(x')}, \quad 2\delta\beta'_{|S^-} \leq \beta(0).$$

The integer  $N$  is that of Theorem 2.1. The functions  $c, C_\tau$  being fixed, it is the same for  $\sigma^2, \underline{K}_c, \bar{K}_c, K_\tau$  and  $\underline{L}$ , which shows this inequality can be achieved by first choosing the value of  $\beta'_{|S^-} > 0$  and then picking a sufficiently large value for  $L$ . The assumption  $2\delta\beta'_{|S^-} \leq \beta(0)$  can easily be fulfilled since  $\beta$  is defined up to a constant.

**Lemma 4.2.** We have

$$\mathcal{B}(u) = s\lambda\varphi_{|S} e^{2s\varphi_{|S}} \left( B_1 |\gamma(u)|^2 + B_2 |s\lambda(\varphi u)_{|S}|^2 \right),$$

with  $\gamma(u) = (c\partial_{x_n} u)_{|S} + c_+\beta'_{|S^-} \frac{L^2 - K_c}{L-1} (s\lambda\varphi u)_{|S}$  and where

$$B_1 = \beta'_{|S^-} (L - 1), \quad B_2(x') = c_+^2(x') (\beta'_{|S^-})^3 \left( 2(L^3 - K_c^2(x')) - \frac{(L^2 - K_c(x'))^2}{L - 1} \right).$$

If  $\beta$  satisfies Assumption 4.1 we have  $B_1 > 0$  and  $B_2(x') \geq \mathbf{B}$ , with  $\mathbf{B}$

$$\mathbf{B} = \mathbf{B}(L) = \left( \inf_{x' \in \Omega'} c_+^2(x') \right) (\beta'_{|S^-})^3 \frac{K_c^2 + L^3(L - \underline{L})}{L - 1}.$$

The idea of the proof of Lemma 4.2 is similar to both proofs of Appendix A.2 and Lemma 4.4 that are in [7].

It now remains to estimate the tangential derivative of  $u$  at the interface  $S$  (the third integral of the second parenthesis in the l.h.s. of (1.2)). The wording of Theorem 2.1 points out an integer  $N$  and its existence implies, see (4.5),

$$\mathcal{I} := s\lambda\varphi_{|S} \int_S \| [c\beta' C_\tau]_S \| |e^{s\varphi} \nabla_\tau u|^2 d\sigma \leq N s\lambda\varphi_{|S} \sum_{j \in J} \int_{S_j} \| [c\beta' C_\tau]_S \| |e^{s\varphi} \nabla_\tau u_j|^2 d\sigma.$$

Using the notation  $C_{\tau\pm}(x') := C_\tau(x', 0^\pm)$  we have  $[c\beta' C_\tau]_S = c_+\beta'_{|S^-} (L \text{Id}_{\mathbb{R}^{n-1}} - K_c C_{\tau-} (C_{\tau+})^{-1}) C_{\tau+}$  and we obtain

$$\| [c\beta' C_\tau]_S \| \leq \beta'_{|S^-} \max_{x' \in \Omega'} (c_+ \|C_{\tau+}\|) \left( L + \bar{K}_c \|C_{\tau-} (C_{\tau+})^{-1}\| \right) \leq D,$$

and, from (2.5), it suffices to estimate  $D s\lambda\varphi_{|S} \sum_{k \geq 1} \mu_{j,k} (0^+)^2 |e^{s\varphi} u_{j,k}|^2$  on  $S_j$  uniformly with respect to  $j$ . For this estimation we shall distinguish small and large values of  $k$  as in [7] and [19].

**Proposition 4.3.** There exists  $C > 0$  such that, for all  $j \in J, k \in \mathbb{N}^*$ , we have

$$D s\lambda\varphi_{|S} |\mu_{j,k}(0^+) e^{s\varphi} u_{j,k}|_{S_j}|^2 \leq \frac{\mathbf{B}}{4N} (s\lambda\varphi_{|S})^3 e^{2s\varphi_{|S}} |u_{j,k}|_{S_j}|^2 + C \left( |e^{s\varphi} f_{j,k}|_{L^2(-\delta,\delta)}^2 + \varphi_{|S} |e^{s\varphi} \varphi^{-1/2} g_{j,k}|_{L^2(-\delta,\delta)}^2 + \varphi_{|S}^{-1} |e^{s\varphi} \varphi^{1/2} h_{j,k}|_{L^2(-\delta,\delta)}^2 + s^{3/2} \lambda\varphi_{|S} e^{2s\varphi_{|S}} |\theta_{j,k}|^2 \right) \quad (4.1)$$

for  $s, \lambda$  and  $L$  large enough.

**Proof.** We shall keep track of the dependency of the constants on  $j$  and  $k$ . For **low frequencies**, a direct computation leads to

$$D s\lambda\varphi_{|S} |\mu_{j,k}(0^+) e^{s\varphi_{|S}} u_{j,k}|_{S_j}|^2 \leq \frac{\mathbf{B}}{4N} (s\lambda\varphi_{|S})^3 e^{2s\varphi_{|S}} |u_{j,k}|_{S_j}|^2, \quad (4.2)$$

if  $\mu_{j,k}(0^+) \leq \frac{1}{2\sqrt{N}} \sqrt{\frac{B}{D}} s\lambda\varphi_{|s}$ .

Now, we consider **high frequencies**. We apply Lemma 3.1 to the solution of (2.4) with  $F = f_{j,k} + g_{j,k} + h_{j,k}$  and  $\theta = \theta_{j,k}$  for  $\mu_{j,k}(0^+) \geq \mu_0$  (definition of  $\mu_0$  given by (3.4)): since the differential equation is linear there exists a constant  $C > 0$  such that,  $\forall \mu_{j,k}(0^+) \geq \mu_0$ , we have

$$Ds\lambda\varphi_{|s} |\mu_{j,k}(0^+) e^{s\varphi_{|s}} u_{j,k}|_{S_j}|^2 \leq C \left( |e^{s\varphi} f_{j,k}|_{L^2(-\delta,\delta)}^2 + \varphi_{|s} |e^{s\varphi} \varphi^{-1/2} g_{j,k}|_{L^2(-\delta,\delta)}^2 + \varphi_{|s}^{-1} |e^{s\varphi} \varphi^{1/2} h_{j,k}|_{L^2(-\delta,\delta)}^2 + s^{3/2} \lambda \varphi_{|s} e^{2s\varphi_{|s}} |\theta_{j,k}|^2 \right). \quad (4.3)$$

Collecting (4.2) and (4.3), we obtain (4.1). Yet it remains to check that our separation between low and high frequencies covers all the spectrum of  $A_\tau(0^+)$ . This will be true if there exists a weight function  $\beta$  that satisfies, in addition to Assumption 2.4,

$$\frac{1}{2\sqrt{N}} \sqrt{\frac{B}{D}} s\lambda\varphi_{|s} \geq \frac{2s\lambda\varphi_{|s}\beta'_{|s^-} + \lambda\beta'_{|s^-}}{\sigma} = \mu_0.$$

For  $s\varphi_{|s} \geq 1/2$ , it suffices to have  $\frac{B}{D} \geq \frac{36N(\beta'_{|s^-})^2}{\sigma^2}$ . This inequality is equivalent to

$$\frac{\inf_{x' \in \Omega'} (c_+^2(x')) (\beta'_{|s^-})^3 \frac{K_c^2 + L^3(L-L)}{L-1}}{\beta'_{|s^-} \sup_{x' \in \Omega'} (c_+(x') \|C_\tau(x', 0^+)\|) (L + \bar{K}_c K_\tau)} \geq \frac{36N(\beta'_{|s^-})^2}{\sigma^2}.$$

As  $\beta$  fulfills Assumption 4.1 the previous inequality holds and the proof is complete.  $\blacksquare$

The summation on  $k \in \mathbb{N}^*$  in each side of (4.1) we lead to prove the two following lemmas.

**Lemma 4.4.** *There exists  $C > 0$  such that, for all  $j \in J$ , we have*

$$\varphi_{|s} \|e^{s\varphi} \varphi^{-1/2} g_j\|^2 + \varphi_{|s}^{-1} \|e^{s\varphi} \varphi^{1/2} h_j\|^2 \leq \frac{C}{s\lambda} \left( \|e^{s\varphi} \varphi^{-1/2} f_j\|^2 + (s\lambda)^2 \|e^{s\varphi} \varphi^{1/2} \nabla u\|^2 + s^3 \lambda^3 \|e^{s\varphi} \varphi^{3/2} u\|^2 \right)$$

for  $s$  and  $\lambda$  large enough.

**Proof.** We recall that  $\|\cdot\|_{L^2(\Omega_{j,\delta})}$  is here noted  $\|\cdot\|$ , that  $h_j = [A_\tau, \chi_j]u + (\partial_{x_n} c) \partial_{x_n} u_j$  and we know that  $[A_\tau, \chi_j] + (\partial_{x_n} c) \partial_{x_n}$  is an operator of order 1 with coefficients depending on  $\nabla_\tau \chi_j$ ,  $\nabla_\tau \cdot C_\tau \nabla_\tau \chi_j$  and  $\partial_{x_n} c$ . So, we can apply Theorem 2.1 to obtain

$$\varphi_{|s}^{-1} \|e^{s\varphi} \varphi^{1/2} h_j\|^2 \leq C \left( s\lambda \|e^{s\varphi} \varphi^{1/2} \nabla u\|^2 + s^2 \lambda^2 \varphi_{|s} \|e^{s\varphi} \varphi^{1/2} u\|^2 \right).$$

Moreover, applying Mean Value Theorem and the first inequality of (2.1) to (2.2), we see  $\|c^j - c\|_\infty \leq C \frac{1}{\sqrt{s\lambda\varphi_{|s}}}$  where the constant  $C$  is independent on  $j \in J$ , and, as  $g_j = -(c^j - c) \partial_{x_n}^2 u_j$ , Lemma B.3 implies that for  $\lambda \geq 1, s \geq 1$ , one has

$$\varphi_{|s} \|e^{s\varphi} \varphi^{-1/2} g_j\|^2 \leq \frac{C}{s\lambda} \left\{ \|e^{s\varphi} \varphi^{-1/2} f_j\|^2 + s\lambda\varphi_{|s} \|e^{s\varphi} \varphi^{-1/2} \nabla u_j\|^2 + (s\lambda\varphi_{|s})^2 \|e^{s\varphi} \varphi^{-1/2} u_j\|^2 + (s\lambda)^2 \|e^{s\varphi} \varphi^{1/2} \partial_{x_n} u_j\|^2 \right\}.$$

Gathering together the previous results we obtain

$$\varphi_{|s} \|e^{s\varphi} \varphi^{-1/2} g_j\|^2 + \varphi_{|s}^{-1} \|e^{s\varphi} \varphi^{1/2} h_j\|^2 \leq \frac{C}{s\lambda} \left( \|e^{s\varphi} \varphi^{-1/2} f_j\|^2 + (s\lambda)^2 \|e^{s\varphi} \varphi^{1/2} \nabla u_j\|^2 + (s\lambda)^2 \|e^{s\varphi} \varphi^{1/2} \nabla u\|^2 + s\lambda\varphi_{|s} \|e^{s\varphi} \varphi^{-1/2} \nabla u_j\|^2 + (s\lambda\varphi_{|s})^2 \|e^{s\varphi} \varphi^{-1/2} u_j\|^2 + s^3 \lambda^3 \varphi_{|s} \|e^{s\varphi} \varphi^{1/2} u\|^2 \right)$$

if  $\lambda \geq 1$ ,  $s \geq 1$ . For the term with  $\nabla u_j$  we have  $\|e^{s\varphi}\varphi^{1/2}\nabla u_j\|^2 \leq C(\|e^{s\varphi}\varphi^{1/2}u\nabla\chi_j\|^2 + \|e^{s\varphi}\varphi^{1/2}\nabla u\|^2)$ . Using Theorem 2.1, we get  $\|e^{s\varphi}\varphi^{1/2}u\nabla\chi_j\|^2 \leq s\lambda\varphi|_S \|e^{s\varphi}\varphi^{1/2}u\|^2$  then  $\|e^{s\varphi}\varphi^{1/2}\nabla u_j\|^2 \leq C(s\lambda\varphi|_S \|e^{s\varphi}\varphi^{1/2}u\|^2 + \|e^{s\varphi}\varphi^{1/2}\nabla u\|^2)$ . We conclude the proof of the lemma by noticing that  $2\delta\beta'_{|S^-} \leq \beta(0)$  by Assumption 4.1 and, therefore, one has  $\beta(0) \leq 2\beta(x_n)$  on  $(-\delta, \delta)$  which implies  $\varphi|_S \leq \varphi^2$ .  $\blacksquare$

**Lemma 4.5.** *There exists a constant  $C > 0$  such that, for all  $j \in J$ , one has*

$$s^{3/2}\lambda\varphi|_S e^{2s\varphi|_S} |\theta_j|_{L^2(S_j)}^2 \leq C s^{1/2} e^{2s\varphi|_S} \int_{S_j} |c\partial_{x_n} u_j|^2 d\sigma.$$

**Proof.** The definition  $\theta_j = [(c^j - c)\partial_{x_n} u_j]_{S_j}$  and (2.2) imply  $|\theta_j|_{L^2(S_j)}^2 \leq \frac{C}{s\lambda\varphi|_S} |c\partial_{x_n} u_j|_{L^2(S_j)}^2$ . Then the conclusion follows.  $\blacksquare$

We now prove the following key result, providing an estimate of the tangential derivative of  $u$ .

**Theorem 4.6.** *There exist  $C, \lambda_0 > 0$  and  $s_0 > 0$  such that, for  $s \geq s_0$  and  $\lambda \geq \lambda_0$ , we have*

$$\begin{aligned} s\lambda\varphi|_S \int_S e^{2s\varphi} |\nabla_\tau u|^2 \|[\beta' c C_\tau]_S\| d\sigma &\leq \frac{B}{4} (s\lambda\varphi|_S)^3 \int_S e^{2s\varphi} |u|^2 d\sigma \\ &+ C \left( \|e^{s\varphi} f\|^2 + s\lambda \|e^{s\varphi}\varphi^{1/2}\nabla u\|^2 + s^2 \lambda^2 \|e^{s\varphi}\varphi^{3/2}u\|^2 + s^{1/2} \int_S e^{2s\varphi} |c\partial_{x_n} u|^2 d\sigma \right). \end{aligned} \quad (4.4)$$

**Proof.** We start from (4.1) and, after summation on  $k \in \mathbb{N}^*$ , use the two previous lemmas. In order to go back to  $u$ , we note successively that

- For each  $x' \in \Omega'$ , we can define  $J(x') := \{j \in J; \chi_j(x') \neq 0\}$  and we know that  $\overline{\overline{J(x')}} \leq N$  ( $N$  is the integer of Theorem 2.1 and  $\overline{\overline{J(x')}}$  is the cardinal of the set  $J(x')$ ).
- As  $u = \sum_j \chi_j u = \sum_j u_j$ , we can write for each  $x \in S$  (the functions  $\chi_j$  do not depend on  $x_n$ )

$$\begin{aligned} |\nabla_\tau u|^2 &= |\nabla_\tau \sum_j \chi_j u|^2 = \left| \sum_j \nabla_\tau (\chi_j u) \right|^2 \\ &= \left| \sum_{j \in J(x)} \nabla_\tau (\chi_j u) \right|^2 \leq N \sum_{j \in J(x)} |\nabla_\tau u_j|^2. \end{aligned} \quad (4.5)$$

- The first and last terms of (4.5) are quantities of the same order, modulo a term not disturbing to conclude. In fact, from  $\nabla_\tau u_j = \chi_j \nabla_\tau u + u \nabla_\tau \chi_j$  we derive that

$$\begin{aligned} \sum_{j \in J(x)} |\nabla_\tau u_j|^2 &\leq 2|\nabla_\tau u|^2 \sum_{j \in J(x)} \chi_j^2 + 2|u|^2 \sum_{j \in J(x)} |\nabla_\tau \chi_j|^2 \\ &\leq 2N|\nabla_\tau u|^2 + 2NCs\lambda\varphi|_S |u|^2. \end{aligned}$$

We used  $0 \leq \chi_j \leq 1$ ,  $\|\nabla_\tau \chi_j\|_\infty \leq C\sqrt{s\lambda\varphi|_S}$  and  $\|\nabla_\tau \cdot (C_\tau \nabla_\tau \chi_j)\|_\infty \leq Cs\lambda\varphi|_S$ . The term  $s\lambda\varphi|_S |u|^2$  will be absorbed by the l.h.s. of (1.2).

- Moreover, for any function  $f$ , we have

$$f^2 = \left( \sum_{j \in J} f\chi_j \right)^2 \leq N \sum_{j \in J} (f\chi_j)^2 \leq N^2 f^2$$

since  $0 \leq \chi_j \leq 1$  and also because of the convexity of the function  $z \rightarrow z^2$ :  $\left( \sum_{1 \leq k \leq M} \frac{1}{M} a_k \right)^2 \leq \frac{1}{M} \sum_{1 \leq k \leq M} a_k^2$ .

■

**Remark 4.7.** From the inequality  $(a + \alpha b)^2 + b^2 \geq (1 + \alpha^2)^{-1} a^2$ , we deduce that  $s^{1/2} e^{2s\varphi_s} |c \partial_{x_n} u|^2$  is less than the product of  $(s^{1/2} \lambda \varphi_s \inf(B_1, \mathbf{B}))^{-1} \mathcal{B}(u)$  and of a constant  $C$  depending on  $L$  but not on  $(s, \lambda)$ , which we shall use to remove the last term in (4.4), absorbing it by the l.h.s. of (1.2) for  $(s, \lambda)$  large enough.

Finally we obtain Theorem 1.2 since the three terms involving  $u$  in the r.h.s. of (4.4) are absorbed by the l.h.s. of (1.2) : use Remark 4.7 for the integral on  $S$  while the two others are absorbed by the l.h.s. of (1.2) by choosing  $s, \lambda$  large enough.

## A Partition of unity

In this appendix, in order to prove Theorem 2.1 and the following results, we construct a suitable lattice in a neighborhood of  $\Omega'$ . The open set  $\Omega'$  is a bounded set of  $\mathbb{R}^{n-1}$  and its closure is included in a cube  $[a, b]^{n-1}$ ,  $-\infty < a < b < \infty$ . For the simplicity of the proof, we shall suppose that  $[a, b] = [0, 1]$  since only the value of the constant  $C$  in (2.1) will change in the general case. We shall proceed gradually: in a first step, we shall suppose that  $\Omega' = (0, 1)$ , then, in a second step, we shall consider the case  $\Omega' = (0, 1)^{n-1}$  and, at last, in a third step, we obtain the generalization to any  $\Omega'$  with the required  $\mathcal{C}^2$  regularity, with some elementary arguments.

**Step 1.** We begin with the case  $n = 2$  which means that the open cube is the interval  $(0, 1)$  and we recall that  $\varphi \geq 1$  and  $\beta \geq 0$  by Assumption 2.4.

Let us denote  $h = h(s, \lambda) := \frac{\Theta}{\sqrt{s\lambda\varphi_s}}$  where  $\frac{1}{2} < \Theta \leq 1$  in such a way that  $\frac{\sqrt{s\lambda\varphi_s}}{\Theta}$  is an even natural integer  $p = p(s, \lambda)$ . For each  $s\lambda > 1$ , there is a finite number of admissible values  $\Theta$ . We shall choose the value nearest to 1.

In order to simplify our explanation, we work on  $\tilde{S} := (-h, 1 + h)$ . On  $\tilde{S}$  we consider the lattice

$$\begin{aligned} x_{-1}(s, \lambda) &= -h \text{ and } x_{p+1}(s, \lambda) = 1 + h, \\ x_j(s, \lambda) &= x_{j-1}(s, \lambda) + h \quad \text{where } j = 0, \dots, p(s, \lambda), \end{aligned}$$

and we set (see Figure 1.)

$$I_j(s, \lambda) = (x_{j-1}(s, \lambda), x_{j+1}(s, \lambda)), \quad 0 \leq j \leq p.$$

Two successive subintervals  $I_j(s, \lambda)$  overlap and the intersection has a measure equal to  $h$ , half of the

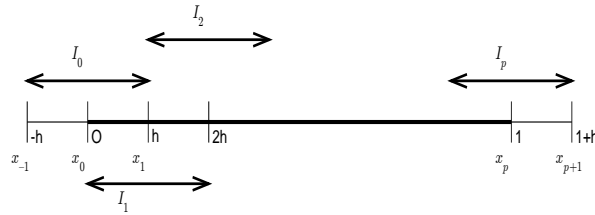


Figure 1: Covering in dimension 1

measure of each interval. Let us choose a function  $\psi \in C_0^\infty(\mathbb{R})$  such that  $0 \leq \psi(x) \leq 1$  and

$$\psi(x) = \begin{cases} 1, & \text{for } -\frac{1}{3} < x < \frac{1}{3}, \\ 0, & \text{for } x \notin (-\frac{2}{3}, \frac{2}{3}), \\ > 0, & \text{for } -\frac{2}{3} < x < \frac{2}{3}. \end{cases}$$

Let us consider now the functions  $\psi_j$  and  $\chi_j$  defined by

$$\psi_j(x) = \psi\left(\frac{x - x_j}{h}\right), \quad \text{and for } -h \leq x \leq 1 + h, 0 \leq j \leq p, \quad \chi_j(x) := \frac{\psi_j(x)}{\sum_{0 \leq k \leq p} \psi_k(x)}. \quad (\text{A.1})$$

**Remark A.1.** The previous sum contains less than two non vanishing terms for each  $x \in [-h, 1 + h]$ , and the family  $(\chi_j)_j$  is a partition of unity on  $(0, 1)$  if we only consider the restrictions on  $(0, 1)$  of each  $\chi_j$ .

**Lemma A.2.** *There exists a constant  $C$ , independant of  $s, \lambda > 0$  and  $j, 0 \leq j \leq p$ , such that*

$$\|(\chi_j)'\|_\infty \leq C \sqrt{s\lambda\varphi_{|s}}, \quad \text{and} \quad \|(\chi_j)''\|_\infty \leq Cs\lambda\varphi_{|s}. \quad (\text{A.2})$$

**Proof.** Differentiating the function  $\psi_j$ , we have  $|\psi_j'(x)| \leq \frac{\|\psi_j'\|_\infty}{h}$  and  $|\psi_j''(x)| \leq \frac{\|\psi_j''\|_\infty}{h^2}$ . From the definition of  $\psi_j$ , there exists a constant  $m > 0$  ( $m = 2$  for an appropriate choice of  $\psi$ ) such that  $\sum_{0 \leq j \leq p} \psi_j(x) > 1/m$ .

Then, using  $\chi_j'(x) = \frac{\psi_j'(x)}{\sum_{0 \leq k \leq p} \psi_k(x)} - \sum_{0 \leq l \leq p} \psi_l'(x) \frac{\psi_j(x)}{(\sum \psi_k)^2}(x)$ , we have  $|\nabla \chi_j(x)| \leq 3m \frac{\|\psi_j'\|_\infty}{h}$ . Using Remark A.1

similarly, the expression of  $\chi_j''(x)$  gives us the last estimation  $|\chi_j''(x)| \leq \frac{3m}{h^2} (\|\psi_j''\|_\infty + 2m\|\psi_j'\|_\infty^2)$ .  $\blacksquare$

**Step 2.** Now, we consider the case of a cube  $(0, 1)^{n-1}$ ,  $n > 2$ . The pair  $(s, \lambda)$  being fixed, we use  $p = p(s, \lambda)$  defined in Step 1, and we build on each axis of coordinate a lattice similar to this one of Step 1. The product gives a lattice indexed by  $j := (j_1, j_2, \dots, j_{n-1})$ ,  $0 \leq j_i \leq p(s, \lambda)$ , which defines the small cubes  $\Omega_j' = I_{j_1} \times I_{j_2} \times \dots \times I_{j_{n-1}}$  where each  $I_{j_i}$  has the form introduced in the previous step. The measure of each cube is  $(2h)^{n-1}$  and the cubes overlap. We associate to each cube  $\Omega_j'$  the function  $\chi_j(x') := \chi_{j_1}(x_1)\chi_{j_2}(x_2)\dots\chi_{j_{n-1}}(x_{n-1})$ . So, in each point  $x' \in \overline{\Omega'}$ , there are  $q(x')$  functions of the partition non vanishing with  $1 \leq q \leq 2^{n-1}$ . The proof of Theorem 2.1 is complete in this case. Moreover, if  $\Omega' = (0, 1)^{n-1}$  we can directly extend the coefficients  $c$  on each side of the boundary as we write it in item 2 of Step 3. The case  $\Omega' \subsetneq (0, 1)^{n-1}$  needs a little more work.

**Step 3.** In what follows, the open set  $\Omega'$  is not a cube but, in order to give a visual explanation, we shall suppose  $n = 3$  meaning that  $\Omega'$  is a bounded open set in  $[0, 1]^2$  and  $\partial\Omega'$  is  $\mathcal{C}^2$ . The reader will see that the extension to  $n > 3$  is easy once this approach is understood. In subsection 1.3, we reduced our problem to the family of problems (1.3) posed in slices, indexed by  $x_n$ , of the cylinder  $\overline{\Omega'} \times (-\delta, \delta)$ , slices with sizes of same order. As the lattice of Step 2 cannot be used if  $\Omega'$  is not a cube, we describe our modified approach in the 6 following items. We build in items 1 and 2 a finite covering  $(R_\kappa)$  of the boundary  $\partial\Omega'$  using the  $\mathcal{C}^2$  regularity of  $\partial\Omega'$ . This covering does not depend on the Carleman parameters  $s, \lambda$  and permits to extend the coefficients in a neighborhood of  $\Omega' \times (-\delta, \delta)$ . Each  $R_\kappa$  corresponds to a true cube  $\tilde{R}_\kappa$  by means of a diffeomorphism. Then, once the pair  $(s, \lambda)$  is chosen, we distinguish the cubes far from the boundary (item 3) which come from the mesh of  $[0, 1]^2 \supset \Omega'$  and the 'cubes' close to the boundary (item 4) which come from a meshing of  $\tilde{R}_\kappa$ . Choosing  $h$  small enough, we show in items 5 and 6 that the covering is complete and the partition of the unity is adapted.

1. We cover the boundary of  $\Omega'$  by a finite family of open sets  $(R_\kappa)_{1 \leq \kappa \leq N_b}$  in such a way that each  $R_\kappa$  is diffeomorphic by  $\varphi_\kappa$  to a rectangle  $\tilde{R}_\kappa := (-\alpha_{\kappa 1}, \alpha_{\kappa 1}) \times (-\alpha_{\kappa 2}, \alpha_{\kappa 2})$ . By example, we can suppose that  $\partial\Omega'$  is locally defined either by a relation  $x_1 = f(x_2)$  or  $x_2 = f(x_1)$ . In the first case,  $\varphi_\kappa(x_1, x_2) := (x_1 - f(x_2), x_2)$  and  $R_\kappa \cap \partial\Omega'$  is the image by  $\varphi_\kappa^{-1}$  of the straight line  $\{0\} \times (-\alpha_{\kappa 2}, \alpha_{\kappa 2})$ . In the other case  $\varphi_\kappa(x_1, x_2) := (x_1, x_2 - f(x_1))$  and  $R_\kappa \cap \partial\Omega'$  is the image by  $\varphi_\kappa^{-1}$  of  $(-\alpha_{\kappa 1}, \alpha_{\kappa 1}) \times \{0\}$ . In all the following items we shall consider the first case, the adaptations will be obvious. We can suppose that the modulus of the determinant of  $J_{\varphi_\kappa}$ , the Jacobian matrix of  $\varphi_\kappa$ , is equal to 1. So,  $\cup_{\kappa=1}^{N_b} R_\kappa$  is an open neighborhood of  $\partial\Omega'$  that contains the tube  $T_{\delta'} := \{x \in \mathbb{R}^2; d(x, \partial\Omega') \leq \delta'\}$  for a some  $\delta' > 0$  which is now fixed. We set  $\tilde{R}_\kappa^\pm := \{y = (y_1, y_2) \in \tilde{R}_\kappa; \pm y_1 > 0\}$  and  $R_\kappa^\pm := \varphi_\kappa^{-1}(\tilde{R}_\kappa^\pm)$ , choosing  $R_\kappa^+ \subset \Omega'$  (if necessary, replace  $x_1 - f(x_2)$  by  $f(x_2) - x_1$ ). Furthermore, without any lose of generality, we suppose that  $\varphi_\kappa$  is a diffeomorphism defined on an open set  $\check{R}_\kappa \ni R_\kappa$ . This will be useful in item 4.

2. To each  $R_\kappa$  and  $\tilde{R}_\kappa$  we associate the cylinders  $R_{\kappa,\delta} := R_\kappa \times (-\delta, \delta)$ ,  $\tilde{R}_{\kappa,\delta} := \tilde{R}_\kappa \times (-\delta, \delta)$  as well as the cylinders  $R_{\kappa,\delta}^\pm$  and  $\tilde{R}_{\kappa,\delta}^\pm$ . For a function  $\hat{w}$ , defined on  $\tilde{R}_{\kappa,\delta}^+$ , we extend it by antisymmetry relatively to the straight line  $\{y_1 = 0\}$ . With regard to the coefficients  $(c_{rl}(\cdot, x_3))_{1 \leq r, l \leq n-1}$  of matrix  $C_\tau(\cdot, x_3)$ , we define their images  $\hat{c}_{rl}(\cdot, x_3) := c_{rl}(\varphi_\kappa^{-1}(\cdot), x_3)$  on  $\tilde{R}_{\kappa,\delta}^+$  first, then we extend these  $\hat{c}_{rl}(\cdot, x_3)$  by 'symmetry' relatively to the line  $\{0\} \times (-\alpha_{\kappa 2}, \alpha_{\kappa 2})$  which gives

$$\tilde{w}(y, x_3) = \begin{cases} \hat{w}(y, x_3) & \text{if } y_1 > 0, \\ -\hat{w}(-y_1, y_2, x_3) & \text{if } y_1 < 0, \end{cases} \quad \tilde{c}_{rl}(y, x_3) = \begin{cases} \hat{c}_{rl}(y, x_3) & \text{if } y_1 > 0, \\ \hat{c}_{rr}(-y_1, y_2, x_3) & \text{if } y_1 < 0, \\ -\hat{c}_{rl}(-y_1, y_2, x_3) & \text{if } y_1 < 0, r + l = 3. \end{cases} \quad (\text{A.3})$$

As a matter of fact,  $\tilde{c}_{rl}$  is  $\mathcal{C}^1$  on the four closed subsets  $\overline{\tilde{R}_{\kappa,\delta}^\pm \cap \{\pm x_3 > 0\}}$  of  $\overline{\tilde{R}_{\kappa,\delta}}$ . Using  $\varphi_\kappa^{-1}$  we obtain an extension of the matrix  $C_\tau$  and we keep the same notation. Using the variational formulation in each slide  $\{x_3 = \text{constant}\}$ , from  $A_\tau(x_3)$  acting in  $L^2(R_\kappa^+)$  with Dirichlet condition on the boundary, we can define the extension  $\tilde{A}_\tau(x_3)$  in  $\tilde{R}_\kappa$ , and not only on  $\tilde{R}_\kappa^+$ , with the tangential matrix  $\tilde{C}_{\tau,\kappa}(\cdot, x_3) = (J_{\varphi_\kappa} \circ \varphi_\kappa^{-1}) C_\tau(\varphi_\kappa^{-1}(\cdot), x_3) (J_{\varphi_\kappa} \circ \varphi_\kappa^{-1})^t$ :

$$\begin{aligned} (A_\tau(x_3)u, v)_{L^2(R_\kappa^+)} &= (C_\tau(\cdot, x_3) \nabla_\tau u, \nabla_\tau v)_{L^2(R_\kappa^+)} = (\tilde{C}_{\tau,\kappa}(\cdot, x_3) \nabla_\tau \hat{u}, \nabla_\tau \hat{v})_{L^2(\tilde{R}_\kappa^+)}, \\ &= \frac{1}{2} (\tilde{C}_{\tau,\kappa}(\cdot, x_3) \nabla_\tau \tilde{u}, \nabla_\tau \tilde{v})_{L^2(\tilde{R}_\kappa)} = \frac{1}{2} (\tilde{A}_\tau(x_3) \tilde{u}, \tilde{v})_{L^2(\tilde{R}_\kappa)}. \end{aligned}$$

Coming back to  $R_\kappa$  we define the extended operator  $A_\tau(x_3)$  acting in  $L^2(R_\kappa)$ , still with Dirichlet condition on the boundary. The new tangential matrix  $\tilde{C}_{\tau,\kappa}(\cdot, x_3)$  defined on  $\tilde{R}_\kappa$  is symmetric and positive defined. Note that  $\tilde{w}$  belongs to the domain of  $\tilde{A}_\tau(x_3)$  if  $w$  belongs to the domain of  $A_\tau(x_3)$  (defined on  $R_\kappa^+$ ).

3. Now, we take into account the parameters  $(s, \lambda)$ . We start by using Step 2 to obtain a mesh of  $[0, 1]^2$ , i.e. a family  $(\Omega'_j) = (I_{j_1} \times I_{j_2})$ . From that covering we keep the cubes  $\Omega'_j$  that satisfy  $\overline{\Omega'_j} \subset \Omega'$ . Let  $J_{int}$  be the set of the corresponding subscripts. Since  $\Omega'$  is not a cube, it is clear that  $\cup_{j \in J_{int}} \Omega'_j$  does not fill  $\Omega'$ . Therefore, we have to complete this family.
4. Using a process similar to Step 1 and 2, we build a covering for each rectangle  $\tilde{R}_\kappa$  with small cubes  $\tilde{R}_{\kappa,l}$  overlapping each other and having a size equivalent to  $2h(s, \lambda)$ . Let  $J_b$  be this family of subscripts  $j = (\kappa, l)$ . As we have supposed in item 1 that  $\tilde{R}_\kappa \ni R_\kappa$ , for small sufficiently  $h$  we deduce functions  $\tilde{\chi}_{\kappa,l}$  adapted to these cubes and take their pullbacks  $\chi_{\kappa,l}$  using the diffeomorphisms  $\varphi_\kappa$ ,  $1 \leq \kappa \leq N_b$  to come back to  $\cup_{1 \leq \kappa \leq N_b} R_\kappa$ . In the same way we obtain  $R_{\kappa,l} = \varphi_\kappa^{-1}(\tilde{R}_{\kappa,l})$ . Recall that the diffeomorphisms  $\varphi_\kappa$  do not depend on  $(s, \lambda)$ . This last remark is important to claim that all the  $R_{\kappa,l}$  have sizes of the same order and that the functions  $\chi_{\kappa,l}$  satisfy (2.1).
5. We have to show that the union of  $(\cup_{j \in J_{int}} \Omega'_j)$  and  $(\cup_{j \in J_b} R_j)$  covers  $\overline{\Omega'}$ . This will be true only if  $h(s, \lambda)$  is small enough, i.e. if  $s$  and  $\lambda$  are large enough. Once this point is checked, it will be sufficient to normalize the functions of the partition similarly to (A.1). However, the overlap must be sufficiently important to avoid that the functions of the partition all vanish at a common point (in Step 1, the size of  $I_{2k} \cap I_{2k+1}$  is half the length of each interval  $I_k$ , for example). Note that  $\cup_{j \in J_b} R_j \supset T_{\delta'}$  (see item 1) and that the partition has all the required properties for each  $x' \in T_{\delta'/2}$ . Let us consider  $x' \in \Omega'$  such that  $d(x', \partial\Omega') > \frac{\delta'}{2}$  which implies that the open disc  $D(x', \frac{\delta'}{2}) \subset \Omega'$ , while the cube  $C(x', \frac{\delta'}{\sqrt{2}})$  of center  $x'$  with sides of length  $L = \frac{\delta'}{\sqrt{2}}$  and parallel to the coordinate axes is inscribed in  $D(x', \frac{\delta'}{\sqrt{2}})$  (see Figure 2.). To conclude, it will be sufficient to prove  
**Claim:** If  $h(s, \lambda) \leq \frac{L}{8}$ , the point  $x'$  is recovered by cubes  $\Omega'_j$  such that at least one of the associated function  $\chi_j$  satisfies  $\chi_j(x') \neq 0$ .
6. In order to prove the claim we suppose that  $h(s, \lambda) = \frac{L}{8}$  (the other situations follow easily). The lattice used in Step 2 does not depend on  $\Omega'$ . We project its vertices on coordinate axes. Let us consider



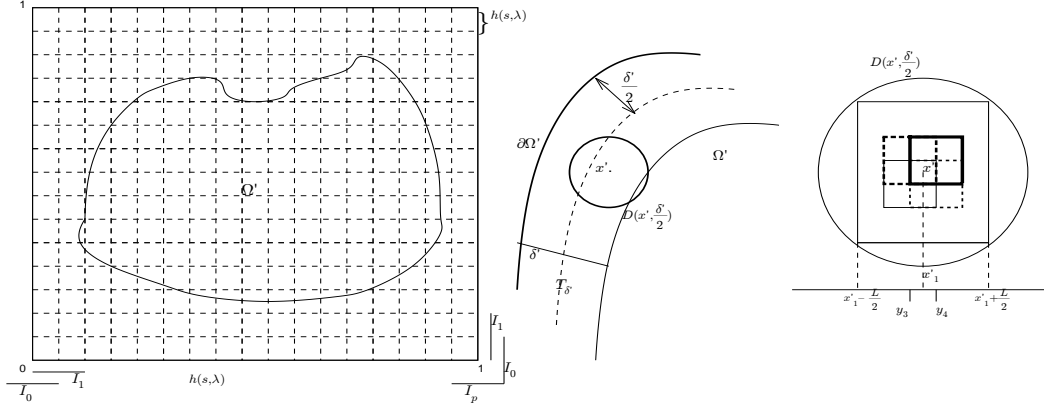


Figure 2: description associated to **Claim**

the first coordinate axis, the projection of the cube  $C(x', L)$  is then the interval  $(x'_1 - \frac{L}{2}, x'_1 + \frac{L}{2})$ . The projection of the lattice give 7 points  $(y_1, \dots, y_7)$  or 8 points  $(y_1, \dots, y_8)$  included in  $(x'_1 - \frac{L}{2}, x'_1 + \frac{L}{2})$ . In the first case,  $x'_1 = y_4$  and in the second case,  $y_3 < x'_1 < y_4$ . We repeat this process for the second coordinate  $x'_2$ . If we now go back to the partition of unity in Step 2 we see that the point  $x'$  is then such that at least one of the functions  $\chi_j$  does not vanish in its neighborhood. This concludes the proof of the Theorem 2.1.

## B Proof of some intermediate results

**Lemma B.1.** *Let*

$$\sigma\mu_{j,k}(0^+) \geq 2s\lambda\varphi_{|s}\beta'_{|s-} + \lambda\beta'_{|s--}. \quad (\text{B.1})$$

*Then, for all function  $g \in L^2(-\delta, \delta)$ , we have*

$$\varphi_{|s} e^{2s\varphi_{|s}} \int_0^\delta e^{-\sigma\mu_{j,k}(0^+)y} g^2(\pm y) dy \leq \int_0^\delta \varphi(\pm y) e^{2s\varphi(\pm y)} g^2(\pm y) dy \quad (\text{B.2})$$

*and*

$$\varphi_{|s}^{-1} e^{2s\varphi_{|s}} \int_0^\delta e^{-\sigma\mu_{j,k}(0^+)y} g^2(\pm y) dy \leq \int_0^\delta \varphi(\pm y)^{-1} e^{2s\varphi(\pm y)} g^2(\pm y) dy. \quad (\text{B.3})$$

**Proof. Step 1.** Let us prove the first estimate. Such kind of estimate is true if  $\varphi_{|s} e^{-\sigma\mu_{j,k}(0^+)y + 2s\varphi_{|s}} \leq \varphi(\pm y) e^{2s\varphi(\pm y)}$ , for all  $y \in (0, \delta)$ , which we write

$$-\sigma\mu_{j,k}(0^+)y + 2s(\varphi_{|s} - \varphi(\pm y)) + \lambda(\beta_{|s} - \beta(\pm y)) \leq 0. \quad (\text{B.4})$$

Since  $\beta' > 0$ ,  $\varphi$  is an increasing function on  $(-\delta, \delta)$  and we have

$$\varphi(0) - \varphi(y) \leq 0 \quad \text{and} \quad (\beta_{|s} - \beta(y)) \leq 0, \quad y \in (0, \delta),$$

which gives the estimate (B.4) for the '+' case. Let us now prove that it also holds with the '-' case. Since the function  $\beta$  is affine on  $(-\delta, 0)$ , we have  $\beta_{|s} - \beta(-y) = \beta'_{|s-}y$  for  $y \in (0, \delta)$  whence

$$\varphi(0) - \varphi(-y) = y \int_0^1 \varphi'(-y + ry) dr = y\lambda\beta'_{|s-} \int_0^1 \varphi(-y + ry) dr.$$

For  $r \in (0, 1)$ ,  $\varphi(-y + ry) \leq \varphi(0)$  since  $\beta' > 0$  and, so, we have  $\varphi(0) - \varphi(-y) \leq y\lambda\beta'_{|S^-} \varphi(0)$ , which gives

$$-\sigma\mu_{j,k}(0^+)y + 2s(\varphi_{|S} - \varphi(-y)) + \lambda(\beta_{|S} - \beta(-y)) \leq -\sigma\mu_{j,k}(0^+)y + 2sy\lambda\beta'_{|S^-} \varphi(0) + \lambda\beta'_{|S^-} y, \quad y \in (0, \delta).$$

Using (B.1), the first estimate (B.2) is proved.

**Step 2.** To prove (B.3), we will prove that  $\varphi_{|S}^{-1} e^{-\sigma\mu_{j,k}(0^+)y + 2s\varphi_{|S}} \leq \varphi(\pm y)^{-1} e^{2s\varphi(\pm y)}$ , for all  $y \in (0, \delta)$ . This is equivalent to proving that

$$-\sigma\mu_{j,k}(0^+)y + 2s(\varphi_{|S} - \varphi(\pm y)) - \lambda(\beta_{|S} - \beta(\pm y)) \leq 0.$$

Let us begin by the  $'-'$  case. As  $\beta_{|S} - \beta(-y) \geq 0$  since  $\beta' > 0$ , it will be sufficient to prove that  $-\sigma\mu_{j,k}(0^+)y + 2s(\varphi_{|S} - \varphi(-y)) \leq 0$ . In the first step we have seen that  $\varphi(0) - \varphi(-y) \leq y\lambda\beta'_{|S^-} \varphi(0)$ , whence

$$-\sigma\mu_{j,k}(0^+)y + 2s(\varphi_{|S} - \varphi(-y)) \leq -\sigma\mu_{j,k}(0^+)y + 2sy\lambda\beta'_{|S^-} \varphi(0) \leq 0$$

with the assumption (B.1) made on  $\mu_{j,k}$ .

Now, let us consider the  $'+'$  case. As  $\beta_{|S} - \beta(y) = -\beta'_{S^+} y$ , we have to prove that  $-\sigma\mu_{j,k}(0^+)y + 2s(\varphi_{|S} - \varphi(y)) + \lambda\beta'_{S^+} y \leq 0$ . Using (B.1) this will be true if we have  $2s(\varphi_{|S} - \varphi(y)) + \lambda\beta'_{S^+} y \leq (2s\lambda\varphi_{|S}\beta'_{|S^-} + \lambda\beta'_{|S^-})y$ . As  $\varphi(y) - \varphi_{|S} \geq \lambda\varphi_{|S}\beta'_{S^+} y$ , we have  $2s(\varphi_{|S} - \varphi(y)) + \lambda\beta'_{S^+} y \leq \lambda\beta'_{S^+} y(1 - 2s\varphi_{|S})$  and  $\lambda\beta'_{S^+} y(1 - 2s\varphi_{|S}) \leq 0$  for  $s \geq 1/2$  since  $\varphi \geq 1$ .  $\blacksquare$

**Lemma B.2.** *The solution of*

$$\begin{cases} v'' - \mu^2 v = F & s \in (-\delta, 0) \cup (0, \delta), \\ v(-\delta) = v(\delta) = 0, \quad v(0^-) = v(0^+), \quad c_+ v'(0^+) = c_- v'(0^-) + \theta \end{cases}$$

satisfies

$$v(0) = -\frac{c_+}{\mu} \int_0^\delta \frac{\sinh(\mu(\delta - \sigma))}{(c_+ + c_-) \cosh(\mu\delta)} F(\sigma) d\sigma - \frac{c_-}{\mu} \int_0^{-\delta} \frac{\sinh(\mu(-\delta - \sigma))}{(c_+ + c_-) \cosh(\mu\delta)} F(\sigma) d\sigma - \frac{\theta \tanh(\mu\delta)}{\mu(c_+ + c_-)}.$$

**Proof.** The solution of this system is of the form

$$v(s) = A_\pm \cosh(\mu s) + B_\pm \sinh(\mu s) + \mu^{-1} \int_0^s \sinh(\mu(s - \sigma)) F(\sigma) d\sigma.$$

For  $s = 0^+$ , we obtain  $A_+ = v(0^+)$ . Similarly,  $s = 0^-$  gives us  $A_- = v(0^-)$ . Taking the derivative of the expression and considering  $s = 0^\pm$ , we also obtain  $\mu B_+ = v'(0^+)$  et  $\mu B_- = v'(0^-)$ . Thus, for  $s \in (-\delta, 0)$ , we have

$$v(s) = v(0) \cosh(\mu s) + \frac{1}{\mu} \frac{c_+ v'(0^+) - \theta}{c_-} \sinh(\mu s) + \mu^{-1} \int_0^s \sinh(\mu(s - \sigma)) F(\sigma) d\sigma.$$

For  $s \in (0, \delta)$ , we obtain  $v(s) = v(0) \cosh(\mu s) + \frac{1}{\mu} v'(0^+) \sinh(\mu s) + \mu^{-1} \int_0^s \sinh(\mu(s - \sigma)) F(\sigma) d\sigma$ . Considering the conditions  $v(-\delta) = v(\delta) = 0$ , we obtain

$$\begin{cases} \left[ \frac{1}{\mu} \cosh(\mu\delta) \left[ \mu v(0) - \frac{c_+ v'(0^+) - \theta}{c_-} \tanh(\mu\delta) + \int_0^{-\delta} \frac{\sinh(\mu(-\delta - \sigma))}{\cosh(\mu\delta)} F(\sigma) d\sigma \right] \right] = 0 \\ \left[ \frac{1}{\mu} \cosh(\mu\delta) \left[ \mu v(0) + v'(0^+) \tanh(\mu\delta) + \int_0^\delta \frac{\sinh(\mu(\delta - \sigma))}{\cosh(\mu\delta)} F(\sigma) d\sigma \right] \right] = 0. \end{cases}$$

This can be written

$$A \begin{pmatrix} v(0) \\ v'(0^+) \end{pmatrix} = \begin{pmatrix} -\int_0^\delta \frac{\sinh(\mu(\delta - \sigma))}{\cosh(\mu\delta)} F(\sigma) d\sigma \\ -\int_0^{-\delta} \frac{\sinh(\mu(-\delta - \sigma))}{\cosh(\mu\delta)} F(\sigma) d\sigma - \frac{\theta}{c_-} \tanh(\mu\delta) \end{pmatrix} \text{ where } A = \begin{pmatrix} \mu & \tanh(\mu\delta) \\ \mu & -\frac{c_+}{c_-} \tanh(\mu\delta) \end{pmatrix}.$$

The determinant of this matrix being  $D = \frac{-\mu}{c_-} \tanh(\mu\delta)(c_+ + c_-)$ , we deduce the value of  $\nu(0)$ :

$$\begin{pmatrix} \nu(0) \\ \nu'(0^+) \end{pmatrix} = \frac{1}{D} \begin{pmatrix} -\frac{c_+}{c_-} \tanh(\mu\delta) & -\tanh(\mu\delta) \\ -\mu & \mu \end{pmatrix} \begin{pmatrix} -\int_0^\delta \frac{\sinh(\mu(\delta - \sigma))}{\cosh(\mu\delta)} F(\sigma) d\sigma \\ -\int_0^{-\delta} \frac{\sinh(\mu(-\delta - \sigma))}{\cosh(\mu\delta)} F(\sigma) d\sigma - \frac{\theta}{c_-} \tanh(\mu\delta) \end{pmatrix}.$$

Finally, we have

$$\nu(0) = -\frac{c_+}{\mu} \int_0^\delta \frac{\sinh(\mu(\delta - \sigma))}{(c_+ + c_-) \cosh(\mu\delta)} F(\sigma) d\sigma - \frac{c_-}{\mu} \int_0^{-\delta} \frac{\sinh(\mu(-\delta - \sigma))}{(c_+ + c_-) \cosh(\mu\delta)} F(\sigma) d\sigma - \frac{\theta \tanh(\mu\delta)}{\mu(c_+ + c_-)}.$$

■

**Lemma B.3.** *There exist  $C > 0$  such that*

$$\|\varphi^{-1/2} e^{s\varphi} \partial_{x_n}^2 u_j\|^2 \leq C \left\{ \|\varphi^{-1/2} e^{s\varphi} f_j\|^2 + s\lambda\varphi_s \|\varphi^{-1/2} e^{s\varphi} \nabla u_j\|^2 + (s\lambda\varphi_s)^2 \|\varphi^{-1/2} e^{s\varphi} u_j\|^2 + (s\lambda)^2 \|\varphi^{1/2} e^{s\varphi} \partial_{x_n} u_j\|^2 \right\}, \quad (\text{B.5})$$

for  $u_j$  verifying (1.3),  $s \geq 1$ ,  $\lambda \geq 1$  and  $j \in J$ .

*Proof.* In  $\Omega^\pm$ , one has

$$\varphi^{-1/2} e^{s\varphi} c \partial_{x_n}^2 u_j = \varphi^{-1/2} e^{s\varphi} \partial_{x_n} (c \partial_{x_n} u_j) - \varphi^{-1/2} (\partial_{x_n} c) e^{s\varphi} \partial_{x_n} u_j. \quad (\text{B.6})$$

As  $\varphi^{-1/2} \leq \varphi^{1/2}$  since  $\beta \geq 0$ , we see straightaway that

$$\|\varphi^{-1/2} (\partial_{x_n} c) e^{s\varphi} \partial_{x_n} u_j\|^2 \leq (\max_{x \in \Omega} |\partial_{x_n} c|)^2 \|\varphi^{1/2} e^{s\varphi} \partial_{x_n} u_j\|^2 \quad (\text{B.7})$$

if  $s \geq 1$  and  $\lambda > 1$ . The other term in (B.6) verifies

$$\begin{aligned} \partial_{x_n} (c \partial_{x_n} u_j) &= (\nabla \cdot B \nabla (\chi_j u) - \nabla_\tau \cdot C_\tau \nabla_\tau u_j) \\ &= -f \chi_j + 2(C_\tau \nabla_\tau u) \cdot \nabla_\tau \chi_j + (\nabla_\tau \cdot C_\tau \nabla_\tau \chi_j) u - \nabla_\tau \cdot C_\tau \nabla_\tau u_j. \end{aligned}$$

Let  $\eta = \varphi^{-1/2} e^{s\varphi}$ . Using (2.1), there exists  $C > 0$  independent of  $s, \lambda, j$  such that

$$\|\eta \partial_{x_n} (c \partial_{x_n} u_j)\|^2 \leq C \left( \|\eta f_j\|^2 + s\lambda\varphi_s \|\eta \nabla_\tau u\|^2 + (s\lambda\varphi_s)^2 \|\eta u\|^2 + \|\eta \nabla_\tau \cdot C_\tau \nabla_\tau u_j\|^2 \right), \quad (\text{B.8})$$

where the norms are taken in  $L^2(\Omega_{j,\delta})$ . To reach the conclusion it remains to estimate the last term of the r.h.s. of (B.8), in other words we will estimate the function  $\eta \partial_{x_k} \partial_{x_l} u_j$  for  $1 \leq k, l \leq n-1$ . Roughly speaking, we shall make this estimation using  $u_j, \nabla u_j, Au_j$  and the weight function  $\eta$ .

Let us denote  $\tilde{B} := \eta^2 B, \tilde{C}_\tau := \eta^2 C_\tau$  and let  $\psi \in H_0^1(\Omega_{j,\delta})$  be a test function. One has

$$\begin{aligned} \int_\Omega \tilde{B} \nabla (\chi_j u) \cdot \nabla \psi &= \int_\Omega \{ (-\nabla \cdot \tilde{B} \nabla u) \chi_j - u \nabla_\tau \cdot (\tilde{C}_\tau \nabla_\tau \chi_j) - 2\tilde{B} \nabla u \cdot \nabla \chi_j \} \psi \\ &= \int_\Omega \tilde{g}_j \psi \end{aligned}$$

where  $\tilde{g}_j = -\nabla \cdot \tilde{B} \nabla (\chi_j u)$ , which also leads to  $\tilde{g}_j = (-\nabla \cdot \tilde{B} \nabla u) \chi_j - u \nabla_\tau \cdot (\tilde{C}_\tau \nabla_\tau \chi_j) - 2\tilde{B} \nabla u \cdot \nabla \chi_j$ . As the function  $\chi_j$  only depends on  $x'$ , we could replace  $2\tilde{B} \nabla u \cdot \nabla \chi_j$  by  $2\tilde{C}_\tau \nabla_\tau u \cdot \nabla_\tau \chi_j$ . From  $-\nabla \cdot (\tilde{B} \nabla u) = \eta^2 f - (\partial_{x_n} \eta^2) c \partial_{x_n} u$ , we deduce

$$\tilde{g}_j = \eta^2 \left\{ f_j - 2B \nabla u \cdot \nabla \chi_j - u \nabla_\tau \cdot (C_\tau \nabla_\tau \chi_j) \right\} - \chi_j (\partial_{x_n} \eta^2) c \partial_{x_n} u. \quad (\text{B.9})$$

For  $h \neq 0$ , parallel to the interface  $S$ , we introduce the difference operator  $D_h \rho(x) = \frac{\rho(x+h) - \rho(x)}{|h|}$ . For  $w \in H_0^1(\Omega_{j,\delta})$  with  $\text{supp } w \Subset \Omega_{j,\delta}$  we set  $\psi = D_{-h}(D_h w)$ . We have  $\psi \in H_0^1(\Omega_{j,\delta})$  if we suppose  $|h|$  small enough. On the one hand, we can write

$$\begin{aligned} \int_{\Omega} \tilde{B} \nabla u_j \cdot \nabla D_{-h}(D_h w) &= \int_{\Omega} \tilde{B} \nabla u_j \cdot D_{-h}(\nabla D_h w) \\ &= \int_{\Omega} D_h(\tilde{B} \nabla u_j) \cdot (\nabla D_h w) = \int_{\Omega} [\tilde{B}(x+h) D_h(\nabla u_j) + D_h(\tilde{B}) \nabla u_j] \cdot \nabla D_h w \end{aligned}$$

and, on the other hand, we have

$$\int_{\Omega} \tilde{B} \nabla u_j \cdot \nabla D_{-h}(D_h w) = - \int_{\Omega} (\nabla \cdot \tilde{B} \nabla u_j) D_{-h}(D_h w)$$

since  $[\eta^2 c \partial_{x_n} u_j]_S = 0$ . As  $\tilde{g}_j = -\nabla \cdot \tilde{B} \nabla u_j$ , we deduce from the two previous expressions the following relation

$$\int_{\Omega} \tilde{B}(x+h) D_h(\nabla u_j) \cdot \nabla D_h w = \int_{\Omega} (\tilde{g}_j D_{-h}(D_h w) - (D_h(\tilde{B}) \nabla u_j) \cdot \nabla D_h w). \quad (\text{B.10})$$

As  $\eta$  does not depend on the tangential variable,  $D_h$  and  $\eta$  commute. This gives

$$\left| \int_{\Omega} (D_h(\tilde{B}) \nabla u_j) \cdot \nabla D_h w \right| \leq C_1(B) \|\eta \nabla u_j\|_{L^2} \|\eta \nabla D_h w\|_{L^2}, \quad \left| \int_{\Omega} \tilde{g}_j D_{-h}(D_h w) \right| \leq \|\eta^{-1} \tilde{g}_j\|_{L^2} \|D_{-h}(\eta D_h w)\|_{L^2}.$$

It is well known that  $\|D_{-h} \tilde{w}\|_{L^2} \leq \|\nabla \tilde{w}\|_{L^2}$  for  $\tilde{w} \in H_0^1(\Omega_{j,\delta})$  with  $\text{supp } \tilde{w} \Subset \Omega_{j,\delta}$  and  $h$  small enough ([8]). Similarly, we have  $\|D_{-h}(\eta D_h w)\|_{L^2} \leq \|\nabla_{\tau}(\eta D_h w)\|_{L^2}$ . Using again that the weight function  $\eta$  does not depend on  $x'$ , we have  $\|D_{-h}(\eta D_h w)\|_{L^2} \leq \|\eta \nabla_{\tau} D_h w\|_{L^2}$  then

$$\left| \int_{\Omega} \tilde{g}_j D_{-h}(D_h w) \right| \leq \|\eta^{-1} \tilde{g}_j\|_{L^2} \|\eta \nabla(D_h w)\|_{L^2}.$$

We plug the two previous estimates into (B.10), which gives

$$\left| \int_{\Omega} \eta^2 (B(x+h) \nabla D_h u_j) \cdot \nabla D_h w \right| \leq (C_1(B) \|\eta \nabla u_j\|_{L^2} + \|\eta^{-1} \tilde{g}_j\|_{L^2}) \|\eta \nabla(D_h w)\|_{L^2}.$$

This inequality is, in particular, true if we choose  $w$  equal to  $u_j$  :

$$C_2(B) \|\eta \nabla D_h u_j\|_{L^2}^2 \leq \left| \int_{\Omega} \eta^2 (B(x+h) \nabla D_h u_j) \cdot \nabla D_h u_j \right| \leq C_3 \|\eta \nabla(D_h u_j)\|_{L^2}$$

where the constants  $C_1(B)$  and  $C_2(B)$  depend on  $B$  but not on  $\eta$  and where  $C_3 := C_1(B) \|\eta \nabla u_j\|_{L^2} + \|\eta^{-1} \tilde{g}_j\|_{L^2}$ . There is no problem to take  $w$  equal to  $u_j$  when  $\bar{\Omega}_{j,\delta} \subset \Omega_{\delta}$  since the projection on  $S$  of the support of  $u_j$  is strongly included in  $\Omega'_j$ . Dividing by  $\|\eta \nabla(D_h u_j)\|_{L^2}$  we obtain  $C_2(B) \|\eta \nabla D_h u_j\|_{L^2} \leq C_3$ . For  $1 \leq l \leq n-1$ , and  $\psi \in H_0^1(\Omega_{j,\delta})$ , one obtains

$$\left| \int_{\Omega} (\partial_{x_l} u_j)(D_{-h} \psi) \right| = \left| \int_{\Omega} (D_h u_j)(\partial_{x_l} \psi) \right| = \left| \int_{\Omega} \eta (\partial_{x_l} D_h u_j) \eta^{-1} \psi \right| \leq \frac{C_3}{C_2(B)} \|\eta^{-1} \psi\|_{L^2}.$$

Letting  $|h|$  go to zero we obtain  $\left| \int_{\Omega} (\partial_{x_l} u_j)(\partial_{x_k} \psi) \right| \leq \frac{C_3}{C_2(B)} \|\eta^{-1} \psi\|_{L^2}$  which leads to

$$\|\eta \partial_{x_k} \partial_{x_l} u_j\|_{L^2} \leq \frac{C_1(B) \|\eta \nabla u_j\|_{L^2} + \|\eta^{-1} \tilde{g}_j\|_{L^2}}{C_2(B)} \text{ for } 1 \leq k, l \leq n-1.$$

Now, we can estimate  $\|\eta \nabla_\tau \cdot C_\tau \nabla_\tau u_j\|^2$  with the help of (B.9), i.e.

$$\begin{aligned}
\|\eta \nabla_\tau \cdot C_\tau \nabla_\tau u_j\|^2 &\leq C \left\{ \|\eta \nabla u_j\|^2 + \|\eta^{-1} \tilde{g}_j\|^2 \right\} \\
&\leq C \left\{ \|\eta \nabla u_j\|^2 + \|\eta f_j\|^2 + s\lambda\varphi_{|s} \|\eta \nabla u\|^2 + (s\lambda\varphi_{|s})^2 \|\eta u\|^2 + \|\eta^{-1} (\partial_{x_n} \eta^2) \partial_{x_n} u\|^2 \right\} \\
&\leq C \left\{ \|\eta f_j\|^2 + s\lambda\varphi_{|s} \|\eta \nabla u\|^2 + (s\lambda\varphi_{|s})^2 \|\eta u\|^2 + \|\eta^{-1} (\partial_{x_n} \eta^2) \partial_{x_n} u\|^2 \right\} \\
&\leq C \left\{ \|\varphi^{-1/2} e^{s\varphi} f_j\|^2 + s\lambda\varphi_{|s} \|\varphi^{-1/2} e^{s\varphi} \nabla u\|^2 + (s\lambda\varphi_{|s})^2 \|\varphi^{-1/2} e^{s\varphi} u\|^2 + (s\lambda)^2 \|\varphi^{1/2} e^{s\varphi} \partial_{x_n} u\|^2 \right\}.
\end{aligned}$$

Plugging this result in (B.8), we deduce (B.5) with the help of (B.6) and (B.7).

Otherwise, for  $\Omega_{j,\delta} \cap \partial\Omega \neq \emptyset$ , the way we proceed to extend functions and coefficients beyond the boundary (see Appendix A) must be modified for the matrix  $B$ : the method does not work for  $(k, l) = (1, 1)$  (if we locally suppose that the first coordinate axis is normal to the boundary). We build an extended matrix  $B$  the coefficients of which are symmetric with regard to the plane  $\{x_1 = 0\}$  and Lipschitz functions. Moreover, the functions  $f_j$  will be replaced by a combination of  $f_j$  and  $\partial_{x_k} \partial_{x_l} u_j$ ,  $1 \leq k, l \leq n-1$ ,  $(k, l) \neq (1, 1)$  which allows us to conclude.  $\blacksquare$

## C About the density in $D(A)$

The main ideas are in [1] where the authors proved the density in  $H_0^1(\Omega)$  of  $\mathfrak{D}(A) := \{u \in \mathcal{C}_0(\overline{\Omega}); u \in \mathcal{C}^2(\overline{\Omega^\pm}), c_+ \partial_{x_n} u = c_- \partial_{x_n} u \text{ on } S\}$ , assuming that the matrix  $B$  is  $\mathcal{C}^2$  on  $\overline{\Omega^\pm}$ . Lemma 3.2 in [1] implies that each  $u \in D(A)$  is the limit in  $H_0^1$  of a sequence  $(u_n) \subset \{u \in \mathcal{C}_0(\overline{\Omega}); u \in \mathcal{C}^2(\overline{\Omega^\pm}), c_+ \partial_{x_n} u = c_- \partial_{x_n} u \text{ on } S\}$  with  $Au_n \rightarrow_{L^2} Au$ . The reader will see here a small extension resulting from a chat with J. Droniou that we thank warmly: the convergence is in norm and not only weak. In other words,  $\mathfrak{D}(A)$  is a core of the operator  $A$ . We start knowing that the restriction to  $\Omega^\pm$  of each  $u \in D(A)$  belongs to  $H^2(\Omega^\pm)$  (this claim is a consequence of Appendix A and the technics developed in the proof of Lemma B.3). We begin by the two following lemmas that are easy:

**Lemma C.1.** *If  $\psi \in H_0^1(\Omega) \cap H^2(\Omega^\pm)$  one has*

$$\psi \in D(A) \iff [c \partial_{x_n} \psi]_S = 0.$$

**Lemma C.2.** *If the sequence  $(\psi_n) \subset D(A) \cap H^2(\Omega^\pm)$  tends to  $\psi \in D(A)$  for the topology of  $H^2(\Omega^\pm)$  then  $A\psi_n \rightarrow A\psi$  in  $L^2$ .*

1. We set  $c_\pm(x') = \lim_{\pm x_n \downarrow 0} c(x', x_n)$  and we call  $n_\pm$  the interior normal to the interface  $S$  relatively to  $\Omega^\pm$  which enables us to introduce a change of variables  $\mathcal{C}^2$  with the function  $\Psi : \Omega' \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}$  setting

$$\Psi(x', t) = \begin{cases} x' + tc_+(x')n_+(x') & \text{if } 0 \leq t \leq \varepsilon, \\ x' + tc_-(x')n_-(x') & \text{if } -\varepsilon \leq t \leq 0. \end{cases}$$

In our case,  $n_\pm(x') = (0, \dots, 0, \pm 1)$  with the initial coordinates. We are seeing to appear the ask in point 2 of Assumption 1.1: the traces  $c_\pm : x' \rightarrow c_\pm(x')$  are  $\mathcal{C}^2$ . So, with the new coordinates, a function belonging to  $H^2(\Omega^+)$  will keep this property in  $\mathcal{U}^+ := \Psi(\Omega^+)$  and if this function is 0 on  $\partial\Omega$ , it will be the same with the change. Idem for  $\Omega^-$  and  $\mathcal{U}^-$ . As a matter of fact, we have replaced  $c$  by the value 1 on  $S$ .

2. Let  $u \in D(A)$  be a function of which the support is included in  $\Omega_\delta$  and such that  $u \in H^2(\Omega^\pm)$ . Setting  $v(x', t) := u(\Psi(x', t))$ , we verify that  $v \in H^1(\mathcal{U})$ ,  $v \in H^2(\mathcal{U}^\pm)$  and

$$\partial_t v(x', 0^\pm) = c_\pm(x') \partial_{x_n} u(x', 0^\pm). \quad (\text{C.1})$$

So, the jump of  $\partial\nu$  at the interface is 0 which implies that  $\partial_t\nu \in H^1(\mathcal{U})$ . The tangential derivatives do not give any problem when we are going through the interface. In conclusion, the function  $\nu$  is belonging to  $H^2(\mathcal{U}) \cap H_0^1(\mathcal{U})$ .

3. There exists a sequence  $(\nu_l) \subset \mathcal{C}^2(\mathcal{U})$  that tends to  $\nu$  for the topology of  $H^2(\mathcal{U})$  and we set  $u_l = \nu_l \circ \Psi^{-1}$ . From the properties of  $\Psi$ , it comes that  $u_l \in H^1(\Omega) \cap \mathcal{C}^2(\overline{\Omega^\pm})$  and  $u_l \rightarrow u$  just as well for  $H^1(\Omega_\delta)$  as for  $H^2(\Omega_\delta^\pm)$ . The relation (C.1) implies the continuity of the trace of  $c\partial_{x_j}$  and from Lemma C.1 one has  $u_l \in D(A)$  while Lemma C.2 gives the convergence of  $Au_l$  to  $Au$  in  $L^2(\Omega_\delta)$ .

### Acknowledgement

The authors are grateful to F. Boyer (Aix Marseille Université) and J. Droniou (Monash University, Australia) for fruitful discussions on Appendix C.

The authors thank warmly the referees whose careful work has improved the presentation of this paper.

### References

- [1] L. Agelas, D. Di Pietro, and J. Droniou, *The G method for heterogeneous anisotropic diffusion on general meshes*, M2AN Math. Model. Numer. Anal. **44** (2010), no. 4, 597–625.
- [2] G. Alessandrini and L. Escauriaza, *Null-controllability of one-dimensional parabolic equations*, ESAIM Control Optim. Calc. Var. **14** (2008), no. 2, 284–293.
- [3] V. Barbu, *Exact controllability of the superlinear heat equation*, Appl. Math. Optim. **42** (2000), 73–89.
- [4] M. Bellassoued, *Carleman estimates and distribution of resonances for the transparent obstacle and application to the stabilization*, Asymptotic Anal. **35** (2003), 257–279.
- [5] A. Benabdallah, Y. Dermenjian, and J. Le Rousseau, *Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications*, Comptes Rendus Mécanique **334** (2006), 582–586.
- [6] ———, *Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications to controllability and an inverse problem*, J. Math. Anal. Appl. **336** (2007), 865–887.
- [7] ———, *Carleman estimates for stratified media*, J. Funct. Anal. **260** (2011), 3645–3677.
- [8] H. Brezis, *Analyse Fonctionnelle*, Masson, Paris, 1983.
- [9] A. L. Bukhgeim and M. V. Klibanov, *Global uniqueness of class of multidimensional inverse problems*, Soviet Math. Dokl. **24** (1981), 244–247.
- [10] T. Carleman, *Sur une problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes*, Ark. Mat. Astr. Fys. **26B** (1939), no. 17, 1–9.
- [11] A. Doubova, A. Osses, and J.-P. Puel, *Exact controllability to trajectories for semilinear heat equations with discontinuous diffusion coefficients*, ESAIM Control Optim. Calc. Var. **8** (2002), 621–661.
- [12] E. Fernández-Cara and E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré, Analyse non lin. **17** (2000), 583–616.
- [13] A. Fursikov and O. Yu. Imanuvilov, *Controllability of evolution equations*, vol. 34, Seoul National University, Korea, 1996, Lecture notes.
- [14] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.
- [15] ———, *The Analysis of Linear Partial Differential Operators*, vol. IV, Springer-Verlag, 1985.
- [16] J. Le Rousseau, *Carleman estimates and controllability results for the one-dimensional heat equation with BV coefficients*, J. Differential Equations **233** (2007), 417–447.
- [17] J. Le Rousseau and N. Lerner, *Carleman estimates for anisotropic elliptic operators with jumps at an interface*, Anal. PDE, to appear.
- [18] ———, *Carleman estimates for elliptic operators with jumps at an interface: Anisotropic case and sharp geometric conditions*, Journées “Équations aux Dérivées Partielles”, 2010, Port d’Albret.

- [19] J. Le Rousseau and L. Robbiano, *Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations*, Arch. Rational Mech. Anal. **105** (2010), 953–990.
- [20] ———, *Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces*, Invent. Math. **183** (2011), 245–336.
- [21] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Comm. Partial Differential Equations **20** (1995), 335–356.
- [22] K. Miller, *Nonunique continuation for uniformly parabolic and elliptic equations in self-adjoint divergence form with Hölder continuous coefficients*, Arch. Rational Mech. Anal. **54** (1974), 105–117.
- [23] A. Pliš, *On non-uniqueness in Cauchy problem for an elliptic second order differential equation*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 95–100.
- [24] O. Poisson, *Carleman estimates for the heat equation with discontinuous diffusion coefficients*, Applicable Anal. **87** (2008), 1129–1144.
- [25] C. Zuily, *Uniqueness and Non Uniqueness in the Cauchy Problem*, Birkhauser, Progress in mathematics, 1983.