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AN INTRODUCTION TO BIENAYMÉ-GALTON-WATSON TREES AND THEIR LOCAL LIMITS

ROMAIN ABRAHAM AND JEAN-FRANÇOIS DELMAS

ABSTRACT. The aim of this lecture is to give an overview of old and new results on Bienaymé-Galton-Watson (BGW) trees. After introducing the framework of discrete trees, we first give alternative proofs of classical results on the extinction probability of BGW processes and on the description of the processes conditioned on extinction or on non-extinction. Then, we study recent local limits of critical or sub-critical BGW trees conditioned to be large.

CONTENTS

1. Introduction	1
2. Bienaymé-Galton-Watson trees and extinction	2
2.1. The set of discrete trees	3
2.2. Bienaymé-Galton-Watson trees	6
2.3. Kesten's tree	10
3. Local limits of Bienaymé-Galton-Watson trees	13
3.1. The topology of local convergence	14
3.2. A criteria for convergence toward Kesten's tree	18
3.3. Applications to various conditioning	20
3.4. Conditioning on the number of nodes with given out-degree, sub-critical case	27
3.5. Other results	31
References	32

1. INTRODUCTION

The first draft of those notes has been written for a course given at Hamamet in 2014, it has then been completed with more examples and up to date references. It concerns the so-called Bienaymé-Galton-Watson (BGW for short) process which can be considered as the first stochastic model for population evolution. It was named after the French mathematician I.-J. Bienaymé (1845) and the British scientists F. Galton and H. W. Watson (1874) who studied it. I.-J. Bienaymé considered the probability of the male population extinction in [15]; its communication, which is reproduced in [39], indicates that he knew the right answer (see also the study [18] on the Bienaymé's proof of the criticality theorem for branching processes). Later on and independently, F. Galton, who was studying human evolution, published in 1873 in Educational Times a question on the probability of extinction of the noble surnames in the UK. It was a very short communication which can be copied integrally here:

“PROBLEM 4001: A large nation, of whom we will only concern ourselves with adult males, N in number, and who each bear separate surnames colonise a district. Their law of population is such that, in each generation, a_0 per cent of the adult males have no male children who reach adult life; a_1 have one such male child; a_2 have two; and

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so on up to a_5 who have five. Find (1) what proportion of their surnames will have become extinct after r generations; and (2) how many instances there will be of the surname being held by m persons.”

In more modern terms, he supposes that all the individuals reproduce independently from each others and have all the same offspring distribution. After receiving no valuable answer to that question, he directly contacted H. W. Watson and worked together on the problem. They published an article one year later [30] where they proved that the probability of extinction is a fixed point of the generating function of the offspring distribution (which is true, see Section 2.2.2) and concluded a bit too rapidly that this probability is always equal to 1 (which is false, see also Section 2.2.2). For further historical comments on BGW processes, we refer to D. Kendall [39] for the “Genealogy of genealogy branching process” up to 1975 as well as the Lecture¹ at the Oberwolfach Symposium on “Random Trees” in 2009 by P. Jagers. In order to track the genealogy of the population of a BGW process, one can consider the so called genealogical trees or BGW trees, which is currently an active domain of research. We refer to T. Harris [33] and K. Athreya and P. Ney [14] for most important results on BGW processes, to M. Kimmel and D. Axelrod [44] and P. Haccou, P. Jagers and V. Vatutin [32] for applications in biology, to M. Drmota [20] and S. Evans [29] on random discrete trees including BGW trees (see also J. Pitman [54] on a more combinatorial aspect and T. Duquesne and J.-F. Le Gall [23] for scaling limits of BGW trees which will not be presented here).

We introduce in the first chapter the framework of discrete random trees, which may be attributed to Neveu [52]. We then use this framework to construct BGW trees that describe the genealogy of a BGW process. It is very easy to recover the BGW process from the BGW tree as it is just the number of individuals at each generation. We then give alternative proofs of classical results on BGW processes using the tree formalism. We focus in particular on the extinction probability (which was the first question of F. Galton) and on the description of the processes conditioned on extinction or non extinction.

In a second chapter, we focus on local limits of conditioned BGW trees. In the critical and sub-critical cases (these terms will be explained in the first chapter), the population becomes a.s. extinct and the associated genealogical tree is finite. However, it has a small but positive probability of being large (this notion must be made precise). The question that arises is to describe the law of the tree conditioned of being large, and to say what exceptional event has occurred so that the tree is not typical. A first answer to this question is due to H. Kesten [41] who conditioned a BGW tree to reach height n and look at the limit in distribution when n tends to infinity. There are however other ways of conditioning a tree to be large: conditioning on having many nodes, see S. Janson [36] and references therein; on having many leaves, see I. Kortchemski [45]; on having nodes with large degrees, see X. He [34] and B. Stuffer [59]; on having large population, see R. Abraham and J.-F. Delmas [7];... In most of those cases, the local limit is an infinite random tree with either an infinite spine (the so called Kesten’s tree); or with an infinite backbone, see for example [3, 7]; or with a node of infinite degree (the so called condensation phenomenon), see for example T. Jonsson and S. O. Stefánsson [38], S. Janson [36] or B. Stuffer [58] and the references therein. The approach presented here is mainly based on our previous work [6, 5].

2. BIENAYMÉ-GALTON-WATSON TREES AND EXTINCTION

We intend to give a short introduction to Bienaymé-Galton-Watson (BGW) trees, which is an elementary model for the genealogy of a branching population. The BGW process, which can be defined directly from the BGW tree, describes the evolution of the size of a branching population. Roughly speaking, each individual of a given generation gives birth to a random number of children in the next generation. The distribution probability of the random number of children, called the offspring distribution, is the same for all the individuals. The offspring distribution is called sub-critical, critical or super-critical if its mean is respectively strictly less than 1, equal to 1, or strictly greater than 1.

¹<http://www.math.chalmers.se/~jagers/BranchingHistory.pdf>

We denote by $\mathbb{N} = \mathbb{Z}_+$ the set of non-negative integers and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ the set of positive integers.

We describe more precisely the BGW process. Let ζ be a random variable taking values in \mathbb{N} with distribution $p = (p(n), n \in \mathbb{N})$: $p(n) = \mathbb{P}(\zeta = n)$. We denote by $m = \mathbb{E}[\zeta]$ the mean of ζ . Let $g(r) = \sum_{k \in \mathbb{N}} p(k) r^k = \mathbb{E}[r^\zeta]$ be the generating function of p defined on $[0, 1]$. We recall that the function g is convex, with $g'(1) = \mathbb{E}[\zeta] \in [0, +\infty]$.

The BGW process $Z = (Z_n, n \in \mathbb{N})$ with offspring distribution p describes the evolution of the size of a population issued from a single individual under the following assumptions:

- Z_n is the size of the population at time or generation n . In particular, $Z_0 = 1$.
- Each individual alive at time n dies at generation $n + 1$ and gives birth to a random number of children at time $n + 1$, which is distributed as ζ and independent of the number of children of other individuals.

We can define the process Z more formally. Let $(\zeta_{i,n}; i \in \mathbb{N}, n \in \mathbb{N})$ be independent random variables distributed as ζ . We set $Z_0 = 1$ and, with the convention $\sum_{\emptyset} = 0$, for $n \in \mathbb{N}^*$:

$$(1) \quad Z_n = \sum_{i=1}^{Z_{n-1}} \zeta_{i,n}.$$

Here the random variable $\zeta_{i,n}$ represents the number of offspring of the i -th individual alive at generation n .

The genealogical tree, or BGW tree, associated with the BGW process will be described in Section 2.2 after an introduction to discrete trees given in Section 2.1.

We say that the population is extinct at time n if $Z_n = 0$ (notice that it is then extinct at any further time, and thus $\{Z_n = 0\} \subset \{Z_{n+1} = 0\}$). The extinction event \mathcal{E} corresponds to:

$$(2) \quad \mathcal{E} = \{\exists n \in \mathbb{N} \text{ s.t. } Z_n = 0\} = \bigcup_{n \in \mathbb{N}} \{Z_n = 0\}.$$

We shall compute the extinction probability $\mathbb{P}(\mathcal{E})$ in Section 2.2.2 using the BGW tree setting (we stress that the usual computation relies on the properties of Z_n and its generating function), see Corollary 2.5 and Lemma 2.6 which state that $\mathbb{P}(\mathcal{E})$ is the smallest root of $g(r) = r$ in $[0, 1]$. In particular the extinction is almost sure (a.s.) in the sub-critical case and critical case (unless $p(1) = 1$). The advantage of the proof provided in Section 2.2.2, is that it directly provides the distribution of the super-critical BGW tree and process conditionally on the extinction event, see Lemma 2.6.

In Section 2.2.3, we describe the distribution of the super-critical BGW tree conditionally on the non-extinction event, see Corollary 2.9. In Section 2.3.2, we study asymptotics of the BGW process in the super-critical case, see Theorem 2.15. We prove this result from Kesten and Stigum [42] by following the proof of Lyons, Pemantle and Peres [48], which relies on a change of measure on the genealogical tree (this proof is also clearly exposed in Alsmeyer's lecture notes²). In particular we shall use Kesten's tree which is an elementary multi-type BGW tree. It is defined in Section 2.3.1 and it will play a central role in Chapter 3.

2.1. The set of discrete trees. We recall Neveu's formalism [52] for ordered rooted trees. We set:

$$\mathcal{U} = \bigcup_{n \geq 0} (\mathbb{N}^*)^n$$

the set of finite sequences of positive integers with the convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. The set \mathcal{U} is sometimes called Ulam's tree. For $n \geq 1$ and $u = (u_1, \dots, u_n) \in \mathcal{U}$, we set $H(u) = n$ the height or generation of u and set $H(\emptyset) = 0$. If u and v are two sequences of \mathcal{U} , we denote by uv the concatenation of the two sequences, with the convention that $uv = vu = v$ if $u = \emptyset$. We define a partial order on \mathcal{U} called the

²<http://wwwmath.uni-muenster.de/statistik/lehre/WS1011/SpezielleStochastischeProzesse/>

genealogical order by: $v \preceq u$ if there exists $w \in \mathcal{U}$ such that $u = vw$. We say that v is an ancestor of u and write $v \prec u$ if $v \preceq u$ and $v \neq u$. The set of ancestors of $u \in \mathcal{U}$ is the set:

$$(3) \quad A_u = \{v \in \mathcal{U}; v \prec u\}.$$

We set $\bar{A}_u = A_u \cup \{u\}$ and notice that $\emptyset \in A_u$ unless u is the root. The most recent common ancestor of a subset s of \mathcal{U} , denoted by $\text{MRCA}(s)$, is the unique element v of $\bigcap_{u \in s} \bar{A}_u$ with maximal height. For the MRCA of two nodes, say u and v , we simply write $v \wedge u$ for $\text{MRCA}(\{v, u\})$. We consider the lexicographic order on \mathcal{U} : for $u, v \in \mathcal{U}$, we set $v < u$ if either $v \prec u$ or $(v = wjv'$ and $u = wiu')$ with $w = u \wedge v$, and $j < i$ for some $i, j \in \mathbb{N}^*$.

A tree \mathbf{t} is a subset of \mathcal{U} that satisfies:

- $\emptyset \in \mathbf{t}$,
- If $u \in \mathbf{t}$ and $v \prec u$, then $v \in \mathbf{t}$.
- For every $u \in \mathbf{t}$, there exists $k_u(\mathbf{t}) \in \mathbb{N} \cup \{+\infty\}$ such that, for every $i \in \mathbb{N}^*$, $ui \in \mathbf{t}$ iff $1 \leq i \leq k_u(\mathbf{t})$.

The integer $k_u(\mathbf{t})$ represents the number of offsprings of the node $u \in \mathbf{t}$. The node $u \in \mathbf{t}$ is called a leaf if $k_u(\mathbf{t}) = 0$ and it is said infinite if $k_u(\mathbf{t}) = +\infty$. By convention, we shall set $k_u(\mathbf{t}) = -1$ if $u \notin \mathbf{t}$. The node \emptyset is called the root of \mathbf{t} . A finite tree is represented in Fig. 1.

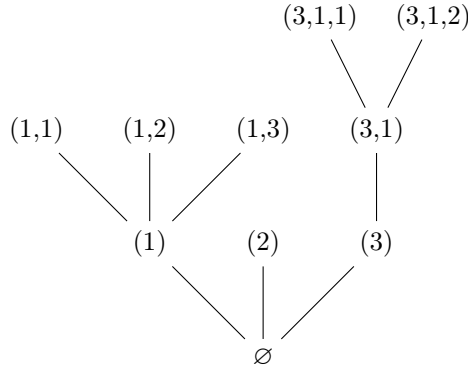


FIGURE 1. A finite tree \mathbf{t} .

We denote by \mathbb{T}_∞ the set of trees and by \mathbb{T} the subset of trees with no infinite node:

$$\mathbb{T} = \{\mathbf{t} \in \mathbb{T}_\infty; k_u(\mathbf{t}) < +\infty, \forall u \in \mathbf{t}\}.$$

Let us stress that the offspring of one individual are ordered; this amounts to consider planar trees. In particular the two trees of Fig. 2 are different.

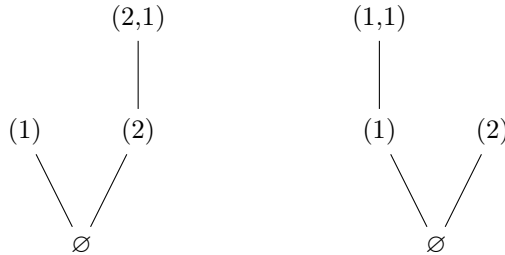


FIGURE 2. Two different planar trees.

Let $\mathbf{t} \in \mathbb{T}_\infty$. We set $\#\mathbf{t} = \text{Card}(\mathbf{t})$ and notice that:

$$(4) \quad \sum_{u \in \mathbf{t}} k_u(\mathbf{t}) = \#\mathbf{t} - 1.$$

The set of its leaves is $\mathcal{L}_0(\mathbf{t}) = \{u \in \mathbf{t}; k_u(\mathbf{t}) = 0\}$. Its height and its width at level $h \in \mathbb{N}$ are respectively defined by:

$$H(\mathbf{t}) = \sup\{H(u); u \in \mathbf{t}\} \quad \text{and} \quad z_h(\mathbf{t}) = \text{Card}(\{u \in \mathbf{t}; (u) = h\});$$

they can be infinite. Notice that $\mathbf{t} \in \mathbb{T}$ if and only if $z_h(\mathbf{t})$ is finite for all $h \in \mathbb{N}$. For $u \in \mathbf{t}$, we define the sub-tree $\mathcal{S}_u(\mathbf{t})$ of \mathbf{t} ‘‘above’’ u as:

$$(5) \quad \mathcal{S}_u(\mathbf{t}) = \{v \in \mathcal{U}; uv \in \mathbf{t}\} = \{v \in \mathcal{U}; u \in \bar{A}_v\}.$$

We will mainly consider trees in \mathbb{T} , but for Section 3.4.3 where we shall consider trees with one infinite node. We denote by \mathbb{T}_0 the countable subset of finite trees and by $\mathbb{T}^{(h)} \subset \mathbb{T}_0$ the subset of finite trees with height less than $h \in \mathbb{N}$:

$$(6) \quad \mathbb{T}_0 = \{\mathbf{t} \in \mathbb{T}; \#\mathbf{t} < +\infty\} \quad \text{and} \quad \mathbb{T}^{(h)} = \{\mathbf{t} \in \mathbb{T}; H(\mathbf{t}) \leq h\}.$$

For $v = (v_k, k \in \mathbb{N}^*) \in (\mathbb{N}^*)^{\mathbb{N}^*}$, we set $\bar{v}_n = (v_1, \dots, v_n)$ for $n \in \mathbb{N}$, with the convention that $\bar{v}_0 = \emptyset$ and $\bar{\mathbf{v}} = \{\bar{v}_n, n \in \mathbb{N}\}$ defines an infinite spine or branch. We denote by \mathbb{T}_1 the subset of trees with only one infinite spine:

$$(7) \quad \mathbb{T}_1 = \{\mathbf{t} \in \mathbb{T}; \text{there exists a unique } v \in (\mathbb{N}^*)^{\mathbb{N}^*} \text{ s.t. } \bar{\mathbf{v}} \subset \mathbf{t}\}.$$

For $h \in \mathbb{N}$, the restriction function r_h from \mathbb{T} to $\mathbb{T}^{(h)}$ is defined by:

$$(8) \quad \forall \mathbf{t} \in \mathbb{T}, r_h(\mathbf{t}) = \{u \in \mathbf{t}; H(u) \leq h\}$$

that is, $r_h(\mathbf{t})$ is the sub-tree of \mathbf{t} obtained by cutting the tree at height h . We endow the set \mathbb{T} with the distance:

$$\delta(\mathbf{t}, \mathbf{t}') = 2^{-\sup\{h \in \mathbb{N}, r_h(\mathbf{t}) = r_h(\mathbf{t}')\}}.$$

It is easy to check that this distance is in fact ultra-metric, that is, for all $\mathbf{t}, \mathbf{t}', \mathbf{t}'' \in \mathbb{T}$:

$$\delta(\mathbf{t}, \mathbf{t}') \leq \max(\delta(\mathbf{t}, \mathbf{t}''), \delta(\mathbf{t}'', \mathbf{t}')).$$

Therefore all the open balls are closed. Furthermore, for $\mathbf{t} \in \mathbb{T}_0$, the singleton $\{\mathbf{t}\}$ is also equal to the open ball centered at \mathbf{t} with radius less than $2^{-H(\mathbf{t})}$. Notice also that for $\mathbf{t} \in \mathbb{T}$ and $h \in \mathbb{N}$, the set:

$$(9) \quad r_h^{-1}(\{r_h(\mathbf{t})\}) = \{\mathbf{t}' \in \mathbb{T}; \delta(\mathbf{t}, \mathbf{t}') \leq 2^{-h}\}$$

is the (open and closed) ball centered at \mathbf{t} with radius h . The restriction functions are contractant with respect to the distance δ and thus continuous.

Let $u \in \mathcal{U}$. Recall $k_u(\mathbf{t})$ is the number of offsprings of the node u in \mathbf{t} , with the convention that $k_u(\mathbf{t}) = -1$ if $u \notin \mathbf{t}$. For $u \in \mathcal{U}$ and $\mathbf{t}, \mathbf{t}' \in \mathbb{T}$ such that $\delta(\mathbf{t}, \mathbf{t}') < 2^{-H(u)}$, we get that $k_u(\mathbf{t}) = k_u(\mathbf{t}')$. This implies that the function $\mathbf{t} \mapsto k_u(\mathbf{t})$ is continuous on \mathbb{T} .

A sequence $(\mathbf{t}_n, n \in \mathbb{N})$ of trees in \mathbb{T} converges to a tree $\mathbf{t} \in \mathbb{T}$ with respect to the distance δ if and only if, for every $h \in \mathbb{N}$, we have $r_h(\mathbf{t}_n) = r_h(\mathbf{t})$ for n large enough, and thus if and only if for all $u \in \mathcal{U}$:

$$\lim_{n \rightarrow +\infty} k_u(\mathbf{t}_n) = k_u(\mathbf{t}) \in \mathbb{N} \cup \{-1\}.$$

We end this section by stating that \mathbb{T} is a Polish metric space (but not compact), that is a complete separable metric space.

Lemma 2.1. *The metric space (\mathbb{T}, δ) is a Polish metric space.*

Proof. Notice that \mathbb{T}_0 , which is countable, is dense in \mathbb{T} as for all $\mathbf{t} \in \mathbb{T}$, the sequence $(r_h(\mathbf{t}), h \in \mathbb{N})$ of elements of \mathbb{T}_0 converges to \mathbf{t} . So the metric space (\mathbb{T}, δ) is separable.

Let $(\mathbf{t}_n, n \in \mathbb{N})$ be a Cauchy sequence in \mathbb{T} . Then for all $h \in \mathbb{N}$, the sequence $(r_h(\mathbf{t}_n), n \in \mathbb{N})$ is a Cauchy sequence in $\mathbb{T}^{(h)}$. Since for $\mathbf{t}, \mathbf{t}' \in \mathbb{T}^{(h)}$, $\delta(\mathbf{t}, \mathbf{t}') \leq 2^{-h}$ implies that $\mathbf{t} = \mathbf{t}'$, we deduce that the sequence $(r_h(\mathbf{t}_n), n \in \mathbb{N})$ is constant for n large enough equal to say \mathbf{t}^h . By continuity of the restriction functions, we deduce that $r_h(\mathbf{t}^{h'}) = \mathbf{t}^h$ for any $h' > h$. This implies that $\mathbf{t} = \bigcup_{h \in \mathbb{N}} \mathbf{t}^h$ is a tree such

that $r_h(\mathbf{t}) = \mathbf{t}^h$ for all $h \in \mathbb{N}$, and that the sequence $(\mathbf{t}_n, n \in \mathbb{N})$ converges to \mathbf{t} . This gives that the metric space (\mathbb{T}, δ) is complete. \square

2.2. Bienaymé-Galton-Watson trees.

2.2.1. Definition. Let $p = (p(n), n \in \mathbb{N})$ be a probability distribution on the set of the non-negative integers and ζ be a random variable with distribution p . Let $g_p(r) = \mathbb{E}[r^\zeta]$, $r \in [0, 1]$, be the generating function of p , $\rho(p)$ its convergence radius and $m(p) = g'_p(1) = \mathbb{E}[\zeta]$ its mean which belongs to $[0, +\infty]$. We will write g , ρ and m for g_p , $\rho(p)$ and $m(p)$ when it is clear from the context. Let $A \subset \mathbb{N}$ be not empty nor reduced to $\{0\}$, and write $\text{GCD}(A)$ for the greatest common divisor of the integers in A . The period of p is defined by:

$$d = \max\{k; \text{supp}(p) \subset k\mathbb{N}\} = \text{GCD}(\text{supp}(p)).$$

We say that p is aperiodic if $d = 1$.

Definition 2.2 (Branching property and BGW tree). *A \mathbb{T} -valued random variable τ is said to satisfy the branching property if for $n \in \mathbb{N}^*$, conditionally on $\{k_\emptyset(\tau) = n\}$, the sub-trees $(\mathcal{S}_1(\tau), \mathcal{S}_2(\tau), \dots, \mathcal{S}_n(\tau))$ are independent and distributed as the original tree τ .*

A \mathbb{T} -valued random variable τ is a BGW tree with offspring distribution p if it satisfies the branching property and the distribution of $k_\emptyset(\tau)$ is p .

It is easy to check that τ is a BGW tree with offspring distribution p if and only if for every $h \in \mathbb{N}^*$ and $\mathbf{t} \in \mathbb{T}^{(h)}$, we have:

$$(10) \quad \mathbb{P}(r_h(\tau) = \mathbf{t}) = \prod_{u \in \mathbf{t}, |u| < h} p(k_u(\mathbf{t})).$$

In particular, the restriction of the distribution of τ on the set \mathbb{T}_0 is given by:

$$(11) \quad \forall \mathbf{t} \in \mathbb{T}_0, \quad \mathbb{P}(\tau = \mathbf{t}) = \prod_{u \in \mathbf{t}} p(k_u(\mathbf{t})).$$

It is easy to check the following lemma. Recall the definition of the BGW process $Z = (Z_h, h \in \mathbb{N})$ given in (1).

Lemma 2.3 (BGW process). *Let τ be a BGW tree. The process $(z_h(\tau), h \in \mathbb{N})$ is distributed as Z .*

The offspring distribution p and the BGW tree are called critical (resp. sub-critical, super-critical) if $m(p) = 1$ (resp. $m(p) < 1$, $m(p) > 1$).

2.2.2. Extinction probability. Let τ be a BGW tree with offspring distribution p . The extinction event of the BGW tree τ is $\mathcal{E}(\tau) = \{\tau \in \mathbb{T}_0\}$, which we shall denote \mathcal{E} when there is no possible confusion. Thanks to Lemma 2.3, this is coherent with Definition (2). We have the following particular cases:

- If $p(0) = 0$, then $\mathbb{P}(\mathcal{E}) = 0$ and a.s. $\tau \notin \mathbb{T}_0$.
- If $p(0) = 1$, then a.s. $\tau = \{\emptyset\}$ and $\mathbb{P}(\mathcal{E}) = 1$.
- If $p(1) = 1$, then $m(p) = 1$ and a.s. the tree $\tau = \bigcup_{n \geq 0} \{1\}^n$, with the convention that $\{1\}^0 = \{\emptyset\}$, is reduced to one infinite spine. In this case $\mathbb{P}(\mathcal{E}) = 0$.
- If $0 < p(0) < 1$ and $p(0) + p(1) = 1$, then $H(\tau) + 1$ is a geometric random variable with parameter $p(0)$ and $\tau = \bigcup_{0 \leq n \leq H(\tau)} \{1\}^n$. In this case $\mathbb{P}(\mathcal{E}) = 1$.

From now on, we shall omit the particular critical case $p(1) = 1$. Let q be the smallest root of the equation $g(r) = r$ in $[0, 1]$ (which exists since $g(1) = 1$ and g is continuous).

Remark 2.4 (Roots of $g(r) = r$). We clearly get that $q = 0$ if and only if $p(0) = 0$. We now suppose that $p(0) > 0$. If $p(0) + p(1) = 1$, then we clearly get $q = 1$.

If $p(0) + p(1) < 1$, we get that g is strictly convex the equation $g(r) = r$ has at most two roots in $[0, 1]$. Since $g(1) = 1$, we get that $q = 1$ if $g'(1) \leq 1$ (in this case the equation $g(r) = r$ has only one root in $[0, 1]$) and, since furthermore $g(0) \geq 0$, we also get that $0 \leq q < 1$ if $g'(1) > 1$, see Fig. 3.

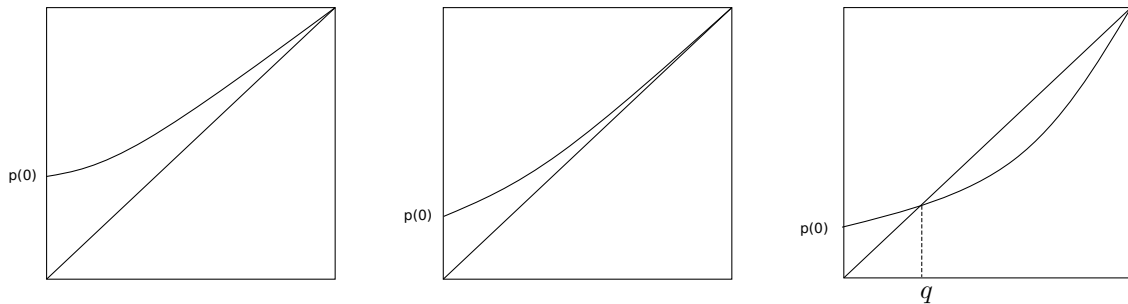


FIGURE 3. Generating function in the sub-critical (left), critical (middle) and super-critical (right) cases.

Using the branching property, we get:

$$\begin{aligned}
 \mathbb{P}(\mathcal{E}) &= \mathbb{P}(\mathcal{E}(\tau)) \\
 &= \sum_{k \in \mathbb{N}} \mathbb{P}(\mathcal{E}(\mathcal{S}_1(\tau)), \dots, \mathcal{E}(\mathcal{S}_k(\tau)) \mid k_{\emptyset}(\tau) = k) p(k) \\
 &= \sum_{k \in \mathbb{N}} \mathbb{P}(\mathcal{E})^k p(k) \\
 &= g(\mathbb{P}(\mathcal{E})).
 \end{aligned}$$

We deduce that $\mathbb{P}(\mathcal{E})$ is a root in $[0, 1]$ of the equation $g(r) = r$. The following corollary is then an immediate consequence of Remark 2.4.

Corollary 2.5 ((Sub-)critical case). *When the offspring distribution p is critical with $p(1) < 1$ or sub-critical, we have a.s. extinction for the associated BGW tree, that is, $\mathbb{P}(\mathcal{E}) = 1$.*

Let p be a super-critical offspring distribution satisfying $p(0) > 0$. In this case we have $0 < q < 1$, and thus $\mathbb{P}(\mathcal{E}) > 0$. For $n \in \mathbb{N}$, we set:

$$(12) \quad \tilde{p}(n) = q^{n-1} p(n).$$

Since $\sum_{n \in \mathbb{N}} \tilde{p}(n) = g(q)/q = 1$, we deduce that $\tilde{p} = (\tilde{p}(n), n \in \mathbb{N})$ is a probability distribution on \mathbb{N} . Since $g_{\tilde{p}}(r) = g(qr)/q$, we deduce that $g'_{\tilde{p}}(1) = g'(q) < 1$. This implies that the offspring distribution \tilde{p} is sub-critical. In particular, if $\tilde{\tau}$ is a BGW tree with offspring distribution \tilde{p} , we have $\mathbb{P}(\mathcal{E}(\tilde{\tau})) = 1$.

Lemma 2.6 (BGW tree conditioned on extinction). *For a super-critical BGW tree τ with offspring distribution p , we have $\mathbb{P}(\mathcal{E}) = q$. Furthermore, if $p(0) > 0$, then conditionally on the extinction event, τ is distributed as a sub-critical BGW tree $\tilde{\tau}$ with offspring distribution \tilde{p} given by (12).*

Proof. If $p(0) = 0$, then we have $\mathbb{P}(\mathcal{E}) = 0$ and $q = 0$. We now assume that $p(0) > 0$. According to Corollary 2.5, a.s. $\tilde{\tau}$ belongs to \mathbb{T}_0 . For $\mathbf{t} \in \mathbb{T}_0$, we have:

$$q \mathbb{P}(\tilde{\tau} = \mathbf{t}) = q \prod_{u \in \mathbf{t}} p(k_u(\mathbf{t})) q^{k_u(\mathbf{t})-1} = q^{1+\sum_{u \in \mathbf{t}} (k_u(\mathbf{t})-1)} \prod_{u \in \mathbf{t}} p(k_u(\mathbf{t})) = \mathbb{P}(\tau = \mathbf{t}),$$

where we used (11) and the definition of \tilde{p} for the first equality and (4) as well as (11) for the last one. We deduce, by summing the previous equality over all finite trees $\mathbf{t} \in \mathbb{T}_0$ that, for any non-negative function \mathcal{H} defined on \mathbb{T}_0 ,

$$\mathbb{E} [\mathcal{H}(\tau) \mathbf{1}_{\{\tau \in \mathbb{T}_0\}}] = q \mathbb{E} [\mathcal{H}(\tilde{\tau})],$$

as $\tilde{\tau}$ is a.s. finite. Taking $\mathcal{H} = 1$, we deduce that $\mathbb{P}(\mathcal{E}(\tau)) = q$. Then we get:

$$\mathbb{E} [\mathcal{H}(\tau) | \mathcal{E}(\tau)] = \mathbb{E} [\mathcal{H}(\tilde{\tau})].$$

Thus, conditionally on the extinction event, τ is distributed as $\tilde{\tau}$. □

We deduce the following corollary on BGW processes.

Corollary 2.7 (BGW process conditioned on extinction). *Let Z be a super-critical BGW process with offspring distribution p satisfying $p(0) > 0$. Conditionally on the extinction event, Z is distributed as a sub-critical BGW process \tilde{Z} with offspring distribution \tilde{p} given by (12).*

2.2.3. *Distribution of the super-critical BGW tree conditionally on the non-extinction event.* Let τ be a super-critical BGW tree with offspring distribution p . We shall present a decomposition of the super-critical BGW tree conditionally on the non-extinction event $\mathcal{E}^c = \{H(\tau) = +\infty\}$. Notice that the event \mathcal{E}^c has positive probability $1 - q$, with q the smallest root of $g(r) = r$ on $[0, 1]$.

We say that $v \in \mathbf{t}$ is a *survivor* in $\mathbf{t} \in \mathbb{T}$ if $\text{Card}(S_v(\mathbf{t})) = +\infty$ and becomes *extinct* otherwise. We define the survivor process $(z_h^s(\mathbf{t}), h \in \mathbb{N})$ by:

$$z_h^s(\mathbf{t}) = \text{Card}(\{u \in \mathbf{t}; H(u) = h \text{ and } u \text{ is a survivor}\}).$$

Notice that the root \emptyset of τ is a survivor with probability $1 - q$. Let S and E denote respectively the numbers of children of the root which are survivors and which become extinct. We define for $r, \ell \in [0, 1]$:

$$G(r, \ell) = \mathbb{E}[r^S \ell^E | \mathcal{E}^c].$$

We have the following lemma.

Lemma 2.8. *Let τ be a super-critical BGW tree with offspring distribution p and let q be the smallest root of $g(r) = r$ on $[0, 1]$. We have for $r, \ell \in [0, 1]$:*

$$(13) \quad G(r, \ell) = \frac{g((1-q)r + q\ell) - g(q\ell)}{1-q}.$$

Proof. Recall $q < 1$. We have:

$$\begin{aligned} \mathbb{E}[r^S \ell^E | \mathcal{E}^c] &= \frac{1}{1-q} \mathbb{E}[r^S \ell^E \mathbf{1}_{\{S \geq 1\}}] \\ &= \frac{1}{1-q} \sum_{n \in \mathbb{N}^*} p(n) \sum_{k=1}^n \binom{n}{k} (1-q)^k r^k q^{n-k} \ell^{n-k} \\ &= \frac{1}{1-q} \sum_{n \in \mathbb{N}^*} p(n) \left(((1-q)r + q\ell)^n - (q\ell)^n \right) \\ &= \frac{g((1-q)r + q\ell) - g(q\ell)}{1-q}, \end{aligned}$$

where we used the branching property and the fact that a BGW tree with offspring distribution p is finite with probability q in the second equality. \square

We consider the following two-type BGW tree $\check{\tau}^s$ distributed as follows:

- Individuals are of type s (for *survivor*) or of type e (for *extinct*).
- The root of $\check{\tau}^s$ is of type s .
- An individual of type e produces only individuals of type e according to the sub-critical offspring distribution \tilde{p} defined by (12).
- An individual of type s produces $S \geq 1$ individuals of type s and E of type e , with generating function $\mathbb{E}[r^S \ell^E] = G(r, \ell)$ given by (13). Furthermore the order of the S individuals of type s and of the E individuals of type e is uniform among the $\binom{E+S}{S}$ possible configurations. Thus the probability for an individual u of type s to have n children and whose children of type s are $\{ui, i \in A\}$, with A a non-empty subset $\{1, \dots, n\}$ of cardinal $\#A$, is (with the convention $0^0 = 1$):

$$(14) \quad p(n)(1-q)^{\#A-1} q^{n-\#A}.$$

This indeed define a probability measure as:

$$\begin{aligned} \sum_{n \in \mathbb{N}^*} \sum_{A \subset \{1, \dots, n\}, A \neq \emptyset} p(n) (1-q)^{\#A-1} q^{n-\#A} &= \sum_{n \in \mathbb{N}^*} \sum_{k=1}^n p(n) \binom{n}{k} (1-q)^{k-1} q^{n-k} \\ &= \sum_{n \in \mathbb{N}^*} p(n) \frac{1-q^n}{1-q} \\ &= \frac{g(1) - g(q)}{1-q} = 1. \end{aligned}$$

Notice that an individual in $\check{\tau}^s$ is a survivor if and only if it is of type s . We write τ^s for the \mathbb{T} -valued random variable defined as $\check{\tau}^s$ when forgetting the types.

Using the branching property, it is easy to deduce the following corollary.

Corollary 2.9 (BGW tree conditioned on non extinction). *Let τ be a super-critical BGW tree with offspring distribution p . Conditionally on \mathcal{E}^c , τ is distributed as τ^s .*

Proof. If $p(0) = 0$, then we have $q = 0$, $\mathbb{P}(\mathcal{E}^c) = 1$ and $\tau^s = \tau$.

We now assume that $q > 0$. We denote by \mathcal{S}_h the set of individuals of τ^s at height h whose type in $\check{\tau}^s$ is s . Because ancestors of an individual of type s are also of type s and that every individual of type s has at least a child of type s , we deduce that $\check{\tau}^s$ truncated at level h is characterized by $r_h(\tau^s)$ and \mathcal{S}_h .

Let $\mathbf{t} \in \mathbb{T}_0$ such that $H(\mathbf{t}) = h$ and $A \subset \{u \in \mathbf{t}, H(u) = h\}$ with $A \neq \emptyset$. Set $n = z_h(\mathbf{t})$. Let $\mathcal{A} = \bigcup_{u \in A} \{v \in \mathcal{U}; v \prec u\}$ be the set of ancestors of elements of A and set $\mathcal{A}^c = r_{h-1}(\mathbf{t}) \setminus \mathcal{A}$. For $u \in \mathcal{A}$, we denote by $k_u^s(\mathbf{t}, A)$ the number of children of u in \mathbf{t} that belong to $\mathcal{A} \cup A$. We have:

$$\begin{aligned} \mathbb{P}(r_h(\tau^s) = \mathbf{t}, \mathcal{S}_h = A) &= \prod_{u \in \mathcal{A}^c} \tilde{p}(k_u(\mathbf{t})) \prod_{u \in A} p(k_u(\mathbf{t})) (1-q)^{k_u^s(\mathbf{t}, A)-1} q^{k_u(\mathbf{t})-k_u^s(\mathbf{t}, A)} \\ &= \left(\prod_{u \in \mathcal{A}^c \cup A} p(k_u(\mathbf{t})) \right) \left(\frac{1-q}{q} \right)^{\sum_{u \in \mathcal{A}} (k_u^s(\mathbf{t}, A)-1)} q^{\sum_{u \in \mathcal{A} \cup A^c} (k_u(\mathbf{t})-1)} \\ &= \mathbb{P}(r_h(\tau) = \mathbf{t}) \left(\frac{1-q}{q} \right)^{\#A-1} q^{n-1} \\ &= \frac{1}{1-q} \mathbb{P}(r_h(\tau) = \mathbf{t}) (1-q)^{\#A} q^{n-\#A}, \end{aligned}$$

where we used (14) for the first equality, Definition (12) of \tilde{p} for the second one and, for the third equality, Formula (4) twice as well as $n = z_h(\mathbf{t})$. Summing the previous equality over all possible choices for A , we get (recall that A is non empty):

$$\begin{aligned} \mathbb{P}(r_h(\tau^s) = \mathbf{t}) &= \sum_A \frac{1}{1-q} \mathbb{P}(r_h(\tau) = \mathbf{t}) (1-q)^{\#A} q^{n-\#A} \\ &= \frac{1}{1-q} \mathbb{P}(r_h(\tau) = \mathbf{t}) \sum_{k=1}^n \binom{n}{k} (1-q)^k q^{n-k} \\ &= \mathbb{P}(r_h(\tau) = \mathbf{t}) \frac{1-q^n}{1-q}. \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} \mathbb{P}(r_h(\tau) = \mathbf{t} | \mathcal{E}^c) &= \frac{\mathbb{P}(r_h(\tau) = \mathbf{t}) - \mathbb{P}(r_h(\tau) = \mathbf{t}, \mathcal{E})}{1 - \mathbb{P}(\mathcal{E})} \\ &= \frac{\mathbb{P}(r_h(\tau) = \mathbf{t}) - \mathbb{P}(r_h(\tau) = \mathbf{t}) q^n}{1-q} = \mathbb{P}(r_h(\tau) = \mathbf{t}) \frac{1-q^n}{1-q}, \end{aligned}$$

where we used the branching property at height h for τ for the second equality. Thus we have obtained that $\mathbb{P}(r_h(\tau^s) = \mathbf{t}) = \mathbb{P}(r_h(\tau) = \mathbf{t} | \mathcal{E}^c)$ for all $\mathbf{t} \in \mathbb{T}_0$, which concludes the proof. \square

In particular, it is easy to deduce from the definition of τ^s , that the survivor process $(z_h^s(\tau), h \in \mathbb{N})$ is conditionally on \mathcal{E}^c a BGW process whose offspring distribution \hat{p} has generating function:

$$g_{\hat{p}}(r) = G(r, 1) = \frac{g((1-q)r + q) - q}{1-q}.$$

The mean of \hat{p} is $g'_{\hat{p}}(1) = g'(1)$ the mean of p . Notice also that $\hat{p}(0) = 0$, so that the BGW process with offspring distribution \hat{p} is super-critical and a.s. does not suffer extinction.

If $p(0) > 0$ (and thus $q > 0$), recall that the BGW tree τ conditionally on the extinction event is a BGW tree with offspring distribution \tilde{p} , whose generating function is:

$$g_{\tilde{p}}(r) = \frac{g(qr)}{q}.$$

We observe that the generating function $g(r)$ of the super-critical offspring distribution p can be recovered from the extinction probability q , the generating functions $g_{\tilde{p}}$ and $g_{\hat{p}}$ of the offspring distribution of the BGW tree conditionally on the extinction event (for $r \leq q$) and of the backbone process (for $r \geq q$):

$$g(r) = qg_{\tilde{p}}\left(\frac{r}{q}\right) \mathbf{1}_{[0,q]}(r) + \left(q + (1-q)g_{\hat{p}}\left(\frac{r-q}{1-q}\right)\right) \mathbf{1}_{(q,1]}(r).$$

We can therefore read from the super-critical generating functions g_p , the sub-critical generating function $g_{\tilde{p}}$ and the super-critical generating function $g_{\hat{p}}$, see Fig. 4.

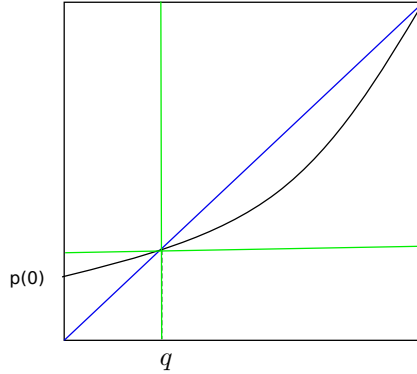


FIGURE 4. In the super-critical case with $p(0) > 0$ (and thus $0 < q < 1$): the generating functions g_p , $g_{\tilde{p}}$ in the lower sub-square and $g_{\hat{p}}$ in the upper sub-square (up to a scaling factor).

2.3. Kesten's tree.

2.3.1. *Definition.* The Kesten's tree is a multi-type Galton-Watson tree, that is a random tree where all individuals reproduce independently of the others, but the offspring distribution depends on the type of the individual. For a probability distribution $p = (p(n), n \in \mathbb{N})$ on \mathbb{N} with finite positive mean m , the corresponding size-biased distribution $p^* = (p^*(n), n \in \mathbb{N})$ is defined by:

$$p^*(n) = \frac{np(n)}{m}.$$

The two-type BGW Kesten's tree $\check{\tau}^*$ associated with the probability distribution p on \mathbb{N} is distributed as follows:

- Individuals are *normal* or *special*.
- The root of $\check{\tau}^*$ is *special*.
- A *normal* individual produces only *normal* individuals according to p .

- A *special* individual produces individuals according to the size-biased distribution p^* (notice that it has always at least one offspring since $p^*(0) = 0$). One of them, chosen uniformly at random, is *special*, the others (if any) are *normal*.

Definition 2.10 (Kesten's tree). *Let p be an offspring distribution with finite positive mean ($m \in (0, +\infty)$). The Kesten's tree τ^* associated with p is the \mathbb{T} -valued random variable defined as $\check{\tau}^*$ when forgetting the types.*

Notice τ^* belongs a.s. to \mathbb{T}_1 , the set of tree with only one infinite spine, if p is sub-critical or critical. In the next lemma we provide a link between the distribution of τ and of τ^* .

Lemma 2.11 (Relation between the BGW tree and the Kesten's tree). *Let p be an offspring distribution with finite positive mean, τ be a BGW tree with offspring distribution p and τ^* be a Kesten's tree associated with p . For all $n \in \mathbb{N}$, $\mathbf{t} \in \mathbb{T}_0$ and $v \in \mathbf{t}$ such that $H(\mathbf{t}) = H(v) = n$, we have:*

$$(15) \quad \begin{aligned} \mathbb{P}(r_n(\tau^*) = \mathbf{t}, v \text{ is special}) &= \frac{1}{m^n} \mathbb{P}(r_n(\tau) = \mathbf{t}), \\ \mathbb{P}(r_n(\tau^*) = \mathbf{t}) &= \frac{z_n(\mathbf{t})}{m^n} \mathbb{P}(r_n(\tau) = \mathbf{t}). \end{aligned}$$

Proof. Notice that if u is special, then the probability that it has k_u children and ui is special (with i given and $1 \leq i \leq k_u$) is just $p^*(k_u)k_u^{-1} = p(k_u)/m$. Let $n \in \mathbb{N}$, $\mathbf{t} \in \mathbb{T}^{(n)}$ and $v \in \mathbf{t}$ such that $H(\mathbf{t}) = H(v) = n$. Using (10), we have:

$$\mathbb{P}(r_n(\tau^*) = \mathbf{t}, v \text{ is special}) = \prod_{u \in \mathbf{t} \setminus A_v, H(u) < n} p(k_u(\mathbf{t})) \prod_{u \in A_v} \frac{p(k_u(\mathbf{t}))}{m} = \frac{1}{m^n} \mathbb{P}(r_n(\tau) = \mathbf{t}).$$

Since there is only one special element of \mathbf{t} at level n among the $z_n(\mathbf{t})$ elements of \mathbf{t} at level n , summing the previous equality over v gives (15). \square

We suppose that $m \in (0, +\infty)$. We consider the filtration $\mathcal{F} = (\mathcal{F}_n, n \in \mathbb{N})$ generated by τ : $\mathcal{F}_n = \sigma(r_n(\tau))$ and the normalized BGW process $(W_n, n \in \mathbb{N})$ defined by:

$$(16) \quad W_n = \frac{z_n(\tau)}{m^n}.$$

Notice that $W_0 = 1$. If necessary, we shall write $W_n(\mathbf{t}) = z_n(\mathbf{t})/m^n$ to stress the dependence in \mathbf{t} .

Corollary 2.12 (The martingale (W_n)). *Let p be an offspring distribution with finite positive mean. The process $(W_n, n \in \mathbb{N})$ is a non-negative martingale adapted to the filtration \mathcal{F} .*

Proof. Let P and P^* denote respectively the distribution on \mathbb{T} of a BGW tree τ with offspring distribution p and Kesten's tree τ^* associated with p . We deduce from Lemma 2.11 that for all $n \geq 0$:

$$(17) \quad dP_{|\mathcal{F}_n}^*(\mathbf{t}) = W_n(\mathbf{t}) dP_{|\mathcal{F}_n}(\mathbf{t}).$$

This implies that $(W_n, n \geq 0)$ is a non-negative P -martingale adapted to the filtration \mathcal{F} . \square

2.3.2. Asymptotics of the BGW process. Let τ be a BGW tree with offspring distribution p having finite positive mean $m = g'(1)$ and $p(1) < 1$. Recall the renormalized BGW process $(W_n, n \in \mathbb{N})$ defined by (16) and q the smallest root of $g(r) = r$ in $[0, 1]$.

Lemma 2.13 (On $W = \lim_{n \rightarrow \infty} W_n$). *Let p be an offspring distribution with finite positive mean. The sequence $(W_n, n \in \mathbb{N})$ converges a.s. to a random variable W such that $\mathbb{E}[W] \leq 1$ and $\mathbb{P}(W = 0) \in \{q, 1\}$.*

Proof. According to Corollary 2.12, $(W_n, n \in \mathbb{N})$ is a non-negative martingale. Thanks to the convergence theorem for martingales, see Theorem 4.2.10 in [25], we get that it converges a.s. to a non-negative random variable W such that $\mathbb{E}[W] \leq 1$.

By decomposing τ with respect to the children of the root, we get:

$$W_n(\tau) = \frac{1}{m} \sum_{i=1}^{k_\emptyset(\tau)} W_{n-1}(\mathcal{S}_i(\tau)).$$

The branching property implies that conditionally on $k_\emptyset(\tau)$, the random trees $\mathcal{S}_i(\tau)$, $1 \leq i \leq k_\emptyset(\tau)$ are independent and distributed as τ . In particular, $(W_n(\mathcal{S}_i(\tau)), n \in \mathbb{N})$ converges a.s. to a limit, say W^i , where $(W^i, i \in \mathbb{N}^*)$ are independent non-negative random variables distributed as W and independent of $k_\emptyset(\tau)$. By taking the limit as n goes to infinity, we deduce that a.s.:

$$W = \frac{1}{m} \sum_{i=1}^{k_\emptyset(\tau)} W^i.$$

This implies that:

$$\mathbb{P}(W = 0) = \sum_{n \in \mathbb{N}} p(n) \mathbb{P}(W^1 = 0, \dots, W^n = 0) = g(\mathbb{P}(W = 0)).$$

This implies that $\mathbb{P}(W = 0)$ is a non-negative solution of $g(r) = r$ and so belongs to $\{q, 1\}$. \square

Remark 2.14 (The (sub-)critical case). When the offspring distribution p is critical with $p(1) < 1$ or sub-critical, we have $q = 0$, a.s. $W_n = 0$ for n large enough, and thus a.s. $W = 0$.

We aim now to compute $\mathbb{P}(W = 0)$ in the super-critical case. The following result goes back to Kesten and Stigum [42] and we present the proof by Lyons, Pemantle and Peres [48]. Recall that ζ is a random variable with distribution p . We use the notation $\log^+(r) = \max(\log(r), 0)$.

Theorem 2.15 (The $L \log L$ condition). *Let p be a super-critical offspring distribution with finite mean ($m \in (1, +\infty)$). Then we have:*

$$\mathbb{P}(W = 0) = q \quad \text{if and only if} \quad \mathbb{E}[\zeta \log^+(\zeta)] < +\infty.$$

In particular, we also have that $\mathbb{P}(W = 0) = 1$ if and only if $\mathbb{E}[\zeta \log^+(\zeta)]$ is infinite.

Remark 2.16 (Exponential growth of the population in the super-critical case). Assume that p is super-critical and satisfy the $L \log L$ condition. Since $\mathcal{E} \subset \{W = 0\}$ and $\mathbb{P}(\mathcal{E}) = q$, we deduce that on the survival event \mathcal{E}^c a.s. $\lim_{n \rightarrow +\infty} z_n(\tau)/m^n = W > 0$. (On the extinction event \mathcal{E} , we have that a.s. $z_n(\tau) = 0$ for n large.) So, a.s. on the survival event, the population size at level n behaves like a positive finite random constant times m^n .

Proof of Theorem 2.15. We use notations from the proof of Corollary 2.12: \mathbb{P} and \mathbb{P}^* denote respectively the distribution of a BGW tree τ with offspring distribution p and of a Kesten's tree τ^* associated with p . According to (17), for all $n \geq 0$:

$$d\mathbb{P}^*_{|\mathcal{F}_n} = W_n d\mathbb{P}_{|\mathcal{F}_n}.$$

This implies that $(W_n, n \geq 0)$ converges \mathbb{P} -a.s. (this is already in Lemma 2.13) and \mathbb{P}^* -a.s. to W taking values in $[0, +\infty]$. According to Theorem 4.3.3 in [25], we get that for any measurable subset B of \mathbb{T} :

$$\mathbb{P}^*(B) = \mathbb{E}[W \mathbf{1}_B] + \mathbb{P}^*(B, W = +\infty).$$

Taking $B = \Omega$ in the previous equality gives:

$$(18) \quad \begin{cases} \mathbb{E}[W] = 1 \Leftrightarrow \mathbb{P}^*(W = +\infty) = 0 & \text{and} \\ \mathbb{P}(W = 0) = 1 \Leftrightarrow \mathbb{P}^*(W = +\infty) = 1. \end{cases}$$

So we shall study the behavior of W under \mathbb{P}^* , which turns out to be (almost) elementary. We first use a similar description as (1) to describe $(z_n(\tau^*), n \in \mathbb{N})$.

Recall that ζ is a random variable with distribution p . Notice that $p^*(0) = 0$, and let Y be a random variable such that $Y + 1$ has distribution p^* . Under \mathbb{P} , let $(\zeta_{i,n}; i \in \mathbb{N}, n \in \mathbb{N})$ be independent random variables distributed as ζ , and let $(Y_n, n \in \mathbb{N}^*)$ be independent random variables distributed as Y and independent of $(\zeta_{i,n}; i \in \mathbb{N}, n \in \mathbb{N})$. We set $Z_0^* = 0$ and for $n \in \mathbb{N}^*$:

$$Z_n^* = Y_n + \sum_{i=1}^{Z_{n-1}^*} \zeta_{i,n},$$

with the convention that $\sum_{\emptyset} = 0$. In particular $(Z_n^*, n \in \mathbb{N})$ is under \mathbb{P} a BGW process with immigration with offspring distribution p and immigration distributed as Y . By construction $(Z_n^* + 1, n \in \mathbb{N})$ under \mathbb{P} is distributed as $(z_n(\tau^*), n \in \mathbb{N})$ under \mathbb{P}^* . We deduce that $(W_n, n \in \mathbb{N})$ is under \mathbb{P}^* distributed as $(W_n^* + m^{-n}, n \in \mathbb{N})$ under \mathbb{P} , with $W_n^* = Z_n^*/m^n$.

Let $(X_n, n \in \mathbb{N})$ be random variables distributed as a non-negative random variable X . We recall the following result, which can be deduced from the Borel-Cantelli lemma using that $\sum_{n \in \mathbb{N}^*} \mathbb{P}(X_n/n \geq \varepsilon) = \mathbb{E}[\lfloor X/\varepsilon \rfloor]$ for $\varepsilon > 0$. We have:

$$(19) \quad \mathbb{E}[X] < +\infty \Rightarrow \text{a.s.} \quad \lim_{n \rightarrow +\infty} \frac{X_n}{n} = 0.$$

Furthermore, if the random variables $(X_n, n \in \mathbb{N})$ are independent, then:

$$(20) \quad \mathbb{E}[X] = +\infty \Rightarrow \text{a.s.} \quad \limsup_{n \rightarrow +\infty} \frac{X_n}{n} = +\infty.$$

We consider the case: $\mathbb{E}[\zeta \log^+(\zeta)] < +\infty$. This implies that $\mathbb{E}[\log^+(Y)] < +\infty$. And according to (19), we deduce that for $\varepsilon > 0$, \mathbb{P}^* -a.s. $Y_n \leq e^{n\varepsilon}$ for n large enough. Denote by \mathcal{Y} the σ -field generated by $(Y_n, n \in \mathbb{N}^*)$ and by $(\mathcal{F}_n^*, n \in \mathbb{N})$ the filtration generated by $(W_n^*, n \in \mathbb{N})$. Using the branching property, it is easy to get:

$$\mathbb{E}[W_n^* | \mathcal{Y}, \mathcal{F}_{n-1}^*] = \frac{1}{m^n} \mathbb{E}[Z_n^* | \mathcal{Y}, \mathcal{F}_{n-1}^*] = \frac{Z_{n-1}^*}{m^{n-1}} + \frac{Y_n}{m^n} \geq W_{n-1}^*.$$

We deduce that $(W_n^*, n \in \mathbb{N})$ is a non-negative sub-martingale with respect to $\mathbb{P}(\cdot | \mathcal{Y})$, that is, with respect to the filtration $(\mathcal{Y} \vee \mathcal{F}_n^*, n \in \mathbb{N})$. We also obtain:

$$\mathbb{E}[W_n^* | \mathcal{Y}] = \sum_{k=1}^n \frac{Y_k}{m^k} \leq \sum_{k=1}^{+\infty} \frac{Y_k}{m^k}.$$

For $\varepsilon > 0$, we have that \mathbb{P} -a.s. $Y_n \leq e^{n\varepsilon}$ for n large enough. We deduce that \mathbb{P} -a.s. we have $\sup_{n \in \mathbb{N}} \mathbb{E}[W_n^* | \mathcal{Y}] < +\infty$, that is, the sub-martingale $(W_n^*, n \in \mathbb{N})$ is bounded in $L^1(\mathbb{P}(\cdot | \mathcal{Y}))$. Adapting the proof of Theorem 4.2.10 in [25] with respect to the probability $\mathbb{P}(\cdot | \mathcal{Y})$, we get that the non-negative sub-martingale $(W_n^*, n \in \mathbb{N})$ converges $\mathbb{P}(\cdot | \mathcal{Y})$ -a.s. to a finite limit. Since $(W_n^* + m^{-n}, n \in \mathbb{N})$ is distributed as $(W_n, n \in \mathbb{N})$ under \mathbb{P}^* , we get that \mathbb{P}^* -a.s. W is finite. Use the first part of (18) to deduce that $\mathbb{E}[W] = 1$. Since $\mathbb{P}(W = 0) \in \{q, 1\}$, see Lemma 2.13, we get that $\mathbb{P}(W = 0) = q$.

We consider the case: $\mathbb{E}[\zeta \log^+(\zeta)] = +\infty$. According to (20), we deduce that for any $\varepsilon > 0$, a.s. $Y_n \geq e^{n/\varepsilon}$ for infinitely many n . Since $Z_n^* \geq Y_n$ and $(W_n^* + m^{-n}, n \in \mathbb{N})$ is distributed as $(W_n, n \in \mathbb{N})$ under \mathbb{P}^* , we deduce that \mathbb{P}^* -a.s. $W_n \geq e^{n(-\log(m)+1/\varepsilon)}$ for infinitely many n . Since the sequence $(W_n, n \in \mathbb{N})$ converges \mathbb{P}^* -a.s. to W taking values in $[0, +\infty]$, we deduce, by taking $\varepsilon > 0$ small enough that \mathbb{P}^* -a.s. $W = +\infty$. Use the second part of (18) to deduce that \mathbb{P} -a.s. $W = 0$. \square

3. LOCAL LIMITS OF BIENAYMÉ-GALTON-WATSON TREES

There are many kinds of limits that can be considered in order to study large trees, among them are the local limits and the scaling limits. The local limits look at the trees up to an arbitrary fixed height and therefore only sees what happen at a finite distance from the root. One can also look at the local limit near a node chosen at random instead of near the root. Scaling limits consider sequences of trees where the branches are scaled by some factor so that all the nodes remain at finite distance from the root. These scaling limits, which lead to the so-called *continuum random trees* where the branches have infinitesimal length, have been intensively studied in recent years, see [12, 22, 23], see also [31] for scaling limits of other random trees and [46] for condensation phenomenon. Other limits have also been considered, see for example [27] for a convergence to dendrons with respect to sampling procedures.

We will focus in this lecture only on local limits of critical or sub-critical BGW trees conditioned on being large. The most famous type of such a conditioning is Kesten's theorem which states that critical or sub-critical BGW trees conditioned on reaching large heights converge to the Kesten's tree

which is a (two-type) BGW tree with a unique infinite spine. This result is recalled in Theorem 3.1. In order to consider other conditionings, we shall give in Section 3.1, see Proposition 3.3, an elementary characterization of the local convergence which is the key ingredient of the method presented here when the limit is the Kesten's tree.

All the conditionings we shall consider can be stated in terms of a functional $A(\mathbf{t})$ of the tree \mathbf{t} and the events we condition on are either of the form $\{A(\tau) \geq n\}$ or $\{A(\tau) = n\}$, with n large. Usually handling the latter conditioning is more delicate. In Section 3.2, we give general assumptions on A so that a **critical** BGW tree conditioned on such an event converges as n goes to infinity, in distribution to the Kesten's tree, see our main result, Theorem 3.7. We then apply this result in Section 3.3 by considering, in the critical case, the following functional:

- 3.3.1 : the height of the tree (Kesten's theorem);
- 3.3.2 : the maximal out-degree;
- 3.3.3 : the Horton-Strahler number;
- 3.3.4 : the largest generation;
- 3.3.5 : the total progeny;
- 3.3.6 : the number of leaves;
- 3.3.7 : and more generally the number of nodes with given out-degree.

Let us stress that in the critical case with finite variance, conditioning on the total population size (for simplicity), the number of leaves is asymptotically equal to the population size times the probability to have zero children, and furthermore the fluctuations are Gaussian, see [37, 60]. It is thus not a surprise that conditioning by the total size or the number of leaves give the same local limit. The conditioning on the total progeny or the number of leaves where already known, but usually under stronger hypothesis on the offspring distribution (higher moments or tail conditions). We stress out that in Theorem 3.7 no further assumptions are needed in the critical case than the non-degeneracy condition $p(1) < 1$.

The material of this section is mainly extracted from [6].

The **sub-critical** case is similar with the functional $A = H$ given by the height, but it is much more involved otherwise and we only present here some results in Section 3.4 without any proofs. All the proofs can be found in [5]. In the sub-critical case, when conditioning on the number of nodes with given out-degree, two cases may appear. In the so-called generic case presented in Section 3.4.2, the limiting tree is still a Kesten's tree but with a modified offspring distribution. In the non-generic case, Section 3.4.3, a condensation phenomenon occurs: intuitively a node that stays at a bounded distance from the root has more and more offsprings as n goes to infinity; and in the limit, the tree has a (unique) node with infinitely many offsprings. This phenomenon has first been pointed out in [38] and then in [36] when conditioning on the total progeny. We end this chapter by giving in Section 3.4.4 a characterization of generic and non-generic offspring distributions, which provides non intuitive behavior.

We eventually give further extensions in Section 3.5.

Take-out message: Most (but not all) of the usual conditionings for a critical BGW tree to be large yield the same local limit given by the Kesten's tree. The picture in the sub-critical case is more complex, with possibly different local limits: Kesten's trees with modified offspring distribution (depending on the conditioning) or random trees with condensation. This latter case might appear in particular when the offspring distribution has an heavy tail.

3.1. The topology of local convergence.

3.1.1. *Kesten's theorem.* We work on the set \mathbb{T} of discrete trees with no infinite node, introduced in Section 2.1. Recall that \mathbb{T}_0 , resp. $\mathbb{T}^{(h)}$, denotes the subset of \mathbb{T} of finite trees, resp. of trees with height less than h , see (6). Recall that the restriction function r_h from \mathbb{T} to $\mathbb{T}^{(h)}$ is defined in (8). When a sequence of random trees $(T_n, n \in \mathbb{N})$ converges in distribution with respect to the distance δ (also

called the local topology) toward a random tree T , we shall write:

$$(21) \quad T_n \xrightarrow{(d)} T.$$

According to (9), the fact that all the open balls are closed, and the Portmanteau theorem (see [16] Theorem 2.1), we deduce that if (21) then:

$$(22) \quad \forall h \in \mathbb{N}, \forall \mathbf{t} \in \mathbb{T}^{(h)}, \quad \lim_{n \rightarrow +\infty} \mathbb{P}(r_h(T_n) = \mathbf{t}) = \mathbb{P}(r_h(T) = \mathbf{t}).$$

Conversely, since δ is an ultra-metric, we get that the intersection of two balls is a ball (possibly empty). Thus the set of balls and the empty set is a π -system. We deduce from Theorem 2.3 in [16] that (22) implies (21). Thus (22) and (21) are equivalent.

Convergence in distribution for the local topology appears in the following Kesten's theorem. Recall that the distribution of the Kesten's tree is given in Definition 2.10.

Theorem 3.1 (Kesten [41]). *Let p be a critical or sub-critical offspring distribution. Let τ be a BGW tree with offspring distribution p and τ^* be a Kesten's tree associated with p . For every non-negative integer n , let τ_n be a random tree distributed as τ conditionally on $\{H(\tau) \geq n\}$. Then we have:*

$$\tau_n \xrightarrow{(d)} \tau^*.$$

This theorem is stated in [41] with an additional second moment condition. The proof of the theorem stated as above is due to Janson [36]. We will give a proof of that theorem in Section 3.3.1 as an application of a more general result, see Theorem 3.7 in the critical case and Theorem 3.11 in the sub-critical case.

3.1.2. *A characterization of the convergence in distribution.* Recall that for a tree $\mathbf{t} \in \mathbb{T}$, we denote by $\mathcal{L}_0(\mathbf{t})$ the set of its leaves. Let $\mathbf{t} \in \mathbb{T}$ be a tree and $x \in \mathcal{L}_0(\mathbf{t})$ be a leaf of the tree \mathbf{t} . For $\mathbf{t}' \in \mathbb{T}$ another tree, we denote by $\mathbf{t} \otimes_x \mathbf{t}'$ the tree obtained by grafting the tree \mathbf{t}' on the leaf x of the tree \mathbf{t} , that is:

$$\mathbf{t} \otimes_x \mathbf{t}' = \mathbf{t} \cup \{xu, u \in \mathbf{t}'\}.$$

For τ a BGW tree with offspring distribution p , we deduce from (11) the following useful formula:

$$(23) \quad \mathbb{P}(\tau = \mathbf{t} \otimes_x \mathbf{t}') = \frac{1}{p(0)} \mathbb{P}(\tau = \mathbf{t}) \mathbb{P}(\tau = \mathbf{t}').$$

We denote by $\mathbb{T}(\mathbf{t}, x)$ the set of all trees obtained by grafting some tree on the leaf $x \in \mathcal{L}_0(\mathbf{t})$ of $\mathbf{t} \in \mathbb{T}$:

$$\mathbb{T}(\mathbf{t}, x) = \{\mathbf{t} \otimes_x \mathbf{t}', \mathbf{t}' \in \mathbb{T}\}.$$

Recall that the maps $\mathbf{t} \mapsto k_u(\mathbf{t})$ are continuous for all $u \in \mathcal{U}$. We deduce that the set $\mathbb{T}(\mathbf{t}, x)$ is closed in \mathbb{T} as $\mathbf{s} \in \mathbb{T}(\mathbf{t}, x)$ if and only if $k_u(\mathbf{s}) = k_u(\mathbf{t})$ for all $u \in \mathbf{t} \setminus \{x\}$. We shall see in Lemma 3.6 below that it is also open.

Computations of the probability of BGW trees (or Kesten's tree) to belong to such sets are very easy and lead to simple formulas. For example, we have for τ a BGW tree with offspring distribution p , and all finite tree $\mathbf{t} \in \mathbb{T}_0$ and leaf $x \in \mathcal{L}_0(\mathbf{t})$:

$$(24) \quad \mathbb{P}(\tau = \mathbf{t}) = \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x), k_x(\tau) = 0) = p(0) \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x)).$$

The next lemma is another example of the simplicity of the formulas.

Lemma 3.2 (Another relation between the BGW tree and the Kesten's tree). *Let p be an offspring distribution with finite positive mean. Let τ be a BGW tree with offspring distribution p and let τ^* be a Kesten's tree associated with p . Then we have, for all finite tree $\mathbf{t} \in \mathbb{T}_0$ and leaf $x \in \mathcal{L}_0(\mathbf{t})$:*

$$(25) \quad \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) = \frac{1}{m^{H(x)}} \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x)).$$

In the particular case of a critical offspring distribution ($m = 1$), we get for all $\mathbf{t} \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{t})$:

$$\mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) = \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x)).$$

However, we have $\mathbb{P}(\tau \in \mathbb{T}_0) = 1$ if $p(1) < 1$, and $\mathbb{P}(\tau^* \in \mathbb{T}_1) = 1$, with \mathbb{T}_1 the set of trees that have one and only one infinite spine, see (7). (Notice that, when $p(1) = 1$, the tree $\tau = \tau^*$ is reduced to one spine.)

Proof. Let $\mathbf{t} \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{t})$. If $\tau^* \in \mathbb{T}(\mathbf{t}, x)$, then the node x must be a special node in τ^* as the tree \mathbf{t} is finite whereas the tree τ^* is a.s. infinite. Therefore, using arguments similar to those used in the proof of Lemma 2.11, we have:

$$\mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) = \prod_{u \in \mathbf{t} \setminus (A_x \cup \{x\})} p(k_u(\mathbf{t})) \prod_{u \in A_x} \frac{p(k_u(\mathbf{t}))}{m} = \frac{1}{m^{H(x)}} \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x)).$$

□

The following key characterization of the local convergence is proved in Section 3.1.3.

Proposition 3.3 (Characterization of the local convergence in $\mathbb{T}_0 \cup \mathbb{T}_1$). *Let $(T_n, n \in \mathbb{N})$ and T be random trees taking values in the set $\mathbb{T}_0 \cup \mathbb{T}_1$. Then the sequence $(T_n, n \geq 0)$ converges in distribution (for the local topology) to T if and only if the two following conditions hold:*

- (i) *For every finite tree $\mathbf{t} \in \mathbb{T}_0$, we have $\lim_{n \rightarrow +\infty} \mathbb{P}(T_n = \mathbf{t}) = \mathbb{P}(T = \mathbf{t})$.*
- (ii) *For every $\mathbf{t} \in \mathbb{T}_0$ and leaf $x \in \mathcal{L}_0(\mathbf{t})$, we have $\liminf_{n \rightarrow +\infty} \mathbb{P}(T_n \in \mathbb{T}(\mathbf{t}, x)) \geq \mathbb{P}(T \in \mathbb{T}(\mathbf{t}, x))$.*

3.1.3. *Proof of Proposition 3.3.* We denote by \mathcal{F} the subclass of Borel sets of \mathbb{T} :

$$\mathcal{F} = \{\{\mathbf{t}\}, \mathbf{t} \in \mathbb{T}_0\} \cup \{\mathbb{T}(\mathbf{t}, x), \mathbf{t} \in \mathbb{T}_0, x \in \mathcal{L}_0(\mathbf{t})\} \cup \{\emptyset\}.$$

Lemma 3.4. *The family \mathcal{F} is a π -system.*

Proof. Recall that a non-empty family of sets is a π -system if it is stable under finite intersection. For every $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{T}_0$ and every $x_1 \in \mathcal{L}_0(\mathbf{t}_1)$, $x_2 \in \mathcal{L}_0(\mathbf{t}_2)$, we have:

$$(26) \quad \mathbb{T}(\mathbf{t}_1, x_1) \cap \mathbb{T}(\mathbf{t}_2, x_2) = \begin{cases} \mathbb{T}(\mathbf{t}_1, x_1) & \text{if } \mathbf{t}_1 = \mathbf{t}_2 \otimes_{x_2} \mathbf{t}'_1 \text{ and } x_1 \wedge x_2 = x_2, \\ \mathbb{T}(\mathbf{t}_2, x_2) & \text{if } \mathbf{t}_2 = \mathbf{t}_1 \otimes_{x_1} \mathbf{t}'_2 \text{ and } x_1 \wedge x_2 = x_1, \\ \{\mathbf{t}_1 \cup \mathbf{t}_2\} & \text{if } \mathbf{t}_1 = \mathbf{t} \otimes_{x_2} \mathbf{t}'_1, \mathbf{t}_2 = \mathbf{t} \otimes_{x_1} \mathbf{t}'_2 \text{ and } x_1 \neq x_2, \\ \emptyset & \text{in the other cases.} \end{cases}$$

Notice that the case $(\mathbf{t}_1, x_1) = (\mathbf{t}_2, x_2)$ belongs to the first two cases. An instance of the third case is represented in Fig. 5.

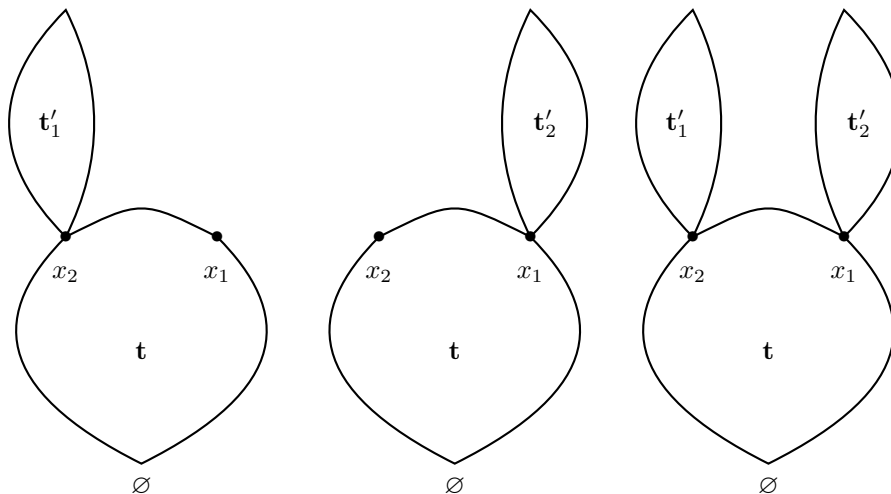


FIGURE 5. Example of the third case in Equation (26) where $x_1 \neq x_2$, $\mathbf{t}_1 = \mathbf{t} \otimes_{x_2} \mathbf{t}'_1$ (on the left), $\mathbf{t}_2 = \mathbf{t} \otimes_{x_1} \mathbf{t}'_2$ (in the middle) and with $\mathbf{t}_1 \cup \mathbf{t}_2$ (on the right).

Thus \mathcal{F} is stable under finite intersection, and is thus a π -system. \square

Remark 3.5. The third case in Equation (26) was omitted in the original paper [6] where only a special case was considered.

We denote by $\mathcal{F}' = \{A \cap (\mathbb{T}_0 \cup \mathbb{T}_1) : A \in \mathcal{F}\}$ the trace of \mathcal{F} on $\mathbb{T}_0 \cup \mathbb{T}_1$.

Lemma 3.6. *All the elements of \mathcal{F} are open and \mathcal{F}' is an open neighborhood system in $\mathbb{T}_0 \cup \mathbb{T}_1$.*

Proof. We first check that all the elements of \mathcal{F} are open. For $\mathbf{t} \in \mathbb{T}$ and $\varepsilon > 0$, let $B(\mathbf{t}, \varepsilon)$ be the ball (which is open and closed) centered at \mathbf{t} with radius ε . If $\mathbf{t} \in \mathbb{T}_0$, we have $\{\mathbf{t}\} = B(\mathbf{t}, 2^{-h})$ for every $h > H(\mathbf{t})$, thus $\{\mathbf{t}\}$ is open. Moreover, for some fixed $x \in \mathcal{L}_0(\mathbf{t})$, for every $\mathbf{s} \in \mathbb{T}(\mathbf{t}, x)$, we have:

$$B(\mathbf{s}, 2^{-H(\mathbf{t})-1}) \subset \mathbb{T}(\mathbf{t}, x),$$

which proves that $\mathbb{T}(\mathbf{t}, x)$ is also open.

We check that \mathcal{F}' is a neighborhood system: that is, since all the elements of \mathcal{F} are open, for all $\mathbf{t} \in \mathbb{T}_0 \cup \mathbb{T}_1$ and $\varepsilon > 0$, there exists an element of \mathcal{F} , say A' , which is a subset of $B(\mathbf{t}, \varepsilon)$ and which contains \mathbf{t} .

If $\mathbf{t} \in \mathbb{T}_0$, it is enough to consider $A' = \{\mathbf{t}\}$.

Let us suppose that $\mathbf{t} \in \mathbb{T}_1$. Let $(u_n, n \in \mathbb{N}^*)$ be the infinite spine of \mathbf{t} so that $\bar{u}_n = u_1 \dots u_n \in \mathbf{t}$ for all $n \in \mathbb{N}^*$. Let $n \in \mathbb{N}^*$ such that $2^{-n} < \varepsilon$ and set \mathbf{t}' the tree obtained by cutting the spine of \mathbf{t} at height n : $\mathbf{t}' = \{v \in \mathbf{t}; \bar{u}_n \notin A_v\}$. Notice that $\bar{u}_n \in \mathbf{t}'$, and set $A' = \mathbb{T}(\mathbf{t}', \bar{u}_n)$ so that A' belongs to the π -system \mathcal{F} . We get $\mathbf{t} \in A' \subset B(\mathbf{t}, \varepsilon)$. Thus, the trace of \mathcal{F} on $\mathbb{T}_0 \cup \mathbb{T}_1$ is a neighborhood system. \square

We are now ready to prove Proposition 3.3, following ideas of the proof of Theorems 2.2 and 2.3 of [16]. As \mathcal{F} is countable, so is its trace \mathcal{F}' on $\mathbb{T}_0 \cup \mathbb{T}_1$. We deduce from Lemma 3.6 that any open set G of $\mathbb{T}_0 \cup \mathbb{T}_1$ can be written as a countable union of some elements of \mathcal{F}' , say $(A_i, i \in \mathbb{N})$. For any $\varepsilon > 0$, there exists n_0 such that $\mathbb{P}(T \in G) \leq \varepsilon + \mathbb{P}(T \in \bigcup_{i \leq n_0} A_i)$. Without loss of generality, we can assume that no A_i is a subset of A_j for $1 \leq i, j \leq n_0$ and $i \neq j$. According to (26), we get that $A_i \cap A_j$ is either empty or reduced to a singleton. We then deduce from the inclusion-exclusion formula that there exists $n_1 \leq n_0$, $\mathbf{t}_j \in \mathbb{T}_0$, $x_j \in \mathcal{L}_0(\mathbf{t}_j)$ for $j \leq n_1$, and $n_2 < \infty$, $\mathbf{t}_\ell \in \mathbb{T}_0$, $\alpha_\ell \in \mathbb{Z}$ for $\ell \leq n_2$ such

that, for any random variable T' taking values in $\mathbb{T}_0 \cup \mathbb{T}_1$:

$$\mathbb{P}\left(T' \in \bigcup_{i \leq n_0} A_i\right) = \sum_{j \leq n_1} \mathbb{P}(T' \in \mathbb{T}(\mathbf{t}_j, x_j)) + \sum_{\ell \leq n_2} \alpha_\ell \mathbb{P}(T' = \mathbf{t}_\ell).$$

We deduce, assuming that (i) and (ii) of Proposition 3.3 hold, that:

$$\liminf_{n \rightarrow +\infty} \mathbb{P}(T_n \in G) \geq \liminf_{n \rightarrow +\infty} \mathbb{P}\left(T_n \in \bigcup_{i \leq n_0} A_i\right) \geq \mathbb{P}\left(T \in \bigcup_{i \leq n_0} A_i\right) \geq \mathbb{P}(T \in G) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\liminf_{n \rightarrow +\infty} \mathbb{P}(T_n \in G) \geq \mathbb{P}(T \in G)$ for every open set G of \mathbb{T} . Thanks to the Portmanteau theorem, see (iv) of Theorem 2.1 in [16], we deduce that $(T_n, n \in \mathbb{N})$ converges in distribution to T .

3.2. A criteria for convergence toward Kesten's tree. Using the previous lemma, we can now state a general result for convergence of conditioned BGW trees toward Kesten's tree.

First, we consider a functional $A : \mathbb{T}_0 \rightarrow \mathbb{N}$ such that $\{\mathbf{t}; A(\mathbf{t}) \geq n\}$ is non empty for all $n \in \mathbb{N}^*$. In the following theorems, we will add some assumptions on A . These assumptions will vary from one theorem to another and in fact we will consider three different properties listed below (from the weaker to the stronger property): for all $\mathbf{t} \in \mathbb{T}_0$ and all leaf $x \in \mathcal{L}_0(\mathbf{t})$, there exists $n_0 \in \mathbb{N}^*$ and $D(\mathbf{t}, x) \geq 0$ (only for the Additivity property) such that for all $\mathbf{t}' \in \mathbb{T}_0$ satisfying $A(\mathbf{t} \otimes_x \mathbf{t}') \geq n_0$:

$$\begin{aligned} \text{(Monotonicity)} \quad & A(\mathbf{t} \otimes_x \mathbf{t}') \geq A(\mathbf{t}'); \\ \text{(Additivity)} \quad & A(\mathbf{t} \otimes_x \mathbf{t}') = A(\mathbf{t}') + D(\mathbf{t}, x); \\ \text{(Identity)} \quad & A(\mathbf{t} \otimes_x \mathbf{t}') = A(\mathbf{t}'). \end{aligned}$$

The Identity property is a particular case of the Additivity property with $D(\mathbf{t}, x) = 0$; and the Additivity property is a particular case of the Monotonicity property. We give examples of such functionals:

- The maximal degree $M(\mathbf{t}) = \max\{k_u(\mathbf{t}), u \in \mathbf{t}\}$ has the Identity property with $n_0 = M(\mathbf{t}) + 1$.
- The cardinal $\#\mathbf{t} = \text{Card}(\mathbf{t})$ has the Monotonicity property with $n_0 = 0$ and has also the Additivity property with $n_0 = 0$ and $D(\mathbf{t}, x) = \#\mathbf{t} - 1 \geq 0$.
- The height $H(\mathbf{t}) = \max\{H(u), u \in \mathbf{t}\}$ has the Additivity property with $n_0 = H(\mathbf{t})$ and $D(\mathbf{t}, x) = H(x) \geq 0$.

Notice that the functional $A(\mathbf{t}) = H(\mathbf{t}) + \#\mathbf{t}$ does not satisfy the Monotonicity, Additivity or Identity properties as $A(\mathbf{t} \otimes_x \mathbf{t}') = A(\mathbf{t}') + A(\mathbf{t}) - 1 - H(\mathbf{t}')$ on $H(\mathbf{t}') \leq H(\mathbf{t}) - H(x)$.

We will condition BGW trees with respect to events \mathbb{A}_n of the form $\mathbb{A}_n = \{A(\tau) \geq n\}$ or $\mathbb{A}_n = \{A(\tau) = n\}$ or in order to avoid periodicity arguments $\mathbb{A}_n = \{A(\tau) \in [n, n + n_1]\}$, for large n . (Notice that the first two cases boil down to the last one with respectively $n_1 = +\infty$ and $n_1 = 1$.)

The next theorem states a general result concerning the local convergence of **critical** BGW tree conditioned on \mathbb{A}_n toward the Kesten's tree. The proof of this theorem is at the end of this section.

Theorem 3.7 (Local convergence of critical BGW to Kesten's tree). *Let τ be a **critical** BGW tree with offspring distribution p such that $p(1) < 1$ and let τ^* be a Kesten's tree associated with p . Let τ_n be a random tree distributed according to τ conditionally on $\mathbb{A}_n = \{A(\tau) \in [n, n + n_1]\}$, where n_1 is fixed and $\mathbb{P}(\mathbb{A}_n) > 0$ for n large enough. If one of the following conditions is satisfied:*

- (i) $n_1 = +\infty$ and A satisfies the Monotonicity property;
- (ii) $n_1 \in \mathbb{N}^* \cup \{+\infty\}$ and A satisfies the Identity property;
- (iii) $n_1 \in \mathbb{N}^* \cup \{+\infty\}$, A satisfies Additivity property and:

$$(27) \quad \limsup_{n \rightarrow +\infty} \frac{\mathbb{P}(\mathbb{A}_{n+1})}{\mathbb{P}(\mathbb{A}_n)} \leq 1,$$

then, we have:

$$\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau^*.$$

Remark 3.8 (On the condition (27)). Assume that the Additivity property holds and that $\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau^*$. As $\mathbb{T}(\mathbf{t}, x)$ is open and closed, we deduce from the Portmanteau theorem that $\lim_{n \rightarrow \infty} \mathbb{P}(\tau_n \in \mathbb{T}(\mathbf{t}, x)) = \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x))$ for all $\mathbf{t} \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{t})$. Thanks to (34) below in the critical case (with $m = 1$), we deduce that $\lim_{n \rightarrow \infty} \mathbb{P}(\mathbb{A}_{n-D(\mathbf{t}, x)}) / \mathbb{P}(\mathbb{A}_n) = 1$ provided that $\mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) > 0$ or, equivalently, that $\mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x)) > 0$ by Equation (25).

We say that the functional D has period d (with respect to p) if the smallest group in \mathbb{Z} containing:

$$\{D(\mathbf{t}, x) : \mathbf{t} \in \mathbb{T}_0, x \in \mathcal{L}_0(\mathbf{t}) \text{ and } \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x)) > 0\}$$

is $d\mathbb{Z}$, and that D is aperiodic if $d = 1$. Assuming either that (27) holds or that the functional D is aperiodic, we deduce that:

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{P}(\mathbb{A}_{n+1})}{\mathbb{P}(\mathbb{A}_n)} = 1.$$

Notice, one can also change the parameter n_1 so that the above convergence holds even if $d > 1$.

Remark 3.9 (Tail conditioning). Let τ be a critical BGW tree with offspring distribution p and let τ^* be a Kesten's tree associated with p . For simplicity, let us assume that $\mathbb{P}(A(\tau) = n) > 0$ for n large enough. Let τ_n be a random tree distributed according to τ conditionally on $\{A(\tau) = n\}$ and assume that:

$$\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau^*.$$

Since the distribution of τ conditionally on $\{A(\tau) \geq n\}$ is a mixture of the distributions of τ conditionally on $\{A(\tau) = k\}$ for $k \geq n$, we deduce that τ conditionally on $\{A(\tau) \geq n\}$ converges in distribution toward τ^* . In particular, as far as Theorem 3.7 is concerned, the cases n_1 finite are the most delicate cases.

Remark 3.10 (On $\mathbb{P}(\mathbb{A}_n) > 0$). Recall that by assumption, $\{\mathbf{t}; A(\mathbf{t}) \geq n\}$ is non empty for all $n \in \mathbb{N}^*$. This is however not enough to ensure that $\mathbb{P}(A(\tau) \geq n) > 0$ for n large enough.

There is an extension of (iii) for a very special case in the sub-critical case.

Theorem 3.11 (Local convergence of sub-critical BGW to Kesten's tree). *Let τ be a sub-critical BGW tree with offspring distribution p with positive mean $m \in (0, 1)$, and let τ^* be a Kesten's tree associated with p . Let τ_n be a random tree distributed according to τ conditionally on $\mathbb{A}_n = \{A(\tau) \in [n, n + n_1]\}$ with $n_1 \in \mathbb{N}^* \cup \{+\infty\}$ fixed, where we assume that $\mathbb{P}(\mathbb{A}_n) > 0$ for n large enough. If A satisfies the Additivity property with $D(\mathbf{t}, x) = H(x)$ and:*

$$(28) \quad \limsup_{n \rightarrow +\infty} \frac{\mathbb{P}(\mathbb{A}_{n+1})}{\mathbb{P}(\mathbb{A}_n)} \leq m,$$

then, we have:

$$\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau^*.$$

Remark 3.12. The condition $D(\mathbf{t}, x) = H(x)$ is very restrictive and holds essentially for $A(\mathbf{t}) = H(\mathbf{t})$ as we will see in the next section.

Proof of Theorems 3.7 and 3.11. As we only consider critical or subcritical trees, the trees τ_n belong a.s. to \mathbb{T}_0 . Moreover, by definition, a.s. the Kesten's tree belongs to \mathbb{T}_1 . Therefore we can use Proposition 3.3 to prove the convergence in distribution of the theorems.

Let $n_1 \in \mathbb{N}^* \cup \{+\infty\}$ and set $\mathbb{A}_n = \{A(\tau) \in [n, n + n_1]\}$ in order to cover all the different cases of the two theorems. Let $\mathbf{t} \in \mathbb{T}_0$. We have:

$$\mathbb{P}(\tau_n = \mathbf{t}) = \frac{\mathbb{P}(\tau = \mathbf{t}, \mathbb{A}_n)}{\mathbb{P}(\mathbb{A}_n)} \leq \frac{1}{\mathbb{P}(\mathbb{A}_n)} \mathbf{1}_{\{A(\mathbf{t}) \in [n, n + n_1]\}}.$$

As $A(\mathbf{t})$ is finite since $\mathbf{t} \in \mathbb{T}_0$, we have $\mathbf{1}_{\{A(\mathbf{t}) \in [n, n + n_1]\}} = 0$ for $n > A(\mathbf{t})$. We deduce that:

$$(29) \quad \lim_{n \rightarrow +\infty} \mathbb{P}(\tau_n = \mathbf{t}) = 0 = \mathbb{P}(\tau^* = \mathbf{t}),$$

as τ^* is a.s. infinite. This gives condition (i) of Proposition 3.3.

To obtain the convergence in distribution of the sequence $(\tau_n, n \in \mathbb{N}^*)$ to τ^* , it is enough to check condition (ii) of Proposition 3.3, that is, for $\mathbf{t} \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{t})$ a leaf of \mathbf{t} :

$$(30) \quad \liminf_{n \rightarrow +\infty} \mathbb{P}(\tau_n \in \mathbb{T}(\mathbf{t}, x)) \geq \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)).$$

Let $\mathbf{t} \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{t})$ be a fixed leaf of \mathbf{t} . Since τ is a.s. finite, we have:

$$(31) \quad \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x), \mathbb{A}_n) = \sum_{\mathbf{t}' \in \mathbb{T}_0} \mathbb{P}(\tau = \mathbf{t} \otimes_x \mathbf{t}') \mathbf{1}_{\{n \leq A(\mathbf{t} \otimes_x \mathbf{t}') < n+n_1\}}.$$

Using (23), (24) and (25), we get that for every tree $\mathbf{t}' \in \mathbb{T}$:

$$(32) \quad \mathbb{P}(\tau = \mathbf{t} \otimes_x \mathbf{t}') = m^{H(x)} \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) \mathbb{P}(\tau = \mathbf{t}').$$

We deduce that:

$$(33) \quad \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x), \mathbb{A}_n) = m^{H(x)} \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) \sum_{\mathbf{t}' \in \mathbb{T}_0} \mathbb{P}(\tau = \mathbf{t}') \mathbf{1}_{\{n \leq A(\mathbf{t} \otimes_x \mathbf{t}') < n+n_1\}}.$$

We now distinguish three different cases.

Assume p critical, $n_1 = +\infty$ and A satisfies property Monotonicity. We deduce from (31) that for $n \geq n_0$:

$$\mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x), \mathbb{A}_n) = \sum_{\mathbf{t}' \in \mathbb{T}_0} \mathbb{P}(\tau = \mathbf{t} \otimes_x \mathbf{t}') \mathbf{1}_{\{n \leq A(\mathbf{t} \otimes_x \mathbf{t}')\}} \geq \sum_{\mathbf{t}' \in \mathbb{T}_0} \mathbb{P}(\tau = \mathbf{t} \otimes_x \mathbf{t}') \mathbf{1}_{\{n \leq A(\mathbf{t}')\}},$$

where we used the Monotonicity property for the inequality. Then, using (32) with $m = 1$, we get that:

$$\mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x), \mathbb{A}_n) \geq \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) \mathbb{P}(\mathbb{A}_n),$$

which gives (30). This proves Theorem 3.7 (i).

Assume p critical and A satisfies property Identity. In that case, we have for $n \geq n_0$:

$$\sum_{\mathbf{t}' \in \mathbb{T}_0} \mathbb{P}(\tau = \mathbf{t}') \mathbf{1}_{\{n \leq A(\mathbf{t} \otimes_x \mathbf{t}') < n+n_1\}} = \sum_{\mathbf{t}' \in \mathbb{T}_0} \mathbb{P}(\tau = \mathbf{t}') \mathbf{1}_{\{n \leq A(\mathbf{t}') < n+n_1\}} = \mathbb{P}(\mathbb{A}_n),$$

and we obtain from (33), with $m = 1$, that $\mathbb{P}(\tau_n \in \mathbb{T}(\mathbf{t}, x)) = \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x))$ for $n \geq n_0$. This yields (30) and proves Theorem 3.7 (ii).

Assume A satisfies property Additivity. We deduce from (33) that for $n \geq n_0$,

$$\begin{aligned} \mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x), \mathbb{A}_n) &= m^{H(x)} \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) \sum_{\mathbf{t}' \in \mathbb{T}_0} \mathbb{P}(\tau = \mathbf{t}') \mathbf{1}_{\{n \leq A(\mathbf{t}') + D(\mathbf{t}, x) < n+n_1\}} \\ &= m^{H(x)} \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) \mathbb{P}(n - D(\mathbf{t}, x) \leq A(\tau) < n - D(\mathbf{t}, x) + n_1) \\ &= m^{H(x)} \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) \mathbb{P}(\mathbb{A}_{n-D(\mathbf{t}, x)}). \end{aligned}$$

Finally, we get:

$$(34) \quad \mathbb{P}(\tau_n \in \mathbb{T}(\mathbf{t}, x)) = \frac{\mathbb{P}(\tau \in \mathbb{T}(\mathbf{t}, x), \mathbb{A}_n)}{\mathbb{P}(\mathbb{A}_n)} = m^{H(x)} \mathbb{P}(\tau^* \in \mathbb{T}(\mathbf{t}, x)) \frac{\mathbb{P}(\mathbb{A}_{n-D(\mathbf{t}, x)})}{\mathbb{P}(\mathbb{A}_n)}.$$

If (27) holds in the critical case ($m = 1$) or (28) and $D(\mathbf{t}, x) = H(x)$ in the sub-critical case, we obtain (30). This proves Theorem 3.7 (iii) and Theorem 3.11. \square

3.3. Applications to various conditioning. We assume that the offspring distribution p satisfies $p(1) < 1$ and is critical ($m = 1$) or sub-critical ($m < 1$).

3.3.1. The height (Kesten's theorem). We give a proof of Kesten's theorem, see Theorem 3.1. We consider the functional of the tree given by its height: $A(\mathbf{t}) = H(\mathbf{t})$. It satisfies the Additivity property, with $n_0 = H(\mathbf{t}) + 1$, as for every tree $\mathbf{t} \in \mathbb{T}_0$, every leaf $x \in \mathcal{L}_0(\mathbf{t})$ and every $\mathbf{t}' \in \mathbb{T}_0$ such that $H(\mathbf{t} \otimes_x \mathbf{t}') \geq H(\mathbf{t}) + 1$, we have:

$$H(\mathbf{t} \otimes_x \mathbf{t}') = H(\mathbf{t}') + H(x).$$

We give a preliminary result.

Lemma 3.13. *Let τ be a critical or sub-critical BGW tree with offspring distribution p satisfying $p(1) < 1$ with mean $m \leq 1$. Let $n_1 \in \mathbb{N}^* \cup \{+\infty\}$. Set $\mathbb{A}_n = \{A(\tau) \in [n, n + n_1)\}$ for $n \in \mathbb{N}^*$. We have:*

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{P}(\mathbb{A}_{n+1})}{\mathbb{P}(\mathbb{A}_n)} = m.$$

We then deduce the following corollary as a direct consequence of Theorem 3.7 (iii) in the critical case or of Theorem 3.11 in the sub-critical case.

Corollary 3.14 (Kesten's theorem). *Let τ be a critical or sub-critical BGW tree with offspring distribution p satisfying $p(1) < 1$, and τ^* a Kesten's tree associated with p . Let τ_n be a random tree distributed according to τ conditionally on $\{H(\tau) = n\}$ (resp. $\{H(\tau) \geq n\}$). Then, we have:*

$$\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau^*.$$

Proof of Lemma 3.13. We shall consider the case $n_1 = 1$, the other cases being deduced using Remark 3.9. So, we have $\mathbb{A}_n = \{H(\tau) = n\}$. Recall that for any tree \mathbf{t} , $z_n(\mathbf{t}) = \text{Card} \{u \in \mathbf{t}, H(u) = n\}$ denotes the size of the n -th generation of the tree \mathbf{t} . We set $Z_n = z_n(\tau)$, so that $(Z_n, n \geq 0)$ is a BGW process. Notice that:

$$\mathbb{A}_n = \{Z_{n+1} = 0\} \setminus \{Z_n = 0\}.$$

Recall that g denotes the generating function of the offspring distribution p . Let g_n be the generating function of Z_n . In particular, we have $g_1 = g$. Using the branching property of the BGW tree, we have that Z_{n+1} is distributed as $\sum_{i=1}^{k_\emptyset(\tau)} z_n(\tau_i)$, where $(\tau_i, i \in \mathbb{N}^*)$ are independent BGW tree with offspring distribution p and independent of $Z_1 = k_\emptyset(\tau)$. This gives:

$$g_{n+1}(s) = \mathbb{E} \left[\prod_{i=1}^{Z_1} s^{z_n(\tau_i)} \right] = \mathbb{E} [g_n(s)^{Z_1}] = g(g_n(s)).$$

We have $\mathbb{P}(\mathbb{A}_n) = \mathbb{P}(Z_{n+1} = 0) - \mathbb{P}(Z_n = 0) = g_{n+1}(0) - g_n(0)$. Since τ is critical or sub-critical, it is a.s. finite and we deduce that $\lim_{n \rightarrow +\infty} g_n(0) = \lim_{n \rightarrow +\infty} \mathbb{P}(z_n(\tau) = 0) = 1$. We have:

$$\frac{\mathbb{P}(\mathbb{A}_{n+1})}{\mathbb{P}(\mathbb{A}_n)} = \frac{g_{n+2}(0) - g_{n+1}(0)}{g_{n+1}(0) - g_n(0)}.$$

Using Taylor formula at $g_n(0)$, we get:

$$\begin{aligned} g_{n+2}(0) &= g(g_n(0) + (g_{n+1}(0) - g_n(0))) \\ &= g_{n+1}(0) + (g_{n+1}(0) - g_n(0)) g'(g_n(0)) + o(g_{n+1}(0) - g_n(0)). \end{aligned}$$

This gives that:

$$\frac{\mathbb{P}(\mathbb{A}_{n+1})}{\mathbb{P}(\mathbb{A}_n)} = g'(g_n(0)) + o(1) \xrightarrow[n \rightarrow \infty]{} m.$$

□

3.3.2. The maximal out-degree, critical case. Following [34], we consider the functional of the tree given by its maximal out-degree: $A(\mathbf{t}) = M(\mathbf{t})$, with:

$$M(\mathbf{t}) = \sup_{u \in \mathbf{t}} k_u(\mathbf{t}).$$

Notice it satisfies the (Identity) property, with $n_0 = M(\mathbf{t}) + 1$, as for every tree $\mathbf{t} \in \mathbb{T}_0$, every leaf $x \in \mathcal{L}_0(\mathbf{t})$ and every $\mathbf{t}' \in \mathbb{T}_0$ such that $M(\mathbf{t} \otimes_x \mathbf{t}') \geq M(\mathbf{t}) + 1$, we have:

$$M(\mathbf{t} \otimes_x \mathbf{t}') = M(\mathbf{t}').$$

The next corollary is then a consequence of Theorem 3.7 (ii) with $n_1 \in \{1, +\infty\}$. For $n_1 = 1$, in the proof of this theorem, the condition $\mathbb{P}(\mathbb{A}_n) > 0$ for n large enough is easily replaced by the convergence along the sub-sequence $\{n; p(n) > 0\}$ given by the support of p .

Corollary 3.15 (Conditioning on the maximal out-degree). *Let τ be a critical BGW tree with offspring distribution p satisfying $p(1) < 1$ and unbounded support, and let τ^* be a Kesten's tree associated with p . Let τ_n be a random tree distributed according to τ conditionally on $\{M(\tau) = n\}$ (resp. $\{M(\tau) \geq n\}$). Then, we have along the sub-sequence $\{n; p(n) > 0\}$ (resp. with $n \in \mathbb{N}^*$):*

$$\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau^*.$$

Intuitively, the height of the largest node in the BGW tree τ_n goes to infinity, so it disappears in the local limit.

3.3.3. The Horton-Strahler number, critical case. The Horton-Strahler number of a finite tree was initially introduced in hydrogeomorphism by Horton (1845) and redefined later by Strahler (1952), see [61, 28, 21, 17] and references therein. It is also known as the register function in computer science. This number measures the unbalance of a tree, and was initially introduced for binary trees ($k_u(\mathbf{t}) \in \{0, 2\}$ for $u \in \mathbf{t}$). Recall the definition of the sub-tree $\mathcal{S}_u(\mathbf{t})$ of \mathbf{t} above the node $u \in \mathbf{t}$ given in (5). The Horton-Strahler number $\Sigma(\mathbf{t})$ of a finite binary tree \mathbf{t} is defined recursively by $\Sigma(\{\emptyset\}) = 0$ and, if $k_\emptyset(\mathbf{t}) = 2$, then $\Sigma(\mathbf{t}) = \Sigma_1 \vee \Sigma_2 + \mathbf{1}_{\{\Sigma_1 = \Sigma_2\}}$ where $\Sigma_u = \Sigma(\mathcal{S}_u(\mathbf{t}))$ for $u \in \{1, 2\}$.

Its generalization, which we still write Σ , to finite general rooted tree also called the register function is defined as follows, for $\mathbf{t} \in \mathbb{T}_0$:

$$\Sigma(\mathbf{t}) = \begin{cases} 0 & \text{if } \mathbf{t} = \{\emptyset\}, \\ \max\{\Sigma_{(1)}, \Sigma_{(2)} + 1, \dots, \Sigma_{(k)} + k - 1\} & \text{if } k_\emptyset(\mathbf{t}) = k \geq 1, \end{cases}$$

where $\Sigma_{(1)} \geq \dots \geq \Sigma_{(k)}$ is the non-increasing reordering of $(\Sigma_1, \dots, \Sigma_k)$ and $\Sigma_u = \Sigma(\mathcal{S}_u(\mathbf{t}))$ for $u \in \{1, \dots, k\}$. Clearly, we have that $\Sigma(\mathbf{t}) \leq \Sigma_{(1)} + k_\emptyset(\mathbf{t}) - 1$. Notice also that the two definitions of Σ coincide on binary trees.

The functional of the tree $A = \Sigma$ has the Identity property, with $n_0 = \Sigma(\mathbf{t}) + M(\mathbf{t}) - 1$, where $M(\mathbf{t})$ is the maximal out-degree of \mathbf{t} . Indeed, for trees $\mathbf{t}, \mathbf{t}' \in \mathbb{T}_0$ and leaf $x \in \mathcal{L}_0(\mathbf{t})$ of \mathbf{t} such that $\Sigma(\mathbf{t} \otimes_x \mathbf{t}') \geq \Sigma(\mathbf{t}) + M(\mathbf{t}) - 1$, it is easy to check that:

$$\Sigma(\mathbf{t} \otimes_x \mathbf{t}') = \Sigma(\mathbf{t}').$$

Notice that $\mathbb{P}(\Sigma(\tau) = n) > 0$ for all $n \geq \inf\{k \geq 1 : p(k+1) > 0\}$. The next corollary is then a consequence of Theorem 3.7 (ii) with $n_1 \in \{1, +\infty\}$.

Corollary 3.16 (Conditioning on the Horton-Strahler number). *Let τ be a critical BGW tree with offspring distribution p satisfying $p(1) < 1$, and let τ^* be a Kesten's tree associated with p . Let τ_n be a random tree distributed according to τ conditionally on $\{\Sigma(\tau) = n\}$ (resp. $\{\Sigma(\tau) \geq n\}$). Then, we have:*

$$\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau^*.$$

Remark 3.17 (Extension to CRT). A very recent result on scaling limit of the Horton-Strahler number for a BGW conditioned to be large appears in [43]. This presentation explains in particular why the fluctuations of the Horton-Strahler number are asymptotically coupled with deterministic oscillations.

3.3.4. The largest generation, critical case. We consider the functional of the tree given by its largest generation: $A(\mathbf{t}) = \mathcal{Z}(\mathbf{t})$ with:

$$\mathcal{Z}(\mathbf{t}) = \sup_{k \geq 0} z_k(\mathbf{t}).$$

Notice the functional has the Monotonicity property as $\mathcal{Z}(\mathbf{t} \otimes_x \mathbf{t}') \geq \mathcal{Z}(\mathbf{t}')$. The next corollary is then a direct consequence of Theorem 3.7 (i); this result appears in [35].

Corollary 3.18 (Conditioning on the largest generation). *Let τ be a critical BGW tree with offspring distribution p satisfying $p(1) < 1$, and let τ^* be a Kesten's tree associated with p . Let τ_n be a random tree distributed according to τ conditionally on $\{\mathcal{Z}(\tau) \geq n\}$. Then, we have:*

$$\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau^*.$$

Remark 3.19 (Conditioning on $\{\mathcal{Z}(\tau) = n\}$). Notice that the functional \mathcal{Z} does not satisfy the Additivity property. Thus, considering the conditioning event $\{\mathcal{Z}(\tau) = n\}$ is still an open problem. However, as noted in [35], if the support of the offspring distribution is bounded, then the functional \mathcal{Z} satisfies the Identity property (take $n_0 = \mathcal{Z}(\mathbf{t}) + \alpha^{H(\mathbf{t}) - H(x)} + 1$ with $\alpha = \sup \text{supp}(p)$ being finite), and then Corollary 3.18 holds with the event $\{\mathcal{Z}(\tau) \geq n\}$ replaced by $\{\mathcal{Z}(\tau) = n\}$ and considering the (infinite) subsequence of n for which this event has positive probability.

3.3.5. The total progeny, critical case. The local convergence in distribution of the critical tree, conditionally on its total size being large, to the Kesten's tree appears implicitly in [40] and was first explicitly stated in [13]. We give here an alternative proof. We consider the functional: $A(\mathbf{t}) = \sharp \mathbf{t}$, with $\sharp \mathbf{t} = \text{Card}(\mathbf{t})$ the total size of \mathbf{t} , which has the Additivity property as for every trees $\mathbf{t}, \mathbf{t}' \in \mathbb{T}_0$ and every leaf $x \in \mathcal{L}_0(\mathbf{t})$:

$$\sharp \mathbf{t} \otimes_x \mathbf{t}' = \sharp \mathbf{t}' + \sharp \mathbf{t} - 1.$$

Recall d is the period of the offspring distribution p . Using Bézout's identity, it is classical to check that:

$$(35) \quad \mathbb{P}(\sharp \tau \in 1 + d\mathbb{N}) = 1$$

and that there exists $n_0 \in \mathbb{N}^*$ such that $\mathbb{P}(\sharp \tau = 1 + dn) > 0$ for all $n \geq n_0$. The next lemma is a direct consequence of Dwass formula and the strong ratio limit theorem. Its proof is at the end of this section.

Lemma 3.20. *Let τ be a critical BGW tree with offspring distribution p satisfying $p(1) < 1$ with period d . Let $n_1 \in \{d, +\infty\}$ and set $\mathbb{A}_n = \{\sharp \tau \in [n, n + n_1)\}$. Then, we have:*

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{P}(\mathbb{A}_{n+1})}{\mathbb{P}(\mathbb{A}_n)} = 1.$$

We then deduce the following corollary as a direct consequence of Theorem 3.7 (iii).

Corollary 3.21 (Conditioning on the total size). *Let τ be a critical BGW tree with offspring distribution p satisfying $p(1) < 1$ with period d , and let τ^* be a Kesten's tree associated with p . Let τ_n be a random tree distributed according to τ conditionally on $\{\sharp \tau = 1 + dn\}$ for n large enough (resp. on $\{\sharp \tau \geq n\}$). Then, we have:*

$$\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau^*.$$

As in Section 3.3.2, we may observe in τ_n a large node whose height and size go to infinity and which disappears in the local limit, see [47] in this direction.

The end of the section is devoted to the proof of Lemma 3.20. We first recall Dwass formula from [26] that links the distribution of the total progeny of BGW trees to the distribution of random walks and then the strong ratio limit theorem.

Let $(\zeta_k, k \in \mathbb{N}^*)$ be a sequence of independent random variables distributed according to p . Set $S_n = \sum_{k=1}^n \zeta_k$ for $n \in \mathbb{N}^*$.

Lemma 3.22 (Dwass formula). *Let τ be a BGW tree with offspring distribution p . Then, for every $n \geq 1$, we have:*

$$\mathbb{P}(\#\tau = n) = \frac{1}{n} \mathbb{P}(S_n = n - 1).$$

We also recall the strong ratio limit theorem that can be found for instance in [51], see also [56] Theorem T1 p.49.

Lemma 3.23 (Strong ratio limit theorem). *Assume that p is critical aperiodic with $p(1) < 1$. Then, we have, for every $\ell \in \mathbb{Z}$,*

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{P}(S_n = n + \ell)}{\mathbb{P}(S_n = n)} = 1 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \frac{\mathbb{P}(S_{n+1} = n + 1)}{\mathbb{P}(S_n = n)} = 1.$$

Remark 3.24 (Strong ratio limit theorem in the periodic case). We first stress that the strong ratio theorem in [56, 51] requires p to be strongly aperiodic, that is, the smallest sub-group G_k which contains $k + \text{supp}(p)$ is \mathbb{Z} for all $k \in \mathbb{Z}$. Taking $k = 0$, we get that p is aperiodic if it is strongly aperiodic. On the other hand, if p is aperiodic and $1 > p(0) > 0$, then we deduce that k belongs to G_k , and thus $-k$ also; this readily implies that $\text{supp}(p) \subset G_k$ and thus $G_k = \mathbb{Z}$, giving that p is strongly aperiodic.

Eventually, if p has period d , we refer to Equation (4.10) in [6] for the extension of the strong ratio limit theorem.

Proof of Lemma 3.20. We shall assume for simplicity that p is aperiodic and take $n_1 = 1$. The cases $d \geq 2$ or $n_1 = +\infty$ are left to the reader. Using Dwass formula, we have:

$$\frac{\mathbb{P}(\#\tau = n + 1)}{\mathbb{P}(\#\tau = n)} = \frac{n}{n + 1} \frac{\mathbb{P}(S_{n+1} = n)}{\mathbb{P}(S_n = n - 1)}.$$

Using the strong ration limit theorem, we get:

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{P}(\#\tau = n + 1)}{\mathbb{P}(\#\tau = n)} = \lim_{n \rightarrow +\infty} \frac{\mathbb{P}(S_{n+1} = n)}{\mathbb{P}(S_n = n - 1)} = 1.$$

This ends the proof of Lemma 3.20. □

3.3.6. The number of leaves, critical case. Recall $\mathcal{L}_0(\mathbf{t})$ denotes the set of leaves of a tree \mathbf{t} and set $L_0(\mathbf{t})$ for its cardinal. We shall consider a critical BGW tree τ conditioned on the number of leaves, that is, on $\{L_0(\tau) = n\}$. Such a conditioning appears first in [19] with a second moment condition. We prove here the convergence in distribution of the conditioned tree to the Kesten's tree in the critical case without any additional assumption using Theorem 3.7.

The functional $A(\mathbf{t}) = L_0(\mathbf{t})$ satisfies the Additivity property with $n_0 = 1$ and $D(\mathbf{t}, x) = L_0(\mathbf{t}) - 1$ as for every trees $\mathbf{t}, \mathbf{t}' \in \mathbb{T}_0$ and every leaf $x \in \mathcal{L}_0(\mathbf{t})$:

$$L_0(\mathbf{t} \otimes_x \mathbf{t}') = L_0(\mathbf{t}') + L_0(\mathbf{t}) - 1.$$

The next lemma due to Minami [49] gives a bijection between the leaves of a finite tree \mathbf{t} and the nodes of a tree $\mathbf{t}_{\{0\}}$. Its proof is given at the end of this section.

Lemma 3.25 (The tree of leaves). *Let τ be a critical BGW tree with offspring distribution p such that $p(1) < 1$. Then $L_0(\tau)$ is distributed as $\#\tau_{\{0\}}$, where $\tau_{\{0\}}$ is a critical BGW tree with offspring distribution $p_{\{0\}}$ such that $p_{\{0\}}(1) < 1$.*

It is elementary to check from the construction of $\tau_{\{0\}}$ in the proof of Lemma 3.25 below that the period d_0 of $p_{\{0\}}$ is given by:

$$d_0 = \max\{k; \text{supp}(p) \subset k\mathbb{N} + 1\}.$$

This, Lemma 3.25 and (35) imply in turn that $\mathbb{P}(L_0(\tau) \in 1 + d_0\mathbb{N}) = 1$ and that there exists $n_0 \in \mathbb{N}^*$ such that $\mathbb{P}(L_0(\tau) = 1 + d_0n) > 0$ for all $n \geq n_0$. The following corollary is then a direct consequence of Lemma 3.20 and Theorem 3.7 (iii).

Corollary 3.26 (Conditioning on the number of leaves). *Let τ be a critical BGW tree with offspring distribution p , such that $p(1) < 1$, and τ^* a Kesten's tree associated with p . Let τ_n be a random tree distributed according to τ conditionally on $\{L_0(\tau) = 1 + d_0 n\}$ (resp. $\{L_0(\tau) \geq n\}$). Then, we have:*

$$\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau^*.$$

Proof of Lemma 3.25. We first describe the correspondence of Minami. The left-most leaf (in the lexicographical order) of \mathbf{t} is mapped on the root of $\mathbf{t}_{\{0\}}$. In the example of Fig. 6, the leaves of the tree \mathbf{t} are labeled from 1 to 9, and the left-most leaf is 1.

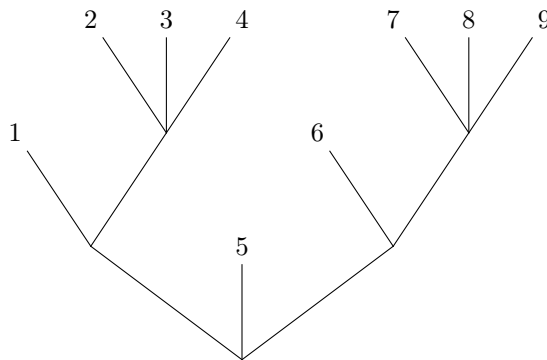


FIGURE 6. A finite tree \mathbf{t} .

Then consider all the subtrees that are attached to the branch between the root and the left-most leaf. All the left-most leaves of these subtrees are mapped on the children of the root of $\mathbf{t}_{\{0\}}$, they form the population at generation 1 of the tree $\mathbf{t}_{\{0\}}$. In Fig. 7, the considered sub-trees are surrounded by dashed lines, and the leaves at generation 1 are labeled $\{2, 5, 6\}$. Remark that the sub-tree that contains the leaf 5 is reduced to a single node (this particular leaf).

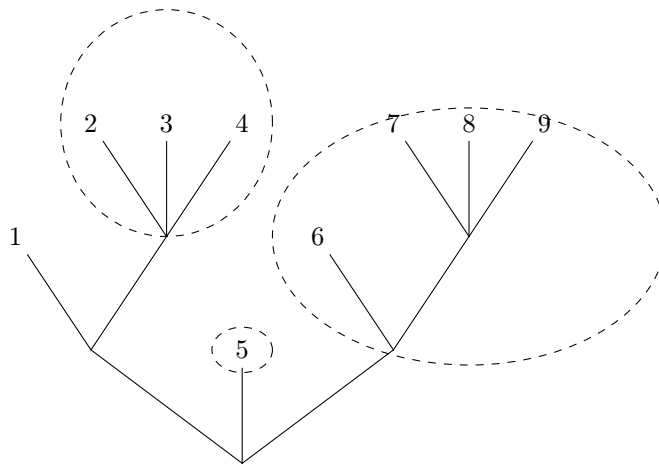
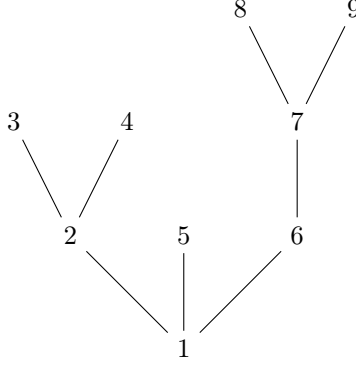


FIGURE 7. The sub-trees attached to the branch between the root and the leaf labeled 1.

Then perform the same procedure inductively at each of these sub-trees to construct the tree $\mathbf{t}_{\{0\}}$. In Fig. 8, we give the tree $\mathbf{t}_{\{0\}}$ associated with the tree \mathbf{t} .

FIGURE 8. The tree $\mathbf{t}_{\{0\}}$ associated with the tree \mathbf{t} from Fig. 6.

Using the branching property, we get that all the sub-trees that are attached to the left-most branch of a BGW tree τ are independent and distributed as τ . Therefore, the tree $\tau_{\{0\}}$ is still a BGW tree.

Next, we compute the offspring distribution, $p_{\{0\}}$, of $\tau_{\{0\}}$. Since p is critical and $p(1) < 1$, we get $p(0) > 0$. Let N denote the generation of the left-most leaf. It is easy to see that this random variable is distributed according to a geometric distribution with parameter $p(0)$, that is, for every $n \geq 1$:

$$\mathbb{P}(N = n) = (1 - p(0))^{n-1} p(0).$$

Let ζ be a random variable with distribution p and mean m , and let X be distributed as ζ conditionally on $\{\zeta > 0\}$, that is, $\mathbb{P}(X = n) = p(n)/(1 - p(0))$ for every $n \geq 1$: In particular, we have:

$$\mathbb{E}[X] = \frac{m}{1 - p(0)}.$$

We denote by (X_1, \dots, X_{N-1}) the respective numbers of offsprings of the nodes on the left-most branch (including the root, excluding the leaf). Then, using again the branching property, these variables are, conditionally on N , independent and distributed as X . Thus, the number of children of the root in the tree $\tau_{\{0\}}$ is the number of the sub-trees attached to the left-most branch that is:

$$\zeta' = \sum_{k=1}^{N-1} (X_k - 1).$$

By construction $p_{\{0\}}$ is the probability distribution of ζ' . Since $p(0) > 0$, we get that that $\mathbb{P}(N = 1) > 0$ and thus $p_{\{0\}}(0) > 0$. We now compute the mean of $p_{\{0\}}$:

$$\mathbb{E}[\zeta'] = \mathbb{E}[N - 1]\mathbb{E}[X - 1] = \left(\frac{1}{p(0)} - 1\right) \left(\frac{m}{1 - p(0)} - 1\right) = \frac{1}{p(0)} (m - (1 - p(0))).$$

In particular, if the BGW tree τ is critical ($m = 1$), then $\mathbb{E}[\zeta'] = 1$, and thus the BGW tree $\tau_{\{0\}}$ is also critical. \square

3.3.7. The number of nodes with given out-degree, critical case. Let \mathcal{A} be a non-empty subset of \mathbb{N} and for a tree \mathbf{t} , we define the subset of nodes with out-degree in \mathcal{A} :

$$\mathcal{L}_{\mathcal{A}}(\mathbf{t}) = \{u \in \mathbf{t}, k_u(\mathbf{t}) \in \mathcal{A}\}$$

and $L_{\mathcal{A}}(\mathbf{t}) = \text{Card}(\mathcal{L}_{\mathcal{A}}(\mathbf{t}))$ its cardinal. (If $\mathcal{A} = \mathbb{N}$, then $L_{\mathcal{A}}(\mathbf{t})$ is the total number of nodes of \mathbf{t} ; and if $\mathcal{A} = \{0\}$, then $L_{\mathcal{A}}(\mathbf{t}) = L_0(\mathbf{t})$ is the total number of leaves of \mathbf{t} .) The functional $L_{\mathcal{A}}$ satisfies Property (Additivity) with $D(\mathbf{t}, x) = L_{\mathcal{A}}(\mathbf{t}) - \mathbf{1}_{\{0 \in \mathcal{A}\}} \geq 0$, that is, for $\mathbf{t}, \mathbf{t}' \in \mathbb{T}_0$ and $x \in \mathcal{L}_0(\mathbf{t})$:

$$L_{\mathcal{A}}(\mathbf{t} \otimes_x \mathbf{t}') = L_{\mathcal{A}}(\mathbf{t}') + L_{\mathcal{A}}(\mathbf{t}) - \mathbf{1}_{\{0 \in \mathcal{A}\}}.$$

Moreover, when $\mathbf{t} \in T_0$ is such that $L_{\mathcal{A}}(\mathbf{t}) > 0$, there exists a bijection, generalizing Minami's correspondence, between the set $\mathcal{L}_{\mathcal{A}}(\mathbf{t})$ and a tree $\mathbf{t}_{\mathcal{A}}$ so that $L_{\mathcal{A}}(\mathbf{t}) = \#\mathbf{t}_{\mathcal{A}}$, see Rizzolo [55]. Let τ be a BGW tree with offspring distribution p such that $p(\mathcal{A}) > 0$, where we set:

$$p(\mathcal{A}) = \sum_{n \in \mathcal{A}} p(n).$$

Without further assumption on p , notice that the event $\{L_{\mathcal{A}}(\tau) > 0\}$ has positive probability and that:

$$(36) \quad \mathbb{P}(\tau \notin \mathbb{T}_0, L_{\mathcal{A}}(\tau) < +\infty) = 0.$$

The latter equality is obvious if p is sub-critical or critical. When p is super-critical with $p(0) > 0$ (and thus $q > 0$), it is an immediate consequence of Corollary 2.9 using the strong law of large number and that there are an infinity of independent BGW sub-critical sub-trees, with offspring distribution \tilde{p} defined by (12), which are grafted on the infinite backbone of the individuals of type survivor. On the other hand, if $p(0) = 0$, then the number of children of the individuals in the infinite left branch of τ , that is $(k_u(\tau), u \in \cup_{\ell \in \mathbb{N}} \{1\}^{\ell})$ are independent random variables distributed as p ; thus a.s. $L_{\mathcal{A}}(\tau)$ is infinite.

Lemma 3.27 (Theorem 6 in [55]). *Let τ be a BGW tree with offspring distribution p such that $p(\mathcal{A}) > 0$, and p is either critical with $p(1) < 1$ or sub-critical. Then, there exists a BGW tree $\tau_{\mathcal{A}}$ with offspring distribution $p_{\mathcal{A}}$ such that $L_{\mathcal{A}}(\tau)$, conditionally on $\{L_{\mathcal{A}}(\tau) > 0\}$, is distributed as $\#\tau_{\mathcal{A}}$. Furthermore if p is critical then $p_{\mathcal{A}}$ is critical with $p_{\mathcal{A}}(1) < 1$, and $p_{\mathcal{A}}$ is sub-critical otherwise.*

According to [55], the period $d_{\mathcal{A}}$ of $p_{\mathcal{A}}$ depends only on the sets \mathcal{A} and $\text{supp}(p)$.

The next result is a generalization of Section 3.3.6. It is a direct consequence of Lemma 3.20 and Theorem 3.7 (iii).

Corollary 3.28 (Conditioning on the number of nodes with given out-degree). *Let τ be a critical BGW tree with offspring distribution p , such that $p(\mathcal{A}) > 0$ and $p(0) > 0$, and let τ^* be Kesten's tree associated with p . Let τ_n be a random tree distributed according to τ conditionally on $\{L_{\mathcal{A}}(\tau) = 1 + d_{\mathcal{A}}n\}$ (resp. $\{L_{\mathcal{A}}(\tau) \geq n\}$). Then, we have:*

$$\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau^*.$$

3.4. Conditioning on the number of nodes with given out-degree, sub-critical case. Theorem 3.11 deals with sub-critical offspring distributions and applies essentially for the conditioning on the height, see Corollary 3.14. We complete the picture of Theorem 3.7 studying the local limit of BGW trees by conditioned on $\{L_{\mathcal{A}}(\tau) = n\}$ in the sub-critical case.

We shall first present in Section 3.4.1 a family of offspring distributions for which the conditioned trees have the same distribution. If this family contains one critical offspring distribution, which is the so-called *generic* case, then we can use the results on the critical case from Section 3.3.7. When this is not the case, this is the so called *non-generic* case. Then the local limit exhibit a *condensation* phenomenon, that is, there exists a (unique) node in the random limiting tree with infinite out-degree, see Proposition 3.36. This result is proved in [5] and is more technical as the limit does not belong to $\mathbb{T}_0 \cup \mathbb{T}_1$, and thus one needs a new characterization of the local convergence. Eventually, we discuss the generic/non-generic properties depending on the set \mathcal{A} in Section 3.4.4.

3.4.1. An equivalent probability. Let p be an offspring distribution and $\mathcal{A} \subset \mathbb{N}$. Recall $p(\mathcal{A}) = \sum_{n \in \mathcal{A}} p(n)$. We assume that $p(\mathcal{A}) > 0$ and we define:

$$I_{\mathcal{A}} = \left\{ \theta \in (0, +\infty) : \sum_{k \in \mathcal{A}} \theta^{k-1} p(k) < +\infty \quad \text{and} \quad \sum_{k \notin \mathcal{A}} \theta^{k-1} p(k) < 1 \right\}.$$

For $\theta \in I_{\mathcal{A}}$, we set for every $k \in \mathbb{N}$:

$$p_{\theta}^{\mathcal{A}}(k) = \begin{cases} \theta^{k-1} p(k) & \text{if } k \notin \mathcal{A}, \\ c_{\mathcal{A}}(\theta) \theta^{k-1} p(k) & \text{if } k \in \mathcal{A}, \end{cases}$$

where $c_{\mathcal{A}}(\theta)$ is a finite positive constant that makes $p_{\theta}^{\mathcal{A}}$ a probability measure on \mathbb{N} , namely:

$$c_{\mathcal{A}}(\theta) = \frac{1 - \sum_{k \notin \mathcal{A}} \theta^{k-1} p(k)}{\sum_{k \in \mathcal{A}} \theta^{k-1} p(k)}.$$

Remark that $I_{\mathcal{A}}$ is exactly the set of θ for which $p_{\theta}^{\mathcal{A}}$ is indeed a probability measure with the same support as p . Remark also that $I_{\mathcal{A}}$ is an interval that contains 1, as $p_1^{\mathcal{A}} = p$.

The following proposition gives the connection between p and $p_{\theta}^{\mathcal{A}}$. It generalizes the results already obtained for the total progeny, $\mathcal{A} = \mathbb{N}$, in [40] and for the number of leaves, $\mathcal{A} = \{0\}$, in [9]. Since p and $p_{\theta}^{\mathcal{A}}$ have the same support, we deduce that $\mathbb{P}(L_{\mathcal{A}}(\tau) = n) > 0$ if and only if $\mathbb{P}(L_{\mathcal{A}}(\tau_{[\theta]}) = n) > 0$.

Proposition 3.29 (Same conditional distribution). *Let τ be a BGW tree with offspring distribution p . Let $\mathcal{A} \subset \mathbb{N}$ such that $p(\mathcal{A}) > 0$ and let $\theta \in I_{\mathcal{A}}$. Let $\tau_{[\theta]}$ be a BGW tree with offspring distribution $p_{\theta}^{\mathcal{A}}$. Then, the conditional laws of τ given $\{L_{\mathcal{A}}(\tau) = n\}$ and of $\tau_{[\theta]}$ given $\{L_{\mathcal{A}}(\tau_{[\theta]}) = n\}$ are the same for all $n \in \mathbb{N}$ such that $\mathbb{P}(L_{\mathcal{A}}(\tau) = n) > 0$.*

Notice that we don't assume that p is critical, sub-critical or super-critical in Proposition 3.29.

Proof. Let $\mathbf{t} \in \mathbb{T}_0$. Using (11) and the definition of $p_{\theta}^{\mathcal{A}}$, we have:

$$\begin{aligned} \mathbb{P}(\tau_{[\theta]} = \mathbf{t}) &= \prod_{v \in \mathbf{t}} p_{\theta}^{\mathcal{A}}(k_v(\mathbf{t})) \\ &= \left(\prod_{k_v(\mathbf{t}) \in \mathcal{A}} c_{\mathcal{A}}(\theta) \theta^{k_v(\mathbf{t})-1} p(k_v(\mathbf{t})) \right) \left(\prod_{k_v(\mathbf{t}) \notin \mathcal{A}} \theta^{k_v(\mathbf{t})-1} p(k_v(\mathbf{t})) \right) \\ &= c_{\mathcal{A}}(\theta)^{L_{\mathcal{A}}(\mathbf{t})} \theta^{\sum_{v \in \mathbf{t}} k_v(\mathbf{t}) - \#\mathbf{t}} \prod_{v \in \mathbf{t}} p(k_v(\mathbf{t})). \end{aligned}$$

Since $\sum_{v \in \mathbf{t}} k_v(\mathbf{t}) = \#\mathbf{t} - 1$, we deduce that:

$$(37) \quad \mathbb{P}(\tau_{[\theta]} = \mathbf{t}) = \frac{c_{\mathcal{A}}(\theta)^{L_{\mathcal{A}}(\mathbf{t})}}{\theta} \mathbb{P}(\tau = \mathbf{t}).$$

By summing (37) on $\{\mathbf{t} \in \mathbb{T}_0, L_{\mathcal{A}}(\mathbf{t}) = n\}$ and using (36), we obtain:

$$\mathbb{P}(L_{\mathcal{A}}(\tau_{[\theta]}) = n) = \frac{c_{\mathcal{A}}(\theta)^n}{\theta} \mathbb{P}(L_{\mathcal{A}}(\tau) = n).$$

Let $n \in \mathbb{N}$ be such that $\mathbb{P}(L_{\mathcal{A}}(\tau) = n) > 0$. Since $c_{\mathcal{A}}(\theta)$ is positive, by dividing (37) with this equation term by term, we get that for $\mathbf{t} \in \mathbb{T}_0$ such that $L_{\mathcal{A}}(\mathbf{t}) = n$, we have:

$$\mathbb{P}(\tau = \mathbf{t} \mid L_{\mathcal{A}}(\tau) = n) = \mathbb{P}(\tau_{[\theta]} = \mathbf{t} \mid L_{\mathcal{A}}(\tau_{[\theta]}) = n).$$

This ends the proof as τ (resp. $\tau_{[\theta]}$) is a.s. finite on $\{L_{\mathcal{A}}(\tau) = n\}$ (resp. $\{L_{\mathcal{A}}(\tau_{[\theta]}) = n\}$) by (36). \square

Remark 3.30 (Offspring distributions with the same distribution of conditioned BGW). Let τ be a BGW tree with offspring distribution p such that:

$$(38) \quad 0 < p(0) \quad \text{and} \quad p(0) + p(1) < 1.$$

Let $\mathcal{A} \subset \mathbb{N}$ such that $p(\mathcal{A}) > 0$. It is possible to characterize the set $\mathcal{P}(p, \mathcal{A})$ of probability p' satisfying the non-degeneracy condition (38), $\text{supp}(p') \subset \text{supp}(p)$ and for all $n \in \mathbb{N}$ such that $\mathbb{P}(L_{\mathcal{A}}(\tau') = n) > 0$, where τ' is the BGW tree with offspring distribution p' , we have that τ' conditionally on $\{L_{\mathcal{A}}(\tau') = n\}$ is distributed as τ conditionally on $\{L_{\mathcal{A}}(\tau) = n\}$. According to [1, Theorem 4.1], we have that $\mathcal{P}(p, \mathcal{A})$ is equal to $\{p_{\theta}^{\mathcal{A}} : \theta \in \dot{I}_{\mathcal{A}}\}$ with eventually (depending on p and \mathcal{A}) one or two more probability distributions associated in some sense to the parameters θ belonging to the boundary of $I_{\mathcal{A}}$ in $[0, +\infty]$.

3.4.2. *The generic sub-critical case.* Let p be a sub-critical offspring distribution such that $p(0)+p(1) < 1$ and let $\mathcal{A} \subset \mathbb{N}$ such that $p(\mathcal{A}) > 0$. For $\theta \in I_{\mathcal{A}}$, we denote by $m^{\mathcal{A}}(\theta)$ the mean value of the probability $p_{\theta}^{\mathcal{A}}$. By hypothesis, we have that $0 < m^{\mathcal{A}}(1) < 1$. According to Corollary 5.7 and Proposition 5.10 in [1], see also [5], the continuous map $\theta \mapsto m^{\mathcal{A}}(\theta)$ defined on $I^{\mathcal{A}}$ is finite on $\overset{\circ}{I}^{\mathcal{A}}$, increasing as long as $m^{\mathcal{A}}(\theta) \leq 1$ (and it might be decreasing at some $\theta \in I^{\mathcal{A}}$ where $m^{\mathcal{A}}(\theta) > 1$, see Remark 5.12 in [1]). We thus deduce the following lemma.

Lemma 3.31 (Existence of at most one critical parameter). *Let p be a sub-critical offspring distribution satisfying $p(0)+p(1) < 1$ and $\mathcal{A} \subset \mathbb{N}$ such that $p(\mathcal{A}) > 0$. There exists at most one critical value $\theta \in I_{\mathcal{A}}$ such that $m^{\mathcal{A}}(\theta) = 1$.*

When it exists, we denote the critical parameter by $\theta_{\mathcal{A}}^c$ given as the unique solution of $m^{\mathcal{A}}(\theta) = 1$ in $I_{\mathcal{A}}$; and we shall consider the critical offspring distribution:

$$(39) \quad p^{\mathcal{A}} = p_{\theta_{\mathcal{A}}^c}^{\mathcal{A}}.$$

Definition 3.32. *The offspring distribution p is said to be generic for the set \mathcal{A} if $\theta_{\mathcal{A}}^c$ exists.*

Let $d_{\mathcal{A}}$ be the period of the offspring distribution associated in Lemma 3.27 to $p^{\mathcal{A}}$ through the Rizzolo's bijection. Notice it depends only on $\text{supp}(p)$ and \mathcal{A} . Using Proposition 3.29 and Corollary 3.28, we immediately deduce the following result in the sub-critical generic case.

Proposition 3.33 (Generic case: the local limit is a Kesten's tree). *Let p be a sub-critical offspring distribution such that $p(0)+p(1) < 1$ and let $\mathcal{A} \subset \mathbb{N}$ such that $p(\mathcal{A}) > 0$. Assume that p is generic for \mathcal{A} . Let τ be a BGW tree with offspring distribution p and let $\tau_{\mathcal{A}}^*$ be a Kesten's tree associated with the offspring distribution $p^{\mathcal{A}}$ given by (39). Let τ_n be a random tree distributed according to τ conditionally on $\{L_{\mathcal{A}}(\tau) = 1 + d_{\mathcal{A}} n\}$ (resp. $\{L_{\mathcal{A}}(\tau) \geq n\}$). Then, we have:*

$$\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau_{\mathcal{A}}^*.$$

3.4.3. *The non-generic sub-critical case.* In order to state precisely the general result, we shall consider the set \mathbb{T}_{∞} of trees that may have infinite nodes and extend the definition of the local convergence on this set.

Let $n \in \mathbb{N}$. For $u = u_1 u_2 \dots u_n \in \mathcal{U}$, we set $|u|_{\infty} = \max\{n, \max\{u_i, 1 \leq i \leq n\}\}$ and we define the associated restriction operator r_n^{∞} on T^{∞} defined by:

$$r_n^{\infty}(\mathbf{t}) = \{u \in \mathbf{t}, |u|_{\infty} \leq n\}.$$

For all tree $\mathbf{t} \in \mathbb{T}_{\infty}$, the restricted tree $r_n^{\infty}(\mathbf{t})$ has height at most n and all the nodes have at most n offsprings (hence the tree $r_n^{\infty}(\mathbf{t})$ is finite). We define also the associated distance, for all $\mathbf{t}, \mathbf{t}' \in \mathbb{T}_{\infty}$:

$$\delta_{\infty}(\mathbf{t}, \mathbf{t}') = 2^{-\sup\{n \in \mathbb{N}, r_n^{\infty}(\mathbf{t}) = r_n^{\infty}(\mathbf{t}')\}}.$$

Remark that, for trees in \mathbb{T} , the topologies induced by the distances δ and δ_{∞} coincide. We will from now-on work on the space \mathbb{T}_{∞} endowed with the distance δ_{∞} . It is clear that the metric space $(\mathbb{T}_{\infty}, \delta_{\infty})$ is compact.

Set $\bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$. If $p = (p(n), n \in \mathbb{N})$ is a sub-critical offspring distribution with mean $m < 1$, we define $p^{**} = (p^{**}(n), n \in \bar{\mathbb{N}})$ a probability distribution on $\bar{\mathbb{N}}$ with:

$$p^{**}(n) = np(n) \quad \text{for } n \in \mathbb{N} \quad \text{and} \quad p^{**}(+\infty) = 1 - m.$$

We define a new random tree on \mathbb{T}_{∞} , denoted τ^{**} , associated to p in a way very similar to the definition of the Kesten's tree.

Definition 3.34. *Let p be a sub-critical offspring distribution. The condensation tree τ^{**} associated with p is a two-type BGW tree taking values in \mathbb{T}_{∞} and distributed as follows:*

- Individuals are normal or special.
- The root of τ^{**} special.
- A normal individual produces only normal individuals according to p .
- A special individual produces individuals according to the distribution p^{**} .

- If it has a finite number of offsprings, then one of them chosen uniformly at random, is special, the others (if any) are normal.
- If it has an infinite number of offsprings, then all of them are normal.

As we suppose that p is sub-critical (that is, $m < 1$), then the condensation tree τ^{**} associated with p has a.s. only one infinite node, and its random height is distributed as $G - 1$, where G has the geometric distribution with parameter $1 - m$.

Recall definitions from Section 3.4.1. Set:

$$\theta_{\mathcal{A}}^* = \sup I_{\mathcal{A}} \in [1, +\infty].$$

The next lemma completes Lemma 3.31, and it is part of Lemma 5.2 in [5]. We provide a short proof based on [1] since $c_{\mathcal{A}}(\theta_{\mathcal{A}}^*)$ in the definition of $I_{\mathcal{A}}$ is assumed to be positive, whereas it is only non-negative in [5].

Lemma 3.35 ($\theta_{\mathcal{A}}^*$ belongs to $I_{\mathcal{A}}$). *Let p be a sub-critical offspring distribution such that $p(0) + p(1) < 1$ and $\mathcal{A} \subset \mathbb{N}$ such that $p(\mathcal{A}) > 0$. If $m^{\mathcal{A}}(\theta) < 1$ for all $\theta \in I_{\mathcal{A}}$ (that is, p is not generic for \mathcal{A}), then the parameter $\theta_{\mathcal{A}}^*$ is finite belongs to $I_{\mathcal{A}}$.*

Proof. By continuity, it is possible that $\sum_{k \notin \mathcal{A}} (\theta_{\mathcal{A}}^*)^{k-1} p(k) = 1$ and $\sum_{k \in \mathcal{A}} (\theta_{\mathcal{A}}^*)^{k-1} p(k) < +\infty$ (and thus $\theta_{\mathcal{A}}^*$ belongs to $(1, +\infty)$). This corresponds to the case $\beta = 0$ in [1, Remark 3.7.d] (with $J = 1$) and thus $m^{\mathcal{A}}(\theta_{\mathcal{A}}^*) > 1$. So assuming that $m^{\mathcal{A}} < 1$ in $I_{\mathcal{A}}$ implies by monotone convergence that $\sum_{k \in \mathcal{A}} (\theta_{\mathcal{A}}^*)^{k-1} p(k)$ is finite and thus that $\sum_{k \notin \mathcal{A}} (\theta_{\mathcal{A}}^*)^{k-1} p(k) < 1$, that is, $\theta_{\mathcal{A}}^*$ belongs to $I_{\mathcal{A}}$. \square

When p is not generic for \mathcal{A} , we shall consider the sub-critical offspring distribution:

$$p^{\mathcal{A}} = p_{\theta_{\mathcal{A}}^*}^{\mathcal{A}}.$$

Let $d_{\mathcal{A}}$ denote the period of the offspring distribution associated in Lemma 3.27 to $p^{\mathcal{A}}$ through Rizzolo's bijection. Notice it depends only on $\text{supp}(p)$ and \mathcal{A} since $\text{supp}(p^{\mathcal{A}}) = \text{supp}(p)$.

Following the idea developed for the critical case, with more involved technicalities, we can prove the following result, see [5, Theorem 1.3]; this extends results on condensation from [36, 38] where only the case $\mathcal{A} = \mathbb{N}$ (i.e. the total population size) was considered.

Proposition 3.36 (Non-generic case: the local limit is a condensation tree). *Let p be a sub-critical offspring distribution such that $p(0) + p(1) < 1$ and let $\mathcal{A} \subset \mathbb{N}$ such that $p(\mathcal{A}) > 0$. Assume that p is not generic for \mathcal{A} . Let τ be a BGW tree with offspring distribution p and let $\tau_{\mathcal{A}}^{**}$ be a condensation tree associated with the sub-critical offspring distribution $p^{\mathcal{A}}$. Let τ_n be a random tree distributed according to τ conditionally on $\{L_{\mathcal{A}}(\tau) = 1 + d_{\mathcal{A}} n\}$ (resp. $\{L_{\mathcal{A}}(\tau) \geq n\}$). Then, we have:*

$$\tau_n \xrightarrow[n \rightarrow \infty]{(d)} \tau_{\mathcal{A}}^{**}.$$

Remark 3.37. Considering the local limit of BGW aims at understanding the shape of the random tree near the root. The local limit of the BGW tree rerooted at a random node is studied in [58] with $\mathcal{A} = \mathbb{N}$ in the generic and non-generic cases; in both cases the limiting tree is similar either to the Kesten's tree or the condensation tree.

The condensation tree appears also as the limit of subcritical BGW with different conditioning: see [34] where the BGW tree is conditioned on the maximal out-degree to be large or [4] where the nodes of the tree are marked independently with a probability depending on their out-degree and the tree is conditioned to have a large number of marks.

3.4.4. Generic and non-generic distributions. Let p be a sub-critical offspring distribution such that $p(0) + p(1) < 1$. We shall give a criterion to say easily for which sets \mathcal{A} the offspring distribution p is generic. As we have $m < 1$, we want to find a θ (which will be greater than 1) such that $m^{\mathcal{A}}(\theta) = 1$. This problem is closely related to the radius $\rho \geq 1$ of convergence of the generating function of p , denoted by g .

We have the following result.

Lemma 3.38 ([5], Lemma 5.4). *Let p be a sub-critical offspring distribution such that $p(0) + p(1) < 1$.*

- (i) If $\rho = +\infty$ or if ($\rho < +\infty$ and $g'(\rho) \geq 1$), then p is generic for any $\mathcal{A} \subset \mathbb{N}$ such that $p(\mathcal{A}) > 0$.
- (ii) If $\rho = 1$ (that is, the probability p admits no exponential moment), then p is non-generic for every $\mathcal{A} \subset \mathbb{N}$ such that $p(\mathcal{A}) > 0$.
- (iii) If $1 < \rho < +\infty$ and $g'(\rho) < 1$, then p is non-generic for $\mathcal{A} \subset \mathbb{N}$, with $p(\mathcal{A}) > 0$, if and only if:

$$\mathbb{E}[Y|Y \in \mathcal{A}] < \frac{\rho - \rho g'(\rho)}{\rho - g(\rho)},$$

where Y is distributed according to $p_\rho^{\mathbb{N}}$, that is:

$$\mathbb{E}[f(Y)] = \frac{\mathbb{E}[f(\zeta)\rho^\zeta]}{g(\rho)}.$$

In particular, p is non-generic for $\mathcal{A} = \{0\}$ but generic of $\mathcal{A} = \{k\}$ for any k large enough such that $p(k) > 0$.

Remark 3.39. In case (iii) of Lemma 3.38, we gave in Remark 5.5 of [5]:

- a sub-critical offspring distribution which is generic for \mathbb{N} but non-generic for $\{0\}$;
- a sub-critical offspring distribution which is non-generic for \mathbb{N} but generic for $\{k\}$ with k large enough.

This shows that the genericity of sets is not monotone with respect to the inclusion.

3.5. Other results. We mention some related problem.

- (1) **Marked BGW tree.** The proof of Theorem 3.7 can be slightly modified to study a tree with randomly marked nodes: conditionally given the tree, we mark its nodes randomly, independently of each others, with a probability that depends only on the out-degree of the node. Then we obtain that a critical BGW tree conditioned on having n marked nodes still converges in distribution toward a Kesten's tree, see [2]. This allows to study a Galton-Watson tree conditioned on the number of protected nodes where a protected node is a node that is neither a leaf nor the parent of a leaf. See also [4] for the sub-critical case in the generic and non-generic case.
- (2) **Very fat BGW trees.** Recall that $Z_n = z_n(\tau)$ represents the population size at generation n of the BGW tree τ . In the same spirit as in Section 3.3.4, we can consider a critical BGW tree τ conditioned on $\{Z_n = a_n\}$ with a sequence $(a_n, n \in \mathbb{N}^*)$ of positive integers. Notice this does not fall into the framework developed in Section 3.2 for a functional A with Monotonicity-Additivity-Identity property. Nevertheless, the proof of Theorem 3.7 can be adapted, see [6], to prove that a critical BGW tree with geometric offspring distribution conditioned on $\{Z_n = a_n\}$ with $\lim_{n \rightarrow \infty} a_n/n^2 = 0$ converges in distribution toward the Kesten's tree. The case $a_n \sim Cn^2$ is more involved as the infinite spine is replaced by an infinite backbone that does not satisfy the usual branching property: the numbers of offspring inside a generation are not independent, see [3] for details. This new limit appears also when considering large BGW trees with n nodes and exponential weight given by its height, see [24].

Let us mention that there are some results, with the same flavor, established in [7] for the super-critical case that can also be adapted to BWG tree with sub-critical offspring distribution p such that $g(r) = r$ has a root strictly larger than 1 (this amounts for p to be of the form \tilde{p} in (12)). The local convergence of the critical BGW tree conditioned to $\{Z_n = a_n\}$ with a sequence $(a_n, n \in \mathbb{N}^*)$ such that $\lim_{n \rightarrow \infty} a_n = +\infty$ is an open question in general.

- (3) **Multi-type BGW tree.** The ideas from Section 3.3.7 for the conditioning on $\{L_{\mathcal{A}}(\tau) = n\}$ in the critical case can also be studied for critical multi-type BGW trees conditioned on the size the population of each type, see [8] where the limit is now a multi-type Kesten's tree. In this setting, the asymptotic proportion of each type is fixed equal to the normalized left-eigen-vector of the mean matrix associated to the eigen-value 1. See also [53, 57, 60] for other results on this topic. Let us mention that conditioning a BGW tree τ on the event $\{L_{\mathcal{A}_1}(\tau) = n_1, \dots, L_{\mathcal{A}_k}(\tau) = n_k\}$, where the sets $\mathcal{A}_1, \dots, \mathcal{A}_k$ are pairwise disjoint, and $n_1 + \dots + n_k$ goes to infinity, is studied in [1] in the generic case using a multi-type BGW tree

approach. The non generic case for this problem and more generally for conditioned multi-type sub-critical BGW tree is still an open question.

- (4) **Lévy tree.** There is also a natural extension of this work when considering scaling limit of conditioned critical BGW: see [22, 45] when conditioning on the total size or the number of leaves, see also [46] for scaling limits in the sub-critical non-generic case (where condensation occurs). It is also possible to consider the same conditioning directly on the Lévy tree which appears as the scaling limit of the BGW tree: see [35, 11] when conditioning on a node to be very large or [10, 50] when conditioning on the total size to be large.

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