A wellposed hypoelastic model derived from a hyperelastic one

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N. Favrie* and S. Gavrilyuk†

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Hypoelastic models are widely used in industrial and military codes for numerical simulation of high strain dynamics of solids. This class of model is often mathematically inconsistent. More exactly, the second principle is not verified on the solutions of the model, and the initial state after a reversible cycle is not recovered. In the past decades, hyperelastic models, which are mathematically consistent, have been intensively studied. For their practical use, ones needs to entirely rewrite the commercial codes. Moreover, calibration of equation of states would be needed. In this paper two hypoelastic models for isotropic solids are derived from equivalent hyperelastic models. The hyperelastic models are hyperbolic for all possible deformations. It allows us to use robust Godunov's schemes for numerical resolution of these models. Two new objective derivatives corresponding to two different equations of state and defining the evolution of the deviatoric part of the stress tensor naturally appear. These derivatives are compatible with the reversibility property of the model : it conserves the specific entropy in a continuous motion. The most used hypoelastic model (Wilkins model) is recovered in the small deformation limit.

Keywords: Hyperelasticity, hypoelasticity, objective derivatives

I. INTRODUCTION

In the literature, two classes of models for a high strain dynamics of solids can be found: *hypoelastic* and *hyperelastic* ones. Hypoelastic models (Wilkins, 1964) are widely used in industrial and military numerical codes (LS-Dyna, CTH (USA), OURANOS (France), EGIDA (Russia), ...). For this class of models an empirical partial differential equation for the deviatoric part of the stress tensor is formulated to closure the governing equations. The deviatoric stress rate depends on the choice of a so called *objective derivative* (cf. Trusdell and Noll (2003) [24], Gurtin *et al.* (2010) [11]). These hypoelastic models presents two main drawbacks :

- in absence of dissipation, the entropy is not in general conserved for continuous motions (see Gavrilyuk *et al.* 2008 [7], and Maire *et al.* 2013(a,b) [16], [17] for details).
- the choice of the objective derivative is not unique, and thus the obtained results will strongly depend on such a choice (Szabo & Balla, 1989 [23], Rouhaud *et al.* (2013) [21], Korobeynikov (2008) [14]).

Hyperelastic models have been intensively studied in the last decades (Godunov and Romenskii (2003) [10], Kulikovskii *et al.* 2000 [15], Miller and Collela (2002)[18], Gavrilyuk *et al.* (2008)[7], Kluth and Deprès (2008) [13], Godunov and Peshkov (2008) [9], Gorsse *et al.* (2014) [12], ...). In these models the stress tensor is obtained by variation of the internal energy. The models are conservative and hyperbolic if the internal energy is rank-one convex (Dafermos, 1999 [1]). An extension of this class of models can be given when viscoplastic effects are present. Also, a multiphase formulation of hyperelasticity allowing us to model solid-fluid interaction can be given (Favrie *et al.* 2009 [2], Favrie and Gavrilyuk 2011(a,b),[3], [4], 2012 [5], Ndanou *et al.* (2015) [20]). When the dissipation is added, these models verify the second law of thermodynamics. In general, the hyperelastic models have better mathematical and numerical properties.

In this paper, we propose a link between hypoelastic and hyperelastic models. This link is in some sense obvious and related to the problem of inversion of stress - strain relation (Romenskii, 1974 [22]). However, an explicit inversion is needed for practical applications. We give in this note an explicit example of a non-linear equation of state where such an inversion is performed. A natural *objective derivative* appears associated with such a law.

This paper is organized as follows. The governing constitutive model for hyperelasticity is presented section II. The derivation of the hypoelastic model is presented in Section III.

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II. A GENERAL HYPERELASTIC MODEL FORMULATION

The governing equations of isotropic elastic solids can be written in the following form :

$$\begin{array}{l}\frac{\partial \rho}{\partial t}+div(\rho \mathbf{v})=0,\\ \frac{\partial \rho \mathbf{v}}{\partial t}+div(\rho \mathbf{v}\otimes\mathbf{v}-\sigma)=0,\\ \frac{\partial \rho E}{\partial t}+div(\rho E\mathbf{v}-\sigma\mathbf{v})=0,\\ \frac{d\mathbf{e}^{\beta}}{dt}+\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^{T}\mathbf{e}^{\beta}=0, \quad rot\mathbf{e}^{\beta}=0,\end{array}$$

with

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{v}\cdot\bigtriangledown$$

Here $\mathbf{e}^{\beta},\beta=1,2,3$ are the columns of $\mathbf{F}^{-T}$, $\mathbf{F}$ is the deformation gradient, $\mathbf{G}=\mathbf{F}^{-T}\mathbf{F}^{-1}=\sum_{\beta=1}^{3}\mathbf{e}^{\beta}\otimes\mathbf{e}^{\beta}$ is the Finger tensor, $\mathbf{B}=\mathbf{G}^{-1}$ is the left Cauchy-Green strain tensor, ρ is the solid density considered is an independent variable, $\mathbf{v}$ is the the velocity field, σ is the stress tensor, $E=e+\frac{1}{2}|\mathbf{v}|^{2}$ is the specific total energy, $e(\mathbf{G},\eta)$ is the specific internal energy depending only on the invariants of $\mathbf{G}$ and the specific entropy η. The symmetric stress tensor is given by :

$$\sigma=-2\rho\frac{\partial e}{\partial \mathbf{G}}\mathbf{G}=-p\mathbf{I}+\mathbf{S} \tag{1}$$

where p is the hydrodynamic pressure, $\mathbf{I}$ is the identity matrix, and $\mathbf{S}$ is the deviatoric part of the stress tensor. This model must be completed by giving the equation of state $e(\mathbf{G},\eta)$. We take the specific energy in separable form :

$$e=e^{h}(\rho,\eta)+e^{e}(\mathbf{g}), \quad \mathbf{g}=\frac{\mathbf{G}}{\det(\mathbf{G})^{1/3}}=\mathbf{b}^{-1}.$$

Thus, the energy is the sum of he hydrodynamic energy $e^{h}(\rho,\eta)$ and the shear energy $e^{e}(\mathbf{g})$ depending only on the reduced Finger tensor $\mathbf{g}$. The shear energy is unaffected by the volume change. For applications, the hydrodynamic part of the energy $e^{h}(\rho,\eta)$ can be taken in the form of stiffened gas equation of state :

$$\begin{array}{c}e^{h}(\rho,p)=\frac{p+\gamma p_{\infty}}{\rho(\gamma-1)}, \quad p+\gamma p_{\infty}=A(\eta)\rho^{\gamma},\\ \gamma=const, \frac{dA}{d\eta}>0, \quad p=\rho^{2}\frac{\partial e^{h}}{\partial \rho}.\end{array}$$

We take the elastic energy $e^{e}(\mathbf{g})$ in the form :

$$e^{e}(\mathbf{g})=\frac{\mu}{4\rho_{0}}\left(\frac{1-2a}{3}j_{1}^{2}+aj_{2}+3(a-1)\right), \tag{2}$$

$$j_{i}=\mathrm{Tr}(\mathbf{g}^{i}), \quad i=1,2,$$

where μ is the shear modulus, and a is a non-linearity parameter. With such an EOS, for any value of a the classical Hooke law is recovered at the limit of small deformations. The model is hyperbolic for any $a\in[-1,0.5]$ (Ndanou *et al.* (2013) [19], Favrie *et al.* (2014)[6], Gavrilyuk *et al.* (2015) [8]). In the following, we will consider two limit cases:

- the case where $a=-1$:

$$e^{e}(\mathbf{g})=\frac{\mu}{4\rho_{0}}\left(j_{1}^{2}-j_{2}-6\right)=\frac{\mu}{2\rho_{0}}\left(i_{1}-3\right), \; i_{1}=\mathrm{Tr}(\mathbf{b}). \tag{3}$$

- the case where $a=0.5$:

$$e^{e}(\mathbf{g})=\frac{\mu}{8\rho_{0}}\left(j_{2}-3\right), \quad j_{2}=\mathrm{Tr}(\mathbf{g}^{2}). \tag{4}$$

To increase the readability of the paper the calculation details will be given only in the case where $a=-1$ which corresponds to the equation of state for neo-hookean solids.

The equations for $\mathbf{e}^{\beta}$ admit the following consequences :

$$\overset{\diamond}{\mathbf{B}}=\frac{d\mathbf{B}}{dt}-\mathbf{B}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^{T}-\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\mathbf{B}=0, \tag{5}$$

$$\overset{\vartriangle}{\mathbf{G}}=\frac{d\mathbf{G}}{dt}+\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^{T}\mathbf{G}+\mathbf{G}\frac{\partial \mathbf{v}}{\partial \mathbf{x}}=0. \tag{6}$$

(5) correspond to Lie derivative of a two times contravariant tensor, and (6) that of a two times covariant tensor (see [11] for definitions). These derivatives are objectives.

III. EVOLUTION EQUATION FOR THE DEVIATORIC PART OF THE STRESS TENSOR

For the energy in the form (3) the deviatoric part of the stress tensor can easily be calculated :

$$\mathbf{S}=\mu\left(\frac{\rho}{\rho_{0}}\right)^{5/3}\left(\mathbf{B}-\frac{1}{3}\mathrm{Tr}(\mathbf{B})\mathbf{I}\right).$$

Applying to this equation the material derivative and using the conservation of mass we have :

$$\frac{d\mathbf{S}}{dt} = -\frac{5}{3}\mathbf{S}div(\mathbf{v}) + \mu\left(\frac{\rho}{\rho_0}\right)^{5/3}\left(\frac{d\mathbf{B}}{dt} - \frac{1}{3}\mathrm{Tr}\left(\frac{d\mathbf{B}}{dt}\right)\mathbf{I}\right).$$

Since

$$\frac{d\mathbf{B}}{dt} = \mathbf{B}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\mathbf{B},$$

we have

$$\mu\left(\frac{\rho}{\rho_0}\right)^{5/3}\frac{d\mathbf{B}}{dt} = \mathbf{S}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\mathbf{S} + \mu\left(\frac{\rho}{\rho_0}\right)^{5/3}\frac{2}{3}\mathrm{Tr}(\mathbf{B})\mathbf{D}$$

with

$$\mathbf{D} = \frac{1}{2}\left(\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)$$

being the rate of deformation tensor. It comes:

$$\overset{\diamond}{\mathbf{S}} + \frac{5}{3}\mathbf{S}div(\mathbf{v}) + \frac{2}{3}\mathrm{Tr}\left(\mathbf{SD}\right)\mathbf{I} = = \mu\left(\frac{\rho}{\rho_0}\right)^{5/3}\frac{2}{3}\mathrm{Tr}(\mathbf{B})(\mathbf{D} - \frac{1}{3}tr(\mathbf{D})\mathbf{I})$$

An equivalent form is :

$$\overset{\diamond}{\mathbf{S}} + \frac{5}{3}\mathbf{S}div(\mathbf{v}) + \frac{2}{3}\mathrm{Tr}\left(\mathbf{SD}\right)\mathbf{I} = \mu\left(\frac{\rho}{\rho_0}\right)^{2/3}\frac{2}{3}\mathrm{Tr}(\mathbf{b})(\mathbf{D} - \frac{1}{3}tr(\mathbf{D})\mathbf{I}). \quad (7)$$

We need now to express $\mathrm{Tr}(\mathbf{b})$ as a function of $\mathbf{S}$ to close (7).

A. Expression of $\mathrm{Tr}(\mathbf{b})$ as a function of the invariants of S

To simplify the notations, we introduce

$$\tilde{\mathbf{S}} = \frac{\mathbf{S}}{\mu\left(\frac{\rho}{\rho_0}\right)^{5/3}}.$$

In the orthogonal basis of eigenvectors of $\mathbf{S}$ we have

$$\tilde{S}_i = b_i - \frac{1}{3}\mathrm{Tr}(\mathbf{b}), \quad i = 1, 2, 3.$$

Here $\tilde{S}_i$ and b_i are the eigenvalues of $\tilde{\mathbf{S}}$ and $\mathbf{b}$. It implies

$$b_i = \frac{1}{3}\mathrm{Tr}(\mathbf{b}) + \tilde{S}_i.$$

Since $\det(\mathbf{b}) = 1$ and $\mathrm{Tr}(\tilde{\mathbf{S}}) = 0$, we obtain :

$$1 = g_1g_2g_3 = (\tfrac{1}{3}\mathrm{Tr}(\mathbf{b}) + \tilde{S}_1)(\tfrac{1}{3}\mathrm{Tr}(\mathbf{b}) + \tilde{S}_2)(\tfrac{1}{3}\mathrm{Tr}(\mathbf{b}) + \tilde{S}_3) = \alpha^3 + p\alpha + q + 1, \quad (8)$$

with

$$\alpha = \frac{1}{3}\mathrm{Tr}(\mathbf{b}), \quad p = -\frac{\mathrm{Tr}\left(\tilde{\mathbf{S}}^2\right)}{2}, \quad q = \det(\tilde{\mathbf{S}}) - 1.$$

If

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27} = \frac{(\det(\tilde{\mathbf{S}}) - 1)^2}{4} - \frac{\left(\mathrm{Tr}\left(\tilde{\mathbf{S}}^2\right)\right)^3}{216} > 0,$$

we will have a unique real solution :

$$\alpha = \frac{1}{3}\mathrm{Tr}(\mathbf{b}) = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}.$$

In applications, Δ is always positive. In Figure 1, we plot in the plane $\tilde{S}_1, \tilde{S}_2$ in thick line the level set $\Delta = 0$. In dashed lines we plot the yield surface for $\frac{Y}{\mu} = 1.6$. For metals or carbon fibers such a ratio is almost vanishing, and the yield surface is reduced to almost a dot. In the case where $a = 0.5$ we introduce

$$\tilde{\mathbf{S}} = -2\frac{\mathbf{S}}{\mu\frac{\rho}{\rho_0}}.$$

An analogous equation (8) is obtained where $Tr(\mathbf{b})$ should be replaced by $Tr(\mathbf{g}^2)$.

B. Hypoelastic formulation

The hypoelastic model can be rewritten under the following form :

$$\begin{aligned} \frac{\partial \rho}{\partial t} + div(\rho\mathbf{v}) &= \mathbf{0} \\ \frac{\partial \rho\mathbf{v}}{\partial t} + div(\rho\mathbf{v}\otimes\mathbf{v} + p - \mathbf{S}) &= 0 \\ \frac{\partial \rho E}{\partial t} + div(\rho E\mathbf{v} + p\mathbf{v} - \mathbf{S}\mathbf{v}) &= 0 \\ \frac{d_O\mathbf{S}}{dt} = 2\mu\left(\frac{\rho}{\rho_0}\right)^{2/3}\frac{\mathrm{Tr}(\mathbf{b})}{3}(\mathbf{D} - \frac{1}{3}\mathrm{Tr}(\mathbf{D})\mathbf{I}) & \end{aligned} \quad (9)$$

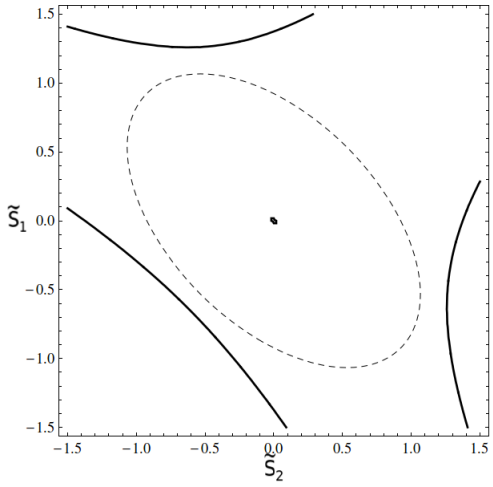

FIG. 1. The limit of the domain where the discriminant is positive $\Delta = 0$ is represented in thick lines. The Von Mises yield surface for Carbon fibbers is almost a dot. In dashed lines we plot the yield surface for $\frac{Y}{\mu} = 1.6$. The validity domain of the unique solution for $\mathrm{Tr}(\mathbf{b})$ is much bigger than the yield limit of usual materials

with a new objective derivative

$$\frac{d_O\mathbf{S}}{dt} = \overset{\diamond}{\mathbf{S}} + \frac{5}{3}\mathbf{S}div(\mathbf{v}) + \frac{2}{3}\mathrm{Tr}\left(\mathbf{SD}\right)\mathbf{I}. \qquad (10)$$

Here

$$E = e^h(p,\rho) + \frac{3\mu}{2\rho_0}\left(\alpha - 1\right) + \frac{1}{2}\mathbf{v}\cdot\mathbf{v},$$

$$\alpha = \frac{\mathrm{Tr}(\mathbf{g})}{3}$$

$$= \sqrt[3]{\frac{\det\left(\frac{\rho_0\mathbf{S}}{\rho\mu}\right)+1}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{\det\left(\frac{\rho_0\mathbf{S}}{\rho\mu}\right)+1}{2} - \sqrt{\Delta}},$$

$$\Delta = \left(\frac{\left(\det\left(\frac{\rho_0\mathbf{S}}{\rho\mu}\right)+1\right)^2}{4} - \frac{\mathrm{Tr}\left(\left(\frac{\rho_0\mathbf{S}}{\rho\mu}\right)^2\right)^3}{216}\right).$$

In the small deformation limit where $\frac{\rho}{\rho_0} \approx 1$ and $\alpha \approx 1$, equation (9) becomes :

$$\frac{d_O\mathbf{S}}{dt} = 2\mu(\mathbf{D} - \frac{1}{3}tr(\mathbf{D})\mathbf{I}),$$

which is similar to the classical Wilkins model (see Wilkins (1964) [25]) with a new objective derivative. An analogous inversion, but with a different objective derivative is obtained in Appendix in the case where $a = 0.5$, i.e. the energy is taken in the form (4). This choice leads to (see details in Appendix A):

$$\overset{\triangle}{\mathbf{S}} - \frac{1}{3}\mathbf{S}div(\mathbf{v}) - \frac{2}{3}\mathrm{Tr}\left(\mathbf{SD}\right)\mathbf{I} =$$
$$= \mu\left(\frac{\rho}{\rho_0}\right)\frac{1}{3}\mathrm{Tr}(\mathbf{g}^2)(\mathbf{D} - \frac{1}{3}tr(\mathbf{D})\mathbf{I}) + \frac{\mu}{2}\frac{\rho}{\rho_0}\mathbf{gDg} \qquad (11)$$

with $\mathbf{g}^2 = \frac{1}{3}Tr(\mathbf{g}^2)\mathbf{I} - 2\frac{\rho_0}{\rho}\frac{\mathbf{S}}{\mu}$. So, $\mathbf{g}$ can be obtained as the square root of $\mathbf{g}^2$, while $\mathrm{Tr}(\mathbf{g}^2)$ is obtained as in the case of $\mathrm{Tr}(\mathbf{b})$.

IV. CONCLUSIONS

From two particular models of nonlinear hyperelasticity, we derive two hypoelastic models which are equivalent to the original hyperelastic ones. New objective derivatives appear which are specific to the equations of state used.

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[1] Dafermos, C. M. (1999). Hyperbolic Conservation Laws in Continuum Physics, Springer. New York.

[2] Favrie, N., Gavrilyuk, S. L., and Saurel, R. (2009). Solid-fluid diffuse interface model in cases of extreme deformations. J. Computational Physics, 228(16), 6037-6077.

[3] Favrie, N., and Gavrilyuk, S. (2011a). Mathematical and numerical model for nonlinear viscoplasticity. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 369(1947), 2864-2880.

[4] Favrie, N., and Gavrilyuk, S. (2011b, November). Dynamics of shock waves in elastic-plastic solids. In ESAIM: Proceedings (Vol. 33, pp. 50-67). EDP Sciences.

[5] Favrie, N., and Gavrilyuk, S. L. (2012). Diffuse interface model for compressible fluid – compressible elastic-plastic solid interaction. J. Computational Physics, 231(7), 2695-2723.

[6] Favrie, N., Gavrilyuk, S., and Ndanou, S. (2014). A thermodynamically compatible splitting procedure in hyperelasticity. J. Computational Physics, 270, 300-324.

[7] Gavrilyuk, S., Favrie, N. and Saurel, R. (2008). Modeling wave dynamics of compressible elastic materials. J. Computational Physics, 227(5), 2941-2969.

[8] Gavrilyuk, S. Ndanou, S. and Hank, S. (2015) One-parameter family of equations of state for isotropic compressible solids. https://hal.archives-ouvertes.fr/hal-01162870

[9] Godunov, S. K. and Peshkov, I. Y. M. (2008). Symmetric hyperbolic equations in the nonlinear elasticity theory. Computational Mathematics and Mathematical Physics, 48(6), 975-995.

[10] Godunov, S. K., and Romenskii, E. I. Elements of Continuum Mechanics and Conservation Laws. 2003.

[11] Gurtin, M. E., Fried, E., and Anand, L. (2010). The mechanics and thermodynamics of continua. Cambridge University Press.

[12] Gorsse, Y., Iollo, A., Milcent, T., and Telib, H. (2014). A simple Cartesian scheme for compressible multimaterials. J. Computational Physics, 272, 772-798.

[13] Kluth, G. and Desprès, B. (2008) Perfect plasticity and hyperelastic models for isotropic materials, Continuum Mechanics and Thermodynamics, Vol. 20, No. 3, 173-192.

[14] Korobeynikov, S. N. (2008). Objective tensor rates and applications in formulation of hyperelastic relations. Journal of Elasticity, 93(2), 105-140.

[15] Kulikovskii, A. G., Pogorelov, N. V., and Semenov, A. Y. (2000). Mathematical aspects of numerical solution of hyperbolic systems. CRC Press.

[16] Maire, P.-H., Le Tallec, P. Rebourcet, B. and Abgrall, R. (2013) A simple elasticity model at large deformations or where does come from the Wilkins model? MULTIMAT Conference, San Francisco, Sept 2-6, 2013, availlable at multimat13.llnl.gov/media/pdf/papers/MultiMat_2013_Maire.pdf

[17] Maire, P.-H., Abgrall, R., Breil, J., Loubère, R. and Rebourcet, B. (2013) A nominally second-order cell-centered Lagrangian scheme for simulating elastic–plastic flows on two-dimensional unstructured grids, J. Computational Physics, Volume 235, Pages 626–665.

[18] Miller, G. H., Colella, P. (2002) A conservative three-dimensional Eulerian method for coupled fluid-solid shock capturing, J. Computational Physics. Vol. 183, pp. 26-82.

[19] Ndanou, S., Favrie, N. and Gavrilyuk, S. (2014). Criterion of hyperbolicity in hyperelasticity in the case of the stored energy in separable form. Journal of Elasticity, 115: 1-25.

[20] Ndanou, S., Favrie, N., and Gavrilyuk, S. (2015). Multi-solid and multi-fluid diffuse interface model: Applications to dynamic fracture and fragmentation. J. Computational Physics, 295, 523-555.

[21] Rouhaud, E., Panicaud, B., and Kerner, R. (2013). Canonical frame-indifferent transport operators with the four-dimensional formalism of differential geometry. Computational Materials Science, 77, 120-130.

[22] Romenskii, E.I, (1974) Hypoelastic form of equations in nonlinear elasticity theory, Zhurnal Prikladnoi Mekhaniki i Tekhnieheskoi Fiziki, No. 2, pp. 133-138

[23] Szabo, L., and Balla, M. (1989). Comparison of some stress rates. International journal of solids and structures, 25(3), 279-297.

[24] Truesdell, C. and Noll, W. (2003) The Non-Linear Field Theories of Mechanics, 3rd Edition, Springer, 2003

[25] Wilkins, M.L. (1964) Calculation of elastic-plastic flow. Methods in Computational physics, Edited by B.Alder, S.Fernbach, and M. Rotenbeg (Academic, New York), Vol. 3, pp. 211-263

Appendix A: case where $a = 0.5$

1. Evolution equation for the deviatoric part of the stress tensor

In the case where $a = 0.5$, i.e. the energy is taken in the form (4) , the deviatoric part of the stress

tensor can be expressed in the form :

$$\mathbf{S} = -\frac{\mu}{2}\left(\frac{\rho}{\rho_0}\right)^{-1/3}\left(\mathbf{G}^2 - \frac{1}{3}\mathrm{Tr}(\mathbf{G}^2)\mathbf{I}\right).$$

Applying the material derivative and using the conservation of mass we have :

$$\frac{d\mathbf{S}}{dt} = \frac{1}{3}\mathbf{S}div(\mathbf{v}) - \frac{\mu}{2}\left(\frac{\rho}{\rho_0}\right)^{-1/3}\left(\frac{d\mathbf{G}^2}{dt} - \frac{1}{3}\mathrm{Tr}\left(\frac{d\mathbf{G}^2}{dt}\right)\mathbf{I}\right). \tag{A1}$$

Since

$$\frac{d\mathbf{G}}{dt} = -\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T\mathbf{G} - \mathbf{G}\frac{\partial \mathbf{v}}{\partial \mathbf{x}},$$

it comes :

$$\frac{d\mathbf{G}^2}{dt} = -2\mathbf{GDG} - \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T\mathbf{G}^2 - \mathbf{G}^2\frac{\partial \mathbf{v}}{\partial \mathbf{x}},$$

we have

$$\begin{aligned}&\frac{\mu}{2}\left(\frac{\rho}{\rho_0}\right)^{-1/3}\frac{d\mathbf{G}}{dt}\\ &= \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T\mathbf{S} + \mathbf{S}\frac{\partial \mathbf{v}}{\partial \mathbf{x}} - \frac{\mu}{2}\left(\frac{\rho}{\rho_0}\right)^{-1/3}\frac{2}{3}\mathrm{Tr}(\mathbf{G}^2)\mathbf{D}\\ &\quad -\frac{\mu}{2}\left(\frac{\rho}{\rho_0}\right)^{-1/3}\mathbf{GDG}\end{aligned}$$

with

$$\mathbf{G} = \sqrt{\frac{1}{3}Tr(\mathbf{G}^2)\mathbf{I} - 2\frac{\mathbf{S}}{\mu}\left(\frac{\rho}{\rho_0}\right)^{1/3}} \tag{A2}$$

It comes:

$$\begin{aligned}&\overset{\triangle}{\mathbf{S}} - \frac{1}{3}\mathbf{S}div(\mathbf{v}) - \frac{2}{3}\mathrm{Tr}\left(\mathbf{SD}\right)\mathbf{I} =\\ &= \mu\left(\frac{\rho_0}{\rho}\right)^{1/3}\frac{1}{3}\mathrm{Tr}(\mathbf{G}^2)(\mathbf{D} - \frac{1}{3}tr(\mathbf{D})\mathbf{I})\\ &\quad +\frac{\mu}{2}\left(\frac{\rho_0}{\rho}\right)^{1/3}\mathbf{GDG}\end{aligned}$$

An alternative form is :

$$\begin{aligned}&\overset{\triangle}{\mathbf{S}} - \frac{1}{3}\mathbf{S}div(\mathbf{v}) - \frac{2}{3}\mathrm{Tr}\left(\mathbf{SD}\right)\mathbf{I} =\\ &= \mu\left(\frac{\rho}{\rho_0}\right)\frac{1}{3}\mathrm{Tr}(\mathbf{g}^2)(\mathbf{D} - \frac{1}{3}tr(\mathbf{D})\mathbf{I}) + \frac{\mu}{2}\frac{\rho}{\rho_0}\mathbf{gDg}\end{aligned} \tag{A3}$$

with

$$\mathbf{g} = \sqrt{\frac{1}{3}Tr(\mathbf{g}^2)\mathbf{I} - 2\frac{\mathbf{S}}{\mu}\frac{\rho_0}{\rho}} \tag{A4}$$

2. Expression of $\mathrm{Tr}(\mathbf{g})$ as a function of the invariants of S

The stress can be rewritten in term of $\mathbf{g}$ under the form:

$$\mathbf{S} = -\frac{\mu}{2}\frac{\rho}{\rho_0}\left(\mathbf{g}^2 - \frac{1}{3}\mathrm{Tr}(\mathbf{g}^2)\mathbf{I}\right).$$

To simplify the notations, we introduce

$$\tilde{\mathbf{S}} = -2\frac{\mathbf{S}}{\mu\frac{\rho}{\rho_0}}.$$

In the orthogonal basis of eigenvectors of $\mathbf{S}$ we have

$$\tilde{S}_i = g_i^2 - \frac{1}{3}\mathrm{Tr}(\mathbf{g}^2), \quad i = 1, 2, 3.$$

Here $\tilde{S}_i$ and g_i^2 are the eigenvalues of $\tilde{\mathbf{S}}$ and $\mathbf{g}^2$. It implies

$$g_i^2 = \frac{1}{3}\mathrm{Tr}(\mathbf{g}^2) + \tilde{S}_i.$$

Since $\det(\mathbf{g}) = 1$ and $\mathrm{Tr}(\tilde{\mathbf{S}}) = 0$, we obtain :

$$\begin{aligned}1 &= g_1^2 g_2^2 g_3^2\\ &= (\tfrac{1}{3}\mathrm{Tr}(\mathbf{g}^2) + \tilde{S}_1)(\tfrac{1}{3}\mathrm{Tr}(\mathbf{g}^2) + \tilde{S}_2)(\tfrac{1}{3}\mathrm{Tr}(\mathbf{g}^2) + \tilde{S}_3)\\ &= \alpha^3 + p\alpha + q + 1,\end{aligned}$$

with

$$\alpha = \frac{1}{3}\mathrm{Tr}(\mathbf{g}), \quad p = -\frac{\mathrm{Tr}\left(\tilde{\mathbf{S}}^2\right)}{2}, \quad q = \det(\tilde{\mathbf{S}}) - 1.$$

This equation is identical to equation (8).