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UNIQUENESS RESULTS FOR DISCRETE SCHRÖDINGER EVOLUTIONS

PHILIPPE JAMING, YURII LYUBARSKII, EUGENIA MALINNIKOVA,
AND KARL-MIKHAEL PERFEKT

ABSTRACT. We prove that if a solution of the discrete time-dependent Schrödinger equation with bounded time-independent real potential decays fast at two distinct times then the solution is trivial. For the free Schrödinger operator or operators with compactly supported potential a sharp analog of the Hardy uncertainty principle is obtained. The argument is based on the theory of entire functions. Logarithmic convexity of weighted norms is employed for the case of general real-valued bounded potential, for this case the result is not optimal.

1. INTRODUCTION

The aim of this paper is to show that a non-trivial solution of semi-discrete Schrödinger equations with bounded potential cannot have arbitrary fast decay at two different times. For the free evolution (with no potential) the result we obtain is precise and it can be interpreted as a discrete version of the dynamical Hardy Uncertainty Principle.

The usual way to formulate the Uncertainty Principle is to say that a function and its Fourier transform can not be both arbitrarily well localized. For example, in Hardy's Uncertainty Principle, the localization is measured in terms of speed of decay at infinity: *if $f \in L^2(\mathbb{R})$ is such that f and its Fourier transform \hat{f} satisfy*

$$|f(x)| \leq C \exp(-\pi|x|^2), \quad |\hat{f}(\xi)| \leq C \exp(-\pi|\xi|^2), \quad x \in \mathbb{R},$$

for some constant $C > 0$, then there is a constant A such that $f(x) = A \exp(-\pi|x|^2)$.

It is known that Uncertainty Principles may also be given dynamical interpretations in terms of solution of the free Schrödinger Equation see e.g. [10, 13, 14] and references therein. Hardy's Uncertainty Principle is equivalent to the following statement:

if $u(t, x)$ is a solution of the free Schrödinger equation $\partial_t u = i\Delta u$ and $|u(0, x)| + |u(1, x)| \leq C \exp(-x^2/4)$, then $u(0, x) = A \exp(-(1+i)x^2/4)$.

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The point is that the free Schrödinger Equation can be solved via a Fourier transform thus a solution of this equation cannot decrease arbitrarily fast at two different times.

In a remarkable series of papers, L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega [9–12] and in collaboration with Cowling in [7] have shown that the Hardy Uncertainty Principle is equivalent to a uniqueness property of solutions to Schrödinger Evolution and extended this uniqueness result to solutions of the Schrödinger equation with a potential, as well as to solutions of a wide class of PDE including some non-linear equations. Results on sharp decay for covariant Schrödinger evolutions were obtained in [2, 4].

In the present work we obtain uniqueness results for solutions of the discrete time-dependent Schrödinger equation,

$$(1) \quad \partial_t u = i(\Delta_d u + V u),$$

where $u : \mathbb{R}_+ \times \mathbb{Z} \rightarrow \mathbb{C}$ and Δ_d is the discrete Laplacian, that is, for a complex valued function $f : \mathbb{Z} \rightarrow \mathbb{C}$,

$$\Delta_d f(n) := f(n+1) + f(n-1) - 2f(n).$$

We assume that the potential $V = V(t, n)$ is a real-valued, bounded function. For this case $H = \Delta_d + V$ is a bounded self-adjoint operator in ℓ^2 and, for any $u(0, \cdot) \in \ell^2$, there exists a unique solution of (1) with $u(0, \cdot)$ as the initial value: $u(t, \cdot) = e^{itH} u(0, \cdot) \in C^1([0, \infty), \ell^2)$. We call such a u a strong solution of (1).

We first mention that a discrete dynamical interpretation of the Heisenberg uncertainty principle was given in [13]. We obtain a uniqueness result for free evolution that can be considered as a discrete version of the dynamical interpretation of Hardy's Uncertainty Principle and then extend it for equations with bounded potentials. The corresponding uniqueness result for discrete Schrödinger evolution has some similarities to the continuous case but it is also different in many respects. First of all, the critical growth is different, though for both situations the optimal growth is that of the heat kernel at time 1. For the continuous case the standard heat kernel is $k(1, 0, x) = (4\pi)^{-1/2} \exp(-x^2/4)$, while for the discrete case the heat kernel is $K(1, 0, n) = e^{-1} |I_n(1)| \asymp e^{-1} (n! 2^n)^{-1}$, where I_n are the modified Bessel functions, $I_n(z) = (-i)^n J_n(iz)$. Computations of the discrete heat kernel for the lattice and asymptotic connecting the two cases can be found in [5, 6]. We note also that discrete heat kernels appeared as weights for convexity results for discrete harmonic functions in a recent work by G. Lippner and D. Mangoubi [16].

To understand better the features of the discrete Schrödinger evolution we start with the free operator and we assume for simplicity that the spacial dimension is one. We prove that if $u(t, n)$ solves the free equation

$$\partial_t u = i\Delta_d u$$

and satisfies

$$(2) \quad |u(0, n)| + |u(1, n)| \leq C \frac{1}{\sqrt{|n|}} \left(\frac{e}{2|n|} \right)^{|n|}, \quad n \in \mathbb{Z} \setminus \{0\},$$

then $u(t, n) = Ai^{-n}e^{-2it}J_n(1-2t)$, where J_n is the Bessel function (the definition is given below). This result is sharp: $|J_n(-1)|$ and $|J_n(1)|$ have precisely the growth of the right hand side in (2) as $|n| \rightarrow \infty$, see Proposition 2.1.

Then we investigate the discrete equation (1) with bounded potential in essentially two different ways using techniques of complex and real analysis. First, applying theory of entire functions, we establish that if $V(n)$ is a compactly supported potential and u is a strong solution of (1) satisfying the one-sided estimates

$$|u(t, n)| \leq C \left(\frac{e}{(2 + \epsilon)n} \right)^n, \quad n > 0, \quad t \in \{0; 1\},$$

for some $\epsilon > 0$, then $u \equiv 0$. In the continuous setting, one-sided Hardy uncertainty principles have previously appeared in works of Nazarov [17] and Demange [8]. The corresponding results for continuous Schrödinger evolutions can be found in the recent survey [12].

In the second part of the paper, we use the real-variable approach following [10]. The main idea is to construct a weight function $\psi(t, n)$ which provides the logarithmic convexity of the weighted ℓ^2 norms $\|\psi(t, \cdot)u(t, \cdot)\|_{\ell^2(\mathbb{Z})}$, where $u(t, n)$ is a strong solution of (1). This line of reasoning has its roots in celebrated results of T. Carleman and S. Agmon, the technique of Carleman estimates goes back to [3] and convexity principles for elliptic operators were described in [1]. The method allows us to consider general bounded potentials V , however so far we have to assume stronger decay of $u(0, n)$ and $u(1, n)$ along both directions $n \rightarrow \pm\infty$. The main result, Theorem 4.2, says that if

$$\left\| (1 + |n|)^{\gamma(1+|n|)} u(0, n) \right\|_{\ell^2(\mathbb{Z})}, \left\| (1 + |n|)^{\gamma(1+|n|)} u(1, n) \right\|_{\ell^2(\mathbb{Z})} < \infty$$

for some $\gamma > (3 + \sqrt{3})/2$, then $u \equiv 0$. We don't expect this result to be sharp. However it provides a universal growth decay that implies uniqueness for solutions of Schrödinger evolution with a general bounded potential.

The paper is organized as follows: in Section 2 we discuss preliminaries of entire functions and use them to obtain our first results. Section 3 contains a precursory energy estimate for solutions of (1), which we need in order to justify the validity of many of our computations. Section 4 splits into several subsections discussing and proving the logarithmic convexity results we require, and in the final subsection the main result is proven.

Note that we will use the symbol C to denote various constants in what follows. Unless otherwise indicated, its value might change from line to line.

Also all $\|\cdot\|_2$ will be $\ell^2(\mathbb{Z})$ -norms in the n -variable.

2. A UNIQUENESS RESULT FOR THE SCHRÖDINGER OPERATOR WITH COMPACTLY SUPPORTED POTENTIAL

In this section, we use methods from complex analysis. For the reader's convenience, we begin by briefly outlining some definitions and facts on entire functions of exponential type that we need. Details can be found in [15] (see in particular

Lectures 8 and 9). Recall that an entire function f is said to be of exponential type if for some $k > 0$

$$(3) \quad |f(z)| \leq \exp(k|z|), \quad \text{for } |z| > r,$$

for some $r > 0$. For this case the type of an entire function f is defined by

$$(4) \quad \sigma = \limsup_{r \rightarrow \infty} \frac{\log \max\{|f(re^{i\phi})|; \phi \in [0, 2\pi]\}}{r} < \infty.$$

In particular, an entire function f is of zero exponential type if for any $k > 0$ there exists $r = r(k)$ such that (3) holds.

Let $f(z)$ be an entire function of exponential type, $f(z) = \sum_{n=0}^{\infty} c_n z^n$, then the type of f can be expressed in terms of the Taylor coefficients in the following way

$$(5) \quad \limsup_{n \rightarrow \infty} n|c_n|^{1/n} = e\sigma.$$

The growth of function f of exponential type along various directions is described by its indicator function

$$h_f(\varphi) = \limsup_{r \rightarrow \infty} \frac{\log |f(re^{i\varphi})|}{r}.$$

This function is the support function of some convex compact set $I_f \subset \mathbb{C}$ which is called the indicator diagram of f , in particular

$$(6) \quad h_f(\varphi) + h_f(\pi - \varphi) \geq 0.$$

For example the indicator function of e^{az} for $a \in \mathbb{C}$ is $h(\varphi) = \Re(ae^{i\varphi})$ and its indicator diagram consists of a single point, \bar{a} .

Clearly, $h_{fg}(\varphi) \leq h_f(\varphi) + h_g(\varphi)$. This implies that

$$I_{fg} \subset I_f + I_g := \{z = z_1 + z_2 : z_1 \in I_f, z_2 \in I_g\}.$$

The optimal decay of a solution to the free discrete Schrödinger equation is given by the Bessel functions; remind that the Bessel functions J_n satisfy

$$\exp(x(z - z^{-1})/2) = \sum_{n=-\infty}^{\infty} J_n(x)z^n, \quad z \neq 0.$$

Moreover, for fixed x ,

$$|J_n(x)| \sim \frac{1}{\sqrt{|n|}} \left(\frac{ex}{2|n|} \right)^{|n|} \quad \text{as } |n| \rightarrow \infty.$$

Our first observation is the following discrete analog of the classical Hardy uncertainty principle.

Proposition 2.1. *Let $u \in C^1([0, 1], \ell^2)$ satisfy the discrete free Schrödinger equation $\partial_t u = i\Delta_d u$, and also*

$$(7) \quad |u(0, n)|, |u(1, n)| \leq C \frac{1}{\sqrt{|n|}} \left(\frac{e}{2|n|} \right)^{|n|}, \quad n \in \mathbb{Z} \setminus \{0\}.$$

for some $C > 0$. Then $u(t, n) = Ai^{-n}e^{-2it}J_n(1 - 2t)$ for all $n \in \mathbb{Z}$ and $0 \leq t \leq 1$, for some constant A .

Proof. Consider the discrete Fourier transforms of $u(t, \cdot)$,

$$\Phi(t, \theta) = \sum_{k=-\infty}^{\infty} u(t, k)\theta^k \in L^2(\mathbb{T}).$$

We have $\partial_t \Phi(t, \theta) = i(\theta + \theta^{-1} - 2)\Phi(t, \theta)$. Thus

$$\Phi(t, \theta) = e^{i(\theta + \theta^{-1} - 2)t}\Phi(0, \theta),$$

in particular

$$(8) \quad \Phi(1, \theta) = e^{i(\theta + \theta^{-1} - 2)}\Phi(0, \theta).$$

It follows from (7) that the functions $\theta \mapsto \Phi(s, \theta)$, for $s = 0$, and $s = 1$ admit holomorphic extensions to $\mathbb{C} \setminus \{0\}$:

$$(9) \quad \Phi(s, \theta) = \sum_{k < 0} u(k, s)\theta^k + \sum_{k \geq 0} u(k, s)\theta^k = \Phi^-(s, \theta) + \Phi^+(s, \theta), \quad s \in \{0; 1\}.$$

and (7) implies $\Phi^+(s, \theta)$ and $\Phi^-(s, 1/\theta)$, $s = 0$ and $s = 1$ are entire functions of exponential type whose indicator diagrams I_s^+ and respectively I_s^- are contained in the disk of radius $1/2$ centered at zero.

Actually one can say more:

$$(10) \quad |\Phi^+(s, \theta)|, |\Phi^-(s, 1/\theta)| \leq Ce^{|\theta|/2}, \quad s \in \{0; 1\}.$$

This follows from the fact that the right-hand side of (7) represent asymptotic of the coefficients in the Taylor expansion of $\exp(z/2)$.

On the other hand it follows from (8) that $I_1^\pm \subset I_0^\pm + i$. Thus $I_0^\pm = \{-i/2\}$ and $I_1^\pm = \{i/2\}$.

Now let

$$(11) \quad g(z) = g^+(z) + g^-(z) = e^{i(z+z^{-1})/2}\Phi(0, z) = e^{-i(z+z^{-1})/2}\Phi(1, z),$$

where, as before, g^\pm are the parts of the Laurent series of g with respectively non-negative and negative powers. It follows that the indicator diagrams I^\pm of the entire functions $g^+(z)$ and $g^-(1/z)$ coincide with $\{0\}$, so $g^+(z)$ and $g^-(1/z)$ are entire functions of type zero.

Relations (10), (11) now yield that $g^+(iy)$ and $g^-(1/iy)$ are bounded for $y \in \mathbb{R} \setminus \{0\}$ and by the Phragmen-Lindelöf principle (see [15], Lecture 6) g^+ , g^- , and hence g , are constants. Finally $\Phi_0(z) = A \exp(-i(z + z^{-1})/2)$ which yields the required expression for $u(t, n)$. \square

Corollary 2.2. *Let u be as in Proposition 2.1 if in addition*

$$|u(0, n)| \left(\frac{2|n|}{e} \right)^{|n|} \sqrt{|n|} = o(1)$$

as $n \rightarrow +\infty$ or $n \rightarrow -\infty$ then $u \equiv 0$.

Assuming a bit stronger decay, one can apply similar techniques in order to obtain uniqueness result for solution of the discrete Schrödinger equations with compactly supported potentials. In this case it suffices to demand that the solution decays just in one direction.

Theorem 2.3. *Let $u(t, n)$, $t > 0$, $n \in \mathbb{Z}$ be a solution of (1), where the potential V does not depend on time and also $V(n) \neq 0$ just for a finite number of n 's. If, for some $\varepsilon > 0$,*

$$(12) \quad |u(t, n)| \leq C \left(\frac{e}{(2 + \varepsilon)n} \right)^n, \quad n > 0, t \in \{0; 1\},$$

then $u = 0$.

Proof. We may assume that $V_n = 0$ for $n > N$ and for $n < 0$. Consider the bounded operator $H = \Delta_d + V : \ell^2 \rightarrow \ell^2$. The solution $u(t, n)$ is then defined by

$$u(\cdot, t) = e^{iHt}u(\cdot, 0)$$

and hence belongs ℓ^2 for all $t > 0$.

The absolutely continuous spectrum of $H : \ell^2 \rightarrow \ell^2$ is the segment $[0, 4]$, each point with multiplicity 2. The continuous spectrum is parametrized naturally by the the unit circle \mathbb{T} :

$$\lambda \in [0, 4] \Rightarrow \lambda = 2 - \theta - \theta^{-1}, \text{ for some } \theta \in \mathbb{T}.$$

For each $\theta \in \mathbb{T}$ set $\lambda(\theta) = 2 - \theta - \theta^{-1}$ and denote by $e^\pm(\theta) = e^\pm(\theta, n)$ the corresponding Jost solutions of the spectral problem

$$(13) \quad Hx = \lambda(\theta)x,$$

i.e. the solutions of (13) satisfying

$$e^+(\theta, n) = \theta^n, \text{ for } n > N, \text{ and } e^-(\theta, n) = \theta^n \text{ for } n < 0.$$

We refer the reader to [18] for precise definition and detailed discussion of Jost solutions.

Except for $\theta = \pm 1$, each of the pairs $\{e^+(\theta), e^+(\theta^{-1})\}$, $\{e^-(\theta), e^-(\theta^{-1})\}$ is a fundamental system of solutions of (13). Hence we have the representations

$$\begin{aligned} e^+(\theta) &= a^-(\theta)e^-(\theta) + b^-(\theta)e^-(\theta^{-1}) \\ e^-(\theta) &= a^+(\theta)e^+(\theta) + b^+(\theta)e^+(\theta^{-1}) \end{aligned}$$

It is known, see e.g. [18], that a^\pm and b^\pm are rational functions of θ , with no poles on \mathbb{T} , and also for $0 \leq n \leq N$ the functions $e^\pm(\theta, n)$ are linear combinations of θ^j , $j \in \{-N, -N + 1, \dots, 2N\}$. In particular,

$$(14) \quad \lim_{|\theta| \rightarrow +\infty} \frac{\log |a^+(\theta)|}{|\theta|} = \lim_{|\theta| \rightarrow +\infty} \frac{\log |b^+(\theta)|}{|\theta|} = 0$$

and

$$(15) \quad \lim_{|\theta| \rightarrow +\infty} \frac{\log |e^\pm(\theta, n)|}{|\theta|} = 0 \quad \text{for } n = 0, \dots, N.$$

Consider the function

$$\begin{aligned}\Phi(\theta, t) &= \sum_{-\infty}^{\infty} u(t, n)e^{-}(\theta, n) = \sum_{-\infty}^{-1} u(t, n)e^{-}(\theta, n) \\ &\quad + a^{+}(\theta) \sum_0^{\infty} u(t, n)e^{+}(\theta, n) + b^{+}(\theta) \sum_0^{\infty} u(t, n)e^{+}(\theta^{-1}, n).\end{aligned}$$

For all $t \geq 0$ these functions are in $L^2(\mathbb{T})$. In addition the first and the third series in the right-hand side converge for $|\theta| > 1$ while the second one converges for $|\theta| < 1$. For $t = 0$ and $t = 1$ the second term also converges for $|\theta| > 1$, by the hypothesis (12), thus the functions $\Phi(\theta, 0)$ and $\Phi(\theta, 1)$ are holomorphic in $\mathbb{C} \setminus \{0\}$ except perhaps at the poles of the functions a^{+} and b^{+} . (Actually by the basic energy estimate in the next section one can extend this convergence property to $\Phi(\theta, t)$ for all $t \in [0, 1]$, see Corollary 3.2. We do not need this fact here.)

We have

$$\begin{aligned}-i \frac{\partial \Phi(\theta, t)}{\partial t} &= \sum_{n=-\infty}^{\infty} (Hu)(t, n)e^{-}(n, \theta) = \sum_{n=-\infty}^{\infty} u(t, n)(He^{-})(n, \theta) \\ &= (2 - \theta - \theta^{-1})\Phi(\theta, t).\end{aligned}$$

Hence $\Phi(\theta, t) = e^{it(2-\theta-\theta^{-1})}\Phi(\theta, 0)$, and in particular

$$(16) \quad \Phi(\theta, 1) = e^{i(2-\theta-\theta^{-1})}\Phi(\theta, 0).$$

This relation can be extended to the whole complex plane.

To derive a contradiction to $\Phi \neq 0$, we write $\Phi(\theta, t)$ as

$$\begin{aligned}\Phi(\theta, t) &= \left(\sum_{n < 0} u(t, n)\theta^n + \sum_{n=0}^N u(t, n)e^{-}(n, \theta) + b^{+}(\theta) \sum_{n > N} u(t, n)\theta^{-n} \right) \\ &\quad + a^{+}(\theta) \left(\sum_{n > N} u(t, n)\theta^n \right) =: A(\theta, t) + a^{+}(\theta)B(\theta, t).\end{aligned}$$

Since $u(t, \cdot) \in \ell^2$, we clearly have

$$\lim_{\theta \rightarrow \infty} \frac{\log |A(\theta, t)|}{|\theta|} = \lim_{\theta \rightarrow \infty} \frac{\log |a^{+}(\theta, t)|}{|\theta|} = 0,$$

while (12) yields that $B(\theta, t)$, $t = 0$ and $t = 1$, are entire functions of exponential type at most $(2 + \varepsilon)^{-1}$. Hence, for each $\alpha \in [0, 2\pi]$ we have

$$\limsup_{r \rightarrow \infty} \frac{\log |B(re^{i\alpha}, t)|}{r} \leq \frac{1}{2 + \varepsilon}, \quad t \in \{0; 1\},$$

and therefore

$$(17) \quad \limsup_{r \rightarrow \infty} \frac{\log |\Phi(re^{i\alpha}, t)|}{r} \leq \frac{1}{2 + \varepsilon}, \quad t \in \{0; 1\}.$$

By (6), we also have

$$\limsup_{r \rightarrow \infty} \frac{\log |\Phi(re^{i\alpha}, t)|}{r} \geq -\frac{1}{2 + \varepsilon}, \quad t \in \{0; 1\}.$$

This leads to a contradiction unless $\Phi \equiv 0$, since according (16) we have

$$\limsup_{y \rightarrow +\infty} \frac{\log |\Phi(iy, 1)|}{y} = 1 + \limsup_{y \rightarrow +\infty} \frac{\log |\Phi(iy, 0)|}{y} > 1 - \frac{1}{2 + \varepsilon} > \frac{1}{2 + \varepsilon}.$$

□

It would be interesting to extend this result to the case of a fast decaying potential for which the Jost solutions exist.

3. FIRST ENERGY ESTIMATE

In the remaining of the paper we will follow ideas of [10], to prove that a solution of the discrete Schrödinger equation which decays sufficiently fast along both half-axes at two different moments of time is trivial.

We begin with an energy estimate for solution of non-homogeneous initial problem and show that if the initial data is well-concentrated, the energy cannot spread out too fast.

Given $\alpha > 0$ and $t \geq 0$ denote

$$\psi_\alpha(t) = \{\psi_\alpha(t, n)\}_{n \in \mathbb{Z}} = \{(1 + |n|)^{\alpha|n|/(1+t)}\}_{n \in \mathbb{Z}}$$

Proposition 3.1. *Let $V = V_1 + iV_2$, with $V_1, V_2 : [0, T] \times \mathbb{Z} \rightarrow \mathbb{R}$ and V_2 bounded and $F : [0, T] \times \mathbb{Z} \rightarrow \mathbb{C}$ bounded. Let $u : [0, T] \times \mathbb{Z} \rightarrow \mathbb{C}$, $u \in \mathcal{C}^1([0, T], \ell^2(\mathbb{Z}))$, satisfy*

$$(18) \quad \partial_t u(t, n) = i(\Delta u(t, n) + V(t, n)u + F(t, n)).$$

Assume that $\{\psi_\alpha(0, n)u(0, n)\} \in \ell^2(\mathbb{Z})$ for some $\alpha \in (0, 1]$. Then, for $T > 0$,

$$(19) \quad \|\psi_\alpha(T, n)u(T, n)\|_2^2 \leq e^{CT} \left(\|\psi_\alpha(0, n)u(0, n)\|_2^2 + \int_0^T \|\psi_\alpha(s, n)F(s, n)\|_2^2 ds \right)$$

Proof. Consider $f(t, n) = \psi_\alpha(t, n)u(t, n)$ and let $H(t) = \|f(t, n)\|_2^2$. We fix α till the end of the proof and write $\psi = \psi_\alpha$.

We will perform several formal computations, assuming that $H(t)$ is finite for all $t \in [0, T]$, and then justify these computations at the end of the proof.

Define

$$\kappa(n, t) = \log \psi(t, n) = \frac{\alpha}{1+t} |n| \log(1 + |n|).$$

Then

$$\partial_t f = i\Delta(\psi^{-1}f) + iVf + \partial_t \kappa f + i\psi F,$$

which we rewrite as $\partial_t f = \mathcal{S}f + \mathcal{A}f + iVf + i\psi F$, where \mathcal{S} and \mathcal{A} are symmetric and anti-symmetric operators, respectively. Explicitly

$$\begin{aligned}\mathcal{S}f &= \frac{i}{2} (\psi \Delta(\psi^{-1}f) - \psi^{-1} \Delta(\psi f)) + \partial_t \kappa f, \\ \mathcal{A}f &= \frac{i}{2} (\psi \Delta(\psi^{-1}f) + \psi^{-1} \Delta(\psi f)).\end{aligned}$$

Denote

$$(20) \quad a_n = \frac{\psi_{n+1}}{\psi_n} - \frac{\psi_n}{\psi_{n+1}}, \quad b_n = \frac{\psi_{n+1}}{\psi_n} + \frac{\psi_n}{\psi_{n+1}}.$$

We then rewrite

$$(21) \quad (\mathcal{S}f)_n = -\frac{i}{2}(a_n f_{n+1} - a_{n-1} f_{n-1}) + (\partial_t \kappa)_n f_n,$$

$$(22) \quad (\mathcal{A}f)_n = \frac{i}{2}(b_n f_{n+1} + b_{n-1} f_{n-1}) - 2i f_n,$$

In what follows we will interchangeably use the notation $a_n = a(t, n)$, similarly for ψ_n etc.

We want to control the growth of $H(t)$. Clearly, $\partial_t H(t) = 2\Re\langle \partial_t f, f \rangle$ and thus

$$\begin{aligned}\partial_t H(t) &= 2\langle \mathcal{S}f, f \rangle - 2\Im\langle Vf, f \rangle - 2\Im\langle \psi F, f \rangle \\ &= 2\Im \sum_n a_n f_{n+1} \bar{f}_n + 2\langle \partial_t \kappa f, f \rangle - 2\langle V_2 f, f \rangle - 2\Im\langle \psi F, f \rangle.\end{aligned}$$

This implies

$$\partial_t H(t) \leq 2\|\psi F\|_2 \|f\|_2 + \|V_2\|_\infty \|f\|_2 + \sum_n (2\partial_t \kappa_n + |a_n| + |a_{n-1}|) |f_n|^2.$$

Our aim is to prove that for all $n \in \mathbb{Z}$

$$(23) \quad 2\partial_t \kappa_n + |a_n| + |a_{n-1}| \leq 2C,$$

where C is a constant. We have

$$\partial_t \kappa_n = -\frac{\alpha}{(1+t)^2} |n| \log(|n|+1).$$

Further, $|a_n| \leq e^\alpha (|n|+1)^\alpha$. Hence, if $\alpha \leq 1$ we obtain (23), for $t \in [0, 1]$.

Therefore $\partial_t \|f\|_2 \leq C\|f\|_2 + \|\psi F\|_2$ and (19) follows.

In order to justify these computations we truncate the weight function ψ to an interval $[-N, N]$:

$$\psi_N(n, t) = \begin{cases} (|n|+1)^{(1+t)^{-1}\alpha|n|}, & |n| \leq N \\ (|N|+1)^{(1+t)^{-1}\alpha|N|}, & |n| > N. \end{cases}$$

Since the solution u is in ℓ^2 , the relevant norms weighted by ψ_N are guaranteed to be finite and by running the above argument we obtain (23) and (19) for the weight ψ_N , this time rigorously. The desired inequality follows by passing to the limit as $N \rightarrow \infty$. \square

Corollary 3.2. *Let $u : [0, 1] \times \mathbb{Z} \rightarrow \mathbb{C}$ be a strong solution of the Schrödinger equation*

$$\partial_t u(t, n) = i(\Delta u(t, n) + V(t, n)u),$$

where $V = V_1 + iV_2$ is as above. Further suppose that

$$\sum_{n>0} n^{2\alpha n} |u(0, n)|^2 < \infty$$

for some $\alpha \leq 1$. Then for each $t \in [0, 1]$ we have

$$\sum_{n>0} n^{\alpha n} |u(t, n)|^2 < \infty.$$

Proof. Define $\tilde{u}(t, n) = 0$ for $n < 0$ and $\tilde{u}(t, n) = u(t, n)$ for $n \geq 0$. Then \tilde{u} satisfies (18) with $F(t, n)$ bounded and vanishing for $n \notin \{-1, 0\}$. If we apply Proposition 3.1 to \tilde{u} we obtain the required estimate \square

4. LOGARITHMIC CONVEXITY OF WEIGHTED ℓ^2 -NORMS

4.1. Preliminary discussion. From now on we fix $\gamma_0 > 0$ and suppose that $V : [0, T] \times \mathbb{Z} \rightarrow \mathbb{R}$ is bounded. Further, we assume that u is a strong solution of

$$\partial_t u = i(\Delta_d u + Vu)$$

such that $\|(1 + |n|)^{\gamma_0(1+|n|)} u(0, n)\|_2$ and $\|(1 + |n|)^{\gamma_0(1+|n|)} u(1, n)\|_2$ are finite.

Following the ideas of [10], we are looking for a weight

$$(24) \quad \psi(t, n) = \exp(\kappa(t, n))$$

to give us a logarithmically convex function $e^{-Ct(1-t)} H(t)$, where

$$H(t) = \|\psi(t, n)u(t, n)\|_2^2$$

and C depends on V and ψ .

We will first use such a convexity argument to show that for any $0 < \gamma < \gamma_0$ and any $t \in [0, 1]$,

$$(25) \quad \|(1 + |n|)^{\gamma(1+|n|)} u(t, n)\|_2 < \infty.$$

This also implies that

$$(26) \quad \|(C_0 + |n| + R_0 t(1-t))^{\gamma(C_0 + |n| + R_0 t(1-t))} u(t, n)\|_2 < +\infty$$

for any $C_0, R_0 > 0$ and $t \in [0, 1]$, and we then set out to prove the logarithmic convexity in t of this latter norm.

In both steps we consider weights of the form (24), with

$$\kappa(t, n) = \gamma(|n| + R(t)) \ln^b(|n| + R(t))$$

where either $1/2 < b < 1$ and $R(t) = 1$, or $b = 1$ and $R(t) = C_0 + R_0 t(1-t)$. As before we set $f(t, n) = \psi(t, n)u(t, n)$.

We will first assume that $b < 1$, prove estimates independent of b , and let $b \rightarrow 1$ to establish (25). This will allow us to justify the computations involved in the second step, when $b = 1$ and we prove the convexity of (26).

4.2. Formal computations. We collect here a number of formal identities which we need in the sequel. First identities are the same as in the continuous case, they can be found in [12], others are specific to the discrete case. The notation has been explained in the proof of Proposition 3.1.

We already know that $\partial_t H(t) = 2\langle \mathcal{S}f, f \rangle$, since V is real-valued, and thus

$$\begin{aligned} \partial_t^2 H(t) &= 2\langle \mathcal{S}_t f, f \rangle + 4\Re\langle \mathcal{S}f, f_t \rangle \\ &= 2\langle \mathcal{S}_t f, f \rangle + 4\|\mathcal{S}f\|^2 + 2\langle [\mathcal{S}, \mathcal{A}]f, f \rangle + 4\Re\langle \mathcal{S}f, iVf \rangle \\ &= 2\langle \mathcal{S}_t f, f \rangle + 2\langle [\mathcal{S}, \mathcal{A}]f, f \rangle + 4\Re\langle \mathcal{S}f + iVf, \mathcal{S}f \rangle \\ &= 2\langle \mathcal{S}_t f, f \rangle + 2\langle [\mathcal{S}, \mathcal{A}]f, f \rangle + \|2\mathcal{S}f + iVf\|^2 - \|Vf\|^2. \end{aligned}$$

Therefore we obtain that

$$\begin{aligned} \|f\|^2 \partial_t^2 (\log H(t)) &= \frac{\|2\mathcal{S}f + iVf\|^2 \|f\|^2 - 4|\langle \mathcal{S}f, f \rangle|^2}{\|f\|^2} \\ &\quad + 2(\langle \mathcal{S}_t f, f \rangle + \langle [\mathcal{S}, \mathcal{A}]f, f \rangle) - \|Vf\|^2 \\ &= \frac{\|2\mathcal{S}f + iVf\|^2 \|f\|^2 - |\Re(2\mathcal{S}f + iVf, f)|^2}{\|f\|^2} \\ &\quad + 2(\langle \mathcal{S}_t f, f \rangle + \langle [\mathcal{S}, \mathcal{A}]f, f \rangle) - \|Vf\|^2 \\ &\geq 2(\langle \mathcal{S}_t f, f \rangle + \langle [\mathcal{S}, \mathcal{A}]f, f \rangle) - \|Vf\|^2. \end{aligned}$$

We reiterate that our aim is to show that

$$(27) \quad \partial_t^2 \log H(t) \geq -2C$$

for some $C \geq 0$, which implies the log-convexity of $e^{-Ct(1-t)}H(t)$. The last term in the right-hand side above is clearly bounded below by $-C\|f\|^2$ since V is bounded. It suffices to establish an estimate of the first two terms of the form

$$(28) \quad \langle \mathcal{S}_t f, f \rangle + \langle [\mathcal{S}, \mathcal{A}]f, f \rangle > -C\|f\|^2.$$

We refer now to (22), (21). It follows that

$$(\mathcal{S}_t f)_n = -i/2(a'_n f_{n+1} - a'_{n-1} f_{n-1}) + \kappa''_n f_n,$$

and finally

$$(2\mathcal{S}_t f + 2[\mathcal{S}, \mathcal{A}]f)_n = \nu_{n+1} f_{n+2} + \lambda_n f_{n+1} + \mu_n f_n + \overline{\lambda_{n-1}} f_{n-1} + \nu_{n-1} f_{n-2},$$

where

$$\begin{aligned} \nu_{n+1} &= \frac{1}{2}(a_n b_{n+1} - a_{n+1} b_n), \\ \lambda_n &= -ib_n(\kappa'_{n+1} - \kappa'_n) - ia'_n, \\ \mu_n &= a_n b_n - a_{n-1} b_{n-1} + 2\kappa''_n, \end{aligned}$$

and, as before, the coefficients a_n, b_n are defined in (20).

Clearly $\psi'_n = \kappa'_n \psi_n$, implying that $a'_n = (\kappa'_{n+1} - \kappa'_n)b_n$ and

$$\lambda_n = -2ib_n(\kappa'_{n+1} - \kappa'_n).$$

4.3. Estimates with an auxiliary weight.

Proposition 4.1. *Let $\gamma > 0$. Assume that u is a strong solution of*

$$\partial_t u = i(\Delta_d u + V u)$$

where the potential V is a bounded real-valued function. Let also

$$(29) \quad \left\| (1 + |n|)^{\gamma(1+|n|)} u(t, n) \right\|_2 < +\infty, \quad t \in \{0; 1\}.$$

Then, for all $t \in [0, 1]$, $\left\| (1 + |n|)^{\gamma(1+|n|)} u(t, n) \right\|_2 < +\infty$.

Proof. Consider the weight function

$$\psi(n) = e^{\kappa_b(n)}, \quad \kappa_b(n) = \gamma(1 + |n|) \ln^b(1 + |n|),$$

where $1/2 < b < 1$. Note that the hypotheses (29) combined with Proposition 3.1 show that $H_b(t) = \left\| \exp(\kappa_b(n)) u(t, n) \right\|_2^2$ is finite for all t , allowing us to justify the computations of the preceding section for this choice of weight. We will show that $H(t) = H_b(t)$ satisfies (27) with some C independent of b , whence

$$\begin{aligned} \left\| \exp(\kappa_b(n)) u(t, n) \right\|_2^2 &\leq e^{\frac{C}{2}t(1-t)} H_b(0)^{1-t} H_b(1)^t \\ &\leq e^{\frac{C}{2}t(1-t)} \left\| (1 + |n|)^{\gamma(1+|n|)} u(0, n) \right\|_2^{2(1-t)} \left\| (1 + |n|)^{\gamma(1+|n|)} u(1, n) \right\|_2^{2t}. \end{aligned}$$

Letting $b \rightarrow 1$ and applying the monotone convergence theorem then concludes the proof.

We refer to computations in the previous section. In the current setting $S_t = 0$ and $\lambda_n = 0$ so relation (28) reduces to

$$(30) \quad \langle 2[\mathcal{S}, \mathcal{A}]f, f \rangle \geq -C\|f\|^2.$$

We have

$$\langle 2[\mathcal{S}, \mathcal{A}]f, f \rangle = \sum_n \mu_n |f_n|^2 + 2\Re \sum_n \nu_{n+1} f_{n+2} \overline{f_n},$$

where

$$\mu_n = a_n b_n - a_{n-1} b_{n-1} = \frac{\psi_{n+1}^2}{\psi_n^2} - \frac{\psi_n^2}{\psi_{n-1}^2} - \frac{\psi_n^2}{\psi_{n+1}^2} + \frac{\psi_{n-1}^2}{\psi_n^2},$$

and

$$\nu_{n+1} = \frac{1}{2}(a_n b_{n+1} - a_{n+1} b_n) = -\frac{\psi_n \psi_{n+2}}{\psi_{n+1}^2} + \frac{\psi_{n+1}^2}{\psi_{n+2} \psi_n},$$

where the coefficients a_n and b_n are defined in (20). By appealing to the second derivative of $x \mapsto (1+x) \ln^b(1+x)$ it is easy to verify that $\kappa_b(n+2) + \kappa_b(n) - 2\kappa_b(n+1)$ is always non-negative and uniformly bounded from above. Thus ν_{n+1} is uniformly bounded and $\mu_n \geq 0$. This implies (30). \square

4.4. Convexity estimate. In this subsection we consider the weight function given by

$$\psi(t, n) = e^{\kappa(t, n)}, \text{ where } \kappa(t, n) = \gamma(|n| + R(t)) \ln(|n| + R(t)),$$

and $R(t) = C_0 + R_0 t(1 - t)$, $R_0 > 0$, C_0 being large enough. As before we define $H(t) = \|u(t, n)\psi(t, n)\|_2^2$.

Lemma 1. *For every $\gamma > (3 + \sqrt{3})/2$ there exists $C(\gamma)$ such that for $C_0 > C(\gamma)$ and $R(t) = C_0 + R_0 t(1 - t)$ we have*

$$\partial_t^2(\log H(t)) \geq -\frac{4\gamma}{2\gamma - 3} R_0 \log R_0 - C_1 R_0 - C_2,$$

where C_1 and C_2 depend on γ and $\|V\|_\infty$ only.

Proof. For $n \geq 0$ we have

$$\frac{\psi(t, n+1)}{\psi(t, n)} = (n+1 + R(t))^\gamma \left(1 + \frac{1}{n+R(t)}\right)^{\gamma(n+R(t))},$$

and $\psi_n = \psi_{-n}$ for $n < 0$. Hence $a_n = -a_{-n-1}$ and $b_n = b_{-n+1}$ for $n < 0$, which in turn implies that $\mu_n = \mu_{-n}$ and $\lambda_n = -\lambda_{-n-1}$ when $n < 0$. We have also $\mu_0 = 2a_0 b_0 + 2\kappa_0''$.

As before, we get

$$|\nu_{n+1}| = \left| \frac{\psi_{n+1}^2}{\psi_n \psi_{n+2}} - \frac{\psi_n \psi_{n+2}}{\psi_{n+1}^2} \right| \leq C_3,$$

where C_3 depends on γ only.

Let $\phi(M) = \gamma M \ln M$ and $M = M(t, n) = |n| + R(t)$. In this notation we have for $n \neq 0$

$$\mu_n \geq \exp(2\phi(M+1) - 2\phi(M)) - \exp(2\phi(M) - 2\phi(M-1)) - C_4 + 2\kappa_n'',$$

where C_4 is a constant that depends only on γ . The derivatives of κ_n are

$$\kappa_n'(t) = R'(t)\phi'(|n| + R(t)),$$

$$\kappa_n''(t) = -2R_0\phi'(|n| + R(t)) + (R'(t))^2\phi''(|n| + R(t)).$$

Then, by the Taylor expansions, we obtain that, for each $\epsilon > 0$ and $C_0 = C_0(\epsilon)$ large enough,

$$\begin{aligned} \mu_n \geq 2\gamma e^{2\gamma} M^{2\gamma-1} + \gamma e^{2\gamma} \left(\frac{(\gamma-1)^2}{3} - \epsilon \right) M^{2\gamma-3} \\ + 2A^2\gamma M^{-1} - 4R_0\gamma(1 + \ln M) - C_4, \end{aligned}$$

where $A = |R'(t)|$ and $n \neq 0$. Further,

$$\mu_0 \geq (2 - \epsilon)M^{2\gamma}e^{2\gamma} + 2A^2\gamma M^{-1} - 4R_0\gamma(1 + \ln M) - C_4.$$

We introduce the notation

$$\sigma_n = 2\gamma e^{2\gamma} M^{2\gamma-1} + \gamma e^{2\gamma} \left(\frac{(\gamma-1)^2}{3} - 2\epsilon \right) M^{2\gamma-3} + 2A^2\gamma M^{-1},$$

and

$$\rho_n = \epsilon\gamma e^{2\gamma} M^{2\gamma-3} - 4R_0\gamma(1 + \ln M)$$

so that $\mu_n \geq \sigma_n + \rho_n - C_4$ for all n . Note that by the inequality of arithmetic and geometric means we have

$$\sigma_n^2 \geq 8A^2\gamma^2 e^{2\gamma} \left(2M^{2\gamma-2} + \left(\frac{(\gamma-1)^2}{3} - 2\epsilon \right) M^{2\gamma-4} \right).$$

For $n \geq 0$ we have also

$$|\kappa'_{n+1} - \kappa'_n| = |R'(t)|(\phi'(M+1) - \phi'(M)) = A\gamma \ln(1 + M^{-1}).$$

Hence, for sufficiently large C_0 ,

$$\begin{aligned} |\lambda_n| = 2|(\kappa'_{n+1} - \kappa'_n)| |b_n| &\leq 2A\gamma e^\gamma M^{\gamma-1} + A\gamma e^\gamma (\gamma-1) M^{\gamma-2} \\ &\quad + A\gamma e^\gamma \left(\frac{3\gamma^2 - 10\gamma + 8}{12} + \epsilon \right) M^{\gamma-3}, \quad n \geq 0. \end{aligned}$$

To estimate $\partial_t^2(\log H(t))$ we note that

$$\begin{aligned} \langle \mathcal{S}_t f + 2[\mathcal{S}, \mathcal{A}]f, f \rangle &= \sum_n \mu_n |f_n|^2 + 2\Re \sum_n \nu_{n+1} f_{n+2} \bar{f}_n + 2\Re \sum_n \lambda_n f_{n+1} \bar{f}_n \\ &\geq \sum_n \sigma_n |f_n|^2 + 2\Re \sum_n \lambda_n f_{n+1} \bar{f}_n + \sum_n \rho_n |f_n|^2 - (C_3 + C_4) \sum_n |f_n|^2 \end{aligned}$$

First, we consider the first two terms. If we show that for any $x, y \geq 0$

$$(31) \quad \sigma_n x^2 + \sigma_{n+1} y^2 \geq 4|\lambda_n| xy,$$

then the summation of these inequalities with $x = f_n, y = f_{n+1}$ yields

$$\sum_n \sigma_n |f_n|^2 + 2\Re \sum_n \lambda_n f_{n+1} \bar{f}_n \geq 0.$$

To show (31) we have to check that

$$(32) \quad \sigma_n \sigma_{n+1} \geq 4|\lambda_n|^2, \quad n \geq 0.$$

Actually we show (31) only for $n \geq 0$. The relations for negative integers given in the beginning of the proof then imply the inequality for all n .

Using the estimates above, we have

$$\begin{aligned} \sigma_n^2 \sigma_{n+1}^2 &\geq 64A^4 \gamma^4 e^{4\gamma} (4M^{4\gamma-4} + 8(\gamma-1)M^{4\gamma-5}) \\ &\quad + 64A^4 \gamma^4 e^{4\gamma} \left(4(\gamma-1)(2\gamma-3) + 4 \left(\frac{(\gamma-1)^2}{3} - 2\epsilon \right) \right) M^{4\gamma-6}. \end{aligned}$$

While

$$\begin{aligned} 16|\lambda_n|^4 &\leq 64A^4 \gamma^4 e^{4\gamma} (4M^{4\gamma-4} + 8(\gamma-1)M^{4\gamma-5}) \\ &\quad + 4A^4 \gamma^4 e^{4\gamma} \left(6(\gamma-1)^2 + 8 \left(\frac{3\gamma^2 - 10\gamma + 8}{12} + \epsilon \right) \right) M^{4\gamma-6}. \end{aligned}$$

Inequality (32) hence follows for sufficiently small ϵ when

$$2(\gamma - 1)(2\gamma - 3) + \frac{2(\gamma - 1)^2}{3} > 3(\gamma - 1)^2 + \frac{3\gamma^2 - 10\gamma + 8}{3}.$$

The last inequality is equivalent to $2\gamma^2 - 6\gamma + 3 > 0$, which holds for $\gamma > (3 + \sqrt{3})/2$.

Finally by minimizing in M one obtains that, for $\gamma > 3/2$,

$$\rho_n \geq \min_{M>0} \{ \epsilon \gamma e^{2\gamma} M^{2\gamma-3} - 4R_0 \gamma (1 + \ln M) \} \geq -\frac{4\gamma}{2\gamma-3} R_0 \ln R_0 - C_1 R_0,$$

where C_1 depends on γ and ϵ . The conclusion of the lemma follows. \square

4.5. Concluding arguments. Using the weight $\psi(n, t, R_0)$ from the last section and Lemma 1, we obtain that

$$H_{R_0}(t) \exp(-d(R_0, \gamma)t(1-t))$$

is logarithmically convex, where

$$d(R_0, \gamma) = \frac{2\gamma}{2\gamma-3} R_0 \ln R_0 + \frac{C_1}{2} R_0 + \frac{C_2}{2}.$$

Hence, for $t = 1/2$ we obtain

$$H_{R_0}(1/2) \leq \exp\left(\frac{\gamma}{2(2\gamma-3)} R_0 \ln R_0 + \frac{C_1}{8} R_0 + \frac{C_2}{8}\right) H_{R_0}(0)^{1/2} H_{R_0}(1)^{1/2}.$$

But since $R(0) = R(1) = C_0$ we see that $H(0)$ and $H(1)$ do not depend on the choice of R_0 . We obtain that

$$\begin{aligned} |u(1/2, n)|^2 \exp(2\gamma(|n| + C_0 + R_0/4) \ln(|n| + C_0 + R_0/4)) \\ \leq D \exp\left(\frac{\gamma}{2(2\gamma-3)} R_0 \ln R_0 + \frac{C_1}{8} R_0\right), \end{aligned}$$

where D is a constant independent of n and R_0 . However, this last inequality is clearly impossible for large R_0 when $\gamma > 2$, unless $u(1/2, \cdot) \equiv 0$, which of course implies that $u \equiv 0$.

Our work of this section can be summarized as follows.

Theorem 4.2. *Assume that $\gamma > (3 + \sqrt{3})/2$ and that $V(t, n)$ is a real-valued bounded function. If u is a strong solution of*

$$\partial_t u = i(\Delta_d u + V u)$$

such that

$$\left\| (1 + |n|)^{\gamma(1+|n|)} u(0, n) \right\|_2, \left\| (1 + |n|)^{\gamma(1+|n|)} u(1, n) \right\|_2 < +\infty,$$

then $u \equiv 0$.

Remark. This result is not sharp. The authors expect that a milder decay condition (with $\gamma = 1 + \epsilon$) and also just one-sided decay should imply uniqueness as in the case of free Schrödinger evolution.

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