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A review of Hybrid High-Order methods: formulations, computational aspects, comparison with other methods

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Abstract

Hybrid High-Order (HHO) methods are formulated in terms of discrete unknowns attached to mesh faces and cells (hence, the term hybrid), and these unknowns are polynomials of arbitrary order $k \geqslant 0$ (hence, the term high-order). HHO methods are devised from local reconstruction operators and a local stabilization term. The discrete problem is assembled cellwise, and cell-based unknowns can be eliminated locally by static condensation. HHO methods are locally conservative, support polyhedral meshes, and allow for a robust treatment of physical parameters in various situations, e.g., heterogeneous/anisotropic diffusion, quasi-incompressible linear elasticity, and advection-dominated transport. This paper reviews HHO methods for a variable-diffusion model problem with nonhomogeneous, mixed Dirichlet–Neumann boundary conditions, including both primal and mixed formulations. Links with other polyhedral discretization methods from the literature are discussed.

1 Introduction

Over the last few years, a significant effort has been devoted to devising and analyzing discretization methods for elliptic PDEs on general polyhedral meshes. Such meshes are encountered, e.g., in the context of subsurface flow simulations in saline aquifers and petroleum basins, where polyhedral elements and nonmatching interfaces appear to account for eroded layers and fractures. In petroleum reservoir modeling, polyhedral elements can also appear in the near-wellbore region, where radial meshes are usually employed to account for the (qualitative) features of the solution. A more recent and original application of meshes composed of polyhedral elements is adaptive mesh coarsening [2, 7], where a coarse mesh is obtained by element agglomeration from a fine mesh accounting for the geometric details of the domain.

Polyhedral discretization methods were first investigated in the framework of lowest-order schemes. In the context of Finite Volume methods, several families of polyhedral methods have resulted from the effort to circumvent the superadmissible mesh condition required for

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the consistency of the classical two-point scheme; cf., in particular, [36, Definition 9.1]. Interestingly, most of these methods possess local conservation properties on the primal mesh and exhibit numerical fluxes without resorting to local reconstructions. We can mention here, e.g., the Mixed and Hybrid Finite Volume (MHFV) schemes of [32, 37] and the Discrete Duality Finite Volume (DDFV) method of [31].

Other families of polyhedral discretization methods have been obtained by trying to reproduce at the discrete level salient features of the continuous problem. The Mimetic Finite Difference (MFD) methods were originally derived by mimicking the Stokes theorem in a discrete setting to formulate discrete counterparts of the usual first-order differential operators combined with constitutive relations and of L^2-products; cf. [14, 15] and also [9] for an overview. Another viewpoint starts from the seminal ideas of Tonti [42] and Bossavit [13] hinging on differential geometry and algebraic topology. Related schemes include the so-called Discrete Geometric Approach (DGA) [21], and more generally, the Compatible Discrete Operator (CDO) framework of [11,12], cf. also [10], where the building blocks are metric-free discrete differential operators combined with a discrete Hodge operator approximating constitutive relations. Another approach consists in reproducing classical properties of nonconforming and penalized methods on general meshes without resorting to a simplicial submesh, as in the Cell Centered Galerkin method [22] and the generalized Crouzeix–Raviart method [30]. The idea is to formulate the method in terms of (possibly incomplete) polynomial spaces so as to re-deploy classical (nonconforming) Finite Element analysis tools.

Recent works have led to unifying frameworks that capture the links among (some of) the above methods. The close relation between MHFV and MFD methods has been investigated in [33], where equivalence at the algebraic level is demonstrated. A unifying viewpoint that encompasses the above and other classical methods has been proposed under the name of Gradient Schemes [34]. Another unifying viewpoint (closely related to Gradient Schemes) is provided by the CDO framework which encompasses vertex-based schemes (such as first-order Lagrange finite elements and nodal MFD) and cell-based schemes (such as MHFV and MFD).

High-order polyhedral discretization methods have received significant attention over the last few years. Increasing the approximation order can significantly speed up convergence when the solution exhibits sufficient (local) regularity. When this is not the case, the better convergence properties of high-order methods can be recovered using mesh adaption (by local refinement or coarsening). High-order polyhedral discretization methods can be obtained by fully nonconforming approaches such as the discontinuous Galerkin (dG) method; cf. [4] and also [5, 16] for a unified presentation for the Poisson problem, [35] for Friedrichs' systems, and [24] for a comprehensive introduction. An interesting class of dG methods is that of Hybridizable Discontinuous Galerkin (HDG) methods [20] (cf. also [18]). Such methods were originally devised as discrete versions of a characterization of the exact solution in terms of solutions of local problems globally matched through transmission conditions.

Very recent works have developed other viewpoints to achieve high-order polyhedral discretizations. A first example is provided by the Virtual Element Method (VEM) introduced in [8, 17]. The H^1-conforming VEM takes the steps from the nodal MFD method recast in a finite element framework, and can be viewed as a generalization of conforming (Lagrange, Hermite) finite element methods. The main idea is to define a local space of basis functions for which only the values of degrees of freedom are known (i.e., no analytical expression is available). Starting from these degrees of freedom, one devises a computable projection onto a polynomial space so as to formulate the local contributions to the discrete problem. Another example of polyhedral discretization method is the Weak Galerkin method [43, 44] based on

so-called weak gradient operators over generalized functions defined as couples of standard functions in the interior and on the boundary of the mesh elements.

Our focus is here on the Hybrid High-Order (HHO) method introduced in [27, 29]. The term hybrid refers to the fact that the method is originally formulated using discrete unknowns attached to mesh faces and cells. These discrete unknowns are polynomial functions, and the cell-based ones can be eliminated locally by static condensation. The term high-order refers to the fact that the order of the polynomial functions can be an arbitrary integer $k \geqslant 0$. The main idea of HHO methods consists in reconstructing high-order differential operators acting on the face- and cell-based unkowns. The guideline underpinning such reconstruction is an integration by parts formula. These reconstructions are then used to formulate the element-wise contributions to the discrete problem including a high-order stabilization term exhibiting a rich structure coupling locally the face- and cell-based unkowns. Local contributions are conceived so that the only globally coupled unknowns after static condensation are discontinuous polynomials on the mesh skeleton. This is a distinctive feature with VEM where H^1-conforming reconstructions are present in the background. A first study of the relations between HHO and HDG methods can be found in [19], which also investigates the links between HHO methods and the recent High-Order MFD method of [6, 41]. We also mention that HHO methods for polynomial order $k = 0$ are closely related to MHFV [29].

HHO methods offer several assets. Besides supporting general meshes, they are locally conservative [26]. Moreover, they allow for a natural treatment of physical parameters [28], and lead to discretizations that are robust over the entire range of variation of physical parameters in various situations, e.g., quasi-incompressible linear elasticity [27] and advection-dominated transport [23]. When compared to interior penalty dG methods, HHO methods are also appealing in terms of computational cost. To achieve an order of convergence of $(k + 1)$ in the energy-norm for a pure diffusion problem in three space dimensions, the globally coupled discrete unknowns for dG grow as $\frac{1}{6}k^3 N_{\mathrm{E}}$ with N_{E} the number of mesh elements, whereas for HHO they only grow as $\frac{1}{2}k^2 N_{\mathrm{F}}$ with N_{F} the number of mesh faces (only leading-order terms are considered in the above computations).

The goal of this paper is to provide an up-to-date review of HHO methods, with a particular focus on the various possible formulations and computational aspects, while discussing existing links with other methods. For the sake of simplicity, we focus on a model elliptic problem with possibly heterogeneous/anisotropic diffusion tensor. Most of the results contained herein can be derived from relatively straightforward adaptations of the proofs contained in previous works [19, 26–29]; for the sake of conciseness, we provide bibliographic references for the most technical proofs, while some details are included for those proofs that allow us to highlight the more practical aspects of the method. The present material contains two novel aspects. First, we discuss the treatment of non-homogeneous mixed Dirichlet–Neumann boundary conditions, while previous work has focused on homogeneous, pure Dirichlet boundary conditions. Second, we also present HHO methods in mixed form, as originally devised in [25], and, proceeding similarly to [1], we identify the link between HHO methods in primal and mixed form. We also observe that another possibility for the mixed form of the HHO method is to eliminate locally the cell-based flux unknowns and the potential unknowns (up to one constant value per element). The resulting global discrete problem still exhibits a saddle-point structure and is closely related to the recent Mixed Hybrid Method of [3, 38].

The material is organized as follows. Section 2 describes the continuous and discrete settings, including the model problem, the notion of admissible mesh sequence, and the assumptions on the data. Section 3 is devoted to the presentation and analysis of the HHO method in

primal form, while Section 4 is concerned with the mixed form of the HHO method. Finally, the links between both forms are studied in Section 5.

2 Continuous and discrete settings

This section presents the model problem, the key definitions and notation concerning meshes, and the assumptions on the data of the model problem.

2.1 Model problem

Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, be an open, connected, bounded, polytopal domain, with boundary Γ and unit outward normal $\boldsymbol{n}$. We assume that there exists a partition of Γ such that $\Gamma := \Gamma_{\mathrm{d}} \cup \Gamma_{\mathrm{n}}$, with $\Gamma_{\mathrm{d}} \cap \Gamma_{\mathrm{n}} = \varnothing$, and such that the measure of Γ_{d} is nonzero. For any open, connected subset $X \subset \overline{\Omega}$ with nonzero Lebesgue measure, the inner product and norm of the Lebesgue space $L^2(X)$ are denoted by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$, respectively, with the convention that the index is omitted if $X = \Omega$.

We consider a variable-diffusion model problem with tensor-valued diffusivity $\mathbb{M}$. Throughout the paper, $\mathbb{M}$ is assumed to be symmetric, piecewise Lipschitz on a partition P_Ω of Ω, and to be uniformly elliptic, in the sense that, for a.e. $\boldsymbol{x} \in \Omega$,

$$0 < \mu_\flat \leqslant \mathbb{M}(\boldsymbol{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \leqslant \mu_\sharp < +\infty, \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d \text{ such that } |\boldsymbol{\xi}| = 1. \tag{1}$$

The model problem reads as follows:

$$-\mathrm{div}(\mathbb{M}\boldsymbol{\nabla} u) = f \qquad \text{in } \Omega, \tag{2a}$$

$$u = u_\partial \qquad \text{on } \Gamma_{\mathrm{d}}, \tag{2b}$$

$$\mathbb{M}\boldsymbol{\nabla} u\cdot\boldsymbol{n} = s_\partial \qquad \text{on } \Gamma_{\mathrm{n}}, \tag{2c}$$

where $f \in L^2(\Omega)$, $u_\partial = (\tilde{u}_\partial)_{|\Gamma_{\mathrm{d}}}$ with $\tilde{u}_\partial \in H^1(\Omega)$, and $s_\partial \in L^2(\Gamma_{\mathrm{n}})$ (whenever the measure of Γ_{n} is nonzero). In what follows, u is termed the potential. Owing to the nonzero assumption on the measure of Γ_{d}, we do not consider pure Neumann boundary conditions; the results presented in what follows can be readily adapted to this case, up to minor modifications. The pure Dirichlet case (i.e., when Γ_{n} has zero measure) is included in the present setting.

2.2 Admissible mesh sequences

Denoting by $\mathcal{H} \subset \mathbb{R}_*^+$ a countable set of meshsizes having 0 as its unique accumulation point, we consider mesh sequences $(\mathcal{T}_h)_{h\in\mathcal{H}}$ where, for all $h \in \mathcal{H}$, $\mathcal{T}_h = \{T\}$ is a finite collection of nonempty disjoint open polytopes (polygons/polyhedra) T, called *elements* or *cells*, such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$ and $h = \max_{T\in\mathcal{T}_h} h_T$ (where h_T stands for the diameter of the element T). Each polyhedral mesh element considered individually is assumed to have planar faces.

A hyperplanar closed connected subset F of $\overline{\Omega}$ is called a *face* if it has positive $(d-1)$-dimensional Lebesgue measure and if either (i) there exist $T_1, T_2 \in \mathcal{T}_h$ such that $F \subset \partial T_1 \cap \partial T_2$ (and F is termed *interface*), or (ii) there exists $T \in \mathcal{T}_h$ such that $F \subset \partial T \cap \partial\Omega$ (and F is termed *boundary face*). Interfaces are collected in the set $\mathcal{F}_h^{\mathrm{i}}$, boundary faces in $\mathcal{F}_h^{\mathrm{b}}$, and we let $\mathcal{F}_h := \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^{\mathrm{b}}$. The diameter of a face $F \in \mathcal{F}_h$ is denoted h_F. For all $T \in \mathcal{T}_h$, $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$ denotes the set of faces lying on the boundary of T and, symmetrically, for all $F \in \mathcal{F}_h$, $\mathcal{T}_F := \{T \in \mathcal{T}_h \mid F \subset \partial T\}$ denotes the set gathering the one (if F is a boundary

face) or two (if F is an interface) element(s) sharing F. For all $F \in \mathcal{F}_T$, we let $\boldsymbol{n}_{T,F}$ be the unit normal vector to F pointing out of T. For every interface $F \subset \partial T_1 \cap \partial T_2$, we adopt the following convention: an orientation is fixed once and for all by means of a unit normal vector $\boldsymbol{n}_F$, and the elements T_1 and T_2 are numbered so that $\boldsymbol{n}_F := \boldsymbol{n}_{T_1,F}$.

We adopt the following notion of admissible mesh sequence, cf. [24, Section 1.4].

Definition 2.1 (Admissible mesh sequence)**.** *The mesh sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is* admissible *if, for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a matching simplicial submesh $\mathfrak{T}_h$ such that any cell and any face in $\mathfrak{T}_h$ belongs to only one cell and face of $\mathcal{T}_h$, respectively, and there exists a real number $\gamma > 0$, called* mesh regularity parameter*, independent of h and such that, for all $h \in \mathcal{H}$,*

(i) for all simplex $S \in \mathfrak{T}_h$ of diameter h_S and inradius r_S, $\gamma h_S \leqslant r_S$;

(ii) for all $T \in \mathcal{T}_h$, and all $S \in \mathfrak{T}_T := \{S \in \mathfrak{T}_h \mid S \subset T\}$, $\gamma h_T \leqslant h_S$.

We add the following notion of compatibility, in order to deal with the partitions associated with the diffusion tensor and with the boundary conditions.

Definition 2.2 (Compatible mesh sequence)**.** *The mesh sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is* compatible *if, for all $h \in \mathcal{H}$,*

(i) $\mathcal{T}_h$ fits the partition P_Ω associated with the diffusion tensor $\mathbb{M}$ meaning that, for all $T \in \mathcal{T}_h$, there is Ω_i in P_Ω containing T;

(ii) $\mathcal{T}_h$ fits the partition $\Gamma = \Gamma_{\mathrm{d}} \cup \Gamma_{\mathrm{n}}$ of the boundary, in the sense that we can define two sets, $\mathcal{F}_h^{\mathrm{d}} := \{F \in \mathcal{F}_h^{\mathrm{b}} \mid F \subset \Gamma_{\mathrm{d}}\}$ and $\mathcal{F}_h^{\mathrm{n}} := \{F \in \mathcal{F}_h^{\mathrm{b}} \mid F \subset \Gamma_{\mathrm{n}}\}$, such that $\mathcal{F}_h^{\mathrm{d}} \cup \mathcal{F}_h^{\mathrm{n}} = \mathcal{F}_h^{\mathrm{b}}$.

2.3 Broken polynomial spaces

For an integer $k \geqslant 0$, the space $\mathbb{P}_d^k$ is composed of d-variate polynomial functions of total degree less than or equal to k. It is well-known that

$$N_{k,d} := \dim(\mathbb{P}_d^k) = \binom{k+d}{k}. \tag{3}$$

For all $T \in \mathcal{T}_h$, $\mathbb{P}_d^k(T)$ denotes the restriction to T of functions in $\mathbb{P}_d^k$. We also introduce the broken polynomial space

$$\mathbb{P}_d^k(\mathcal{T}_h) := \{v \in L^2(\Omega) \mid v_{|T} \in \mathbb{P}_d^k(T) \text{ for all } T \in \mathcal{T}_h\}. \tag{4}$$

Broken polynomial spaces are special instances of broken Sobolev spaces, for an integer $m \geqslant 1$:

$$H^m(\mathcal{T}_h) := \{v \in L^2(\Omega) \mid v_{|T} \in H^m(T) \text{ for all } T \in \mathcal{T}_h\}. \tag{5}$$

We use the notation $\boldsymbol{\nabla}_h$ to denote the broken gradient operator acting elementwise on functions from broken Sobolev spaces.

We denote by π_h^k the L^2-orthogonal projector onto $\mathbb{P}_d^k(\mathcal{T}_h)$ such that, introducing π_T^k the L^2-orthogonal projector onto $\mathbb{P}_d^k(T)$, for all $v \in L^2(\Omega)$ and all $T \in \mathcal{T}_h$, $(\pi_h^k v)_{|T} := \pi_T^k v_{|T}$. Additionally, for all $F \in \mathcal{F}_h$ and all $v \in L^2(F)$, we denote by $\pi_F^k v$ the L^2-orthogonal projection of v onto $\mathbb{P}_{d-1}^k(F)$.

2.4 Diffusion tensor

We assume, for the sake of simplicity, that $\mathbb{M}$ is piecewise constant on P_Ω, and thus on $\mathcal{T}_h$ for every $h \in \mathcal{H}$ (cf. Definition 2.2). For $T \in \mathcal{T}_h$, we let $\mathbb{M}_T := \mathbb{M}_{|T}$ (owing to the above assumption, $\mathbb{M}_T$ is a constant matrix), and we denote by $\mu_{\flat,T}$ and $\mu_{\sharp,T}$, respectively, the lowest and largest eigenvalues of $\mathbb{M}_T$. We also introduce the local anisotropy ratio $\rho_T := \mu_{\sharp,T}/\mu_{\flat,T} \geqslant 1$; the global ratio is defined as $\rho := \max_{T\in\mathcal{T}_h} \rho_T$. Finally, for all $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, we denote by $\mu_{T,F}$ the real number $\mu_{T,F} := \mathbb{M}_T \boldsymbol{n}_F \cdot \boldsymbol{n}_F > 0$.

In what follows, we often abbreviate as $a \lesssim b$ the inequality $a \leqslant Cb$, with $C > 0$ independent of the meshsize h and of the diffusion tensor $\mathbb{M}$, but possibly depending on the mesh regularity parameter γ and on the polynomial degree k.

3 The HHO method in primal form

Let $U := H^1(\Omega)$ and $U_0 := \{v \in U \mid v_{|\Gamma_\mathrm{d}} = 0\}$. The starting point of the HHO method in primal form is the following weak problem: Find $\tilde{u} \in U_0$ such that

$$(\mathbb{M}\boldsymbol{\nabla}\tilde{u}, \boldsymbol{\nabla}v) = (f, v) - (\mathbb{M}\boldsymbol{\nabla}\tilde{u}_\partial, \boldsymbol{\nabla}v) + (s_\partial, v)_{\Gamma_\mathrm{n}} \qquad \forall v \in U_0. \tag{6}$$

The solution $u \in U$ is then computed as $u = \tilde{u} + \tilde{u}_\partial$ with $\tilde{u}_\partial$ defined in Section 2.1.

3.1 Discrete setting

Let an integer $k \geqslant 0$ be fixed, and let us consider an admissible and compatible mesh sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ in the sense of Definitions 2.1 and 2.2. We further suppose that the assumptions of Section 2.4 concerning the diffusion tensor hold.

3.1.1 Discrete unknowns

We adopt the convention that underlined quantities (sets, elements from these sets) are hybrid quantities, i.e., quantities featuring both a cell-based and a face-based contribution. We introduce, first locally, then globally, the discrete unknowns associated with the potential.

Local definition For $T \in \mathcal{T}_h$, letting

$$U_T^k := \mathbb{P}_d^k(T), \qquad \mathfrak{U}_F^k := \mathbb{P}_{d-1}^k(F) \text{ for all } F \in \mathcal{F}_T, \tag{7}$$

we define the local set of hybrid potential unknowns, cf. Figure 1, as

$$\underline{\mathrm{U}}_T^k := U_T^k \times \mathfrak{U}_T^k, \qquad \mathfrak{U}_T^k := \bigtimes_{F\in\mathcal{F}_T} \mathfrak{U}_F^k. \tag{8}$$

In the sequel, any element $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$ is decomposed as $\underline{\mathrm{v}}_T := (v_T \in U_T^k, \mathfrak{v}_T \in \mathfrak{U}_T^k)$, with $\mathfrak{v}_T := (\mathfrak{v}_F \in \mathfrak{U}_F^k)_{F\in\mathcal{F}_T}$. We also introduce the local reduction operator $\underline{\mathrm{I}}_T^k : H^1(T) \to \underline{\mathrm{U}}_T^k$ such that, for all $v \in H^1(T)$, $\underline{\mathrm{I}}_T^k v := \left(\pi_T^k v, (\pi_F^k v)_{F\in\mathcal{F}_T}\right)$.

Remark 3.1 (Variant on cell-based unknowns)**.** *A variant in the definition of cell-based unknowns is studied in [19] where these unknowns belong to the polynomial space $\mathbb{P}_d^l(T)$ with $l \in \{k-1, k, k+1\}$ (up to some minor adaptations if $k = 0$ and $l = -1$). The choice*

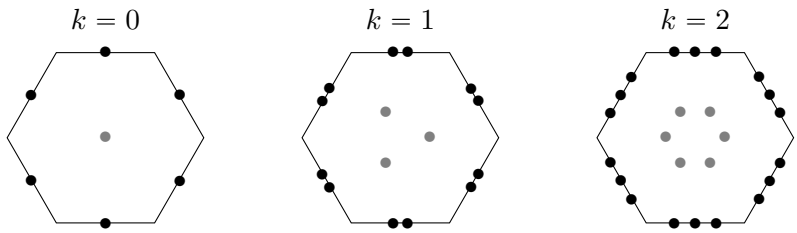


Figure 1: Hybrid (cell- and face-based) potential discrete unknowns, $d = 2$, $k \in \{0, 1, 2\}$.

$l = k - 1$ allows one to establish a link (up to equivalent stabilizations) with the High-Order MFD method of [6, 41], while the choice $l = k + 1$ is related to a variant of the HDG method introduced in [40].

Global definition We define the global set of hybrid potential unknowns as

$$\underline{\mathrm{U}}_h^k := U_h^k \times \mathfrak{U}_h^k, \tag{9}$$

with

$$U_h^k := \bigtimes_{T \in \mathcal{T}_h} U_T^k, \qquad \mathfrak{U}_h^k := \bigtimes_{F \in \mathcal{F}_h} \mathfrak{U}_F^k.$$

Observe that $U_h^k = \mathbb{P}_d^k(\mathcal{T}_h)$ and that potential unknowns attached to interfaces are single-valued. Given an element $\underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_h^k$, we denote v_h and $\mathfrak{v}_h$ its restrictions to U_h^k and $\mathfrak{U}_h^k$, respectively, while, for any $T \in \mathcal{T}_h$, we denote by $\underline{\mathrm{v}}_T = (v_T, (\mathfrak{v}_F)_{F \in \mathcal{F}_T}) \in \underline{\mathrm{U}}_T^k$ its restriction to the element T. To account for homogeneous Dirichlet boundary conditions in a strong manner, we introduce the following subspace of $\underline{\mathrm{U}}_h^k$:

$$\underline{\mathrm{U}}_{h,0}^k := U_h^k \times \mathfrak{U}_{h,0}^k, \qquad \text{with} \qquad \mathfrak{U}_{h,0}^k := \left\{ \mathfrak{v}_h \in \mathfrak{U}_h^k \mid \mathfrak{v}_F \equiv 0, \forall F \in \mathcal{F}_h^{\mathrm{d}} \right\}. \tag{10}$$

Finally, we introduce the global reduction operator $\underline{\mathrm{I}}_h^k : U \to \underline{\mathrm{U}}_h^k$ such that, for all $v \in U$, and for all $T \in \mathcal{T}_h$, $(\underline{\mathrm{I}}_h^k v)_{|T} := \underline{\mathrm{I}}_T^k v_{|T}$. Single-valuedness at interfaces is ensured by the regularity provided by U.

3.1.2 Potential reconstruction operator

Let $T \in \mathcal{T}_h$. The local potential reconstruction operator $p_T^{k+1} : \underline{\mathrm{U}}_T^k \to \mathbb{P}_d^{k+1}(T)$ is defined, for all $\underline{\mathrm{v}}_T = (v_T, (\mathfrak{v}_F)_{F \in \mathcal{F}_T}) \in \underline{\mathrm{U}}_T^k$, as the solution $p_T^{k+1} \underline{\mathrm{v}}_T \in \mathbb{P}_d^{k+1}(T)$ of the well-defined Neumann problem

$$(\mathbb{M}_T \boldsymbol{\nabla} p_T^{k+1} \underline{\mathrm{v}}_T, \boldsymbol{\nabla} w)_T = -(v_T, \mathrm{div}(\mathbb{M}_T \boldsymbol{\nabla} w))_T + \sum_{F \in \mathcal{F}_T} (\mathfrak{v}_F, \mathbb{M}_T \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{T,F})_F \quad \forall w \in \mathbb{P}_d^{k+1}(T), \tag{11}$$

which further satisfies $\int_T p_T^{k+1} \underline{\mathrm{v}}_T = \int_T v_T$ (the usual compatibility condition on the right-hand side is verified). Computing the operator p_T^{k+1} requires to invert a symmetric positive-definite system of size $N_{k+1,d}$, cf. (3), which can be performed effectively via a Cholesky factorization (the cost of such a factorization is roughly $N_{k+1,d}^3/3$ flops). The following result shows, in particular, that $p_T^{k+1} \underline{\mathrm{I}}_T^k$ is the $\mathbb{M}_T$-weighted elliptic projector onto $\mathbb{P}_d^{k+1}(T)$.

Lemma 3.1 (Polynomial consistency)**.** *The following holds for all $v \in H^1(T)$:*

$$(\mathbb{M}_T \boldsymbol{\nabla}(v - p_T^{k+1}\underline{\mathrm{I}}_T^k v), \boldsymbol{\nabla} w)_T = 0 \qquad \forall w \in \mathbb{P}_d^{k+1}(T). \tag{12}$$

Consequently, for all $v \in \mathbb{P}_d^{k+1}(T)$, we infer that

$$p_T^{k+1}\underline{\mathrm{I}}_T^k v = v. \tag{13}$$

Proof. For $v \in H^1(T)$, let us plug $\underline{\mathrm{v}}_T := \underline{\mathrm{I}}_T^k v = \left(\pi_T^k v, (\pi_F^k v)_{F\in\mathcal{F}_T}\right)$ into (11). Since $\mathbb{M}_T$ is constant tensor and since $w \in \mathbb{P}_d^{k+1}(T)$, we infer that $\mathrm{div}(\mathbb{M}_T\boldsymbol{\nabla} w) \in \mathbb{P}_d^{k-1}(T) \subset \mathbb{P}_d^k(T)$ and that $\mathbb{M}_T\boldsymbol{\nabla} w_{|F}\cdot\boldsymbol{n}_{T,F} \in \mathbb{P}_{d-1}^k(F)$, which means that, for all $w \in \mathbb{P}_d^{k+1}(T)$,

$$(\mathbb{M}_T\boldsymbol{\nabla} p_T^{k+1}\underline{\mathrm{I}}_T^k v, \boldsymbol{\nabla} w)_T = -(v, \mathrm{div}(\mathbb{M}_T\boldsymbol{\nabla} w))_T + \sum_{F\in\mathcal{F}_T}(v, \mathbb{M}_T\boldsymbol{\nabla} w\cdot\boldsymbol{n}_{T,F})_F = (\mathbb{M}_T\boldsymbol{\nabla} v, \boldsymbol{\nabla} w)_T,$$

hence concluding the proof of (12). For $v \in \mathbb{P}_d^{k+1}(T)$, we deduce from (12) that $(v - p_T^{k+1}\underline{\mathrm{I}}_T^k v) \in \mathbb{P}_d^0(T)$, and we conclude by invoking the relation $\int_T p_T^{k+1}\underline{\mathrm{I}}_T^k v = \int_T \pi_T^k v = \int_T v$. □

The next result can be found in [28, Lemma 2.1].

Lemma 3.2 (Approximation)**.** *For all $v \in H^{k+2}(T)$, the following holds:*

$$\|v - p_T^{k+1}\underline{\mathrm{I}}_T^k v\|_T + h_T^{1/2}\|v - p_T^{k+1}\underline{\mathrm{I}}_T^k v\|_{\partial T} + h_T\|\boldsymbol{\nabla}(v - p_T^{k+1}\underline{\mathrm{I}}_T^k v)\|_T \\ + h_T^{3/2}\|\boldsymbol{\nabla}(v - p_T^{k+1}\underline{\mathrm{I}}_T^k v)\|_{\partial T} \lesssim \rho_T^{1/2} h_T^{k+2}\|v\|_{H^{k+2}(T)}. \tag{14}$$

In the more general case of a piecewise Lipschitz diffusivity, the estimate (14) still holds with a factor ρ_T instead of $\rho_T^{1/2}$ (cf. [28]).

For further use, we define the global potential reconstruction operator

$$p_h^{k+1} : \underline{\mathrm{U}}_h^k \to \mathbb{P}_d^{k+1}(\mathcal{T}_h)$$

such that, for all $\underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_h^k$, and for all $T \in \mathcal{T}_h$, $(p_h^{k+1}\underline{\mathrm{v}}_h)_{|T} := p_T^{k+1}\underline{\mathrm{v}}_T$.

3.1.3 Stabilization

For all $T \in \mathcal{T}_h$, we define the stabilization bilinear form $j_T : \underline{\mathrm{U}}_T^k \times \underline{\mathrm{U}}_T^k \to \mathbb{R}$ such that

$$j_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) := \sum_{F\in\mathcal{F}_T}\frac{\mu_{T,F}}{h_F}(\pi_F^k(\hat{p}_T^{k+1}\underline{\mathrm{u}}_T - \mathfrak{u}_F), \pi_F^k(\hat{p}_T^{k+1}\underline{\mathrm{v}}_T - \mathfrak{v}_F))_F, \tag{15}$$

with $\hat{p}_T^{k+1} : \underline{\mathrm{U}}_T^k \to \mathbb{P}_d^{k+1}(T)$ such that, for all $\underline{\mathrm{w}}_T \in \underline{\mathrm{U}}_T^k$,

$$\hat{p}_T^{k+1}\underline{\mathrm{w}}_T := w_T + (p_T^{k+1}\underline{\mathrm{w}}_T - \pi_T^k p_T^{k+1}\underline{\mathrm{w}}_T).$$

Notice that j_T is symmetric, positive semi-definite, and polynomially consistent (as a consequence of (13)) in the sense that, for all $v \in \mathbb{P}_d^{k+1}(T)$,

$$j_T(\underline{\mathrm{I}}_T^k v, \underline{\mathrm{w}}_T) = 0 \qquad \forall \underline{\mathrm{w}}_T \in \underline{\mathrm{U}}_T^k. \tag{16}$$

Another important property of j_T is the following approximation property: For all $v \in H^{k+2}(T)$, the following bound holds:

$$j_T(\underline{\mathrm{I}}_T^k v, \underline{\mathrm{I}}_T^k v)^{1/2} \lesssim \mu_{\sharp,T}^{1/2} h_T^{k+1}\|v\|_{H^{k+2}(T)}, \tag{17}$$

showing that j_T matches the approximation properties of the gradient of p_T^{k+1}; cf. Lemma 3.2.

3.2 Discrete problem: formulation and key properties

3.2.1 Formulation

For all $T \in \mathcal{T}_h$, we define the following local bilinear form:

$$a_T : \underline{\mathrm{U}}_T^k \times \underline{\mathrm{U}}_T^k \to \mathbb{R}; \qquad (\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) \mapsto (\mathbb{M}_T \boldsymbol{\nabla} p_T^{k+1} \underline{\mathrm{u}}_T, \boldsymbol{\nabla} p_T^{k+1} \underline{\mathrm{v}}_T)_T + j_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T), \tag{18}$$

with potential reconstruction operator p_T^{k+1} defined by (11) and stabilization bilinear form j_T defined by (15). Introduce now the following global bilinear form obtained by a standard element-by-element assembly procedure:

$$a_h : \underline{\mathrm{U}}_h^k \times \underline{\mathrm{U}}_h^k \to \mathbb{R}; \qquad (\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) \mapsto \sum_{T \in \mathcal{T}_h} a_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T). \tag{19}$$

Then, the (primal) HHO discretization of problem (6) reads: Find $\underline{\tilde{\mathrm{u}}}_h \in \underline{\mathrm{U}}_{h,0}^k$ such that

$$a_h(\underline{\tilde{\mathrm{u}}}_h, \underline{\mathrm{v}}_h) = (f, v_h) - a_h(\underline{\tilde{\mathrm{u}}}_{h,\partial}, \underline{\mathrm{v}}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{n}}} (s_\partial, \mathfrak{v}_F)_F \qquad \forall \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k, \tag{20}$$

where $\underline{\tilde{\mathrm{u}}}_{h,\partial} := \underline{\mathrm{I}}_h^k \tilde{u}_\partial \in \underline{\mathrm{U}}_h^k$ is a discrete lifting of u_∂ obtained by applying the global reduction operator $\underline{\mathrm{I}}_h^k$ to the continuous lifting $\tilde{u}_\partial$ of u_∂. The discrete solution $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_h^k$ is finally computed as $\underline{\mathrm{u}}_h = \underline{\tilde{\mathrm{u}}}_h + \underline{\tilde{\mathrm{u}}}_{h,\partial}$.

3.2.2 Stability

Let us introduce, for all $T \in \mathcal{T}_h$, the following diffusion-dependent seminorm on $\underline{\mathrm{U}}_T^k$:

$$\|\underline{\mathrm{v}}_T\|_{U,T}^2 := \|\mathbb{M}_T^{1/2} \boldsymbol{\nabla} v_T\|_T^2 + \sum_{F \in \mathcal{F}_T} \frac{\mu_{T,F}}{h_F} \|v_T - \mathfrak{v}_F\|_F^2. \tag{21}$$

Setting, for all $\underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_h^k$,

$$\|\underline{\mathrm{v}}_h\|_{U,h}^2 := \sum_{T \in \mathcal{T}_h} \rho_T^{-1} \|\underline{\mathrm{v}}_T\|_{U,T}^2,$$

we observe that $\|\cdot\|_{U,h}$ is a norm on $\underline{\mathrm{U}}_{h,0}^k$. Stability for problem (20) is expressed by the following result (cf. [28, Lemma 3.1]).

Lemma 3.3 (Stability)**.** *For all $T \in \mathcal{T}_h$, and for all $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$, the following inequalities hold:*

$$\rho_T^{-1} \|\underline{\mathrm{v}}_T\|_{U,T}^2 \lesssim a_T(\underline{\mathrm{v}}_T, \underline{\mathrm{v}}_T) \lesssim \rho_T \|\underline{\mathrm{v}}_T\|_{U,T}^2. \tag{22}$$

As a consequence, we infer that

$$\|\underline{\mathrm{v}}_h\|_{U,h}^2 \lesssim a_h(\underline{\mathrm{v}}_h, \underline{\mathrm{v}}_h) \quad \forall \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_h^k, \tag{23}$$

implying that problem (20) is well-posed.

3.3 Error estimates

Let $u \in U$ be such that $u = \tilde{u} + \tilde{u}_\partial$, where $\tilde{u} \in U_0$ is the solution to (6), and $\tilde{u}_\partial \in U$ is defined in Section 2.1. Let $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_h^k$ be such that $\underline{\mathrm{u}}_h = \tilde{\underline{\mathrm{u}}}_h + \tilde{\underline{\mathrm{u}}}_{h,\partial}$, where $\tilde{\underline{\mathrm{u}}}_h \in \underline{\mathrm{U}}_{h,0}^k$ is the solution to (20), and $\tilde{\underline{\mathrm{u}}}_{h,\partial} \in \underline{\mathrm{U}}_h^k$ is defined in Section 3.2.1. Then, we can state the following result, whose proof can be easily adapted from the one of [28, Theorem 4.1].

Theorem 3.1 (Energy-norm error estimate)**.** *Assume that u further belongs to $H^{k+2}(\mathcal{T}_h)$. Then, the following holds:*

$$\|\underline{\mathrm{I}}_h^k u - \underline{\mathrm{u}}_h\|_{U,h} \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\sharp,T} \rho_T^2 h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right\}^{1/2}, \tag{24}$$

which implies, by an additional use of Lemma 3.2, that

$$\|\mathbb{M}^{1/2}(\boldsymbol{\nabla} u - \boldsymbol{\nabla}_h p_h^{k+1} \underline{\mathrm{u}}_h)\| \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\sharp,T} \rho_T^2 h_T^{2(k+1)} \|u\|_{H^{k+2}(T)}^2 \right\}^{1/2}. \tag{25}$$

In the more general case of a piecewise polynomial diffusivity, estimates (24) and (25) still hold with a factor ρ_T^3 instead of ρ_T^2 (cf. [28]). For non-polynomial diffusivity, an additional quadrature error has to be accounted for. Finally, whenever elliptic regularity holds, an L^2-error estimate of order h^{k+2} can be established, cf. [29, Theorem 10].

3.4 Local conservativity

For all $T \in \mathcal{T}_h$, let us first introduce the local bilinear form $\tilde{a}_T : \underline{\mathrm{U}}_T^k \times \underline{\mathrm{U}}_T^k \to \mathbb{R}$ such that, for all $\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$,

$$\tilde{a}_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) := (\mathbb{M}_T \boldsymbol{\nabla} p_T^{k+1} \underline{\mathrm{w}}_T, \boldsymbol{\nabla} p_T^{k+1} \underline{\mathrm{v}}_T)_T + \sum_{F \in \mathcal{F}_T} \frac{\mu_{T,F}}{h_F} (w_T - \mathfrak{w}_F, v_T - \mathfrak{v}_F)_F. \tag{26}$$

Then, we use (26) to define the local isomorphism $\underline{\mathrm{c}}_T^k : \underline{\mathrm{U}}_T^k \to \underline{\mathrm{U}}_T^k$ such that, for all $\underline{\mathrm{w}}_T \in \underline{\mathrm{U}}_T^k$, $\underline{\mathrm{c}}_T^k \underline{\mathrm{w}}_T$ is defined from the following local problem:

$$\tilde{a}_T(\underline{\mathrm{c}}_T^k \underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) = a_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) + \sum_{F \in \mathcal{F}_T} \frac{\mu_{T,F}}{h_F} (w_T - \mathfrak{w}_F, v_T - \mathfrak{v}_F)_F \qquad \forall\, \underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k, \tag{27}$$

and $\int_T c_T^k \underline{\mathrm{w}}_T = \int_T w_T$. Finally, we define the local gradient reconstruction $\mathfrak{G}_T^{k+1} : \underline{\mathrm{U}}_T^k \to \mathbb{M}_T \boldsymbol{\nabla} \mathbb{P}_d^{k+1}(T)$ such that

$$\mathfrak{G}_T^{k+1} := \mathbb{M}_T \boldsymbol{\nabla} (p_T^{k+1} \circ \underline{\mathrm{c}}_T^k).$$

Adapting the arguments of [26, Lemmata 2 and 3], one can show the following result.

Lemma 3.4 (Conservativity)**.** *Let $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_h^k$ denote the unique solution to (20). Then, for all $T \in \mathcal{T}_h$, the following local equilibrium relation holds:*

$$(\mathfrak{G}_T^{k+1} \underline{\mathrm{u}}_T, \boldsymbol{\nabla} v_T)_T - \sum_{F \in \mathcal{F}_T} (\Phi_{T,F}(\underline{\mathrm{u}}_T), v_T)_F = (f, v_T)_T \qquad \forall\, v_T \in \mathbb{P}_d^k(T), \tag{28}$$

where the numerical flux $\Phi_{T,F}(\underline{\mathfrak{u}}_T) \in \mathbb{P}_{d-1}^k(F)$ *is given by*

$$\Phi_{T,F}(\underline{\mathfrak{u}}_T) := \mathfrak{G}_T^{k+1}\underline{\mathfrak{u}}_T \cdot \boldsymbol{n}_{T,F} - \frac{\mu_{T,F}}{h_F}\left[(c_T^k\underline{\mathfrak{u}}_T - u_T) - (\mathfrak{c}_F^k\underline{\mathfrak{u}}_T - \mathfrak{u}_F)\right]. \tag{29}$$

In addition, the numerical fluxes are equilibrated in the following sense: For all $F \in \mathcal{F}_h^{\mathrm{i}}$ *such that* $F \subset \partial T_1 \cap \partial T_2$,

$$\Phi_{T_1,F}(\underline{\mathfrak{u}}_T) + \Phi_{T_2,F}(\underline{\mathfrak{u}}_T) = 0,$$

and $\Phi_F(\underline{\mathfrak{u}}_T) = \pi_F^k s_\partial$ *for all* $F \in \mathcal{F}_h^{\mathrm{n}}$.

3.5 Computational aspects

Following [19, Section 2.4], we show how to eliminate locally the cell-based unknowns from problem (20). Introducing the notation $f_T := f_{|T}$ for all $T \in \mathcal{T}_h$, we begin by remarking that $\underline{\mathfrak{u}}_h = \tilde{\underline{\mathfrak{u}}}_h + \tilde{\underline{\mathfrak{u}}}_{h,\partial}$ solves (20) if and only if

$$a_T((u_T,0),(v_T,0)) = (f_T,v_T)_T - a_T((0,\mathfrak{u}_T),(v_T,0)) \qquad \forall v_T \in U_T^k,\ \forall T \in \mathcal{T}_h, \tag{30a}$$

$$a_h(\underline{\mathfrak{u}}_h,(0,\mathfrak{v}_h)) = \sum_{F\in\mathcal{F}_h^{\mathrm{n}}} (s_\partial,\mathfrak{v}_F)_F \qquad \forall \mathfrak{v}_h \in \mathfrak{U}_{h,0}^k, \tag{30b}$$

that is to say, system (20) can be split into card($\mathcal{T}_h$) local problems (30a) that allow one to express, for all $T \in \mathcal{T}_h$, u_T in terms of $\mathfrak{u}_T$ and f_T, and one global problem (30b) written in terms of face-based unknowns only.

Next, we introduce two local cell-based potential lifting operators:

- a trace-based lifting $l_T^{\mathrm{tr},k} : \mathfrak{U}_T^k \to U_T^k$ such that, for all $\mathfrak{z}_T \in \mathfrak{U}_T^k$, $l_T^{\mathrm{tr},k}(\mathfrak{z}_T) \in U_T^k$ solves

$$a_T((l_T^{\mathrm{tr},k}(\mathfrak{z}_T),0),(v_T,0)) = -a_T((0,\mathfrak{z}_T),(v_T,0)) \qquad \forall v_T \in U_T^k; \tag{31}$$

- a datum-based lifting $l_T^{\mathrm{dt},k} : L^2(T) \to U_T^k$ such that, for all $\varphi \in L^2(T)$, $l_T^{\mathrm{dt},k}(\varphi) \in U_T^k$ solves

$$a_T((l_T^{\mathrm{dt},k}(\varphi),0),(v_T,0)) = (\varphi,v_T)_T \qquad \forall v_T \in U_T^k. \tag{32}$$

Note that problems (31) and (32) are well-posed owing to the first inequality in (22) and the fact that $\|\cdot\|_{U,T}$ is a norm on the zero-trace subspace of $\underline{\mathrm{U}}_T^k$, cf. (21). Note also that problem (31) can be rewritten as

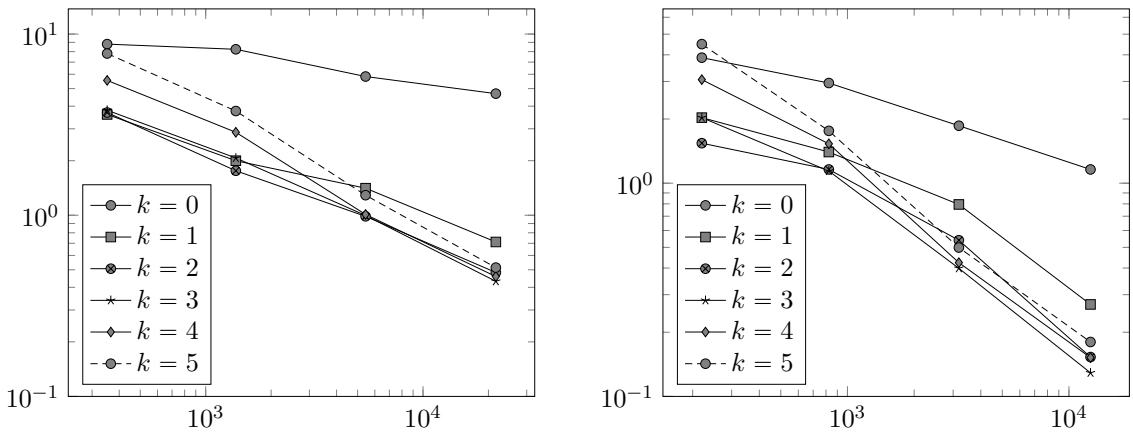
$$a_T((l_T^{\mathrm{tr},k}(\mathfrak{z}_T),\mathfrak{z}_T),(v_T,0)) = 0 \qquad \forall v_T \in U_T^k. \tag{33}$$

Using (30a), (33), and (32), we infer that

$$\underline{\mathfrak{u}}_T = (l_T^{\mathrm{tr},k}(\mathfrak{u}_T) + l_T^{\mathrm{dt},k}(f_T),\mathfrak{u}_T). \tag{34}$$

Introducing the global operators $l_h^{\mathrm{tr},k} : \mathfrak{U}_h^k \to U_h^k$ and $l_h^{\mathrm{dt},k} : L^2(\Omega) \to U_h^k$ such that, for all $\mathfrak{z}_h \in \mathfrak{U}_h^k$, all $\varphi \in L^2(\Omega)$, and all $T \in \mathcal{T}_h$, $(l_h^{\mathrm{tr},k}(\mathfrak{z}_h))_{|T} := l_T^{\mathrm{tr},k}(\mathfrak{z}_T)$ and $(l_h^{\mathrm{dt},k}(\varphi))_{|T} := l_T^{\mathrm{dt},k}(\varphi_{|T})$, we can rewrite (34) globally as follows:

$$\underline{\mathfrak{u}}_h = (l_h^{\mathrm{tr},k}(\mathfrak{u}_h) + l_h^{\mathrm{dt},k}(f),\mathfrak{u}_h). \tag{35}$$



(a) Triangular mesh family

(b) Hexagonal mesh family

Figure 2: Assembly/solution time ratio vs. $\mathrm{card}(\mathcal{F}_h)$

Finally, we reformulate the global problem (30b) under an equivalent form. We remark, using (35), that

$$\begin{aligned} a_h(\underline{\mathfrak{u}}_h, (0, \mathfrak{v}_h)) &= a_h(\underline{\mathfrak{u}}_h, (l_h^{\mathrm{tr},k}(\mathfrak{v}_h), \mathfrak{v}_h)) - a_h(\underline{\mathfrak{u}}_h, (l_h^{\mathrm{tr},k}(\mathfrak{v}_h), 0)) \\ &= a_h((l_h^{\mathrm{tr},k}(\mathfrak{u}_h), \mathfrak{u}_h), (l_h^{\mathrm{tr},k}(\mathfrak{v}_h), \mathfrak{v}_h)) + a_h((l_h^{\mathrm{dt},k}(f), 0), (l_h^{\mathrm{tr},k}(\mathfrak{v}_h), \mathfrak{v}_h)) \\ &\quad - a_h((l_h^{\mathrm{tr},k}(\mathfrak{u}_h), \mathfrak{u}_h), (l_h^{\mathrm{tr},k}(\mathfrak{v}_h), 0)) - a_h((l_h^{\mathrm{dt},k}(f), 0), (l_h^{\mathrm{tr},k}(\mathfrak{v}_h), 0)) \\ &:= \mathfrak{T}_1 + \mathfrak{T}_2 - \mathfrak{T}_3 - \mathfrak{T}_4, \end{aligned}$$

where $\mathfrak{T}_2 = \mathfrak{T}_3 = 0$ owing to (33) and to the symmetry of a_h, and where $\mathfrak{T}_4 = (f, l_h^{\mathrm{tr},k}(\mathfrak{v}_h))$ owing to (32). Introducing for all $\mathfrak{z}_h \in \mathfrak{U}_h^k$ the notation $\underline{l}_h^{\mathrm{tr},k}\mathfrak{z}_h := (l_h^{\mathrm{tr},k}\mathfrak{z}_h, \mathfrak{z}_h)$ and introducing the decomposition $\mathfrak{u}_h = \tilde{\mathfrak{u}}_h + \tilde{\mathfrak{u}}_{h,\partial}$ for the face-based unknowns, the previous relation enables us to rewrite the global problem (30b) as follows: Find $\tilde{\mathfrak{u}}_h \in \mathfrak{U}_{h,0}^k$ such that

$$a_h(\underline{l}_h^{\mathrm{tr},k}\tilde{\mathfrak{u}}_h, \underline{l}_h^{\mathrm{tr},k}\mathfrak{v}_h) = (f, l_h^{\mathrm{tr},k}\mathfrak{v}_h) - a_h(\underline{l}_h^{\mathrm{tr},k}\tilde{\mathfrak{u}}_{h,\partial}, \underline{l}_h^{\mathrm{tr},k}\mathfrak{v}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{n}}} (s_\partial, \mathfrak{v}_F)_F \qquad \forall \mathfrak{v}_h \in \mathfrak{U}_{h,0}^k. \tag{37}$$

Problem (37) is well-posed owing to (23) and to the fact that $\|\cdot\|_{U,h}$ defines a norm on $\underline{\mathrm{U}}_{h,0}^k$. The above results are summarized in the following proposition.

Proposition 3.1 (Static condensation)**.** *The solution* $\underline{\mathrm{U}}_h^k \ni \underline{\mathfrak{u}}_h = \tilde{\underline{\mathfrak{u}}}_h + \tilde{\underline{\mathfrak{u}}}_{h,\partial}$ *with* $\tilde{\underline{\mathfrak{u}}}_h \in \underline{\mathrm{U}}_{h,0}^k$ *solving* (20) *can be expressed as* (35)*, where the operator* $l_h^{\mathrm{tr},k}$ *and the vector of cell-based unknowns* $l_h^{\mathrm{dt},k}(f)$ *are defined cell-wise as the solutions of the local problems* (31) *and* (32)*, respectively, and where* $\mathfrak{u}_h \in \mathfrak{U}_h^k$ *is such that* $\mathfrak{u}_h = \tilde{\mathfrak{u}}_h + \tilde{\mathfrak{u}}_{h,\partial}$ *and* $\tilde{\mathfrak{u}}_h \in \mathfrak{U}_{h,0}^k$ *is the unique solution of the global problem* (37)*.*

As far as numerical costs are concerned, we first have to solve the local problems (31) to compute the operator $l_h^{\mathrm{tr},k}$, which requires to invert $\mathrm{card}(\mathcal{T}_h)$ symmetric positive-definite systems of size $N_{k,d}$, cf. (3). This can be done effectively in parallel and using Cholesky factorization. Then, given a right-hand side $f \in L^2(\Omega)$, one has to solve additionally $\mathrm{card}(\mathcal{T}_h)$ local problems (32) of size $N_{k,d}$, and one global problem (37) of size approximately equal

to card$(\mathcal{F}_h) \times N_{k,d-1}$. Note that a modification of the boundary conditions only affects the global problem. Even for a naive implementation that does not exploit parallelism, the overall cost of the assembly time (computation of the local contributions, static condensation, and matrix assembly) becomes quickly negligible with mesh refinement (except for $k = 0$). This is illustrated in Figure 2, where we plot the assembly/solution time ratio as a function of the number of mesh faces for two families of meshes corresponding, respectively, to the first mesh family of the FVCA5 benchmark [39] and to the (predominantly) hexagonal mesh family introduced in [30, Section 4.2.3] for polynomial degrees up to 5.

4 The HHO method in mixed form

In this section, we study the HHO method in mixed formulation. The starting point is the following mixed form of the model problem (2):

$$\boldsymbol{s} = \mathbb{M}\boldsymbol{\nabla} u \quad \text{in } \Omega, \tag{38a}$$
$$-\mathrm{div}\boldsymbol{s} = f \quad \text{in } \Omega, \tag{38b}$$
$$u = u_\partial \quad \text{on } \Gamma_{\mathrm{d}}, \tag{38c}$$
$$\boldsymbol{s}\cdot\boldsymbol{n} = s_\partial \quad \text{on } \Gamma_{\mathrm{n}}. \tag{38d}$$

To write this problem in weak form, we introduce the functional spaces

$$\boldsymbol{S} := \boldsymbol{H}(\mathrm{div}, \Omega), \qquad \boldsymbol{S}_0 := \left\{\boldsymbol{t} \in \boldsymbol{S} \mid \boldsymbol{t}\cdot\boldsymbol{n}_{|\Gamma_{\mathrm{n}}} = 0\right\}, \qquad V := L^2(\Omega), \tag{39}$$

so that the weak problem reads: Find $(\tilde{\boldsymbol{s}}, u) \in \boldsymbol{S}_0 \times V$ such that

$$(\mathbb{M}^{-1}\tilde{\boldsymbol{s}}, \boldsymbol{t}) + (u, \mathrm{div}\,\boldsymbol{t}) = \langle \boldsymbol{t}\cdot\boldsymbol{n}, \tilde{u}_\partial\rangle_\Gamma - (\mathbb{M}^{-1}\tilde{\boldsymbol{s}}_\partial, \boldsymbol{t}) \qquad \forall \boldsymbol{t} \in \boldsymbol{S}_0, \tag{40a}$$
$$-(\mathrm{div}\tilde{\boldsymbol{s}}, v) = (f, v) + (\mathrm{div}\tilde{\boldsymbol{s}}_\partial, v) \qquad \forall v \in V, \tag{40b}$$

where $\tilde{\boldsymbol{s}}_\partial \in \boldsymbol{S}$ is a lifting of the Neumann datum such that $(\tilde{\boldsymbol{s}}_\partial\cdot\boldsymbol{n})_{|\Gamma_{\mathrm{n}}} = s_\partial$ (which can be taken to be $\tilde{\boldsymbol{s}}_\partial = \nabla\phi$ where $\phi \in H^1(\Omega)$ solves $\phi - \Delta\phi = 0$ in Ω with $\nabla\phi\cdot\boldsymbol{n} = \overline{s}_\partial$ on Γ where $\overline{s}_\partial$ is the zero-extension of s_∂ to Γ), and $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ (note that $\langle \boldsymbol{t}\cdot\boldsymbol{n}, \tilde{u}_\partial\rangle_\Gamma$ does not depend on the choice of the lifting $\tilde{u}_\partial$ of u_∂). The solution $(\boldsymbol{s}, u) \in \boldsymbol{S} \times V$ is then computed as $(\boldsymbol{s}, u) = (\tilde{\boldsymbol{s}} + \tilde{\boldsymbol{s}}_\partial, u)$.

4.1 Discrete setting

Let us fix an integer $k \geqslant 0$ and consider an admissible and compatible mesh sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ in the sense of Definitions 2.1 and 2.2. We suppose that the assumptions of Section 2.4 concerning the diffusivity hold.

4.1.1 Discrete unknowns

We adopt the same notation as in Section 3.1.1, to which we add the use of boldface to denote vector-valued quantities. We introduce, first locally, then globally, the discrete unknowns associated with the flux and with the potential. For the flux, we consider hybrid unknowns, in the sense that they consist of both cell- and face-based unknowns. The cell-based flux unknowns are vector-valued while the face-based ones are scalar-valued.

Local definition Let $T \in \mathcal{T}_h$. Setting

$$\boldsymbol{S}_T^k := \mathbb{M}_T \boldsymbol{\nabla} \mathbb{P}_d^k(T), \qquad \mathfrak{S}_F^k := \mathbb{P}_{d-1}^k(F) \text{ for all } F \in \mathcal{F}_T, \tag{41}$$

we define the local set of hybrid flux unknowns, cf. Figure 3, as

$$\underline{\mathbf{S}}_T^k := \boldsymbol{S}_T^k \times \mathfrak{S}_T^k, \qquad \text{where } \mathfrak{S}_T^k := \bigtimes_{F \in \mathcal{F}_T} \mathfrak{S}_F^k. \tag{42}$$

In the lowest-order case $k = 0$, cell-based flux unknowns are unnecessary and $\boldsymbol{S}_T^k$ has zero dimension. Any element $\underline{\mathbf{t}}_T \in \underline{\mathbf{S}}_T^k$ can be decomposed as $\underline{\mathbf{t}}_T := (\boldsymbol{t}_T \in \boldsymbol{S}_T^k, \mathfrak{t}_T \in \mathfrak{S}_T^k)$, with $\mathfrak{t}_T := (\mathfrak{t}_F \in \mathfrak{S}_F^k)_{F \in \mathcal{F}_T}$.

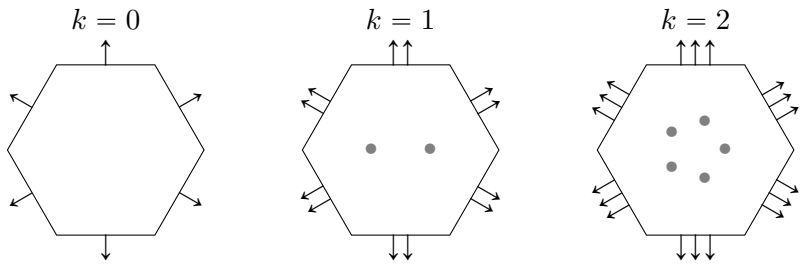


Figure 3: Hybrid flux discrete unknowns, $d = 2$, $k \in \{0, 1, 2\}$.

Letting, for $q > 2$,

$$\boldsymbol{S}^+(T) := \{\boldsymbol{t} \in \boldsymbol{L}^q(T) \mid \operatorname{div} \boldsymbol{t} \in L^2(T)\},$$

and recalling that functions in this space have integrable normal component on all faces of T, we introduce the local reduction operator $\underline{\mathbf{I}}_T^k : \boldsymbol{S}^+(T) \to \underline{\mathbf{S}}_T^k$ such that, for all $\boldsymbol{t} \in \boldsymbol{S}^+(T)$,

$$\underline{\mathbf{I}}_T^k \boldsymbol{t} := \left(\mathbb{M}_T \boldsymbol{\nabla} y, (\pi_F^k(\boldsymbol{t} \cdot \boldsymbol{n}_F))_{F \in \mathcal{F}_T} \right),$$

where $y \in \mathbb{P}_d^k(T)$ is a solution (defined up to an additive constant) of the Neumann problem

$$(\mathbb{M}_T \boldsymbol{\nabla} y, \boldsymbol{\nabla} w)_T = (\boldsymbol{t}, \boldsymbol{\nabla} w)_T \qquad \forall w \in \mathbb{P}_d^k(T), \tag{43}$$

observing that the required compatibility condition on the right-hand side is verified.

As far as the potential is concerned, we let U_T^k, introduced in (7), be the associated local set of discrete unknowns.

Global definition We define the global set of hybrid flux unknowns as

$$\underline{\mathbf{S}}_h^k := \boldsymbol{S}_h^k \times \left\{ \bigtimes_{F \in \mathcal{F}_h} \mathfrak{S}_F^k \right\}, \tag{44}$$

where $\boldsymbol{S}_h^k := \bigtimes_{T \in \mathcal{T}_h} \boldsymbol{S}_T^k$. Observe that the flux unknowns attached to interfaces are single-valued. Given an element $\underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_h^k$, for any $T \in \mathcal{T}_h$, we denote by $\underline{\mathbf{t}}_T = (\boldsymbol{t}_T, (\mathfrak{t}_F)_{F \in \mathcal{F}_T}) \in \underline{\mathbf{S}}_T^k$ its restriction to the element T. We introduce the following subspace of $\underline{\mathbf{S}}_h^k$, that allows us to account for the Neumann boundary condition in a strong manner:

$$\underline{\mathbf{S}}_{h,0}^k := \left\{ \underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_h^k \mid \mathfrak{t}_F \equiv 0, \quad \forall F \in \mathcal{F}_h^{\mathrm{n}} \right\}. \tag{45}$$

We also define the global reduction operator $\underline{\mathbf{I}}_h^k : \boldsymbol{S} \cap \boldsymbol{S}^+(\mathcal{T}_h) \to \underline{\mathbf{S}}_h^k$ such that, for all $\boldsymbol{t} \in \boldsymbol{S} \cap \boldsymbol{S}^+(\mathcal{T}_h)$, and for all $T \in \mathcal{T}_h$, $(\underline{\mathbf{I}}_h^k \boldsymbol{t})_{|T} := \underline{\mathbf{I}}_T^k \boldsymbol{t}_{|T}$. Single-valuedness at interfaces is ensured owing to the regularity provided by $\boldsymbol{S} \cap \boldsymbol{S}^+(\mathcal{T}_h)$.

We finally define U_h^k, cf. (9), as the global set of discrete potential unknowns, and we denote by $v_T \in U_T^k$ the restriction of any $v_h \in U_h^k$ to the element T.

4.1.2 Divergence reconstruction operator

Let $T \in \mathcal{T}_h$. We define the local divergence reconstruction operator $D_T^k : \underline{\mathbf{S}}_T^k \to U_T^k$ as the operator such that, for all $\underline{\mathbf{t}}_T = (\boldsymbol{t}_T, (\mathrm{t}_F)_{F \in \mathcal{F}_T}) \in \underline{\mathbf{S}}_T^k$,

$$(D_T^k \underline{\mathbf{t}}_T, v_T)_T := -(\boldsymbol{t}_T, \boldsymbol{\nabla} v_T)_T + \sum_{F \in \mathcal{F}_T} (\mathrm{t}_F \varepsilon_{T,F}, v_T)_F \qquad \forall v_T \in U_T^k, \tag{46}$$

where $\varepsilon_{T,F} := \boldsymbol{n}_{T,F} \cdot \boldsymbol{n}_F$ for all $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$. This definition reproduces at the discrete level an integration by parts formula, that brings into action the local hybrid flux unknowns. The following property is crucial for stability, cf. [25, Lemma 1].

Lemma 4.1 (Commuting property)**.** *The following holds for all $\boldsymbol{t} \in \boldsymbol{S}^+(T)$:*

$$D_T^k \underline{\mathbf{I}}_T^k \boldsymbol{t} = \pi_T^k (\operatorname{div} \boldsymbol{t}). \tag{47}$$

Proof. For all $\boldsymbol{t} \in \boldsymbol{S}^+(T)$, let us plug the quantity $\underline{\mathbf{t}}_T := \underline{\mathbf{I}}_T^k \boldsymbol{t} = \left(\mathbb{M}_T \boldsymbol{\nabla} y, (\pi_F^k(\boldsymbol{t} \cdot \boldsymbol{n}_F))_{F \in \mathcal{F}_T}\right)$ into (46), where $y \in \mathbb{P}_d^k(T)$ is a solution to (43). Let $v_T \in U_T^k$, and observe that $v_T \in \mathbb{P}_d^k(T)$ and $v_{T|F} \in \mathbb{P}_{d-1}^k(F)$. Hence,

$$(D_T^k \underline{\mathbf{I}}_T^k \boldsymbol{t}, v_T)_T = -(\boldsymbol{t}, \boldsymbol{\nabla} v_T)_T + \sum_{F \in \mathcal{F}_T} (\boldsymbol{t} \cdot \boldsymbol{n}_{T,F}, v_T)_F = (\operatorname{div} \boldsymbol{t}, v_T)_T,$$

which concludes the proof. □

For further use, we introduce the global divergence reconstruction operator $D_h^k : \underline{\mathbf{S}}_h^k \to U_h^k$ such that, for all $\underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_h^k$, and for all $T \in \mathcal{T}_h$, $(D_h^k \underline{\mathbf{t}}_h)_{|T} := D_T^k \underline{\mathbf{t}}_T$.

4.1.3 Flux reconstruction operator

Let $T \in \mathcal{T}_h$. The local flux reconstruction operator $\boldsymbol{F}_T^{k+1} : \underline{\mathbf{S}}_T^k \to \boldsymbol{S}_T^{k+1}$ is defined, for all $\underline{\mathbf{t}}_T = (\boldsymbol{t}_T, (\mathrm{t}_F)_{F \in \mathcal{F}_T}) \in \underline{\mathbf{S}}_T^k$, as $\boldsymbol{F}_T^{k+1} \underline{\mathbf{t}}_T := \mathbb{M}_T \boldsymbol{\nabla} z$, where $z \in \mathbb{P}_d^{k+1}(T)$ is a solution (defined up to an additive constant) of the Neumann problem

$$(\mathbb{M}_T \boldsymbol{\nabla} z, \boldsymbol{\nabla} w)_T = (\boldsymbol{t}_T, \boldsymbol{\nabla} \pi_T^k w)_T + \sum_{F \in \mathcal{F}_T} (\mathrm{t}_F \varepsilon_{T,F}, \pi_F^k w - \pi_T^k w)_F \qquad \forall w \in \mathbb{P}_d^{k+1}(T), \tag{48}$$

observing that the required compatibility condition on the right-hand side is verified. The definition of $\boldsymbol{F}_T^{k+1} \underline{\mathbf{t}}_T$ is motivated by the following link between $\boldsymbol{F}_T^{k+1} \underline{\mathbf{t}}_T$ and the divergence reconstruction operator defined in (46): For all $\underline{\mathbf{t}}_T = (\boldsymbol{t}_T, (\mathrm{t}_F)_{F \in \mathcal{F}_T}) \in \underline{\mathbf{S}}_T^k$,

$$(\boldsymbol{F}_T^{k+1} \underline{\mathbf{t}}_T, \boldsymbol{\nabla} w)_T = -(D_T^k \underline{\mathbf{t}}_T, w)_T + \sum_{F \in \mathcal{F}_T} (\mathrm{t}_F \varepsilon_{T,F}, w)_F \qquad \forall w \in \mathbb{P}_d^{k+1}(T). \tag{49}$$

As in Section 3.1.2, computing the operator $\boldsymbol{F}_T^{k+1}$ using (48) requires to invert a symmetric positive-definite system of size $N_{k+1,d}$, cf. (3), which can be performed effectively via Cholesky factorization. The following result can be found in [25, Lemma 2].

Lemma 4.2 (Polynomial consistency)**.** *The following holds for all* $\boldsymbol{t} \in \boldsymbol{S}_T^{k+1}$*:*

$$\boldsymbol{F}_T^{k+1} \underline{\mathbf{I}}_T^k \boldsymbol{t} = \boldsymbol{t}. \tag{50}$$

Proof. Let $\boldsymbol{t} \in \boldsymbol{S}_T^{k+1}$ and plug $\underline{\mathbf{t}}_T := \underline{\mathbf{I}}_T^k \boldsymbol{t}$ into (49). Using the commuting property (47) leads to $D_T^k \underline{\mathbf{I}}_T^k \boldsymbol{t} = \pi_T^k(\operatorname{div} \boldsymbol{t}) = \operatorname{div} \boldsymbol{t}$ since $\boldsymbol{t} \in \boldsymbol{S}_T^{k+1} \subset \mathbb{P}_d^k(T)$ ($\mathbb{M}_T$ is a constant tensor), which combined with the fact that $\pi_F^k(\boldsymbol{t}\cdot\boldsymbol{n}_F) = \boldsymbol{t}\cdot\boldsymbol{n}_F$ (since faces are planar), allows us to infer that, for all $w \in \mathbb{P}_d^{k+1}(T)$,

$$(\boldsymbol{F}_T^{k+1} \underline{\mathbf{I}}_T^k \boldsymbol{t}, \boldsymbol{\nabla} w)_T = -(\operatorname{div} \boldsymbol{t}, w)_T + \sum_{F \in \mathcal{F}_T} (\boldsymbol{t}\cdot\boldsymbol{n}_{T,F}, w)_F = (\boldsymbol{t}, \boldsymbol{\nabla} w)_T.$$

This last relation proves (50) since $(\boldsymbol{F}_T^{k+1} \underline{\mathbf{I}}_T^k \boldsymbol{t} - \boldsymbol{t}) \in \boldsymbol{S}_T^{k+1} = \mathbb{M}_T \boldsymbol{\nabla} \mathbb{P}_d^{k+1}(T)$. □

The next result is adapted from [25, Lemma 8], and can be wiewed, in the light of Lemma 5.1 below, as a consequence of Lemma 3.2.

Lemma 4.3 (Approximation)**.** *For all* $v \in H^{k+2}(T)$*, letting* $\boldsymbol{t} := \mathbb{M}_T \boldsymbol{\nabla} v$*, the following holds for all* $F \in \mathcal{F}_T$*:*

$$\|\mathbb{M}_T^{-1/2}(\boldsymbol{t} - \boldsymbol{F}_T^{k+1} \underline{\mathbf{I}}_T^k \boldsymbol{t})\|_T + h_F^{1/2} \mu_{T,F}^{-1/2} \|(\boldsymbol{t} - \boldsymbol{F}_T^{k+1} \underline{\mathbf{I}}_T^k \boldsymbol{t})\cdot\boldsymbol{n}_F\|_F \lesssim \rho_T^{1/2} \mu_{\sharp,T}^{1/2} h_T^{k+1} \|v\|_{H^{k+2}(T)}. \tag{51}$$

For further use, we define the global flux reconstruction operator $\boldsymbol{F}_h^{k+1} : \underline{\mathbf{S}}_h^k \to \boldsymbol{S}_h^{k+1}$ such that, for all $\underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_h^k$, and for all $T \in \mathcal{T}_h$, $(\boldsymbol{F}_h^{k+1} \underline{\mathbf{t}}_h)_{|T} := \boldsymbol{F}_T^{k+1} \underline{\mathbf{t}}_T$.

4.1.4 Stabilization

For all $T \in \mathcal{T}_h$, we define the stabilization bilinear form $J_T : \underline{\mathbf{S}}_T^k \times \underline{\mathbf{S}}_T^k \to \mathbb{R}$ such that

$$J_T(\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T) := \sum_{F \in \mathcal{F}_T} \frac{h_F}{\mu_{T,F}} ((\boldsymbol{F}_T^{k+1} \underline{\mathbf{s}}_T)\cdot\boldsymbol{n}_F - \mathfrak{s}_F, (\boldsymbol{F}_T^{k+1} \underline{\mathbf{t}}_T)\cdot\boldsymbol{n}_F - \mathfrak{t}_F)_F. \tag{52}$$

Notice that J_T is symmetric, positive semi-definite, and polynomially consistent (as a consequence of Lemma 4.2) in the sense that, for all $\boldsymbol{t} \in \boldsymbol{S}_T^{k+1}$ (this result can be found in [25, Eq. (18)]),

$$J_T(\underline{\mathbf{I}}_T^k \boldsymbol{t}, \underline{\mathbf{r}}_T) = 0 \qquad \forall \underline{\mathbf{r}}_T \in \underline{\mathbf{S}}_T^k. \tag{53}$$

Another important property of J_T is the following approximation property (see [25, Lemma 8] and Lemma 4.3 above): For all $v \in H^{k+2}(T)$, the following holds with $\boldsymbol{t} := \mathbb{M}_T \boldsymbol{\nabla} v$:

$$J_T(\underline{\mathbf{I}}_T^k \boldsymbol{t}, \underline{\mathbf{I}}_T^k \boldsymbol{t})^{1/2} \lesssim \rho_T^{1/2} \mu_{\sharp,T}^{1/2} h_T^{k+1} \|v\|_{H^{k+2}(T)}. \tag{54}$$

4.2 Discrete problem: formulation and key properties

4.2.1 Formulation

For $T \in \mathcal{T}_h$, we define the following local bilinear form:

$$H_T : \underline{\mathbf{S}}_T^k \times \underline{\mathbf{S}}_T^k \to \mathbb{R}; \qquad (\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T) \mapsto (\mathbb{M}_T^{-1} \boldsymbol{F}_T^{k+1} \underline{\mathbf{s}}_T, \boldsymbol{F}_T^{k+1} \underline{\mathbf{t}}_T)_T + J_T(\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T), \tag{55}$$

where the notation H_T is reminiscent of the similarity with the discrete Hodge operator considered in the CDO framework in the lowest-order case [11]. Introduce now the following global bilinear form:

$$H_h : \underline{\mathbf{S}}_h^k \times \underline{\mathbf{S}}_h^k \to \mathbb{R}; \qquad (\underline{\mathbf{s}}_h, \underline{\mathbf{t}}_h) \mapsto \sum_{T \in \mathcal{T}_h} H_T(\underline{\mathbf{s}}_T, \underline{\mathbf{t}}_T). \tag{56}$$

Then, the mixed form of the HHO method for problem (40) reads: Find $(\tilde{\underline{\mathbf{s}}}_h, u_h) \in \underline{\mathbf{S}}_{h,0}^k \times U_h^k$ such that

$$H_h(\tilde{\underline{\mathbf{s}}}_h, \underline{\mathbf{t}}_h) + (u_h, D_h^k \underline{\mathbf{t}}_h) = \sum_{F \in \mathcal{F}_h^{\mathrm{d}}} (\mathbf{t}_F, u_\partial)_F - H_h(\tilde{\underline{\mathbf{s}}}_{h,\partial}, \underline{\mathbf{t}}_h) \qquad \forall \underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_{h,0}^k, \tag{57a}$$

$$-(D_h^k \tilde{\underline{\mathbf{s}}}_h, v_h) = (f, v_h) + (D_h^k \tilde{\underline{\mathbf{s}}}_{h,\partial}, v_h) \qquad \forall v_h \in U_h^k, \tag{57b}$$

where $\tilde{\underline{\mathbf{s}}}_{h,\partial} := \underline{\mathbf{I}}_h^k \tilde{\boldsymbol{s}}_\partial \in \underline{\mathbf{S}}_h^k$ is a discrete lifting of s_∂. The discrete solution $(\underline{\mathbf{s}}_h, u_h) \in \underline{\mathbf{S}}_h^k \times U_h^k$ is finally computed as $(\underline{\mathbf{s}}_h, u_h) = (\tilde{\underline{\mathbf{s}}}_h + \tilde{\underline{\mathbf{s}}}_{h,\partial}, u_h)$.

4.2.2 Stability

Let us introduce, for all $T \in \mathcal{T}_h$, the following norm on $\underline{\mathbf{S}}_T^k$:

$$\|\underline{\mathbf{t}}_T\|_{\boldsymbol{S},T}^2 := \|\boldsymbol{t}_T\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \|\mathbf{t}_F\|_F^2. \tag{58}$$

Setting $\|\underline{\mathbf{t}}_h\|_{\boldsymbol{S},h}^2 := \sum_{T \in \mathcal{T}_h} \mu_{\sharp,T}^{-1} \|\underline{\mathbf{t}}_T\|_{\boldsymbol{S},T}^2$ for all $\underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_h^k$, it follows that $\|\cdot\|_{\boldsymbol{S},h}$ defines a norm on $\underline{\mathbf{S}}_h^k$. The coercivity of H_h can be expressed in terms of this norm, cf. [25, Lemma 3].

Lemma 4.4 (Stability for H_h)**.** *For all $T \in \mathcal{T}_h$, and for all $\underline{\mathbf{t}}_T \in \underline{\mathbf{S}}_T^k$, the following holds:*

$$\mu_{\sharp,T}^{-1} \|\underline{\mathbf{t}}_T\|_{\boldsymbol{S},T}^2 \lesssim H_T(\underline{\mathbf{t}}_T, \underline{\mathbf{t}}_T) \lesssim \mu_{\flat,T}^{-1} \|\underline{\mathbf{t}}_T\|_{\boldsymbol{S},T}^2. \tag{59}$$

Consequently, we infer that

$$\|\underline{\mathbf{t}}_h\|_{\boldsymbol{S},h}^2 \lesssim H_h(\underline{\mathbf{t}}_h, \underline{\mathbf{t}}_h) \qquad \forall \underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_h^k.$$

We can then state the following result, whose proof hinges on Lemma 4.1, and which is a slightly modified version of [25, Lemma 4].

Lemma 4.5 (Well-posedness of (57))**.** *For all $v_h \in U_h^k$, the following holds:*

$$\mu_\flat^{1/2} \|v_h\| \lesssim \sup_{\underline{\mathbf{t}}_h \in \underline{\mathbf{S}}_{h,0}^k, \|\underline{\mathbf{t}}_h\|_{\boldsymbol{S},h} = 1} (D_h^k \underline{\mathbf{t}}_h, v_h). \tag{60}$$

Combining (60) with Lemma 4.4, we infer that problem (57) is well-posed.

4.2.3 Error estimates

Let $(\boldsymbol{s}, u) \in \boldsymbol{S} \times V$ be such that $(\boldsymbol{s}, u) = (\tilde{\boldsymbol{s}} + \tilde{\boldsymbol{s}}_\partial, u)$, where $(\tilde{\boldsymbol{s}}, u) \in \boldsymbol{S}_0 \times V$ is the solution to (40), and $\tilde{\boldsymbol{s}}_\partial \in \boldsymbol{S}$ is defined above. We further assume that $\boldsymbol{s} \in \boldsymbol{S}$ fulfills the additional regularity $\boldsymbol{s} \in \boldsymbol{S}^+(\mathcal{T}_h)$. Similarly, let $(\underline{\mathbf{s}}_h, u_h) \in \underline{\mathbf{S}}_h^k \times U_h^k$ be such that $(\underline{\mathbf{s}}_h, u_h) = (\tilde{\underline{\mathbf{s}}}_h + \tilde{\underline{\mathbf{s}}}_{h,\partial}, u_h)$, where $(\tilde{\underline{\mathbf{s}}}_h, u_h) \in \underline{\mathbf{S}}_{h,0}^k \times U_h^k$ is the solution to (57), and $\tilde{\underline{\mathbf{s}}}_{h,\partial} \in \underline{\mathbf{S}}_h^k$ is defined in Section 4.2.1. Then, we can state the following result, whose proof can be easily adapted from the one of [25, Theorem 5].

Theorem 4.1 (Error estimate for the flux)**.** *Assume the additional regularity $u \in H^{k+2}(\mathcal{T}_h)$. Then, the following holds:*

$$\|\underline{\mathbf{I}}_h^k \boldsymbol{s} - \underline{\mathbf{s}}_h\|_{\boldsymbol{S},h} \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\sharp,T} \rho_T h_T^{2(k+1)} \|u\|^2_{H^{k+2}(T)} \right\}^{1/2}, \tag{61}$$

which implies, by an additional use of Lemma 4.3,

$$\|\mathbb{M}^{-1/2}(\boldsymbol{s} - \boldsymbol{F}_h^{k+1} \underline{\mathbf{s}}_h)\| \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\sharp,T} \rho_T h_T^{2(k+1)} \|u\|^2_{H^{k+2}(T)} \right\}^{1/2}. \tag{62}$$

Finally, whenever elliptic regularity holds, a supercloseness result for the potential can be established, cf. [25, Theorem 6].

4.3 Computational aspects

As far as static condensation is concerned, there are two ways of reducing the size of the discrete problem (57).

First, as detailed in [25, Section 3.4], it is possible to eliminate locally the cell-based flux unknowns and the potential unknowns, up to one constant value per element. Thus, the global system to solve only writes in terms of the face-based flux unknowns and of the mean value of the potential in each element. For all $T \in \mathcal{T}_h$, let $U_T^{k,0}$ be the space of polynomials of degree at most k having zero mean value in T, so that $U_T^k = U_T^0 \oplus U_T^{k,0}$. Hence, any function $v_T \in U_T^k$ can be written $v_T = v_T^0 + \hat{v}_T$ with $v_T^0 \in U_T^0$ and $\hat{v}_T \in U_T^{k,0}$. Then, we infer from (57) that, for all $T \in \mathcal{T}_h$, $(\boldsymbol{s}_T, \hat{u}_T) \in \boldsymbol{S}_T^k \times U_T^{k,0}$ can be eliminated locally by solving the following saddle-point type problem:

$$\tilde{H}_T(\boldsymbol{s}_T, \boldsymbol{t}_T) + (\boldsymbol{t}_T, \boldsymbol{\nabla} \hat{u}_T)_T = g_1(\boldsymbol{t}_T) \qquad \forall\, \boldsymbol{t}_T \in \boldsymbol{S}_T^k, \tag{63a}$$

$$-(\boldsymbol{s}_T, \boldsymbol{\nabla} \hat{v}_T)_T = g_2(\hat{v}_T) \qquad \forall\, \hat{v}_T \in U_T^{k,0}, \tag{63b}$$

where g_1, g_2 are suitable linear forms and $\tilde{H}_T(\boldsymbol{s}_T, \boldsymbol{t}_T) := H_T((\boldsymbol{s}_T, 0_{\mathfrak{U}_T^k}), (\boldsymbol{t}_T, 0_{\mathfrak{U}_T^k}))$. Problem (63) is well-posed, since, according to (59) and (58), $\tilde{H}_T(\boldsymbol{t}_T, \boldsymbol{t}_T)$ is uniformly equivalent to $\|\boldsymbol{t}_T\|_T^2$.

The second static condensation approach is based on a reformulation of the mixed problem (57) into a primal problem. The reformulation is based on the introduction of Lagrange multipliers that enforce the continuity of interface-based flux unknowns and that can be interpreted as potential traces on mesh faces. One can eliminate the cell- and face-based flux unknowns, and, once the reformulation has been performed, one can adapt the arguments of Section 3.5 to further eliminate locally the cell-based potential unknowns, ending up with a global system only depending on the Lagrange multipliers (face-based potential unknowns). This static condensation approach has the double advantage that it requires to solve local coercive problems (as opposed to local saddle point problems) and that it yields a coercive global problem. For this reason, we discuss it in more detail in the next Section.

5 Bridging the primal and mixed forms of HHO methods

The goal of this section is to bridge the primal and mixed forms of the HHO method studied in Sections 3 and 4, respectively. As discussed in the previous section, this can be exploited in practice to implement the mixed form of the HHO method in terms of a coercive problem posed on the Lagrange multipliers only.

5.1 Unpatching interface-based flux unknowns

We introduce a global set of hybrid flux unknowns where interface-based unknowns are two-valued; we refer to these unknowns as unpatched. Then, the unpatched global set of hybrid flux unknowns is defined as

$$\check{\underline{\mathbf{S}}}_h^k := \bigtimes_{T\in\mathcal{T}_h} \underline{\mathbf{S}}_T^k, \tag{64}$$

with subset

$$\check{\underline{\mathbf{S}}}_{h,0}^k := \left\{ \check{\underline{\mathbf{t}}}_h \in \check{\underline{\mathbf{S}}}_h^k \mid \check{\mathfrak{t}}_F \equiv 0, \quad \forall F \in \mathcal{F}_h^{\mathrm{n}} \right\}. \tag{65}$$

Given an element $\check{\underline{\mathbf{t}}}_h \in \check{\underline{\mathbf{S}}}_h^k$, for any $T \in \mathcal{T}_h$, we denote as before by $\check{\underline{\mathbf{t}}}_T = (\check{\boldsymbol{t}}_T, (\check{\mathfrak{t}}_{T,F})_{F\in\mathcal{F}_T}) \in \underline{\mathbf{S}}_T^k$ its restriction to the element T. For boundary faces $F \in \mathcal{F}_h^{\mathrm{b}}$, the subscript T in $\check{\mathfrak{t}}_{T,F}$ can be omitted, and we simply write $\check{\mathfrak{t}}_F$.

Owing to the definition of $\check{\underline{\mathbf{S}}}_h^k$ (respectively, $\check{\underline{\mathbf{S}}}_{h,0}^k$), it can be easily seen that there exists a natural isomorphism between the subspace of $\check{\underline{\mathbf{S}}}_h^k$ (respectively, $\check{\underline{\mathbf{S}}}_{h,0}^k$) spanned by those elements $\check{\underline{\mathbf{t}}}_h$ satisfying $\sum_{T\in\mathcal{T}_F} \check{\mathfrak{t}}_{T,F} = 0$ for all $F \in \mathcal{F}_h^{\mathrm{i}}$, and the space $\underline{\mathbf{S}}_h^k$ (respectively $\underline{\mathbf{S}}_{h,0}^k$).

5.2 Unpatched mixed formulation

We begin by extending to $\check{\underline{\mathbf{S}}}_h^k$ the definitions, respectively built from (46) and (49), of the divergence operator D_h^k and of the flux reconstruction operator $\boldsymbol{F}_h^{k+1}$, for which we keep the same notation (locally, the definitions are unchanged up to the replacement of $\mathfrak{t}_F\varepsilon_{T,F}$ by $\check{\mathfrak{t}}_{T,F}$ in face terms). We can then naturally extend the bilinear form H_h, defined in (56) and built from (55), to the product space $\check{\underline{\mathbf{S}}}_h^k \times \check{\underline{\mathbf{S}}}_h^k$.

We next introduce, for all $T \in \mathcal{T}_h$, the additional bilinear form

$$B_T : \underline{\mathbf{S}}_T^k \times \underline{\mathrm{U}}_T^k \to \mathbb{R}; \qquad (\check{\underline{\mathbf{t}}}_T, \underline{\mathrm{v}}_T) \mapsto (v_T, D_T^k \check{\underline{\mathbf{t}}}_T)_T - \sum_{F\in\mathcal{F}_T\cap\mathcal{F}_h^{\mathrm{i}}} (\mathfrak{v}_F, \check{\mathfrak{t}}_{T,F})_F, \tag{66}$$

whose global version is as usual obtained by element-by-element assembly:

$$B_h : \check{\underline{\mathbf{S}}}_h^k \times \underline{\mathrm{U}}_h^k \to \mathbb{R}; \qquad (\check{\underline{\mathbf{t}}}_h, \underline{\mathrm{v}}_h) \mapsto \sum_{T\in\mathcal{T}_h} B_T(\check{\underline{\mathbf{t}}}_T, \underline{\mathrm{v}}_T). \tag{67}$$

This bilinear form includes interface terms that can be viewed as continuity constraints on the face-based flux unknowns. In that vision, the face-based potential unknowns can be seen as Lagrange multipliers.

The unpatched (mixed) HHO discretization of problem (40) then reads: Find $(\underline{\tilde{\mathbf{s}}}_h, \underline{\tilde{\mathbf{u}}}_h) \in \underline{\check{\mathbf{S}}}_{h,0}^k \times \underline{\mathrm{U}}_{h,0}^k$ such that

$$H_h(\underline{\tilde{\mathbf{s}}}_h, \underline{\check{\mathbf{t}}}_h) + B_h(\underline{\check{\mathbf{t}}}_h, \underline{\tilde{\mathbf{u}}}_h) = \sum_{F\in\mathcal{F}_h^{\mathrm{d}}} (\check{\mathfrak{t}}_F, u_\partial)_F - H_h(\underline{\check{\mathbf{s}}}_{h,\partial}, \underline{\check{\mathbf{t}}}_h) - B_h(\underline{\check{\mathbf{t}}}_h, \underline{\tilde{\mathbf{u}}}_{h,\partial}) \qquad \forall \underline{\check{\mathbf{t}}}_h \in \underline{\check{\mathbf{S}}}_{h,0}^k, \quad (68a)$$

$$-B_h(\underline{\tilde{\mathbf{s}}}_h, \underline{\mathrm{v}}_h) = (f, v_h) + B_h(\underline{\check{\mathbf{s}}}_{h,\partial}, \underline{\mathrm{v}}_h) \qquad \forall \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k, \quad (68b)$$

where $\underline{\check{\mathbf{s}}}_{h,\partial} \in \underline{\check{\mathbf{S}}}_h^k$ is such that, for all $T \in \mathcal{T}_h$, $\underline{\check{\mathbf{s}}}_{T,\partial} := (\tilde{\boldsymbol{s}}_{T,\partial}, (\tilde{\mathfrak{s}}_{F,\partial}\, \varepsilon_{T,F})_{F\in\mathcal{F}_T})$, with $\underline{\tilde{\mathbf{s}}}_{h,\partial} \in \underline{\mathbf{S}}_h^k$ defined in Section 4.2.1. The discrete solution $(\underline{\check{\mathbf{s}}}_h, \underline{\check{\mathbf{u}}}_h) \in \underline{\check{\mathbf{S}}}_h^k \times \underline{\mathrm{U}}_h^k$ is finally computed as $(\underline{\check{\mathbf{s}}}_h, \underline{\check{\mathbf{u}}}_h) = (\underline{\tilde{\mathbf{s}}}_h + \underline{\check{\mathbf{s}}}_{h,\partial}, \underline{\tilde{\mathbf{u}}}_h + \underline{\tilde{\mathbf{u}}}_{h,\partial})$.

5.3 Equivalence of primal and mixed formulations

The bridge between primal- and mixed-form HHO methods is built in two steps: first, we prove the equivalence between the mixed and unpatched mixed formulations; then, we prove that the unpatched mixed formulation can be recast into a primal formulation. The following result is an adaptation of [1, Lemma 3.3].

Theorem 5.1 (Equivalence between (57) and (68))**.** *Denote by* $(\underline{\mathbf{s}}_h, u_h) \in \underline{\mathbf{S}}_h^k \times U_h^k$ *and* $(\underline{\check{\mathbf{s}}}_h, \underline{\check{\mathbf{u}}}_h) \in \underline{\check{\mathbf{S}}}_h^k \times \underline{\mathrm{U}}_h^k$ *the solutions to* (57) *and* (68)*, respectively. Then,* $(\underline{\mathbf{s}}_h, u_h) = (\underline{\check{\mathbf{s}}}_h, \check{u}_h)$*, where* $\check{u}_h \in U_h^k$ *is the cell-based part of* $\underline{\check{\mathbf{u}}}_h$*.*

Following [1, Section 3.3], let us now introduce, for all $T \in \mathcal{T}_h$, the local potential-to-flux mapping operator $\underline{\check{\boldsymbol{\varsigma}}}_T^k : \underline{\mathrm{U}}_T^k \to \underline{\mathbf{S}}_T^k$ such that, for all $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$,

$$H_T(\underline{\check{\boldsymbol{\varsigma}}}_T^k \underline{\mathrm{v}}_T, \underline{\check{\mathbf{t}}}_T) = -B_T(\underline{\check{\mathbf{t}}}_T, \underline{\mathrm{v}}_T) + \sum_{F\in\mathcal{F}_T\cap\mathcal{F}_h^{\mathrm{b}}} (\check{\mathfrak{t}}_F, \mathfrak{v}_F)_F \qquad \forall\, \underline{\check{\mathbf{t}}}_T \in \underline{\mathbf{S}}_T^k. \quad (69)$$

This yields a well-posed problem owing to the first inequality in (59). Defining next the local consistent gradient reconstruction operator $\boldsymbol{G}_T^{k+1} : \underline{\mathrm{U}}_T^k \to \boldsymbol{S}_T^{k+1}$ as

$$\boldsymbol{G}_T^{k+1} := \boldsymbol{F}_T^{k+1} \circ \underline{\check{\boldsymbol{\varsigma}}}_T^k, \quad (70)$$

one can prove the following result.

Lemma 5.1 (Link between $\boldsymbol{F}_T^{k+1}$ and p_T^{k+1})**.** *For all* $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$*, the following holds:*

$$\boldsymbol{G}_T^{k+1} \underline{\mathrm{v}}_T = \mathbb{M}_T \boldsymbol{\nabla} p_T^{k+1} \underline{\mathrm{v}}_T. \quad (71)$$

Proof. Let $\underline{\mathrm{v}}_T \in \underline{\mathrm{U}}_T^k$, and let us plug, for $w \in \mathbb{P}_d^{k+1}(T)$, $\underline{\check{\mathbf{t}}}_T := \underline{\mathbf{I}}_T^k(\mathbb{M}_T \boldsymbol{\nabla} w)$ into (69). Using (66), (46), the polynomial consistency property of Lemma 4.2 coupled to (70), and the one of (53), we get

$$(\boldsymbol{G}_T^{k+1} \underline{\mathrm{v}}_T, \boldsymbol{\nabla} w)_T = (\boldsymbol{\nabla} v_T, \mathbb{M}_T \boldsymbol{\nabla} w)_T + \sum_{F\in\mathcal{F}_T} (\mathfrak{v}_F - v_T, \mathbb{M}_T \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{T,F})_F, \quad (72)$$

where we have used that $\check{\boldsymbol{t}}_T = \mathbb{M}_T \boldsymbol{\nabla} w$ and $\check{\mathfrak{t}}_{T,F} = \mathbb{M}_T \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{T,F}$, owing to (43) and to the fact that $w \in \mathbb{P}_d^{k+1}(T)$. Finally, performing a last integration by parts in (72), and comparing to the definition (11) of p_T^{k+1}, we prove (71). □

Now, defining $\check{\underline{\boldsymbol{\varsigma}}}_h^k : \underline{\mathrm{U}}_h^k \to \check{\underline{\mathbf{S}}}_h^k$ such that, for all $\underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_h^k$, and for all $T \in \mathcal{T}_h$, $(\check{\underline{\boldsymbol{\varsigma}}}_h^k \underline{\mathrm{v}}_h)_{|T} := \check{\underline{\boldsymbol{\varsigma}}}_T^k \underline{\mathrm{v}}_T$, we infer from (69) that

$$H_h(\check{\underline{\boldsymbol{\varsigma}}}_h^k \check{\underline{\mathrm{u}}}_h, \check{\underline{\mathbf{t}}}_h) = -B_h(\check{\underline{\mathbf{t}}}_h, \check{\underline{\mathrm{u}}}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{d}}} (\check{\mathfrak{t}}_F, u_\partial)_F \qquad \forall\, \check{\underline{\mathbf{t}}}_h \in \check{\underline{\mathbf{S}}}_{h,0}^k, \tag{73}$$

where we have used the fact that $\check{\mathfrak{t}}_F \equiv 0$ for all $F \in \mathcal{F}_h^{\mathrm{n}}$ and that $\check{\mathfrak{u}}_F = \pi_F^k u_\partial$ for all $F \in \mathcal{F}_h^{\mathrm{d}}$. Comparing (73) with (68a), it is readily inferred that $\check{\underline{\mathbf{s}}}_h = \check{\underline{\boldsymbol{\varsigma}}}_h^k \check{\underline{\mathrm{u}}}_h$. Plugging this relation into (68b), we get that

$$-B_h(\check{\underline{\boldsymbol{\varsigma}}}_h^k \check{\underline{\mathrm{u}}}_h, \underline{\mathrm{v}}_h) = (f, v_h) \qquad \forall \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k. \tag{74}$$

Using again (69), we additionally prove that

$$H_h(\check{\underline{\boldsymbol{\varsigma}}}_h^k \underline{\mathrm{v}}_h, \check{\underline{\boldsymbol{\varsigma}}}_h^k \check{\underline{\mathrm{u}}}_h) = -B_h(\check{\underline{\boldsymbol{\varsigma}}}_h^k \check{\underline{\mathrm{u}}}_h, \underline{\mathrm{v}}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{n}}} (\check{\mathfrak{s}}_F, \mathfrak{v}_F)_F \qquad \forall\, \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k, \tag{75}$$

where we have used the fact that $\mathfrak{v}_F \equiv 0$ for all $F \in \mathcal{F}_h^{\mathrm{d}}$. Plugging (75) into (74), using the symmetry of H_h, the decomposition $\check{\underline{\mathrm{u}}}_h = \tilde{\underline{\mathrm{u}}}_h + \tilde{\underline{\mathrm{u}}}_{h,\partial}$, and the fact that $\check{\mathfrak{s}}_F = \pi_F^k s_\partial$ for all $F \in \mathcal{F}_h^{\mathrm{n}}$, we obtain

$$H_h(\check{\underline{\boldsymbol{\varsigma}}}_h^k \tilde{\underline{\mathrm{u}}}_h, \check{\underline{\boldsymbol{\varsigma}}}_h^k \underline{\mathrm{v}}_h) = (f, v_h) - H_h(\check{\underline{\boldsymbol{\varsigma}}}_h^k \tilde{\underline{\mathrm{u}}}_{h,\partial}, \check{\underline{\boldsymbol{\varsigma}}}_h^k \underline{\mathrm{v}}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{n}}} (s_\partial, \mathfrak{v}_F)_F \qquad \forall\, \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k. \tag{76}$$

Finally, introducing the global bilinear form $A_h : \underline{\mathrm{U}}_h^k \times \underline{\mathrm{U}}_h^k \to \mathbb{R}$ such that $A_h(\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) := H_h(\check{\underline{\boldsymbol{\varsigma}}}_h^k \underline{\mathrm{u}}_h, \check{\underline{\boldsymbol{\varsigma}}}_h^k \underline{\mathrm{v}}_h)$, problem (76) can be rewritten under the form

$$A_h(\tilde{\underline{\mathrm{u}}}_h, \underline{\mathrm{v}}_h) = (f, v_h) - A_h(\tilde{\underline{\mathrm{u}}}_{h,\partial}, \underline{\mathrm{v}}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{n}}} (s_\partial, \mathfrak{v}_F)_F \qquad \forall\, \underline{\mathrm{v}}_h \in \underline{\mathrm{U}}_{h,0}^k. \tag{77}$$

Using (56), (55), (70), and (71), we also infer that

$$A_h(\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) = \sum_{T \in \mathcal{T}_h} (\mathbb{M}_T \boldsymbol{\nabla} p_T^{k+1} \underline{\mathrm{u}}_T, \boldsymbol{\nabla} p_T^{k+1} \underline{\mathrm{v}}_T)_T + \sum_{T \in \mathcal{T}_h} J_T(\check{\underline{\boldsymbol{\varsigma}}}_T^k \underline{\mathrm{u}}_T, \check{\underline{\boldsymbol{\varsigma}}}_T^k \underline{\mathrm{v}}_T). \tag{78}$$

Finally, owing to (78), the comparison of problem (77) to problem (20) allows to infer the following result, cf. [1, Section 3.3.4].

Theorem 5.2 (Equivalence between (20) and (68))**.** *Let us denote by* $\underline{\mathrm{u}}_h \in \underline{\mathrm{U}}_h^k$ *and* $(\check{\underline{\mathbf{s}}}_h, \check{\underline{\mathrm{u}}}_h) \in \check{\underline{\mathbf{S}}}_h^k \times \underline{\mathrm{U}}_h^k$ *the solutions to* (20) *and* (68)*, respectively. Then, up to a choice of stabilization* $j_T(\cdot,\cdot) := J_T(\check{\underline{\boldsymbol{\varsigma}}}_T^k \cdot, \check{\underline{\boldsymbol{\varsigma}}}_T^k \cdot)$ *in* (18) *for problem* (20)*,* $\underline{\mathrm{u}}_h = \check{\underline{\mathrm{u}}}_h$ *holds.*

The combination of Theorems 5.1 and 5.2 states the equivalence between primal- and mixed-form HHO methods, up to an appropriate choice of stabilization.

References

[1] J. Aghili, S. Boyaval, and D. A. Di Pietro. Hybridization of mixed high-order methods on general meshes and application to the Stokes equations. *Comput. Meth. Appl. Math.*, 15(2):111–134, 2015.

[2] P. F. Antonietti, S. Giani, and P. Houston. *hp*-version composite discontinuous Galerkin methods for elliptic problems on complicated domains. *SIAM J. Sci. Comput*, 35(3):A1417–A1439, 2013.

[3] R. Araya, C. Harder, D. Paredes, and F. Valentin. Multiscale hybrid-mixed method. *SIAM J. Numer. Anal.*, 51(6):3505–3531, 2013.

[4] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19:742–760, 1982.

[5] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2002.

[6] B. Ayuso de Dios, K. Lipnikov, and G. Manzini. The nonconforming virtual element method. Submitted. Preprint arXiv:1405.3741, May 2014.

[7] F. Bassi, L. Botti, A. Colombo, D. A. Di Pietro, and P. Tesini. On the flexibility of agglomeration based physical space discontinuous Galerkin discretizations. *J. Comput. Phys.*, 231(1):45–65, 2012.

[8] L. Beirão da Veiga, F. Brezzi, and L.D. Marini. Virtual elements for linear elasticity problems. *SIAM J. Numer. Anal.*, 51(2):794–812, 2013.

[9] L. Beirão da Veiga, K. Lipnikov, and G. Manzini. *The Mimetic Finite Difference Method for Elliptic Problems*, volume 11 of *MS&A*. Springer, New York, 2014.

[10] J. Bonelle, D. A. Di Pietro, and A. Ern. Low-order reconstruction operators on polyhedral meshes: Application to Compatible Discrete Operator schemes. *Computer Aided Geometric Design*, 35–36:27–41, 2015.

[11] J. Bonelle and A. Ern. Analysis of compatible discrete operator schemes for elliptic problems on polyhedral meshes. *M2AN Math. Model. Numer. Anal.*, 48:553–581, 2014.

[12] J. Bonelle and A. Ern. Analysis of Compatible Discrete Operator schemes for the Stokes equations on polyhedral meshes. *IMA J. Numer. Anal.*, 2014. Published online. DOI: 10.1093/imanum/dru051.

[13] A. Bossavit. Computational electromagnetism and geometry. *J. Japan Soc. Appl. Electromagn. & Mech.*, 7-8:150–9 (no 1), 294–301 (no 2), 401–8 (no 3), 102–9 (no 4), 203–9 (no 5), 372–7 (no 6), 1999-2000.

[14] F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of the mimetic finite difference method for diffusion problems on polyhedral meshes. *SIAM J. Numer. Anal.*, 43(5):1872–1896, 2005.

[15] F. Brezzi, K. Lipnikov, and V. Simoncini. A family of mimetic finite difference methods on polygonal and polyhedral meshes. *M3AS Math. Models Methods Appl. Sci.*, 15(10):1533–1551, 2005.

[16] F. Brezzi, G. Manzini, L. D. Marini, P. Pietra, and A. Russo. Discontinuous Galerkin approximations for elliptic problems. *Numer. Methods Partial Differential Equations*, 16(4):365–378, 2000.

[17] F. Brezzi and L. D. Marini. Virtual elements for plate bending problems. *Comput. Methods Appl. Mech. Engrg.*, 253:455–462, 2013.

[18] P. Castillo, B. Cockburn, I. Perugia, and D. Schötzau. An a priori error analysis of the local discontinuous Galerkin method for elliptic problems. *SIAM J. Numer. Anal.*, 38:1676–1706, 2000.

[19] B. Cockburn, D. A. Di Pietro, and A. Ern. Bridging the Hybrid High-Order and Hybridizable Discontinuous Galerkin methods. *ESAIM: Mathematical Modelling and Numerical Analysis*, 2015. Accepted for publication. Preprint hal-01115318.

[20] B. Cockburn, J. Gopalakrishnan, and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.*, 47(2):1319–1365, 2009.

[21] L. Codecasa, R. Specogna, and F. Trevisan. A new set of basis functions for the discrete geometric approach. *J. Comput. Phys.*, 19(299):7401–7410, 2010.

[22] D. A. Di Pietro. Cell centered Galerkin methods for diffusive problems. *M2AN Math. Model. Numer. Anal.*, 46(1):111–144, 2012.

[23] D. A. Di Pietro, J. Droniou, and A. Ern. A discontinuous-skeletal method for advection-diffusion-reaction on general meshes. Submitted. Preprint arXiv:1411.0098, November 2014.

[24] D. A. Di Pietro and A. Ern. *Mathematical Aspects of Discontinuous Galerkin Methods*, volume 69 of *Mathématiques & Applications*. Springer, Berlin Heidelberg, 2012.

[25] D. A. Di Pietro and A. Ern. Arbitrary-order mixed methods for heterogeneous anisotropic diffusion on general meshes. Submitted. Preprint hal-00918482, December 2013.

[26] D. A. Di Pietro and A. Ern. Equilibrated tractions for the Hybrid High-Order method. *C. R. Acad. Sci. Paris, Ser. I*, 353:279–282, 2015.

[27] D. A. Di Pietro and A. Ern. A Hybrid High-Order locking-free method for linear elasticity on general meshes. *Comput. Meth. Appl. Mech. Engrg.*, 283:1–21, 2015.

[28] D. A. Di Pietro and A. Ern. Hybrid High-Order methods for variable-diffusion problems on general meshes. *C. R. Acad. Sci Paris, Ser. I*, 353:31–34, 2015.

[29] D. A. Di Pietro, A. Ern, and S. Lemaire. An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators. *Comput. Methods Appl. Math.*, 14(4):461–472, 2014.

[30] D. A. Di Pietro and S. Lemaire. An extension of the Crouzeix–Raviart space to general meshes with application to quasi-incompressible linear elasticity and Stokes flow. *Math. Comp.*, 84:1–31, 2015.

[31] K. Domelevo and P. Omnes. A finite volume method for the Laplace equation on almost arbitrary two-dimensional grids. *ESAIM: Mathematical Modelling and Numerical Analysis*, 39(6):1203–1249, 2005.

[32] J. Droniou and R. Eymard. A mixed finite volume scheme for anisotropic diffusion problems on any grid. *Numer. Math.*, 105:35–71, 2006.

[33] J. Droniou, R. Eymard, T. Gallouët, and R. Herbin. A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods. *M3AS Mathematical Models and Methods in Applied Sciences*, 20(2):1–31, 2010.

[34] J. Droniou, R. Eymard, T. Gallouët, and R. Herbin. Gradient schemes: a generic framework for the discretisation of linear, nonlinear and nonlocal elliptic and parabolic equations. *Math. Models Methods Appl. Sci. (M3AS)*, 23:2395–2432, 2013.

[35] A. Ern and J.-L. Guermond. Discontinuous Galerkin methods for Friedrichs' systems. I. General theory. *SIAM J. Numer. Anal.*, 44(2):753–778, 2006.

[36] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. In P. G. Ciarlet and J.-L. Lions, editors, *Techniques of Scientific Computing, Part III*, pages 713–1020. 2000.

[37] R. Eymard, T. Gallouët, and R. Herbin. Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. SUSHI: a scheme using stabilization and hybrid interfaces. *IMA J. Numer. Anal.*, 30(4):1009–1043, 2010.

[38] C. Harder, D. Paredes, and F. Valentin. A family of multiscale hybrid-mixed finite element methods for the Darcy equation with rough coefficients. *J. Comput. Phys.*, 245:107–130, 2013.

[39] R. Herbin and F. Hubert. Benchmark on discretization schemes for anisotropic diffusion problems on general grids. In R. Eymard and J.-M. Hérard, editors, *Finite Volumes for Complex Applications V*, pages 659–692. John Wiley & Sons, 2008.

[40] C. Lehrenfeld. *Hybrid Discontinuous Galerkin methods for solving incompressible flow problems.* PhD thesis, Rheinisch-Westfälischen Technischen Hochschule Aachen, 2010.

[41] K. Lipnikov and G. Manzini. A high-order mimetic method on unstructured polyhedral meshes for the diffusion equation. *J. Comput. Phys.*, 272:360–385, 2014.

[42] E. Tonti. On the formal structure of physical theories. *Quaderni dei Gruppi di Ricerca Matematica del CNR*, 1975.

[43] J. Wang and X. Ye. A weak Galerkin element method for second-order elliptic problems. *J. Comput. Appl. Math.*, 241:103–115, 2013.

[44] J. Wang and X. Ye. A weak Galerkin mixed finite element method for second order elliptic problems. *Math. Comp.*, 83(289):2101–2126, 2014.