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# An Adaptive Spectral Algorithm for the Recovery of Overlapping Communities in Networks

Emilie Kaufmann<sup>1\*</sup>, Thomas Bonald<sup>2\*</sup> and Marc Lelarge<sup>1\*</sup>  
Inria & Ecole Normale Supérieure<sup>1</sup>, Telecom ParisTech<sup>2</sup>

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## Abstract

This paper presents *combinatorial spectral clustering*, a simple spectral algorithm designed to identify overlapping communities in networks. The algorithm is based on geometric properties of the spectrum of the expected adjacency matrix in a random graph model that we call *stochastic blockmodel with overlap* (SBMO). An adaptive version of the algorithm, that does not require the knowledge of the number of hidden communities, is proved to be consistent under the SBMO when the degrees in the graph are (slightly more than) logarithmic. The algorithm is shown to perform well on simulated data and on real-world graphs with known overlapping communities.

## 1 Introduction

Many datasets (e.g., social networks, communication networks, gene regulation networks) take the form of graphs whose structure depends on some underlying *communities*. The commonly accepted definition of a community is that nodes tend to be more densely connected within a community than with the rest of the graph. Communities are often hidden in practice and recovering the community structure directly from the graph is a key step in the analysis of these datasets.

Spectral algorithms are popular methods for detecting communities [Von Luxburg, 2007]. First, a *spectral embedding* is built, where the  $n$  nodes of the graph are projected onto some low dimensional space generated by well-chosen eigenvectors of some matrix related to the graph (e.g., the adjacency matrix or the Laplacian matrix). Then, a *clustering algorithm* (e.g.,  $k$ -means or  $k$ -median) is applied to the  $n$  embedded vectors to obtain a partition of the nodes into communities.

It turns out that the structure of many real datasets are better explained by *overlapping* communities. This is the case for instance for co-authorship networks, in which authors often belong to several scientific communities, and for protein-protein interaction networks, in which a given protein may belong to several protein complexes [Palla et al., 2005]. The communities do not form a partition of the graph and new algorithms need to be designed. This paper presents a simple spectral algorithm called combinatorial spectral clustering (CSC). The algorithm consists in a spectral embedding based on the adjacency matrix of the graph, coupled with a clustering phase suited to overlapping communities. The proposed algorithm does not require the knowledge of the number of communities present in the graph and can thus be qualified as adaptive.

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\*The authors are members of the LINCS, Paris, France. See [www.lincs.fr](http://www.lincs.fr).

To evaluate the performance of a community detection algorithm, there are two possible approaches. The first is to test the algorithm on real datasets for which ground-truth communities are known. The second is to consider a random graph whose probability distribution depends on some underlying set of communities and to prove that the communities are recovered with high probability, provided the network is sufficiently large and dense. Such consistency results have been obtained for non-overlapping community detection under variants of the stochastic block model (SBM), introduced by [Holland and Leinhardt, 1983]. Several extensions of the SBM to overlapping communities have been proposed in the literature (e.g., [Airoldi et al., 2008, Latouche et al., 2011, Yang and Leskovec, 2012, Ball et al., 2011]), as well as some algorithms for recovering the communities. Our algorithm is inspired by the simplest possible extension of the SBM, we refer to as the *stochastic blockmodel with overlaps* (SBMO).

We prove the consistency of our algorithm under this model and evaluate its performance on both simulated and real datasets for which ground-truth communities are known.

**Notation.** We denote by  $\|x\|$  the Euclidean norm of a vector  $x \in \mathbb{R}^d$ . For any matrix  $M \in \mathbb{R}^{n \times d}$ , we let  $M_{i,\cdot}$  denote its  $i$ -th row and  $M_{\cdot,j}$  its  $j$ -th column. To ease the notation, we will often simply denote by  $M_i$  the  $i$ -th row of matrix  $M$ . For any  $\mathcal{S} \subset \{1, \dots, d\}$ ,  $|\mathcal{S}|$  denotes its cardinality and  $\mathbb{1}_{\mathcal{S}} \in \{0, 1\}^{1 \times d}$  is a row vector such that  $(\mathbb{1}_{\mathcal{S}})_{1,i} = \mathbb{1}_{\{i \in \mathcal{S}\}}$ . The Frobenius norm of a matrix  $M \in \mathbb{R}^{n \times d}$  is

$$\|M\|_F^2 = \sum_{i=1}^n \|M_{i,\cdot}\|^2 = \sum_{j=1}^d \|M_{\cdot,j}\|^2 = \sum_{1 \leq i,j \leq n} M_{i,j}^2.$$

The spectral norm of a symmetric matrix  $M \in \mathbb{R}^{d \times d}$  with eigenvalues  $\lambda_1, \dots, \lambda_d$  is  $\|M\| = \max_{i=1..d} |\lambda_i|$ . For  $\sigma \in \mathfrak{S}_K$ , we let  $P_\sigma \in \mathbb{R}^{K \times K}$  the permutation matrix associated to  $\sigma$ , defined by  $(P_\sigma)_{k,l} = \delta_{\sigma(k),l}$ .

## 2 The stochastic blockmodel with overlaps (SBMO)

### 2.1 The model

For any symmetric matrix  $A \in [0, 1]^{n \times n}$ , let  $\hat{A}$  be some random symmetric binary matrix whose entries  $(\hat{A}_{i,j})_{i \leq j}$  are independent Bernoulli random variables with respective parameters  $(A_{i,j})_{i \leq j}$ . Then  $\hat{A}$  is the adjacency matrix of an undirected random graph with expected adjacency matrix  $A$ . In all the paper, we restrict the hat notation to variables that depend on this random graph. For example, the empirical degree of node  $i$  observed on the random graph and the expected degree of node  $i$  are respectively denoted by

$$\hat{d}_i = \sum_{j=1}^n \hat{A}_{i,j} \quad \text{and} \quad d_i = \sum_{j=1}^n A_{i,j}.$$

Similarly, we write  $\hat{D} = \text{Diag}(\hat{d}_i)$ ,  $D = \text{Diag}(d_i)$ , and

$$\hat{d}_{\max} := \max_i \sum_{j=1}^n \hat{A}_{i,j}, \quad d_{\max} = \max_i \sum_{j=1}^n A_{i,j}.$$

The stochastic block model (SBM) with  $n$  nodes and  $K$  communities depends on some mapping  $k : \{1, \dots, n\} \rightarrow \{1, \dots, K\}$  that associates nodes to communities and on some symmetric community connectivity matrix  $B \in [0, 1]^{K \times K}$ . In this model, two nodes  $i$  and  $j$  are connected with probability

$$A_{i,j} = B_{k(i),k(j)} = B_{k(j),k(i)}.$$

Introducing a membership matrix  $Z \in \{0, 1\}^{n \times K}$  such that  $Z_{i,k} = \mathbb{1}_{\{k(i)=k\}}$ , the expected adjacency matrix can be written

$$A = ZBZ^T. \quad (1)$$

The stochastic blockmodel with overlap (SBMO) is a slight extension of this model, in which  $Z$  is only assumed to be in  $\{0, 1\}^{n \times K}$  and  $Z_i \neq 0$  for all  $i$ . Compared to the SBM, the rows of the membership matrix  $Z$  are no longer constrained to have only one non-zero entry. Since these  $n$  rows give the communities of the respective  $n$  nodes of the graph, this means that each node can now belong to several communities.

## 2.2 Performance metrics

Given some adjacency matrix  $\hat{A}$  drawn under the SBMO, our goal is to recover the underlying communities, that is to build an estimate  $\hat{Z}$  of the membership matrix  $Z$ , up to some permutation of its columns (corresponding to a permutation of the community labels). We denote by  $\hat{K}$  the estimate of the number of communities ( $K$  is in general unknown), so that  $\hat{Z} \in \{0, 1\}^{n \times \hat{K}}$ .

We introduce two performance metrics for this problem. The first is related to the number of nodes that are “well classified”, in the sense that there is no error in the estimate of their membership vector. The objective is to minimize the number of misclassified nodes of an estimate  $\hat{Z}$  of  $Z$ , defined by  $\text{MisC}(\hat{Z}, Z) = n$  if  $\hat{K} \neq K$  and

$$\text{MisC}(\hat{Z}, Z) = \min_{\sigma \in \mathfrak{S}_K} |\{i \in \{1, \dots, n\} : \exists k \in \{1, \dots, K\}, \hat{Z}_{i, \sigma(k)} \neq Z_{i,k}\}|$$

otherwise. The second performance metric is the fraction of wrong predictions in the membership matrix (again, up to a permutation of the community labels). We define the estimation error of  $\hat{Z}$  as  $\text{Error}(\hat{Z}, Z) = 1$  if  $\hat{K} \neq K$  and otherwise by

$$\text{Error}(\hat{Z}, Z) = \frac{1}{nK} \inf_{\sigma \in \mathfrak{S}_K} \|\hat{Z}P_\sigma - Z\|_F^2 \leq \frac{\text{MisC}(\hat{Z}, Z)}{n}.$$

## 2.3 Identifiability

The communities of a SBMO can only be recovered if the model is *identifiable* in that the equality  $Z'B'Z'^T = ZBZ^T$ , for some integer  $K'$  and matrices  $Z' \in \{0, 1\}^{n \times K'}$ ,  $B' \in [0, 1]^{K' \times K'}$ , implies  $\text{MisC}(Z', Z) = 0$  (and thus  $K' = K$ ): two SBMO with the same expected adjacency matrices have the same communities, up to a permutation of the community labels. In this section, we derive sufficient conditions for identifiability.

**Example 1.** Consider the following SBMO with  $n$  nodes and 3 overlapping communities:

$$B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad Z = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad (2)$$

where  $a, b, c > 0$  and  $\mathbf{1}$  (resp.  $\mathbf{0}$ ) is a vector of length  $n/3$  with all coordinates equal to 1 (resp. 0). This SBMO is not identifiable since  $ZBZ^T = Z'B'Z'^T$  with

$$B' = \begin{pmatrix} a+b & b & a \\ b & b+c & c \\ a & c & a+c \end{pmatrix}, \quad Z' = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Observe that this is a SBM with 3 non-overlapping communities.

In view of the above example, some additional assumptions are required to ensure identifiability. A first approach is to restrict the analysis to SBM. The following result is proved in Appendix A.

**Proposition 2.** *The SBMO is identifiable under the following assumptions:*

(SBM1) *for all  $\ell \neq k$ , the rows  $B_\ell$  and  $B_k$  are different;*

(SBM2) *for all  $i = 1, \dots, n$ ,  $\sum_{\ell=1}^K Z_{i,\ell} = 1$ .*

Assumption (SBM1) is the usual condition for identifiability of a SBM; the absence of overlap is enforced by assumption (SBM2). Note that the SBM of Example 1 clearly satisfies both assumptions and thus is identifiable: this is the only SBM with expected adjacency matrix  $A = ZBZ^T$ . One may wonder whether the SBMO is identifiable if we impose an overlap, that is the existence of some node  $i$  such that  $\sum_{\ell=1}^K Z_{i,\ell} \geq 2$ . The answer is negative, as shown by the following example.

**Example 1 (continued).** *Without loss of generality, we assume that  $c \leq \min(a, b)$ . Consider the following SBMO with  $n$  nodes and 4 overlapping communities:*

$$B'' = \begin{pmatrix} a+b-c & b-c & a-c & 0 \\ b-c & b & 0 & 0 \\ a-c & 0 & a & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, \quad Z'' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

We have  $ZBZ^T = Z''B''Z''^T$ .

Thus some additional assumptions are required to make the SBMO identifiable. It is in fact sufficient that the community connectivity matrix is invertible and that each community contains at least one pure node (that is, belonging to this community only). The following result is proved in Appendix A.

**Theorem 3.** *The SBMO is identifiable under the following assumptions:*

(SBMO1)  *$B$  is invertible;*

(SBMO2) *for each  $k = 1, \dots, K$ , there exists  $i$  such that  $Z_{i,k} = \sum_{\ell=1}^K Z_{i,\ell} = 1$ ,*

Observe that the two SBMO of Example 1, with membership matrices  $Z$  and  $Z''$ , violate (SBMO2). Only the SBM is identifiable. In particular, if we generate a SBMO with 3 overlapping communities based on the matrices  $B$  and  $Z$ , our algorithm will return at best 3 non-overlapping communities corresponding to the SBM with membership matrix  $Z'$ . To recover the model (2), some additional information is required on the community structure. For instance, one may impose  $K = 3$  and that each node belongs to exactly two communities. Note that this last condition alone is not sufficient, in view of the third model of Example 1. In the rest of the paper, we assume that the identifiability conditions (SBMO1) and (SBMO2) are satisfied.

## 2.4 Subcommunity detection

Any SBMO with  $K$  overlapping communities may be viewed as a SBM with up to  $2^K$  non-overlapping communities, corresponding to groups of nodes sharing exactly the same communities in the SBMO and that we refer to as *subcommunities*.

Let  $K'$  be the number of subcommunities in the SBMO:

$$K' = |\mathcal{S}|, \text{ where } \mathcal{S} = \{z \in \{0, 1\}^{1 \times K} : \exists i \in \{1, \dots, n\} : Z_i = z\}.$$

The corresponding SBM has  $K'$  communities indexed by  $z \in \mathcal{S}$ , with community connectivity matrix  $B'$  given by  $B'_{y,z} = yBz^T$ . The SBM of Example 1 can be derived from the first SBMO in this way for instance. More interestingly, it is easy to check that if the initial SBMO satisfies (SBMO1)-(SBMO2) then the corresponding SBM satisfies (SBM1)-(SBM2).

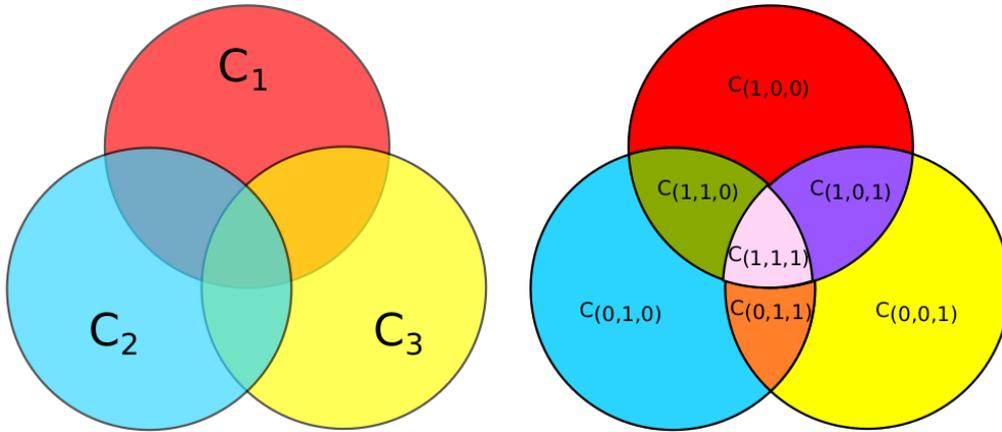


Figure 1: Three overlapping communities of a SBMO (left) and the subcommunities of the associated SBM (right).

This suggests that community detection in the SBMO reduces to community detection in the corresponding SBM, for which many efficient algorithms are known. However, the notion of performance for a SBM is different from the that for the underlying SBMO: the knowledge of the subcommunities is not sufficient to recover the initial overlapping communities, that is to obtain an estimate  $\hat{Z}$  such that  $\text{MisC}(\hat{Z}, Z)$  is small. It is indeed necessary to map these subcommunities to elements of  $\{0, 1\}^K$ , which is not an easy task: first, the number of communities  $K$  is unknown; second, assuming  $K$  is known, there are up to  $2^K!$  such mappings so that a simple approach by enumeration is not feasible in general. Moreover, the performance of clustering algorithms degrades rapidly with the number of communities so that it is preferable to work directly on the  $K$  overlapping communities rather than on the  $K'$  subcommunities, with  $K'$  possibly as large as  $2^K$ .

Our algorithm detects directly the  $K$  overlapping communities using the specific geometry of the eigenvectors of the expected adjacency matrix,  $A$ . We provide conditions under which these geometric properties hold for the observed adjacency matrix,  $\hat{A}$ , which guarantees the consistency of our algorithm: the  $K$  communities are recovered with probability tending to 1 in the limit of a large number of nodes  $n$ .

## 2.5 Scaling

To study the performance of our algorithm when the number of nodes  $n$  grows, we introduce a degree parameter  $\alpha_n$  so that the expected adjacency matrix of a graph with  $n$  nodes is in fact given by

$$A = \frac{\alpha_n}{n} Z B Z^T,$$

with  $B \in [0, 1]^{K \times K}$  independent of  $n$  and  $Z \in \{0, 1\}^{n \times K}$ . Although  $Z$  depends on  $n$ , we do not make it explicit in the notation. Observe that the expected degree of each node grows like  $\alpha_n$ , since

$$d_i = \alpha_n \left( \frac{1}{n} Z_i B Z^T \mathbf{1} \right),$$

where  $\mathbf{1}$  is the vector of one's of dimension  $n$ .

We assume that the set of subcommunities  $\mathcal{S}$  does not depend on  $n$  and that for all  $z \in \mathcal{S}$ , there exists a positive constant (independent of  $n$ )  $\beta_z$  such that:

$$\frac{|\{i : Z_i = z\}|}{n} \rightarrow \beta_z. \quad (3)$$

This implies the existence of positive constants  $L_z$  and of a matrix  $O \in \mathbb{R}^{K \times K}$ , such that

$$\forall z \in \mathcal{S}, \quad \frac{1}{n} z B Z^T \mathbf{1} \rightarrow L_z, \quad \text{and} \quad \frac{1}{n} Z^T Z \rightarrow O.$$

One has  $d_i \sim \alpha_n L_z$  for any  $i$  such that  $Z_i = z$ . In the sequel, we assume that the graph is sparse in the sense that  $\alpha_n \rightarrow \infty$  with  $\alpha_n/n \rightarrow 0$ . Observe also that  $O_{k,k}$  is the (limit) proportion of nodes that belong to community  $k$  while  $O_{k,l}$  is the (limit) proportion of nodes that belong to communities  $k$  and  $l$ , for any  $k \neq l$ . Hence we refer to  $O$  as the *overlap matrix*.

In the following, we will slightly abuse notation by writing  $O = \frac{1}{n} Z^T Z$  and  $d_i = \alpha_n L_z$  if  $Z_i = z$ , although these equalities in fact hold only in the limit.

## 3 Related work

**Models.** Several random graph models have been proposed in the literature to model graphs with overlapping communities. In these models, each node  $i$  is characterized by some community membership vector  $Z_i$  that is not always a binary vector, as in the SBMO.

In the Mixed-Membership Stochastic Block Model, introduced by [Airoldi et al., 2008] as the first model with overlaps, membership vectors are probability vectors drawn from a Dirichlet distribution. In this model, given  $Z_i$  and  $Z_j$ , the probability that nodes  $i$  and  $j$  are connected is  $Z_i B Z_j^T$  for some community connectivity matrix  $B$ , just like in the SBMO. However, as  $Z_i$  and  $Z_j$  are probability vectors, the probability of connection of two nodes does not necessarily increase with the number of communities that they have in common, as pointed out by [Yang and Leskovec, 2012].

A first model that relies on binary membership vectors was introduced by [Latouche et al., 2011] as the Overlapping Stochastic Block Model (OSBM), in which two nodes  $i, j$  are connected with probability  $\sigma(Z_i W Z_j^T + Z_i V + Z_j U + w)$ , where  $W \in \mathbb{R}_+^{K \times K}$ ,  $U, V \in \mathbb{R}_+^K$ ,  $w \in \mathbb{R}$ , and  $\sigma(x) = (1 + e^{-x})^{-1}$  is the sigmoid function. Now the probability of connectivity of two nodes increases with the number of communities they have in common but the particular form of the probability of connection makes the model hard to analysis.

Given some community connectivity matrix  $B$ , another natural way to build a random graph model that depends on binary membership vectors is to assume that nodes  $i$  and  $j$  are connected if any pair of communities  $k, l$  to which nodes  $i, j$  respectively belong can explain the connection. In other words,  $i$  and  $j$  are connected with probability  $1 - \prod_{k,l=1}^K (1 - B_{k,l})^{Z_{i,k}Z_{j,l}}$ . Denoting by  $Q$  the matrix with entries  $Q_{k,l} = -\log(1 - B_{k,l})$ , this probability can be written  $1 - \exp(-Z_i Q Z_j^T) \simeq Z_i Q Z_j^T$ , where the approximation is valid for sparse networks. In this case, the model is very close to the SBMO, with community connectivity matrix  $Q$ . The Community-Affiliation Graph Model (AGM) proposed by [Yang and Leskovec, 2012] is a particular case of this model in which  $B$  is diagonal. The SBMO with a diagonal connectivity matrix is also related to the ‘colored edges’ model introduced by [Ball et al., 2011]. In this model, a multi-graph is observed:  $\hat{A}_{i,j}$  is drawn from a Poisson distribution with mean  $\theta_i \theta_j^T$ , where  $\theta_i \in \mathbb{R}^{1,K}$  is the (non-binary) membership vector of node  $i$ . Letting  $\theta_i = \sqrt{B_{i,i}} Z_i$  and approximating the Poisson distribution by a Bernoulli distribution, we recover the SBMO.

[Zhang et al., 2014] recently proposed the Overlapping Continuous Community Assignment Model (OCCAM), that relies on overlapping communities and on individual degree parameters, which generalizes the degree-corrected stochastic block model [Karrer and Newman, 2011]. In the OCCAM, there is a degree parameter  $\theta_i$  associated to each node  $i$ . Letting  $\Theta = \text{Diag}(\theta_i) \in \mathbb{R}^{n \times n}$ , the expected adjacency matrix is  $A = \Theta Z B Z^T \Theta$ , with  $Z \in \mathbb{R}^{n \times K}$  a membership matrix. The identifiability of this model is proved under the assumptions that  $B$  is positive definite, that each row  $Z_i$  satisfies  $\|Z_i\| = 1$ , and that the degree parameters satisfy  $n^{-1} \sum_{i=1}^n \theta_i = 1$ . The SBMO can be viewed as a particular instance of the OCCAM, for which we provide new identifiability conditions when the membership vectors are binary.

**Algorithms.** Several algorithmic methods have been proposed in the literature to identify overlapping community structure in networks. [Xie et al., 2013] provides a review of some of them. Among the model-based methods, i.e. the algorithms that rely on the assumption that the observed network is drawn under some random graph model, some are approximation of the maximum likelihood or maximum a posteriori estimate of the membership vectors under one of the random graph models discussed above. For example, under the MMSB or the OSBM the membership vectors are assumed to be drawn from a probability (prior) distribution, and variational EM algorithms are proposed to approximate the posterior distributions [Airoldi et al., 2008, Latouche et al., 2011]. However, there is no proof of consistency of the proposed algorithms. The consistency of such variational algorithms has only been established for the SBM, with the extra assumption that the communities are drawn at random [Celisse et al., 2013]. In the MMSB, a different approach that uses tensor power iteration is proposed in [Anandkumar et al., 2014] to compute an estimator derived using the moments method, for which the first consistency results are provided.

The first occurrence of a spectral algorithm to find overlapping communities goes back to [Zhang et al., 2007]. The proposed method is an adaptation of spectral clustering with the normalized Laplacian (see e.g., [Newman, 2013]) with a fuzzy clustering algorithm in place of  $k$ -means, and its justification is rather heuristic. Another spectral algorithm has been recently proposed by [Zhang et al., 2014], as an estimation procedure for the (non-binary) membership matrix under the OCCAM. The spectral embedding is a row-normalized version of  $\hat{U} \hat{\Lambda}^{1/2} \in \mathbb{R}^{n \times K}$ , with  $\hat{\Lambda}$  the diagonal matrix containing  $K$  leading eigenvalues of  $\hat{A}$  and  $\hat{U}$  the matrix of associated eigenvectors. The centroids obtained by a  $k$ -median clustering algorithm are then used to estimate  $Z$ . This algorithm is proved to be consistent under the OCCAM, when moreover degree parameters and membership vectors are drawn according to some distributions. Similar assumptions have appeared before in the proof of consistency of some community detection algorithms in the SBM or DC-SBM (e.g., [Zhao et al., 2012]). Our consistency results are established for fixed parameters of the model.

## 4 Spectral analysis

In this section, we describe the spectral structure of the adjacency matrix in the SBMO.

### 4.1 Expected adjacency matrix

Let  $\mathcal{Z}$  be the set of membership matrices that contains at least one pure node per community:

$$\mathcal{Z} = \{Z \in \{0, 1\}^{n \times K}, \forall k = 1, \dots, K, \exists i \in \{1, \dots, n\}, Z_{i,k} = \sum_{\ell} Z_{i,\ell} = 1\}.$$

From the identifiability conditions (SBMO1) and (SBMO2),  $A = ZBZ^T$  is of rank  $K$  (refer to the proof of Theorem 3) and  $Z$  belongs to  $\mathcal{Z}$ . Let  $U \in \mathbb{R}^{n \times K}$  be a matrix whose columns  $u_1, \dots, u_K \in \mathbb{R}^n$  are normalized orthogonal eigenvectors associated to the  $K$  non-zero eigenvalues of  $A$ . The structure of  $U$  is described in the following proposition. Its first statement follows from the fact that the eigenvectors  $u_1, \dots, u_K$  form a basis of  $\text{Im}(A)$  and that  $\text{Im}(A) \subseteq \text{Im}(Z)$ . Its second statement follows from the proof of Theorem 3.

**Proposition 4.** 1. *There exists  $X \in \mathbb{R}^{K \times K}$  such that  $U = ZX$ .*

2. *If  $U = Z'X'$  for some  $Z' \in \mathcal{Z}$ ,  $X' \in \mathbb{R}^{K \times K}$ , then there exists  $\sigma \in \mathfrak{S}_K$  such that  $Z = Z'P_\sigma$ .*

Proposition 5, proved in Appendix A, gives a more precise description of the eigenvectors of  $A$  based on the overlap matrix  $O$  introduced in the previous section. Note that for any  $x \in \mathbb{R}^K$ , we have  $x^T O x = |Zx|^2/n$  so that  $O$  has the same rank as  $Z$ , equal to  $K$ . Hence  $O$  is invertible and positive definite, thus the matrix  $O^{1/2}$  (and its inverse) is well defined.

**Proposition 5.** *Let  $\mu \neq 0$ . The following statements are equivalent:*

1.  *$x$  is an eigenvector of  $BO$  associated to the eigenvalue  $\mu$ ;*
2.  *$O^{1/2}x$  is an eigenvector of  $O^{1/2}BO^{1/2}$  associated to the eigenvalue  $\mu$ ;*
3.  *$u = Zx$  is an eigenvector of  $A$  associated to the eigenvalue  $\alpha_n \mu$ .*

*In particular, the non-zero eigenvalues of  $A$  are of the same order as  $\alpha_n$ .*

The following corollary is particularly useful in the sequel.

**Proposition 6.** *There exists a matrix  $V \in \mathcal{O}_K(\mathbb{R})$  of eigenvectors of  $M_0 = O^{1/2}BO^{1/2}$  such that*

$$U = ZX \quad \text{with} \quad X = \frac{1}{\sqrt{n}} O^{-1/2} V.$$

### 4.2 Observed adjacency matrix

In practice, we observe the adjacency matrix  $\hat{A}$ , which can be viewed as some noisy version of  $A$ . Our hope is that the  $K$  leading eigenvectors of  $\hat{A}$  are not too far from the  $K$  leading eigenvectors of  $A$ , so that, in view of Proposition 4, the solution in  $Z'$  the following optimization problem provides a good estimate of  $Z$ :

$$\min_{Z' \in \mathcal{Z}, X' \in \mathbb{R}^{K \times K}} \|Z'X' - \hat{U}\|_F,$$

where  $\hat{U}$  is the matrix of the  $K$  normalized eigenvectors of  $\hat{A}$  associated to the  $K$  largest eigenvalues.

This hope is supported by the following result on the perturbation of the largest eigenvectors of the adjacency matrix of any random graph, proved in Appendix D. In practice, the number of communities  $K$  is unknown and this result also provides an adaptive procedure to select the eigenvectors to use in the spectral embedding. We denote by  $\lambda_{\min}(A)$  the smallest absolute value of a non-zero eigenvalue of  $A$ .

**Lemma 7.** *Let  $\delta \in ]0, 1[$  and  $\eta \in ]0, 1/2[$ . Let  $\hat{U}$  be a matrix formed by orthogonal eigenvectors of  $\hat{A}$  with an associated eigenvalue  $\lambda$  that satisfy*

$$|\lambda| \geq \sqrt{2(1+\eta)\hat{d}_{\max}\log(4n/\delta)}.$$

Let  $\hat{K}$  be the number of such eigenvectors. Let  $U$  be matrix of  $\hat{K}$  largest eigenvectors of  $A$ . If

$$d_{\max} \geq \frac{4(2\eta+3)(2+\eta)}{3\eta^2} \log\left(\frac{4n}{\delta}\right) \quad \text{and} \quad \frac{\lambda_{\min}(A)^2}{d_{\max}} > \sqrt{2\left(1+\frac{\eta}{2+\eta}\right)}(1+\sqrt{1+\eta}) \log\left(\frac{4n}{\delta}\right),$$

then with probability larger than  $1-\delta$ ,  $\hat{K} = \text{rank}(A)$  and there exists a matrix  $\hat{P} \in \mathcal{O}_n(\mathbb{R})$  such that

$$\|\hat{U} - U\hat{P}\|_F^2 \leq 32\left(1+\frac{\eta}{\eta+2}\right)\left(\frac{d_{\max}}{\lambda_{\min}(A)^2}\right)\log\left(\frac{4n}{\delta}\right).$$

This result requires that  $\lambda_{\min}(A)^2/(d_{\max}\log(n)) \rightarrow \infty$  to show that  $\hat{U}$  is a good estimate of  $U$ . In view of Proposition 5, we have  $\lambda_{\min}(A) = \Theta(\alpha_n)$ ; since  $d_{\max} = \Theta(\alpha_n)$ , we need  $\alpha_n/\log(n) \rightarrow +\infty$ . We show in the next section that this condition is sufficient to obtain asymptotically exact recovery of the communities.

## 5 Combinatorial spectral clustering

The spectral structure of the adjacency matrix suggests that  $\hat{Z}$  defined below is a good estimate of the membership matrix  $Z$  in the SBMO:

$$(\mathcal{P}) : (\hat{Z}, \hat{X}) \in \underset{Z' \in \mathcal{Z}, X' \in \mathbb{R}^{K \times K}}{\text{argmin}} \|Z'X' - \hat{U}\|_F^2, \quad (4)$$

where  $\hat{U} \in \mathbb{R}^{n \times K}$  is the matrix of the  $K$  normalized leading eigenvectors of  $\hat{A}$ . In practice, solving  $(\mathcal{P})$  is very hard, and the algorithm introduced in Section 5.1 solves a relaxation of  $(\mathcal{P})$  in which  $Z'$  is only constrained to have binary entries. In Section 5.2, we prove that an adaptive version of the estimate  $\hat{Z}$  given by (4) is consistent.

### 5.1 Combinatorial Spectral Clustering (CSC)

The *combinatorial spectral clustering* (CSC) algorithm consists in first computing a matrix  $\hat{U} \in \mathbb{R}^{n \times K}$  whose columns are normalized eigenvectors of  $\hat{A}$  associated to the  $K$  largest eigenvalues (in absolute value), and then computing the solution of the following optimization problem:

$$(\mathcal{P})' : (\hat{Z}, \hat{X}) \in \underset{\substack{Z' \in \{0,1\}^{n \times K} : \forall i, Z'_i \neq 0 \\ X' \in \mathbb{R}^{K \times K}}}{\text{argmin}} \|Z'X' - \hat{U}\|_F^2.$$

$(\mathcal{P})'$  is reminiscent of the (NP-hard)  $k$ -means problem, in which the same objective function is minimized under the additional constraint that  $\|Z_i\| = 1$  for all  $i$ . The name of the algorithm highlights the fact that compared to spectral clustering based on the adjacency matrix, the  $k$ -means objective is replaced by a different (more complex) combinatorial optimization problem, that permits to recover overlapping communities.

In practice, just like  $k$ -means, we propose to solve  $(\mathcal{P})'$  by an alternate minimization over  $Z'$  and  $X'$ . The proposed implementation of the adaptive version of the algorithm, inspired by Theorem 9, is presented as Algorithm 1. An upper bound  $O_{\max}$  on the maximum overlap  $\max\{\|z\|, z \in \mathcal{S}\}$  is provided to limit the combinatorial complexity of the algorithm. If  $K$  is known, the selection phase can be removed, and one use directly the matrix  $\hat{U} \in \mathbb{R}^{n \times K}$  of  $K$  leading eigenvectors.

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**Algorithm 1** Adaptive Combinatorial Spectral Clustering for Overlapping Community Detection

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**Require:** Parameters  $\epsilon, r, \eta > 0$ . Upper bound on the maximum overlap  $O_{\max}$ .

**Require:**  $\hat{A}$ , the adjacency matrix of the observed graph.

- 1:  $\#$  Selection of the eigenvectors
- 2: Form  $\hat{U}$  a matrix whose columns are  $\hat{K}$  eigenvectors of  $\hat{A}$  associated to eigenvalues  $\lambda$  satisfying

$$|\lambda| > \sqrt{2(1 + \eta)\hat{d}_{\max}(n) \log(4n^{1+r})}$$

- 3:  $\#$  Initialization
  - 4:  $\hat{Z} = 0 \in \mathbb{R}^{n \times \hat{K}}$
  - 5:  $\hat{X} \in \mathbb{R}^{\hat{K} \times \hat{K}}$  initialized with  $k$ -means++ applied to  $\hat{U}$ , the first centroid being chosen at random among nodes with degree smaller than the median degree
  - 6:  $Loss = +\infty$
  - 7:  $\#$  Alternating minimization
  - 8: **while**  $(Loss - \|\hat{Z}\hat{X} - \hat{U}\|_F^2 > \epsilon)$  **do**
  - 9:  $Loss = \|\hat{Z}\hat{X} - \hat{U}\|_F^2$
  - 10: Update membership vectors:  $\forall i, \hat{Z}_{i,\cdot} = \arg \min_{z \in \{0,1\}^{1 \times \hat{K}}: 1 \leq \|z\|_1 \leq O_{\max}} \|\hat{U}_{i,\cdot} - z\hat{X}\|$ .
  - 11: Update centroids:  $\hat{X} = (\hat{Z}^T \hat{Z})^{-1} \hat{Z}^T \hat{U}$ .
  - 12: **end while**
- 

Alternate minimization is guaranteed to converge, in a finite number of steps, towards a local minimum of  $\|\hat{Z}\hat{X} - \hat{U}\|_F^2$ . However, the convergence is very sensitive to initialization. We use a  $k$ -means++ initialization (see [Arthur and Vassilvitskii, 2007]), which is a randomized procedure that picks as initial centroids rows from  $\hat{U}$  that should be far from each other. For the first centroid, we choose at random a row in  $\hat{U}$  corresponding to a node whose degree is smaller than the median degree in the network. We do so because in the SBMO model, pure nodes tend to have smaller degrees and we expect the algorithm to work well if the initial centroids are chosen not too far from rows in  $\hat{U}$  corresponding to pure nodes.

Given  $\hat{Z}$ , as long as the matrix  $\hat{Z}^T \hat{Z}$  is invertible, there is a closed form solution to the minimization of  $\|\hat{Z}\hat{X} - \hat{U}\|_F$  in  $\hat{X}$ , which is  $\hat{X} = (\hat{Z}^T \hat{Z})^{-1} \hat{Z}^T \hat{U}$ . The fact that  $\hat{Z}^T \hat{Z}$  is not invertible implies in particular that  $\hat{Z}$  does not contain a pure node for each community. If this happens, we re-initialize the centroids, using again the  $k$ -means++ procedure.

## 5.2 Consistency of an adaptive estimator

We give in Theorem 9 theoretical properties for a slight variant of the estimate  $\hat{Z}$  in (4), that is solution of the optimization problem  $(\mathcal{P}_\epsilon)$  defined therein, that features the set of membership matrices for which the proportion of pure nodes in each community is larger than  $\epsilon$ :

$$\mathcal{Z}_\epsilon = \left\{ Z' \in \{0, 1\}^{n \times K}, \forall k \in \{1, \dots, K\}, \frac{|\{i : Z'_i = \mathbb{1}_{\{k\}}\}|}{n} > \epsilon \right\}.$$

Recall the notation introduced in (3). We assume that  $\epsilon$  is smaller than the smallest proportion of pure nodes (in the limit), given by  $\min_k \beta_{\mathbb{1}_{\{k\}}}$ , and let  $L_{\max} = \max_z L_z$ .

The estimator analyzed is adaptive, for it relies on an estimate of the number of communities. We prove the asymptotic consistency of this adaptive estimation procedure for any fixed community connectivity matrix  $B$  and membership matrix  $Z$  satisfying the assumptions of Theorem 3. Moreover, it is to be noted that while the consistency result of [Zhang et al., 2014] applies to moderately dense graphs ( $\alpha_n$  has to be of order  $n^\alpha$  for some  $\alpha > 0$ ), our result handle relatively sparse graphs, in which  $\alpha_n$  is of order  $(\log(n))^{1+c}$  for some  $c > 0$ . Our result involves some constants, defined below, than are related to the overlap matrix  $O$  and to the matrix  $O^{1/2}BO^{1/2}$  introduced in Proposition 5.

**Definition 8.** *The core matrix is the  $K \times K$  symmetric matrix  $M_0 := O^{1/2}BO^{1/2}$ . We let*

$$\begin{aligned} \mu_0 &:= \min\{|\lambda| : \lambda \neq 0 \text{ is an eigenvalue of } M_0\}, \\ d_0 &:= \min_{\substack{z \in \{-1, 0, 1, 2\}^{1 \times K} \\ z \neq 0}} \|zO^{-1/2}\|. \end{aligned}$$

Note that  $d_0$  is positive as seen by the following argument: if  $d_0 = 0$ , then there would exist a linear combination of the rows of  $O^{-1/2}$  which is zero; this is impossible because the matrix  $O^{-1/2}$  is invertible.

**Theorem 9.** *Let  $\eta \in ]0, 1/2[$  and  $r > 0$ . Let  $\hat{U}$  be a matrix whose columns are orthogonal eigenvectors of  $\hat{A}$  associated to an eigenvalue  $\hat{\lambda}$  satisfying*

$$|\hat{\lambda}| \geq \sqrt{2(1 + \eta) \hat{d}_{\max} \log(4n^{1+r})}.$$

*Let  $\hat{K}$  be the number of such eigenvectors. Let*

$$(\mathcal{P}_\epsilon) : (\hat{Z}, \hat{X}) \in \underset{Z' \in \mathcal{Z}_\epsilon, X' \in \mathbb{R}^{\hat{K} \times \hat{K}}}{\operatorname{argmin}} \|Z'X' - \hat{U}\|_F^2.$$

*Assume that  $\frac{\alpha_n}{\log n} \rightarrow \infty$  and  $\epsilon < \min_k \beta_{\mathbb{1}_{\{k\}}}$ . There exists some constant  $c_1 > 0$  such that, if*

$$\alpha_n \geq \max \left[ \frac{4(2\eta + 3)(2 + \eta)}{3\eta^2 L_{\max}}; \sqrt{2 \left(1 + \frac{\eta}{2 + \eta}\right) \frac{1 + \sqrt{1 + \eta}}{\mu_0^2}} \right] \log(4n^{1+r})$$

*then, for  $n$  large enough, with probability larger than  $1 - n^{-r}$ ,  $\hat{K} = K$  and*

$$\frac{\operatorname{MisC}(\hat{Z}, Z)}{n} \leq c_1 \frac{K^2 L_{\max} \log(4n^{1+r})}{d_0^2 \mu_0^2 \alpha_n}.$$

In particular, assuming that  $\alpha_n/\log(n) \rightarrow \infty$  when  $n$  goes to infinity, it can be shown (using the Borel-Cantelli Lemma) that the estimation procedure described in Theorem 9 with a parameter  $r \geq 2$ , satisfies

$$\frac{\text{MisC}(\hat{Z}, Z)}{n} \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

The proof of Theorem 9, given in Appendix B strongly relies on the decomposition of  $U$  given in Proposition 6. In particular, it leads to the following important result, that characterizes the sensitivity to noise of a decomposition  $U = ZX$ . This result, whose proof can be found in Appendix A.4, features the quantity  $d_0$  introduced in Definition 8.

**Proposition 10.** (*Robustness to noise*) Let  $Z' \in \mathbb{R}^{n \times K}$ ,  $X' \in \mathbb{R}^{K \times K}$  and  $\mathcal{N} \subset \{1, \dots, n\}$ . Assume that

1.  $\forall i \in \mathcal{N}, \|Z'_i X' - U_i\| \leq \frac{d_0}{4K\sqrt{n}}$
2. there exists  $(i_1, \dots, i_K) \in \mathcal{N}^K$  and  $(j_1, \dots, j_K) \in \mathcal{N}^K: \forall k \in [1, K], Z_{i_k} = Z'_{j_k} = \mathbf{1}_{\{k\}}$   
( $\mathcal{N}$  contains a pure node for each community relatively to  $Z$  and to  $Z'$ ).

Then there exists a permutation matrix  $P_\sigma$  such that for all  $i \in \mathcal{N}$ ,  $Z_i = (Z' P_\sigma)_i$ .

**Theoretical guarantees for other estimates.** Theorem 9 leads to an upper bound on the estimation error of  $\hat{Z}$  a solution to  $(\mathcal{P}_\epsilon)$ . In some cases, it is also possible to prove directly that the solution of  $(\mathcal{P})'$  leads to a consistent estimate of  $Z$ . This is the case for instance in an identifiable SBMO with two overlapping communities or with three communities with pairwise overlaps.

If  $K$  is known, tighter results can be obtained for non-adaptive procedures in which  $\hat{U} \in \mathbb{R}^{n \times \hat{K}}$  is replaced by  $\hat{U} \in \mathbb{R}^{n \times K}$ . These results are stated in Appendix C, where two non-adaptive estimation procedures are shown to be consistent under the (looser) condition  $\alpha \geq c_0 \log(n)$  for some constant  $c_0$  precised therein.

## 6 Experimental results

We mostly use the estimation error to evaluate the quality of an estimate  $\hat{Z}$  of some membership matrix  $Z$ , that we recall is defined by

$$\text{Error}(\hat{Z}, Z) = \frac{1}{nK} \min_{\sigma \in \mathfrak{S}_K} \|\hat{Z} P_\sigma - Z\|_F^2.$$

This error can be split into two kinds of errors: entries that are ones in  $\hat{Z} P_{\sigma^*}$  (where  $\sigma^*$  realizes the minimum above) but zeros in  $Z$ , called false positive, and entries that are zeros in  $\hat{Z} P_{\sigma^*}$  but ones in  $Z$ , called false negative. We define the false positive and false negative rates as

$$\text{FP}(\hat{Z}, Z) = \frac{|(i, k) : \hat{Z}_{i, \sigma^*(k)} = 1 \text{ and } Z_{i, k} = 0|}{|(i, k) : Z_{i, k} = 1|}, \quad \text{FN}(\hat{Z}, Z) = \frac{|(i, k) : \hat{Z}_{i, \sigma^*(k)} = 0 \text{ and } Z_{i, k} = 1|}{|(i, k) : Z_{i, k} = 0|}.$$

An extension of the *normalized variation of information* (NVI) introduced by [Lancichinetti et al., 2009] is also used as a measure of performance in several papers. This indicator compares the distribution

of two random vectors  $\mathbf{X} = (X_1, \dots, X_K)$  and  $\mathbf{Y} = (Y_1, \dots, Y_K)$  in  $\{0, 1\}^K$  associated to  $\hat{Z}$  and  $Z$  respectively, such that the joint distribution of any two marginal is given by

$$\mathbb{P}(X_k = x_k, Y_l = y_l) = \frac{|i : \hat{Z}_{i,k} = x_k \text{ and } Z_{i,l} = y_l|}{n}.$$

The NVI is defined by

$$\text{NVI}(\hat{Z}, Z) = 1 - \min_{\sigma \in \mathfrak{S}_K} \frac{1}{2K} \sum_{k=1}^K \left[ \frac{H(X_k | Y_{\sigma(k)})}{H(X_k)} + \frac{H(Y_{\sigma(k)} | X_k)}{H(Y_{\sigma(k)})} \right],$$

where  $H(V)$  and  $H(V|W)$  denote respectively the entropy of a random variable  $V$  and the conditional entropy of  $V$  given  $W$  (see e.g. [Cover and Thomas, 2006] for definition). Unlike the other performance measures that we consider, the NVI should be maximized.

Our analysis shows that for a graph drawn under the SBMO the error of Combinatorial Spectral Clustering goes to zero almost surely when the number of nodes  $n$  grows large, and the degrees are large enough, more precisely (slightly more than) logarithmic in  $n$ . We illustrate this fact on simulated data, and compare CSC to other (spectral) algorithms on simulated data and on two kinds of real-world graphs with overlapping communities : ego networks and co-authorship networks.

## 6.1 Simulated data

We compare CSC to (normalized) spectral clustering using the adjacency matrix, referred to as SC and to the spectral algorithm proposed by [Zhang et al., 2014] to fit the random graph model called OCCAM. We refer to this algorithm as the OCCAM spectral method.

First, we generate networks from SBMO models with  $n = 500$  nodes,  $K = 5$  communities,  $\alpha_n = \log^{1.5}(n)$ ,  $B = \text{Diag}([5, 4, 3, 3, 3])$  and  $Z$  drawn at random in such a way that each community has a fraction of pure nodes equal to  $p/K$  for some parameter  $p$  and the size of the maximum overlap  $O_{\max}$  is smaller than 3. The left part of Figure 2 shows the error of each method as a function of  $p$ , averaged over 100 networks. CSC significantly outperforms OCCAM, especially when there is a large overlap between communities. As expected, both methods outperform SC, which is designed to handle non-overlapping communities, except when the overlap gets really small.

To have a more fair comparison with the OCCAM spectral algorithm, we then draw networks under a modified version of the model used before, in which the rows of  $Z$  are normalized, so that for all  $i$ , one has  $\|Z_i\| = 1$ : this random graph model is a particular instance of the OCCAM. Results are displayed on the right part of Figure 2. The OCCAM spectral algorithm, designed to fit this model, performs most of the time slightly better than the other methods, but the gap between OCCAM and CSC is very narrow.

## 6.2 Real networks

[Zhang et al., 2014] compare the performance of the OCCAM spectral algorithm to that of other algorithms on both simulated data and real data, namely ego networks [Mc Auley and Leskovec, 2012]. Nodes in an ego network are the set of friends of a given central node in a social network, and edges indicate friendship relationship between these nodes. We first apply CSC on networks from this dataset, that naturally contain overlap. To do so, we use the pre-processing of the networks described in [Zhang et al., 2014], that especially keeps communities if they have at least a fraction of pure nodes equal to 10% of the network. Additionally, because the focus is on overlapping communities, we keep only networks

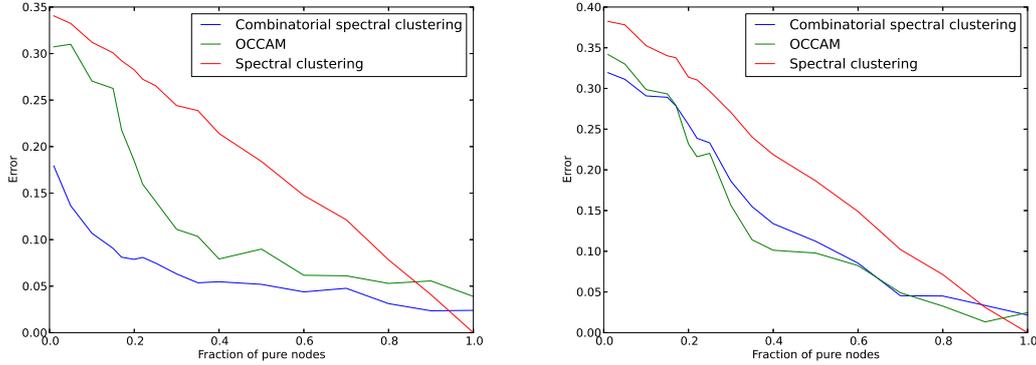


Figure 2: Comparison of SC, CSC and the OCCAM spectral algorithm under instances of SBMO (left) and OCCAM (right) random graph models.

for which the fraction of nodes that belong to more than one community is larger than 1%. This leads us to keep only 6 (out of 10) Facebook networks (labeled 0, 414, 686, 698, 1912 and 3437 in the dataset) and 26 (out of 133) Google Plus networks from the original dataset.

Table 1 presents the characteristics of the Facebook networks used, and the performance of SC, CSC and OCCAM, averaged over the 6 networks used (with the standard deviation added). For each algorithm, the estimation error is displayed but also the fraction of false positive (FP) and false negative (FN) entries in  $\hat{Z}$ , and the extended normalized variation of information (NVI). The parameter  $c$  corresponds to the average number of communities per node,  $c = \sum_{i,k} Z_{i,k}/n$  and  $O_{\max}$  is the maximum size of an overlap. OCCAM and CSC have comparable performance, but there is no significant improvement over spectral clustering. This can be explained by the fact that the amount of overlap ( $c$ ) is very small in this dataset. The same tendency was observed on the Google Plus networks.

	$n$	$K$	$c$	$O_{\max}$	FP	FN	Error	NVI
SC	190 (173)	3.17 (1.07)	1.09 (0.06)	2.17 (0.37)	0.200 (0.110)	0.139 (0.107)	0.120 (0.083)	0.556 (0.256)
OCCAM	190 (173)	3.17 (1.07)	1.09 (0.06)	2.17 (0.37)	0.176 (0.176)	0.113 (0.084)	0.127 (0.102)	0.556 (0.280)
CSC	190 (173)	3.17 (1.07)	1.09 (0.06)	2.17 (0.37)	0.125 (0.067)	0.101 (0.062)	0.102 (0.049)	0.544 (0.217)

Table 1: Spectral algorithms recovering overlapping friend circles in ego-networks from Facebook.

We then try CSC on co-authorship networks built from DBLP in the following way. Nodes correspond to authors and we fix as ground-truth communities some conferences (or group of conferences): an author belongs to some community if she/he has published at least one paper in the corresponding conference(s). We then build the network of authors by putting an edge between authors if they have published a paper together in one of the considered conferences. We present results with machine learning conferences : Neural Information Processing Systems (NIPS), International Conference on Machine Learning (ICML) and two more theory-oriented conferences that we group together, Conference On Learning Theory (COLT) and Algorithmic Learning Theory (ALT). We compare the three spectral al-

gorithms in terms of estimation error and false positive / false negative rates in two cases, one with two communities (in which the amount of overlap  $c$  is quite small) and one with three communities (in which the amount of overlap is a bit larger). Results are presented in Table 2, in which the estimated amount of overlap  $\hat{c} = \sum_{i,k} \hat{Z}_{i,k}/n$  is also reported. Although the error is relatively high, and the amount of overlap is clearly under-estimated, CSC significantly outperforms its two competitors. The difficulty of recovering communities in that case may come from the fact that the networks constructed are very sparse. Indeed, we propose in Appendix E preliminary experiments that illustrate the difficulty of recovering overlapping communities in very sparse networks.

$$\mathcal{C}_1 = \{NIPS\}, \mathcal{C}_2 = \{ICML\}, \mathcal{C}_3 = \{COLT, ALT\}$$

$$n = 9272, K = 3, d_{\text{mean}} = 4.5$$

	$c$	$\hat{c}$	FP	FN	Error
SC	1.22	1.	0.38	0.39	0.39
OCCAM	1.22	1.02	0.43	0.41	0.42
CSC	1.22	1.04	0.26	0.28	0.27

$$\mathcal{C}_1 = \{ICML\}, \mathcal{C}_2 = \{COLT, ALT\}.$$

$$n = 4374, K = 2, d_{\text{mean}} = 3.8$$

	$c$	$\hat{c}$	FP	FN	Error
SC	1.09	1.	0.39	0.55	0.46
OCCAM	1.09	1.01	0.29	0.44	0.36
CSC	1.09	1.03	0.21	0.31	0.25

Table 2: Spectral algorithms recovering overlapping machine learning conferences

## 7 Conclusion

Most existing algorithms for community detection assume non overlapping communities. Although they may in principle be used to detect all *subcommunities* generated by the various overlaps, this is not sufficient to recover the initial communities due to the combinatorial complexity of the corresponding mapping. We have proposed a spectral algorithm that works directly on the overlapping communities, using the specific geometry of the eigenvectors of the adjacency matrix under the SBMO. We have proved the consistency of this algorithm under the SBMO, provided each community has some positive fraction of pure nodes and the expected node degree is at least logarithmic, and tested its performance on both simulated and real data. This work has raised many interesting issues. First, it would be worth relaxing the assumption that each community has some positive fraction of pure nodes. Next, the experiments on simulated data have shown threshold phenomena in the very sparse regime that should be further explored (see Appendix E). Finally, the proof of consistency actually assumes that the underlying (NP-hard) optimization problem is solved exactly while this is not feasible in practice and heuristics need to be applied (like the proposed alternate optimization procedure). The impact of these heuristics on the performance of the algorithm is currently not well understood.

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## A Properties of the SBMO

### A.1 Identifiability: proof of Theorem 3

First note that  $A = ZBZ^T$  implies  $\text{rank}(A) \leq \text{rank}(B)$ . Now condition (SBMO2) means that the restriction of  $Z$  to its  $K$  first rows is equal to  $I_K$ , up to some reordering of the nodes. This gives  $\text{rank}(A) \geq \text{rank}(B)$ , and thus  $\text{rank}(A) = \text{rank}(B)$ . If  $B$  satisfies (SBMO1), then  $\text{rank}(A) = \text{rank}(B) = K$ : the parameter  $K$  is identifiable.

Now let  $Z, Z' \in \mathcal{Z}$  and  $B, B'$  invertible matrices such that  $A = ZBZ^T = Z'B'Z'^T$ . We show that there exists some permutation  $\sigma \in \mathfrak{S}_K$  such that  $Z = Z'P_\sigma$  and  $B = P_\sigma B' P_\sigma^T$ .

Let  $U$  be a matrix containing  $K$  independent normalized eigenvalues of  $A$  associated to non-zero eigenvalues. The columns of  $U$  form a basis of  $\text{Im}(A)$ . As  $\text{Im}(A) \subset \text{Im}(Z)$  and  $\text{Im}(A) \subset \text{Im}(Z')$ , there exist invertible matrices  $X, X'$  such that  $U = ZX = Z'X'$ . As for all  $k = 1, \dots, K$  there exists some  $i$  such that  $Z_{i,k} = \delta_{i,k}$ , the  $k$ -th row of  $X$  is a sum of rows in  $X'$ , namely

$$X_k = \sum_{l \in \mathcal{S}_k} X'_l,$$

where  $\mathcal{S}_k \subset \{1, \dots, K\}$ . Similarly, each row of  $X'$  is a sum of rows in  $X$ . In particular, for any  $k \neq l$ , there exist  $K$  integers  $a_1, \dots, a_K$  such that:

$$X_k + X_l = \sum_{m=1}^K a_m X_m.$$

If  $\mathcal{S}_k \cap \mathcal{S}_l \neq \emptyset$ , there exists some  $m$  such that  $a_m \geq 2$ . But this is in contradiction with the fact that  $X$  is invertible. Hence,  $\mathcal{S}_k \cap \mathcal{S}_l = \emptyset$  for all  $k \neq l$ . The only way for the  $\mathcal{S}_k$  to be pairwise disjoint is that there exists a permutation  $\sigma$  such that  $X' = P_\sigma X$ . Since  $ZX = Z'X'$  and  $X$  is invertible, this implies  $Z = Z'P_\sigma$ . We deduce that  $ZBZ^T = ZP_{\sigma^{-1}}B'P_{\sigma^{-1}}^T Z^T$  and  $B = P_{\sigma^{-1}}B'P_{\sigma^{-1}}^T$ , by the injectivity of  $Z$ .

### A.2 Identifiability for SBM: proof of Proposition 2

We simply prove that two nodes  $i, j$  are in the same community if and only if  $A_i = A_j$ . This implies the identifiability of the model: it is indeed sufficient to group nodes whose rows in  $A$  are identical. Let  $i, j$  be such that  $A_i = A_j$ . If  $Z_i \neq Z_j$  then  $BZ_i^T \neq BZ_j^T$  by assumption (SBM1) and  $A_i = ZBZ_i^T \neq ZBZ_j^T = A_j$  by assumption (SBM2), a contradiction. Conversely,  $Z_i = Z_j$  clearly implies  $A_i = A_j$ .

### A.3 Spectrum of the adjacency matrix

**Proof of Proposition 5.** As any non zero-eigenvector of  $A$  belongs to  $\text{Im}(A) \subseteq \text{Im}(Z)$ , if  $u$  is an eigenvector of  $A$  associated to  $\alpha_n \mu \neq 0$ , there exists  $x \in \mathbb{R}^K$  such that  $u = Zx$ . One has

$$\begin{aligned} \frac{\alpha_n}{n} ZBZ^T Zx &= \alpha_n \mu Zx \\ ZB \left( \frac{1}{n} Z Z^T \right) x &= \mu Zx \\ BOx &= \mu x. \end{aligned}$$

Hence  $x$  is an eigenvector of  $BO$  associated to  $\mu$ . This shows that 3.  $\Rightarrow$  1. As  $O$  is positive definite,  $O^{1/2}$  is well-defined. Let  $x$  be an eigenvector of  $BO$  associated to  $\mu$ . One has

$$\begin{aligned} BOx &= \mu x \\ OBOx &= \mu Ox \\ O^{-1/2}OBO^{1/2}O^{1/2}x &= \mu O^{1/2}x \\ O^{1/2}BO^{1/2}(O^{1/2}x) &= \mu(O^{1/2}x), \end{aligned}$$

which proves that 1.  $\Rightarrow$  2. Let  $O^{1/2}x$  be an eigenvector of  $O^{1/2}BO^{1/2}$  associated to  $\mu$ . One has

$$\begin{aligned} A(Zx) &= \frac{\alpha_n}{n} ZBZ^T Zx = \alpha_n ZBOx = \alpha_n ZO^{-1/2}O^{1/2}BO^{1/2}(O^{1/2}x) = \alpha_n ZO^{-1/2}(\mu O^{1/2}x) \\ &= \alpha_n \mu(Zx), \end{aligned}$$

thus  $Zx$  is an eigenvector of  $A$  associated to  $\alpha_n \mu$ , and 2.  $\Rightarrow$  3.

**Proof of Proposition 6.**  $\sqrt{n}U$  contains independent eigenvectors of  $A$  associated to non-zero eigenvalues. From the first statement in Proposition 5, there exists a matrix  $V$  of eigenvectors of  $M_0$  such that  $\sqrt{n}U = ZO^{-1/2}V$ . As  $U$  contains normalized eigenvectors,  $U^T U = I_K$ , which yields  $V^T V = I_K$  and  $V \in \mathcal{O}_K(\mathbb{R})$ .

#### A.4 Sensitivity to noise: proof of Proposition 10

Recall that from Proposition 6, there exists  $V \in \mathcal{O}_K(\mathbb{R})$  such that the matrix of leading eigenvectors  $U$  can be written

$$U = ZX \quad \text{with} \quad X = \frac{1}{\sqrt{n}} ZO^{-1/2}V.$$

Using that  $\|zX\| = \|zO^{-1/2}\|/\sqrt{n}$ , the following inequality is a consequence of the definition of  $d_0$  (Definition 8):

$$\forall z \in \{-1, 0, 1, 2\}^{1 \times K} \setminus \{0\}, \quad \|zX\| \geq \frac{d_0}{\sqrt{n}}. \quad (5)$$

The above decomposition of  $U$  and its consequence will be used as well in the proof of Theorem 9 in Appendix B.

Let  $i_1, \dots, i_K$  (resp.  $j_1, \dots, j_K$ ) be pure nodes in  $\mathcal{N}$  relatively to  $Z$  (resp.  $Z'$ ) that belong to communities  $1, \dots, K$ :  $Z_{i_k} = \mathbb{1}_{\{k\}}$  (resp.  $Z'_{j_k} = \mathbb{1}_{\{k\}}$ ). We first prove that  $i_1, \dots, i_K$  are also pure nodes relatively to  $Z'$ . For any  $k = 1, \dots, K$ ,  $Z'_{i_k}$  can be written as a sum of pure nodes: there exists a set  $\mathcal{S}_k \subset \{1, \dots, n\}$  such that

$$Z'_{i_k} = \sum_{m \in \mathcal{S}_k} Z'_{j_m}$$

Let  $k \neq l$ . As  $i_k$  and  $i_l$  belong to  $\mathcal{N}$ ,

$$\|(Z'_{i_k} + Z'_{i_l})X' - (Z_{i_k} + Z_{i_l})X\| \leq \|Z'_{i_k}X' - Z_{i_k}X\| + \|Z'_{i_l}X' - Z_{i_l}X\| \leq \frac{d_0}{2\sqrt{n}}$$

and

$$\begin{aligned}
\|(Z'_{i_k} + Z'_{i_l})X' - (\sum_{m \in \mathcal{S}_k} Z_{j_m} + \sum_{m \in \mathcal{S}_l} Z_{j_m})X\| &= \|(\sum_{m \in \mathcal{S}_k} Z'_{j_m} + \sum_{m \in \mathcal{S}_l} Z'_{j_m})X' - (\sum_{m \in \mathcal{S}_k} Z_{j_m} + \sum_{m \in \mathcal{S}_l} Z_{j_m})X\| \\
&\leq \sum_{m \in \mathcal{S}_k} \|Z'_{j_m}X' - Z_{j_m}X\| + \sum_{m \in \mathcal{S}_l} \|Z'_{j_m}X' - Z_{j_m}X\| \\
&\leq \frac{d_0}{2\sqrt{n}}.
\end{aligned}$$

This proves that

$$\|(\sum_{m \in \mathcal{S}_k} Z_{j_m} + \sum_{m \in \mathcal{S}_l} Z_{j_m})X - (Z_{i_k} + Z_{i_l})X\| \leq d_0/\sqrt{n}.$$

If  $\mathcal{S}_k \cap \mathcal{S}_l \neq \emptyset$ , there exists  $z \in \{0, 1, 2, -1\} \setminus \{0\}$  such that  $\|zX\| \leq d_0/\sqrt{n}$ , which contradicts (5). Thus  $\mathcal{S}_k \cap \mathcal{S}_l = \emptyset$ . Hence, the support of the  $Z'_{i_k}$  are all disjoint, thus they must be distinct pure nodes. There exists a permutation  $\sigma \in \mathfrak{S}_K$  such that

$$\forall k = 1, \dots, K, \quad Z_{i_k} = \mathbb{1}_{\{k\}} \quad \text{and} \quad Z'_{i_k} = \mathbb{1}_{\{\sigma(k)\}}.$$

To conclude the proof, we show that for  $\sigma$  the permutation defined above, it holds that

$$\forall i \in \mathcal{N}, \forall k \in \{1, K\}, \quad Z'_{i, \sigma(k)} = Z_{i, k}.$$

Let  $i \in \mathcal{N}$ . There exists a set  $\mathcal{S} \subset \{1, \dots, n\}$  such that  $Z_i = \sum_{k \in \mathcal{S}} \mathbb{1}_{\{k\}}$ . It is sufficient to prove that  $Z'_i = \sum_{k \in \mathcal{S}} \mathbb{1}_{\{\sigma(k)\}}$ . To do so, we first introduce  $\mathcal{C} = \{0, 1\}^{1 \times K} \setminus \{0\}$  and the following important mapping:

$$\begin{aligned}
\Phi: \mathcal{C} &\longrightarrow \mathcal{C} \\
z &\longmapsto y: zX' \in \mathcal{R}_y,
\end{aligned}$$

where  $\mathbb{R}^K$  is partitioned into the following  $2^K - 1$  regions indexed by  $y \in \mathcal{C}$ ,

$$\mathcal{R}_y = \{x \in \mathbb{R}^{1 \times K} : \|x - yX\| < \|x - y'X\| \text{ for all } y' \in \mathcal{C}, y' \neq y\}.$$

The following lemma gathers useful properties of the mapping  $\Phi$ . Its proof is given below.

**Lemma 11.**  $\Phi$  is a one-to-one mapping satisfying  $\|zX' - yX\| \leq \frac{d_0}{2\sqrt{n}} \Rightarrow \Phi(z) = y$ .

As  $i \in \mathcal{N}$ , from assumption 1.,

$$\|Z'_i X' - Z_i X\| \leq \frac{d_0}{2\sqrt{n}}.$$

Moreover, using that  $Z_i = \sum_{k \in \mathcal{S}} \mathbb{1}_{\{k\}} = \sum_{k \in \mathcal{S}} Z_{i_k}$  and  $\sum_{k \in \mathcal{S}} \mathbb{1}_{\{\sigma(k)\}} = \sum_{k \in \mathcal{S}} Z'_{i_k}$ , one has

$$\begin{aligned}
\left\| \left( \sum_{k \in \mathcal{S}} \mathbb{1}_{\{\sigma(k)\}} \right) X' - Z_i X \right\| &= \left\| \left( \sum_{k \in \mathcal{S}} Z'_{i_k} \right) X' - \left( \sum_{k \in \mathcal{S}} Z_{i_k} \right) X \right\| \\
&\leq \sum_{k \in \mathcal{S}} \|Z'_{i_k} X' - Z_{i_k} X\| \leq \frac{d_0}{2\sqrt{n}}.
\end{aligned}$$

Using Lemma 11, the last two inequalities yield  $\Phi(Z'_i) = Z_i$  and  $\Phi(\sum_{k \in \mathcal{S}} \mathbb{1}_{\{\sigma(k)\}}) = Z_i$  respectively. Using that  $\Phi$  is one-to-one (again from Lemma 11) concludes the proof:

$$Z'_i = \sum_{k \in \mathcal{S}} \mathbb{1}_{\{\sigma(k)\}}.$$

**Proof of Lemma 11.** Let  $z, y \in \mathcal{C}$  be such that  $\|zX' - yX\| \leq \frac{d_0}{2\sqrt{n}}$ . Let  $y' \in \mathcal{C} : y' \neq y$ . Using (5),

$$\|zX' - y'X\| > \|zX - y'X\| - \|zX' - yX\| \geq \frac{d_0}{\sqrt{n}} - \frac{d_0}{2\sqrt{n}} > \frac{d_0}{2\sqrt{n}} \geq \|zX' - yX\|.$$

Hence,  $zX' \in \mathcal{R}_y$  and  $\Phi(z) = y$ , which proves the second part of the result.

We now prove that  $\Phi$  is one-to-one. Let  $y \in \mathcal{C}$ : there exists a set  $\mathcal{S} \subseteq \{1, \dots, K\}$  such that  $y = \sum_{m \in \mathcal{S}} \mathbb{1}_{\{k\}} = \sum_{m \in \mathcal{S}} Z_{i_m}$ . Let  $z = \sum_{m \in \mathcal{S}} Z'_{i_m}$ . As the  $Z'_{i_m}$  are disjoint indicators, one has  $z \in \mathcal{C}$ . Moreover,

$$\|zX' - yX\| \leq \sum_{m \in \mathcal{S}} \|Z'_{i_m}X' - Z_{i_m}X\| \leq \frac{d_0}{2\sqrt{n}}.$$

From what we've just proved, this implies  $\Phi(z) = y$ . As  $\mathcal{C}$  is finite and  $\forall y \in \mathcal{C}, \exists z \in \mathcal{C} : \Phi(z) = y$ ,  $\Phi$  is one-to-one.

## B Proof of Theorem 9

Let  $U \in \mathbb{R}^{n \times K}$  be a matrix whose columns are  $K$  independent normalized eigenvectors of  $A$  associated to the non-zero eigenvalues. From Proposition 6, there exists  $V \in \mathcal{O}_K(\mathbb{R})$  such that

$$U = ZX \quad \text{with} \quad X = \frac{1}{\sqrt{n}}ZO^{-1/2}V.$$

We use again (5), which follows from the definition of  $d_0$ :

$$\forall z \in \{-1, 0, 1, 2\}^{1 \times K} \setminus \{0\}, \quad \|zX\| \geq \frac{d_0}{\sqrt{n}}.$$

Let  $\hat{Z}$  the estimate defined by

$$(\hat{Z}, \hat{X}) \in \underset{Z' \in \mathcal{Z}_\epsilon, X' \in \mathbb{R}^{K \times K}}{\operatorname{argmin}} \|Z'X' - \hat{U}\|_F^2.$$

Let  $\hat{P} \in \mathcal{O}_K(\mathbb{R})$  be a fixed rotation (that may depend on the observed graph), and let  $\hat{X}_1 = \hat{X}\hat{P}^{-1}$ . The following inequality shows that  $\hat{Z}\hat{X}_1$  is a good estimate of  $U$ , provided that  $\hat{U}$  is:

$$\|\hat{Z}\hat{X}_1 - U\|_F \leq 2\|U\hat{P} - \hat{U}\|_F. \quad (6)$$

It can be obtained in the following way. First, by definition of  $\hat{Z}$  and  $\hat{X}$  and the fact that  $Z \in \mathcal{Z}_\epsilon$  (since  $\epsilon < \max_k \beta_{\mathbb{1}_{\{k\}}}$ ),

$$\|\hat{Z}\hat{X} - \hat{U}\|_F^2 \leq \|ZX\hat{P} - \hat{U}\|_F^2 = \|U\hat{P} - \hat{U}\|_F^2.$$

Then, one has

$$\begin{aligned} \|\hat{Z}\hat{X}\hat{P}^{-1} - U\|_F &\leq \|\hat{Z}\hat{X}\hat{P}^{-1} - \hat{U}\hat{P}^{-1}\|_F + \|\hat{U}\hat{P}^{-1} - U\|_F = \|\hat{Z}\hat{X} - \hat{U}\|_F + \|\hat{U} - U\hat{P}\|_F \\ &\leq 2\|\hat{U} - U\hat{P}\|_F. \end{aligned}$$

We now upper bound the number of misclassified nodes using Proposition 10: we introduce a set of nodes  $\mathcal{N}_n$  for which assumption 1. and 2. are satisfied for the pair  $(\hat{Z}, \hat{X}_1)$ , and give an upper bound on the cardinality of this set, that leads to an upper bound on  $\operatorname{MisC}(\hat{Z}, Z)$ .

**Step 1: Introducing a set of well-estimated nodes.** We introduce the set

$$\mathcal{N}_n = \left\{ i : \|\hat{Z}_i \hat{X}_1 - U_i\| \leq \frac{d_0}{4K\sqrt{n}} \right\}.$$

By definition of  $\mathcal{N}_n$ , assumption 1. in Proposition 10 is satisfied for the pair  $(\hat{Z}, \hat{X}_1)$ . Moreover, by (6),

$$\begin{aligned} |\mathcal{N}_n^c| &= \sum_{i \in \mathcal{N}_n^c} 1 \leq \frac{16K^2 n}{d_0^2} \sum_{i=1}^n \|\hat{Z}_i \hat{X}_1 - U_i\|^2 = \frac{16K^2 n}{d_0^2} \|\hat{Z} \hat{X}_1 - U\|_F^2 \\ \frac{|\mathcal{N}_n^c|}{n} &\leq \frac{64K^2}{d_0^2} \|\hat{U} - U \hat{P}\|_F^2. \end{aligned} \quad (7)$$

**Step 2:  $\mathcal{N}_n$  contains pure nodes** Under the assumption

$$\frac{64K^2}{d_0^2} \|\hat{U} - U \hat{P}\|_F^2 \leq \epsilon, \quad (8)$$

$|\mathcal{N}_n^c| \leq \epsilon n$ . As  $\hat{Z} \in \mathcal{Z}_\epsilon$ , for all  $k$ , the cardinality of the set of nodes  $i$  such that  $\hat{Z}_i = \mathbb{1}_{\{k\}}$  is strictly larger than  $\epsilon n$ , hence this set cannot be included in  $\mathcal{N}_n^c$ . Thus, for all  $k \in \{1, \dots, K\}$ , there exists  $j_k \in \mathcal{N}_n$  such that  $\hat{Z}_{j_k} = \mathbb{1}_{\{k\}}$ .  $\min_k \beta_{\mathbb{1}_{\{k\}}}$  is the minimal proportion of pure nodes in a community. As  $\epsilon$  is smaller than this quantity, by a similar argument the set of nodes  $i$  such that  $Z_i = \mathbb{1}_{\{k\}}$  cannot be included in  $\mathcal{N}_n$  either. Thus for all  $k \in \{1, \dots, K\}$ , there exists  $i_k \in \mathcal{N}_n$  such that  $Z_{i_k} = \mathbb{1}_{\{k\}}$ .

**Step 3: upper bound on  $|\mathcal{N}_n^c|$  and conclusion.** So far, we proved that for any  $\hat{P} \in \mathcal{O}_K(\mathbb{R})$ , if (8) holds, the pair  $(\hat{Z}, \hat{X} \hat{P}^{-1})$  satisfies assumption 1. and 2. in Proposition 10. Hence, there exists  $\sigma \in \mathfrak{S}_K$  such that  $\forall i \in \mathcal{N}_n$ ,  $\hat{Z}_{i, \sigma(k)} = Z_{i, k}$ : up to a permutation of the community labels, all the communities of nodes in  $\mathcal{N}_n$  are recovered. This implies that, if  $\hat{K} = K$ ,

$$\frac{\text{MisC}(\hat{Z}, Z)}{n} \leq \frac{|\mathcal{N}_n^c|}{n} \leq \frac{64K^2}{d_0^2} \|\hat{U} - U \hat{P}\|_F^2.$$

To conclude, we use Lemma 7 that provides a high-probability upper bound on  $\|\hat{U} - U \hat{P}\|_F^2$  for some rotation  $\hat{P}$ . The conditions on  $d_{\max}$  and  $\lambda_{\min}(A)$  in this result can be expressed in terms of the degree parameter  $\alpha_n$ . Indeed, from Proposition 5, one has  $\lambda_{\min}(A) = \alpha_n \mu_0$ , with  $\mu_0$  in Definition 8. One also has  $d_{\max} = \alpha_n L_{\max}$ , with

$$L_{\max} = \max_{i=1 \dots n} \left( \frac{1}{n} Z_i B Z^T \mathbb{1}_{n,1} \right).$$

From Lemma 7, choosing  $\delta = n^{-r}$ , under the condition

$$\alpha_n \geq \max \left[ \frac{4(2\eta + 3)(2 + \eta)}{3\eta^2 L_{\max}}; \sqrt{2 \left( 1 + \frac{\eta}{2 + \eta} \right) \frac{1 + \sqrt{1 + \eta}}{\mu_0^2}} \right] \log(4n^{1+r}),$$

with probability larger than  $1 - n^{-r}$ , one has  $\hat{K} = K$  and

$$\frac{\text{MisC}(\hat{Z}, Z)}{n} \leq \frac{2048K^2 L_{\max}}{d_0^2 \mu_0^2} \left( 1 + \frac{\eta}{\eta + 2} \right) \frac{\log(4n^{1+r})}{\alpha_n},$$

if the right-hand side is smaller than  $\epsilon$ , which is the case for  $n$  large enough. This yields Theorem 9.

## C Results for non-adaptive procedures

We present here tighter upper bounds on the fraction of nodes that are misclassified by some non-adaptive estimation procedures, based on  $\hat{U} \in \mathbb{R}^{n \times K}$  rather than on  $\hat{U} \in \mathbb{R}^{n \times \hat{K}}$  (with  $\hat{K}$  given in Theorem 9). In this case, it is possible to analyze the solution of  $(\mathcal{P}_\epsilon)$ , defined in Section 5.2, as well as the solution of the following optimization problem:

$$(\mathcal{P}_S) : \min_{\substack{Z' \in \{0,1\}^{n,K} : \forall i, Z'_i \in \mathcal{S} \\ X' \in \mathbb{R}^{K \times K}}} \|Z' X' - \hat{U}\|_F^2.$$

$(\mathcal{P}_S)$  relies on the knowledge of  $\mathcal{S}$ , the set of subcommunities that are present in the network. If one has this knowledge, note that the above estimate can be computed using alternate minimization, just like the solution of  $(\mathcal{P})'$ . Theorem 12 below gathers the theoretical guarantees obtained for these two estimators. Compared to Theorem 9, a logarithmic factor is removed in the upper bound on the number of misclassified nodes: both estimates are consistent provided that

$$\alpha_n \geq \max \left[ \frac{1}{L_{\max}} \log(n); \frac{C_r}{\mu_0^2} \right].$$

**Theorem 12.** *Let  $\hat{U}$  be a matrix formed by  $K$  independent eigenvectors associated to the eigenvalues of  $\hat{A}$  that are largest in absolute value. Let  $(\hat{Z}, \hat{C})$  be the solution of  $(\mathcal{P}_\epsilon)$  or of  $(\mathcal{P}_S)$ . There exists a constant  $C_2 > 0$  such that for all  $r > 0$ , there exists a constant  $C_r$  such that if*

$$\alpha_n \geq \max \left[ \frac{1}{L_{\max}} \log(n); \frac{C_r}{\mu_0^2} \right],$$

*then, for  $n$  large enough, with probability larger than  $1 - n^{-r}$ ,*

$$\frac{\text{MisC}(\hat{Z}, Z)}{n} \leq C_2 \frac{C_r^2 K^2 L_{\max}}{d_0^2 \mu_0^2} \frac{1}{\alpha_n}.$$

The proof of Theorem 12 is very similar to that of Theorem 9 given in the previous section. The main difference is that in the non-adaptive case it is possible to use a tighter eigenvectors perturbation result (specific to SBMO), that we state below as Lemma 13. Compared to Lemma 7, in Lemma 13 an extra logarithmic factor is removed, but at the price of non-explicit constants, that do not permit to propose an adaptive version of the result. The proof of both Lemma 7 and Lemma 13 are given in the next section.

**Lemma 13.** *Let  $\hat{A}$  be drawn under a SBMO model with expected adjacency matrix  $A$ . Let  $K$  be the rank of  $A$ . Let  $U$  (resp.  $\hat{U}$ ) be a matrix whose columns are  $K$  independent eigenvectors associated to the  $K$  eigenvalues of  $A$  (resp.  $\hat{A}$ ) with largest absolute values.*

*For all  $r > 0$ , there exists a constant  $C_r$  such that under the conditions*

$$\frac{\lambda_{\min}(A)^2}{d_{\max}} > C_r \quad \text{and} \quad d_{\max} \geq \log(n),$$

*with probability larger than  $1 - n^{-r}$ , there exists a matrix  $\hat{P} \in \mathcal{O}_n(\mathbb{R})$  such that*

$$\|\hat{U} - U \hat{P}\|_F^2 \leq 4C_r^2 \left( \frac{d_{\max}}{\lambda_{\min}(A)^2} \right).$$

Also, compared to that of  $(\mathcal{P}_\epsilon)$ , the analysis of the solution of  $(\mathcal{P}_S)$  requires a more complex argument to prove that the set  $\mathcal{N}_n$  defined in the proof of Theorem 9 contains one pure nodes per community in  $Z$  and  $\hat{Z}$ . We present below the argument that replace Step 2 in the previous section in that case.

**Step 2:  $\mathcal{N}_n$  contains pure nodes** Under the assumption

$$\frac{64K^2}{d_0^2} \|\hat{U} - U\hat{P}\|_F^2 \leq \min_{z \in \mathcal{S}} \beta_z,$$

$|\mathcal{N}_n^c|/n \leq \beta_z$  for each possible membership vector  $z \in \mathcal{S}$ . Thus, for all  $z \in \mathcal{S}$ , the set of nodes  $i$  such that  $Z_i = z$  cannot be included in  $\mathcal{N}_n^c$  and there exists  $i_z \in \mathcal{N}_n$  such that  $Z_{i_z} = z$ . In particular,  $\mathcal{N}_n$  contains pure nodes relatively to  $Z$ . Now we need to prove that it also contains pure nodes relatively to  $\hat{Z}$ .

To do so, we introduce the following mapping and prove it is one-to-one:

$$\begin{aligned} \Psi : \mathcal{S} &\longrightarrow \mathcal{S} \\ z &\longmapsto y : z\hat{X} \in \mathcal{R}_y, \end{aligned}$$

where  $\mathbb{R}^K$  is partitioned into  $\tilde{K} = |\mathcal{S}|$  regions, indexed by  $y \in \mathcal{S}$ ,

$$\mathcal{R}_y = \{x \in \mathbb{R}^K : \|x - yX\| < \|x - y'X\| \text{ for all } y' \in \mathcal{S} : y' \neq y\}.$$

For all  $i \in \mathcal{N}_n$ ,  $\Psi(\hat{Z}_i) = Z_i$ . Indeed,  $\hat{Z}_i\hat{X} \in \tilde{\mathcal{R}}_{Z_i}$  for if  $y' \in \mathcal{S}$  is such that  $y' \neq Z_i$ , using (5) and the fact that  $i$  belongs to  $\mathcal{N}_n$  yields

$$\|\hat{Z}_i\hat{X} - y'X\| > \|Z_iX - y'X\| - \|\hat{Z}_i\hat{X} - Z_iX\| \geq \frac{d_0}{\sqrt{n}} - \frac{d_0}{2\sqrt{n}} > \frac{d_0}{2\sqrt{n}} \geq \|\hat{Z}_i\hat{X} - Z_iX\|.$$

It follows that for all  $y \in \mathcal{S}$ , there exists  $z \in \mathcal{S}$  such that  $y = \Psi(z)$ . Indeed, there exists  $i_y \in \mathcal{N}_n$  such that  $Z_{i_y} = y$ , thus  $\Psi(\hat{Z}_{i_y}) = y$  and  $z = \hat{Z}_{i_y}$  belongs to  $\mathcal{S}$  by definition of the optimization problem that  $\hat{Z}$  solves. As  $\mathcal{S}$  is a finite set,  $\Psi$  is one-to-one. Thus, one has

$$\{\hat{Z}_{i_z} : z \in \mathcal{S}\} = \Psi^{-1}(\{Z_{i_z} : z \in \mathcal{S}\}) = \Psi^{-1}(\mathcal{S}) = \mathcal{S}.$$

In particular, there exists  $i_1, \dots, i_K$  (resp.  $j_1, \dots, j_K$ ) such that  $\forall k \in \{1, \dots, K\}$ ,  $Z_{i_k} = \hat{Z}_{j_k} = \mathbf{1}_{\{k\}}$ .

## D Proof of the eigenvectors perturbation results

Lemma 7 and Lemma 13 rely on two main ingredients, described below: a high-probability bound on the spectral norm of  $\hat{A} - A$  (i.e. a concentration result), and results from linear algebra, mostly the Davis-Kahan theorem.

### D.1 Some technical tools

We state here the matrix concentration result that is used to prove Lemma 7, which is of interest in its own. This result is not specific to the DC-SBM, it holds for any random graph model. It follows from a Bernstein inequality for sum of independent matrices, and its proof is given in Section F.1.

**Theorem 14.** *Let  $\delta \in ]0, 1[$ . Let  $\epsilon > 0$  be fixed. If*

$$d_{\max} \geq \frac{2}{9} \frac{1 + \epsilon}{\epsilon^2} \log \frac{2n}{\delta},$$

one has

$$\mathbb{P} \left( \|\hat{A} - A\| > \sqrt{2(1 + \epsilon)d_{\max} \log \left( \frac{2n}{\delta} \right)} \right) \leq \delta.$$

Another concentration result, given below, is used to prove Lemma 13. This result, recently obtained by [Lei and Rinaldo, 2015] improves the dependency in  $n$  in the high-probability upper bound on  $\|\hat{A} - A\|$ , since a logarithmic term is removed compared to Theorem 14. However, the constants in the upper bound are non-explicit.

**Theorem 15.** [Theorem 5.2 of [Lei and Rinaldo, 2015]] *In a random graph model, if  $d$  is such that  $d \geq n \max_{i,j} A_{i,j}$  and  $d \geq c_0 \log(n)$ , for every  $r > 0$  there exists a constant  $C = C(r, c_0)$  such that*

$$\mathbb{P}\left(\|\hat{A} - A\| > C\sqrt{d}\right) \leq \delta.$$

In the proof of Lemma 7, another concentration result is needed to control the deviations of the empirical degrees from the mean degrees. The following result follows from Bernstein inequalities (for independent random variables) and is proved in Appendix F.2.

**Lemma 16.** *Let  $\alpha \in ]0, 1[$ .*

$$\begin{aligned} \mathbb{P}\left(\hat{d}_{\max} \leq (1 + \alpha)d_{\max}\right) &\geq 1 - ne^{-d_{\max} \frac{\alpha^2}{2(1+\alpha/3)}} \\ \mathbb{P}\left(\hat{d}_{\max} \geq (1 - \alpha)d_{\max}\right) &\geq 1 - e^{-d_{\max} \frac{\alpha^2}{2(1+\alpha/3)}} \end{aligned}$$

We state here two useful results from linear algebra, that relate the eigenvalues and eigenvectors of two matrices  $A$  and  $B$  to the difference in spectral norm between the two matrices.

**Lemma 17** (Weyl's inequalities). *Let  $\lambda_1(A) \geq \dots \geq \lambda_n(A)$  denote the ordered eigenvalues of a symmetric matrix  $A$  of size  $n$ . For any two symmetric matrices  $A$  et  $B$  of size  $n$ ,*

$$\text{for all } i = 1, \dots, n \quad |\lambda_i(A) - \lambda_i(B)| \leq \|A - B\|.$$

**Theorem 18** (Davis-Kahan theorem). *Let  $A$  and  $B$  be two symmetric matrices of size  $n$ . Let  $I \subset \mathbb{R}$  be an interval that contains exactly  $k$  eigenvalues of  $A$  and  $B$ . Let  $X_A$  (resp.  $X_B$ ) be a matrix in  $\mathbb{R}^{n \times k}$  whose columns are  $k$  independent normalized eigenvectors associated to the eigenvalues of  $A$  (resp.  $B$ ) in  $I$ .*

*Then there exists a rotation  $P \in \mathcal{O}_k(\mathbb{R})$  such that, with  $\delta := \inf\{|\lambda - s|, \lambda \in \text{sp}(B), \lambda \notin I, s \in I\}$ ,*

$$\|X_A - X_B P\|_F \leq \frac{\sqrt{2}}{\delta} \|A - B\|.$$

The usual statement of the Davis-Kahan theorem involves principal angles between the column spaces of  $X_A$  and  $X_B$ , but the above formulation can be easily obtained from Proposition B.1 of [Rohe et al., 2011] and the following explanation therein relating the principal angles to the Frobenius norm of  $X_A - X_B P$  for some rotation  $P$ .

## D.2 Proof of Lemma 7 and Lemma 13

Let  $\lambda_k(A)$  be the eigenvalues of  $A$ , and  $\lambda_k(\hat{A})$  be the eigenvalues of  $\hat{A}$ , sorted in non-increasing order. Let  $s$  (resp.  $r$ ) be the number of eigenvalues of  $A$  that are strictly positive (resp. negative), so that  $K = s + r$  (where  $K$  is the rank of  $A$ ). Using Weyl's inequalities (Lemma 17), one can write

$$\begin{aligned} \text{for } k = 1, \dots, s, \quad \lambda_k(\hat{A}) &\geq \lambda_k(A) - \|\hat{A} - A\|, \\ \text{for } k = s + 1, \dots, n - r, \quad |\lambda_k(\hat{A})| &\leq \|\hat{A} - A\|, \\ \text{for } k = n - r + 1, \dots, n, \quad \lambda_k(\hat{A}) &\leq \lambda_k(A) + \|\hat{A} - A\|. \end{aligned}$$

Let  $U_K$  (resp.  $\hat{U}_K$ ) be a matrix whose columns are  $K$  orthogonal eigenvectors associated to the largest eigenvalues (in absolute value) of matrix  $A$  (resp.  $\hat{A}$ ). The proof of the two lemmas rely on the following important statement:

$$\lambda_{\min}(A) > 2\|\hat{A} - A\| \Rightarrow \|\hat{U}_K - U_K\|_F^2 \leq \frac{16}{\lambda_{\min}(A)^2} \|\hat{A} - A\|^2. \quad (9)$$

We prove (9). If  $\lambda_{\min}(A) > 2\|\hat{A} - A\|$ , letting  $a_n = \lambda_{\min}(A)/2$ , one has

$$\begin{aligned} k \in \{1, s\} &\Rightarrow \lambda_k(A) \in [a_n, +\infty[ \\ k \in \{s+1, n-r\} &\Rightarrow \lambda_k(A) \in ]-a_n, a_n[ \\ k \in \{n-r+1, n\} &\Rightarrow \lambda_k(A) \in ]-\infty, -a_n] \end{aligned}$$

In particular, the  $K$  eigenvalues of  $\hat{A}$  with largest absolute values are  $\lambda_k(\hat{A})$ , for  $k \in \{1, s\}$  (positive eigenvalues) and  $k \in \{n-r+1, n\}$  (negative eigenvalues). The matrix  $\hat{U}_K$  can thus be written (up to a permutation of the columns)  $\hat{U}_K = [\hat{U}^+ | \hat{U}^-]$ , where the  $s$  columns of  $U^+$  are normalized eigenvectors associated to positive eigenvalues and the  $r$  columns of  $U^-$  are normalized eigenvectors associated to negative eigenvalues. Let  $U_K = [U^+ | U^-]$  be a matrix of normalized eigenvectors of  $A$  decomposed similarly (up to the same permutation of columns).

As  $I^+ := [a_n, +\infty[$  contains exactly  $s$  eigenvalues of  $A$  and  $\hat{A}$ , from Theorem 18, there exists a matrix  $\hat{P}^+ \in \mathcal{O}_s(\mathbb{R})$  such that

$$\|\hat{U}^+ - U^+ \hat{P}^+\|_F \leq \frac{2\sqrt{2}}{\lambda_{\min}(A)} \|\hat{A} - A\|.$$

Similarly, as  $I^- := ]-\infty, -a_n]$  contains exactly  $r$  eigenvalues of  $A$  and  $\hat{A}$ , from Theorem 18, there exists a matrix  $\hat{P}^- \in \mathcal{O}_r(\mathbb{R})$  such that

$$\|\hat{U}^- - U^- \hat{P}^-\|_F \leq \frac{2\sqrt{2}}{\lambda_{\min}(A)} \|\hat{A} - A\|.$$

Let  $\hat{P}$  be the block diagonal matrix of size  $r+s = K$  with  $\hat{P}^+$  and  $\hat{P}^-$  as first and second block. One has

$$\|\hat{U}_K - U_K \hat{P}\|_F^2 = \|\hat{U}^+ - U^+ \hat{P}^+\|_F^2 + \|\hat{U}^- - U^- \hat{P}^-\|_F^2 \leq \frac{16}{\lambda_{\min}(A)^2} \|\hat{A} - A\|^2.$$

This proves (9).

**Proof of Lemma 7.** Let  $\eta$  be fixed and let  $\epsilon = \eta/(2+\eta)$ , so that  $(1+\epsilon)/(1-\epsilon) = 1+\eta$ . Let  $\mathcal{E}, \mathcal{F}, \mathcal{G}$  be the three events

$$\begin{aligned} \mathcal{E} &= \left( \|\hat{A} - A\| \leq \sqrt{2(1+\epsilon)d_{\max} \log(4n/\delta)} \right) \\ \mathcal{F} &= \left( \hat{d}_{\max}(n) \leq (1+\epsilon)d_{\max}(n) \right) \\ \mathcal{G} &= \left( \hat{d}_{\max}(n) \geq (1-\epsilon)d_{\max}(n) \right) \end{aligned}$$

and  $\mathcal{H} = \mathcal{E} \cap \mathcal{F} \cap \mathcal{G}$ . We first show that  $\mathbb{P}(\mathcal{H}) \geq 1 - \delta$  under the assumption

$$d_{\max} \geq \frac{2(1+\epsilon/3)}{\epsilon^2} \log\left(\frac{4n}{\delta}\right). \quad (10)$$

From Theorem 14, this condition implies  $\mathbb{P}(\mathcal{E}^c) \leq \delta/2$ . From Lemma 16, one has

$$\begin{aligned}\mathbb{P}(\mathcal{F}^c) &\leq ne^{-d_{\max} \frac{\epsilon^2}{2(1+\epsilon/3)}} \leq \delta/4, \\ \mathbb{P}(\mathcal{G}^c) &\leq e^{-d_{\max} \frac{\epsilon^2}{2(1+\epsilon/3)}} \leq ne^{-d_{\max} \frac{\epsilon^2}{2(1+\epsilon/3)}} \leq \delta/4.\end{aligned}$$

A union bound then yields  $\mathbb{P}(\mathcal{H}) \geq 1 - \delta$ .

From now on, we assume that the event  $\mathcal{H}$  holds. We first prove that under the extra assumption

$$\lambda_{\min}(A) \geq C_\epsilon \sqrt{d_{\max} \log(4n/\delta)} \quad \text{with} \quad C_\epsilon = \sqrt{2(1+\epsilon)} \left(1 + \sqrt{\frac{1+\epsilon}{1-\epsilon}}\right), \quad (11)$$

the set

$$\hat{S}_n^\eta = \left\{ k : |\lambda_k(\hat{A})| > \sqrt{2 \frac{1+\epsilon}{1-\epsilon} \hat{d}_{\max} \log(4n/\delta)} \right\}$$

coincides with  $\{1, s\} \cup \{n-r+1, n\}$ , thus its cardinality is  $K$ . The consequences of Weyl's inequality yield in this particular case, using that  $\mathcal{E}$  holds

$$\text{for } k \notin \{s+1, n-r\}, \quad |\lambda_k(\hat{A})| > \lambda_{\min}(A) - \sqrt{2(1+\epsilon)d_{\max}(n) \log(4n/\delta)} \quad (12)$$

$$\text{for } k \in \{s+1, n-r\} \quad |\lambda_k(\hat{A})| < \sqrt{2(1+\epsilon)d_{\max} \log(4n/\delta)}. \quad (13)$$

For every  $k \in \hat{S}_n$ , using that  $\mathcal{G}$  holds, one has

$$|\lambda_k(\hat{A})| > \sqrt{2 \frac{1+\epsilon}{1-\epsilon} \hat{d}_{\max} \log(4n/\delta)} \geq \sqrt{2(1+\epsilon)d_{\max}(n) \log(4n/\delta)}.$$

From (13), this proves that  $k \in \{1, n\} \setminus \{s+1, n-r\}$ . Conversely, Let  $k \in \{1, n\} \setminus \{s+1, n-r\}$ . Using (12),

$$\begin{aligned}|\lambda_k(\hat{A})| &\geq C_\epsilon \sqrt{d_{\max} \log(4n/\delta)} - \sqrt{2(1+\epsilon)d_{\max} \log(4n/\delta)} \\ &\geq (C_\epsilon - \sqrt{2(1+\epsilon)}) \sqrt{(\hat{d}_{\max}/(1+\epsilon)) \log(4n/\delta)} = \frac{C_\epsilon - \sqrt{2(1+\epsilon)}}{\sqrt{1+\epsilon}} \sqrt{\hat{d}_{\max} \log(4n/\delta)} \\ &> \sqrt{2 \frac{1+\epsilon}{1-\epsilon} \hat{d}_{\max} \log(4n/\delta)},\end{aligned}$$

where we use that  $\mathcal{F}$  holds for the second inequality. Hence  $k \in \hat{S}_n$ . Thus  $\hat{S}_n = \{1, n\} \setminus \{s+1, n-r\}$ .

As the set  $\hat{S}_n^\eta$  is of cardinality  $K$ , the matrix  $\hat{U}$  in the statement of Lemma 7 coincides with  $\hat{U}_K$ , the matrix formed by the  $K$  leading vectors of  $\hat{A}$ . Assumption (11) implies in particular (using additionally that  $\mathcal{E}$  holds) that

$$\lambda_{\min}(A) \geq 2\sqrt{2(1+\epsilon)d_{\max} \log(4n/\delta)} \geq 2\|\hat{A} - A\|.$$

From (9), one obtains

$$\|\hat{U} - U\| = \|\hat{U}_K - U_K\| \leq \frac{16}{\lambda_{\min}(A)^2} \|\hat{A} - A\|^2 \leq \frac{32(1+\epsilon)}{\lambda_{\min}(A)^2} d_{\max} \log\left(\frac{4n}{\delta}\right).$$

The result follow by substituting  $\epsilon$  with  $\eta$  in this last equation and in assumptions (10) and (11).

**Proof of Lemma 13.** In the SBMO model, there exists a constant  $c$  such that

$$d_{\max} \leq n \max_{i,j} A_{i,j} \leq cd_{\max}.$$

Let  $r > 0$ . From Theorem 15, there exists a constant  $\tilde{C}_r$  such that if  $d_{\max} \geq \log(n)$ , with probability larger than  $1 - n^{-r}$ ,

$$\|\hat{A} - A\| \leq \tilde{C}_r \sqrt{n \max_{i,j} A_{i,j}} \leq (\sqrt{c}\tilde{C}_r) \sqrt{d_{\max}}.$$

Letting  $C_r = \sqrt{c}\tilde{C}_r$ , under the assumption that  $\lambda_{\min}(A) \geq 2C_r \sqrt{d_{\max}}$ , one has  $\lambda_{\min}(A) > 2\|\hat{A} - A\|$ . From (9), this yields

$$\|\hat{U}_K - U_K\|_F^2 \leq \frac{16}{\lambda_{\min}(A)^2} \|\hat{A} - A\|^2 \leq \frac{16}{\lambda_{\min}(A)^2} C_r^2 d_{\max}.$$

Lemma 13 follows by rescaling the constant  $C_r$ .

## E The sparse case: uncovering a phase transition

Consider the following simple SBMO with two communities and a diagonal connectivity matrix such that if  $\mathbf{1}_r \in \mathbb{R}^{r \times 1}$  is a vector containing only ones, the expected adjacency matrix is

$$A = \frac{\alpha_n}{n} Z B Z^T, \text{ with } B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \text{ and } Z = \begin{pmatrix} \mathbf{1}_{sn} & 0 \\ \mathbf{1}_{(1-2s)n} & \mathbf{1}_{(1-2s)n} \\ 0 & \mathbf{1}_{sn} \end{pmatrix},$$

where  $0 < s < 1/2$  is the fraction of pure nodes in each of the two communities: the smaller  $s$ , the larger the overlap, whereas  $s = 1/2$  corresponds to pure nodes only, i.e. a SBM without overlap. We now elaborate on this example. The matrix  $A$  has rank 2 with eigenvalues  $\alpha_n a(2 - 3s) > \alpha_n s a$  and associated eigenvectors that are respectively

$$X = \begin{pmatrix} \mathbf{1}_{sn} \\ \mathbf{2}_{(1-2s)n} \\ \mathbf{1}_{sn} \end{pmatrix} \text{ and } Y = \begin{pmatrix} -\mathbf{1}_{sn} \\ \mathbf{0}_{(1-2s)n} \\ \mathbf{1}_{sn} \end{pmatrix}.$$

Each node  $i$  of the network has a spectral embedding given by  $(X_i, Y_i)$ , i.e. pure nodes in community one correspond to  $P_1 = (1, -1)$ , pure nodes in community two correspond to  $P_2 = (1, 1)$  and mixed nodes correspond to  $M = (2, 0)$ . As expected, we have  $M = P_1 + P_2$  and if  $\alpha_n$  is sufficiently large, i.e.  $\alpha_n \gg \log n$ , then Theorems 9 and 12 apply: the eigenvectors of the empirical adjacency matrix  $\hat{A}$  will be close to the eigenvectors  $X$  and  $Y$  and as a consequence, the number of nodes that are misclassified by CSC will vanish as  $n$  tends to infinity.

Let now consider the very sparse case where  $\alpha_n = 1$ . In this case our theoretical results are not valid and indeed we believe that there is a range of parameters where only partial recovery is possible. Note that if you have only access to the eigenvector  $X$  (associated to largest eigenvalue  $a(2 - 3s)$ ), then it is possible to distinguish pure nodes from mixed nodes but it is impossible to distinguish pure nodes of community one from pure nodes of community two. Observe that the second eigenvalue of  $A$ ,  $sa$  can be very small in which case, this eigenvalue will be ‘hidden’ in the noise of the model. In the sparse regime, it is known that high-degree nodes induce a lot of noise on the spectrum of the adjacency matrix. We now

give a quantitative conjecture based on recent results obtained on the non-backtracking matrix [Krzakala et al., 2013, Bordenave et al., 2015, Saade et al., 2015] which can be seen as a way to regularize the adjacency matrix. We refer to the works cited above for a precise description of the non-backtracking matrix and its spectral analysis. We should stress that the rigorous results obtained so far for the non-backtracking matrix do not allow us to cover the present framework. However, it is believed that the largest eigenvalue of the non-backtracking matrix for our graph will be  $a(2 - 3s) + o_n(1)$  and that the noise, i.e. the eigenvalue  $\lambda$  corresponding to eigenvectors not correlated with the communities will be of modulus  $|\lambda| < \sqrt{a(2 - 3s)}$ . In particular, if  $sa > \sqrt{a(2 - 3s)}$ , then a second eigenvalue appears on the real axis at  $sa + o_n(1)$ . Moreover the eigenvector associated to these 2 eigenvalues are correlated with the true communities. To summarize, we claim:

**Conjecture 19.** *If  $s^2a > 2 - 3s$  then a spectral algorithm based on the non-backtracking matrix will classify a positive fraction of the pure nodes.*

*If  $s^2a < 2 - 3s$  then it is impossible to classify the pure nodes better than by random guessing.*

The spectral algorithm will be similar to CSC except that we replace  $\hat{U}$  containing the eigenvectors of  $\hat{A}$  by the corresponding matrix computed from the eigenvectors of the empirical non-backtracking matrix. The rest of the algorithm is unchanged. Figure 3 shows the spectrum of the non-backtracking matrix for three values of  $a$  around the phase transition at  $\frac{2-3s}{s^2}$  which in this particular case is 9 as  $s = 1/3$ . Note that in this example, it is always possible to distinguish the pure nodes from the mixed nodes. Only the classification of the pure nodes is non-trivial.

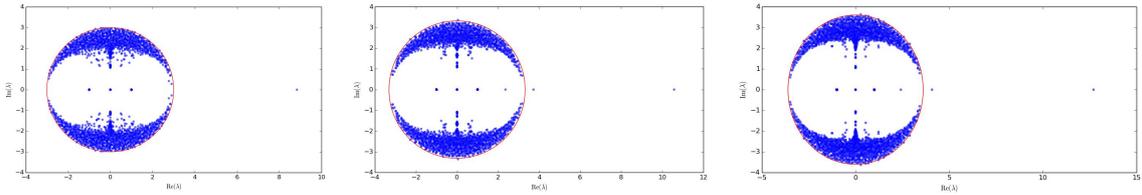


Figure 3: Spectrum of the non-backtracking operator with  $n = 1200$ ,  $sn = 400$  and  $a = 9, 11, 13$ . The circle has radius  $\sqrt{a(2 - 3s)}$  in each case.

Figure 4 illustrates the behavior, in the sparse case ( $\alpha_n = 1$ ), of the CSC algorithm as described in this paper (i.e. with a spectral embedding based on the adjacency matrix, not on the non-backtracking matrix). The fraction of correct entries ( $1 - \text{Error}(\hat{Z}, Z)$ ) is displayed as a function of the parameter  $s$ . The number of nodes is fixed to  $n = 1000$  and the curves in different colors correspond to different values of  $a$ . For each value of  $s$ , the error is averaged over 200 networks drawn under the corresponding SBMO. Overall, the algorithm performs best with large values of  $a$  (that correspond to larger degrees). For each value of  $a$ , the case  $s = 1/2$  corresponds to a standard SBM without overlap and we see that our algorithm performs well. As  $s$  decreases, we see that below a certain value of  $s$  the performance of the algorithm deteriorates greatly which is in accordance with the phase transition conjectured above.

## F Concentration results

### F.1 Proof of Theorem 14

Our proof is based on the following result by [Tropp, 2012].

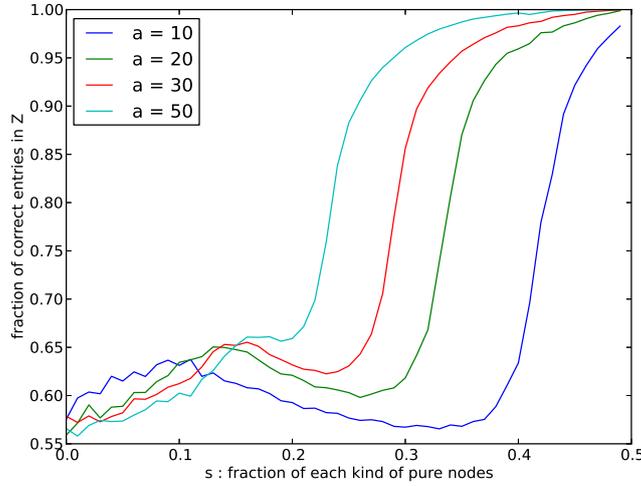


Figure 4: Error of CSC as a function of the fraction of each type of pure nodes in a SBMO model with two-by-two overlap between  $K = 2$  communities

**Lemma 20** (Theorem 1.4, [Tropp, 2012]). *Let  $(X_k)$  be a sequence of independent, random, symmetric matrices with dimension  $d$ . Assume that each random matrix satisfies*

$$\mathbb{E}[X_k] = 0 \text{ and } \lambda_{\max}(X_k) \leq R \text{ almost surely}$$

*and let  $\sigma^2$  be such that  $\|\sum_{k=1}^n \mathbb{E}[X_k^2]\| \leq \sigma^2$ . Then, for all  $t \geq 0$ ,*

$$\mathbb{P}\left(\lambda_{\max}\left(\sum_{k=1}^n X_k\right) \geq t\right) \leq d \exp\left(-\frac{t^2}{2(\sigma^2 + Rt/3)}\right).$$

One has

$$\hat{A} - A = \sum_{i \leq j} X_{i,j},$$

where  $X_{i,j}$  is a matrix of size  $n$  defined by

$$X_{i,j} := (\hat{A}_{i,j} - A_{i,j}) \times \begin{cases} e_i e_j^T + e_j e_i^T & \text{if } i < j \\ e_i e_i^T & \text{if } i = j. \end{cases}$$

One has  $\|X_{i,j}\| \leq |\hat{A}_{i,j} - A_{i,j}| \leq 1$  and

$$\left\| \sum_{i \leq j} \mathbb{E}[X_{i,j}^2] \right\| = \left\| \text{Diag}_i \left( \sum_{j=1}^n \mathbb{E}[X_{i,j}^2] \right) \right\| = \max_i \sum_{j=1}^n A_{i,j} (1 - A_{i,j}) \leq \max_i \sum_{j=1}^n A_{i,j} \leq d_{\max}.$$

From Lemma 20,

$$\mathbb{P}(\|\hat{A} - A\| > \alpha d_{\max}) \leq 2n \exp\left(-d_{\max} \frac{\alpha^2}{2(1 + \alpha/3)}\right)$$

Let  $\epsilon > 0$ . Choosing  $\alpha = \sqrt{2(1+\epsilon)\log(2n/\delta)/d_{\max}}$ , for

$$d_{\max} \geq \frac{2}{9} \frac{1+\epsilon}{\epsilon^2} \log \frac{2n}{\delta}$$

(which is equivalent to  $\alpha/3 \leq \epsilon$ ), one has

$$\mathbb{P}\left(\|\hat{A} - A\| > \sqrt{2(1+\epsilon)d_{\max} \log\left(\frac{2n}{\delta}\right)}\right) \leq 2n \exp\left(-\frac{2(1+\epsilon)\log(2n/\delta)}{2(1+\alpha/3)}\right) \leq \delta.$$

## F.2 Proof of the deviation results for the empirical degrees

For all  $i \in \{1, n\}$ ,

$$\hat{d}_i - d_i = \sum_{j=1}^n (\hat{A}_{i,j} - A_{i,j}).$$

As  $\mathbb{E}[\hat{A}_{i,j}] = A_{i,j}$ ,  $|A_{i,j}| \leq 1$  and  $\sum_{j=1}^n \mathbb{E}[\hat{A}_{i,j}^2] = \sum_{j=1}^n A_{i,j} = d_i$ , Bennett's inequality ([Boucheron et al., 2013]) yields, for all  $t > 0$

$$\begin{aligned} \mathbb{P}(\hat{d}_i - d_i > t) &\leq \exp\left(-d_i h\left(\frac{\alpha}{d_i}\right)\right), \\ \mathbb{P}(\hat{d}_i - d_i < -t) &\leq \exp\left(-d_i h\left(\frac{\alpha}{d_i}\right)\right), \end{aligned}$$

where  $h$  is the function defined by  $h(u) = (u+1)\log(u+1) - u$ .

The first two inequalities follow from the fact that  $v \mapsto vh(t/v)$  is decreasing for all  $t$ , hence

$$\mathbb{P}(\hat{d}_i - d_i > \alpha d_{\max}) \leq \exp(-d_{\max} h(\alpha)), \quad (14)$$

$$\mathbb{P}(\hat{d}_i - d_i < -\alpha d_{\max}) \leq \exp(-d_{\max} h(\alpha)). \quad (15)$$

Let  $i_0$  be such that  $d_{i_0} = d_{\max}$ . From (15),

$$\begin{aligned} \mathbb{P}(\hat{d}_{i_0} \geq d_{i_0} - \alpha d_{\max}) &\geq 1 - e^{-d_{\max} h(\alpha)} \\ \mathbb{P}(\hat{d}_{\max} \geq (1-\alpha)d_{\max}) &\geq 1 - e^{-d_{\max} h(\alpha)}, \end{aligned}$$

using that in particular  $\hat{d}_{\max} \geq \hat{d}_{i_0}$ . From (14) and a union bound,

$$\begin{aligned} \mathbb{P}(\forall i \in \{1, n\}, \hat{d}_i \leq d_i + \alpha d_{\max}) &\geq 1 - n \exp(-d_{\max} h(\alpha)), \\ \mathbb{P}(\forall i \in \{1, n\}, \hat{d}_i \leq (1+\alpha)d_{\max}) &\geq 1 - n \exp(-d_{\max} h(\alpha)), \\ \mathbb{P}(\hat{d}_{\max} \leq (1+\alpha)d_{\max}) &\geq 1 - n \exp(-d_{\max} h(\alpha)), \end{aligned}$$

by definition of  $\hat{d}_{\max}$ . The statements in Lemma 16 follow from the lower bound

$$h(u) \geq \frac{u^2}{2(1+u/3)},$$

that can be found in [Boucheron et al., 2013].