



HAL
open science

Queuing for an infinite bus line and aging branching process

Vincent Bansaye, Alain Camanes

► **To cite this version:**

Vincent Bansaye, Alain Camanes. Queuing for an infinite bus line and aging branching process. 2017. hal-01162541v3

HAL Id: hal-01162541

<https://hal.science/hal-01162541v3>

Preprint submitted on 31 Aug 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

QUEUEING FOR AN INFINITE BUS LINE AND AGING BRANCHING PROCESS

VINCENT BANSAYE & ALAIN CAMANES

ABSTRACT. We study a queueing system with Poisson arrivals on a bus line indexed by integers. The buses move at constant speed to the right and the time of service per customer getting on the bus is fixed. The customers arriving at station i wait for a bus if this latter is less than d_i stations before, where d_i is non-decreasing. We determine the asymptotic behavior of a single bus and when two buses eventually coalesce almost surely by coupling arguments. Three regimes appear, two of which leading to a.s. coalescing of the buses. The approach relies on a connection with aged structured branching processes with immigration and varying environment. We need to prove a Kesten Stigum type theorem, i.e. the a.s. convergence of the successive size of the branching process normalized by its mean. The technics developed combines a spine approach for multitype branching process in varying environment and geometric ergodicity along the spine to control the increments of the normalized process.

Key words. Queueing systems, Polling, Multitype Branching processes, Immigration, Inhomogeneity, Coupling.

A.M.S. Classification. 60J80, 60K25, 60F05, 60F10.

CONTENTS

1. Introduction & Main results	2
1.1. Queueing system for a bus line	2
1.2. Main results	4
1.3. Aging branching processes	5
1.4. Connection between queueing and branching process and discussion	6
1.5. Organization of the paper	7
2. Aging branching process with maximal age going to infinity	8
2.1. Aging branching process without immigration	8
2.2. Aging branching process with immigration	18
3. From the motion of one single bus to aging branching process	21
4. Queueing system for a bus line	23
4.1. The motion of one single bus	23
4.2. Coalescence criteria for two buses with the same discipline	25
Appendix A. Mean behaviors of aging branching process	28
Appendix B. Dichotomy for branching processes	30
Appendix C. M/G/1 queues	32
References	32

1. INTRODUCTION & MAIN RESULTS

The main motivation of this paper is the study of queueing systems where customers arrive on the half line of integers (a bus line) and the traveling servers (the buses) move at a constant speed to the right. The time service for each customer (the time required to get on the bus) is assumed to be constant for simplicity. In the model considered here, a customer arriving in station $i \in \mathbb{N}$ is waiting at this station if the distance from this customer to the next bus is less than d_i . We describe now this process, its link with aging branching processes and the main results of the paper.

1.1. Queuing system for a bus line. We consider an infinite bus line, where stations are labeled by $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. The buses all start from station 0, the depot, and no customer get on at this station. To each station $i \geq 1$, we associate an independent Poisson Point Processes $\mathbf{T}^{(i)} = (T_k^{(i)} : k \geq 1)$ on $[0, \infty)$ with intensity $\alpha \in (0, 1)$: $T_k^{(i)}$ is the time arrival of the k -th customer at station i .

The discipline $(d_i : i \geq 1)$ of the queuing is a sequence of positive integers such that

$$(A) \quad d_1 = 1, \quad d_2 = 2, \quad d_i \leq d_{i+1} \leq d_i + 1 \quad (i \geq 2).$$

When a customer arrives at station i , if the next bus is d_i , or more, stations far from him, then he decides not to take the bus and he leaves the queue. Thus, the customers who are indeed waiting at the station i are those arrived at this station after a bus has left the station $i - d_i$ and before it has left station i . Notice that, when $d_i = 1$, customers wait at station i only if the bus has already left the previous station and we have assumed $d_i > 1$ for $i > 1$ to simplify the links with aging branching processes and avoid the degenerate case when only switch over times matter.

The discipline $d_i = i$ is a first relevant example. Then, the customers wait the first bus at the station i as soon as the first bus has left the depot (time 0) and has not already left station i . This polling system for one server with infinitely many stations is a sequential clearing process and has been studied in [19] when Poisson arrivals happen on the half line $[0, \infty)$ with rate λ . Then, the position P_t of the bus (server) at time t is equivalent to $\log(t)/\lambda$ when $t \rightarrow \infty$. When the bus can move to the left, analogous transient behavior occur and we refer to [20, 14] for the study of the greedy walker respectively on \mathbb{Z} and \mathbb{R} .

The model when $d_i = d \wedge i$ is an other interesting discipline. Then, the customers wait at station i if the distance to the next bus is strictly less than d stations, soon as $i \geq d$. Many works have been dedicated to models with a finite number of queues and servers moving from one queue to another. In particular, a vast literature deal with customers arriving on a circle while the servers visit the queues cyclically. We refer to [28, 8, 9, 34] and reference therein for an overview about polling models, see also the discussion below. The stability of the queueing then depends on the number of queues and time service. In this case, ballistic and logarithm regimes appear, which are linked to the subcritical and supercritical regimes of the associated branching structure. This happens also in our model.

The class of discipline (A) that we consider allows to describe intermediate regimes between the two previous examples. Then, the farther the station is from the depot, the more customers may wait. Since sequence (d_i) is non-decreasing from Assumption (A), the case when d_i is bounded amounts to consider, for i large

enough, d_i constant and the main difficulties arise when d_i goes to infinity. Several extensions of discipline (A) could be considered and we make a few more precise comments along the paper. Another motivation for considering the discipline (A) is the study of the motion of a bus behind another bus, when the distance between them increases like the sequence $(d_i)_i$, see the last part (Section 4.2) of the paper. The position of the bus $k \in \{1, 2, \dots\}$ at time t is denoted by $P_t^{(k)}$ and the time when it leaves the station i by $H_i^{(k)}$. Between two stations, the buses move at constant speed τ and for any $i \geq 0$:

$$\begin{aligned} P_t^{(k)} &= i + (t - H_i^{(k)})/\tau \quad \text{for } t \in [H_i^{(k)}, H_i^{(k)} + \tau]; \\ P_t^{(k)} &= i + 1 \quad \text{for } t \in [H_i^{(k)} + \tau, H_{i+1}^{(k)}]. \end{aligned}$$

For simplicity, we assume that the time service for each customer (to get on a bus) is constant and equal to one, so the time spent in a station is the number of customers who have got on.

In this paper, we consider only two buses and $k \in \{1, 2\}$. We assume that bus 1 starts from the depot at time 0 and bus 2 starts from the depot at time $\mu > 0$: $P_0^{(1)} = 0$, $P_t^{(2)} = 0$ for any $t \in [0, \mu]$. The successive positions of the buses can be defined recursively. To do so, we introduce the label $F_i^{(k)}$ of the first potential customer waiting for bus k at the station i : this customer arrived after the departure of the bus k from station $i - d_i$ and after the passage of the previous bus (if there is one, i.e. for the second bus). We then consider the set of customers $\{F_i^{(k)} + j : j \in \mathcal{I}_i^{(k)}\}$ arriving after this first customer (including himself) and before the departure of the bus from the station. This set may be empty if the first potential customer arrived too late. It provides the label of the last (potential) customer $M_i^{(k)}$ getting on the bus and the number of customers getting on the bus, which yields the time spent by the bus in station i .

More precisely, the times $(H_i^{(1)} : i \geq 0)$ are defined recursively by

$$\begin{aligned} H_0^{(1)} &= 0 \text{ a.s.} \\ F_i^{(1)} &= \min \left(j \geq 1 : T_j^{(i)} > H_{i-d_i}^{(1)} \right) \\ \mathcal{I}_i^{(1)} &= \{k \in \mathbb{N} : \forall j \leq k, T_{F_i^{(1)}+j}^{(i)} \leq H_{i-1}^{(1)} + \tau + j\} \\ M_i^{(1)} &= F_i^{(1)} - 1 + \# \mathcal{I}_i^{(1)} \\ H_i^{(1)} &= H_{i-1}^{(1)} + \tau + (M_i^{(1)} - F_i^{(1)} + 1), \end{aligned} \tag{1}$$

where $\# \mathcal{I}_i^{(1)}$ denotes the number of elements of $\mathcal{I}_i^{(1)}$. First note that $0 \leq i - d_i < i$ under (A). Moreover, since $\alpha > 0$, then $(T_j^{(i)})_j$ goes to infinity a.s. and $F_i^{(1)}$ is a.s. finite. Finally, since $\alpha < 1$, we will explain in Section 3 and Appendix C, using a standard link with Galton-Watson process, that $\mathcal{I}_i^{(1)}$ is bounded and $M_i^{(1)}$ is also finite a.s.

Let us define similarly the motion of the second bus but now customers may have been taken by the first bus. Using the label $M_i^{(1)}$ of the last customer getting

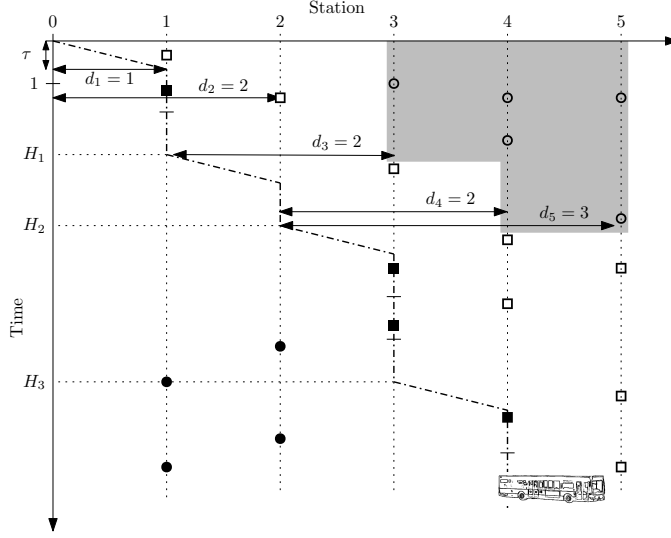


FIGURE 1. Successive positions of a single bus following discipline (d_i) . Customers who are not getting on the bus, either arrive when the bus was too far (o), or when the bus has already left (•). Customers getting on the bus arrived at the station either before the bus arrival (□), or when another customer of this station was getting on (■).

on the bus 1, we set for $i \geq 1$,

$$\begin{aligned}
 H_0^{(2)} &= \mu \text{ a.s.} \\
 F_i^{(2)} &= \min \left(j > M_i^{(1)} : T_j^{(i)} > H_{i-d_i}^{(2)} \right) \\
 \mathcal{I}_i^{(2)} &= \{k \in \mathbb{N} : \forall j \leq k, T_{F_i^{(2)}+j}^{(i)} \leq H_{i-1}^{(2)} + \tau + j\} \\
 M_i^{(2)} &= F_i^{(2)} - 1 + \# \mathcal{I}_i^{(2)} \\
 H_i^{(2)} &= H_{i-1}^{(2)} + \tau + (M_i^{(2)} - F_i^{(2)} + 1).
 \end{aligned}
 \tag{2}$$

1.2. Main results. The position of the bus 1 (a bus alone on the line) is described using the following theorem, whose proof is deferred to Section 4.1. When $(d_i)_i$ is bounded (and non decreasing from Assumption (A)), it is stationary and for simplicity we focus in that case on $d_i = d \wedge i$. For any integer $a \geq 0$, let $\varrho(a) \in (0, \infty]$ be the the largest root of

$$P_a = X^{a+1} - m \sum_{k=0}^a X^k, \quad \text{where } m = \frac{\alpha}{1-\alpha}.$$

Thus, $\varrho(a)$ is the largest eigenvalue of the Leslie matrix of size $a + 1$, see Appendix A. In the next statement, $d \geq 2$ is a constant.

Theorem. (i) If $d_i = d \wedge i$ and $\varrho(d-2) < 1$, then there exists $c > \tau$ such that

$$H_i^{(1)} \sim_{i \rightarrow \infty} ci \quad \text{a.s.}$$

Moreover, $(H_i^{(1)} - ic)/\sqrt{i}$ converges weakly to a Gaussian r.v. with positive variance.

(ii) If $d_i = d \wedge i$ with $\varrho(d - 2) > 1$ or $d_i \rightarrow \infty$, then

$$H_i^{(1)} \sim_{i \rightarrow \infty} m_i W \quad \text{a.s.},$$

where W is a finite positive random variable and m_i satisfies the following recursion

$$(3) \quad m_0 = 1; \quad m_n = m \sum_{i=0}^{d_{n+1}-2} m_{n-1-i} \quad (n \geq 1).$$

This dichotomy ($i - ii$) between the ballistic and logarithm speed is inherited from the subcritical and supercritical behavior of the underlying branching structure and is classical for polling models, see in particular [25, 28] and discussion below. Here the main difficulty arises when d_i goes to infinity and in particular when it does not go to infinity fast enough to behave as the greedy walker. The critical case and the heavy traffic limit are not investigated in this paper, see in particular [29] for cycling polling models.

In the case (ii), if $d_i = d \wedge i$ then m_i grows geometrically with rate $\varrho(d - 2) \in (1, m + 1)$; while if $d_i \rightarrow \infty$, then m_{i+1}/m_i goes to $m + 1$. We refer to Lemma A.3 for finer results in that case. In particular, we give an analytical criterion on d_i to ensure that $m_i \sim c(m + 1)^i$ for some $c > 0$, which corresponds to the asymptotic behavior of the clearing sequential process and greedy walk [19, 20] ($d_i = i$).

The station where the two buses coalesce is defined by

$$N = \min\{i \geq 0 : H_i^{(1)} \geq H_i^{(2)}\} = \min\{i \geq 1; \exists t > 0 : P_t^{(1)} = P_t^{(2)} = i\},$$

which is infinite when the buses never coalesce. Obviously, $\mathbf{P}(N < \infty) > 0$ and we say that the two buses coalesce a.s. iff $\mathbf{P}(N < \infty) = 1$. In Section 4.2, we prove the following result using coupling arguments and the discipline of the buses:

Theorem. (i) If $d_i = d \wedge i$ and $\varrho(d - 2) < 1$, then the two buses coalesce a.s.
(ii) If $d_i = d \wedge i$ and $\varrho(d - 2) > 1$, then the two buses do not coalesce a.s.
(iii) If $d_i \rightarrow \infty$ as $i \rightarrow \infty$, then the two buses coalesce a.s.

1.3. Aging branching processes. The behaviors of the buses rely on the study of the branching structure associated to the motion of a single bus. Indeed, during the time a customer gets on the bus, new customers may arrive both at the same station and at the following stations. . . Customers waiting at station i take a bus and then provoke some queuing (only) in a set of stations following their position, whose size depends on i through the discipline $(d_k)_k$. These customers queuing in these successive stations $i + k$ can be seen as offsprings and the integer k as an age for the original customer. That leads us to study an aging branching structure where individuals (customers) give birth during a life period whose length depends on their birth time (their station of arrival). We describe in this section this branching process, which may be interesting for itself. The link with the bus line is specified in Proposition 1, Section 3.

We consider a branching process where each individual reproduces independently with reproduction law given by an integer valued random variable R and may die depending on his age. We write a_n the maximal age one individual may have in generation n and we assume that

$$(B) \quad \forall n \in \mathbb{N}, \quad 0 \leq a_n \leq a_{n+1} \leq a_n + 1.$$

This provides an aging structure to the population. This is also the counterpart of Assumption (A) above for the discipline of the queueing system. More specifically, both sequences are linked by $a_n = d_{n+1} - 2$ for $n \geq 2$ (see Section 3 for details) as soon as the right hand side is non-negative.

For each $n, a \in \mathbb{N}$, we denote by $Z_n(a)$ the number of individuals in generation n whose age is equal to a and Z_n the total size of the population in generation n . The aging branching process with immigration ($Z_n : n \in \mathbb{N}$) is initialized with a bounded random vector ($Z_0(a) : a \in \mathbb{N}$) where $Z_0(a) = 0$ if $a > a_0$ and defined recursively as follows:

$$(4a) \quad Z_n(a) = Z_{n-1}(a-1)1_{a \leq a_n} \quad \text{for } a \geq 1;$$

$$(4b) \quad Z_n(0) = \sum_{j=1}^{Z_{n-1}} R_{j,n} + I_n;$$

$$(4c) \quad Z_n := \sum_{a=0}^{a_n} Z_n(a);$$

where $(R_{j,n}, j, n \in \mathbb{N})$ are i.i.d. integer valued random variables distributed as R and $(I_n, n \in \mathbb{N})$ are independent random variables independent of $(R_{j,n}, j, n \in \mathbb{N})$.

In Section 2.2, we prove a Kesten Stigum theorem and specify the distribution of traits among the population. We write $m = \mathbf{E}(R)$ and assume that

$$(C) \quad \mathbf{E}(R^2) < \infty, \quad \limsup_{n \rightarrow \infty} \frac{\log(\mathbf{E}(I_n))}{n} < 1 + m, \quad \mathbf{P}(I_n \geq x) \geq \mathbf{P}(I \geq x),$$

for any $x \geq 0$, where I is a non-negative random variable and $\mathbf{P}(I > 0) > 0$. These assumptions allow to use L^2 computations and guarantees that the process survives but the immigration term is not exploding fast enough to change its growth.

Theorem. *Assume that $a_n \rightarrow \infty$.*

- (i) *The process $(Z_n/m_n : n \in \mathbb{N})$ converges a.s. to a finite positive random variable.*
- (ii) *For each $a \geq 0$, the process $(Z_n(a)/Z_n : n \in \mathbb{N})$ converges a.s. to $m/(m+1)^{a+1}$.*

The proof is first achieved without immigration and the quantity m_n is the mean number of individuals living at generation n in that case, see Section 2.1.2. Of course, m_n is determined by Equation (3) replacing $d_{n+1} - 2$ by a_n . Let us stress that when a_n goes to infinity, the asymptotical distribution of age in the population coincide with the immortal case $a_n = \infty$ but the growth of the population $(m_n)_n$ may differ, see Lemma A.3 in Appendix for details.

1.4. Connection between queuing and branching process and discussion.

The connection between branching processes and queueing systems is known from a long time and can be found for example in [13] (chapter XIV). Basically the offspring of an individual is the number of customers arriving during the time when this individual is served.

Multitype branching processes have played a key role in the analysis of polling models and immigration allows to capture polling models with non-zero switch-over times, see e.g. [25]. Using multitype branching process with final product allows to describe the busy period time and could provide an efficient way to extend this work to random time service, see [31] with zero switch-over time and [33] with non-zero switch-over time, both in random environment. The stable case corresponds to subcritical branching process, and the heavy traffic limit (when the load tends

to 1) has also been investigated using near critical branching processes [29]. This would provide another stimulating regime for the model considered here.

A vast literature is dedicated to multitype branching processes (see e.g. [23, 5]) with a finite number of types and their applications to population dynamics (see e.g. [15, 18]) and queueing systems (see e.g. [30]). In particular, they give a convenient way to model population without interactions and take into account a bounded age structure influencing the reproduction law of the individuals. The associated mean matrix is then called Leslie matrix and Perron Frobenius theory enables to describe the asymptotic behavior of the process. In particular, the convergence of the renormalized size of a branching process can be found in [5, 21].

Much less is known about infinite number of types and varying environment since the spectral approach and martingale techniques cannot be extended easily. The pioneering works of e.g. Moy [24] and Kesten [17] provide some theoretical extensions of limit theorems and extinction criteria for a denumerable number of types, while multitype branching processes with a finite number of types have been well studied for stationary ergodic environment (see e.g. [4, 3]).

In varying environment and finite dimension, Coale Lopez theorem [27] about product of matrices can be used to describe the mean behavior of the process and prove asymptotic results on the branching process via weak ergodicity, see [16] and [12]. Here we have to deal both with varying environment and infinite dimension. Necessary and sufficient conditions for a.s. and L^p convergence of martingales associated to time and space harmonic vectors have been obtained in [7] in the case of countable set of types. But they rely on the ergodic behavior of the first moment semigroup and their application in our framework seems delicate. Nevertheless, it could help for example to relax the moment assumptions made here.

We develop here a slightly different approach and control the L^2 moments of the successive increments of the renormalized process $Z_n/\mathbf{E}(Z_n)$ without immigration. This technique is inspired by the spine approach initiated by [11, 22, 21] (and the so-called many-to-one formula and formula for forks). We provide an expression of the transition matrix of the Markov process along the spine following [6]. Thanks to a Doeblin's condition, we get the geometric ergodicity of this Markov chain which enables us to control finely the first moment semigroups and then the increments of the renormalized process. We believe this method is rather efficient and could be generalizable in several ways.

Let us finally mention that taking into account time non-homogeneity and immigration is more generally motivated by self excited process (like Hawkes process) where the immigration is the excitation and the branching structure is the cascade effect provoked by this excitation. Here letting a_n goes to infinity makes the range of memory of the process be very large.

1.5. Organization of the paper. The first part (Section 2) of this work is dedicated to the study of the aging structured branching population Z_n . Branching process without immigration is first studied and Theorem 1 states the L^2 convergence of the normalized process and the asymptotic distribution of the ages among the population. Then, to derive the behavior in the presence of immigration (see Theorem 3), we sum the descendants of each immigrant. We conclude this section with the behavior of the cumulative number of newborns (see Corollary 4), which is the key quantity used for the study of the motion of the buses. All along the paper, bold letters will be dedicated to processes with immigration, e.g. Z_n is for

the size of the population in generation n without immigration and Z_n is for the one with immigration.

In Section 3, we specify the link between bus line and the aging branching process. In Section 4, we first study the progression of one single bus (Theorem 5). We then apply these results to determine when two successive buses coalesce a.s. (Theorem 7) by coupling arguments.

2. AGING BRANCHING PROCESS WITH MAXIMAL AGE GOING TO INFINITY

In this section, we assume (B) and focus on maximal ages going to infinity:

$$\lim_{n \rightarrow \infty} a_n = \infty.$$

We first note that the means of Z_n and Z_n go to infinity and the process is supercritical. Indeed, for any $a_0 \geq 0$ and n large enough, $Z_n(\cdot)$ is stochastically larger than a multitype branching process whose mean matrix is the squared Leslie's matrix of size $a_0 + 1$; see Appendix A and below for details and finer results.

2.1. Aging branching process without immigration. First define the aging branching process without immigration. The process $(Z_n(\cdot) : n \in \mathbb{N})$ is defined by $Z_0(0) = 1$, $Z_0(a) = 0$ for $a \geq 1$ and for $n \geq 1$,

$$(5a) \quad Z_n(a) = Z_{n-1}(a-1)1_{a \leq a_n} \quad \text{for } a \geq 1;$$

$$(5b) \quad Z_n(0) = \sum_{j=1}^{Z_{n-1}} R_{j,n};$$

$$(5c) \quad Z_n := \sum_{a=0}^{a_n} Z_n(a);$$

where $(R_{j,n}, j, n \in \mathbb{N})$ are i.i.d. random variables distributed as R . The non-extinction event is defined by

$$\text{Nonext} := \{\forall n \geq 0 : Z_n > 0\}.$$

We give now the asymptotic behavior of Z_n under the L^2 condition

$$\sigma^2 = \mathbf{E}[(R - m)^2] < \infty.$$

It is used in next Section 2.2 to derive the counterpart when the immigration is non zero.

Theorem 1. (i) *The event Nonext has positive probability.*

(ii) *The process $(Z_n/\mathbf{E}(Z_n) : n \in \mathbb{N})$ converges a.s. and in L^2 to a non-negative r.v. W , which is positive a.s. on Nonext.*

(iii) *For each $a \in \mathbb{N}$, the process $(Z_n(a)/Z_n : n \in \mathbb{N})$ converges a.s. to $m/(m+1)^{a+1}$ on Nonext.*

We prove first (i). The two following subsections 2.1.3 and 2.1.4 are respectively devoted to the proofs of (ii) and (iii).

Proof of Theorem 1 (i). Since (a_n) goes to infinity, for n_0 large enough $\varrho(a_{n_0}) > 1$, where $\varrho(a)$ has been defined in the Introduction and is the greatest eigenvalue of the Leslie Matrix sized $a + 1$. Moreover, with positive probability, $Z_{n_0}(0) \neq 0$. Then, $(Z_n)_{n \geq n_0}$ is lower bounded by a multitype branching process starting from one newborn, with fixed maximal age a_{n_0} and reproduction law R . Recalling

that the mean matrix of this latter is a primitive matrix with maximal eigenvalue greater than 1 and that the second moment of R is finite ensures that with positive probability, it grows to infinity, see the supercritical regime in [5], Theorem 1, Chapter V - Section 6. This proves (i). \square

We prove in Section 2.1.2 that $m_{n+1}/m_n \rightarrow 1$, so Theorem 1 ensures that the ratio of consecutive population sizes Z_{n+1}/Z_n goes to $m + 1$ on Nonext. However, the process may grow slower than $(m+1)^n$. In fact, two asymptotic regimes appear, depending whether (a_n) goes fast enough to infinity or not.

Corollary 2. *The process $M_n = (m+1)^{-n}Z_n$ is a supermartingale which converges a.s. to a finite non-negative random variable M . Furthermore,*

- (i) *if $\liminf_{n \rightarrow \infty} a_n / \log n > 1 / \log(m+1)$, then $\{M > 0\} = \text{Nonext}$.*
- (ii) *if $\limsup_{n \rightarrow \infty} a_n / \log n < 1 / \log(m+1)$, then $M = 0$ a.s.*

Notice that, when $a_i = i$, each individual is immortal and we obtain a Galton-Watson process with mean $m + 1$, which grows geometrically with rate $m + 1$ as $n \rightarrow \infty$. The result above shows at which speed (a_n) has to go infinity so that such a growth still holds. The proof of this statement is postponed to Section 2.1.5.

2.1.1. *Geneological description.* To achieve the proof of Theorem 1, we use a geneological description of the aging branching process. Following notations of Ulam-Harris-Neveu, we introduce

$$\mathcal{I} = \{\emptyset\} \cup \bigcup_{n \geq 1} \{0, 1, 2, \dots\}^n$$

endowed with the natural order \preceq : $u \preceq v$ if there exists $w \in \mathcal{I}$ such that $v = uw$. Let $|u| = n$ denotes the length of u when $u = u_1 \dots u_n \in \mathcal{I}$ and $|\emptyset| = 0$.

We define now $(\mathcal{T}, \mathcal{Z})$ the geneological tree labeled by the age of the individuals where \mathcal{T} is a random subset of \mathcal{I} and $\mathcal{Z} = (\mathcal{Z}(u) : u \in \mathcal{T})$ is a collection of integers giving the ages. This can be achieved recursively as follows:

- $\emptyset \in \mathcal{T}$ and $\mathcal{Z}(\emptyset) = 0$.
- $u0 \in \mathcal{T}$ iff $u \in \mathcal{T}$ and $\mathcal{Z}(u) + 1 \leq a_{|u|+1}$. Then, $\mathcal{Z}(u0) = \mathcal{Z}(u) + 1$.
- For any $j \geq 1$ and $u \in \mathcal{T}$, $uj \in \mathcal{T}$ iff $j \leq R_{\text{Cous}_\ell(u), |u|}$ where

$$\text{Cous}_\ell(u) = \#\{v \in \mathcal{T} : v = v_1 \dots v_{|u|}, \exists j \leq |u|, \forall k \leq j, v_k \leq u_k\}.$$

Then $\mathcal{Z}(uj) = 0$.

Thus, the offspring of an individual u are labeled by ui with $1 \leq i \leq R_{\text{Cous}_\ell(u), |u|}$, while u becomes $u0$ in the next generation if it survives. The set of individuals in generation n is denoted \mathcal{G}_n and the set of the offspring of u in the next generation (including himself if it survives) is denoted $X(u)$. Finally, in each generation n , the part of the population which survives in the next generation is denoted by \mathcal{G}_n^+ , while the part of the population which dies is denoted \mathcal{G}_n^- . More precisely,

$$\begin{aligned} \mathcal{G}_n &:= \{u \in \mathcal{T} \text{ s.t. } |u| = n\}, \\ X(u) &:= \{v \in \mathcal{T} : |v| = |u| + 1, u \preceq v\}, \\ \mathcal{G}_n^+ &:= \{u \in \mathcal{G}_n \text{ s.t. } \mathcal{Z}(u) + 1 \leq a_{n+1}\}, \\ \mathcal{G}_n^- &:= \{u \in \mathcal{G}_n \text{ s.t. } \mathcal{Z}(u) + 1 > a_{n+1}\}. \end{aligned}$$

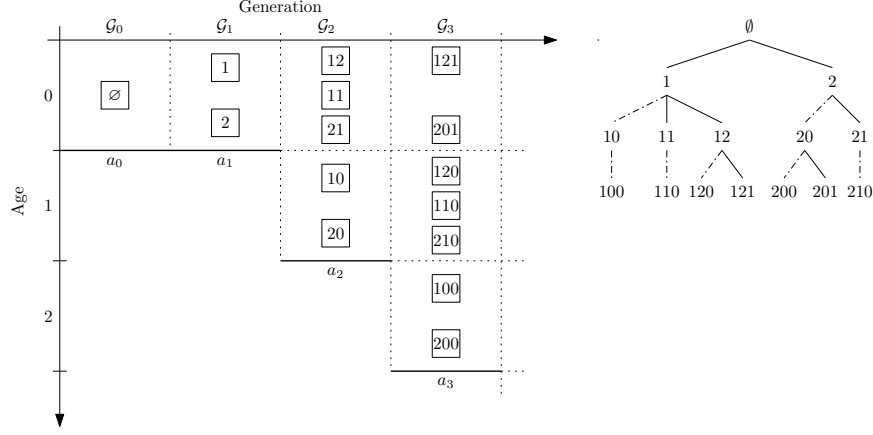


FIGURE 2. Example of a population evolution with labeled individuals, starting with a single individual and with maximal age $a_0 = a_1 = 0$, $a_2 = 1$ and $a_3 = 2$. In the tree description, dash-dotted lines are used to denote surviving individuals.

Then, the population $(Z_n(a) : a \geq 0)$ in generation n is described by the measure:

$$\mathcal{Z}_n = \sum_{u \in \mathcal{G}_n} \delta_{\mathcal{Z}(u)}.$$

Indeed, $Z_n(a) = \mathcal{Z}_n(\{a\}) = \#\{u \in \mathcal{G}_n : \mathcal{Z}(u) = a\}$ is the size of the population with age a in generation n and $Z_n = \mathcal{Z}_n(\mathbb{N}) = \#\mathcal{G}_n$. Finally, we write \mathcal{F}_n the σ algebra generated by $((u, X(u)) : u \in \mathcal{T}, |u| \leq n)$.

2.1.2. Moment estimates. Before the proofs, we start with some simple estimates on $\mathbf{E}[Z_n]$. To evaluate the mean behavior of the process, we consider the mean number of individuals at generation n which are descendants of a single individual aged a at generation i . Thus, let us note for every $0 \leq i \leq n$ and $0 \leq a \leq a_i$,

$$(6) \quad m_{i,n}(a) = \mathbf{E}[Z_n | \mathcal{Z}_i = \delta_a],$$

For sake of simplicity, when the initial individual is a newborn, we write

$$(7) \quad m_{i,n} = m_{i,n}(0), \quad m_n = m_{0,n} = m_{0,n}(0) = \mathbf{E}[Z_n | \mathcal{Z}_0 = \delta_0].$$

Finally, we consider the mean number of individuals aged b in generation n , which are descendants from a single individual with age a in generation i . For $0 \leq i \leq n$, $0 \leq a \leq a_i$, $0 \leq b \leq a_n$, let

$$m_{i,n}(a, b) = \mathbf{E}[Z_n(b) | \mathcal{Z}_i = \delta_a].$$

We provide here some simple but useful estimates on these mean quantities. The proofs rely on the semi group property of the first moment operators $(m_{i,n}(\cdot, \cdot))_{i \leq n}$, which is inherited from the Markov and branching properties of the process $(Z_n)_n$.

Lemma 2.1. (i) For every $n > 0$, $m_n(0, 0) = m_n m_{n-1}$.

(ii) For all $0 \leq i < n$, $i < k < n$ and $a \leq a_i$,

$$(8) \quad m_{i,n}(a) = \sum_{0 \leq \ell \leq a_k} m_{i,k}(a, \ell) m_{k,n}(\ell).$$

Proof. (i) Each individual in generation $n - 1$ reproduces independently with the same mean number of offspring m and gives birth to the newborns of generation n .
 (ii) Conditioning by the distribution of traits in generation k , we get the equality using the branching and Markov property of $(Z_n(\cdot) : n \in \mathbb{N})$. \square

Let us describe the behavior of the mean growth of the number of individuals.

Lemma 2.2. (i) For every $0 \leq k \leq n$, $m_{n-k}(0, 0) m_{n-k, n} \leq m_n$.
 (ii) $m_{n+1}/m_n \rightarrow m + 1$ as $n \rightarrow \infty$.
 More generally, for any $k \geq 0$, $m_{n-k}/m_n \rightarrow (m + 1)^{-k}$ as $n \rightarrow \infty$.
 (iii) For all $0 \leq i \leq n - 1$ and $0 \leq a \leq a_i$,

$$m m_{i+1, n} \leq m_{i, n}(a) \leq m_{i, n} \leq (m + 1)m_{i+1, n}.$$

(iv) For each $\varrho \in (1, m + 1)$, there exists a positive constant α such that for n large enough, $m_n \geq \alpha \varrho^n$.

Proof. (i) The first inequality is obtained from (8) by replacing k by $n - k$ and considering $i = 0$, $a = 0$ and focusing on newborn of generation $n - k$ ($\ell = 0$).

(ii) We first observe that from generation n to generation $n + 1$, only individuals aged a_n may die. Furthermore, each individual gives birth, on average, to m children. Thus,

$$(m + 1)m_n - m_n(0, a_n) \leq m_{n+1} \leq (m + 1)m_n.$$

This yields the upper bound. For the lower bound, note that individuals aged a_n in generation n were newborn at generation $n - a_n$, so $m_n(0, a_n) = m_{n-a_n}(0, 0)$. Thus, using (i) with $k = a_n$,

$$\frac{m_n(0, a_n)}{m_n} = \frac{m_{n-a_n}(0, 0)}{m_n} \leq \frac{1}{m_{n-a_n, n}}.$$

Moreover, $m_{n-a_n, n} = (m + 1)^{a_n}$ since starting from an individual aged 0 in generation $n - a_n$, nobody goes beyond the maximal age (and thus die) in the time interval $[n - a_n, n]$. So, each individual is replaced by $m + 1$ individuals in average. Adding that a_n goes to infinity, $m_n(0, a_n)/m_n$ goes to 0 and this yields the lower bound for (ii). By iteration, we obtain the second part of (ii).

(iii) The bound $m_{i, n}(a) \leq m_{i, n}$ comes simply from the fact that an individual with age 0 lives longer than an individual with age $a > 0$. Then (8) yields the two last inequalities recalling that $m_{i, i+1}(a, 0) = m$ and $m_{i, i+1}(a) \leq m + 1$.

(iv) Recall that $\varrho(a)$ is the greatest eigenvalue of Leslie Matrix sized $a + 1$. Since (a_n) goes to infinity, $(\varrho(a_n))_n$ goes to $m + 1$, see Lemma A.1. Then, there exists n_0 such that $\varrho(a_{n_0}) > \varrho$. Considering again the coupling with a multitype branching process with $a_{n_0} + 1$ types starting at generation n_0 with individuals aged 0 yields

$$m_n \geq m_{n_0}(0, 0)M_{n_0, n},$$

where $M_{n_0, n}$ is the mean number of individuals of this multitype branching process. Using Perron-Frobenius Theorem, there exists a positive constant c such that $M_{n_0, n} \sim c\varrho(a_{n_0})^n$, which gives the expected result. \square

2.1.3. *Proof of Theorem 1 (ii).* The proof relies on the following lemma, which focuses on a typical individual, following the inspiration of spine decomposition (see e.g. [11, 22, 21]) of branching processes. For every $0 \leq i < n$ and $0 \leq a \leq a_i$, we define

$$P_{i,n}(a, b) = m_{i,i+1}(a, b) \frac{m_{i+1,n}(b)}{m_{i,n}(a)},$$

Then by recursion we can set $Q_{n,n}(a, a) = 1$ and $Q_{n,n}(a, b) = 0$ if $a \neq b$ and

$$Q_{i,n}(a, b) := \sum_{0 \leq k \leq a_{i+1}} P_{i,n}(a, k) Q_{i+1,n}(k, b).$$

Identity (8) ensures that $P_{i,n}$ is a Markov kernel. Then, $Q_{i,n}(\cdot, \cdot)$ is the semigroup of the associated inhomogeneous Markov Chain between generations i and n . We are proving in next Lemma 2.3 that it satisfies Doeblin's condition and we are deriving geometric estimates for asymptotic behavior of the means $m_{i,n}$ via a many-to-one formula. Let us note that this technic is inspired by probabilistic methods, while the result is analytical. It can be extended for more general product of (non-negative) matrices in finite or infinite dimension using the ergodicity of the Markov chain Q .

Lemma 2.3. (*i*-Many-to-one formula). *For all $0 \leq i \leq n$ and $0 \leq a \leq a_i$,*

$$m_{i,n}(a, \cdot) = m_{i,n}(a) Q_{i,n}(a, \cdot).$$

(*ii*-Doeblin's condition). *For all $0 \leq i < n$ and $0 \leq a \leq a_i$,*

$$P_{i,n}(a, \cdot) \geq c \delta_0(\cdot),$$

where $c = m/(m+1)$ and δ_0 is the Dirac mass at point 0.

(*iii*-Geometrical Ergodicity). *For any $i \leq n$ and probability measures μ and ν on $\{0, \dots, a_i\}$,*

$$d_{TV}(Q_{i,n}(\mu, \cdot), Q_{i,n}(\nu, \cdot)) \leq (1-c)^{n-i} d_{TV}(\mu, \nu),$$

where $d_{TV}(\cdot, \cdot)$ is the total variation distance on $\mathcal{P}(\mathbb{N})$ defined by

$$d_{TV}(\mu, \nu) = \frac{1}{2} \sum_{k \in \mathbb{N}} |\mu_k - \nu_k|.$$

(*iv*) *Setting*

$$E_{i,n}(a) := \frac{m_{i,n+1}(a)}{m_{n+1}} - \frac{m_{i,n}(a)}{m_n},$$

there exists $C_m > 0$, such that for all $0 \leq i < n$, $0 \leq a \leq a_i$,

$$|E_{i,n}(a)| \leq C_m \frac{(1-c)^{n-i}}{m_i}.$$

Proof. We prove (*i*) by induction and we assume that $m_{i+1,n}(k, b) = m_{i+1,n}(k) Q_{i+1,n}(k, b)$. Using (8), we get

$$\begin{aligned} \frac{m_{i,n}(a, b)}{m_{i,n}(a)} &= \sum_{k \geq 0} \frac{m_{i,i+1}(a, k) m_{i+1,n}(k)}{m_{i,n}(a)} Q_{i+1,n}(k, b) \\ &= \sum_{k \geq 0} P_{i,n}(a, k) Q_{i+1,n}(k, b), \end{aligned}$$

which proves (*i*), since the right hand side is equal to $Q_{i,n}(a, b)$ by definition.

(ii) From (8) applied to $k = i + 1$,

$$\begin{aligned} m_{i,n}(a) &\leq m_{i,i+1}(a,0)m_{i+1,n}(0) + m_{i,i+1}(a,a+1)m_{i+1,n}(a+1) \\ &\leq mm_{i+1,n}(0) + m_{i+1,n}(a+1) \\ &\leq (m+1)m_{i+1,n}(0), \end{aligned}$$

since an individual of age 0 lives longer than an individual of age $a + 1$. Thus, thanks to Lemma 2.2 (iii),

$$P_{i,n}(a,0) = m_{i,i+1}(a,0) \frac{m_{i+1,n}(0)}{m_{i,n}(a)} = m \frac{m_{i+1,n}}{m_{i,n}(a)} \geq \frac{m}{m+1}.$$

(iii) Doeblin's condition obtained in (ii) yields a contraction for the Total Variation distance (see e.g. Lemma 4.3.13 in [10])

$$d_{TV}(P_{i,n}(\mu, \cdot), P_{i,n}(\nu, \cdot)) \leq (1-c)d_{TV}(\mu, \nu),$$

for any $0 \leq i \leq n$ and by a classical induction

$$d_{TV}(Q_{i,n}(\mu, \cdot), Q_{i,n}(\nu, \cdot)) \leq (1-c)^{n-i}d_{TV}(\mu, \nu).$$

(iv) Using again (8), we get

$$\begin{aligned} m_{i,n+1}(a) &= \sum_{0 \leq k \leq a_n} m_{i,n}(a,k)m_{n,n+1}(k) \\ &= m_{i,n}(a) \sum_{0 \leq k \leq a_n} Q_{i,n}(a,k)m_{n,n+1}(k). \end{aligned}$$

In particular,

$$m_{n+1} = m_{0,n+1}(0) = m_n \sum_{0 \leq k \leq a_n} Q_{0,n}(0,k)m_{n,n+1}(k).$$

Thus,

$$\begin{aligned} E_{i,n}(a) &= \frac{m_{i,n}(a)}{m_n} \left[\frac{\sum_{0 \leq k \leq a_n} Q_{i,n}(a,k)m_{n,n+1}(k)}{\sum_{0 \leq k \leq a_n} Q_{0,n}(0,k)m_{n,n+1}(k)} - 1 \right] \\ &= \frac{m_{i,n}(a)}{m_n} \cdot \frac{\sum_{0 \leq k \leq a_n} (Q_{i,n}(a,k) - Q_{0,n}(0,k))m_{n,n+1}(k)}{\sum_{0 \leq k \leq a_n} Q_{0,n}(0,k)m_{n,n+1}(k)}. \end{aligned}$$

Recalling that $m_{n,n+1}(k)$ is equal to m or $m + 1$ whether individual aged k belongs to \mathcal{G}_n^- or \mathcal{G}_n^+ , we get

$$|E_{i,n}(a)| \leq \frac{(m+1)m_{i,n}(a)}{mm_n} d_{TV}(Q_{i,n}(a, \cdot), Q_{0,n}(0, \cdot)).$$

Now, from the triangular inequality,

$$d_{TV}(Q_{i,n}(a, \cdot), Q_{0,n}(0, \cdot)) \leq d_{TV}(Q_{i,n}(a, \cdot), Q_{i,n}(0, \cdot)) + \sum_{k=1}^i d_{TV}(Q_{k,n}(0, \cdot), Q_{k-1,n}(0, \cdot)).$$

Then, using previous points (ii) and (iii),

$$\begin{aligned} d_{TV}(Q_{k,n}(0, \cdot), Q_{k-1,n}(0, \cdot)) &= d_{TV}(Q_{k,n}(0, \cdot), Q_{k,n}(P_{k-1,n}(0, \cdot), \cdot)) \\ &\leq (1-c)^{n-k} d_{TV}(\delta_0, P_{k-1,n}(0, \cdot)) \\ &\leq (1-c)^{n-k}, \end{aligned}$$

and from Lemma 2.2 (i) and (iii), there exists a constant $C'_m > 0$ s.t.

$$|E_{i,n}(a)| \leq C'_m (1-c)^{n-i} \frac{m_{i,n}(a)}{m_n} \leq C'_m (1-c)^{n-i} \frac{m_{i,n}}{m_n} \leq C'_m (1-c)^{n-i} \frac{1}{m_i(0,0)}.$$

Recalling that $m_i(0,0) = m_{i-1}m \geq m_i m / (m+1)$, we obtain (iv). \square

We prove now the convergence of Z_n/m_n using the L^2 moments of its successive increments. This method has already been powerfully used for branching processes and also allows to obtain estimates on the speed of convergence, see e.g. [1].

Proof of Theorem 1 (ii). To show that (Z_n/m_n) converges a.s., we prove that

$$(9) \quad \sum_{n \geq 0} \mathbf{E} \left[\left(\frac{Z_{n+1}}{m_{n+1}} - \frac{Z_n}{m_n} \right)^2 \right]^{1/2} < \infty$$

by splitting this increment into the martingale increment and the drift term. First, we recall that

$$Z_{n+1} = \sum_{u \in \mathcal{G}_n} \#X(u) \quad \text{and} \quad \mathcal{G}_n = \mathcal{G}_n^+ \sqcup \mathcal{G}_n^-.$$

For $u \in \mathcal{G}_n$, either u belongs to \mathcal{G}_n^+ and $\mathbf{E}(\#X(u) \mid \mathcal{F}_n) = m_{n,n+1}(\mathcal{Z}(u)) = m+1$, or $u \in \mathcal{G}_n^-$ and $\mathbf{E}(\#X(u) \mid \mathcal{F}_n) = m_{n,n+1}(\mathcal{Z}(u)) = m$. Then

$$(10) \quad \mathbf{E}(Z_{n+1} \mid \mathcal{F}_n) = \sum_{u \in \mathcal{G}_n} m_{n,n+1}(\mathcal{Z}(u)) = (m+1)\#\mathcal{G}_n^+ + m\#\mathcal{G}_n^-$$

and

$$m_{n+1} = (m+1)\mathbf{E}(\#\mathcal{G}_n^+) + m\mathbf{E}(\#\mathcal{G}_n^-).$$

Using Minkowski's inequality and denoting $\|\cdot\|_2$ the L^2 norm on r.v., we obtain

$$(11) \quad \begin{aligned} \left\| \frac{Z_{n+1}}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2 &= \left\| \frac{\sum_{u \in \mathcal{G}_n^+ \sqcup \mathcal{G}_n^-} \#X(u)}{m_{n+1}} - \frac{\mathbf{E}(Z_{n+1} \mid \mathcal{F}_n)}{m_{n+1}} + \frac{\mathbf{E}(Z_{n+1} \mid \mathcal{F}_n)}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2 \\ &\leq \left\| \frac{\sum_{u \in \mathcal{G}_n^+} [\#X(u) - (m+1)]}{m_{n+1}} \right\|_2 + \left\| \frac{\sum_{u \in \mathcal{G}_n^-} [\#X(u) - m]}{m_{n+1}} \right\|_2 \\ &\quad + \left\| \frac{\mathbf{E}(Z_{n+1} \mid \mathcal{F}_n)}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2. \end{aligned}$$

First remark that, when u belongs to \mathcal{G}_n^+ , then $\#X(u)$ is distributed as $R+1$. Thus, $(\#X(u) - (m+1))_{u \in \mathcal{G}_n^+}$ are independent, centered r.v. and

$$\mathbf{E} \left[\left(\sum_{u \in \mathcal{G}_n^+} [\#X(u) - (m+1)] \right)^2 \right] = \mathbf{E}[(R-m)^2] \mathbf{E}[\#\mathcal{G}_n^+].$$

Adding that the individuals in \mathcal{G}_n^+ survive, $\mathbf{E}[\#\mathcal{G}_n^+] \leq m_{n+1}$ and

$$(12) \quad \left\| \frac{\sum_{u \in \mathcal{G}_n^+} [\#X(u) - (m+1)]}{m_{n+1}} \right\|_2 \leq \frac{\sigma}{\sqrt{m_{n+1}}}.$$

In the same manner, recalling that $m_{n+1} \geq m_n m$ from Lemma 2.2 (iii),

$$(13) \quad \left\| \frac{\sum_{u \in \mathcal{G}_n^-} [\#X(u) - m]}{m_{n+1}} \right\|_2 \leq \frac{\sqrt{m_n \mathbf{E}[(R-m)^2]}}{m_{n+1}} \leq \frac{\sigma}{\sqrt{m m_{n+1}}}.$$

To study the last term, we write

$$\frac{\mathbf{E}(Z_{n+1} \mid \mathcal{F}_n)}{m_{n+1}} - \frac{Z_n}{m_n} = \sum_{u \in \mathcal{G}_n} F_n(u),$$

where using (10),

$$F_n(u) = \frac{m_{n,n+1}(\mathcal{Z}(u))}{m_{n+1}} - \frac{1}{m_n}.$$

We use now standard arguments involving the genealogy of the population to exploit the branching property in the lineages. Given two individuals u and v living in generation n , let us consider their youngest common ancestor ω at generation $i-1$. Then, individuals u and v arise from two independent branching processes starting at time i from descendants of ω . We denote pairs of siblings born at generation i :

$$\mathcal{V}_{i,n} = \{(u, v) \in \mathcal{G}_n : \exists \omega \in \mathcal{G}_{i-1}, a \succ 0, b \succ 0 \text{ s.t. } u \succ \omega a, v \succ \omega b, a_0 \neq b_0\}.$$

Then,

$$\begin{aligned} \mathbf{E} \left[\left(\sum_{u \in \mathcal{G}_n} F_n(u) \right)^2 \right] &= \mathbf{E} \left[\sum_{(u,v) \in \mathcal{G}_n^2} F_n(u) F_n(v) \right] \\ &= \sum_{i=1}^n \mathbf{E} \left[\sum_{(u,v) \in \mathcal{V}_{i,n}} F_n(u) F_n(v) \right] + \mathbf{E} \left[\sum_{u \in \mathcal{G}_n} F_n(u)^2 \right]. \end{aligned}$$

On the one side,

$$\begin{aligned} \mathbf{E} \left[\sum_{u \in \mathcal{G}_n} F_n(u)^2 \right] &\leq \frac{1}{m_n} + \frac{1}{m_{n+1}^2} \mathbf{E} \left[\sum_{u \in \mathcal{G}_n} m_{n,n+1}(\mathcal{Z}(u))^2 \right] \\ &\leq \frac{1}{m_n} + \frac{(m+1)^2 m_n}{m_{n+1}^2} \\ &\leq \left(1 + \frac{(m+1)^2}{m^2} \right) \frac{1}{m_n}, \end{aligned}$$

recalling that $m_{n+1} \geq m_n m$ from Lemma 2.2 (iii).

On the other side, using

$$\begin{aligned} E_{i,n}(a) &= \mathbf{E} \left(\sum_{u \in \mathcal{G}_n} F_n(u) \mid \mathcal{Z}_i = \delta_a \right) \\ &= \mathbf{E} \left(\frac{\sum_{u \in \mathcal{G}_n^+} (m+1) + \sum_{u \in \mathcal{G}_n^-} m}{m_{n+1}} - \frac{\#\mathcal{G}_n}{m_n} \mid \mathcal{Z}_i = \delta_a \right) \\ &= \frac{\mathbf{E}(\#\mathcal{G}_{n+1} \mid \mathcal{Z}_i = \delta_a)}{m_{n+1}} - \frac{\mathbf{E}(\#\mathcal{G}_n \mid \mathcal{Z}_i = \delta_a)}{m_n} \\ &= \frac{m_{i,n+1}(a)}{m_{n+1}} - \frac{m_{i,n}(a)}{m_n} \end{aligned}$$

and writing

$$V_{i,a,a',a''} := \mathbf{E}(Z_i(a') Z_i(a'') \mid \mathcal{Z}_{i-1} = \delta_a),$$

we condition on the ages of the individuals at generation $i - 1$ and i to get

$$\begin{aligned} & \mathbf{E} \left[\sum_{(u,v) \in \mathcal{V}_{i,n}} F_n(u) F_n(v) \right] \\ &= \mathbf{E} \left[\mathbf{E} \left[\sum_{(u,v) \in \mathcal{V}_{i,n}} F_n(u) F_n(v) \mid \mathcal{F}_i \right] \right] \\ &= \sum_{a=0}^{a_{i-1}} m_{0,i-1}(0, a) \sum_{a', a'' \in \{0, a+1\}} V_{i,a,a',a''} E_{i,n}(a') E_{i,n}(a''). \end{aligned}$$

Finally, using Lemma 2.3 (iv) and the fact that $V_{i,a,a',a''}$ is bounded by $\mathbf{E}((R+1)^2)$,

$$\left\| \frac{\mathbf{E}(Z_{n+1} \mid \mathcal{F}_n)}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2^2 \leq \mathbf{E}((R+1)^2) C_m^2 \sum_{i=1}^n \frac{(1-c)^{2(n-i)}}{m_{i+1}} + C_R \frac{1}{m_n},$$

where the constant C_R depends only on m and σ and may change from line to line. Then, splitting the sum in two parts and recalling that (m_i) goes to infinity so is bounded from below,

$$(14) \quad \left\| \frac{\mathbf{E}(Z_{n+1} \mid \mathcal{F}_n)}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2^2 \leq C_R \left[\sum_{n/2 \leq k \leq n} \frac{1}{m_k} + (1-c)^n \right].$$

Finally, putting (12), (13) and (14) in (11) yields

$$(15) \quad \left\| \frac{Z_{n+1}}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2 \leq C_R \sum_{n/2 \leq k \leq n} \frac{1}{\sqrt{m_k}} + (1-c)^{n/2}.$$

Adding that m_k grows geometrically from Lemma 2.2 (iv), ensures that the right hand side is summable and (9) holds. We get the L^2 convergence of Z_n/m_n and Cauchy Schwarz inequality also ensures that

$$(16) \quad \sum_{n \in \mathbb{N}} \mathbf{E}(|Z_{n+1}/m_{n+1} - Z_n/m_n|) < \infty.$$

Then Z_n/m_n converges a.s. and in L^1 to a non negative finite r.v. W . As $\mathbf{E}(Z_n/m_n) = 1$, $\mathbf{P}(W > 0) > 0$. Adding that on Nonext, Z_n and then $Z_n(0)$ have to go to infinity, a standard branching argument (see Lemma B.1 in Appendix for details) ensures that $\{W > 0\} = \text{Nonext}$. This ends up the proof. \square

2.1.4. *Proof of Theorem 1 (iii).* To prove the third part of Theorem 1, we use the following standard law of large numbers, in the vein of [2], whose proof is provided for safe of completeness.

Lemma 2.4. *Let $(X_n)_n$ be a sequence of integer valued r.v. We assume that for each integer n , $(X_{i,n} : i \geq 0)$ are identically distributed centered r.v., which are independent of X_n . Moreover, we assume that $(X_{i,n} : i, n \geq 0)$ are bounded in L^2 . Then,*

$$\frac{1}{X_n} \sum_{i=1}^{X_n} X_{i,n} \rightarrow 0 \quad \text{a.s. on } \left\{ \liminf_{n \rightarrow \infty} \frac{1}{n} \log(X_n) > 0 \right\}.$$

Proof. Fix $c \in (0, \infty)$ and $\epsilon > 0$ and denote by

$$E_n^{(c)} = \left\{ \left| \frac{1}{X_n} \sum_{i=1}^{X_n} X_{i,n} \right| \geq \epsilon \right\} \cap \{X_n \geq cn^2\}.$$

Using Bienaymé Tchebychev inequality and $\mathbf{E}(X_{i,n}X_{j,n} | X_n) = \mathbf{E}(X_{i,n} | X_n)\mathbf{E}(X_{j,n} | X_n) = 0$ when $i \neq j$, we get that

$$\mathbf{P}(E_n^{(c)}) \leq \mathbf{E} \left(\frac{1}{\epsilon^2 X_n} 1_{X_n \geq cn^2} \sup_i \mathbf{E}(X_{i,n}^2) \right) \leq \frac{1}{c\epsilon^2 n^2} \sup_i \mathbf{E}(X_{i,n}^2),$$

where the supremum is bounded with respect to n by assumption. As a consequence,

$$\sum_{n \geq 0} \mathbf{P}(E_n^{(c)}) < \infty$$

and $\mathbf{P}(E_n^{(c)} \text{ occurs i.o.}) = 0$ by Borel-Cantelli lemma. Then, a.s. on the event $\{\forall n \geq 0 : X_n \geq cn^2\}$,

$$\left| \frac{1}{X_n} \sum_{i=1}^{X_n} X_{i,n} \right| \xrightarrow{n \rightarrow \infty} 0.$$

Adding that

$$\left\{ \liminf_{n \rightarrow \infty} \frac{1}{n} \log(X_n) > 0 \right\} \subset \bigcup_{c \in \mathbb{Q} \cap (0,1)} \{\forall n \geq 0 : X_n \geq cn^2\}$$

ends up the proof. \square

Proof of Theorem 1 (iii). First we prove that a.s on Nonext,

$$(17) \quad \frac{Z_{n+1}(0)}{Z_n} \xrightarrow{n \rightarrow \infty} m$$

using the previous law of large numbers. Indeed, recall from equation (5b) that

$$Z_{n+1}(0) = \sum_{j=1}^{Z_n} R_{j,n+1},$$

where $(R_{j,n+1}, j \in \mathbb{N})$ are i.i.d. and distributed as R , whose variance is finite. Moreover, Theorem 1 (ii) and the fact that $m_n \geq C\varrho^n$ for some $\varrho > 1$ ensures that, on Nonext, $\liminf_{n \rightarrow \infty} \frac{1}{n} \log(Z_n) > 0$. Then Lemma 2.4 ensures that

$$\frac{Z_{n+1}(0)}{Z_n} \xrightarrow{n \rightarrow \infty} \mathbf{E}(R)$$

a.s. on Nonext. Now we just note that $Z_n(a) = Z_{n-a}(0)$ to write

$$\frac{Z_n(a)}{Z_n} = \frac{Z_{n-a}(0)}{Z_{n-a-1}} \frac{Z_{n-a-1}}{Z_n}.$$

Using Theorem 1 (ii) and Lemma 2.2 (ii) yields

$$\lim_{n \rightarrow \infty} \frac{Z_{n-a-1}}{Z_n} = \lim_{n \rightarrow \infty} \frac{m_{n-(a+1)}}{m_n} = (m+1)^{-a-1}$$

a.s. on Nonext, which ends up the proof. \square

2.1.5. *Proof of Corollary 2.* First, recalling (5), we have

$$\begin{aligned} \mathbf{E}[Z_{n+1}|\mathcal{F}_n] &= \mathbf{E}\left[\sum_{k=0}^{a_{n+1}} Z_{n+1}(k)|\mathcal{F}_n\right] \\ &= \mathbf{E}\left[\sum_{k=0}^{a_{n+1}-1} Z_n(k) + \sum_{j=0}^{Z_n} R_{j,n+1}|\mathcal{F}_n\right] \\ &= \sum_{k=0}^{a_{n+1}-1} Z_n(k) + mZ_n \\ &\leq (m+1)Z_n, \end{aligned}$$

since $a_{n+1} - 1 \leq a_n$. Thus, $M_n = (m+1)^{-n} Z_n$ is a supermartingale.

Since M_n is a non-negative supermartingale, it converges a.s. to a non-negative finite random variable M .

First, when $\liminf_{n \rightarrow \infty} a_n / \log n > 1 / \log(m+1)$, using Lemma A.3 (i), we get that m_n is of the same magnitude as $(m+1)^n$. Part (i) of Corollary 2 is then a consequence of Theorem 1.

Similarly, Lemma A.3 (ii) ensures that $m_n / (m+1)^n$ goes to 0 as $n \rightarrow \infty$ as soon as $\limsup_{i \rightarrow \infty} a_n / \log n < 1 / \log(m+1)$ and we get (ii).

2.2. Aging branching process with immigration. We now consider the aging branching process with immigration $(\mathbf{Z}_n : n \in \mathbb{N})$ defined in (4) under Assumptions (B) and (C). Recall that m_n , defined in Equation (7) and satisfying (3), is the mean number of individuals in absence of immigration.

Theorem 3. (i) *The process $(\mathbf{Z}_n / m_n : n \in \mathbb{N})$ converges a.s. to a finite positive random variable.*

(ii) *For each $a \geq 0$, $(\mathbf{Z}_n(a) / \mathbf{Z}_n : n \in \mathbb{N})$ converges a.s. to $m / (m+1)^{a+1}$.*

We remark that, contrary to the previous results, we are not needing the L^2 convergence of \mathbf{Z}_n / m_n in the sequel, whose proof seems quite technical and is not considered in this paper. Before the proof, we provide the asymptotic behavior of the cumulative number of newborn individuals until generation n , which will be useful in next Section 4:

$$\overline{\mathbf{Z}}_n(0) = \sum_{k=0}^n \mathbf{Z}_k(0).$$

Corollary 4. *The process $(\overline{\mathbf{Z}}_n(0) / m_n : n \in \mathbb{N})$ converges a.s. to a finite positive r.v.*

2.2.1. *Proof of Theorem 3.* We start with some technical additional results on the mean number m_n of individuals in generation n . Lemma 2.3 (iv) ensures that for any $k \geq 0$,

$$\sum_{n \geq k} \sum_{a \leq a_n} |E(a, k, n)| < \infty$$

since (a_n) grows at most linearly. Then

$$\sum_{n \geq k} \left| \frac{m_{k,n+1}}{m_{n+1}} - \frac{m_{k,n}}{m_n} \right| < \infty$$

and the following limit exists for $k \geq 0$,

$$(18) \quad \alpha_k = \lim_{n \rightarrow \infty} \frac{m_{k,n}}{m_n}.$$

We also remark that from Lemma 2.2 (i), $m_{k,n}/m_n \leq 1/m_k(0,0)$. Thus, $\alpha_k \leq 1/m_k(0,0)$. Adding that $m_k(0,0) \geq mm_{k-1}$, Lemma 2.2 (iv) ensures that $(\alpha_k)_k$ goes to 0 geometrically and for any $\varrho \in (1, m+1)$, there exists $C > 0$ such that

$$(19) \quad \alpha_k \leq C\varrho^k.$$

Proof of Theorem 3 (i). We use that \mathbf{Z}_n is the sum over $k \leq n$ of the descendants of immigrants arrived in generation k . We recall that notations m_n and $m_{k,n}$ are introduced in Equation (7).

Moreover, the successive number of descendants of an immigrant i in generation k is equal to an aging branching process $(Z_{k,n}^{(i)} : n \geq k)$, without immigration, started at generation k , with maximal age $(a_n)_{n \geq k}$. Thus,

$$(20) \quad \frac{\mathbf{Z}_n}{m_n} = \sum_{k=0}^n \sum_{i=1}^{I_k} \frac{m_{k,n}}{m_n} W_{k,n}^{(i)} \quad \text{a.s.},$$

where for each $k \leq n$,

$$(21) \quad W_{k,n}^{(i)} := \frac{Z_{k,n}^{(i)}}{m_{k,n}}.$$

The sequence $(W_{k,n}^{(i)}, i \geq 1)$ is a sequence of i.i.d. r.v.

Theorem 1 obtained in previous Section 2.1 ensures that for each $k \geq 0, i \geq 1$,

$$(22) \quad W_{k,n}^{(i)} \xrightarrow{n \rightarrow \infty} W_k^{(i)} \quad \text{a.s. and in } L^2.$$

We also note that $W_k^{(i)}$ are independent random variables for $k \geq 0, i \geq 1$.

Using now the estimates (15) established in the proof of Theorem 1, there exists a constant C_R (which depends neither of k, i or n) such that

$$\left\| W_{k+1,n}^{(i)} - W_{k,n}^{(i)} \right\|_2 \leq C_R \sum_{(n-k)/2 \leq i \leq n-k} \frac{1}{\sqrt{m_{k,k+i}}} + (1-c)^{(n-k)/2}.$$

Since (a_n) is non-decreasing, $m_{k,k+i} \geq m_i$ and m_i grows geometrically. Thus,

$$C_k := \sum_{n \geq k} \left\| W_{k+1,n}^{(i)} - W_{k,n}^{(i)} \right\|_2$$

is bounded for $k \geq 0$. Then, by Cauchy Schwarz inequality,

$$\sup_k \mathbf{E} \left(\sum_{n \geq k} |W_{k+1,n}^{(i)} - W_{k,n}^{(i)}| \right) < \infty$$

and

$$(23) \quad C := \sup_{k \geq 0} \mathbf{E}(\sup_{n \geq k} W_{k,n}^{(i)}) < \infty.$$

We can now prove that

$$\frac{\mathbf{Z}_n}{m_n} \xrightarrow{n \rightarrow \infty} S := \sum_{k=0}^{+\infty} \sum_{i=1}^{I_k} \alpha_k W_k^{(i)}.$$

From the triangular inequality and (20), for $1 \leq K \leq n$,

$$\begin{aligned} \left| \frac{\mathbf{Z}_n}{m_n} - S \right| &\leq \sum_{k=0}^{K-1} \sum_{i=1}^{I_k} \left| \frac{m_{k,n}}{m_n} W_{k,n}^{(i)} - \alpha_k W_k^{(i)} \right| \\ &\quad + \sum_{k=K}^n \sum_{i=1}^{I_k} \frac{m_{k,n}}{m_n} W_{k,n}^{(i)} + \sum_{k=K}^{+\infty} \sum_{i=1}^{I_k} \alpha_k W_k^{(i)}. \end{aligned}$$

The first term goes a.s. to 0 from the definitions of α_k given in (18) and $W_k^{(i)}$ in (21). The last term is the rest of a convergent serie that goes to 0 with K . This can be seen taking its expectation and combining (C) and (19). Then,

$$\limsup_{n \rightarrow +\infty} \left| \frac{\mathbf{Z}_n}{m_n} - S \right| \leq \mathcal{R},$$

where \mathcal{R} is the limit of the non-negative and non-increasing sequence

$$(24) \quad \mathcal{R}_K := \sup_{n \geq K} \sum_{k=K}^n \sum_{i=1}^{I_k} \frac{m_{k,n}}{m_n} W_{k,n}^{(i)}.$$

Thus, it remains to show that $\mathcal{R} = 0$. But,

$$\begin{aligned} \mathbf{E}[\mathcal{R}_K] &\leq \mathbf{E} \left[\sum_{k \geq K} \sum_{i=1}^{I_k} \frac{1}{m_k(0,0)} \sup_{n \geq k} W_{k,n}^{(i)} \right] \\ &\leq \sum_{k \geq K} \frac{\mathbf{E}[I_k]}{m_k(0,0)} \mathbf{E}[\sup_{n \geq k} W_{k,n}^{(i)}] \\ &\leq C \sum_{k \geq K} \frac{\mathbf{E}[I_k]}{m_k(0,0)}, \end{aligned}$$

where constant C is defined above in Equation (23). Finally, since $m_k(0,0)$ goes to infinity faster than ϱ^k for every $\varrho \in (1, m+1)$, Assumption (C) on I_k ensures that the rest of the serie goes to 0 and $\mathcal{R} = 0$ a.s. Then \mathbf{Z}_n/m_n converges a.s. to a finite random variable \mathbf{W} which is non-negative (see Appendix B). \square

Proof of Theorem 3 (ii). Following the proof of Theorem 1 (ii), we only prove that $\mathbf{Z}_{n+1}(0)/\mathbf{Z}_n$ goes to m as n goes to ∞ . First we recall from (4) that

$$\frac{\mathbf{Z}_{n+1}(0)}{\mathbf{Z}_n} = \frac{1}{\mathbf{Z}_n} \sum_{j=1}^{\mathbf{Z}_n} R_{j,n+1} + \frac{I_n}{\mathbf{Z}_n}.$$

From (i), the process $(\mathbf{Z}_n : n \in \mathbb{N})$ satisfies the assumptions of the law of large numbers in Lemma 2.4 and $\frac{1}{\mathbf{Z}_n} \sum_{j=1}^{\mathbf{Z}_n} R_{j,n+1}$ goes to m a.s.

It remains to show that I_n/m_n goes to 0. To do so, we use Borel-Cantelli Lemma. Indeed, for any $\varepsilon > 0$,

$$\mathbf{P}(I_n/m_n \geq \varepsilon) \leq \frac{\mathbf{E}(I_n)}{\varepsilon m_n}.$$

Moreover Assumption (C) ensures that there exists $\ell < m+1$ such that for n large enough, $\mathbf{E}[I_n] \leq \ell^n$. On the other side, from mean behavior Lemma 2.2, for $\varrho \in (\ell, m+1)$ and n large enough, there exists a positive constant α such that $m_n \geq \alpha \varrho^n$. Thus, $\sum_n \mathbf{P}(I_n/m_n \geq \varepsilon) < \infty$ and I_n/m_n goes to 0 a.s. This ends up the proof. \square

2.2.2. *Proof of Corollary 4.* The proof is in the same vein as the previous one and we just give the main lines. We note that

$$\begin{aligned} \frac{\bar{Z}_n(0)}{m_n} &= \frac{Z_0(0)}{m_n} + \sum_{k=0}^{n-1} \frac{Z_{n-k}(0)}{m_n} \\ &= \frac{Z_0(0)}{m_n} + \sum_{k=0}^{n-1} \frac{Z_{n-k}(0)}{m_{n-k-1}} \frac{m_{n-k-1}}{m_n}. \end{aligned}$$

From previous Theorem 3 (ii) and writing S the limit of Z_n/m_n , for each $k \geq 0$,

$$\lim_{n \rightarrow \infty} \frac{Z_{n-k}(0)}{m_{n-k-1}} = \lim_{n \rightarrow \infty} \frac{Z_{n-k}(0)}{Z_{n-k-1}} \frac{Z_{n-k-1}}{m_{n-k-1}} = mS.$$

Recalling from Lemma 2.2 (ii) that $m_{n-k-1}/m_n \rightarrow (m+1)^{-k-1}$ as $n \rightarrow \infty$, we obtain, for each $K > 0$,

$$\lim_{n \rightarrow \infty} \sum_{k=0}^K \frac{Z_{n-k}(0)}{m_n} = Sm \sum_{k=0}^K (m+1)^{-k-1} = (1 - 1/(m+1)^{K+1})S.$$

Then, letting $M := \sup_{n \geq 0} \frac{Z_n}{m_n}$,

$$\sum_{k=K}^{n-1} \frac{Z_{n-k}(0)}{m_n} \leq M \sum_{k=K}^n \frac{m_{n-k}}{m_n} \leq M \sum_{k=K}^n \frac{1}{m_k(0,0)}$$

using Lemma 2.2 (i). Using $M < \infty$ a.s. from the previous Theorem and $m_k(0,0) = mm_{k-1} \geq m\alpha q^k$ from Lemma 2.1 (i) and Lemma 2.2 (iii), we get

$$\lim_{K \rightarrow \infty} \sup_{n \geq K} \sum_{k=K}^{n-1} \frac{Z_{n-k}(0)}{m_n} = 0.$$

This proves the a.s. convergence.

3. FROM THE MOTION OF ONE SINGLE BUS TO AGING BRANCHING PROCESS

Links between polling systems and multitype branching processes with immigration have been successfully used for a finite number of queues, see e.g. [25, 29, 32, 31, 33]. We specify in this section the link between the bus line from Section 1.1 and the aging branching process from Section 1.3. To simplify notations, we get rid of the dependance of the bus number and will denote respectively M_n and F_n the labels of the first and the last customers getting on the bus at the station n . Then, H_n and P_t are the time the bus leaves the station n and its position at time t .

We write $R(t)$ the number of customers getting on the bus in a station when customers have been queuing during t units of time before the bus arrival. The number of customers waiting at the station when the bus arrives is distributed as a Poisson distribution with parameter αt . During the time these customers get on the bus, new customers may arrive and while these customers get on the bus, other customers arrive. . . We get a subcritical Galton-Watson Process whose reproduction law is a Poisson law with parameter α . Since $\alpha < 1$, it will become extinct a.s. in finite time. Then, the number $R(t)$ of customers the bus takes at this station is a finite random variable, which is distributed as the total size of a subcritical branching process whose initial distribution is a Poisson r.v. with parameter αt .

First and second moments of this r.v. $R(t)$ can be found in Appendix C. Here, a key role is played by the mean number m of customers getting on the bus due to the time the bus has waited in a station while one customer was getting on:

$$m = \mathbf{E}[R(1)] = \frac{\alpha}{1 - \alpha}.$$

To construct the aging branching process with immigration associated with the bus queuing defined in (1), we define the number of newborns $\mathbf{Z}_n(0)$ in generation n as the number of customers getting on the bus at the station n :

$$\mathbf{Z}_n(0) = M_n - F_n + 1 \quad (n \geq 1).$$

Letting $\mathbf{Z}_0(0) = 0$, we observe that (1) ensures that

$$(25) \quad H_n = n\tau + \sum_{i=0}^n \mathbf{Z}_i(0).$$

Then, we introduce the maximal age

$$a_0 = 0, \quad a_n = d_{n+1} - 2 \quad (n \geq 1)$$

and we set, for any $a \leq a_n$,

$$\mathbf{Z}_n(a) = \mathbf{Z}_{n-a}(0)$$

while $\mathbf{Z}_n(a) = 0$ if $a > a_n$. Observe also that $a_1 = 1$. Moreover,

$$\mathbf{Z}_n = \sum_{i=n-a_n}^n \mathbf{Z}_i(0) = \sum_{i=0}^{a_n} \mathbf{Z}_n(i),$$

The following result shows that \mathbf{Z}_n is an aging branching process with immigration.

Proposition 1. *For every $n \geq 1$,*

$$\mathbf{Z}_n(0) = \sum_{j=1}^{\mathbf{Z}_{n-1}} R_{j,n} + I_n,$$

where $(R_{j,n}, I_n : j, n \in \mathbb{N})$ are independent r.v., $R_{j,n}$ are distributed as $R(1)$ and I_n is distributed as $R(d_n\tau)$.

Thus, $H_n - n\tau$ is the cumulated sum of the number of newborn until time n for an aging branching process with maximal age (a_n) , with reproduction law $R(1)$ and with immigration in generation n distributed as $R(d_n\tau)$.

Proof of Proposition 1. The number customers who get on the bus at the station n can be split as

$$\mathbf{Z}_n(0) = N_n + I_n$$

where

- N_n is the size of the population of customers in station n who arrived when the bus was staying in the stations $i = n - d_n + 1, \dots, n - 1$ and their descendants, i.e. customers who arrived at the station n while these customers are getting on the bus and so on...
- I_n is the size of the population of customers who arrived at the station n during the time intervals when the bus was traveling between two stations i and $i + 1$, where $i = n - d_n, \dots, n - 1$, and their descendants.

Let us first characterize N_n . The cumulative time the bus spent among the stations $i = n - d_n + 1, \dots, n - 1$ is equal to the number of customers who got on the bus in one of these stations, so it is equal to

$$\sum_{k=n-d_n+1}^{n-1} \mathbf{Z}_k(0) = \mathbf{Z}_{n-1}.$$

We write $R_{j,n}$ the number of customers getting on the bus in station n whose arrival is associated to the j th unit time interval spent by the bus among the stations $i = n - d_n + 1, \dots, n - 1$. Then the r.v. $R_{j,n}$ are distributed as $R(1)$ and conditionally on \mathbf{Z}_{n-1} , $(R_{j,n} : j = 1, \dots, \mathbf{Z}_{n-1})$ is independent of $(R_{j,k} : j \geq 0, k < n)$ since arrivals are independent in each station. Moreover,

$$N_n = \sum_{j=1}^{\mathbf{Z}_{n-1}} R_{j,n}, \quad N_n \stackrel{d}{=} R(\mathbf{Z}_{n-1})$$

We give now the distribution of I_n . The cumulative time spent by the bus between the stations $i = n - d_n, \dots, n$ is equal to $d_n \tau$. Then the number of customers I_n who are getting on the bus in the station n and who are associated to this cumulative time $d_n \tau$ is distributed as $R(d_n \tau)$. Finally, I_n is independent of $(R_{k,n} : k, n \geq 0)$ using the Poisson structure of arrivals in station n .

Summing the sizes of these two subpopulations gives the result. \square

4. QUEUEING SYSTEM FOR A BUS LINE

We combine now the results of the two previous sections to study the time departures $(H_n)_n$ and motion $(P_t)_t$ of one single bus and then of two buses on the same line. Recall the connection $a_n = d_{n+1} - 2$ between the maximal age and the queueing discipline.

4.1. The motion of one single bus. We prove the first statement given in Introduction using aging branching process with immigration. In the sequel, $d \geq 2$ is an integer and we denote $\mathbf{a} = d - 2 \geq 0$.

Theorem 5. (i) *If $d_n = d \wedge n$ and $\varrho(d - 2) < 1$, then there exists $c > \tau$ such that*

$$H_n \sim_{n \rightarrow \infty} cn \quad a.s.$$

Moreover, $(H_n - nc)/\sqrt{n}$ converges weakly to a Gaussian r.v. with positive variance, as $n \rightarrow \infty$.

(ii) *If $d_n = d \wedge n$ and $\varrho(d - 2) > 1$ or $d_n \rightarrow \infty$, then there exists a finite positive random variable W s.t.*

$$H_n \sim_{n \rightarrow \infty} m_n W \quad a.s.$$

Proof. For (i), let $d_n = d \wedge n$ and $\varrho(d - 2) < 1$. The process defined for $n \in \mathbb{N}$ by $\mathcal{Z}_n = (\mathbf{Z}_n(a) : 0 \leq a \leq \mathbf{a})$ is a multitype branching process (with a finite number of types) and immigration I_n in generation n distributed as $R((d \wedge n)\tau)$. Let M be the reproduction matrix associated to this process (see Appendix A) and $\bar{I} = (\mathbf{E}(R(d\tau)), 0, \dots, 0)^*$. Since $\varrho(\mathbf{a}) < 1$, Theorems 1.2 and 1.3 respectively in [26] ensure that,

$$\frac{\sum_{k=0}^n \mathcal{Z}_k}{n} \xrightarrow[n \rightarrow \infty]{} b := \sum_{k=0}^{\infty} M^k \bar{I} \quad a.s.$$

and $(\sum_{k=0}^n \mathcal{Z}_k - nb)/\sqrt{n}$ converges in distribution to a Gaussian r.v. Then we use this limit for the first coordinate of $\sum_{k=0}^n \mathcal{Z}_k$, which is $\sum_{k=0}^n \mathbf{Z}_k(0)$ that goes to b_1 . We get that H_n/n goes a.s. to $\tau + b_1$ and the central limit theorem, thanks to its description (25).

The point (ii) for $a_n \rightarrow \infty$ is a direct consequence of Corollary 4. Indeed, using the notation from this corollary for the aging branching process \mathbf{Z} and (25),

$$H_n = n\tau + \overline{\mathbf{Z}}_n.$$

Moreover, from Proposition 1, the immigration term I_n is distributed as $R(d_n\tau)$, so

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log(\mathbf{E}(I_n)) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log(d_n\tau m) = 0 < m + 1,$$

since d_n is bounded by n . Finally, since $d_n \geq 2$,

$$\mathbf{P}(I_n \geq x) = \mathbf{P}(R(d_n\tau) \geq x) \geq \mathbf{P}(R(\tau) \geq x).$$

We can now apply Corollary 4 to complete the proof.

The point (ii) when $d_n = d \wedge n$ and $\varrho(d-2) > 1$ can be handled similarly and is simpler since $\mathcal{Z}_n = (\mathbf{Z}_n(a), 0 \leq a \leq \mathbf{a})$ is a multitype branching process (with a finite number of types). The counterpart Theorem 1 (i-ii-iii) is actually the classical Kesten Stigum theorem in the supercritical regime, see Theorem 1, Chapter V - Section 6 in [5]; while the proofs of Theorem 3 and Corollary 4 in that finite dimension cases are the same. That completes the proof. \square

Remark 1. We stress that [26] deals with multitype branching process in random environment with immigration and we just apply here Theorems 1.2 and 1.3 in the constant environment case, where the assumptions can be easily verified since we have L^2 moment assumptions. The results of next Corollary 6 could be extended to the case of a_n random, which follows a stationary ergodic sequence. Let us note that in that case b can have infinite components and new regimes appear, which could be of interest for a future work.

Corollary 6. (i) If $d_n = d \wedge n$ and $\varrho(d-2) < 1$, then

$$P_t \sim_{t \rightarrow \infty} \frac{t}{c} \quad \text{a.s.}$$

(ii) If $d_n = d \wedge n$ and $\varrho(d-2) > 1$ or $d_n \rightarrow \infty$, then

$$\limsup_{t \rightarrow \infty} |P_t - u_t| < \infty \quad \text{a.s.},$$

where $u_t = \min(n \in \mathbb{N} : m_n \geq t)$ is the inverse function of m_n .

Remark 2. In the case (ii), let us note that m_n tends to infinite, so u_t is well defined (finite) and goes to infinity. If $d_n = d \wedge n$ and $\varrho(d-2) > 1$, then $u_t \sim \log(t)/\log(\varrho(d-2))$. If $\liminf_{n \rightarrow \infty} d_n/\log n > 1/\log(m+1)$, then m_n is of the same magnitude as $(m+1)^n$, so $u_t \sim \log(t)/\log(m+1)$ when $t \rightarrow \infty$.

Proof. The point (i) is a direct consequence of Theorem 5 (i).

For the point (ii), we first use Theorem 5 (ii) to get that

$$H_n \sim_{n \rightarrow \infty} m_n W \quad \text{a.s.}$$

with $W \in (0, \infty)$ a.s. We recall from Lemma 2.2 (ii) that

$$\liminf_{n \rightarrow +\infty} \frac{m_{n+1}}{m_n} > 1.$$

To conclude, we prove now that for any sequences v_n and \tilde{v}_n , if there exists $c > 0$ such that

$$v_n \sim c\tilde{v}_n \quad (n \rightarrow \infty), \quad \lim v_n = +\infty, \quad \liminf \tilde{v}_{n+1}/\tilde{v}_n > 1,$$

then $|\tilde{u}_t - u_t|$ is bounded where u_t (resp. \tilde{u}_t) is the inverse function of v_n (resp. \tilde{v}_n). Indeed, there exists $\ell > 1$, $c_1, c_2 > 0$ such that for n large enough,

$$\frac{\tilde{v}_{n+1}}{\tilde{v}_n} > \ell, \quad \frac{v_{n+1}}{v_n} > \ell, \quad c_1\tilde{v}_n \leq v_n \leq c_2\tilde{v}_n.$$

Then, for t large enough, $v_{u_t-1}/\tilde{v}_{\tilde{u}_t} \geq c_1\tilde{v}_{u_t-1}/\tilde{v}_{\tilde{u}_t}$ and if additionally $u_t \geq \tilde{u}_t + 1$,

$$\frac{v_{u_t-1}}{\tilde{v}_{\tilde{u}_t}} \geq c_1\ell^{u_t-1-\tilde{u}_t}.$$

Furthermore, from the definition of the inverse functions, $v_{u_t-1}/\tilde{v}_{\tilde{u}_t} \leq t/t = 1$, so

$$1 \leq u_t - \tilde{u}_t \leq -\frac{\log c_1}{\log \ell} + 1.$$

which end up the proof, since the case $u_t - 1 \leq \tilde{u}_t$ is handled by symmetry. \square

4.2. Coalescence criteria for two buses with the same discipline. We can now determine when the two buses defined in the introduction coalesce a.s. The proof relies on a coupling argument, which enables to compare these buses to two independent buses following the same discipline, whose behavior are described in the previous part.

Theorem 7. (i) If $d_n = d \wedge n$ and $\varrho(d-2) < 1$, then $\mathbf{P}(N < \infty) = 1$.
 (ii) If $d_n = d \wedge n$ and $\varrho(d-2) > 1$, then $\mathbf{P}(N = \infty) > 0$.
 (iii) If $d_n \rightarrow \infty$ as $n \rightarrow \infty$, then $\mathbf{P}(N < \infty) = 1$.

First notice that the number of customers waiting for the bus 2 at the successive stations is stochastically less than the number of customers for the bus 1, since the customers obey the same discipline for the two buses, but some of them may have taken the bus 1 instead of waiting for the bus 2. Then, recalling that μ is the initial time delay between the two buses and using the Poissonian structure of arrivals, we have for each $n \geq 0$,

$$(26) \quad \{N > n\} \subset \{H_n^{(2)} \leq \tilde{H}_n^{(1)} + \mu\} \quad \text{a.s.},$$

where $(\tilde{H}_n^{(1)} : n \in \mathbb{N})$ and $(H_n^{(1)} : n \in \mathbb{N})$ are independent and identically distributed. Roughly speaking, the proof works now as follows. In the first case (i) the two buses move linearly, but the fluctuations of their positions make them coalesce. This can already be seen on the two independent buses. In the second case, the two buses may keep a gap larger than the discipline, which again can be seen using the positions of two independent buses. An event with an initial gap large enough is introduced for the coupling. Finally, in case (iii), the second bus will be closer to the first bus than an independent bus would be; this ensures that the distance between them is bounded, so the second has much less customers to take than the first one; the two buses eventually coalesce.

Proof of (i). In this regime, we can use the central limit theorem of Theorem 5 (i) for $(\tilde{H}_n^{(1)} : n \in \mathbb{N})$ and $(H_n^{(1)} : n \in \mathbb{N})$. So there exists $n_1 \in \mathbb{N}$ such that

$$\mathbf{P}(\tilde{H}_{n_1}^{(1)} + \mu < n_1 c) \geq 1/4 \quad \text{and} \quad \mathbf{P}(H_{n_1}^{(1)} > n_1 c) \geq 1/4.$$

Thus, the coupling (26) ensures that

$$\mathbf{P}(N \leq n_1) \geq \mathbf{P}(\tilde{H}_{n_1}^{(1)} + \mu < n_1 c, H_{n_1}^{(1)} > n_1 c) \geq 1/16.$$

We observe that μ can be taken random in the previous inequality, as soon as it is independent of $(\mathbf{T}(i) : i \geq 1)$. Moreover this result also holds if the discipline for the second bus is equal to d for any station n , instead of $d \wedge n$, which means that any customer arriving in the stations $i \in \{0, \dots, d\}$ before the arrival of bus 2 stays in the queue. Indeed, this simply adds some customers for bus 2 in the first d stations, which yields an additional immigration term (only) in the first d generations of the branching structure, with finite second moment.

We can then iterate the procedure and choose $n_2 \in \mathbb{N}$ such that

$$\mathbf{P}(N \leq n_2 | N \geq n_1) \geq 1/16$$

and then inductively find $n_k \geq n_{k-1}$ for $k \geq 3$ such that

$$\mathbf{P}(N \geq n_k) \leq (1 - 1/16)^k.$$

Letting k go to infinity ends up the proof. \square

Proof of (ii). We show that

$$(27) \quad \mathbf{P}(\exists t_0 > 0, \forall t \geq t_0 : P_t^{(1)} - P_t^{(2)} > d) > 0,$$

so that the two buses can travel with the same discipline d after time t_0 , independently, as if they were on two different lines. We introduce the stopping time

$$T_d := \min(t \geq H_1^{(2)}, P_t^{(1)} - P_t^{(2)} \leq d),$$

with $\min \emptyset = \infty$ and we aim at proving that $\mathbf{P}(T_d = \infty) > 0$. First notice that, before time T_d , the bus 2 is independent of the bus 1 and with the same dynamics. The motion of the bus 2 is indeed not (yet) accelerated by the fact that customers are taken by the bus 1 instead of waiting for the bus 2. Thus,

$$P_{t-\mu}^{(2)} \stackrel{d}{=} P_t^{(1)}, \quad \text{for } t \in [\mu, T_d].$$

By Corollary 6 (ii), there exists an integer ℓ such that

$$(28) \quad \mathbf{P}\left(\exists t \geq 0, |P_t^{(1)} - u_t| \geq \ell/2\right) \leq 1/2.$$

We introduce the bus $\hat{P}^{(1)}$ which takes the customers waiting at the stations $i \geq d + \ell$ with discipline $d_i = d \wedge i$, but those arrived at the station $d + \ell + j$ before time $\tau(d + \ell)$ for any $j \leq d$. It is defined recursively as follows. For any $i \geq 1$,

$$\begin{aligned} \hat{H}_0^{(1)} &= 0 \text{ a.s.} \\ \hat{F}_i^{(1)} &= \min\left(j \geq 1 : T_j^{(d+\ell+i)} > \hat{H}_{i-d_i}^{(1)} + \tau(d + \ell)\right) \\ \hat{\mathcal{I}}_i^{(1)} &= \{k \in \mathbb{N} : \forall j \leq k, T_{\hat{F}_i^{(1)}+j}^{(d+\ell+i)} \leq \hat{H}_{i-1}^{(1)} + \tau + j\} \\ \hat{M}_i^{(1)} &= \hat{F}_i^{(1)} - 1 + \#\hat{\mathcal{I}}_i^{(1)} \\ \hat{H}_i^{(1)} &= \hat{H}_{i-1}^{(1)} + \tau + (\hat{M}_i^{(1)} - \hat{F}_i^{(1)} + 1). \end{aligned}$$

By independence and stationarity of the Poisson Point Processes,

$$\hat{P}^{(1)} \stackrel{d}{=} P^{(1)}$$

Let N be a Poisson random variable of parameter $\alpha\tau$ which is independent of the process of time arrivals $(\mathbf{T}^{(i)} : i \geq 1)$ of customers. Without loss of generality, we assume that τ is an integer; otherwise one can replace τ by an integer larger than τ . Denote by E_ℓ the event

$$E_\ell = \bigcap_{1 \leq i \leq d+\ell} \{T_1^{(i)} \geq i\tau\} \bigcap_{d+\ell+1 \leq i \leq 2d+\ell} \{T_1^{(i)} \geq (d+\ell)\tau\} \bigcap \{T_1^{(1)} > \mu, T_{(d+\ell)\tau+N}^{(1)} \leq \mu+\tau\}.$$

which is the event when

- there are no customers at the stations $i = 1, 2, \dots, d + \ell$ when the bus 1 arrives in these stations;
- there are no arrivals of customers at the stations $i = d + \ell + 1, \dots, 2d + \ell$ before the bus arrives at the station $d + \ell$;
- the number of customers at the station 1 arriving during the time interval $[\max(\mu, \tau), \mu + \tau]$ (so between the arrival of buses 1 and 2) is at least $(d + \ell)\tau + N$.

We first observe that E_ℓ has a positive probability and a.s. on E_ℓ , we have

$$H_{d+\ell}^{(1)} = (d + \ell)\tau.$$

Adding that $\hat{P}^{(1)}$ and E_ℓ are constructed by restriction of the Poisson Point Process of arrivals to two disjoint domains, $\hat{P}^{(1)}$ and E_ℓ are independent. Moreover, by construction, a.s. on the event E_ℓ , for any $t \geq 0$,

$$P_{t+(d+\ell)\tau}^{(1)} = \hat{P}_t^{(1)} + d + \ell.$$

Let $\hat{P}^{(2)}$ be a new bus, alone on the line. It starts from the station 0 at time μ , takes exactly N customers in the station 1 and then takes the customers waiting at the stations $i \geq 2$ which are given by the Poisson Point Process $(\mathbf{T}^{(i)} - (d+\ell)\tau : i \geq 2) \cap [0, \infty)$ with discipline d_i (associated to a_i). Then, $(\hat{P}_{\mu+t}^{(2)})_{t \geq 0}$ is distributed as $(P_t^{(1)})_{t \geq 0}$. Finally, since the bus $\hat{P}^{(2)}$ has the same customers than $P^{(2)}$ to take at the stations $i \geq 2$ before time T_d and $(d + \ell)\tau$ customers less to take in the station 1 on the event E_ℓ , we have

$$P_{t+(d+\ell)\tau}^{(2)} = \hat{P}_t^{(2)} \quad \text{a.s. on } E_\ell$$

for any $t \in [H_1^{(2)}, T_d]$. Thus

$$\begin{aligned} & \{E_\ell, \forall t \geq 0, \hat{P}_t^{(1)} - \hat{P}_t^{(2)} > -\ell\} \\ &= \{E_\ell, \forall t \geq (d + \ell)\tau, (\hat{P}_{t-(d+\ell)\tau}^{(1)} + d + \ell) - \hat{P}_{t-(d+\ell)\tau}^{(2)} > d\} \\ &\subset \{E_\ell, \forall t \geq (d + \ell)\tau, P_t^{(1)} - P_t^{(2)} > d, T_d = \infty\} \subset \{T_d = \infty\}. \end{aligned}$$

Finally,

$$\mathbf{P}(T_d = \infty) \geq \mathbf{P}(E_\ell)\mathbf{P}(\forall t \geq 0, \hat{P}_t^{(1)} - \hat{P}_t^{(2)} > -\ell),$$

and

$$\begin{aligned} & \mathbf{P}(\forall t \geq 0, \hat{P}_t^{(1)} - \hat{P}_t^{(2)} > -\ell) \\ & \geq 1 - \mathbf{P}(\exists t \geq 0 : \hat{P}_t^{(1)} - u_t \leq -\ell/2 \text{ or } u_t - \hat{P}_t^{(2)} \leq -\ell/2) \\ & \geq 1 - \mathbf{P}(\exists t \geq 0 : |P_t^{(1)} - u_t| \geq \ell/2) > 0 \end{aligned}$$

using (28). We obtain that $\mathbf{P}(T_d = \infty) > 0$, which ends up the proof. \square

Proof of (iii). We use again the coupling (26) and Corollary 6 (ii) to get

$$\limsup_{t \rightarrow \infty} P_t^{(1)} - P_t^{(2)} \leq \limsup_{t \rightarrow \infty} P_t^{(1)} - \tilde{P}_t^{(1)} < \infty \quad a.s.$$

Then the non-decreasing sequence of events

$$A_K := \{\forall t > 0 : P_t^{(1)} - P_t^{(2)} \leq K\}$$

satisfies $\mathbf{P}(\cup_{K \in \mathbb{N}} A_K) = 1$. Moreover, a coupling for the second bus ensures that

$$\mathbf{P}(A_K, \forall t > 0 : P_t^{(1)} > P_t^{(2)}) \leq \mathbf{P}(\forall t > 0 : P_t^{(1)} > \hat{P}_t^{(2)}),$$

where $\hat{P}^{(2)}$ is independent of $P^{(1)}$ and gives the position of a single bus on the line with discipline $\widehat{d}_n = K \wedge n$. Using again Corollary 6 (ii) with this discipline $\widehat{d}_n = K \wedge n$, we obtain a logarithm speed $u_t^{(2)} \sim \log(t)/\log(\varrho(K-2))$ for $\hat{P}^{(2)}$ as $t \rightarrow \infty$, see Remark 2. Adding that $P^{(1)}$ goes slower and more precisely that $u_t^{(1)} \sim \log(t)/\log(\varrho)$ as $t \rightarrow \infty$ for some $\varrho > \varrho(K-2)$, we get

$$\lim_{t \rightarrow \infty} \hat{P}_t^{(2)} - P_t^{(1)} = \infty \quad a.s.$$

So $\mathbf{P}(A_K, \forall t > 0 : P_t^{(1)} > P_t^{(2)}) = 0$ and letting $K \rightarrow \infty$,

$$\mathbf{P}(\forall t > 0 : P_t^{(1)} > P_t^{(2)}) = 0,$$

which ends up the proof. \square

APPENDIX A. MEAN BEHAVIORS OF AGING BRANCHING PROCESS

Let $\varrho(a)$ be the greatest eigenvalue of the squared Leslie's matrix of size $a+1$ governing a population with maximal age a

$$\begin{pmatrix} m & m & \cdots & m \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix},$$

i.e. the greatest root of $P_a = X^{a+1} - m \sum_{k=0}^a X^k$. Thus,

$$(29) \quad \varrho(a)^{a+1} = m \sum_{k=0}^a \varrho(a)^k.$$

Obviously, $(\varrho(a))_{a \geq 1}$ is non-decreasing. Let us specify its asymptotic behavior. When a goes to infinity, individuals become immortal and we recover a Galton-Watson process with mean $m+1$. We give here the second order term.

Lemma A.1. *When a goes to infinity,*

$$\varrho(a) = (m+1) - m(m+1)^{-a-1} + O(a(m+1)^{-2a}).$$

Proof. Notice that $P_a(X) = Q_a(X)/(X-1)$ where $Q_a(X) = X^{a+1}(X-m-1)+m$. Adding that $Q'_a((a+1)(m+1)/(a+2)) = 0$ and $Q_a(m+1) > 0$ yields

$$\frac{a+1}{a+2}(m+1) \leq \varrho(a) \leq m+1.$$

Thus, $\varrho(a)$ goes to $m+1$.

Writing $\varrho(a) = (m+1)(1-\varepsilon_a)$ and raising the previous inequality at exponent $a+1$ yields

$$\left(1 - \frac{1}{a+2}\right)^{a+1} \leq (1-\varepsilon_a)^{a+1} \leq 1$$

and $((1-\varepsilon_a)^{a+1} : a \geq 1)$ is bounded. Since $Q_a(\varrho(a)) = 0$, then $(m+1)^{a+2}(1-\varepsilon_a)^{a+1}\varepsilon_a = m$. First,

$$(a+1)\varepsilon_a = m \cdot (a+1) \cdot (m+1)^{-a-2} \cdot (1-\varepsilon_a)^{-a-1} \rightarrow 0.$$

Then,

$$\varepsilon_a(m+1)^{a+2} = me^{-(a+1)\ln(1-\varepsilon_a)} \rightarrow m.$$

Going one step further,

$$\begin{aligned} \varepsilon_a(m+1)^{a+2} - m &= me^{(a+1)\varepsilon_a + o((a+1)\varepsilon_a)} \\ &= m(a+1)\varepsilon_a + o((a+1)\varepsilon_a) \sim m^2(a+1)(m+1)^{-a-2}. \end{aligned}$$

Finally,

$$(m+1) - \varrho(a) - m(m+1)^{-a-1} \sim m^2(a+1)(m+1)^{-2a-3}$$

as $a \rightarrow \infty$. □

Lemma A.2. (i) If $\sum(m+1)^{-a_i} = +\infty$, then

$$\prod_{i=1}^n \varrho(a_i) = o((m+1)^n) \quad (n \rightarrow \infty).$$

(ii) If $\sum(m+1)^{-a_i} < +\infty$, then there exists $\alpha > 0$ s.t.

$$\prod_{i=1}^n \varrho(a_i) \sim \alpha(m+1)^n \quad (n \rightarrow \infty).$$

Proof. Let us write

$$(m+1)^{-n} \prod_{i=1}^n \varrho(a_i) = \exp \left\{ \sum_{i=1}^n \log \frac{\varrho(a_i)}{m+1} \right\}.$$

Using the expansion from Lemma A.1, recall that $(m+1)^{-1}\varrho(a_i) = 1 - m \cdot (m+1)^{-a_i-2} + O(a_i(m+1)^{-2a_i})$ and its logarithm behaves like $(m+1)^{-a_i}$. □

Lemma A.3. (i) If $\sum(m+1)^{-a_i} < +\infty$, then there exists $\beta > 0$ s.t.

$$\beta(m+1)^n \leq m_n \leq (m+1)^n.$$

(ii) If $\limsup_{i \rightarrow \infty} a_i / \log i < 1 / \log(m+1)$, then

$$m_n = o((m+1)^n).$$

Remark 3. The first condition is valid as soon as $\liminf a_i / \log i > 1 / \log(m+1)$.

Proof of (i) - Upper bound. Branching process $(Z_n : n \in \mathbb{N})$ is stochastically bounded by a Galton-Watson process with reproduction law $R+1$. Thus, for every integer n , $m_n \leq (1+m)^n$. □

Proof of (i) - Lower bound. Since $m_{n+1}(0,0) = mm_n$, we just need to consider $m_n(0,0)$. We prove by induction that $\prod_{j=0}^{n-1} \varrho(a_j) \leq m_n(0,0)$ and conclude using Lemma A.2. Recall that $m_0(0) = 1 = \varrho(0)$. Then, for $k \in \llbracket n - a_n, n \rrbracket$, by induction

$$\begin{aligned} m_k(0,0) &\geq \prod_{j=0}^{k-1} \varrho(a_j) \\ &\geq \prod_{j=0}^{n-1} \varrho(a_j) \left[\prod_{j=k}^{n-1} \varrho(a_j) \right]^{-1} \geq \prod_{j=0}^{n-1} \varrho(a_j) \cdot \varrho(a_n)^{k-n}, \end{aligned}$$

since sequence (a_j) and function ϱ are non decreasing. We obtain

$$m_{n+1}(0,0) = m \sum_{k=n-a_n}^n m_k(0) \geq \prod_{j=0}^{n-1} \varrho(a_j) \left[m \sum_{k=n-a_n}^n \varrho(a_n)^{k-n} \right] = \prod_{j=0}^n \varrho(a_j).$$

which ends up the induction. \square

Proof of (ii). Let us show, by induction, that

$$\forall n \in \mathbb{N}, m_n(0,0) \leq \varrho(a_n)^n.$$

First, $m_0(0) = \varrho(0) = 1$. Then, since ϱ and $(a_n)_n$ are increasing, for every $k \in \llbracket n - a_n, n \rrbracket$,

$$m_k(0) \leq \varrho(a_k)^k \leq \varrho(a_n)^k.$$

Thus, $m \sum_{k=n-a_n}^n m_k(0) \leq m \sum_{k=n-a_n}^n \varrho(a_n)^k$ and

$$m_{n+1}(0) \leq \varrho(a_n)^{n+1} \leq \varrho(a_{n+1})^{n+1}.$$

Finally, from Lemma A.1, $(m+1)^{-n} \varrho(a_n)^n = \exp\{n \log(1 - m(m+1)^{-a_n-2} + \varrho(a_n(m+1)^{-2a_n})\}$, which gives the desired result. \square

APPENDIX B. DICHOTOMY FOR BRANCHING PROCESSES

Let $(Z_n : n \in \mathbb{N})$ be an aging branching process (without immigration) with maximal age (a_n) as defined in Section 2 and introduce the stopping times

$$T_K = \inf\{n \in \mathbb{N} : Z_n(0) \geq K\}$$

where $\inf \emptyset = +\infty$ by convention and

$$S = \bigcap_{K \geq 1} \{T_K < \infty\} = \left\{ \limsup_{n \rightarrow \infty} Z_n(0) = +\infty \right\}.$$

We recall that $\text{Nonext} = \{\forall n \in \mathbb{N} : Z_n \neq 0\}$ and we know from Section 2.1 that Z_n/m_n converges to a finite random variable W which is positive with positive probability.

Lemma B.1. (i) $S = \text{Nonext}$.

(ii) A.s. on the event Nonext , we have

$$\lim_{n \rightarrow \infty} \frac{Z_n}{m_n} > 0.$$

Proof. (i) is a classical result of dichotomy for branching processes. Either $\mathbf{P}(R = 0) = 0$ and 0 is not accessible for $Z_n(\cdot)$ and $\mathbf{P}(\text{Nonext}) = 1$. Then we use the same coupling as in Section 2.1, i.e. a multitype branching process with fixed maximal age a , such that $\varrho(a) > 1$. Then, the branching process Z_n is larger than a supercritical multitype branching process with finite variances and primitive mean matrix and no extinction. It ensures that each of its coordinates goes to ∞ a.s. Or $\mathbf{P}(R = 0) = 0$ and 0 is accessible for $Z_n(\cdot)$. Then

$$\limsup_{n \rightarrow \infty} Z_n = \infty \quad \text{a.s. on Nonext.}$$

which ensures that $\limsup_{n \rightarrow \infty} Z_n(0) = \infty$ a.s. on Nonext. Finally, the converse inclusion is obvious.

(ii) By branching property and monotonicity of the sequence (a_n) ,

$$Z_{T_K+n} \geq \sum_{i=1}^K Z_n^{(i)},$$

a.s. on $\{T_K < \infty\}$, where $(Z^{(i)} : i \geq 1)$ are i.i.d. aging branching processeses with maximal age is given by (a_n) . Adding that

$$\mathbf{P}\left(\lim_{n \rightarrow \infty} \frac{Z_n}{m_n} > 0\right) = \mathbf{P}(W > 0) > 0$$

we get

$$\mathbf{P}\left(\liminf_{n \rightarrow \infty} \frac{Z_{T_K+n}}{m_n} > 0 \mid T_K < \infty\right) \geq 1 - (1 - \mathbf{P}(W > 0))^K$$

Finally $T_K < \infty$ and m_n/m_{n+T_K} has a positive lower bound as $n \rightarrow \infty$ thanks to Lemma 2.2 (ii), so

$$\mathbf{P}\left(\liminf_{n \rightarrow \infty} \frac{Z_n}{m_n} > 0 \mid T_K < \infty\right) = \mathbf{P}\left(\liminf_{n \rightarrow \infty} \frac{Z_{T_K+n}}{m_n} > 0 \mid T_K < \infty\right)$$

and we get

$$\mathbf{P}\left(\liminf_{n \rightarrow \infty} \frac{Z_n}{m_n} > 0 \mid T_K < \infty\right) \geq 1 - (1 - \mathbf{P}(W > 0))^K.$$

Adding that the right hand goes to 0 as K goes to infinity ensures that

$$\mathbf{P}\left(\liminf_{n \rightarrow \infty} \frac{Z_n}{m_n} = 0; S\right) = 0$$

and ends up the proof. \square

We have the following counterpart for aging branching process with immigration $(Z_n)_{n \in \mathbb{N}}$. The proof is similar and omitted.

Lemma B.2. *We have $\mathbf{P}(\text{Nonext}) = 1$ and*

$$\lim_{n \rightarrow \infty} \frac{Z_n}{m_n} > 0 \quad \text{a.s.}$$

APPENDIX C. M/G/1 QUEUES

To be self-contained, we recall below some well known facts around queues. We want to determine the time R a bus wait at a station where one customer was waiting when it arrived. We recall that, more generally, $R(t)$ is the time a bus wait at a station where customers gathered from t units of times when the bus arrived. We use the classical connection with Galton-Watson processes. For each customer getting on, his children are the customers arriving during his boarding. Since the customers arrivals are led by a Poisson Point process, the number of children is distributed as i.i.d. r.v. with Poisson distribution with parameter α . Intuitively, $R < +\infty$ a.s. if and only if customers arrive at the station slower than they board.

Proposition 2. $R < \infty$ a.s. iff $\alpha \leq 1$. Then,

$$\mathbf{E}[R(t)] = \frac{\alpha}{1-\alpha}t.$$

Furthermore,

$$E [R(t)^2] = \frac{\alpha}{(1-\alpha)^3}t + \frac{\alpha^2}{(1-\alpha)^2}t^2.$$

Proof. To enlight notations, let us still denote $(Z_n : n \in \mathbb{N})$ the standard Galton-Watson process described just above. Then, $R = \sum_{i=0}^{\infty} Z_i$. Thus, $R < +\infty$ a.s. if and only if $\alpha \leq 1$. Furthermore,

$$\mathbf{E}[R] = \sum_{i=0}^{\infty} \mathbf{E}(Z_i) = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}.$$

The number of customers gathering during a time interval of length t when the bus arrives is distributed as $N(t)$, a r.v. with Poisson distribution with parameter αt . Then,

$$\mathbf{E}[R(t)] = \mathbf{E}[N(t)]\mathbf{E}[R] = \frac{\alpha t}{1-\alpha}.$$

In the same way, using independence and denoting $(R^{(i)}, i \in \mathbb{N})$ a sequence of i.i.d. r.v. with common law R , we can compute the second moment:

$$\mathbf{E}[R(t)^2] = \mathbf{E} \left[\left(\sum_{i=1}^{N(t)} R^{(i)} \right)^2 \right] = \mathbf{E}[N(t)]\mathbf{E}[R^2] + \mathbf{E}[N(t)(N(t)-1)]\mathbf{E}[R]^2.$$

Finally, using that R and $1 + R(1)$ are equal in distribution, $\mathbf{E}[R^2] = 1/(1-\alpha)^3$, and the proof is complete. \square

Acknowledgement. This work was partially funded by Chair Modelisation Mathematique et Biodiversite VEOLIA-Ecole Polytechnique-MNHN-F.X. and the professorial chair Jean Marjoulet.

REFERENCES

- [1] S. Asmussen. Convergence rates for branching processes. *Ann. Probab.*, 4:139–146, 1976.
- [2] K. B. Athreya and H.-J. Kang. Some limit theorems for positive recurrent branching Markov chains. II. *Adv. Appl. Probab.*, 30(3):711–722, 1998.
- [3] K. B. Athreya and S. Karlin. Branching processes with random environments, ii: Limit theorems. *Ann. Math. Statist.*, 42(6):1843–1858, 12 1971.

- [4] K. B. Athreya and S. Karlin. On branching processes with random environments: I: Extinction probabilities. *Ann. Math. Statist.*, 42(5):1499–1520, 10 1971.
- [5] K. B. Athreya and P. E. Ney. *Branching processes. Reprint of the 1972 original*. Mineola, NY: Dover Publications, reprint of the 1972 original edition, 2004.
- [6] V. Bansaye. Ancestral lineages and limit theorems for branching Markov chains. *To appear in Journal of Theoretical Probability*, <http://www.cmap.polytechnique.fr/~bansaye/Ancestral.pdf>.
- [7] J. Biggins, H. Cohn, and O. Nerman. Multi-type branching in varying environment. *Stochastic Processes Appl.*, 83(2):357–400, 1999.
- [8] S. Borst. *Polling systems*. Amsterdam: CWI, 1996.
- [9] O. J. Boxma. *Analysis and optimization of polling systems*. CWI, 1991.
- [10] O. Cappé, E. Moulines, and T. Rydén. *Inference in hidden Markov models*. Springer Science & Business Media, 2006.
- [11] B. Chauvin and A. Rouault. KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Probab. Theory Relat. Fields*, 80(2):299–314, 1988.
- [12] H. Cohn. On the growth of the multitype supercritical branching process in a random environment. *The Annals of Probability*, pages 1118–1123, 1989.
- [13] W. Feller. *An introduction to probability theory and its applications*, volume 2. John Wiley & Sons, 2008.
- [14] S. Foss, L. T. Rolla, and V. Sidoravicius. Greedy walk on the real line. *The Annals of Probability*, 43(3):1399–1418, 2015.
- [15] P. Haccou, P. Jagers, and V. A. Vatutin. *Branching processes: variation, growth, and extinction of populations*. Number 5. Cambridge university press, 2005.
- [16] O. D. Jones. On the convergence of multitype branching processes with varying environments. *The Annals of Applied Probability*, 7(3):772–801, 1997.
- [17] H. Kesten. Supercritical branching processes with countably many types and the size of random cantor sets. *Probability, Statistics, and Mathematics: Papers in Honor of Samuel Karlin (Anderson, TW, Athreya, KB and Iglehart, DL, eds.)*. Academic Press, Boston, pages 103–121, 1989.
- [18] M. Kimmel and D. A. Axelrod. *Branching processes in biology. 2nd extended ed*. New York, NY: Springer, 2nd extended ed. edition, 2015.
- [19] I. Kurkova. A sequential clearing process. *Fundam. Prikl. Mat.*, 2:619–624, 1996.
- [20] I. Kurkova and M. Menshikov. Greedy algorithm, \mathbf{z}^1 case. *Markov Process. Relat. Fields*, 3(2):243, 1997.
- [21] T. Kurtz, R. Lyons, R. Pemantle, and Y. Peres. A conceptual proof of the kesten-stigum theorem for multi-type branching processes. *IMA Volumes in Mathematics and its Applications*, 84:181–186, 1996.
- [22] R. Lyons, R. Pemantle, and Y. Peres. Conceptual proofs of $l \log l$ criteria for mean behavior of branching processes. *The Annals of Probability*, 23(3):1125–1138, 1995.
- [23] C. J. Mode. *Multitype branching processes: theory and applications*, volume 34. American Elsevier Pub. Co., 1971.
- [24] S.-T. C. Moy. Extensions of a limit theorem of everett, ulam and harris on multitype branching processes to a branching process with countably many types. *The Annals of Mathematical Statistics*, 38(4):992–999, 1967.
- [25] J. Resing. Polling systems and multitype branching processes. *Queueing Systems*, 13(4):409–426, 1993.
- [26] A. Roitershtein. A note on multitype branching processes with immigration in a random environment. *The Annals of Probability*, 35(4):1573–1592, 2007.
- [27] E. Seneta. *Non-negative matrices and Markov chains*. Springer Science & Business Media, 2006.
- [28] H. Takagi. *Analysis of polling systems*. MIT press, 1986.
- [29] R. D. van der Mei. Towards a unifying theory on branching-type polling systems in heavy traffic. *Queueing Systems*, 57(1):29–46, 2007.
- [30] V. Vatutin. Lectures (edinburgh): Bpre and properites of random walk. www.maths.bath.ac.uk/~ak257/vatutin/vatutin_lecture4.pdf and www.maths.bath.ac.uk/~ak257/vatutin/vatutin_lecture5.pdf, 2006.

- [31] V. Vatutin. Polling systems and multitype branching processes in random environment with final product. *Theory of Probability & Its Applications*, 55(4):631–660, 2011.
- [32] V. Vatutin and E. Dyakonova. Multitype branching processes and some queueing systems. *Journal of Mathematical Sciences*, 111(6):3901–3911, 2002.
- [33] V. A. Vatutin. Multitype branching processes with immigration in random environment, and polling systems. *Siberian Advances in Mathematics*, 21(1):42–72, 2011.
- [34] V. Vishnevskii and O. Semenova. Mathematical methods to study the polling systems. *Automation and Remote Control*, 67(2):173–220, 2006.