



HAL
open science

Aging branching process and queuing for an infinite bus line

Vincent Bansaye, Alain Camanes

► **To cite this version:**

Vincent Bansaye, Alain Camanes. Aging branching process and queuing for an infinite bus line. 2015. hal-01162541v1

HAL Id: hal-01162541

<https://hal.science/hal-01162541v1>

Preprint submitted on 10 Jun 2015 (v1), last revised 31 Aug 2017 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

AGING BRANCHING PROCESS AND QUEUING FOR AN INFINITE BUS LINE

VINCENT BANSAYE & ALAIN CAMANES

ABSTRACT. We study a multitype branching process in varying environment which describes an aged structure population, when the maximal age of individuals may vary over generations and go to infinity. We prove a Kesten Stigum type theorem, namely the a.s. convergence of the successive size of the population normalized by its mean. The technic developed is inspired by the spine approach for multitype branching processes and from geometric ergodicity along the spine using a Doeblin condition. We also obtain the a.s. convergence of the distribution of the ages among the population.

Adding an immigration, this process is connected to a queuing system for buses which serve stations indexed by \mathbb{N} . We can then determine the asymptotic behavior of a single bus and when two buses are going to merge with probability one.

Key words. Multitype Branching processes, Queuing systems, Immigration, Inhomogeneity, Coupling

A.M.S. Classification. 60J80, 60K25, 60F05, 60F10.

CONTENTS

1. Introduction	1
1.1. Model	2
1.2. Main results and motivations	3
1.3. Organization of the paper and notations	4
2. Aging branching process	5
2.1. Preliminaries : moment estimates without immigration	5
2.2. Aging branching process without immigration	6
2.3. Aging branching process with immigration	13
3. Queuing system for bus line	15
3.1. From Aging Branching Process to the motion of one single bus	16
3.2. Coalescence criteria for two buses with the same discipline	19
Appendix A. Mean's Regimes	21
Appendix B. M/G/1 queues	24
References	26

1. INTRODUCTION

We are interested in a model of age structured population where the maximal age of individuals increase with time. Thus the associated branching process is both

Date: Notes révisées le June 10, 2015.

time non-homogeneous and infinite dimensional if the age is unbounded. A vast literature is dedicated to multitype branching processes (see e.g. [16, 4]) and their applications to population dynamics (see e.g. [10, 5]) and queuing systems (see e.g. [22]). In particular, they give a convenient way to model population without interactions and take into account a bounded age structure influencing the reproduction law of the individuals. The associated mean matrix is then called Leslie matrix and Perron Frobenius theory yields the asymptotic behavior of the process.

Much less is known about infinite number of types and varying environment since the spectral approach and martingale techniques cannot be extended easily. The pioneering works of Moy [17], Kesten [13] and Seneta provide some theoretical extension of limit theorems and extinction criteria for a denumerable number of types, while multitype branching processes with a finite number of types (MBP in the sequel) have been well studied for stationary ergodic environment, see e.g. [1, 2].

1.1. Model. In each generation n , each individual reproduces independently with reproduction law R . His offsprings are newborns in generation $n + 1$. Moreover each individual with age a in generation n becomes an individual aged $a + 1$ in generation $n + 1$ if $a + 1 \leq a_{n+1}$ and dies otherwise. Thus a_n gives the maximal age which is allowed in generation n . We take also into account a potential immigration and our process is defined as follows. For each n , $a \in \mathbb{N}$, we denote by $Z_n(a)$ the number of individuals in generation n whose age is equal to a . We denote by Z_n the total size of the population. Thus,

$$(1) \quad Z_n := \sum_{a=0}^{a_n} Z_n(a);$$

$$(2) \quad Z_n(a) = Z_{n-1}(a-1)1_{1 \leq a \leq a_n};$$

$$(3) \quad Z_n(0) = \sum_{j=1}^{Z_{n-1}} R_{j,n} + I_n;$$

where $(R_{j,n}, j, n \in \mathbb{N})$ are i.i.d. distributed as R and $(I_n, n \in \mathbb{N})$ are i.i.d. independent of $(R_{j,n}, j, n \in \mathbb{N})$.

The fact that a_n is the maximal age among the population in generation n makes us assume that

$$\forall n \in \mathbb{N}, a_{n+1} \leq a_n + 1.$$

Remark that each individual born in generation n lives during ℓ_n generations, where ℓ_n is defined by

$$\ell_n := \max\{k : a_{n+k} + 1 \geq k\},$$

and it satisfies $\ell_{n+1} \geq \ell_n - 1$.

When a_n is constant, we recover the classical age structured branching process. Let us also note that when a_n is stationary or periodic, the behavior of the process can be again derived from branching processes with finite number of types. Our motivations for aging and queuing given below make us focus on the case when the maximal age is non-decreasing. Then either it is bounded and then it is stationary and the study is derived from the finite dimensional case; or it goes to infinity and we determine in this paper how the process behaves for large times.

1.2. Main results and motivations. A first motivation is to describe population dynamics when the life span of individuals is non decreasing, so that the maximal age a_n is non decreasing. When a_n goes to infinity, the aging branching process is supercritical and the size of the population grows geometrically. Our main result deals with the asymptotic behavior of the number of individuals and the distribution of age among the population. More precisely, we prove the a.s. convergence of $\mathbf{Z}_n/\mathbf{E}(\mathbf{Z}_n)$ and $\mathbf{Z}_n(a)/\mathbf{E}(\mathbf{Z}_n)$ to non-degenerated random variable as $n \rightarrow \infty$. We also provide estimates of $\mathbf{E}(\mathbf{Z}_n)$ to illustrate our results. Such convergences are well known in the homogeneous framework and finite dimension [4, 14]. In varying environment and finite dimension, Coale Lopez theorem about product of matrices can be used to describe the mean behavior of the process and prove such results via weak ergodicity. We refer to [20] and the works of Harry Cohn for more details. Here we have to deal both with varying environment and infinite dimension. Necessary and sufficient conditions for *a.s.* and L^p convergence of martingales associated to harmonic vectors have been obtained in [8] in the case of countable set of types. But it seems rather delicate to apply such results in our framework and to capture the behavior of the quantities we consider in this paper.

We develop a different technic relying on the control of the L^2 moments of the successive increments of the renormalized process using the spine approach initiated by [15, 14] and Doeblin condition for geometric ergodicity of Markov chains which enables us to achieve the computations. We believe this method is rather efficient and could be interesting for itself and generalizable in several ways. In particular, it can be extended to continuum set of types, Lyapounov functions may be invoked in more general cases and the L^2 bounds could be relaxed following [8].

The connection between branching processes and queuing systems is known from a long time and can be found for example in [9] (chapter XIV). Basically the offspring of an individual is the number of customers arriving during the time when this individual is served. Our original motivation, which is handled in the second part of this article, is the study of some queuing systems where customers arrive on the half line of integers (a bus line) and the traveling server (the bus) moves at a constant speed to the right and serve each customer (the time required to enter the bus). In our model, a customer is waiting at station i if and only if the distance from this customer to the next bus is at most d_i when this customer arrives at the station. The sequence d_i will both a way to model a discipline for the queues and take into account the distance between two successive buses. As a first example, if customers are waiting at a station wherever the buses are, then the first bus of the line is associated with the (aging) sequence $d_i = i$, which actually amounts to take $d_i = \infty$.

Many works have been dedicated to models with a finite number of queues and in particular to customers arriving on a circle while the servers visit the queues cyclically. We refer to polling models [23] and works of Sergey Foss and al.

In this work, we extend the a.s. convergence for the renormalized number of individuals of an aging branching process with maximal age given by the sequence d_i by adding an immigration and proving some functional convergence result. We derive then the speed of a single bus with discipline d_i travels and make appear two regimes following a_i goes fast enough to infinity or not. It also enables us to derive qualitative results on the dynamics where several buses serve.

The links between polling systems and multitype branching processes with immigration is well known and finely used for a finite number of queues, see e.g. [18, 21]. Here the queues are indexed by \mathbb{N} and one can see the set of buses as a particle system on \mathbb{N} . The more a bus is far from the previous one, the more customers are waiting for him and the slower it travels. This is well known phenomenon which make delay increase in traffic. As a consequence, the large distance between buses tend to increase fast. Such a model then intrinsically differs from particles systems such as the q -TASEP or Hammersley-Aldous-Diaconis process where the distance between two particles tend to be reduced. If the distance between the two buses increases fast, the bus behind, which advances very slowly should eventually be reached by the previous one. This makes two buses merge. One question we deal here is “do any two buses should eventually merge (or coalesce) ?”.

Let us finally mention that taking into account immigration and time non homogeneity is more general motivated by self excited process (like Hawkes process) where the immigration is the excitation and the branching structure is the cascade effect provoked by this excitation.

1.3. Organization of the paper and notations. In Section 2, we consider branching processes with an aged structure and first deal with the case where there is no immigration. We then derive asymptotic behavior of this branching process with immigration by summing the descendants of each immigrant.

In Section 3, we introduce the queuing system for an infinite bus line. We then link this model with aging branching process with immigration by proving that the time at which a bus leaves some station is obtained by summing the number of new born individuals of such a branching process. We can then study the progression of the bus and determine when two successive buses will merge a.s.

The population can be described by a discrete tree \mathcal{T} . Following notations of Ulam-Harris-Neveu, the individuals are labeled by sequences of integers which give their location in the tree and \mathcal{T} is a random subset of $\cup_{n \geq 0} \{1, 2, \dots\}^n$. We denote $\mathcal{Z}(u)$ the age of the individual $u \in \mathcal{T}$ and $|u|$ its generation in the tree. The set of individuals in generation n is denoted \mathcal{G}_n and the set of the descendants of u in the next generation (including himself if it survives) is denoted $X(u)$. Finally, in each generation n , the part of the population which survives is denoted by \mathcal{G}_n^+ , while the part of the population which dies is denoted \mathcal{G}_n^- . More precisely, denoting \leq the natural order on \mathcal{T} ,

$$(4) \quad \mathcal{G}_n := \{u \in \mathcal{T} \text{ s.t. } |u| = n\},$$

$$(5) \quad X(u) := \{v \in \mathcal{T} : |v| = |u| + 1, v \geq u\},$$

$$(6) \quad \mathcal{G}_n^+ := \{u \in \mathcal{G}_n \text{ s.t. } \mathcal{Z}(u) + 1 \leq a_{n+1}\},$$

$$(7) \quad \mathcal{G}_n^- := \{u \in \mathcal{G}_n \text{ s.t. } \mathcal{Z}(u) + 1 > a_{n+1}\}.$$

Then, the population in generation n can be viewed as measure

$$\mathcal{Z}_n := \sum_{u \in \mathcal{G}_n} \delta_{\mathcal{Z}(u)}$$

so that $\mathbf{Z}_n = \mathcal{Z}_n(\mathbb{N}) = \#\mathcal{G}_n$ and $\mathbf{Z}_n(a) = \mathcal{Z}_n(\{a\}) = \#\{u \in \mathcal{G}_n : \mathcal{Z}(u) = a\}$.

2. AGING BRANCHING PROCESS

We assume in the whole section that

$$a_n \leq a_{n+1} \leq a_n + 1, \quad a_n \rightarrow \infty$$

and

$$m = \mathbf{E}(R) < \infty, \quad \mathbf{E}(R^2) < \infty.$$

In the two first subsections, we will deal with the aging branching process without immigration, i.e. the process $(Z_n : n \in \mathbb{N})$ defined by $Z_0 = 0$ and

$$(8) \quad Z_n := \sum_{a=0}^{a_n} Z_n(a);$$

$$(9) \quad Z_n(a) = Z_{n-1}(a-1)1_{1 \leq a \leq a_n};$$

$$(10) \quad Z_n(0) = \sum_{j=1}^{Z_{n-1}} R_{j,n};$$

where $(R_{j,n}, j, n \in \mathbb{N})$ are i.i.d. distributed as R .

2.1. Preliminaries : moment estimates without immigration. In this section, we use coupling with well chosen (multitype) branching processes to give some first estimates on the two first moment of the aging Galton Watson process without immigration, i.e. we assume here that for each $n \in \mathbb{N}$,

$$I_n = 0 \text{ a.s.}$$

To evaluate the mean behavior of the process, let us note for every $0 \leq i \leq n$ and $0 \leq a \leq a_i$,

$$m_{i,n}(a) = \mathbf{E}[Z_n | \mathcal{Z}_i = \delta_a],$$

which gives the mean number of individuals at generation n which are descendants of an individual aged a at generation i . For sake of simplicity, we will denote

$$m_{i,n} = m_{i,n}(0), \quad m_n = m_{0,n} = m_{0,n}(0) = \mathbf{E}[Z_n].$$

We are also using, for $1 \leq i \leq n$, $0 \leq a \leq a_i$, $0 \leq b \leq a_n$,

$$m_{i,n}(a, b) = \mathbf{E}[Z_n(b) | \mathcal{Z}_i = \delta_a]$$

the mean number of individuals of age b in generation n , which are descendant from a single individual with age a in generation i .

Lemma 2.1 (Mean behavior). *(i) We have $(m_{n+1}/m_n) \rightarrow m + 1$ as $n \rightarrow \infty$. Hence, for any $0 \leq k \leq n$, (m_{n-k}/m_n) goes to $(m + 1)^{-k}$.*

(ii) For each $\rho \in (1, m + 1)$, there exists a positive constant α s.t. for n large enough, $m_n \geq \alpha \rho^n$.

(iii) For all $0 \leq i \leq n - 1$ and $0 \leq a \leq a_i$,

$$mm_{i+1,n} \leq m_{i,n}(a) \leq m_{i,n} \leq (m + 1)m_{i+1,n}.$$

Proof. (i) We first observe that from generation n to generation $n + 1$, only individuals aged a_n may die. Furthermore, each individual gives birth, on average, to m children. Thus,

$$(m + 1)m_n - m_n(0, a_n) \leq m_{n+1} \leq (m + 1)m_n.$$

This yields the upper bound. For the lower bound, note that individuals aged k in generation i were newborns of generation $i - k$, so $m_n(0, a_n) = m_{n-a_n}(0, 0)$. Using the Markov and branching properties,

$$m_n \geq m_{n-a_n}(0, 0)m_{n-a_n, n}.$$

Moreover $m_{n-a_n, n} = (m+1)^{a_n}$ since starting from an individual of age 0 in generation $n - a_n$, each individual is replaced by $R+1$ individuals in the next generation during the time interval $[n - a_n, n]$. Then,

$$\frac{m_n(0, a_n)}{m_n} = \frac{m_{n-a_n}(0, 0)}{m_n} \leq \frac{1}{m_{n-a_n, n}}$$

and the fact that the right hand side goes to 0 as $n \rightarrow \infty$ yields the lower bound for (i).

(ii) Let $\rho(a)$ be the greatest eigenvalue of Leslie Matrix sized $a + 1$ (whose definition is recalled in the Appendix) and $\rho \in (1, m + 1)$. Since (a_n) goes to infinity, $(\rho(a_n))$ goes to $m + 1$, see Lemma A. Then there exists n_0 s.t. $\rho(a_{n_0}) > \rho$. Considering the new borns in generation n_0 yields

$$m_n \geq m_{n_0}(0, 0)M_{n_0, n},$$

where $M_{n_0, n}$ is the mean number of individuals in a MBP with $a_{n_0} + 1$ types starting in generation n_0 and living till time n . Using Perron-Frobenius Theorem, there exists a positive constant c s.t. $M_{n_0, n} \sim c\rho(a_{n_0})^n$, which leads to the expected result.

(iii) The bound $m_{i, n}(a) \leq m_{i, n}$ comes simply from the fact that an individual with age 0 lives longer than an individual with age $a > 0$. Using the distribution of traits in generation $i + 1$, we get by branching and Markov property

$$(11) \quad m_{i, n}(a) = \sum_{0 \leq \ell \leq a_{i+1}} m_{i, i+1}(a, \ell)m_{i+1, n}(\ell).$$

It yields the two last inequalities recalling that $m_{i, i+1}(a, 0) = m$, $m_{i, n}(a) \leq m_{i, n}$ and $m_{i, i+1}(a) \leq m + 1$. □

2.2. Aging branching process without immigration. We study the asymptotic behavior of the process without immigration and we recall that it means that for each $n \in \mathbb{N}$, $I_n = 0$ a.s. We will derive in the next Section 2.3 the counterpart when the immigration I is non zero.

We first note that $a_n \rightarrow \infty$ ensures that the event

$$\text{Nonext} := \{\forall n \geq 0 : Z_n > 0\}$$

has positive probability. We refer to forthcoming Lemma 2.5 for details.

Theorem 1. (i) *The sequence of random variable $(Z_n/m_n)_n$ converges a.s. and in L^2 to a non-negative r.v. W , which is positive a.s. on Nonext.*

(ii) *For each $k \in \mathbb{N}$, $(Z_n(k)/Z_n)_n$ converges a.s. to $m/(m+1)^k$ on the event Nonext.*

Let us recall from Section 2.1 that $m_{n+1}/m_n \rightarrow 1$, so the theorem ensures that ratio of consecutive size of the population Z_{n+1}/Z_n goes to $m + 1$ on Nonext. But the process may grow slower than $(m+1)^n$. Two asymptotic regimes appear, which we put in light in the following result. These two regimes will be relevant when studying the queuing system in Section 3.

Corollary 2. *The process $M_n = (m + 1)^{-n} Z_n$ is a supermartingale which converges a.s. to a finite non-negative r.v. M and*

- (i) *If $\liminf_{i \rightarrow \infty} a_i / \log i > 1 / \log(m + 1)$, then $\{M > 0\} = \text{Nonext.}$*
 (ii) *If $\limsup_{i \rightarrow \infty} a_i / \log i < 1 / \log(m + 1)$, then $M = 0$ a.s.*

When $a_n = n$, each individual survives in each generation and we simply obtain a GW process with mean $m + 1$, which grows geometrically with rate $m + 1$ as $n \rightarrow \infty$. The result above shows at which speed a_n has to go infinity so that such a growth still holds. Let us proceed with the proof of the two last results.

2.2.1. *Proof of Theorem 1 (i).* The proof relies on the following lemma, which focuses on the evolution of types on a spine of the process.

For every $i \leq n$ and $a \in \{0, \dots, a_i\}$, we define

$$P_{i,n}(a, b) = m_{i,i+1}(a, b) \frac{m_{i+1,n}(b)}{m_{i,n}(a)}$$

and then by recursion we can set

$$Q_{i,n}(a, b) := \sum_{0 \leq k \leq a_{i+1}} P_{i,n}(a, k) Q_{i+1,n}(k, b).$$

Moreover (11) ensures that $P_{i,n}$ is a Markov kernel. Then $Q_{i,n}(\cdot, \cdot)$ is the semigroup of an inhomogeneous Markov Chain between generations i and n . We are proving in the next lemma that it satisfies the Doeblin's condition and we are deriving geometric estimates for the means $m_{i,n}$ via a many-to-one formula. Let us note that this technic is inspired by probabilistic methods, while the result here is purely analytical. Such an approach can be extended for more general product of (non-negative) matrices in finite or infinite dimension.

Lemma 2.2. (*i*-Many-to-one formula). *For all $0 \leq i \leq n$ and $a \geq 0$,*

$$m_{i,n}(a, \cdot) = m_{i,n}(a) Q_{i,n}(a, \cdot).$$

(*ii*-Doeblin's condition). *For all $0 \leq i \leq n$ and $0 \leq a \leq a_i$,*

$$P_{i,n}(a, \cdot) \geq \frac{m}{m+1} \delta_0(\cdot).$$

(*iii*-Geometrical Ergodicity). *For any $i \leq n$ and probability measures μ and ν on $\{0, \dots, a_i\}$,*

$$d_{TV}(Q_{i,n}(\mu, \cdot), Q_{i,n}(\nu, \cdot)) \leq (1 - c)^{n-i} d_{TV}(\mu, \nu),$$

where $c = m/(m + 1)$ and $d_{TV}(\cdot, \cdot)$ is the total variation distance on the space of probabilities on \mathbb{N} defined by

$$d_{TV}(\mu, \nu) = \frac{1}{2} \sum_{k \in \mathbb{N}} |\mu_k - \nu_k|.$$

(*iv*) *We set*

$$E(a, i, n) := \frac{m_{i,n}(a)}{m_n} - \frac{m_{i,n+1}(a)}{m_{n+1}}.$$

There exists $C_m > 0$, which only depends on m such that for all $i, n \in \mathbb{N}$,

$$|E(a, i, n)| \leq C_m \frac{(1 - c)^{n-i}}{m_i}.$$

Proof. We prove (i) by induction. Using (11) and $m_{i,n}(\ell, k) = m_{i,n}(\ell)Q_{i+1,n}(\ell, k)$, we get

$$\begin{aligned} \frac{m_{i,n}(a, k)}{m_{i,n}(a)} &= \sum_{\ell \geq 0} \frac{m_{i,i+1}(a, \ell)m_{i+1,n}(a)}{m_{i,n}(\ell)} Q_{i+1,n}(\ell, k) \\ &= \sum_{\ell \geq 0} P_{i,n}(a, \ell) Q_{i+1,n}(\ell, k), \end{aligned}$$

which proves (i), since the right hand side is equal to $Q_{i,n}(a, k)$.

(ii) It is a consequence of Lemma 2.1 (iii) and $m_{i,i+1}(a, 0) = m$.

(iii) Doeblin's condition obtained above yields a contraction for the Total Variation distance

$$d_{TV}(P_{i,n}(\mu, \cdot), P_{i,n}(\nu, \cdot)) \leq (1 - c)d_{TV}(\mu, \nu).$$

for any $0 \leq i \leq n$ and by a classical induction

$$d_{TV}(Q_{i,n}(\mu, \cdot), Q_{i,n}(\nu, \cdot)) \leq (1 - c)^{n-i} d_{TV}(\mu, \nu).$$

(iv) Using again the Markov and branching property but now in generation n and recalling (i), we get

$$\begin{aligned} m_{i,n+1}(a) &= \sum_{0 \leq k \leq a_n} m_{i,n}(a, k)m_{n,n+1}(k) \\ &= m_{i,n}(a) \sum_{0 \leq k \leq a_n} Q_{i,n}(a, k)m_{n,n+1}(k). \end{aligned}$$

In the same way,

$$m_{n+1} = m_{0,n+1}(0) = m_n \sum_{0 \leq k \leq a_n} Q_{0,n}(0, k)m_{n,n+1}(k).$$

Thus,

$$E(a, i, n) = \frac{m_{i,n}(a)}{m_n} \cdot \frac{\sum_{0 \leq k \leq a_n} (Q_{i,n}(a, k) - Q_{0,n}(0, k))m_{n,n+1}(k)}{\sum_{0 \leq k \leq a_n} Q_{0,n}(0, k)m_{n,n+1}(k)}.$$

Recalling that $m_{n,n+1}(k)$ is equal to m if $k \in \mathcal{G}_n^-$ and to $m + 1$ if $k \in \mathcal{G}_n^+$, we get

$$|E(a, i, n)| \leq \frac{(m + 1)m_{i,n}(a)}{mm_n} d_{TV}(Q_{i,n}(a, \cdot), Q_{0,n}(0, \cdot)).$$

Now, from the triangular inequality,

$$d_{TV}(Q_{i,n}(a, \cdot), Q_{0,n}(0, \cdot)) \leq d_{TV}(Q_{i,n}(a, \cdot), Q_{i,n}(0, \cdot)) + \sum_{k=1}^i d_{TV}(Q_{k,n}(0, \cdot), Q_{k-1,n}(0, \cdot)).$$

Then, thanks to previous points (ii) and (iii),

$$\begin{aligned} d_{TV}(Q_{k,n}(0, \cdot), Q_{k-1,n}(0, \cdot)) &= d_{TV}(Q_{k,n}(0, \cdot), Q_{k,n}(P_{k-1,n}(0, \cdot), \cdot)) \\ &\leq (1 - c)^{n-k} d_{TV}(\delta_0, P_{k,n}(0, \cdot)) \\ &\leq (1 - c)^{n-k+1}, \end{aligned}$$

and there exists a constant $C'_m = \left(\frac{m+1}{m}\right)^2 > 0$ s.t. and

$$|E(a, i, n)| \leq C'_m (1-c)^{n-i} \frac{m_{i,n}(a)}{m_n} \leq C'_m (1-c)^{n-i} \frac{m_{i,n}}{m_n} \leq C'_m (1-c)^{n-i} \frac{1}{m_{0,i}(0,0)}.$$

using again the new borns in generation i . Recalling that $m_{0,i}(0,0) = m_{i-1}m \geq m_i m / (m+1)$, we obtain (iv). \square

Proof of Theorem 1 (i). To show that (Z_n/m_n) converges a.s., we prove that

$$(12) \quad \sum_{n \geq 0} \mathbf{E} \left[\left(\frac{Z_{n+1}}{m_{n+1}} - \frac{Z_n}{m_n} \right)^2 \right]^{1/2} < \infty.$$

Thanks to Minkowski's inequality and recalling that

$$(13) \quad m_{n+1} = \mathbf{E}(\#\mathcal{G}_n^+)(m+1) + \mathbf{E}(\#\mathcal{G}_n^-)m,$$

we get, denoting $\|\cdot\|_2$ the L^2 norm on r.v.,

$$\begin{aligned} \left\| \frac{Z_{n+1}}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2 &= \left\| \frac{\sum_{u \in \mathcal{G}_n} \#X(u)}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2 \\ &\leq \left\| \frac{\sum_{u \in \mathcal{G}_n^+} [\#X(u) - (m+1)]}{m_{n+1}} \right\|_2 + \left\| \frac{\sum_{u \in \mathcal{G}_n^-} [\#X(u) - m]}{m_{n+1}} \right\|_2 \\ &\quad + \left\| \frac{(m+1)\#\mathcal{G}_n^+ + m\#\mathcal{G}_n^-}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2 \end{aligned}$$

First remark that, when u belongs to \mathcal{G}_n^+ , then $\#X(u)$ is distributed as $R+1$. Thus, $(\#X(u) - (m+1))_{u \in \mathcal{G}_n^+}$ are independent, centered r.v. and

$$\mathbf{E} \left[\left(\sum_{u \in \mathcal{G}_n^+} \#X(u) - (m+1) \right)^2 \right] = \mathbf{E}[(R+m)^2] \mathbf{E}[\#\mathcal{G}_n^+].$$

Adding that the individuals in \mathcal{G}_n^+ survive, $\mathbf{E}[\#\mathcal{G}_n^+] \leq m_{n+1}$ and

$$\left\| \frac{\sum_{u \in \mathcal{G}_n^+} [\#X(u) - (m+1)]}{m_{n+1}} \right\|_2 \leq \frac{\sqrt{\mathbf{E}[(R-m)^2]}}{\sqrt{m_{n+1}}}.$$

In the same manner, recalling that $m_{n+1} \geq m_n m$ (Lemma 2.1 (iii))

$$\left\| \frac{\sum_{u \in \mathcal{G}_n^-} [\#X(u) - m]}{m_{n+1}} \right\|_2 \leq \frac{\sqrt{m_n \mathbf{E}[(R-m)^2]}}{m_{n+1}} \leq \sqrt{\frac{\mathbf{E}[(R-m)^2]}{m m_n}}.$$

To study the last term, let

$$F_n(u) = \frac{m_{n,n+1}(\mathcal{Z}(u))}{m_{n+1}} - \frac{1}{m_n}.$$

For $u \in \mathcal{G}_n$, either u belongs to \mathcal{G}_n^+ and $m_{n,n+1}(\mathcal{Z}(u)) = m+1$, or $u \in \mathcal{G}_n^-$ and $m_{n,n+1}(\mathcal{Z}(u)) = m$.

Using this notation,

$$\left\| \frac{(m+1)\#\mathcal{G}_n^+ + m\#\mathcal{G}_n^-}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2 = \left\| \sum_{u \in \mathcal{G}_n} F_n(u) \right\|_2.$$

The proof follows now standard arguments (see e.g. Lemma 3 in [6]). We use the genealogy of the population to exploit the branching property in the lineages.

Given two individuals u and v living in generation n , let us consider their youngest common ancestor ω at generation $i - 1$. Then, individuals u and v arise from two independent branching processes starting at time i from descendants of ω . Thus, we denote pairs of siblings born at generation i :

$$\mathcal{V}_i = \{(\omega a, \omega b) ; a \neq b, |\omega a| = |\omega b| = i\}.$$

Then,

$$\begin{aligned} & \mathbf{E} \left[\left(\sum_{u \in \mathcal{G}_n} F_n(u) \right)^2 \right] \\ = & \mathbf{E} \left[\sum_{(u,v) \in \mathcal{G}_n^2} F_n(u) F_n(v) \right] \\ = & \sum_{i=1}^{n-1} \mathbf{E} \left[\sum_{(u,v) \in \mathcal{V}_i} F_n(u) F_n(v) \right] + \mathbf{E} \left[\sum_{u \in \mathcal{G}_n} F_n(u)^2 \right] \\ \leq & \sum_{i \leq n-1} \sum_{a \leq a_i} m_{0,i}(a) \sum_{a', a''} V_{i,a,a',a''} E(a', i+1, n) E(a'', i+1, n) + 2m_n \left[\frac{(m+1)^2}{m_{n+1}^2} + \frac{1}{m_n^2} \right], \end{aligned}$$

where

$$V_{i,a,a',a''} := \mathbf{E}(Z_{i+1}(a') Z_{i+1}(a'') | Z_i = \delta_a)$$

is bounded by $\mathbf{E}((R+1)^2)$ and

$$\begin{aligned} E(a, i, n) &= \mathbf{E} \left(\sum_{u \in \mathcal{G}_n} F_n(u) \mid Z_i = \delta_a \right) \\ &= \mathbf{E} \left(\frac{\sum_{u \in \mathcal{G}_n^+} (m+1) + \sum_{u \in \mathcal{G}_n^-} m}{m_{n+1}} - \frac{\#\mathcal{G}_n}{m_n} \mid Z_i = \delta_a \right) \\ &= \frac{\mathbf{E}(\#\mathcal{G}_{n+1} \mid Z_i = \delta_a)}{m_{n+1}} - \frac{\mathbf{E}(\#\mathcal{G}_n \mid Z_i = \delta_a)}{m_n} \\ &= \frac{m_{i,n+1}(a)}{m_{n+1}} - \frac{m_{i,n}(a)}{m_n}. \end{aligned}$$

Using Lemma 2.2 (iv), we get

$$\begin{aligned} & \left\| \frac{\#\mathcal{G}_n^+(m+1) + \mathcal{G}_n^- m}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2^2 \\ & \leq \mathbf{E}((R+1)^2) \cdot C_m^2 \cdot \sum_{i \leq n-1} \frac{(1-c)^{2(n-i)}}{m_i} + 2 \left[\frac{(m+1)^2 m_n}{m_{n+1}^2} + \frac{1}{m_n} \right] \end{aligned}$$

Thus, there exists a constant C_R which only depends on $m = \mathbf{E}(R)$ and $\mathbf{E}(R^2)$ such that

$$\left\| \frac{\#\mathcal{G}_n^+(m+1) + \mathcal{G}_n^- m}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2^2 \leq C_R \left[\sum_{n/2 \leq k \leq n} \frac{1}{m_k} + (1-c)^n \right].$$

Putting pieces together yields

$$(14) \quad \left\| \frac{Z_{n+1}}{m_{n+1}} - \frac{Z_n}{m_n} \right\|_2 \leq C'_R \sqrt{\sum_{n/2 \leq k \leq n} \frac{1}{m_k} + \left(1 - \frac{m}{m+1}\right)^n}$$

for some constant C'_R which only depends on m and $\mathbf{E}(R^2)$. Adding that m_k grows geometrically, see Lemma 2.1 (ii), ensures that the right hand side is summable and (12). We get the L^2 convergence of Z_n/m_n and Cauchy Schwarz inequality also ensures that

$$(15) \quad \sum_{n \in \mathbb{N}} \mathbf{E}(|Z_{n+1}/m_{n+1} - Z_n/m_n|) < \infty.$$

Then Z_n/m_n converges a.s. and in L^1 to a non negative finite r.v. W . As $\mathbf{E}(Z_n/m_n) = 1$, $\mathbf{P}(W > 0) > 0$. Adding that on Nonext, Z_n and then $Z_n(0)$ have to go to infinity, standard branching argument ensures that $\{W > 0\} = \text{Nonext}$, which ends up the proof. \square

2.2.2. *Proof of Theorem (ii).* To prove the second part of the Theorem, we use the following simple law of large numbers, in the vein of [3].

Lemma 2.3. *Let $(X_n)_n$ be a sequence of integer valued r.v. We assume that for each $n \in \mathbb{N}$, $(X_{i,n} : i \geq 0)$ are identically distributed centered r.v., which are independent of X_n . Moreover $(X_{i,n} : i, n \geq 0)$ are bounded in L^2 . Then,*

$$\frac{1}{X_n} \sum_{i=1}^{X_n} X_{i,n} \rightarrow 0 \quad \text{a.s. on } \left\{ \liminf_{n \rightarrow \infty} \frac{1}{n} \log(X_n) > 0 \right\}.$$

Proof. Fix $c \in (0, \infty)$ and $\epsilon > 0$ and denote by

$$E_n^{(c)} = \left\{ \left| \frac{1}{X_n} \sum_{i=1}^{X_n} X_{i,n} \right| \geq \epsilon \right\} \cap \{X_n \geq cn^2\}.$$

Using Bienaymé Tchebychev inequality and conditioning by X_n , we get that

$$\mathbf{P}(E_n^{(c)}) \leq \mathbf{E} \left(\frac{1}{\epsilon^2 X_n} 1_{X_n \geq cn^2} \sup_{i,n} \mathbf{E}(X_{i,n}^2) \right) \leq \frac{1}{c\epsilon^2 n^2} \sup_{i,n} \mathbf{E}(X_{i,n}^2),$$

where the last term is bounded by assumption. As a consequence,

$$\sum_{n \geq 0} \mathbf{P}(E_n^{(c)}) < \infty$$

and $\mathbf{P}(E_n^{(c)} \text{ occurs i.o.}) = 0$ by Borel-Cantelli lemma. Then, a.s. on the event $\{\forall n \geq 0 : X_n \geq cn^2\}$,

$$\left| \frac{1}{X_n} \sum_{i=1}^{X_n} X_{i,n} \right| \xrightarrow{n \rightarrow \infty} 0.$$

Adding that

$$\left\{ \liminf_{n \rightarrow \infty} \frac{1}{n} \log(X_n) > 0 \right\} \subset \cup_{c \in \mathbb{Q} \cap (0,1)} \{\forall n \geq 0 : X_n \geq cn^2\}$$

ends up the proof. \square

Proof of Theorem 1 (ii). First we prove that a.s on Nonext,

$$(16) \quad \frac{Z_{n+1}(0)}{Z_n} \xrightarrow{n \rightarrow \infty} m$$

using the previous law of large numbers. Indeed, recall that

$$Z_{n+1}(0) = \sum_{k=1}^{Z_n} R_{k,n},$$

where $R_{k,n}$ are i.i.d. and distributed as R , whose variance is finite. Moreover, Theorem 1 and the fact that $m_n \geq C\rho^n$ for some $\rho > 1$ ensures that, on Nonext, $\liminf_{n \rightarrow \infty} \frac{1}{n} \log(Z_n) > 0$. Then Lemma 2.3 ensures that

$$\frac{Z_{n+1}(0)}{Z_n} \xrightarrow{n \rightarrow \infty} \mathbf{E}(R)$$

a.s. on Nonext.

Now we just note that $Z_n(k) = Z_{n-k}(0)$ to write

$$\frac{Z_n(k)}{Z_n} = \frac{Z_{n-k}(0)}{Z_{n-k}} \frac{Z_{n-k}}{Z_n}.$$

Adding that the first part of Theorem 1 yields

$$\lim_{n \rightarrow \infty} \frac{Z_{n-k}}{Z_n} = \lim_{n \rightarrow \infty} \frac{m_{n-k}}{m_n} = (m+1)^k$$

a.s. on Nonext ends up the proof. \square

2.2.3. Proof of Corollary 2.

Proof. First, using (8),

$$\begin{aligned} \mathbf{E}[Z_{n+1}|\mathcal{F}_n] &= \mathbf{E} \left[\sum_{k=0}^{a_{n+1}} Z_{n+1}(k) | \mathcal{F}_n \right] \\ &= \mathbf{E} \left[\sum_{k=0}^{a_{n+1}-1} Z_n(k) + \sum_{i=0}^{Z_n} R_{i,n} | \mathcal{F}_n \right] \\ &= \sum_{k=0}^{a_{n+1}-1} Z_n(k) + mZ_n \\ &\leq (m+1)Z_n, \end{aligned}$$

since $a_{n+1} - 1 \leq a_n$. Thus, $M_n = (m+1)^{-n} Z_n$ is a supermartingale.

Since M_n is a non-negative supermartingale, it converges a.s. to a non-negative finite r.v. M .

Moreover, using Lemma A.3 (i), we get that m_n is of the same magnitude as $(m+1)^n$ when $\liminf_{i \rightarrow \infty} a_i / \log i > 1 / \log(m+1)$. The part (i) of the Corollary is then a consequence of Theorem 1.

Similarly, the fact that $m_n / (m+1)^n$ goes to 0 as $n \rightarrow \infty$ soon as $\limsup_{i \rightarrow \infty} a_i / \log i < 1 / \log(m+1)$ yields (ii). \square

2.3. Aging branching process with immigration. We now consider the aging branching process with immigration $(\mathbf{Z}_n : n \in \mathbb{N})$ defined in (1). We still denote $m_n = \mathbf{E}[\mathbf{Z}_n]$. We also assume that

$$(17) \quad \limsup_{n \rightarrow \infty} \frac{\log(\mathbf{E}(I_n))}{n} < (m+1).$$

We derive now from the previous section the following counterpart for aging branching process with immigration:

Theorem 3. *The sequence $(\mathbf{Z}_n/m_n)_n$ converges a.s. to a positive r.v. \mathbf{W} .*

We are not needing the L^2 convergence in the sequence, whose proof seems rather technical. Thus, it is not considered in this paper. We focus finally on the cumulative number of new born individuals until generation n , which will be useful in the next section:

$$\mathfrak{Z}_n(0) = \sum_{k=0}^n \mathbf{Z}_k(0).$$

Corollary 4. *The sequence of random variable $(\mathfrak{Z}_n(0)/m_n)_n$ converges a.s.*

2.3.1. Proof of Theorem 3 and Corollary 4. We use that \mathbf{Z}_n is the sum over $k \leq n$ of the descendants of immigrants arrived at generation k . Moreover the successive number of descendants of an immigrant i in generation k is equal to an aging branching process $(Z_{k,n}^{(i)} : n \in \mathbb{N})$ without immigration started at generation k with maximal age $(a_n)_{n \geq k}$. Thus,

$$\frac{\mathbf{Z}_n}{m_n} = \sum_{k=0}^n \sum_{i=1}^{I_k} \frac{m_{k,n}}{m_n} W_{k,n}^{(i)} \text{ a.s. ,}$$

where for each $k \leq n$,

$$W_{k,n}^{(i)} := \frac{Z_{k,n}^{(i)}}{m_{k,n}}$$

is a squence of i.i.d. r.v. for $i \geq 0$, distributed as a r.v. $W_{k,n}$.

Theorem 1 obtained in the previous Section ensures that for each $(k, i) \in \mathbb{N}^2$,

$$(18) \quad W_{k,n}^{(i)} \xrightarrow{n \rightarrow \infty} W_k^{(i)} \text{ a.s. and in } L^2.$$

We also note that $W_k^{(i)}$ are independent random variables. Moreover, Lemma 2.2 (iv) ensures that the following limit exists for $k \geq 0$,

$$(19) \quad \frac{m_{k,n}}{m_n} \xrightarrow{n \rightarrow \infty} \alpha_k,$$

while Lemma 2.1 ensures that α_k goes geometrically to 0 as $k \rightarrow \infty$.

Proof of Theorem 3. We use (14) to get

$$\|W_{k+1,n} - W_{k,n}\|_2 \leq C'_R \sqrt{\sum_{(n-k)/2 \leq i \leq n-k} \frac{1}{m_{k,k+i}} + \left(1 - \frac{m}{m+1}\right)^{n-k}}.$$

Adding that $m_{k,k+i} \geq m_i$ since a_n is non decreasing, we obtain that the sequence

$$C_k := \sum_{n \geq k} \|W_{k+1,n} - W_{k,n}\|_2$$

is bounded for $k \geq 0$. We get by Cauchy Schwarz inequality

$$\sup_k \mathbf{E} \left(\sum_{n \geq k} |W_{k+1,n} - W_{k,n}| \right) < \infty.$$

As a consequence,

$$\sup_{k \geq 0} \mathbf{E} \left(\sup_{n \geq k} W_{k,n} \right) < \infty$$

and it ensures that there exist $C > 0$ such that

$$\mathbf{E}(\mathcal{R}_k) \leq C \sum_{k \geq K} \mathbf{E}(I_k) \frac{m_{k,n}}{m_n},$$

where

$$\mathcal{R}_K := \sup_{n \geq K} \sum_{k=K}^n \sum_{i=1}^{I_k} \frac{m_{k,n}}{m_n} W_{k,n}^{(i)}$$

Adding thanks to Lemma 2.1 that $m_n \geq m_k(0,0)m_{k,n}$ and for each $\varrho \in (1, m+1)$, there exists $C > 0$ such that $m_n \geq C\varrho^k m_{k,n}$ for any $k \leq n$, Assumption (17) ensures that

$$\sum_{k \geq K} \mathbf{E}(I_k) \frac{m_{k,n}}{m_n} \longrightarrow 0$$

as $K \rightarrow \infty$. Using now that \mathcal{R}_K is a.s. non increasing when $K \rightarrow \infty$, we obtain

$$\lim_{K \rightarrow \infty} \mathcal{R}_K = 0 \quad \text{a.s.}$$

Finally we combine (18) and (19) to get

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{K-1} \sum_{i=1}^{I_k} \frac{m_{k,n}}{m_n} W_{k,n}^{(i)} = \sum_{k=0}^{K-1} \sum_{i=1}^{I_k} \alpha_k W_k^{(i)}$$

and conclude for the a.s. convergence. \square

Proof of Corollary 4. The proof is in the same vein as the previous one and we just give the main lines. We note that

$$\begin{aligned} \frac{\mathfrak{Z}_n(0)}{m_n} &= \frac{\mathbf{Z}_0}{m_n} + \sum_{k=0}^{n-1} \frac{\mathbf{Z}_{n-k}(0)}{m_n} \\ &= \frac{\mathbf{Z}_0}{m_n} + \sum_{k=0}^{n-1} \frac{\mathbf{Z}_{n-k}(0)}{m_{n-k-1}} \frac{m_{n-k-1}}{m_n}. \end{aligned}$$

First, following the law of large numbers (16) and using Theorem 3 ensures that for each $k \geq 0$,

$$\lim_{n \rightarrow \infty} \frac{\mathbf{Z}_{n-k}(0)}{m_{n-k-1}} = \lim_{n \rightarrow \infty} \frac{\mathbf{Z}_{n-k}(0)}{\mathbf{Z}_{n-k-1}} \frac{\mathbf{Z}_{n-k-1}}{m_{n-k-1}} = m\mathbf{W}.$$

Recalling from Lemma 2.1 (i) that $m_{n-k-1}/m_n \rightarrow (m+1)^{-k-1}$ as $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \sum_{k=0}^K \frac{\mathbf{Z}_{n-k}(0)}{m_n} = \mathbf{W}m \sum_{k=0}^K (m+1)^{-k-1}$$

We then use Lemma 2.1 (ii) to write

$$\frac{m_{n-k}}{m_n} \leq C\alpha^k$$

for some $C > 0$ and $\alpha \in (0, 1/(m+1))$ and control the rest of the serie

$$\begin{aligned} \sum_{k=K}^{n-1} \frac{Z_{n-k}(0)}{m_n} &\leq C \sum_{k=K}^n \alpha^k \sup_{n \geq k} \frac{Z_{n-k}}{m_{n-k}} \\ &\leq \frac{C}{1-\alpha} \alpha^K S, \end{aligned}$$

where

$$S := \sup_{n \geq 0} \frac{Z_n}{m_n}$$

is a.s. finite thanks to Theorem 3. This proves the a.s. convergence. \square

3. QUEUING SYSTEM FOR BUS LINE

We are now able to study the model for a bus line introduced in the beginning. We consider an infinite bus line, where stations are labeled by $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. Let τ be the time spent by the bus to go from station i to station $i+1$. To describe the position of the bus, we will denote by H_i the time when he leaves from station i and by P_t the station he is occupying or he has just left at time t . Then,

$$P_t = \inf\{i / H_i \geq t\} = H_t^{-1}.$$

The customers arrive to station i with a Poisson process with parameter $\alpha \in (0, 1)$. We suppose that the time for each customer to get on the bus is equal to one. So, the bus waits in a station during a time which is equal to the number of customers which have got on. Furthermore, the bus line obeys the following discipline, with d_i a non-decreasing sequence. When a customer arrives at station i , if the bus is further than d_i stations, then he decides not to take the bus and he leaves the queue. Thus, the customers who are waiting at station i when the bus arrives came at this station when the bus was between stations $i - d_i$ and i . The fact that d_i may be unbounded is motivated in particular by the study of the distance between two consecutive buses, see below. Note also that $d_i = i$ amounts to get rid of the discipline : any customer arriving at a station waits for the bus.

Let us give a more precise definition of the process and recall some classic results on queuing systems. We associate to each station $i \in \mathbb{N}$ independent Poisson Point Process $\{T_k^{(i)} : k \geq 0\}$ on $[0, \infty)$ with intensity α . The following quantities can be defined recursively with $H_0 = 0$ a.s.

We first note that the time when the bus arrives at station $i \in \{1, 2, \dots\}$ is equal to $H_{i-1} + \tau$. We denote by

$$F_i = \min(j \in \mathbb{N} : T_j^{(i)} > H_{i-d_i-1} + \tau)$$

the label of the first customer which is not leaving the station because the bus is too far from station i .

First, the customers are waiting at station i if they arrive just after the bus arrived in station $i - d_i$ and their arrival times $T_k^{(i)}$ are such that $T_k^{(i)} > H_{i-d_i-1} + \tau$. Then, customers will get on the bus if the bus is still at station i when they arrive, i.e.

$$T_k^{(i)} \leq (H_{i-1} + \tau) + k - F_i.$$

Thus, the last customer getting on the bus at station i is labeled by

$$M_i = \max(k \geq F_i : T_k^{(i)} - (H_{i-1} + \tau) \leq k - F_i)$$

and we take the convention $\max \emptyset = -\infty$. Thus, since $(M_i - F_i + 1)^+$ customers get on the bus, the sequence H_i is recursively defined as

$$(20) \quad H_i = H_{i-1} + \tau + (M_i - F_i + 1)^+.$$

Let us now focus on what happens when only one customer is waiting when the bus arrives. During the time he gets on the bus, new customers arrive and while these customers get on the bus, other customers arrive, ... Thus, different generations of customers can be defined and the number of customers at generation i is the sum of independent Poisson r.v. with parameter α . Since $\alpha < 1$, we get a subcritical Galton-Watson Process and it will extinct a.s. in finite time. More generally, if the customers have accumulated during a time interval of size t at a station before the bus arrives, then the number of customers the bus will take at this station is a finite r.v. $R(t)$, which is distributed as a subcritical branching process with initial population distributed as a Poisson distribution with parameter αt . Moreover, we recall in Appendix B that

$$\mathbf{E}[R(t)] = \frac{\alpha t}{1 - \alpha} \text{ and } \mathbf{E}[R(t)^2] = \frac{\alpha}{(1 - \alpha)^3} t + \frac{\alpha^2}{(1 - \alpha)^2} t^2.$$

3.1. From Aging Branching Process to the motion of one single bus. To highlight the link with aging branching process, we denote by

$$\mathbf{Z}_n(0) = (M_n - F_n + 1)^+ \quad (n \geq 1)$$

the number of customers getting on at station n . We observe that (20) ensures that

$$H_n = n\tau + \sum_{i=0}^n \mathbf{Z}_i(0).$$

Then, the maximal age and the mean reproduction number will be denoted

$$a_n = d_{n+1} - 1, \quad m = \mathbf{E}[R(1)] = \frac{\alpha}{1 - \alpha},$$

and

$$\mathbf{Z}_n(a) = \mathbf{Z}_{n-a}(0) \quad \text{and} \quad \mathbf{Z}_n = \sum_{i=n-a_n}^n \mathbf{Z}_i(0) = \sum_{i=0}^{a_n} \mathbf{Z}_n(i).$$

and the following result shows that $H_n - n\tau$ is the cumulated sum of the number of newborns until time n in an Aging Branching Process with maximal age (a_n) , with reproduction law $R(1)$ and with immigration in generation n distributed as $R(a_n\tau)$.

Proposition 3.1. *For every $n \geq 1$,*

$$\mathbf{Z}_n(0) = \sum_{j=1}^{\mathbf{Z}_{n-1}} R_{j,n} + I_n,$$

where $(R_{j,n}, I_n)$ are independent r.v., $R_{j,n}$ are distributed as $R(1)$ and I_n are distributed as $R(a_n\tau)$.

Proof. The customers which get on the bus at station n can be split in the two following subpopulations, depending whether they correspond to customers which arrived at the station during the time when the bus was in a previous station or during the time when the bus was moving between two stations.

- The customers which are arrived at the station n when the bus was staying in station $i = n - d_n, \dots, n - 1$ and the customers which then arrived at station n while these customers are getting on the bus and so on...

The time that the bus has stayed in station $i = n - d_n, \dots, n - 1$ is equal to the number of customers which have get on the bus in these stations, so it is equal to

$$\sum_{k=n-d_n}^{n-1} Z_k(0) = Z_{n-1}.$$

Then the number of customers which want to get on the bus at station is n is given by

$$\sum_{k=1}^{Z_{n-1}} R_{k,n} \stackrel{d}{=} R(Z_{n-1}),$$

where $R_{k,n}$ are i.i.d. and distributed as $R(1)$.

- The customers which are arrived at station n during the time intervals when the bus was traveling between the stations $i = n - d_n, \dots, n$ and the customers which then arrived at station n while these customers are getting on the bus and so on...

During each travel between two such successive stations, the numbers of customers arriving at station n is a random variable of parameter and the whole number of customers associated which are getting on the bus in station n is distributed as $R(\tau)$. Then the sum over $i = n - d_n, \dots, n$, which we note I_n , is distributed as $R(d_n \tau)$ and independent of $(R_{k,n} : k, n \geq 0)$.

Summing these two numbers of customers gives the result. \square

We can now describe the motion of the bus for large time, recalling $m_n = \mathbf{E}[Z_n]$.

Theorem 5. (i) *If $a_n = a$ is constant and $\rho(a) < 1$, then there exists $c > \tau$ such that*

$$H_n \sim_{n \rightarrow \infty} cn \text{ a.s.}$$

Moreover, $(H_n - nc)/\sqrt{n}$ converges to a Gaussian r.v. with positive variance.

(ii) *If $a_n = a$ is constant with $\rho(a) > 1$ or $a_n \rightarrow \infty$, then there exists a finite positive r.v. W s.t.*

$$H_n \sim_{n \rightarrow \infty} m_n W \text{ a.s.}$$

Let us recall that if $a_n = a$ and $\rho(a) > 1$, then $m_n \sim C\rho(a)^n$ for some $C > 0$ since we get a classical MBP with immigration. Moreover we recall that $\liminf_{n \rightarrow \infty} a_n / \log i > 1 / \log(m + 1)$ ensures that m_n is of the same magnitude as $(m + 1)^n$.

Proof. For (i), let $a_n = a$ constant and $\rho(a) < 1$. The process $(Z_n : n \in \mathbb{N})$ is a MBP with finite space types and immigration. Let M be the reproduction matrix associated to this process. Since $\rho(a) < 1$, thanks to Theorems 1.2 and 1.3 in [19],

$$\frac{\sum_{k=0}^n Z_k}{n} \xrightarrow{n \rightarrow \infty} b := \sum_{k=0}^{\infty} \mathbf{E}(M^k I) \text{ a.s.}$$

and $(\sum_{k=0}^n \mathbf{Z}_k - nb)/\sqrt{n}$ converges in distribution to a Gaussian r.v. Then we use this limit for the first coordinate of $\sum_{k=0}^n \mathbf{Z}_k$, which is $\mathfrak{Z}_n(0)$. We get then the asymptotic behavior of H_n via Proposition 3.1.

The point (ii) when $a_n = a$ and $\rho(a) > 1$ is similar, but now the MBP is supercritical.

The point (ii) for $a_n \rightarrow \infty$ is a direct consequence of Corollary 4 and Proposition 3.1, since

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log(\mathbf{E}(I_n)) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log(a_n \tau \alpha / (1 - \alpha)) = 0 < m + 1.$$

It completes the proof. \square

Remark 1. We stress that [19] deal with multitype branching process in random environment with immigration and we just apply here Theorems 1.2 and 1.3 in the constant environment case, where the assumptions can be easily verified. The results of the Corollary could be extended to the case of a_n random, which follows a stationary ergodic sequence. Let us note that in that case b can have infinite components be infinite and new regimes appear, which could be of interest for a future work.

Corollary 6. (i) If $a_n = a$ is constant and $\rho(a) < 1$, then

$$P_t \sim_{t \rightarrow \infty} \frac{t}{c} \text{ a.s.}$$

(ii) If $a_n = a$ is constant with $\rho(a) > 1$ or $a_n \rightarrow \infty$, then

$$\limsup_{t \rightarrow \infty} |P_t - u_t| < \infty \text{ a.s.}$$

where $u_t = \inf\{n \in \mathbb{N} : m_n \geq t\}$ is the inverse function of m_n .

Recall that when $\liminf_{n \rightarrow \infty} a_n / \log i > 1 / \log(m + 1)$, then m_n is of the same magnitude as $(m + 1)^n$. So in that case, the proof below ensures that $u_t \sim \log(t) / \log(m + 1)$ when $t \rightarrow \infty$.

Proof. The point (i) is a direct consequence of Theorem 5 (i).

For the point (ii), we first use Theorem 5 (ii) to get that

$$H_n \sim_{n \rightarrow \infty} W m_n \quad \text{a.s.}$$

with $W \in (0, \infty)$ a.s. So we just need to prove that for any sequence v_n such that

$$v_n \sim_{n \rightarrow \infty} c m_n$$

for some $c > 0$, then $w_t - u_t$ is bounded for $t \geq 0$, where w is the inverse of v . Indeed, recall that $\rho(a) > 1$ or $a_n \rightarrow \infty$ ensures that

$$\liminf_{n \rightarrow \infty} \frac{m_{n+1}}{m_n} > 1,$$

so there exists $\ell > 1$ such that for n large enough, $m_{n+1}/m_n \geq \ell$. Moreover there exists $C > 0$ such that $v_n \geq C m_n$ for any $n \in \mathbb{N}$. Then for t large enough

$$\frac{v_{w_t-1}}{m_{u_t}} \geq C \frac{m_{w_t-1}}{m_{u_t}} \geq C \ell^{w_t-1-u_t}.$$

Adding that by definition of the inverse functions u and w ,

$$\frac{v_{w_t-1}}{m_{u_t}} \leq \frac{t}{t} = 1$$

ensures that $w_t - u_t$ is upper bounded and we conclude by symmetry. \square

3.2. Coalescence criteria for two buses with the same discipline. We now consider two buses on the same line. We assume that the first bus starts from station 0 at time 0 and the second bus starts at time $\mu > 0$ from the station 0. Then, the second follows the same discipline as the first one. But some customers may have already been got on. In other words, the customers arrived at station i after the departure of the first bus wait for the second bus if and (only if) it is between station $i - d_i$ and i . Let us denote by $P^{(1)}$ (resp. $P^{(2)}$) the position of the first (resp. the second) bus. We say that the two buses merge if there exists $t > 0$ such that $P_t^{(1)} = P_t^{(2)}$. After this time, we consider that the two buses stay together and travel as a single bus. The following results shows that several regimes exist, depending on the discipline $(a_i : i \geq 0)$. Quite surprisingly for us, they do not only depend on the stability of the queue (i.e. the criticality of MBP).

Corollary 7. (i) If $a_n = a$ is constant and $\rho(a) < 1$, then the two buses merge a.s.
 (ii) If $a_n = a$ is constant and $\rho(a) > 1$, then with positive probability the two buses do not merge.
 (iii) If $a_n \rightarrow \infty$ as $n \rightarrow \infty$, then the two buses merge a.s.

The proof relies on a coupling argument, which enables us to compare the process with two buses which travel independently following the discipline (d_n) . Then, we use the previous results.

Proof. We denote by $H_n^{(1)}$ (resp. $H_n^{(2)}$) the time when the bus 1 (resp the bus 2) leaves station n . The number of customers waiting for the bus 2 at the successive stations is stochastically less than the number of customers for the bus 1, since the customers obey the same discipline for the two buses, but some of them may have taken the bus 1 instead of waiting for the bus 2. Then, recalling that μ is the initial time delay between the two buses, we have for each $n \geq 0$,

$$(21) \quad H_n^{(2)} \leq \tilde{H}_n^{(1)} + \mu \quad \text{a.s.},$$

where $(H_n^{(1)})_n$ and $(\tilde{H}_n^{(1)})_n$ are independent and identically distributed as the process $(H_n)_n$ considered in the previous Section . We denote by

$$N = \inf\{n \geq 0 : H_n^{(1)} \geq H_n^{(2)}\}$$

the station at which the two buses merge, where $\inf \emptyset = \infty$ and the event $\{N = +\infty\}$ means that the two buses do not merge.

(i) In this regime, we can use the central limit theorem of Theorem 5 (i) to get that there exists $n_1 \in \mathbb{N}$ such that

$$\mathbf{P}(H_{n_1} + \mu < n_1 c) \geq 1/4, \quad \mathbf{P}(H_{n_1} > n_1 c) \geq 1/4.$$

Thus, the coupling (21) ensures that

$$\mathbf{P}(N \leq n_1) \geq \mathbf{P}(\tilde{H}_{n_1}^{(1)} + \mu < n_1 c, H_{n_1}^{(1)} > n_1 c) \geq 1/16$$

We can iterate this procedure and conclude as follows. Choose $n_2 \in \mathbb{N}$ such that

$$\mathbf{P}(N \leq n_2 | N \geq n_1) \geq 1/16$$

and inductively find $n_k \geq n_{k-1}$ for $k \geq 3$ such that

$$\mathbf{P}(N \geq n_k) \leq (1 - 1/16)^k.$$

Letting k go to infinity ends up the proof.

(ii) We show that

$$(22) \quad \mathbf{P}(\exists t_0 > 0, \forall t \geq t_0 : P_t^{(1)} - P_t^{(2)} > a) > 0,$$

so that the two buses can travel with the same discipline a after time t_0 , independently, as if they were on two different lines.

For that purpose, we denote by E_{ℓ, n_0} the event when there are no customers at stations $i = 1, 2, \dots, \ell + a$ when the bus 1 arrives and the number of customers at station 0 arriving during the time interval $[0, \tau]$ (so before bus 2 arrives) is equal to n_0 . We first observe that E_{ℓ, n_0} has a positive probability and a.s. on E_{ℓ, n_0} , we have

$$H_{a+\ell}^{(1)} = (a + \ell)\tau.$$

For $i \geq 0$, we consider the following point processes on \mathbb{R}^+ :

$$\{\hat{T}_k^{(i)} : k \geq 0\} := \{T_k^{(i+a+\ell)} - \ell\tau\} \cap \mathbb{R}^+.$$

These are independent Poisson Point processes. Let us note by $\hat{P}_t^{(1)}$ the position of the (single) bus at time t associated with these arrival times on stations $i \geq 0$. We note that $\hat{P}^{(1)}$ is independent of E_{ℓ, n_0} . Moreover, these arrival times $\hat{T}^{(i)}$ correspond to the customers which may go inside the bus 1 on the event E_{ℓ, n_0} . Thus, a.s. on the event E_{ℓ, n_0} , for any $t \geq 0$,

$$P_{t+(a+\ell)\tau}^{(1)} = \hat{P}_t^{(1)} + a + \ell.$$

We take $\tilde{P}^{(1)}$ independent of $\hat{P}^{(1)}$ and distributed as $P^{(1)}$. By Corollary 6 (ii), we have

$$\limsup_{t \rightarrow \infty} |\tilde{P}_t^{(1)} - u_t| < \infty \quad a.s.$$

A slight adaptation of Corollary 6 (ii) ensures that

$$\limsup_{t \rightarrow \infty} |\hat{P}_t^{(1)} - u_t| < \infty \quad a.s.$$

Adding that the distribution of $\hat{P}^{(1)}$ does not depend on ℓ and $\hat{P}^{(1)}$ and $\tilde{P}^{(1)}$ are independent, we can now fix $\ell \in \mathbb{N}$ such that

$$\eta := \mathbf{P}\left(\forall t \geq 0, \hat{P}_t^{(1)} - \tilde{P}_t^{(1)} > -\ell\right) > 0.$$

Finally, we introduce the stopping time

$$T_a := \inf\{t \geq H_1^{(2)}, P_t^{(1)} - P_t^{(2)} \leq a\}.$$

We note that one can find n_0 large enough and a process $\hat{P}^{(2)}$ independent of $\hat{P}^{(1)}$ and distributed as $\tilde{P}^{(1)}$ such that for all $t \in [0, T_a]$,

$$P_{t+(a+\ell)\tau}^{(2)} \leq \hat{P}_t^{(2)} \quad a.s. \text{ on } E_{\ell, n_0},$$

since before T_a the second bus is not affected by the first one, while its departure is delayed. Then,

$$\begin{aligned} \mathbf{P}(T_a = \infty) &\geq \mathbf{P}\left(E_{\ell, n_0}, \forall t \geq (a + \ell)\tau, P_t^{(1)} - P_t^{(2)} > a\right) \\ &\geq \mathbf{P}\left(E_{\ell, n_0}, \forall t \geq 0, \hat{P}_t^{(1)} + a + \ell - \hat{P}_t^{(2)} > a\right) \\ &\geq \mathbf{P}(E_{\ell, n_0})\mathbf{P}\left(\forall t \geq 0, \hat{P}_t^{(1)} - \hat{P}_t^{(2)} > -\ell\right) = \mathbf{P}(E_{\ell, n_0})\eta > 0, \end{aligned}$$

so that (22) hold and the proof of (ii) is complete.

(iii) We use again the coupling (21) and Corollary 6 (ii) to get

$$\limsup_{t \rightarrow \infty} P_t^{(1)} - P_t^{(2)} \leq \limsup_{t \rightarrow \infty} P_t^{(1)} - \tilde{P}_t^{(1)} < \infty \text{ a.s.}$$

Then the non-decreasing sequence of events

$$A_K := \{\forall t > 0 : P_t^{(1)} - P_t^{(2)} \leq K\}$$

satisfies $\mathbf{P}(\cup_{K \in \mathbb{N}} A_K) = 1$. Moreover, a new coupling for the second bus ensures that

$$\mathbf{P}(A_K, \forall t > 0 : P_t^{(1)} > P_t^{(2)}) \leq \mathbf{P}(\forall t > 0 : P_t^{(1)} > \hat{P}_t^{(2)}),$$

where $\hat{P}^{(2)}$ is independent of $P^{(1)}$ and gives the position of a single bus with discipline K . Using again Corollary 6 (ii), but now with $a_n = K$ constant, we obtain

$$\lim_{t \rightarrow \infty} \hat{P}_t^{(2)} - P_t^{(1)} = -\infty \quad \text{a.s.}$$

so $\mathbf{P}(A_K, \forall t > 0 : P_t^{(1)} > P_t^{(2)}) = 0$ and letting $K \rightarrow \infty$,

$$\mathbf{P}(\forall t > 0 : P_t^{(1)} > P_t^{(2)}) = 0,$$

which ends up the proof. \square

APPENDIX A. MEAN'S REGIMES

Let $\rho(a)$ be the greatest eigenvalue of the Leslie's matrix governing a population with maximal age a

$$\begin{pmatrix} m & m & \cdots & m \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0, \end{pmatrix}$$

i.e. the greatest root of $P_a = X^{a+1} - m \sum_{k=0}^a X^k$.

Lemma A.1. *We have*

$$\rho(a) = (m + 1) - m(m + 1)^{-a-1} + O(a(m + 1)^{-2a}).$$

Moreover, the sequence $(\rho(a))$ is non decreasing.

Proof. Notice that we can rewrite $P_a = \frac{X^{a+1}(X-m-1)+m}{X-1}$. Using monotonicity of $x \mapsto x^{a+1}(x-m-1) + m$,

$$\frac{a+1}{a+2}(m+1) \leq \rho(a) \leq m+1.$$

Thus, $\rho(a)$ goes to $m+1$. This corresponds to the behavior of a Galton-Watson process without death.

Moreover, $P_{a+1}(\rho(a)) = \rho(a)^{a+1}(\rho(a) - (m+1))$. From the first point, $\rho(a) \leq \rho(a+1)$ and $(\rho(a))$ is non decreasing.

Noting $\rho(a) = (m+1)(1 - \varepsilon_a)$, we get

$$\left(1 - \frac{1}{a+2}\right)^{a+1} \leq (1 - \varepsilon_a)^{a+1} \leq 1$$

and $\left((1 - \varepsilon_a)^{a+1}\right)_a$ is bounded.

Since $P_a(\rho(a)) = 0$, $(m+1)^{a+2}(1 - \varepsilon_a)^{a+1}\varepsilon_a = m$.

First,

$$(a+1)\varepsilon_a = m \cdot (a+1) \cdot (m+1)^{-a-2} \cdot (1 - \varepsilon_a)^{-a-1} \rightarrow 0.$$

Then,

$$\varepsilon_a(m+1)^{a+2} = me^{-(a+1)\ln(1-\varepsilon_a)} \rightarrow m.$$

Going one step further,

$$\begin{aligned} \varepsilon_a(m+1)^{a+2} - m &= me^{(a+1)\varepsilon_a + o((a+1)\varepsilon_a)} \\ &= m(a+1)\varepsilon_a + o((a+1)\varepsilon_a) \\ &\sim m^2(a+1)(m+1)^{-a-2}. \end{aligned}$$

Finally,

$$(m+1) - \rho(a) - m(m+1)^{-a-1} \sim m^2(a+1)(m+1)^{-2a-3}.$$

□

The following Lemma quantifies the behavior of the product of Leslie's matrices.

Lemma A.2. *We observe the following regimes.*

(i) *If $\sum(m+1)^{-a_i} = +\infty$, then*

$$\prod_{i=1}^n \rho(a_i) = o((m+1)^n).$$

(ii) *If $\sum(m+1)^{-a_i} < +\infty$, then there exists $\alpha > 0$ s.t.*

$$\prod_{i=1}^n \rho(a_i) \sim \alpha \cdot (m+1)^n.$$

Remark 2. One can consider both particular cases :

- If $\limsup \frac{a_i}{\log i} < \frac{1}{\log(m+1)}$, then $\prod \rho(a_i) = o((m+1)^n)$.
- If $\liminf \frac{a_i}{\log i} > \frac{1}{\log(m+1)}$, then $\prod \rho(a_i) \sim \alpha(m+1)^n$.

Proof. Let us write

$$(m+1)^{-n} \prod_{i=1}^n \rho(a_i) = \exp \left\{ \sum_{i=1}^n \log \frac{\rho(a_i)}{m+1} \right\}.$$

From Technical Lemma A, recall that $\frac{\rho(a_i)}{m+1} = 1 - m \cdot (m+1)^{-a_i-2} + O(a_i(m+1)^{-2a_i})$ and this term is equivalent to $(m+1)^{-a_i}$. □

Lemma A.3 (Mean Asymptotic Regime). *The mean number m_n of individuals at generation n asymptotically are lead by the following dichotomy.*

(i) *If $\sum_{i \geq 0} (m+1)^{-a_i} < +\infty$, then there exists $\alpha > 0$ s.t.*

$$\alpha(m+1)^n \leq m_n \leq (m+1)^n.$$

(ii) *If $\limsup_{i \rightarrow \infty} a_i / \log i < 1 / \log(m+1)$, then*

$$m_n = o((m+1)^n).$$

Remark 3. First, from the preceding remark, there is a gap between the two regimes. We expect that, in each regime, (m_n) behaves like $\prod \rho(a_i)$, but we cannot yet prove it. Then, the first condition is valid as soon as $\liminf a_i / \log i > 1 / \log(m+1)$.

Proof of (i) - Upper bound. branching process (Z_n) is stochastically bounded by a Galton-Watson process (\bar{Z}_n) where no one dies, i.e. with reproduction law $R+1$. Thus, for every integer n , $Z_n \leq \bar{Z}_n$ and $m_n \leq (1+m)^n$. \square

Proof of (i) - Lower bound. Since $m_{n+1}(0) = m m_n$, we prove the lower bound for the mean number of newborns $m_n(0, 0)$.

Let $x_n = \prod_{j=0}^{n-1} \rho(a_j)$. We'll prove that $x_n \leq m_n(0, 0)$ which leads the asymptotic result, thanks to Lemma A.2. Recall that $m_0(0) = 1 = \rho(0)$.

Then, for $k \in \llbracket n - a_n, n \rrbracket$, using induction hypothesis,

$$\begin{aligned} m_k(0) &\geq \prod_{j=0}^{k-1} \rho(a_j) \\ &\geq \prod_{j=0}^{n-1} \rho(a_j) \left[\prod_{j=k}^{n-1} \rho(a_j) \right]^{-1} \geq \prod_{j=0}^{n-1} \rho(a_j) \cdot \rho(a_n)^{k-n}, \end{aligned}$$

since sequence (a_j) and function ρ are increasing.

Thus, by induction,

$$m \sum_{k=n-a_n}^n m_k(0) \geq \prod_{j=0}^{n-1} \rho(a_j) \left[m \sum_{k=n-a_n}^n \rho(a_n)^{k-n} \right] = \prod_{j=0}^n \rho(a_j).$$

\square

Proof of (ii). Let us show, by induction, that

$$\forall n \in \mathbb{N}, m_n(0, 0) \leq \rho(a_n)^n.$$

First, $m_0(0) = \rho(0) = 1$. Then, since ρ and (a_n) are increasing, for every $k \in \llbracket n - a_n, n \rrbracket$,

$$\begin{aligned} m_k(0) &\leq \rho(a_k)^k \\ &\leq \rho(a_n)^k. \end{aligned}$$

Thus,

$$\begin{aligned} m \sum_{k=n-a_n}^n m_k(0) &\leq m \sum_{k=n-a_n}^n \rho(a_n)^k \\ m_{n+1}(0) &\leq \rho(a_n)^{n+1} \\ &\leq \rho(a_{n+1})^{n+1}. \end{aligned}$$

Finally, from Technical Lemma , $\rho(a_n)^n = \exp\{n \log(1 - m(m+1)^{-a_n-2} + o((m+1)^{-2a_n})\}$, which gives the desired result. \square

APPENDIX B. M/G/1 QUEUES

To be self-contained, we recall below some well known facts around queues.

We want to determine the time R a bus wait at a station where one passenger was waiting when its arrived. We recall that, more generally, $R(t)$ is the time a bus wait at a station where passengers arrived from t unit of times when its arrived. To do so, we model passengers by two means :

- a *Galton-Watson process* ($Z_n : n \in \mathbb{N}$). For each passengers getting on, his children are the passengers arriving during his boarding. From the hypothesis, the number of children is distributed as i.i.d. r.v. with Poisson distribution with parameter α .
- a *Lévy process* ($Y_x : x \geq 0$). Starting from 0 with a drift -1 , Y makes jumps of height 1 when a passenger arrives at the station. The time the bus spend at the station where N passengers are waiting is equal to the time Y reaches $-N$.

From the first description, the bus starts a finite time at the station if and only if $\alpha \leq 1$. Intuitively, passengers arrive at the station slower than they board.

Proposition B.1. $R < \infty$ a.s. iff $\alpha \leq 1$. Then,

$$\mathbf{E}[R(t)] = \frac{\alpha}{1 - \alpha} t.$$

Furthermore,

$$E[R(t)^2] = \frac{\alpha}{(1 - \alpha)^3} t + \frac{\alpha^2}{(1 - \alpha)^2} t^2.$$

Proof. Given the description with the Galton-Watson process ($Z_n : n \in \mathbb{N}$),

$$R = \sum_{i=0}^{\infty} Z_i.$$

Thus, $R < +\infty$ if and only if $\alpha \leq 1$. Furthermore,

$$\mathbf{E}[R] = \sum_{i=0}^{\infty} \mathbf{E}(Z_i) = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1 - \alpha}.$$

When the bus arrives after t times of accumulation of customers, the initial number of passengers is distributed as $N(t)$ a r.v. with Poisson distribution with parameter αt . Then,

$$\mathbf{E}[R(t)] = \mathbf{E}[N(t)]\mathbf{E}[R] = \frac{\alpha t}{1 - \alpha}.$$

In the same way, using independence, we can reach the L^2 -moment values,

$$\mathbf{E}[R(t)^2] = \mathbf{E} \left[\left(\sum_{i=1}^{N(t)} R^i \right)^2 \right] = \mathbf{E}[N(t)]\mathbf{E}[R^2] + \mathbf{E}[N(t)(N(t) - 1)]\mathbf{E}[R]^2.$$

Finally, using that R and $1 + R(1)$ are equal in distribution, we have

$$\mathbf{E}[R^2] = \frac{1}{(1 - \alpha)^3}$$

and the proof is complete. \square

Using the Lévy Process representation, one can even get the distribution of $R(t)$.

Proposition B.2. *For every $\alpha < 1$ and $n \in \mathbb{N}$,*

$$\mathbf{P}(R = n) = \frac{1}{n} e^{-\alpha n} \frac{(\alpha n)^{n-1}}{(n-1)!},$$

and for every $t \geq 0$,

$$\mathbf{P}(R(t) = n) = \frac{te^{-t-\alpha n} (\alpha n)^n}{n \cdot n!} \left(1 + \frac{t}{\alpha n}\right)^{n-1} \quad (n \in \mathbb{N}).$$

Proof. Let $(T_i)_{i \in \mathbb{N}}$ be the arrival times of the customers. We construct the corresponding Lévy process $(Y_x)_{x \geq 0}$:

$$Y_x = -x + \sum_{i \in \mathbb{N}} 1_{T_i \leq x}.$$

If N customers are waiting when the bus arrive, the time the bus spend at the station is equal to the time τ_N where the Lévy process reaches $-N$,

$$\tau_N := \inf\{t \geq 0 : Y_t < -N\}.$$

From Chapters VI and VII in [7], $(\tau_N)_{N \geq 0}$ is a subordinator and

$$x\mathbf{P}(\tau_\ell = x) = \ell\mathbf{P}(-Y_x = \ell).$$

Since, for all $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$,

$$\begin{aligned} \mathbf{P}(Y_x + x = n) &= e^{-\alpha x} \frac{(\alpha x)^n}{n!}, \\ \mathbf{P}(\tau_x = x + n) &= \frac{x}{x+n} e^{-\alpha(x+n)} \frac{(\alpha(n+x))^n}{n!}. \end{aligned}$$

Finally, since $R(t) = \tau_{N(t)}$, where $N(t)$ is the number of passengers at the station when the bus arrives,

$$\begin{aligned} \mathbf{P}(R(t) = n) &= \sum_{0 \leq k \leq n} \mathbf{P}(N(t) = k) \mathbf{P}(\tau_k = n) \\ &= \sum_{1 \leq k \leq n} \frac{e^{-\alpha t} (\alpha t)^k}{k!} \frac{k}{n} e^{-\alpha n} \frac{(\alpha n)^{n-k}}{(n-k)!} \\ &= \frac{te^{-\alpha(t+n)} (\alpha n)^n}{n \cdot n!} \sum_{1 \leq k \leq n} \frac{(n-1)!}{(n-k)!(k-1)!} \left(\frac{t}{n}\right)^{k-1} \\ &= \frac{te^{-\alpha(t+n)} (\alpha n)^n}{n \cdot n!} \left(1 + \frac{t}{n}\right)^{n-1}. \end{aligned}$$

It ends up the proof. \square

Acknowledgement. This work was partially funded by Chair Modelisation Mathématique et Biodiversité VEOLIA-Ecole Polytechnique-MNHN-F.X. and the professorial chair Jean Marjoulet.

REFERENCES

- [1] K. B. Athreya, S. Karlin (1971). On branching processes with random environments I : Extinction probability. *Ann. Math. Stat.* 42. 1499-1520.
- [2] K. B. Athreya, S. Karlin (1971). On branching processes with random environments II : Limit theorems. *Ann. Math. Stat.* 42. 1843-1858.
- [3] K. B. Athreya, H.J. Kang (1998). Some limit theorems for positive recurrent branching Markov chains II. *Adv. Appl. Prob.* 30(3). 711-722.
- [4] K. B. Athreya, P. E. Ney (2004). *Branching processes*. Dover Publications, Inc., Mineola, NY.
- [5] D. Axelrod, M. Kimmel (2002). *Branching processes for biology*. Springer 2002.
- [6] V. Bansaye (2014). Ancestral lineage and limit theorems for branching Markov chains. Available via <https://hal.archives-ouvertes.fr/hal-00851284v4>.
- [7] J. Bertoin (1998). *Lévy processes*. Cambridge University Press.
- [8] J. D. Biggins, H. Cohn, O. Nerman (1999). Multi-type branching in varying environment. *Stoch. Proc. Appl.* 83, 357-400.
- [9] W. Feller (1966). *An introduction to probability theory and its applications*. Vol. II. John Wiley & Sons, Inc., New York-London-Sydney xviii+636 pp.
- [10] P. Haccou, P. Jagers, V. Vladimirov (2007). *Branching processes: variation, growth, and extinction of populations*. Cambridge Studies in Adaptive Dynamics. Cambridge University Press, Cambridge; IIASA, Laxenburg.
- [11] T.E. Harris (1963) The Theory of Branching Processes. Berlin: Springer.
- [12] H. Kesten, B. Stigum (1966). A limit theorem for multidimensional Galton Watson processes. *Ann. Math. Statist.* 37, 1211-1223.
- [13] H. Kesten (1989). Supercritical branching processes with countably many types and the size of random Cantor sets. In: Anderson, T.E., Athreya, K. (Eds.), *Probability, Statistics and Mathematics. Papers in honour of Sam Karlin*. Academic Press, New York.
- [14] T. Kurtz, R. Lyons, R. Pemantle, Y. Peres (1997). A conceptual proof of the Kesten-Stigum theorem for multi-type branching processes. In *Classical and Modern Branching Processes*, ed. K. B. Athreya and P. Jagers. Springer, New York. 181-185.
- [15] R. Lyons, R. Pemantle, Y. Peres (1995). Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.* 23 (3), 1125-1138.
- [16] C. J. Mode (1971). *Multi-Type Branching Processes-Theory and Application*, (1971) American Elsevier Publishing Company, Inc., New York, 300 pages, 1971.
- [17] S-T. C. Moy (1967). Extensions of a limit theorem of Everett, Ulam and Harris on multitype branching processes to a branching process with countably many types. *Ann. Math. Statist.* 38, 992-999.
- [18] J. A. C. Resing (1993). Polling systems and multitype branching processes. *Queueing Systems Theory Appl.* 13 (1993), no. 4, 409-426
- [19] A. Roithershtein (2007). A note on multitype branching processes with immigration in a random environment. *Ann. Probab.* Vol. 35, No. 4, 1573-1592.
- [20] E. Seneta (1981). *Non-Negative Matrices and Markov Chains*. Springer, New York.
- [21] R.D. van der Mei (2007). Towards a unifying theory on branching-type polling systems in heavy traffic. *Queueing Systems*. Volume 57, Issue 1, pp 29-46
- [22] V. Vatutin. Lectures in Edinburgh. Available via http://www.maths.bath.ac.uk/~ak257/vatutin/vatutin_lecture4.pdf and http://www.maths.bath.ac.uk/~ak257/vatutin/vatutin_lecture5.pdf.
- [23] V. M. Vishnevskii, O. V. Semenova (2006). Mathematical Methods to Study the Polling Systems. *Automation and Remote Control*, Vol. 67, No. 2, 173-220.