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Abstract: If $\Omega$ is a bounded domain in $\mathbb{R}^n$, $1 \leq q < p \leq \infty$ and $s = 0, 1, 2, \ldots$, then we clearly have $W^{s,p}(\Omega) \subset W^{s,q}(\Omega)$. We prove that this property does not hold when $s$ is not an integer.
A Sobolev non embedding

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Abstract - If Ω is a bounded domain in \( \mathbb{R}^n \), \( 1 \leq q < p \leq \infty \) and \( s = 0, 1, 2, \ldots \), then we clearly have \( W^{s,p}(\Omega) \subset W^{s,q}(\Omega) \). We prove that this property does not hold when \( s \) is not an integer.

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1 A non embedding

In connection with his work on distributional Jacobians [3], H. Brezis asked us whether

the inclusion \( W^{1/2,3}((0,1)) \subset W^{1/2,2}((0,1)) \) holds. (1)

The answer is negative. This is counterintuitive at first sight, since \( L^3((0,1)) \subset L^2((0,1)) \) and \( W^{1,3}((0,1)) \subset W^{1,2}((0,1)) \); thus, by "1/2 interpolation", we would expect (1) to hold.

Below we shall formulate our main result in a little bit greater generality. The class of fractional Sobolev spaces we have in mind is defined as follows. Let \( \Omega \) be a nontrivial open subset of \( \mathbb{R}^n \). Let \( 1 \leq p \leq \infty \). With \( s = m + \sigma, m \in \mathbb{N}_0 \) (the natural numbers including 0), and \( 0 < \sigma < 1 \), the fractional Sobolev space \( W^{s,p}(\Omega) \) is the collection of all \( f \in L^p(\Omega) \) such that its distributional derivatives \( D^\alpha f, \alpha \leq m \), are regular and

\[
\max_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+\sigma p}} \, dx \, dy < \infty.
\]

(2)

In this note, we give several proofs of the following

Theorem 1.1 Let \( s > 0 \) be a non integer, and let \( 1 \leq q < p \leq \infty \). Then there exists some in \( \Omega \) compactly supported function \( f \) such that \( f \in W^{s,p}(\Omega) \) but \( f \notin W^{s,q}(\Omega) \).
The same result was obtained independently by J. Van Schaftingen [12], using a proof similar to our second one.

Below we shall discuss three examples, all having their own advantages and disadvantages. In two examples we shall work with a periodic background, in the remaining with a non-periodic one. In the first example we shall work with the Gagliardo semi-norm itself (see (2)). In the other cases our computations will rely on norm equivalences whose proofs are sometimes delicate.

2 The first example

We shall work with the Gagliardo semi-norm. In some sense the first example is elementary.

Before proceeding, let us note that it suffices to establish the following fact:

with \( s, p, q \) as above and with \( T = \mathbb{R}/(2\pi\mathbb{Z}) \) the standard torus, there exists some \( g \in W^{s,p}(T) \setminus W^{s,q}(T) \). (3)

Proof of "(3) implies Theorem 1.1". Let \( g \in W^{s,p}(T) \setminus W^{s,q}(T) \). Using a partition of unity on \( T \), we find that for some \( \varphi \in C^\infty \) supported in some interval of length \(< 2\pi\), the function \( h := \varphi f \) is in \( W^{s,p}(T) \setminus W^{s,q}(T) \). By the choice of \( \varphi \), \( h \) can be identified with a compactly supported function in \( W^{s,p}(\mathbb{R}) \setminus W^{s,q}(\mathbb{R}) \).

Consider next some function \( \psi \in C^\infty_c(\mathbb{R}^{n-1}) \), \( \psi \not\equiv 0 \). Then clearly \( f := \psi \otimes h \) is compactly supported, and belongs to \( W^{s,p}(\mathbb{R}^n) \setminus W^{s,q}(\mathbb{R}^n) \).

For all \( \lambda > 0 \) and all \( x_0 \in \mathbb{R}^n \), the mapping \( f \mapsto f(\lambda(\cdot - x_0)) \) leaves the space \( W^{s,p}(\mathbb{R}^n) \) invariant. Applying this argument our construction yields a function supported in a ball whose radius and centre are at our disposal. □

For \( s = m + \sigma \), \( m \in \mathbb{N}_0 \) and \( 0 < \sigma < 1 \), the periodic fractional order Sobolev space \( W^{s,p}(T) \) can be normed with

\[
\|f\|_{W^{s,p}(T)} := \|f\|_{L^p} + \left( \int_T \int_T \frac{|\Delta_h f(n)(x)|^p}{|h|_{\sigma,p+1}} \; dhdx \right)^{1/p} 
\]

(obvious modification when \( p = \infty \)). Here, \( \Delta_h g(x) := g(x + h) - g(x) \).

We will rely on the Brezis-Lieb lemma [2] that we recall here: if \( 1 \leq p < \infty \), \( f_\ell \to f \) a.e. and \( \|f_\ell\|_{L^p} \leq C \), then

\[
\|f_\ell\|_{L^p}^p = \|f\|_{L^p}^p + \|f_\ell - f\|_{L^p}^p + o(1) \quad \text{as } \ell \to \infty.
\]

We also rely on the following straightforward

Lemma 2.1 We have

\[
\|x \mapsto e^{\ell x}\|_{W^{s,p}} \sim \ell^s \quad \text{as } \ell \to \infty.
\]
Proof. The case $p = \infty$ being left to the reader, we assume that $1 \leq p < \infty$. Clearly, it is enough to consider $0 < s < 1$. Set $f_\ell(x) = e^{i \ell x}$. Since $\|f_\ell\|_{L^p} \sim 1$, in order to prove the lemma it suffices to prove that

$$I_\ell := \int_T \int_T |\Delta h f_\ell(x)|^p |h|^s |h|^{sp+1} dxdh \sim \ell^p.$$  

This follows from the identity

$$I_\ell = 2\pi \ell^p \int_0^{2\pi} \frac{|e^{i\xi} - 1|^p}{|\xi|^{sp+1}} d\xi$$  

and the fact that the integral in (6) has a positive finite limit as $\ell \to \infty$. □

First proof of Theorem 1.1. We let to the reader the case where $p = \infty$, which is obtained by a rather straightforward modification of the argument below. We thus assume that $p < \infty$.

We will construct by induction on $j$ sequences $\lambda_j$ and $\ell_j$ such that

$$x \mapsto g(x) := \sum_{j \geq 1} \lambda_j e^{i \ell_j x} \text{ belongs to } W^s,p \text{ but not to } W^s,q.$$  

We pick $\lambda_1 = 1$, $\ell_1 = 1$. Assuming $\lambda_1, \ldots, \lambda_j, \ell_1, \ldots, \ell_j$ already constructed, let

$$f_\ell(x) := \frac{1}{j^{1/p} \ell^s} e^{i \ell x}.$$  

By Lemma 5, we have

$$\|f_\ell\|_{W^s,r} \sim \frac{1}{j^{1/q} \ell^s} \forall 1 \leq r < \infty.$$  

On the other hand, if we write $s = m + \sigma$ then we have $f_\ell \to 0$ and $f_\ell^{(m)} \to 0$ pointwise as $\ell \to \infty$. By the Brezis-Lieb lemma, for $1 \leq r < \infty$ we have, as $\ell \to \infty$,

$$\left\| x \mapsto \sum_{k=1}^j \lambda_k e^{i \ell_k x} + f_\ell(x) \right\|_{W^s,r}^r = \left\| x \mapsto \sum_{k=1}^j \lambda_k e^{i \ell_k x} \right\|_{W^s,r}^r + \|f_\ell\|_{W^s,r}^r + o(1).$$  

Thus, for large $\ell$, we have

$$\left\| x \mapsto \sum_{k=1}^j \lambda_k e^{i \ell_k x} + f_\ell(x) \right\|_{W^s,p}^p \leq \left\| x \mapsto \sum_{k=1}^j \lambda_k e^{i \ell_k x} \right\|_{W^s,p}^p + \frac{K_1}{j^{p/q}}.$$  

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and
\[ \left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{i \ell_k x} + f_t(x) \right\|_{W^{s,q}}^q \geq \left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{i \ell_k x} \right\|_{W^{s,q}}^q + \frac{K_2}{j}. \] (9)

Using (8) and (9), we construct \( \lambda_j \) and \( \ell_j \) such that
\[ \|g\|_{W^{s,p}(\mathbb{T})} \leq C_p + K_1 \sum_{j \geq 2} \frac{1}{j^{p/q}} \]
and
\[ \|g\|_{W^{s,q}(\mathbb{T})} \geq K_2 \sum_{j \geq 2} \frac{1}{j} \]
and thus \( g \) satisfies (7). \( \square \)

3 The second example

We shall work with lacunary series and Fourier-analytical characterizations of \( W^{s,p}(\mathbb{T}) \).

Therefore we recall the following characterization of \( W^{s,p}(\mathbb{T}) \) in terms of Fourier series, see [6, Theorem 3.5.3]. If \( f(x) = \sum f_\ell e^{i \ell x} \), set
\[ f^0 = f_0, \quad f^j(x) = \sum_{2^{j-1} < |\ell| \leq 2^j} f_\ell e^{i \ell x}, \quad \forall j \geq 1. \]

If \( 1 < p < \infty \), then
\[ \|f\|_{W^{s,p}(\mathbb{T})} \sim \left( \sum_{j \geq 0} 2^{jsp} \|f^j\|_{L^p}^p \right)^{1/p}. \] (10)

To incorporate the extremal cases \( p = 1 \) and \( p = \infty \) we need the following a little bit more technical modification. Let \( \psi \) be an infinitely differentiable compactly supported function such that \( \psi(x) = 1 \) if \( |x| \leq 1 \). We define
\[ \varphi_0(x) := \psi(x), \quad \varphi_j(x) := \psi(2^{-j} x) - \psi(2^{-j+1} x), \quad j = 1, 2, \ldots. \]
This results in a smooth dyadic decomposition of unity, i.e.,
\[ \sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}. \]
If we assume in addition supp $\psi \subset [-2, 2]$, then $\varphi_j(2^j) = 1$ and

$$\text{supp } \varphi_j \subset [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}], \quad j = 1, 2, \ldots,$$

follow. Just from the Fourier-analytic definition used in [6, Chapt. 3] we derive

$$\|f\|_{W^{s,p}(T)} \sim \left( \sum_{j \geq 0} 2^{jsp} \|\tilde{f}^j\|_{L^p}^{p} \right)^{1/p}, \quad (11)$$

where

$$\tilde{f}^j(x) = \sum_{\ell=-\infty}^{\infty} f_\ell \varphi_j(\ell) e^{i\ell x}, \quad j = 0, 1, \ldots,$$

and (11) holds for all $p \in [1, \infty]$.

**Second proof of Theorem 1.1.** We choose

$$\lambda_j := \frac{1}{2^{sj} j^{1/q}}, \quad \forall j \geq 1,$$

and put

$$g(x) := \sum_{j \geq 1} \lambda_j e^{i2^j x}.$$

Using either (10) (if $1 < p < \infty$) or (11) (for $p = 1$ or $p = \infty$), we clearly have $g \in W^{s,p}(T) \setminus W^{s,q}(T)$. \(\square\)

Note that the above yields an explicit version of our first example, in the sense that the $\lambda_j$’s and the $\ell_j$’s are given by explicit formulas.

### 4 The third example

In this example we apply wavelets. We follow [11, Section 1.7], but see also Meyer [5].

In this perspective, it will be more convenient to construct some

$$g$$

such that $g \in W_{c}^{s,p}(\mathbb{R})$ but $g \notin W_{c}^{s,q}(\mathbb{R}), \quad (12)$

i.e., we work in the non-periodic context from the very beginning.

Let $k > s+2$ be an integer, and consider father and mother Daubechies wavelets $\psi_F$ and $\psi_M$, compactly supported and of class $C^k$. Let, for $j \in \mathbb{N}$ and $m \in \mathbb{Z}$,

$$\psi^j_m(x) = \begin{cases} 
\psi_F(x - m), & \text{if } j = 0 \text{ and } m \in \mathbb{Z} \\
2^{(j-1)/2} \psi_M(2^{j-1} x - m), & \text{if } j \geq 1 \text{ and } m \in \mathbb{Z}.
\end{cases}$$
Set (assuming say $g \in L^1_{loc}$)
\[ g^m_j = \int_{\mathbb{R}} \psi^m_j(x) g(x) \, dx. \]

Then
\[ \|g\|_{W^{s,p}} \sim \left( \sum_{j=0}^{\infty} 2^{(sp+p/2-1)} \sum_{m \in \mathbb{Z}} |g^m_j|^p \right)^{1/p}, \] (13)
with the obvious modification when $p = \infty$.

**Third proof of Theorem 1.1.** The generators of the wavelet basis are compactly supported. Without loss of generality we may assume
\[ \text{supp} \, \psi_M \subset [0, N] \] (14)
for some $N = N(s)$ sufficiently large. We put
\[ \lambda_j := \frac{1}{2^{j(s+1/2)} j^{1/q}}, \quad j = 1, 2, \ldots \] (15)

Define
\[ g := \sum_{j=1}^{\infty} \lambda_j \sum_{m=0}^{2^j-1} \psi^m_j. \] (16)

By (15) and the fact that the $\psi^m_j$'s define an orthonormal basis in $L^2(\mathbb{R})$, we find that $g \in L^2(\mathbb{R})$, and in particular we have
\[ g^m_j = \begin{cases} \lambda_j, & \text{if } j \geq 1 \text{ and } 0 \leq m \leq 2^j - 1, \\ 0, & \text{otherwise}. \end{cases} \] (17)

By (14) and (16) we have $\text{supp} \, g \subset [0, N + 1]$. Finally, by (13), (15) and (17) we find that $g$ satisfies (12).

\[ \square \]

5 Besov spaces and the interpolation argument

Unlike the first proof, the second and the third one are suited to the scale of Besov or Triebel-Lizorkin spaces. This goes beyond the scope of this note. However, we would like to mention that in Example 2 and 3 we already used the identification of our fractional Sobolev spaces as special cases of Besov spaces. More exactly
\[ W^{\alpha,p}(\mathbb{T}) = B_{p,p}^\alpha(\mathbb{T}) \quad \text{and} \quad W^{\alpha,p}(\mathbb{R}^n) = B_{p,p}^\alpha(\mathbb{R}^n), \]
$s > 0, s \notin \mathbb{N}, 1 \leq p \leq \infty$, see [6, 3.5.4] and [10, 2.5.12].

In the framework of Besov spaces a straightforward adaptation of the second proof lead to the following improvement of (3):

$$W^{s,p}(\mathbb{T}) \nsubseteq B^{r}_{q,r}(\mathbb{T}) \quad \text{if} \quad p \geq q \quad \text{and} \quad r < p.$$  \hfill (18)

Completely analogous, Example 3 yields the following counterpart for non-periodic spaces

$$W^{s,p}(\Omega) \nsubseteq B^{r}_{q,r}(\Omega) \quad \text{if} \quad p \geq q \quad \text{and} \quad r < p.$$  \hfill (19)

Here the Besov space on the domain $\Omega$ is defined by restriction, i.e., $f \in L^{q}(\Omega)$ belongs to $B^{r}_{q,r}(\Omega)$ if there exists some $g \in B^{r}_{q,r}(\mathbb{R}^{n})$ such that $f = g$ on $\Omega$.

Some comments to the literature. Necessary and sufficient conditions for embeddings of one Besov space into another can be found in Taibleson [8], S., Triebel [7] and Haroske, Skrzypczak [4]. Whereas in [7] the authors were dealing with the situation on $\mathbb{R}^{n}$, Taibleson [8] also considered the periodic case. E.g., (18) can be found in [8, Thm. 19(b)]. For smooth domains $\Omega$ Haroske and Skrzypczak [4] have proved (19) in the much more general context of Besov-Morrey spaces.

Finally, for convenience of the reader, we will comment on the "interpolation argument" from page 1. We restrict ourselves to real and complex interpolation. It is known that

$$(L^{u}(0,1),W^{1,u}(0,1))_{1/2,r} = B^{1/2}_{u,r}(0,1), \quad 1 \leq r \leq \infty.$$  \hfill (1)

Now, choosing $u = r = 3$ we conclude

$$W^{1/2,3}(0,1) = (L^{3}(0,1),W^{1,3}(0,1))_{1/2,3} \quad \mapsto (L^{2}(0,1),W^{1,2}(0,1))_{1/2,3} = B^{1/2}_{3,3}(0,1).$$

The Besov space $B^{1/2}_{2,3}(0,1)$ does not belong to the scale of fractional Sobolev spaces under consideration, it is just a space containing $W^{1/2,2}(0,1) = B^{1/2}_{2,2}(0,1)$. Similarly for the complex method we obtain that

$$[L^{u}(0,1),W^{1,u}(0,1)]_{1/2} = F^{1/2}_{u,2}(0,1), \quad 1 < u < \infty.$$  \hfill (2)

Here $F^{1/2}_{u,2}(0,1)$ denotes a Lizorkin-Triebel space. Again choosing $u = 3$ we conclude

$$F^{1/2}_{3,2}(0,1) = [L^{3}(0,1),W^{1,3}(0,1)]_{1/2} \quad \mapsto [L^{2}(0,1),W^{1,2}(0,1)]_{1/2} = W^{1/2,2}(0,1).$$

The Lizorkin-Triebel space $F^{1/2}_{3,2}(0,1)$ does also not belong to the scale of fractional Sobolev spaces, it is just a space embedded into $W^{1/2,2}(0,1)$. For all this we refer to [1, 6.4] and [9, 2.4].
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