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# A two-phase two-layer model for fluidized granular flows with dilatancy effects

François Bouchut\*, Enrique D. Fernández-Nieto†  
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## Abstract

We propose a two-phase two-thin-layer model for fluidized debris flows that takes into account dilatancy effects, based on the closure relation proposed by Roux and Radjai (1998). This relation implies that the occurrence of dilation or contraction of the granular material depends on whether the solid volume fraction is respectively higher or lower than a critical value. When dilation occurs, the fluid is sucked into the granular material, the pore pressure decreases and the friction force on the granular phase increases. On the contrary, in the case of contraction, the fluid is expelled from the mixture, the pore pressure increases and the friction force diminishes. To account for this transfer of fluid into and out of the mixture, a two-layer model is proposed with a fluid layer on top of the two-phase mixture layer. Mass and momentum conservation are satisfied for the two phases, and mass and momentum are transferred between the two layers. A thin-layer approximation is used to derive average equations. Special attention is paid to the drag friction terms that are responsible for the transfer of momentum between the two phases and for the appearance of an excess pore pressure with respect to the hydrostatic pressure. We obtain a depth-averaged model with a dissipative energy balance in accordance with the corresponding 3D initial system.

**Keywords:** Fluidized granular flows, two-phase, dilatancy, two-layer, depth-averaged model, critical volume fraction, excess pore pressure

## 1 Introduction

Gravity driven flows such as debris flows, sub-aerial and submarine landslides play a key role in erosion processes on the Earth's surface. They represent one of the major natural hazards threatening life and property in mountainous, volcanic, seismic and coastal areas, as shown recently by the debris flows that occurred in Uganda and Brazil in 2010, causing 400 and 350 deaths respectively and displacing several hundred thousand inhabitants.

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One of the ultimate goals of landslide studies is to produce tools for the prediction of velocity and runout extent of rapid landslides. Developing a theoretical description and physical understanding of the associated processes in a natural environment remains an unsolved and extremely challenging problem in Earth science, mechanics and mathematics. Recent progress in the mathematical, physical and numerical modelling of gravity driven flows has led to the development and use of numerical models for investigating geomorphological processes and assessing risks related to such natural hazards. However, severe limitations prevent us from fully understanding the physical processes acting in natural flows and from predicting landslide dynamics and deposition. One of the important issues is that existing models do not accurately account for the co-existence and interaction of fluid (water and gas) and solid granular phases within the flowing mass, which play a key role in natural gravity related instabilities. Water is almost always present in natural landslides and the frequently resulting debris flows (mixture of water and grains) are often highly destructive.

The interaction between the fluid and granular phases within a saturated mixture essentially depends on the fluid pressure, also called pore pressure, that determines the effective friction force acting on the granular medium [17, 12, 13]. A change in the fluid pressure may result from a dilation of the granular phase, that induces a sucking of the fluid within the mixture and a diminution of the fluid pressure, thereby increasing the effective friction on the granular phase. On the other hand, a contraction of the granular phase induces an expulsion of the fluid from the mixture and an increase of the fluid pressure, thereby decreasing the effective friction. This process is sometimes called “pore pressure feedback” [13]. Contraction of a grain-fluid mixture may lead to liquefaction of the mixture, see e.g. [3]. Dilation and contraction occur in particular in response to the shearing of the granular medium. Indeed, a densely packed granular assembly (high solid volume fraction) must dilate to be sheared, in order for the grains to have room enough to move one with respect to the other. On the other hand, a loosely packed assembly contracts in response to shearing [3]. These processes play a dramatic role in the dynamics of fluidized granular flows, from their initial destabilization to their final deposition [2, 15, 8, 21, 31].

Taking into account dilatancy effects in numerical models of granular flows is a crucial issue. However, solving the complete 3D equations of granular mass motion, with sufficient resolution to describe the real topography, requires prohibitive computational costs. For this reason, it is necessary to write simplified models. A class of efficient techniques, developed and successfully employed to reproduce a large range of experimental and geological observations, makes use of a depth-averaged continuum description, based on the thin-layer approximation (i.e. the thickness of the flowing mass is assumed to be small compared to its downslope extension) [Savage and Hutter, 1989]. This leads to the assumption that the velocity normal to the topography is small compared to the downslope velocity. Taking into account two-phase grain-fluid mixtures and dilatancy in the thin-layer approximation raises significant mathematical difficulties because of the need for a consistent description of these effects within this approximation. In particular, contraction-dilation induces a relative motion of the fluid and solid phases in the direction normal to the topography, that is formally small in the thin-layer asymptotic expansion. The drag friction force between the fluid and solid phases is however strong enough to make it important to take this relative motion into account in the asymptotic model as detailed in this paper.

The solid-fluid mixture models described in the literature are generally based on Jackson’s model [17] that describes the main interactions between the two phases, such as buoyancy and drag frictional

forces. The system of four equations of mass and momentum conservation for the two phases has five unknowns: the solid volume fraction, the solid and fluid pressures and the solid and fluid velocities. As a result, a scalar closure equation is necessary to complete the model. Several depth-averaged thin-layer models have been deduced from Jackson’s model (e.g. [29], [27], [26], [16]). Pitman and Le in [29] followed by Pelanti et al. in [27] replaced the closure relation by an extra boundary condition for the pressure at the free surface. This leads to an overdetermined problem at the free surface, and to an underdetermined problem inside the domain. However, given the hydrostatic pressure assumption, a depth-averaged model can be obtained since the disappearance of the normal variable gives a kind of equivalence between a boundary condition and a closure relation inside the domain. The lack of a relevant closure equation leads to a non-dissipative energy balance in the Pitman and Le model, as well as in its variants. Moreover, these models do not take into account dilatancy effects. See [4] for more details on the different methods used to tackle this problem and on the validity of the proposed closure relations.

A crucial point in order to obtain a realistic model is that the energy balance associated with the model must be physically relevant. A main objective here is to propose a closure equation that gives such an energy balance. Along this line, in our previous work [4] we proposed a depth-averaged two-phase debris flow model that gives a dissipative energy balance. In that model, the closure equation is simply the incompressibility of the solid phase, so that dilatancy is not accounted for. Moreover, in order to avoid overdetermined boundary conditions, only the sum of the solid and fluid normal stresses is set to zero at the free surface, instead of both separately. We propose here to close Jackson’s model by including dilatancy effects, based on the model proposed by Roux and Radjai [32] for dry granular flows. In this model, the dilation rate is directly related to the volume fraction and is taken to be equal to  $\dot{\gamma} \tan \psi$ , where  $\dot{\gamma}$  is the shear rate and  $\psi$  is the “dilation angle” that depends on the volume fraction. This description of dilatancy has been used in [26] to develop a thin-layer depth-averaged two-phase model for immersed granular flows. However, in that model, the authors followed the ideas of Pitman and Le and imposed too many boundary conditions at the free surface. They therefore had to drop mass conservation. Furthermore, they made strong assumptions to approximate the velocities in the direction normal to the slope from the tangential velocities. In their final model, the dilatancy effect appears through an excess pore pressure term, in addition to the hydrostatic pressure.

Other kinds of debris-flow models are based on the idea of a single-phase mixture model. The first such model, also deduced from Jackson’s equations, was presented by Iverson in [11]. Other versions have since been proposed in [14, 8, 16, 9], still based on a single-phase mixture model. As a result, the relative motion between the solid and fluid phases does not appear explicitly. The mass and momentum equations for the mixture are coupled to an advection-diffusion equation to describe the changes in pore pressure. To close the system, they assume that the mixture obeys a Darcy law and they use a closure relation that takes into account the dilatancy effects. More precisely, George and Iverson [16] considered a modification of the Roux and Radjai dilatancy law in order to introduce the variations of the effective stress, already proposed in [14]. In this case the dilation rate is given by  $\dot{\gamma} \tan \psi - \alpha \partial_t(\sigma - p^f)$ , where  $\alpha$  is the compressibility of the mixture,  $\sigma$  the total stress and  $p^f$  the fluid pressure. The definition of  $\alpha$  is discussed in [2].

The aim of this paper is to establish a depth-averaged two-phase thin-layer model including dilatancy effects from Jackson’s model with the Roux and Radjai closure. As opposed to previously cited works, and in order to be consistent with the physical processes described above, we consider an extra upper fluid layer, that allows the fluid to be expelled or sucked in from the mixture at its upper boundary. This also allows us to resolve the overdetermination at the boundary, because now there

are two moving surfaces. This is a key point in our approach. An asymptotic analysis is performed to derive the depth-averaged system. We show that the effect of dilatancy on the fluid pressure appears through an extra contribution to the hydrostatic pressure, the so-called excess pore pressure. It is strongly related to the normal relative motion between the granular and fluid phases. We prove additionally that the proposed model satisfies a dissipative energy balance equation, as well as the initial 3D starting system. This is obtained via a compressible interpretation of our model.

The paper is organized as follows. Section 2 describes the 3D starting mixture system together with closure equation and boundary conditions. The thin-layer model is derived in Section 3, where the scaling assumptions are specified. In Section 4, we discuss the properties of our thin-layer model and the differences with other models in the literature. Section 5 presents our conclusions. In the Appendix, we provide the technical calculations justifying the boundary conditions at the interface.

## 2 Two-phase mixture model

### 2.1 Jackson's model

The starting point of our derivation is the same as in [4], i.e. the 3D model proposed by Jackson [17] for flows of solid granular materials filled (saturated) with fluid. The two mass conservation equations for the solid and fluid phases are, respectively,

$$\partial_t(\rho_s\varphi) + \nabla \cdot (\rho_s\varphi v) = 0, \quad (2.1a)$$

$$\partial_t(\rho_f(1 - \varphi)) + \nabla \cdot (\rho_f(1 - \varphi)u) = 0, \quad (2.1b)$$

and equations of momentum conservation for each phase are

$$\rho_s\varphi(\partial_tv + (v \cdot \nabla)v) = -\nabla \cdot T_s + f_0 + \rho_s\varphi \mathbf{g}, \quad (2.2a)$$

$$\rho_f(1 - \varphi)(\partial_tu + (u \cdot \nabla)u) = -\nabla \cdot T_{f_m} - f_0 + \rho_f(1 - \varphi)\mathbf{g}. \quad (2.2b)$$

The velocities are  $v$  for the solid phase and  $u$  for the fluid phase, while  $T_s$  and  $T_{f_m}$  denote the (symmetric) stress tensors for the solid and the fluid, respectively. Moreover, the constant densities are denoted by  $\rho_s$  and  $\rho_f$ . Acceleration due to gravity is denoted by  $\mathbf{g}$ , and  $f_0$  represents the average value of the resultant force exerted by the fluid on a solid particle. The solid volume fraction is  $\varphi$ . The combination of (2.1a) and (2.1b) yields the mass conservation for the mixture

$$\partial_t(\rho_m) + \nabla \cdot (\rho_m V_m) = 0, \quad (2.3)$$

where

$$\rho_m = \rho_s\varphi + \rho_f(1 - \varphi), \quad V_m = \frac{\rho_s\varphi v + \rho_f(1 - \varphi)u}{\rho_s\varphi + \rho_f(1 - \varphi)}, \quad (2.4)$$

are the density and velocity of the mixture, respectively. Dividing (2.1a) by  $\rho_s$ , (2.1b) by  $\rho_f$  and adding the results gives

$$\nabla \cdot (\varphi v + (1 - \varphi)u) = 0, \quad (2.5)$$

that can be written also  $\nabla \cdot v = \nabla \cdot ((1 - \varphi)(v - u))$ . Note that this relation does not imply that  $\nabla \cdot V_m$  is equal to zero.

According to Jackson [1] and as in [4], the force  $f_0$  is decomposed into the sum of the buoyancy force  $f_B$  and all remaining contributions  $f$ ,

$$f_0 = f_B + f = -\varphi \nabla p_{f_m} + f, \quad (2.6)$$

where  $p_{f_m}$  is the fluid pressure in the mixture (pore pressure). The term  $f$  combines the drag force, the lift force and the virtual mass force (see [17] for details). Here, we assume that  $f$  can be expressed simply by the drag force, thus

$$f = \tilde{\beta}(u - v), \quad (2.7)$$

$\tilde{\beta}$  being the drag coefficient given by

$$\tilde{\beta} = \frac{(\rho_s - \rho_f)\varphi g}{v_T(1 - \varphi)^{m-1}}, \quad (2.8)$$

where  $v_T$  is the terminal velocity of an isolated representative solid particle falling in the fluid under gravity, see [30, 29] for details.

By substituting (2.6) into (2.2a) and (2.2b), we obtain

$$\rho_s \varphi (\partial_t v + (v \cdot \nabla)v) = -\nabla \cdot T_s - \varphi \nabla p_{f_m} + f + \rho_s \varphi \mathbf{g}, \quad (2.9a)$$

$$\rho_f (1 - \varphi) (\partial_t u + (u \cdot \nabla)u) = -\nabla \cdot T_{f_m} + \varphi \nabla p_{f_m} - f + \rho_f (1 - \varphi) \mathbf{g}. \quad (2.9b)$$

Note that adding (2.9a) and (2.9b) and taking into account (2.1a), (2.1b) yields the conservation of total momentum

$$\begin{aligned} & \partial_t (\rho_s \varphi v + \rho_f (1 - \varphi)u) + \nabla \cdot (\rho_s \varphi v \otimes v + \rho_f (1 - \varphi)u \otimes u + T_s + T_{f_m}) \\ & = (\rho_s \varphi + \rho_f (1 - \varphi)) \mathbf{g}. \end{aligned} \quad (2.10)$$

We shall assume rheologies of the form

$$T_s = p_s \text{Id} + \widetilde{T}_s, \quad T_{f_m} = p_{f_m} \text{Id} + \widetilde{T}_{f_m}, \quad (2.11)$$

where  $p_s$  and  $p_{f_m}$  are the total pressures for the solid and fluid within the mixture, respectively, and  $\widetilde{T}_s, \widetilde{T}_{f_m}$  need to be defined, according to rheological assumptions. The system of four equations (2.1a), (2.1b), (2.9a), (2.9b) has five unknowns  $\varphi, p_s, p_{f_m}, u$  and  $v$ . Thus, as exposed in [4], it is not closed, and this is due to the averaging process used for its deduction (see [17] for details). Therefore, a closure relation is needed, under the form of an additional scalar equation that should be imposed, based on the physical processes involved. A possible closure is to impose the incompressibility of the solid phase,  $\nabla \cdot v = 0$ , considered in the previous work [4]. But in real granular materials the dilatancy effects, due to geometrical congestion, may induce changes on the solid dilation rate  $\nabla \cdot v$ , even if the mass of the granular material remains constant. This effect has to be included in the model instead of incompressibility.

## 2.2 Closure and energy balance

The energy balance associated to Jackson's system can be written, as in [4],

$$\begin{aligned} & \partial_t \left( \rho_s \varphi \frac{|v|^2}{2} + \rho_f (1 - \varphi) \frac{|u|^2}{2} - (\mathbf{g} \cdot X)(\rho_s \varphi + \rho_f (1 - \varphi)) \right) \\ & + \nabla \cdot \left( \rho_s \varphi \frac{|v|^2}{2} v + \rho_f (1 - \varphi) \frac{|u|^2}{2} u - (\mathbf{g} \cdot X)(\rho_s \varphi v + \rho_f (1 - \varphi)u) \right. \\ & \quad \left. + p_{f_m} (\varphi v + (1 - \varphi)u) + \widetilde{T}_{f_m} u + T_s v \right) \\ & = T_s : \nabla v + \widetilde{T}_{f_m} : \nabla u + f \cdot (v - u), \end{aligned} \quad (2.12)$$

where  $X$  denotes the space position. The friction effects give naturally a dissipative term  $f \cdot (v - u) \leq 0$ , and it is also natural to assume that  $\widetilde{T}_{f_m} : \nabla u \leq 0$ . The sign of  $T_s : \nabla v$  remains however undetermined. Since by (2.11)

$$T_s : \nabla v = p_s \nabla \cdot v + \widetilde{T}_s : \nabla v, \quad (2.13)$$

and it is also natural to have  $\widetilde{T}_s : \nabla v \leq 0$ , it remains the term  $p_s \nabla \cdot v$ . As mentioned above, the closure relation that states the incompressibility of the solid phase  $\nabla \cdot v = 0$  gives a consistent energy balance and the model of [4], but does not take into account dilatancy. Thus we consider the following closure equation to Jackson's model, involving the solid dilation rate  $\nabla \cdot v$ ,

$$\nabla \cdot v = \Phi, \quad (2.14)$$

with  $\Phi$  a function to be determined, that may depend on the unknowns of the system, as discussed in the next subsection. This kind of “weakly compressible” closure is considered in low Mach number flows, see for example [28]. This equation (2.14) together with (2.1a), (2.1b), (2.9a), (2.9b), (and (2.11) with suitable definitions of  $\widetilde{T}_s$ ,  $\widetilde{T}_{f_m}$ ), gives a closed system. Then in the right-hand side of (2.12) with the decomposition (2.13), only the first term  $p_s \Phi$  is not always nonpositive. This term is further analyzed in Subsection 2.4.

### 2.3 Dilatancy in dense granular flows

In the work of ROUX & RADJAI [32], a model for introducing dilatancy effects into the behaviour of dry granular media is proposed. This effect is directly related to the changes experimented by the solid volume fraction. In particular, the rate of volume change is given by  $\dot{\gamma} \tan \psi$ , where  $\dot{\gamma} = |Dv|$  is the norm of the strain rate  $Dv = (\nabla v + \nabla v^t)/2$ , and  $\psi$  is the so called “dilation angle”. This means more explicitly that

$$\partial_t \varphi + v \cdot \nabla \varphi = -\varphi \dot{\gamma} \tan \psi. \quad (2.15)$$

From the mass equation (2.1a) we have  $\partial_t \varphi + v \cdot \nabla \varphi = -\varphi \nabla \cdot v$ , thus we can reformulate (2.15) as a relation between the solid dilation rate  $\nabla \cdot v$  and the dilation angle  $\psi$ , as

$$\nabla \cdot v = \dot{\gamma} \tan \psi. \quad (2.16)$$

The dilation angle  $\psi$  is in turn related to the solid volume fraction  $\varphi$ , and a linear approximation can be written  $\psi = a(\varphi - \varphi_c)$ , with  $a > 0$ , and  $\varphi_c$  the critical-state compacity, that corresponds to the volume fraction obtained when a steady-state regime is reached. This critical-state compacity is in general an increasing function of the pressure  $p_s$ , see [3]. This approach allows to recover the different behaviours of loose and dense granular media, according to the sign of  $\varphi - \varphi_c$ . Namely, for a dense packing  $\varphi > \varphi_c$ , one has a positive dilation angle,  $\psi > 0$ , that induces dilation of the granular medium,  $\nabla \cdot v > 0$ , while for a loose packing  $\varphi < \varphi_c$ , one has a negative dilation angle,  $\psi < 0$ , that induces contraction of the granular medium,  $\nabla \cdot v < 0$ . This is valid as soon as  $\dot{\gamma} > 0$ , i.e. when a deformation occurs.

PAILHA & POULIQUEN deal in [26] with the immersed granular flows system. In that case, the critical solid volume value  $\varphi_c$  is modified in order to take into account the viscosity of the fluid in the mixture. Furthermore, they consider the precedent model where a linearization of  $\tan \psi$  is proposed. In the case when the viscosity of the fluid is neglected, we can write this approximation as follows,

$$\tan \psi = K(\varphi - \varphi_c), \quad (2.17)$$

$K > 0$  being a calibration constant (dilation constant). We adopt this dilation model to write

$$\nabla \cdot v = K\dot{\gamma}(\varphi - \varphi_c). \quad (2.18)$$

Thus the closure considered in this work for (2.14) is

$$\Phi = K\dot{\gamma}(\varphi - \varphi_c). \quad (2.19)$$

As exposed by IVERSON in [13], [33], there is a coupling between the dilatancy and the pore pressure, called ‘‘pore pressure feedback’’. This effect plays an important role in the way a landslide starts, and then dramatically affects the flow dynamics. The formula (2.18) well reproduces the contraction-dilation effects (see [3], [26]), which are

- If  $\varphi > \varphi_c$  then the granular medium dilates ( $\nabla \cdot v > 0$ ) as soon as there is a deformation ( $\dot{\gamma} > 0$ ). Consequently,
  - the fluid must be sucked into the mixture,
  - the pore pressure decreases.
- If  $\varphi < \varphi_c$  then the granular medium contracts ( $\nabla \cdot v < 0$ ) as soon as there is a deformation ( $\dot{\gamma} > 0$ ). Consequently,
  - the fluid must be expelled from the mixture,
  - the pore pressure increases.

The type of closure (2.18) entails a modification of the coefficient of the Coulomb friction law that becomes  $\tan(\delta + \psi)$  instead of  $\tan \delta$ . By linearization, we can write an effective friction coefficient as

$$\tan \delta_{\text{eff}} = \tan \delta + \tan \psi. \quad (2.20)$$

In the thin-layer expansion performed below, we neglect the deviatoric solid stress  $\tilde{T}_s$  inside the mixture, and only consider the bottom solid friction with the friction coefficient  $\tan \delta_{\text{eff}}$ .

## 2.4 Interpretation as a compressible model

We would like here to propose an interpretation of the ROUX & RADJAI dilatancy relation under the form (2.18) as a compressible model, that enables to write down a fully dissipative energy equation.

We consider the critical volume fraction  $\varphi_c$  to be an increasing function of the solid pressure,  $\varphi_c = \varphi_c(p_s)$ , bounded by some maximal value  $\varphi_{\text{max}} \sim 0.6$ . This function  $\varphi = \varphi_c(p_s)$  can be defined by its inverse  $p = p_c(\varphi)$  ( $p_c(\varphi)$  being called the critical pressure), as for example  $p_c(\varphi) = \mathcal{K}\varphi^\gamma / (\varphi_{\text{max}} - \varphi)^\iota$ , for some coefficient  $\mathcal{K}$ , and some exponents  $\gamma, \iota$ . Particular dependencies of  $p_c(\varphi)$  in  $\varphi$  appear for example in [18], [19]. With further linear approximation  $K(\varphi - \varphi_c(p_s)) \simeq K_p(p_c(\varphi) - p_s)$ , the relation (2.18) could be written then as

$$\nabla \cdot v = K_p \dot{\gamma}(p_c(\varphi) - p_s). \quad (2.21)$$

Classically in thermodynamics, the mechanical internal energy  $U$  is related to the pressure  $p$  and volume  $V$  by the relation  $dU = -pdV$ . Here the specific volume (i.e. volume per mass unit) is  $1/(\rho_s \varphi)$ , thus to the critical pressure  $p_c(\varphi)$  one can associate by this relation a specific internal energy (i.e. internal energy per mass unit)  $e_c(\varphi)$ . Since  $d(1/\varphi) = -d\varphi/\varphi^2$ , we obtain the differential relation

$$\frac{de_c}{d\varphi} = \frac{p_c}{\rho_s \varphi^2}. \quad (2.22)$$



Then writing the mass equation (2.1a) as  $\partial_t \varphi + v \cdot \nabla \varphi + \varphi \nabla \cdot v = 0$ , and multiplying it by  $de_c/d\varphi$ , we get

$$\partial_t e_c + v \cdot \nabla e_c + \frac{p_c}{\rho_s \varphi} \nabla \cdot v = 0. \quad (2.23)$$

Multiplying this by  $\varphi$  and using again (2.1a) yields

$$\partial_t(\varphi e_c) + \nabla \cdot (\varphi e_c v) + \frac{p_c}{\rho_s} \nabla \cdot v = 0. \quad (2.24)$$

Adding this times  $\rho_s$  to the energy equation (2.12) gives

$$\begin{aligned} & \partial_t \left( \rho_s \varphi \frac{|v|^2}{2} + \rho_f (1 - \varphi) \frac{|u|^2}{2} - (\mathbf{g} \cdot X) (\rho_s \varphi + \rho_f (1 - \varphi)) + \rho_s \varphi e_c \right) \\ & + \nabla \cdot \left( \rho_s \varphi \frac{|v|^2}{2} v + \rho_f (1 - \varphi) \frac{|u|^2}{2} u - (\mathbf{g} \cdot X) (\rho_s \varphi v + \rho_f (1 - \varphi) u) \right. \\ & \quad \left. + p_{f_m} (\varphi v + (1 - \varphi) u) + \widetilde{T}_{f_m} u + T_s v + \rho_s \varphi e_c v \right) \\ & = (p_s - p_c) \nabla \cdot v + \widetilde{T}_s : \nabla v + \widetilde{T}_{f_m} : \nabla u + f \cdot (v - u). \end{aligned} \quad (2.25)$$

Now, according to (2.18) or (2.21), one has  $(p_s - p_c) \nabla \cdot v \leq 0$ , and the energy balance equation (2.25) has a nonpositive right-hand side. This means that, as required by the laws of physics, the total mechanical energy of the system is dissipated. In the case of (2.21), one can indeed write this dilatancy law as

$$p_s = p_c(\varphi) - \frac{\nabla \cdot v}{K_p \dot{\gamma}}, \quad (2.26)$$

which appears clearly as a compressible rheological law with bulk viscoplastic term, that can be compared with (4) in [18].

## 2.5 Domain and boundary conditions

We assume that the mixture ( $0 < \varphi < 1$ ) lies between a fixed bottom and an interface, and that between the interface and an upper free surface, there is only fluid ( $\varphi \equiv 0$ ), see Figure 1. The thickness of the mixture layer is denoted by  $h_m$ , the thickness of the fluid-only layer by  $h_f$ , and the fixed bottom is defined by a function  $b$ .

The fluid velocity in the top layer is denoted by  $u_f$ , and in the mixture layer by  $u$ , while  $v$  denotes the velocity of the solid phase. For other terms, we will use as general notation the subscript  $(\ )_s$  for the solid phase,  $(\ )_{f_m}$  for the fluid in the mixture and just  $(\ )_f$  for the fluid-only layer.

Then the solid equations (2.1a), (2.9a) are set in the mixture domain, while the fluid equations (2.1b), (2.9b) must hold within both domains. This yields for the fluid-only domain

$$\nabla \cdot u_f = 0, \quad (2.27a)$$

$$\rho_f (\partial_t u_f + (u_f \cdot \nabla) u_f) = -\nabla \cdot T_f + \rho_f \mathbf{g}, \quad (2.27b)$$

with the energy equation

$$\partial_t \left( \rho_f \frac{|u_f|^2}{2} - \rho_f (\mathbf{g} \cdot X) \right) + \nabla \cdot \left( \rho_f \frac{|u_f|^2}{2} u_f - \rho_f (\mathbf{g} \cdot X) u_f + T_f u_f \right) = \widetilde{T}_f : \nabla u_f. \quad (2.28)$$

We can also consider that (2.1a), (2.9a) hold in the upper domain with the convention that there  $\varphi = 0$  and  $T_s = 0$ . The closure equation (2.14) holds in the mixture domain.

The boundary conditions are taken as follows.

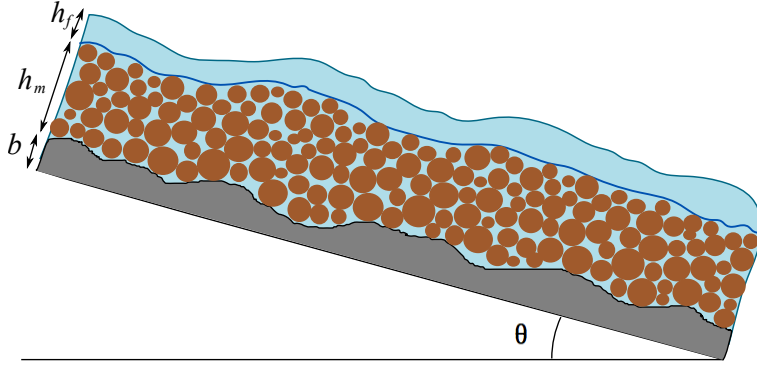


Figure 1: Domain and geometrical parameters. The solid-fluid mixture lies between a fixed bottom and an upper pure fluid layer. The width  $h_m$  of the mixture layer and the width  $h_f$  of the pure fluid layer evolve with time.

- At the bottom we consider the non penetration conditions

$$u \cdot n = 0, \quad v \cdot n = 0 \quad \text{at the bottom,} \quad (2.29)$$

where  $n$  is the upward space unit normal (i.e. the normal to the topography). This is completed with friction conditions. At first, a solid Coulomb friction law is applied,

$$(T_s n)_\tau = -\tan \delta_{\text{eff}} \text{sgn}(v)(T_s n) \cdot n \quad \text{at the bottom,} \quad (2.30)$$

where  $\delta_{\text{eff}}$  is the effective intergranular Coulomb friction angle from (2.20),  $\text{sgn}(v) = v/|v|$ , and the subscript  $\tau$  denotes the tangential projection,  $v_\tau = v - (v \cdot n)n$  for any vector  $v$ . Moreover, a Navier friction condition for the fluid phase is applied,

$$(T_{f_m} n)_\tau = -k_b u \quad \text{at the bottom,} \quad (2.31)$$

for some coefficient  $k_b \geq 0$ .

- At the free surface we assume no tension for the fluid

$$T_f N_X = 0 \quad \text{at the free surface,} \quad (2.32)$$

together with the kinematic condition

$$N_t + u_f \cdot N_X = 0 \quad \text{at the free surface,} \quad (2.33)$$

where  $N = (N_t, N_X)$  is a time-space normal to the free surface.

- At the interface, we consider the kinematic condition for the solid phase

$$\tilde{N}_t + v \cdot \tilde{N}_X = 0 \quad \text{at the interface,} \quad (2.34)$$

where we denote by  $\tilde{N} = (\tilde{N}_t, \tilde{N}_X)$  a time-space upward normal to the interface. Additional jump relations have to be prescribed. These relations state that the fluxes on both sides of the interface are related through transfer conditions. These are determined by global conservation properties, under the form of Rankine-Hugoniot conditions. We must first ensure that the total

fluid mass is conserved. The Rankine-Hugoniot condition associated to (2.1b), where  $\varphi$  vanishes in the fluid-only region, leads to

$$\tilde{N}_t + u_f \cdot \tilde{N}_X = (1 - \varphi^*)(\tilde{N}_t + u \cdot \tilde{N}_X) \equiv \mathcal{V}_f \quad \text{at the interface,} \quad (2.35)$$

where  $\varphi^*$  is the value of the solid volume fraction at the interface (the limit is taken from the mixture side). Note that  $\varphi$  is discontinuous at the interface. The term  $\mathcal{V}_f$  defines the fluid mass that is transferred from the mixture to the fluid-only layer ( $\mathcal{V}_f < 0$  means that the fluid is transferred from the fluid-only region to the mixture region). The equation (2.35) says that the amount of fluid that is entering in the fluid-only region is the same as the amount of fluid that leaves the mixture.

The conservation of the total momentum gives (see Appendix),

$$\rho_f \mathcal{V}_f (u - u_f) + (T_s + T_{f_m}) \tilde{N}_X = T_f \tilde{N}_X \quad \text{at the interface.} \quad (2.36)$$

The energy balance through the interface (see Appendix) yields the stress transfer condition

$$\begin{aligned} T_s \tilde{N}_X = & \left( \frac{\rho_f}{2} \left( (u - u_f) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|} \right)^2 \right. \\ & \left. + \left( (T_{f_m} \tilde{N}_X) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|^2} - p_{f_m} \right) \frac{\varphi^*}{1 - \varphi^*} \right) \tilde{N}_X \quad \text{at the interface.} \end{aligned} \quad (2.37)$$

These conditions are completed by a Navier fluid friction condition

$$\left( \frac{T_{f_m} + T_f}{2} \tilde{N}_X \right)_\tau = -k_i (u_f - u)_\tau \quad \text{at the interface,} \quad (2.38)$$

where  $k_i \geq 0$  is a friction coefficient. Note that we also have the relation (A.3), deduced from (2.35).

### 3 Derivation of the thin-layer depth-averaged model

In this section we derive a depth-integrated thin-layer model from the Jackson model with the closure stated in Section 2.

The geometrical setting is as follows. We have two layers, the one below being filled with the mixture of grains and fluid and the one above only with fluid (see Figure 1). The equations of mass and momentum in the mixture region are given by (2.1a), (2.1b), (2.9a) and (2.9b), closed by the relation (2.14) with  $\Phi$  defined by (2.19). The equations for the fluid-only layer are defined by (2.27a), (2.27b). The stress tensors for the solid and fluid phases in the mixture are given by (2.11). The boundary conditions are written in the previous subsection, as (2.29)-(2.38).

#### 3.1 Local coordinates

We now write the equations in local coordinates. We use a decomposition of the velocities and the derivatives in their longitudinal and normal components. We denote by  $\mathbf{x} = (x, y)$  a vector variable in a fixed plane inclined at angle  $\theta$ ,  $x$  being in the direction of the slope, and by  $z$  the variable normal

to this plane (see Figure 1). The equation of the bottom is thus given by  $z = b(\mathbf{x})$ , the interface by  $z = b(\mathbf{x}) + h_m(t, \mathbf{x})$  and the free surface by  $z = b(\mathbf{x}) + h_m(t, \mathbf{x}) + h_f(t, \mathbf{x})$ . The gravity vector is then  $\mathbf{g} = (-g \sin \theta, 0, -g \cos \theta)^t$  (the slope angle  $\theta$  is indeed negative on Figure 1). The velocities are written as  $u_f = (u_f^x, u_f^z)$ ,  $u_f^x = (u_f^x, u_f^y)$ ;  $u = (u^x, u^z)$ ,  $u^x = (u^x, u^y)$ ;  $v = (v^x, v^z)$ ,  $v^x = (v^x, v^y)$  and the gradient is  $\nabla = (\nabla_{\mathbf{x}}, \partial_z)$  with  $\nabla_{\mathbf{x}} = (\partial_x, \partial_y)$ . The equations can then be written as follows.

- In the mixture layer  $b < z < b + h_m$ :

$$\partial_t \varphi + \nabla_{\mathbf{x}} \cdot (\varphi v^{\mathbf{x}}) + \partial_z (\varphi v^z) = 0, \quad (3.1a)$$

$$\partial_t (1 - \varphi) + \nabla_{\mathbf{x}} \cdot ((1 - \varphi) u^{\mathbf{x}}) + \partial_z ((1 - \varphi) u^z) = 0, \quad (3.1b)$$

$$\rho_s \varphi (\partial_t v^{\mathbf{x}} + v^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} v^{\mathbf{x}} + v^z \partial_z v^{\mathbf{x}}) = -\nabla_{\mathbf{x}} \cdot T_s^{\mathbf{xx}} - \partial_z T_s^{\mathbf{xz}} - \varphi \nabla_{\mathbf{x}} p_{f_m} + f_{\mathbf{x}} - \varphi \rho_s g \sin \theta (1, 0)^t, \quad (3.2a)$$

$$\rho_s \varphi (\partial_t v^z + v^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} v^z + v^z \partial_z v^z) = -\nabla_{\mathbf{x}} \cdot T_s^{\mathbf{xz}} - \partial_z T_s^{zz} - \varphi \partial_z p_{f_m} + f_z - \varphi \rho_s g \cos \theta, \quad (3.2b)$$

$$\rho_f (1 - \varphi) (\partial_t u^{\mathbf{x}} + u^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} u^{\mathbf{x}} + u^z \partial_z u^{\mathbf{x}}) = -\nabla_{\mathbf{x}} \cdot T_{f_m}^{\mathbf{xx}} - \partial_z T_{f_m}^{\mathbf{xz}} + \varphi \nabla_{\mathbf{x}} p_{f_m} - f_{\mathbf{x}} - (1 - \varphi) \rho_f g \sin \theta (1, 0)^t, \quad (3.3a)$$

$$\rho_f (1 - \varphi) (\partial_t u^z + u^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} u^z + u^z \partial_z u^z) = -\nabla_{\mathbf{x}} \cdot T_{f_m}^{\mathbf{xz}} - \partial_z T_{f_m}^{zz} + \varphi \partial_z p_{f_m} - f_z - (1 - \varphi) \rho_f g \cos \theta, \quad (3.3b)$$

$$\nabla_{\mathbf{x}} \cdot v^{\mathbf{x}} + \partial_z v^z = \Phi. \quad (3.4)$$

- In the fluid-only layer  $b + h_m < z < b + h_m + h_f$ :

$$\nabla_{\mathbf{x}} \cdot u_f^{\mathbf{x}} + \partial_z u_f^z = 0, \quad (3.5)$$

$$\rho_f (\partial_t u_f^{\mathbf{x}} + u_f^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} u_f^{\mathbf{x}} + u_f^z \partial_z u_f^{\mathbf{x}}) = -\nabla_{\mathbf{x}} \cdot T_f^{\mathbf{xx}} - \partial_z T_f^{\mathbf{xz}} - \rho_f g \sin \theta (1, 0)^t, \quad (3.6a)$$

$$\rho_f (\partial_t u_f^z + u_f^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} u_f^z + u_f^z \partial_z u_f^z) = -\nabla_{\mathbf{x}} \cdot T_f^{\mathbf{xz}} - \partial_z T_f^{zz} - \rho_f g \cos \theta. \quad (3.6b)$$

The boundary conditions can be written as follows.

- At the bottom  $z = b$ , with  $n = (-\nabla_{\mathbf{x}} b, 1) / \sqrt{1 + |\nabla_{\mathbf{x}} b|^2}$ :

- Non-penetration condition for each phase

$$v^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} b = v^z \quad \text{at } z = b, \quad (3.7)$$

$$u^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} b = u^z \quad \text{at } z = b. \quad (3.8)$$

- Coulomb friction law

$$\frac{T_s^{\mathbf{xz}} - T_s^{\mathbf{xx}} \nabla_{\mathbf{x}} b + \nabla_{\mathbf{x}} b (T_s n) \cdot n}{\sqrt{1 + |\nabla_{\mathbf{x}} b|^2}} = -\tan \delta_{\text{eff}} \frac{v^{\mathbf{x}}}{\sqrt{|v^{\mathbf{x}}|^2 + (v^z)^2}} (T_s n) \cdot n \quad \text{at } z = b, \quad (3.9)$$

with

$$(T_s n) \cdot n = \frac{(T_s^{\mathbf{xx}} \nabla_{\mathbf{x}} b) \cdot \nabla_{\mathbf{x}} b - 2T_s^{\mathbf{xz}} \cdot \nabla_{\mathbf{x}} b + T_s^{zz}}{1 + |\nabla_{\mathbf{x}} b|^2}. \quad (3.10)$$

- Navier friction condition for the fluid phase

$$\frac{T_f^{xz} - T_f^{xx} \nabla_x b + \nabla_x b (T_f n) \cdot n}{\sqrt{1 + |\nabla_x b|^2}} = -k_b u^x \quad \text{at } z = b. \quad (3.11)$$

- At the free surface  $z = b + h_m + h_f$ , with  $N_X = (-\nabla_x(b + h_m + h_f), 1)$ ,  $N_t = -\partial_t(b + h_m + h_f)$ :

- Stress free condition

$$-T_f^{xx} \nabla_x(b + h_m + h_f) + T_f^{xz} = 0 \quad \text{at } z = b + h_m + h_f, \quad (3.12)$$

$$-T_f^{xz} \cdot \nabla_x(b + h_m + h_f) + T_f^{zz} = 0 \quad \text{at } z = b + h_m + h_f. \quad (3.13)$$

- Kinematic condition

$$\partial_t(h_m + h_f) + u_f^x \cdot \nabla_x(b + h_m + h_f) = u_f^z \quad \text{at } z = b + h_m + h_f. \quad (3.14)$$

- At the interface  $z = b + h_m$ , with  $\tilde{N}_X = (-\nabla_x(b + h_m), 1)$ ,  $\tilde{N}_t = -\partial_t(b + h_m)$ :

- Kinematic condition

$$\partial_t h_m + v^x \cdot \nabla_x(b + h_m) = v^z \quad \text{at } z = b + h_m. \quad (3.15)$$

- Conservation of fluid mass

$$\partial_t h_m + u_f^x \cdot \nabla_x(b + h_m) - u_f^z = (1 - \varphi^*)(\partial_t h_m + u^x \cdot \nabla_x(b + h_m) - u^z) \equiv -\mathcal{V}_f \quad (3.16)$$

at  $z = b + h_m$ .

- Conservation of total momentum

$$\rho_f \mathcal{V}_f(u^x - u_f^x) - (T_s^{xx} + T_f^{xx} - T_f^{xx}) \nabla_x(b + h_m) + T_s^{xz} + T_f^{xz} - T_f^{xz} = 0, \quad (3.17)$$

$$\rho_f \mathcal{V}_f(u^z - u_f^z) - (T_s^{zz} + T_f^{zz} - T_f^{zz}) \cdot \nabla_x(b + h_m) + T_s^{zz} + T_f^{zz} - T_f^{zz} = 0, \quad (3.18)$$

at  $z = b + h_m$ .

- Stress transfer

$$-T_s^{xx} \nabla_x(b + h_m) + T_s^{xz} = -p_s^* \nabla_x(b + h_m) \quad \text{at } z = b + h_m, \quad (3.19a)$$

$$-T_s^{xz} \cdot \nabla_x(b + h_m) + T_s^{zz} = p_s^* \quad \text{at } z = b + h_m, \quad (3.19b)$$

with

$$p_s^* = \frac{\rho_f}{2} \frac{1}{1 + |\nabla_x(b + h_m)|^2} \left( u^z - u_f^z - (u^x - u_f^x) \cdot \nabla_x(b + h_m) \right)^2 + \frac{\varphi^*}{1 - \varphi^*} \left( \frac{(T_f^{xx} \nabla_x(b + h_m)) \cdot \nabla_x(b + h_m) - 2T_f^{xz} \cdot \nabla_x(b + h_m) + T_f^{zz}}{1 + |\nabla_x(b + h_m)|^2} - p_{f_m} \right). \quad (3.20)$$

- Navier fluid friction

$$\begin{aligned} & T_f^{xz} + T_f^{xz} - (T_f^{xx} + T_f^{xx}) \nabla_x(b + h_m) \\ & + \nabla_x(b + h_m) \left( ((T_f^{xx} + T_f^{xx}) \nabla_x(b + h_m)) \cdot \nabla_x(b + h_m) \right. \\ & \quad \left. - 2(T_f^{xz} + T_f^{xz}) \cdot \nabla_x(b + h_m) + T_f^{zz} + T_f^{zz} \right) / (1 + |\nabla_x(b + h_m)|^2) \\ & = -2k_i \left( u_f^x - u^x + \nabla_x(b + h_m) \frac{u_f^z - u^z - (u_f^x - u^x) \cdot \nabla_x(b + h_m)}{1 + |\nabla_x(b + h_m)|^2} \right) \quad \text{at } z = b + h_m. \end{aligned} \quad (3.21)$$

## 3.2 Averaged mass equations

In order to get the averaged solid mass equation, we integrate (3.1a) with respect to  $z$  in the mixture layer  $b < z < b + h_m$ . Using (3.7) and (3.15) we obtain

$$\partial_t \int_b^{b+h_m} \varphi dz + \nabla_{\mathbf{x}} \cdot \int_b^{b+h_m} \varphi v^{\mathbf{x}} dz = 0. \quad (3.22)$$

Similarly, the fluid averaged mass equation in the mixture is obtained by integrating (3.1b) for  $b < z < b + h_m$ . According to (3.8) and (3.16) it gives

$$\partial_t \int_b^{b+h_m} (1 - \varphi) dz + \nabla_{\mathbf{x}} \cdot \int_b^{b+h_m} (1 - \varphi) u^{\mathbf{x}} dz = -\mathcal{V}_f. \quad (3.23)$$

Finally, the fluid averaged mass equation in the fluid-only layer is obtained by integrating (3.5) for  $b + h_m < z < b + h_m + h_f$  together with the conditions (3.14) and (3.16). It yields

$$\partial_t h_f + \nabla_{\mathbf{x}} \cdot \int_{b+h_m}^{b+h_m+h_f} u_f^{\mathbf{x}} dz = \mathcal{V}_f. \quad (3.24)$$

The sum of (3.23) and (3.24) gives indeed the total fluid mass conservation.

## 3.3 Asymptotic hypothesis

We introduce the characteristic width and length of the domain,  $H$  and  $L$  respectively, and the aspect ratio  $\epsilon = H/L$ , supposed to be small in agreement with the thin-layer framework. Then, we assume the following asymptotic scales in terms of  $\epsilon$ ,

$$\begin{aligned} h_m &\sim \epsilon, \quad h_f \sim \epsilon, \quad \nabla_{\mathbf{x}} b = \mathcal{O}(\epsilon), \quad T_s = \mathcal{O}(\epsilon), \quad T_{f_m} = \mathcal{O}(\epsilon), \quad T_f = \mathcal{O}(\epsilon), \\ v^{\mathbf{x}} &= \mathcal{O}(1), \quad u^{\mathbf{x}} = \mathcal{O}(1), \quad u_f^{\mathbf{x}} = \mathcal{O}(1), \quad \varphi = \mathcal{O}(1), \quad \Phi = \mathcal{O}(1), \\ k_b &= \mathcal{O}(\epsilon), \quad k_i = \mathcal{O}(\epsilon). \end{aligned} \quad (3.25)$$

These orders of magnitude have indeed to be expressed in the natural units of each quantity. Taking  $L$  as typical length unit,  $\tau = \sqrt{L/g}$  as typical time unit, all these natural units can be expressed in terms of  $L$ ,  $\tau$ , and  $\rho_s$  (or  $\rho_f$ , that is assumed of the same order of magnitude as  $\rho_s$ ). We assume that the unknowns vary at the scales  $L$  in the downslope direction,  $\epsilon L$  in the normal direction, and  $\tau$  in time, which means formally that  $\nabla_{\mathbf{x}} = \mathcal{O}(1)$ ,  $\partial_z = \mathcal{O}(\epsilon^{-1})$ ,  $\partial_t = \mathcal{O}(1)$ .

These scaling assumptions deserve some comments. First, the scaling in the downslope direction means that we are describing the observable phenomenon at the typical scale  $L$  where the collective phenomenon take place, this scale being much larger than the size of the grains. Second, the scaling in the normal direction means that there could be normal variations at the scale of the layer. Third, the time assumption means that we are describing transient flows typical in avalanche dynamics, that occur for example when an initial mass at rest is entrained by gravity. The time scale at which gravity comes into play is exactly  $\tau$ . In particular, this scaling does not describe well-established almost steady flows for which gravity balances viscoplastic effects.

Then, (3.24) implies that  $\mathcal{V}_f = \mathcal{O}(\epsilon)$ . As in [5, 6] we shall assume that the tangential velocities and the solid volume fraction do not depend on  $z$  up to errors in  $\mathcal{O}(\epsilon^2)$ ,

$$v^{\mathbf{x}} = \overline{v^{\mathbf{x}}}(t, \mathbf{x}) + \mathcal{O}(\epsilon^2), \quad (3.26)$$

$$u^{\mathbf{x}} = \overline{u^{\mathbf{x}}}(t, \mathbf{x}) + \mathcal{O}(\epsilon^2), \quad (3.27)$$

$$u_f^{\mathbf{x}} = \overline{u_f^{\mathbf{x}}}(t, \mathbf{x}) + \mathcal{O}(\epsilon^2), \quad (3.28)$$

$$\varphi = \overline{\varphi}(t, \mathbf{x}) + \mathcal{O}(\epsilon^2). \quad (3.29)$$

Then, from (3.4) and the boundary condition (3.7) we get that  $v^z = \mathcal{O}(\epsilon)$ . Similarly, from (3.1b) and (3.8), we get  $(1 - \varphi)u^z = \mathcal{O}(\epsilon)$ , thus  $u^z = \mathcal{O}(\epsilon)$ . Finally, from (3.5) and (3.16) we obtain  $u_f^z = \mathcal{O}(\epsilon)$ . We assume also for the closure function (2.19) an expansion as

$$\Phi = \bar{\Phi}(t, \mathbf{x}) + \mathcal{O}(\epsilon^2), \quad (3.30)$$

with

$$\bar{\Phi} = K\bar{\gamma}(\bar{\varphi} - \bar{\varphi}_c). \quad (3.31)$$

We adopt this approximation in order to make the derivation possible, even if it looks not appropriate because of the dependency on the pressure of  $\varphi_c$ , and of the nonlinear coupling of  $\dot{\gamma}$ . Without (3.30), one should analyze the dependency in  $z$  of  $\varphi$  and  $\Phi$ , as done in [22] in the dry case. The values for  $\bar{\gamma}$  and  $\bar{\varphi}_c$  are discussed in Subsection 4.6. Then using the closure equation (3.4), the equation (3.1a) for  $\varphi$  gives

$$\partial_t \bar{\varphi} + \bar{v}^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{\varphi} = -\bar{\varphi} \bar{\Phi} + \mathcal{O}(\epsilon^2). \quad (3.32)$$

About the stress tensors  $T_k$  ( $k = s, f_m, f$ ), they are decomposed as

$$T_k = p_k \text{Id} + \tilde{T}_k, \quad (3.33)$$

and suitable rheological assumptions should be made to define  $\tilde{T}_k$ . A general approach has been proposed in [7] to deal with velocity profiles in the thin-layer asymptotics and in the case of Newtonian or non-Newtonian rheologies. Here, as in [6], since we aim to represent only depth-average effects, we prefer to simplify the rheologies and replace the effect of the stress tensors inside the domain by boundary layers due to the friction conditions, namely (3.9), (3.11), (3.21), and also due to the momentum conservation (3.17), while we neglect viscous effects. Thus we shall assume that the stresses  $\tilde{T}_k$  are  $\mathcal{O}(\epsilon^2)$  far from the boundaries  $z = b, b + h_m$  and can just be nonzero close to these boundaries. Indeed, because of the particular form of (3.9), (3.11), (3.21), (3.17), we assume that

$$\begin{aligned} \tilde{T}_s^{\mathbf{x}z}, \tilde{T}_{f_m}^{\mathbf{x}z}, \tilde{T}_f^{\mathbf{x}z} &\text{ can be } \mathcal{O}(\epsilon) \text{ close to the boundaries } z = b, b + h_m, \\ &\text{ but are } \mathcal{O}(\epsilon^2) \text{ far from these boundaries,} \end{aligned} \quad (3.34)$$

while the other components satisfy

$$\tilde{T}_k^{\mathbf{x}\mathbf{x}} = \tilde{T}_k^{zz} = \mathcal{O}(\epsilon^2) \quad \text{everywhere.} \quad (3.35)$$

Regarding the drag term defined in (2.7), we have according to (2.8)

$$\tilde{\beta} = \bar{\beta}(t, \mathbf{x})(1 + \mathcal{O}(\epsilon^2)), \quad (3.36)$$

with

$$\bar{\beta} = \frac{(\rho_s - \rho_f)\bar{\varphi}g}{v_T(1 - \bar{\varphi})^{m-1}}. \quad (3.37)$$

We shall consider two possible sets of assumptions.

(I) The drag term is quite strong, that is

$$\bar{\beta} \sim \epsilon^{-1}. \quad (3.38)$$

Then since the drag force  $\tilde{\beta}(u - v)$  has to balance gravity terms, it necessarily remains bounded. This implies that after an eventual initial layer (i.e. a short time interval during which the initial value of  $u^{\mathbf{x}} - v^{\mathbf{x}}$  is damped), one has

$$u^{\mathbf{x}} - v^{\mathbf{x}} = \mathcal{O}(\epsilon). \quad (3.39)$$

(II) The drag term is moderate, that is

$$\bar{\beta} = \mathcal{O}(1). \quad (3.40)$$

In this case one has just  $u^{\mathbf{x}} - v^{\mathbf{x}} = \mathcal{O}(1)$ , according to (3.25).

Note that in both cases one has  $\bar{\beta}(u^{\mathbf{x}} - v^{\mathbf{x}}) = \mathcal{O}(1)$ . The relevance of the assumptions (3.38) or (3.40) can be evaluated as follows. According to (2.9a), the effective drag friction coefficient for the solid phase is  $\tilde{\beta}/\rho_s\varphi$ . The assumption (3.38) or (3.40) has to be evaluated in the corresponding unit, which means that we must evaluate the dimensionless number  $\bar{\beta}\tau/\rho_s\bar{\varphi}$ , with  $\tau = \sqrt{L/g}$  the reference time unit (see above). We compute using (3.37)

$$\frac{\bar{\beta}\tau}{\rho_s\bar{\varphi}} = \frac{(1 - \rho_f/\rho_s)g\tau}{v_T(1 - \bar{\varphi})^{m-1}} = \frac{(1 - \rho_f/\rho_s)}{(1 - \bar{\varphi})^{m-1}} \frac{L}{\tau v_T}. \quad (3.41)$$

The terminal velocity  $v_T$  can be expressed according to [25] as

$$v_T = \frac{\rho_s - \rho_f}{\rho_f} \frac{gd^2}{18\nu_f}, \quad (3.42)$$

where  $\nu_f$  is the kinematic viscosity of the fluid, and  $d$  is the diameter of the solid grains. This leads to another expression of (3.41) as

$$\frac{\bar{\beta}\tau}{\rho_s\bar{\varphi}} = \frac{\rho_f/\rho_s}{(1 - \bar{\varphi})^{m-1}} \frac{18\nu_f\tau}{d^2}, \quad (3.43)$$

which is proportional to  $\sqrt{L}/d^2$ . We can consider the values  $g = 9.81\text{m/s}^2$ ,  $\rho_f = 1000\text{kg/m}^3$  for water,  $\rho_s = 2500\text{kg/m}^3$ ,  $\nu_f = 10^{-6}\text{m}^2/\text{s}$ ,  $m = 1$ . In the typical laboratory scale experimental context, one has  $d = 5 \times 10^{-4}\text{m}$ , which gives  $v_T = 0.2\text{m/s}$ , and  $L \approx 0.2\text{m}$ , which gives a slightly strong dimensionless drag coefficient  $\bar{\beta}\tau/\rho_s\bar{\varphi} \approx 4.1$ . For natural landslides or large scale USGS debris flows [15], one can take  $d = 10^{-2}\text{m}$ , which gives  $v_T = 82\text{m/s}$ , and  $L \approx 20\text{m}$ , which gives a small dimensionless drag coefficient  $\bar{\beta}\tau/\rho_s\bar{\varphi} \approx 0.1$ . We conclude that the assumption (3.38) is more valid for small grains in an experimental context, while (3.40) is more valid in the natural context. However, (3.38) could be valid for a very viscous fluid, according to (3.43). This assumption (3.38) is also useful in order to focus our discussion on the effect of drag.

### 3.4 Averaged momentum equations

In order to get the averaged momentum equations, we have first to get expressions for the pressures. For the fluid-only layer we integrate the normal momentum equation (3.6b) with respect to  $z$  and use (3.13), (3.35) to get for  $b + h_m < z < b + h_m + h_f$

$$p_f = T_f^{zz} + \mathcal{O}(\epsilon^2) = \rho_f g \cos \theta (b + h_m + h_f - z) + \mathcal{O}(\epsilon^2). \quad (3.44)$$

In the mixture, the normal fluid momentum equation (3.3b) gives with (3.35)

$$\partial_z p_{f_m} = -\rho_f g \cos \theta - \frac{\bar{\beta}}{1 - \bar{\varphi}} (u^z - v^z) + \mathcal{O}(\epsilon). \quad (3.45)$$

Integrating with respect to  $z$ , we obtain for  $b < z < b + h_m$

$$p_{f_m} = p_{f_m|_{b+h_m}} + \rho_f g \cos \theta (b + h_m - z) + \frac{\bar{\beta}}{1 - \bar{\varphi}} \int_z^{b+h_m} (u^z - v^z)(z') dz' + \mathcal{O}(\epsilon^2), \quad (3.46)$$



where the notation  $|b + h_m$  means that the quantity is evaluated at  $z = b + h_m$ . From (3.18), we have  $p_{f_m|b+h_m} = p_f|b+h_m - p_s|b+h_m + \mathcal{O}(\epsilon^2)$ . Also from (3.19b) we have  $p_s|b+h_m = p_s^* + \mathcal{O}(\epsilon^2)$ , with according to (3.20),  $p_s^* = \mathcal{O}(\epsilon^2)$ . Thus

$$p_s^* = \mathcal{O}(\epsilon^2), \quad p_s|b+h_m = \mathcal{O}(\epsilon^2), \quad p_{f_m|b+h_m} = p_f|b+h_m + \mathcal{O}(\epsilon^2). \quad (3.47)$$

Then from (3.44) we obtain the pressure for the fluid in the mixture at the interface,

$$p_{f_m|b+h_m} = \rho_f g \cos \theta h_f + \mathcal{O}(\epsilon^2). \quad (3.48)$$

Finally with (3.46) we deduce the fluid pressure for the mixture layer,

$$p_{f_m} = \rho_f g \cos \theta (b + h_m + h_f - z) + p_{f_m}^e + \mathcal{O}(\epsilon^2), \quad (3.49)$$

where

$$p_{f_m}^e \equiv \frac{\bar{\beta}}{1 - \bar{\varphi}} \int_z^{b+h_m} (u^z - v^z)(z') dz' \quad (3.50)$$

is the excess pore pressure. In the expression (3.49) of the fluid pressure we can see that there is an extra contribution  $p_{f_m}^e$  to the commonly found hydrostatic pressure (3.44). A similar contribution to the hydrostatic pressure of the fluid phase is found in [26]. This excess pore pressure term is induced by the normal displacement produced by the dilation-compaction of the granular material immersed into the fluid. It is negative if the granular material goes up with respect to the fluid ( $v^z > u^z$ ), and positive in the converse case. It vanishes at  $z = b + h_m$ . Then, the solid normal momentum equation (3.2b) gives

$$\partial_z p_s = -\bar{\varphi} \partial_z p_{f_m} - \bar{\varphi} \rho_s g \cos \theta + \bar{\beta} (u^z - v^z) + \mathcal{O}(\epsilon). \quad (3.51)$$

Integrating with respect to  $z$  gives the expression of the solid pressure,

$$p_s = p_s|b+h_m - \bar{\varphi} (p_{f_m} - p_{f_m|b+h_m}) + \bar{\varphi} \rho_s g \cos \theta (b + h_m - z) - \bar{\beta} \int_z^{b+h_m} (u^z - v^z)(z') dz' + \mathcal{O}(\epsilon^2). \quad (3.52)$$

Using (3.47), (3.48), (3.49) and the notation (3.50), we finally obtain

$$p_s = \bar{\varphi} (\rho_s - \rho_f) g \cos \theta (b + h_m - z) - p_{f_m}^e + \mathcal{O}(\epsilon^2). \quad (3.53)$$

Let us now focus on the tangential components of momentum equations. For the fluid-only layer (3.6a), taking into account (3.44), we have

$$\rho_f (\partial_t \bar{u}_f^{\mathbf{x}} + \bar{u}_f^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{u}_f^{\mathbf{x}}) = -\rho_f g \cos \theta \nabla_{\mathbf{x}} (b + h_m + h_f) - \partial_z T_f^{\mathbf{x}z} - \rho_f g \sin \theta (1, 0)^t + \mathcal{O}(\epsilon^2). \quad (3.54)$$

Next we write the mixture tangential fluid momentum equation (3.3a), using (3.49),

$$\begin{aligned} \rho_f (1 - \bar{\varphi}) (\partial_t \bar{u}^{\mathbf{x}} + \bar{u}^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{u}^{\mathbf{x}}) &= -(1 - \bar{\varphi}) \rho_f g \cos \theta \nabla_{\mathbf{x}} (b + h_m + h_f) - (1 - \bar{\varphi}) \nabla_{\mathbf{x}} p_{f_m}^e \\ &\quad - \partial_z T_f^{\mathbf{x}z} - \bar{\beta} (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) - (1 - \bar{\varphi}) \rho_f g \sin \theta (1, 0)^t + \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.55)$$

Similarly, the tangential solid momentum equation (3.2a) gives with (3.53)

$$\begin{aligned} \rho_s \bar{\varphi} (\partial_t \bar{v}^{\mathbf{x}} + \bar{v}^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{v}^{\mathbf{x}}) &= -(\rho_s - \rho_f) g \cos \theta \nabla_{\mathbf{x}} (\bar{\varphi} (b + h_m - z)) + (1 - \bar{\varphi}) \nabla_{\mathbf{x}} p_{f_m}^e \\ &\quad - \bar{\varphi} \rho_f g \cos \theta \nabla_{\mathbf{x}} (b + h_m + h_f) - \partial_z T_s^{\mathbf{x}z} \\ &\quad + \bar{\beta} (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) - \bar{\varphi} \rho_s g \sin \theta (1, 0)^t + \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.56)$$

We are going now to average (3.54) over the fluid layer, and (3.55), (3.56) over the mixture layer, so that the effects of the rheology are only taken into account by the boundary values of  $T_f^{\mathbf{x}z}$ ,  $T_{f_m}^{\mathbf{x}z}$ ,  $T_s^{\mathbf{x}z}$ . According to (3.33), (3.35), the equation (3.13) gives  $p_f = \mathcal{O}(\epsilon^2)$  at the free surface, and then (3.12) yields

$$T_f^{\mathbf{x}z} = \mathcal{O}(\epsilon^3) \quad \text{at } z = b + h_m + h_f. \quad (3.57)$$

Next, using (3.47), the equation (3.19a) gives  $T_s^{\mathbf{x}z} = \mathcal{O}(\epsilon^3)$  at the interface, while (3.17) gives  $T_f^{\mathbf{x}z} - T_{f_m}^{\mathbf{x}z} = \rho_f \mathcal{V}_f(u^{\mathbf{x}} - u_f^{\mathbf{x}}) + \mathcal{O}(\epsilon^3)$  at the interface. But (3.21) gives  $T_f^{\mathbf{x}z} + T_{f_m}^{\mathbf{x}z} = -2k_i(u_f^{\mathbf{x}} - u^{\mathbf{x}}) + \mathcal{O}(\epsilon^3)$  at the interface. We conclude that

$$\begin{aligned} T_s^{\mathbf{x}z} &= \mathcal{O}(\epsilon^3) \quad \text{at } z = b + h_m, \\ T_f^{\mathbf{x}z} &= -(k_i + \frac{1}{2}\rho_f \mathcal{V}_f)(u_f^{\mathbf{x}} - u^{\mathbf{x}}) + \mathcal{O}(\epsilon^3) \quad \text{at } z = b + h_m, \\ T_{f_m}^{\mathbf{x}z} &= -(k_i - \frac{1}{2}\rho_f \mathcal{V}_f)(u_f^{\mathbf{x}} - u^{\mathbf{x}}) + \mathcal{O}(\epsilon^3) \quad \text{at } z = b + h_m. \end{aligned} \quad (3.58)$$

Finally, the conditions (3.9), (3.11) at the bottom give

$$\begin{aligned} T_s^{\mathbf{x}z} &= -\tan \delta_{\text{eff}} \frac{v^{\mathbf{x}}}{|v^{\mathbf{x}}|} \left( T_s^{zz} - 2T_s^{\mathbf{x}z} \cdot \nabla_{\mathbf{x}} b \right) + \mathcal{O}(\epsilon^3) \quad \text{at } z = b, \\ T_{f_m}^{\mathbf{x}z} &= -k_b u^{\mathbf{x}} + \mathcal{O}(\epsilon^3) \quad \text{at } z = b. \end{aligned} \quad (3.59)$$

Now, to go further, one would need an information on  $T_s^{zz} - p_s$  up to  $\mathcal{O}(\epsilon^3)$  error terms, i.e. on the rheology, and an expansion of  $p_s$  up to  $\mathcal{O}(\epsilon^3)$ , that we do not have in (3.53). This should lead to complementary terms in the expression of  $p_{s|b}$ , as for example the term proportional to the curvature of the bottom and quadratic in  $v^{\mathbf{x}}$  as in [6].

We prefer here to avoid further expansions, and to assume as in [6] that the solid friction is small,

$$\tan \delta_{\text{eff}} = \mathcal{O}(\epsilon), \quad (3.60)$$

even if this assumption is not satisfactory. We can then drop the term  $T_s^{\mathbf{x}z} \cdot \nabla_{\mathbf{x}} b$  in (3.59), and replace  $T_s^{zz}$  by  $p_s$  without lowering the order of accuracy. Then, when averaging the mixture momentum equations (3.55), (3.56), one comes up with the average excess pore pressure, that we can express with (3.50) as

$$\overline{p_{f_m}^e} \equiv \frac{1}{h_m} \int_b^{b+h_m} p_{f_m}^e(z) dz = \frac{\bar{\beta}}{1-\bar{\varphi}} \int_b^{b+h_m} \frac{z'-b}{h_m} (u^z - v^z)(z') dz'. \quad (3.61)$$

Then one computes

$$\int_b^{b+h_m} \nabla_{\mathbf{x}} p_{f_m}^e dz = \nabla_{\mathbf{x}} \int_b^{b+h_m} p_{f_m}^e dz - (p_{f_m}^e)_{|b+h_m} \nabla_{\mathbf{x}}(b+h_m) + (p_{f_m}^e)_{|b} \nabla_{\mathbf{x}} b. \quad (3.62)$$

Since  $(p_{f_m}^e)_{|b+h_m} = 0$ , we deduce the expression of the average excess pore pressure force,

$$\overline{\nabla_{\mathbf{x}} p_{f_m}^e} \equiv \frac{1}{h_m} \int_b^{b+h_m} \nabla_{\mathbf{x}} p_{f_m}^e dz = \frac{1}{h_m} \left( \nabla_{\mathbf{x}}(h_m \overline{p_{f_m}^e}) + (p_{f_m}^e)_{|b} \nabla_{\mathbf{x}} b \right). \quad (3.63)$$

We have also to average in (3.56) the term  $\nabla_{\mathbf{x}}(\bar{\varphi}(b+h_m-z))$ , which gives

$$\begin{aligned} & \frac{1}{h_m} \int_b^{b+h_m} \nabla_{\mathbf{x}}(\bar{\varphi}(b+h_m-z)) dz \\ &= \frac{1}{h_m} \int_b^{b+h_m} \left( \bar{\varphi} \nabla_{\mathbf{x}}(b+h_m) + (b+h_m-z) \nabla_{\mathbf{x}} \bar{\varphi} \right) dz \\ &= \bar{\varphi} \nabla_{\mathbf{x}}(b+h_m) + \frac{h_m}{2} \nabla_{\mathbf{x}} \bar{\varphi}. \end{aligned} \quad (3.64)$$

Therefore, averaging (3.54) over the fluid layer and using (3.57), (3.58), we obtain the momentum equation for the fluid-only layer

$$\begin{aligned} \rho_f(\partial_t \overline{u_f^{\mathbf{x}}} + \overline{u_f^{\mathbf{x}}} \cdot \nabla_{\mathbf{x}} \overline{u_f^{\mathbf{x}}}) &= -\rho_f g \cos \theta \nabla_{\mathbf{x}}(b + h_m + h_f) - \frac{1}{h_f} \left( \frac{1}{2} \rho_f \mathcal{V}_f + k_i \right) (\overline{u_f^{\mathbf{x}}} - \overline{u^{\mathbf{x}}}) \\ &\quad - \rho_f g \sin \theta (1, 0)^t + \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.65)$$

For the fluid phase in the mixture, averaging (3.55) and using (3.58), (3.59), we obtain

$$\begin{aligned} \rho_f(1 - \bar{\varphi})(\partial_t \overline{u^{\mathbf{x}}} + \overline{u^{\mathbf{x}}} \cdot \nabla_{\mathbf{x}} \overline{u^{\mathbf{x}}}) &= -(1 - \bar{\varphi}) \rho_f g \cos \theta \nabla_{\mathbf{x}}(b + h_m + h_f) - (1 - \bar{\varphi}) \overline{\nabla_{\mathbf{x}} p_{f_m}^e} \\ &\quad - \frac{1}{h_m} \left( \left( \frac{1}{2} \rho_f \mathcal{V}_f - k_i \right) (\overline{u_f^{\mathbf{x}}} - \overline{u^{\mathbf{x}}}) + k_b \overline{u^{\mathbf{x}}} \right) \\ &\quad - \bar{\beta}(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) - (1 - \bar{\varphi}) \rho_f g \sin \theta (1, 0)^t + \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.66)$$

For the solid phase, averaging (3.56) with again (3.58), (3.59), and (3.64), we get

$$\begin{aligned} \rho_s \bar{\varphi}(\partial_t \overline{v^{\mathbf{x}}} + \overline{v^{\mathbf{x}}} \cdot \nabla_{\mathbf{x}} \overline{v^{\mathbf{x}}}) &= -\bar{\varphi} g \cos \theta (\rho_s \nabla_{\mathbf{x}}(b + h_m) + \rho_f \nabla_{\mathbf{x}} h_f) - (\rho_s - \rho_f) g \cos \theta \frac{h_m}{2} \nabla_{\mathbf{x}} \bar{\varphi} \\ &\quad + (1 - \bar{\varphi}) \overline{\nabla_{\mathbf{x}} p_{f_m}^e} - \text{sgn}(\overline{v^{\mathbf{x}}}) \tan \delta_{\text{eff}} \frac{p_{s|b}}{h_m} \\ &\quad + \bar{\beta}(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) - \bar{\varphi} \rho_s g \sin \theta (1, 0)^t + \mathcal{O}(\epsilon^2), \end{aligned} \quad (3.67)$$

where according to (3.53), the bottom value of the solid pressure is given by

$$p_{s|b} = \bar{\varphi}(\rho_s - \rho_f) g \cos \theta h_m - (p_{f_m}^e)_{|b} + \mathcal{O}(\epsilon^2), \quad (3.68)$$

and by (3.50),

$$(p_{f_m}^e)_{|b} = \frac{\bar{\beta}}{1 - \bar{\varphi}} \int_b^{b+h_m} (u^z - v^z)(z') dz', \quad (3.69)$$

while  $\overline{\nabla_{\mathbf{x}} p_{f_m}^e}$  is given by (3.63).

### 3.5 Evaluation of the excess pore pressure

The excess pore pressure  $p_{f_m}^e$  is involved in (3.66), (3.67) and represents physically important effects. Thus it is necessary to derive an expansion of  $p_{f_m}^e$  up to  $\mathcal{O}(\epsilon^2)$  error terms. Recalling the definition (3.50) of  $p_{f_m}^e$ , we have thus to evaluate  $u^z - v^z$  up to  $\mathcal{O}(\epsilon^2)$  errors. We use equations (3.4) and (3.7) to get the solid normal velocity,

$$v^z = \overline{v^{\mathbf{x}}} \cdot \nabla_{\mathbf{x}} b + (z - b)(\bar{\Phi} - \nabla_{\mathbf{x}} \cdot \overline{v^{\mathbf{x}}}) + \mathcal{O}(\epsilon^3). \quad (3.70)$$

Next, adding the mass equations in the mixture (3.1a), (3.1b), we find

$$\nabla_{\mathbf{x}} \cdot (\varphi v^{\mathbf{x}} + (1 - \varphi) u^{\mathbf{x}}) + \partial_z (\varphi v^z + (1 - \varphi) u^z) = 0, \quad (3.71)$$

and using (3.7) and (3.8), we get

$$\varphi v^z + (1 - \varphi) u^z = (\bar{\varphi} \overline{v^{\mathbf{x}}} + (1 - \bar{\varphi}) \overline{u^{\mathbf{x}}}) \cdot \nabla_{\mathbf{x}} b - (z - b) \nabla_{\mathbf{x}} \cdot (\bar{\varphi} \overline{v^{\mathbf{x}}} + (1 - \bar{\varphi}) \overline{u^{\mathbf{x}}}) + \mathcal{O}(\epsilon^3). \quad (3.72)$$

Then, subtracting (3.70) to (3.72) yields

$$u^z - v^z = (\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \cdot \nabla_{\mathbf{x}} b - \frac{z - b}{1 - \bar{\varphi}} \left( \bar{\Phi} + \nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}})) \right) + \mathcal{O}(\epsilon^3). \quad (3.73)$$

We can then expand the bottom value  $(p_{f_m}^e)|_b$  from (3.69),

$$(p_{f_m}^e)|_b = \frac{\bar{\beta}}{1 - \bar{\varphi}} \left( h_m (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) \cdot \nabla_{\mathbf{x}} b - \frac{h_m^2}{2(1 - \bar{\varphi})} \left( \bar{\Phi} + \nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}})) \right) \right) + \mathcal{O}(\epsilon^3). \quad (3.74)$$

Similarly, we get from (3.61)

$$\overline{p_{f_m}^e} = \frac{\bar{\beta}}{1 - \bar{\varphi}} \left( \frac{h_m}{2} (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) \cdot \nabla_{\mathbf{x}} b - \frac{h_m^2}{3(1 - \bar{\varphi})} \left( \bar{\Phi} + \nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}})) \right) \right) + \mathcal{O}(\epsilon^3). \quad (3.75)$$

Finally, taking into account (3.39) for case (I), we obtain

$$(p_{f_m}^e)|_b = -\frac{\bar{\beta}}{(1 - \bar{\varphi})^2} \frac{h_m^2}{2} \bar{\Phi} + \mathcal{O}(\epsilon^2), \quad \overline{p_{f_m}^e} = -\frac{\bar{\beta}}{(1 - \bar{\varphi})^2} \frac{h_m^2}{3} \bar{\Phi} + \mathcal{O}(\epsilon^2) \quad \text{for case (I),} \quad (3.76)$$

while (3.74) and (3.75) have to be taken for case (II), for which we keep the terms of order  $\epsilon^2$  and the errors are indeed  $\mathcal{O}(\epsilon^4)$ .

We observe on (3.73) and (3.76) that at leading order, as explained in the introduction, the relative velocity  $u^z - v^z$  and the excess pore pressure  $p_{f_m}^e$  have sign opposite to  $\bar{\Phi}$ .

## 4 The two-phase two-layer model

In the previous section we have established a complete set of equations for our two-phase two-layer model. In this section we give the main properties of this system.

### 4.1 System and first properties

The system of equations derived in Section 3 has three scalar unknowns  $\bar{\varphi}$ ,  $h_m$ ,  $h_f$ , and three vector unknowns  $\bar{v}^{\mathbf{x}}$ ,  $\bar{u}^{\mathbf{x}}$ ,  $\bar{u}_f^{\mathbf{x}}$ . Dropping the error terms, it can be written as follows. The mass conservation equations follow from (3.22)-(3.24) by dropping  $\mathcal{O}(\epsilon^3)$  terms,

$$\partial_t(\bar{\varphi}h_m) + \nabla_{\mathbf{x}} \cdot (\bar{\varphi}h_m\bar{v}^{\mathbf{x}}) = 0, \quad (4.1)$$

$$\partial_t((1 - \bar{\varphi})h_m) + \nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})h_m\bar{u}^{\mathbf{x}}) = -\mathcal{V}_f, \quad (4.2)$$

$$\partial_t h_f + \nabla_{\mathbf{x}} \cdot (h_f\bar{u}_f^{\mathbf{x}}) = \mathcal{V}_f. \quad (4.3)$$

We can eliminate the fluid mass exchange term  $\mathcal{V}_f$  by writing the fluid total mass conservation. Adding the two last equations yields

$$\partial_t((1 - \bar{\varphi})h_m + h_f) + \nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})h_m\bar{u}^{\mathbf{x}} + h_f\bar{u}_f^{\mathbf{x}}) = 0. \quad (4.4)$$

Adding (4.1) we deduce also whole system volume conservation as

$$\partial_t(h_m + h_f) + \nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})h_m\bar{u}^{\mathbf{x}} + \bar{\varphi}h_m\bar{v}^{\mathbf{x}} + h_f\bar{u}_f^{\mathbf{x}}) = 0. \quad (4.5)$$

The evolution equation (3.32) for  $\bar{\varphi}$  is

$$\partial_t \bar{\varphi} + \bar{v}^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{\varphi} = -\bar{\varphi} \bar{\Phi}. \quad (4.6)$$

Multiplying it by  $h_m$  and subtracting the result to (4.1), it yields

$$\partial_t h_m + \nabla_{\mathbf{x}} \cdot (h_m \bar{v}^{\mathbf{x}}) = h_m \bar{\Phi}. \quad (4.7)$$

Finally, combining it with (4.5) gives

$$\partial_t h_f + \nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})h_m(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) + h_f \bar{u}_f^{\mathbf{x}}) = -h_m \bar{\Phi}. \quad (4.8)$$

Thus, regarding scalar equations we have to keep a set of three independent equations for the three independent unknowns  $\bar{\varphi}$ ,  $h_m$ ,  $h_f$ . This can be either (4.1), (4.4), (4.6), or (4.1), (4.4), (4.8), or (4.6), (4.7), (4.8), or (4.1), (4.7), (4.8). This has to be completed by (4.2) or (4.3) to define  $\mathcal{V}_f$ , that can in fact be expressed without time derivative, since subtracting (4.8) to (4.3) yields

$$\mathcal{V}_f = -h_m \bar{\Phi} - \nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})h_m(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}})). \quad (4.9)$$

The momentum equations are given by (3.65), (3.66), and (3.67). Thus the model is reduced to the following set of equations:

$$\partial_t(\bar{\varphi}h_m) + \nabla_{\mathbf{x}} \cdot (\bar{\varphi}h_m \bar{v}^{\mathbf{x}}) = 0, \quad (4.10a)$$

$$\begin{aligned} \rho_s \bar{\varphi}(\partial_t \bar{v}^{\mathbf{x}} + \bar{v}^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{v}^{\mathbf{x}}) &= -\bar{\varphi}g \cos \theta (\rho_s \nabla_{\mathbf{x}}(b + h_m) + \rho_f \nabla_{\mathbf{x}} h_f) \\ &\quad - (\rho_s - \rho_f)g \cos \theta \frac{h_m}{2} \nabla_{\mathbf{x}} \bar{\varphi} + (1 - \bar{\varphi}) \overline{\nabla_{\mathbf{x}} p_{f_m}^e} \\ &\quad - \text{sgn}(\bar{v}^{\mathbf{x}}) \tan \delta_{\text{eff}} \frac{(\bar{\varphi}(\rho_s - \rho_f)g \cos \theta h_m - (p_{f_m}^e)_{|b})_+}{h_m} \\ &\quad + \bar{\beta}(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) - \bar{\varphi} \rho_s g \sin \theta (1, 0)^t, \end{aligned} \quad (4.10b)$$

$$\partial_t((1 - \bar{\varphi})h_m) + \nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})h_m \bar{u}^{\mathbf{x}}) = -\mathcal{V}_f, \quad (4.11a)$$

$$\begin{aligned} \rho_f(1 - \bar{\varphi})(\partial_t \bar{u}^{\mathbf{x}} + \bar{u}^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{u}^{\mathbf{x}}) &= -(1 - \bar{\varphi})\rho_f g \cos \theta \nabla_{\mathbf{x}}(b + h_m + h_f) \\ &\quad - (1 - \bar{\varphi}) \overline{\nabla_{\mathbf{x}} p_{f_m}^e} \\ &\quad - \frac{1}{h_m} \left( \left( \frac{1}{2} \rho_f \mathcal{V}_f - k_i \right) (\bar{u}_f^{\mathbf{x}} - \bar{u}^{\mathbf{x}}) + k_b \bar{u}^{\mathbf{x}} \right) \\ &\quad - \bar{\beta}(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) - (1 - \bar{\varphi})\rho_f g \sin \theta (1, 0)^t, \end{aligned} \quad (4.11b)$$

$$\partial_t h_f + \nabla_{\mathbf{x}} \cdot (h_f \bar{u}_f^{\mathbf{x}}) = \mathcal{V}_f, \quad (4.12a)$$

$$\begin{aligned} \rho_f(\partial_t \bar{u}_f^{\mathbf{x}} + \bar{u}_f^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{u}_f^{\mathbf{x}}) &= -\rho_f g \cos \theta \nabla_{\mathbf{x}}(b + h_m + h_f) \\ &\quad - \frac{1}{h_f} \left( \frac{1}{2} \rho_f \mathcal{V}_f + k_i \right) (\bar{u}_f^{\mathbf{x}} - \bar{u}^{\mathbf{x}}) - \rho_f g \sin \theta (1, 0)^t, \end{aligned} \quad (4.12b)$$

$$\partial_t \bar{\varphi} + \bar{v}^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{\varphi} = -\bar{\varphi} \bar{\Phi}, \quad (4.13)$$

where we used (3.68) for the value of  $p_{s|b}$ , the average  $\overline{\nabla_{\mathbf{x}} p_{f_m}^e}$  is defined by (3.63), and according to (3.76) and (3.74), (3.75),

$$(p_{f_m}^e)_{|b} = -\frac{\bar{\beta}}{(1 - \bar{\varphi})^2} \frac{h_m^2}{2} \bar{\Phi}, \quad \overline{p_{f_m}^e} = -\frac{\bar{\beta}}{(1 - \bar{\varphi})^2} \frac{h_m^2}{3} \bar{\Phi} \quad \text{for case (I)}, \quad (4.14)$$

$$\left. \begin{aligned} (p_{f_m}^e)_{|b} &= -\frac{\bar{\beta}}{1-\bar{\varphi}} \left( \frac{h_m^2 \bar{\Phi} + \nabla_{\mathbf{x}} \cdot ((1-\bar{\varphi})(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}))}{2} - h_m(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) \cdot \nabla_{\mathbf{x}} b \right), \\ \bar{p}_{f_m}^e &= -\frac{\bar{\beta}}{1-\bar{\varphi}} \left( \frac{h_m^2 \bar{\Phi} + \nabla_{\mathbf{x}} \cdot ((1-\bar{\varphi})(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}))}{3} - \frac{h_m}{2}(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) \cdot \nabla_{\mathbf{x}} b \right) \end{aligned} \right\} \text{for case (II)}. \quad (4.15)$$

We put a positive part (we denote the positive part of a number  $x$  by  $x_+ \equiv \max(0, x)$ ) in the bottom solid friction term in (4.10b) because otherwise we could have a negative value for  $p_{s|b}$ . The coefficient  $\bar{\beta}$  is defined in (3.37), and the closure function  $\bar{\Phi}$  is defined in (3.31).

We observe that writing the linear combination  $\rho_s \bar{v}^{\mathbf{x}} \times (4.10a) + h_m \times (4.10b) + \rho_f \bar{u}^{\mathbf{x}} \times (4.11a) + h_m \times (4.11b) + \rho_f \bar{u}_f^{\mathbf{x}} \times (4.12a) + h_f \times (4.12b)$  we obtain the total momentum conservation

$$\begin{aligned} & \partial_t \left( \rho_s \bar{\varphi} h_m \bar{v}^{\mathbf{x}} + \rho_f (1-\bar{\varphi}) h_m \bar{u}^{\mathbf{x}} + \rho_f h_f \bar{u}_f^{\mathbf{x}} \right) + \nabla_{\mathbf{x}} \cdot \left( \rho_s \bar{\varphi} h_m \bar{v}^{\mathbf{x}} \otimes \bar{v}^{\mathbf{x}} \right. \\ & \quad \left. + \rho_f (1-\bar{\varphi}) h_m \bar{u}^{\mathbf{x}} \otimes \bar{u}^{\mathbf{x}} + \rho_f h_f \bar{u}_f^{\mathbf{x}} \otimes \bar{u}_f^{\mathbf{x}} \right) + g \cos \theta \nabla_{\mathbf{x}} \cdot \left( (\rho_s - \rho_f) \bar{\varphi} \frac{h_m^2}{2} + \rho_f \frac{(h_m + h_f)^2}{2} \right) \\ &= -\text{sgn}(\bar{v}^{\mathbf{x}}) \tan \bar{\delta}_{\text{eff}} \left( \bar{\varphi} (\rho_s - \rho_f) g \cos \theta h_m - (p_{f_m}^e)_{|b} \right)_+ - k_b \bar{u}^{\mathbf{x}} \\ & \quad - \left( \rho_s \bar{\varphi} h_m + \rho_f ((1-\bar{\varphi}) h_m + h_f) \right) (g \cos \theta \nabla_{\mathbf{x}} b + g \sin \theta (1, 0)^t). \end{aligned} \quad (4.16)$$

The system (4.10)-(4.13) has the following other properties. It is a quasilinear system in case (I), while in case (II) it has an extra second-order term involving  $\nabla_{\mathbf{x}} \cdot ((1-\bar{\varphi})(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}))$  due to the term  $\nabla_{\mathbf{x}} \cdot (h_m \bar{p}_{f_m}^e)$  in (3.63), and also a nonlinearity in terms of  $\nabla_{\mathbf{x}} \cdot ((1-\bar{\varphi})(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}))$  in the bottom solid friction term. Next, solid and fluid masses conserved, according to (4.10a) and (4.11a)+(4.12a). The width of the mixture  $h_m$  remains nonnegative because of (4.10a). The solid volume fraction  $\bar{\varphi}$  remains between 0 and 1 because of (4.6) and (3.31), indeed the value  $\bar{\varphi}_c$  is an attractive value for  $\bar{\varphi}$ . However, there is no reason for the width of the fluid-only layer  $h_f$  to remain nonnegative, and this is due to the fact that the fluid could be fully sucked into the granular material. Therefore, our model is valid as long as  $h_f$  remains nonnegative. Otherwise, one should write down equations that include the case of a mixture layer topped by a dry granular layer, what we have not done here. The system has the rest solution characterized by  $\bar{v}^{\mathbf{x}} = \bar{u}^{\mathbf{x}} = \bar{u}_f^{\mathbf{x}} = 0$ ,  $\bar{\Phi} = 0$ ,  $b + \tilde{b} + h_m = cst$ ,  $h_f = cst$ ,  $\bar{\varphi} = cst$ , with  $\tilde{b} \equiv x \tan \theta$ .

## 4.2 Comparison with other debris flows models

In this subsection we would like to explain the main differences between our model and other debris flow models in the literature that include excess pore pressure effects, namely those of [26] and [16].

### The Pailha and Pouliquen model

In [26] a two-phase debris flows model is proposed. As in our model, it is based on the dilatancy law proposed by Roux and Radjai (2.16). However, they consider some extra simplifications regarding the velocities. Namely, they use

$$\varphi v + (1-\varphi)u = 0 \quad (4.17)$$

as a particular solution to the mass conservation (2.5), and the following relations for the solid velocity,

$$\bar{v}^z = K_{\text{PP}} |\bar{v}^{\mathbf{x}}| \tan \psi_{\text{PP}}, \quad \dot{\gamma} = \frac{3|\bar{v}^{\mathbf{x}}|}{h}, \quad (4.18)$$

for some constant  $K_{\text{PP}}$  of the order of unity. This gives  $\bar{v}^z = \frac{1}{3}K_{\text{PP}}h\dot{\gamma}\tan\psi_{\text{PP}}$ , and this is indeed related to (3.70) where only the term in  $\bar{\Phi}$  is considered, with  $\bar{\Phi} = \frac{2}{3}K_{\text{PP}}\dot{\gamma}\tan\psi_{\text{PP}}$ . The relation with our notations is thus  $\frac{2}{3}K_{\text{PP}}\tan\psi_{\text{PP}} = \tan\psi$ . Then, the solid pressure at the bottom is given by

$$p_{\text{bed}}^s = \varphi(\rho_s - \rho_f)gh \cos\theta + \frac{K_{\text{PP}}\mu}{3\kappa}h^2\dot{\gamma}\tan\psi_{\text{PP}}, \quad (4.19)$$

where the coefficients  $\kappa$  and  $\mu$  are respectively the hydraulic permeability and the dynamic viscosity of the granular debris. One has indeed

$$\frac{\mu}{\kappa} = \frac{\bar{\beta}}{(1-\varphi)^2}, \quad (4.20)$$

and we refer to [4] for a discussion on the definition of the drag friction coefficient for two-phase debris flows. If we consider  $h_m \equiv h$  ( $h_f = 0$ ) in the value of  $p_{s|b}$  in (3.68) with (4.14), i.e. case (I), we obtain the same equation (4.19). Therefore, there are two main differences between our model and that of Pailha and Pouliquen. First we do not make the assumption (4.17) that prevents to have mass conservation. Instead we consider an upper fluid-only layer to fully describe the mass exchanges. Second we consider in case (II) all the terms involving the difference  $\bar{u}^x - \bar{v}^x$  in the evaluation of the excess pore pressure. Another difference is that they consider constitutive relations for shear stresses. Instead we have neglected these effects in (3.34), (3.35). Depth-averaged models with shear stresses are also considered in [10].

### The George and Iverson model

The dynamics of debris flows is described in [16] by a single phase model including dilatancy effects. An evolution equation for the fluid pore pressure is established using a dilatancy law and a Darcy law. Namely, these two relations are used:

$$\text{Dilatancy empirical law:} \quad \nabla \cdot v = \dot{\gamma}\tan\psi - \alpha\partial_t(\sigma - p^f), \quad (4.21)$$

$$\text{Darcy law:} \quad (1-\varphi)(u-v) = -\frac{\kappa}{\mu}\nabla p_e^f, \quad (4.22)$$

with  $\alpha$  the mixture compressibility,  $\sigma$  the total stress and  $p^f = \rho_f g \cos\theta(h-z) + p_e^f$  the pore fluid pressure. The coefficients  $\kappa$  and  $\mu$  are again the hydraulic permeability and the dynamic viscosity of the granular debris. The Darcy law enables to express  $\Phi = \nabla \cdot v = \nabla \cdot ((1-\varphi)(v-u)) = \nabla \cdot (\frac{\kappa}{\mu}\nabla p_e^f)$ , as in [23, 20, 24]. Combining both equations, the following evolution equation for the fluid pore pressure is deduced,

$$\partial_t p_e^f - \nabla \cdot \left( \frac{\kappa}{\alpha\mu} \nabla p_e^f \right) = -\frac{\dot{\gamma}\tan\psi}{\alpha} + \partial_t \left( \sigma - \rho_f g \cos\theta(h-z) \right). \quad (4.23)$$

Thus, the excess pore pressure  $p_e^f$  obeys a diffusion like equation with diffusion coefficient given by  $\kappa/\mu\alpha$ . Valid ranges for the mixture compressibility  $\alpha$  are discussed in [2]. This coefficient strongly affects the pore pressure diffusion in the sense that large values of  $\alpha$  translate into delayed pore pressure diffusion. The value considered by George and Iverson in [16] is  $\alpha = 5 \times 10^{-5} \text{ Pa}^{-1}$ , which corresponds to a typical value for air-particle compressibility, while the typical value for water-particle compressibility that should be used is of the order of  $5 \times 10^{-10} \text{ Pa}^{-1}$ , see [2, 21]. Note that for this value, the dilatancy law (4.21) is very close to that of Roux and Radjai.

An evolution equation for the pore fluid pressure at the bottom  $p_{\text{bed}}^f$  can be deduced from (4.23), see [16] for details. It can be seen that in the case  $\alpha = 0$  it simplifies to

$$p_{\text{bed}}^f = \rho_f g \cos\theta h - \frac{1}{2}\frac{\mu}{\kappa}h^2\dot{\gamma}\tan\psi. \quad (4.24)$$

As for the Pailha and Pouliquen model above, with (3.49) this corresponds again to our formula (4.14) for  $(p_{f_m}^e)_{|b}$ , i.e. case (I), with  $h_m \equiv h$ ,  $h_f = 0$ . Thus, as in the comparison to the Pailha and Pouliquen model, there are two main differences between our model and that of George and Iverson. First we consider an upper fluid-only layer to fully describe the mass exchanges. Second we consider in case (II) all the terms involving the difference  $\overline{u^x} - \overline{v^x}$  in the evaluation of the excess pore pressure. The additional term involving  $\nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})(\overline{u^x} - \overline{v^x}))$  in (4.15) induces a diffusion term in our system. Its strength can be evaluated by writing the equation on  $(1 - \bar{\varphi})(\overline{u^x} - \overline{v^x})$  that can be deduced from (4.10b), (4.11b). The result is a diffusion coefficient given by

$$\mathcal{D}_{\text{two-layer}} = \bar{\beta} \frac{h_m^2}{3} \left( \frac{1}{\rho_f(1 - \bar{\varphi})} + \frac{1}{\rho_s \bar{\varphi}} \right). \quad (4.25)$$

It should be compared with the diffusion coefficient of the George and Iverson model from (4.23), which gives with (4.20)

$$\mathcal{D}_{\text{GI}} = \frac{(1 - \bar{\varphi})^2}{\alpha \bar{\beta}}. \quad (4.26)$$

In our model the diffusion equation is on  $(1 - \bar{\varphi})(\overline{u^x} - \overline{v^x})$ , while in the George and Iverson model it is stated on  $p_e^f$ . But the Darcy law (4.22) enables to perform the change of unknowns from  $p_e^f$  to  $(1 - \bar{\varphi})(\overline{u^x} - \overline{v^x})$ , thus the two formulations are very close. Indeed the George and Iverson model contains an equation on  $\overline{v^x}$  and one on  $p_e^f$ , while our model has two velocity equations, that can be taken  $\overline{v^x}$ ,  $\overline{u^x} - \overline{v^x}$ . However the two approaches differ by the physical interpretation of the diffusion. In our model the diffusion comes directly from the relative momentum equation. In the George and Iverson model, the Darcy law (4.22) means that the time derivative in the equation on the relative velocity  $u - v$  is neglected. It is reintroduced via the mixture compressibility  $\alpha$ . This difference results in different diffusion coefficients (4.25), (4.26). The proportionality to  $\bar{\beta}$  in (4.25) is quite natural, it means that the largest is the drag  $\bar{\beta}$ , the strongest is the diffusion coefficient and the damping of the relative velocity. A strong drag thus leads to a fast convergence to the hydrostatic equilibrium.

Note that the Darcy law (4.22) can be recovered from our model in the limit of large drag, see Subsection 4.4.

If we look more precisely at conservation equations, we can describe the differences between the model proposed by George and Iverson in [16] and the one proposed in this work in case (I). Differences lie in the form of dilatancy terms in the mass and momentum conservation laws.

Let us compare the continuity equations. The model that we propose contains two mass conservation equations, (4.10a), (4.11a)+(4.12a), and a closure equation (4.13), or equivalently (4.7) or (4.8). The George-Iverson model is defined by a continuity equation for the mixture and another one for the solid phase (see [16]), defined as follows,

$$\partial_t h_m + \nabla_{\mathbf{x}} \cdot (h_m \overline{v^x}) = \frac{\rho_m - \rho_f}{\rho_m} D, \quad (4.27)$$

$$\partial_t (\bar{\varphi} h_m) + \nabla_{\mathbf{x}} \cdot (\bar{\varphi} h_m \overline{v^x}) = -\frac{\rho_f}{\rho_m} \bar{\varphi} D, \quad (4.28)$$

with  $D = \int_0^{h_m} (\nabla \cdot v) dz$ . In [16], by considering the Darcy law,  $D$  is approximated by a quantity proportional to  $p_e^f$ . In our case we have (4.7), which is equation (4.27) with a different right-hand side  $h_m \bar{\Phi}$ , that can be obtained directly as an approximation of  $D$  with (2.14). Thus somehow in our model we do not have the term  $-(\rho_f/\rho_m)D$  that is in (4.27). Moreover, in our model, (4.10a)



is (4.28) but without right-hand side. Thus again we do not have the term  $(\rho_f/\rho_m)D$ . Therefore, in our model the solid mass is conserved, while in that of George and Iverson it is not. Following the derivation of [16], we can identify why the right-hand side of (4.28) does not vanish. The problem is that George and Iverson replaced the kinematic condition (2.34) considered in this paper by the assumption that  $\varphi$  has a constant normal profile in all the domain. Indeed the kinematic condition (2.34) is strongly related to the solid mass conservation, since it says that no solid particle can cross the interface. Writing down a model without this condition means that a source of solid phase exists above the interface, in relation with the source term in (4.28).

Finally, in order to compare the velocity equations of the George-Iverson model and the ones proposed here, it is necessary to consider anisotropy of the solid phase. The velocity equation for the case of a single phase considered in [16] can be obtained by summing up the equations (4.10b) and (4.11b). Then the excess pore pressure terms cancel. However, if we modify our system by considering anisotropy of the solid phase and we sum up both equations, it remains roughly a term  $(K_{\text{anis}} - 1)(1 - \bar{\varphi})\overline{\nabla_{\mathbf{x}} p_{f_m}^e}$ , if we denote the anisotropy coefficient by  $K_{\text{anis}}$ . This is the only extra term appearing in the model proposed in [16].

Nevertheless, the momentum balance equations differ in both models because they are obtained by multiplying the mass conservation equations by the velocity equations, and the mass conservation equations differ.

### 4.3 Simplified two-velocity model

In this subsection we propose a simplified model having only two unknown velocities, one for the granular phase and one for the fluid phase, instead of three unknown velocities for the model of Subsection 4.1. The two-velocity model is obtained as the limit of the model of Subsection 4.1 when the friction coefficient  $k_i$  between the two parts of the fluid phase tends to infinity. It leads to the relation  $\overline{u_f^{\mathbf{x}}} = \overline{u^{\mathbf{x}}}$ , while we remain with the sum  $h_m \times (4.11b) + h_f \times (4.12b)$  as momentum equation for the fluid phase (that can be normalized by the fluid volume  $(1 - \bar{\varphi})h_m + h_f$ ). We thus have now the unknowns  $\bar{\varphi}$ ,  $h_m$ ,  $h_f$ ,  $\overline{v^{\mathbf{x}}}$ ,  $\overline{u^{\mathbf{x}}}$ , and we obtain the following model:

$$\partial_t(\bar{\varphi}h_m) + \nabla_{\mathbf{x}} \cdot (\bar{\varphi}h_m\overline{v^{\mathbf{x}}}) = 0, \quad (4.29a)$$

$$\partial_t((1 - \bar{\varphi})h_m + h_f) + \nabla_{\mathbf{x}} \cdot \left( ((1 - \bar{\varphi})h_m + h_f)\overline{u^{\mathbf{x}}} \right) = 0, \quad (4.29b)$$

$$\partial_t\bar{\varphi} + \overline{v^{\mathbf{x}}} \cdot \nabla_{\mathbf{x}}\bar{\varphi} = -\bar{\varphi}\bar{\Phi}, \quad (4.29c)$$

where as in Subsection 4.1 the set of three independent equations can be chosen differently, for example by replacing (4.29c) by (4.7) or (4.8). We can skip the definition of  $\mathcal{V}_f$ , since it disappears from the

momentum equations, that are

$$\begin{aligned}
\rho_s \bar{\varphi} (\partial_t \bar{v}^{\mathbf{x}} + \bar{v}^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{v}^{\mathbf{x}}) &= -\bar{\varphi} g \cos \theta (\rho_s \nabla_{\mathbf{x}} (b + h_m) + \rho_f \nabla_{\mathbf{x}} h_f) \\
&\quad - (\rho_s - \rho_f) g \cos \theta \frac{h_m}{2} \nabla_{\mathbf{x}} \bar{\varphi} + (1 - \bar{\varphi}) \overline{\nabla_{\mathbf{x}} p_{f_m}^e} \\
&\quad - \operatorname{sgn}(\bar{v}^{\mathbf{x}}) \tan \bar{\delta}_{\text{eff}} \frac{(\bar{\varphi} (\rho_s - \rho_f) g \cos \theta h_m - (p_{f_m}^e)_{|b})_+}{h_m} \\
&\quad + \bar{\beta} (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) - \bar{\varphi} \rho_s g \sin \theta (1, 0)^t,
\end{aligned} \tag{4.30a}$$

$$\begin{aligned}
\rho_f (\partial_t \bar{u}^{\mathbf{x}} + \bar{u}^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{u}^{\mathbf{x}}) &= -\rho_f g \cos \theta \nabla_{\mathbf{x}} (b + h_m + h_f) \\
&\quad - \frac{1 - \bar{\varphi}}{(1 - \bar{\varphi}) h_m + h_f} h_m \overline{\nabla_{\mathbf{x}} p_{f_m}^e} \\
&\quad - \frac{k_b \bar{u}^{\mathbf{x}} + \bar{\beta} h_m (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}})}{(1 - \bar{\varphi}) h_m + h_f} - \rho_f g \sin \theta (1, 0)^t,
\end{aligned} \tag{4.30b}$$

with  $\overline{\nabla_{\mathbf{x}} p_{f_m}^e}$  defined by (3.63),  $(p_{f_m}^e)_{|b}$  and  $\bar{p}_{f_m}^e$  defined by (4.14) in case (I), or by (4.15) in case (II),  $\bar{\beta}$  defined by (3.37), and  $\bar{\Phi}$  defined in (3.31). The system satisfies the total momentum conservation, obtained by writing the linear combination  $\rho_s \bar{v}^{\mathbf{x}} \times (4.29a) + h_m \times (4.30a) + \rho_f \bar{u}^{\mathbf{x}} \times (4.29b) + ((1 - \bar{\varphi}) h_m + h_f) \times (4.30b)$ ,

$$\begin{aligned}
&\partial_t \left( \rho_s \bar{\varphi} h_m \bar{v}^{\mathbf{x}} + \rho_f ((1 - \bar{\varphi}) h_m + h_f) \bar{u}^{\mathbf{x}} \right) + \nabla_{\mathbf{x}} \cdot \left( \rho_s \bar{\varphi} h_m \bar{v}^{\mathbf{x}} \otimes \bar{v}^{\mathbf{x}} \right. \\
&\quad \left. + \rho_f ((1 - \bar{\varphi}) h_m + h_f) \bar{u}^{\mathbf{x}} \otimes \bar{u}^{\mathbf{x}} \right) + g \cos \theta \nabla_{\mathbf{x}} \cdot \left( (\rho_s - \rho_f) \bar{\varphi} \frac{h_m^2}{2} + \rho_f \frac{(h_m + h_f)^2}{2} \right) \\
&= -\operatorname{sgn}(\bar{v}^{\mathbf{x}}) \tan \bar{\delta}_{\text{eff}} (\bar{\varphi} (\rho_s - \rho_f) g \cos \theta h_m - (p_{f_m}^e)_{|b})_+ - k_b \bar{u}^{\mathbf{x}} \\
&\quad - (\rho_s \bar{\varphi} h_m + \rho_f ((1 - \bar{\varphi}) h_m + h_f)) (g \cos \theta \nabla_{\mathbf{x}} b + g \sin \theta (1, 0)^t).
\end{aligned} \tag{4.31}$$

As in Subsection 4.1, the system (4.29), (4.30) is a quasilinear system with an extra second-order term in case (II), with solid and fluid masses conserved, the width of the mixture  $h_m$  remains nonnegative, and the solid volume fraction  $\bar{\varphi}$  remains between 0 and 1.

#### 4.4 Oversimplified single-velocity model and Darcy law

An even more simplified model can be obtained by taking the limit of the previous two-velocity model as  $v_T$  tends to 0. According to (3.37),  $\bar{\beta}$  contains a factor  $1/v_T$  thus tends to infinity. This is in contradiction with (3.40) and even with (3.38), but nevertheless the limit model is worthwhile to state since it includes the Darcy law. Ignoring the blow up of the bottom solid friction in (4.30a), the finiteness in equations (4.30a), (4.30b) yields by taking into account (3.63) that

$$\bar{\beta} h_m (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) = -(1 - \bar{\varphi}) \left( \nabla_{\mathbf{x}} (h_m \bar{p}_{f_m}^e) + (p_{f_m}^e)_{|b} \nabla_{\mathbf{x}} b \right), \tag{4.32}$$

with the convention that  $1/v_T$  has to be factorized out in all terms in this relation, noticing that the formulas (4.14) or (4.15) all contain the factor  $\bar{\beta}$ . Therefore, we remain with a system with three scalars unknowns  $\bar{\varphi}$ ,  $h_m$ ,  $h_f$  and a single unknown velocity  $\bar{v}^{\mathbf{x}}$  (or  $\bar{u}^{\mathbf{x}}$ ). The equations are (4.29) and (4.31), with the relation (4.32) that enables to eliminate one velocity. This relation involves  $\nabla_{\mathbf{x}} (h_m \bar{p}_{f_m}^e)$  with  $\bar{p}_{f_m}^e$  defined by (4.14) or (4.15), thus (4.32) involves first-order derivatives in case (I),

and second-order derivatives in case (II). Therefore, plugging this relation in (4.29), (4.31), we see that the single-velocity system includes second-order derivatives in case (I), and third-order derivatives in case (II). In the latter case the system thus includes dispersive effects, and it is quite reminiscent of the Green-Naghdi shallow water model.

We note the identification between (4.32) and the Darcy law (4.22) with (4.20), indeed (4.32) appears as the average of the  $\mathbf{x}$  component of (4.22) (recall (3.63)), while the  $z$  component of (4.22) is simply the definition (3.50) of  $p_{f_m}^e$ .

## 4.5 Local energy balance

We would like here to establish and discuss the local energy balance for our two-phase two-layer model. We consider the three-velocity system (4.10)-(4.13). In order to simplify a bit the expressions, we write

$$\sin \theta(1, 0)^t = \cos \theta \nabla_{\mathbf{x}} \tilde{b}, \quad \text{with } \tilde{b} = x \tan \theta, \quad (4.33)$$

so that the topography and gravity terms can be grouped according to the formula  $\cos \theta \nabla_{\mathbf{x}} b + \sin \theta(1, 0)^t = \cos \theta \nabla_{\mathbf{x}}(b + \tilde{b})$ .

We proceed as in [4], by first performing the linear combination  $\frac{1}{2}\rho_s|\overline{v^{\mathbf{x}}}|^2 \times (4.10a) + h_m \overline{v^{\mathbf{x}}} \cdot (4.10b) + \frac{1}{2}\rho_f|\overline{u^{\mathbf{x}}}|^2 \times (4.11a) + h_m \overline{u^{\mathbf{x}}} \cdot (4.11b) + \frac{1}{2}\rho_f|\overline{u_f^{\mathbf{x}}}|^2 \times (4.12a) + h_f \overline{u_f^{\mathbf{x}}} \cdot (4.12b)$ . Noticing that the terms in  $\mathcal{V}_f$  cancel out, we obtain

$$\begin{aligned} & \partial_t \left( \rho_s \bar{\varphi} h_m \frac{|\overline{v^{\mathbf{x}}}|^2}{2} + \rho_f (1 - \bar{\varphi}) h_m \frac{|\overline{u^{\mathbf{x}}}|^2}{2} + \rho_f h_f \frac{|\overline{u_f^{\mathbf{x}}}|^2}{2} \right) \\ & + \nabla_{\mathbf{x}} \cdot \left( \rho_s \bar{\varphi} h_m \frac{|\overline{v^{\mathbf{x}}}|^2}{2} \overline{v^{\mathbf{x}}} + \rho_f (1 - \bar{\varphi}) h_m \frac{|\overline{u^{\mathbf{x}}}|^2}{2} \overline{u^{\mathbf{x}}} + \rho_f h_f \frac{|\overline{u_f^{\mathbf{x}}}|^2}{2} \overline{u_f^{\mathbf{x}}} \right) \\ = & -\rho_s \bar{\varphi} g \cos \theta h_m \overline{v^{\mathbf{x}}} \cdot \nabla_{\mathbf{x}}(b + \tilde{b} + h_m) - \rho_f \bar{\varphi} g \cos \theta h_m \overline{v^{\mathbf{x}}} \cdot \nabla_{\mathbf{x}} h_f - (\rho_s - \rho_f) g \cos \theta \frac{h_m^2}{2} \overline{v^{\mathbf{x}}} \cdot \nabla_{\mathbf{x}} \bar{\varphi} \\ & - \rho_f g \cos \theta \left( (1 - \bar{\varphi}) h_m \overline{u^{\mathbf{x}}} + h_f \overline{u_f^{\mathbf{x}}} \right) \cdot \nabla_{\mathbf{x}}(b + \tilde{b} + h_m + h_f) - (1 - \bar{\varphi}) h_m (\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \cdot \nabla_{\mathbf{x}} p_{f_m}^e \\ & - \bar{\beta} h_m |\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}|^2 - k_i |\overline{u_f^{\mathbf{x}}} - \overline{u^{\mathbf{x}}}|^2 - |\overline{v^{\mathbf{x}}}| \tan \delta_{\text{eff}} \left( \bar{\varphi} (\rho_s - \rho_f) g \cos \theta h_m - (p_{f_m}^e)_b \right)_+ - k_b |\overline{u^{\mathbf{x}}}|^2. \end{aligned} \quad (4.34)$$

The terms in  $\nabla_{\mathbf{x}}(b + \tilde{b} + h_m)$  in the right-hand side of (4.34) are written using the mass equations (4.1), (4.4) as

$$\begin{aligned} & -g \cos \theta \left( \rho_s \bar{\varphi} h_m \overline{v^{\mathbf{x}}} + \rho_f \left( (1 - \bar{\varphi}) h_m \overline{u^{\mathbf{x}}} + h_f \overline{u_f^{\mathbf{x}}} \right) \right) \cdot \nabla_{\mathbf{x}}(b + \tilde{b} + h_m) \\ = & -\nabla_{\mathbf{x}} \cdot \left( g \cos \theta \left( \rho_s \bar{\varphi} h_m \overline{v^{\mathbf{x}}} + \rho_f \left( (1 - \bar{\varphi}) h_m \overline{u^{\mathbf{x}}} + h_f \overline{u_f^{\mathbf{x}}} \right) \right) (b + \tilde{b} + h_m) \right) \\ & + g \cos \theta (b + \tilde{b} + h_m) \nabla_{\mathbf{x}} \cdot \left( \rho_s \bar{\varphi} h_m \overline{v^{\mathbf{x}}} + \rho_f \left( (1 - \bar{\varphi}) h_m \overline{u^{\mathbf{x}}} + h_f \overline{u_f^{\mathbf{x}}} \right) \right) \\ = & -\nabla_{\mathbf{x}} \cdot \left( g \cos \theta \left( \rho_s \bar{\varphi} h_m \overline{v^{\mathbf{x}}} + \rho_f \left( (1 - \bar{\varphi}) h_m \overline{u^{\mathbf{x}}} + h_f \overline{u_f^{\mathbf{x}}} \right) \right) (b + \tilde{b} + h_m) \right) \\ & - g \cos \theta (b + \tilde{b} + h_m) \partial_t \left( \rho_s \bar{\varphi} h_m + \rho_f \left( (1 - \bar{\varphi}) h_m + h_f \right) \right) \\ = & -\nabla_{\mathbf{x}} \cdot \left( g \cos \theta \left( \rho_s \bar{\varphi} h_m \overline{v^{\mathbf{x}}} + \rho_f \left( (1 - \bar{\varphi}) h_m \overline{u^{\mathbf{x}}} + h_f \overline{u_f^{\mathbf{x}}} \right) \right) (b + \tilde{b} + h_m) \right) \\ & - g \cos \theta \partial_t \left( \left( \rho_s \bar{\varphi} h_m + \rho_f \left( (1 - \bar{\varphi}) h_m + h_f \right) \right) (b + \tilde{b}) \right) \\ & - \rho_f g \cos \theta h_m \partial_t (h_m + h_f) - (\rho_s - \rho_f) g \cos \theta h_m \partial_t (\bar{\varphi} h_m). \end{aligned} \quad (4.35)$$

Similarly, the terms in  $\nabla_{\mathbf{x}}h_f$  in the right-hand side of (4.34) are written

$$\begin{aligned} & -\rho_f g \cos \theta \left( \bar{\varphi} h_m \bar{v}^{\mathbf{x}} + (1 - \bar{\varphi}) h_m \bar{u}^{\mathbf{x}} + h_f \bar{u}_f^{\mathbf{x}} \right) \cdot \nabla_{\mathbf{x}} h_f \\ &= -\nabla_{\mathbf{x}} \cdot \left( \rho_f g \cos \theta \left( \bar{\varphi} h_m \bar{v}^{\mathbf{x}} + (1 - \bar{\varphi}) h_m \bar{u}^{\mathbf{x}} + h_f \bar{u}_f^{\mathbf{x}} \right) h_f \right) \\ & \quad - \rho_f g \cos \theta h_f \partial_t \left( \bar{\varphi} h_m + (1 - \bar{\varphi}) h_m + h_f \right). \end{aligned} \quad (4.36)$$

Then, the last term in the right-hand side of (4.35) is combined with the term in  $\nabla_{\mathbf{x}}\bar{\varphi}$  in the right-hand side of (4.34), according to the identity

$$-h_m \partial_t (\bar{\varphi} h_m) - \frac{h_m^2}{2} \bar{v}^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{\varphi} = -\partial_t \left( \bar{\varphi} \frac{h_m^2}{2} \right) - \frac{h_m^2}{2} \left( \partial_t \bar{\varphi} + \bar{v}^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \bar{\varphi} \right), \quad (4.37)$$

where the right-hand side can be expressed with (4.6). Next, the excess pore pressure term in (4.34) can be written using (3.63)

$$\begin{aligned} & -(1 - \bar{\varphi}) h_m (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) \cdot \overline{\nabla_{\mathbf{x}} p_{f_m}^e} \\ &= -\nabla_{\mathbf{x}} \cdot \left( (1 - \bar{\varphi}) h_m \overline{p_{f_m}^e} (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) \right) \\ & \quad + h_m \overline{p_{f_m}^e} \nabla_{\mathbf{x}} \cdot \left( (1 - \bar{\varphi}) (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) \right) - (1 - \bar{\varphi}) (p_{f_m}^e)_{|b} (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) \cdot \nabla_{\mathbf{x}} b. \end{aligned} \quad (4.38)$$

Finally, multiplying (4.6) by  $\frac{de_c}{d\varphi}(\bar{\varphi})$  and using (2.22), we obtain

$$\partial_t (e_c(\bar{\varphi})) + \bar{v}^{\mathbf{x}} \cdot \nabla_{\mathbf{x}} (e_c(\bar{\varphi})) = -\frac{p_c(\bar{\varphi})}{\rho_s \bar{\varphi}} \bar{\Phi}. \quad (4.39)$$

Multiplying this by  $h_m \bar{\varphi}$  and using (4.10a), we deduce

$$\partial_t (h_m \bar{\varphi} e_c(\bar{\varphi})) + \nabla_{\mathbf{x}} \cdot (h_m \bar{\varphi} e_c(\bar{\varphi}) \bar{v}^{\mathbf{x}}) = -h_m \frac{p_c(\bar{\varphi})}{\rho_s} \bar{\Phi}. \quad (4.40)$$

Using the formulas (4.35)-(4.38), (4.40) in (4.34), this yields the energy balance equation

$$\begin{aligned} & \partial_t \left( \rho_s \bar{\varphi} h_m \frac{|\bar{v}^{\mathbf{x}}|^2}{2} + \rho_f (1 - \bar{\varphi}) h_m \frac{|\bar{u}^{\mathbf{x}}|^2}{2} + \rho_f h_f \frac{|\bar{u}_f^{\mathbf{x}}|^2}{2} + \rho_s h_m \bar{\varphi} e_c(\bar{\varphi}) \right) \\ & \quad + g \cos \theta \left( \rho_s \bar{\varphi} h_m + \rho_f ((1 - \bar{\varphi}) h_m + h_f) \right) (b + \tilde{b}) \\ & \quad + \left( \rho_s - \rho_f \right) g \cos \theta \bar{\varphi} \frac{h_m^2}{2} + \rho_f g \cos \theta \frac{(h_m + h_f)^2}{2} \\ & \quad + \nabla_{\mathbf{x}} \cdot \left( \rho_s \bar{\varphi} h_m \frac{|\bar{v}^{\mathbf{x}}|^2}{2} \bar{v}^{\mathbf{x}} + \rho_f (1 - \bar{\varphi}) h_m \frac{|\bar{u}^{\mathbf{x}}|^2}{2} \bar{u}^{\mathbf{x}} + \rho_f h_f \frac{|\bar{u}_f^{\mathbf{x}}|^2}{2} \bar{u}_f^{\mathbf{x}} + \rho_s h_m \bar{\varphi} e_c(\bar{\varphi}) \bar{v}^{\mathbf{x}} \right) \\ & \quad + g \cos \theta \left( \rho_s \bar{\varphi} h_m \bar{v}^{\mathbf{x}} + \rho_f ((1 - \bar{\varphi}) h_m \bar{u}^{\mathbf{x}} + h_f \bar{u}_f^{\mathbf{x}}) \right) (b + \tilde{b} + h_m) \\ & \quad + \rho_f g \cos \theta \left( \bar{\varphi} h_m \bar{v}^{\mathbf{x}} + (1 - \bar{\varphi}) h_m \bar{u}^{\mathbf{x}} + h_f \bar{u}_f^{\mathbf{x}} \right) h_f + (1 - \bar{\varphi}) h_m \overline{p_{f_m}^e} (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) \\ &= \frac{1}{2} (\rho_s - \rho_f) \bar{\varphi} g \cos \theta h_m^2 \bar{\Phi} - h_m p_c(\bar{\varphi}) \bar{\Phi} + R_e - |\bar{v}^{\mathbf{x}}| \tan \bar{\delta}_{\text{eff}} (\bar{\varphi} (\rho_s - \rho_f) g \cos \theta h_m - (p_{f_m}^e)_{|b})_+ \\ & \quad - \bar{\beta} h_m |\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}|^2 - k_i |\bar{u}_f^{\mathbf{x}} - \bar{u}^{\mathbf{x}}|^2 - k_b |\bar{u}^{\mathbf{x}}|^2 \equiv R, \end{aligned} \quad (4.41)$$

with

$$R_e = h_m \overline{p_{f_m}^e} \nabla_{\mathbf{x}} \cdot \left( (1 - \bar{\varphi})(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \right) - (1 - \bar{\varphi})(p_{f_m}^e)_{|b} (\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \cdot \nabla_{\mathbf{x}} b. \quad (4.42)$$

One can check that this energy equation indeed corresponds to the integral of the mixture energy equation (2.25) with respect to  $z$  from  $z = b$  to  $z = b + h_m$  to which we add the integral of the energy equation (2.28) of the fluid-only layer from  $z = b + h_m$  to  $z = b + h_m + h_f$ . The first term in the right-hand side of (4.41) corresponds to the integral over the mixture layer of  $p_s^{\text{hydro}} \Phi$ , where  $p_s^{\text{hydro}}$  is the hydrostatic part of the solid pressure from (3.53). Then three terms in the right-hand side of (4.41) are dissipation terms associated to boundaries, and there is the dissipation of drag friction.

The term  $R_e$  needs to be explained. We claim that it represents the integral over the mixture layer of the excess term  $-p_{f_m}^e \Phi$  plus the  $z$  part of the drag  $-\bar{\beta}(u^z - v^z)^2$ , both from the right-hand side of (2.25). To see this, let us consider separately the cases (I) or (II), corresponding to assumptions (4.14) or (4.15) respectively.

**Case (I).** In this case the sum of the terms from the right-hand side of (2.25) cancel out in average, according to

$$\int_b^{b+h_m} \left( -p_{f_m}^e \Phi - \tilde{\beta}(u^z - v^z)^2 \right) dz = \mathcal{O}(\epsilon^3), \quad (4.43)$$

as can be checked with the expansion of  $\overline{p_{f_m}^e}$  in (3.76) and the expansion (3.73) of  $u^z - v^z$  where we retain only the  $\bar{\Phi}$  term, the other being negligible. We have also  $R_e = \mathcal{O}(\epsilon^3)$  with the same approximation arguments.

**Case (II).** In this case,  $R_e$  and all terms in (4.43) are  $\mathcal{O}(\epsilon^3)$ . However we can write an exact identity, thereby achieving higher order accuracy. Writing (3.73) without error gives

$$u^z - v^z = B - (z - b)A, \quad (4.44)$$

with

$$A = \frac{\bar{\Phi} + \nabla_{\mathbf{x}} \cdot \left( (1 - \bar{\varphi})(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \right)}{1 - \bar{\varphi}}, \quad B = (\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \cdot \nabla_{\mathbf{x}} b. \quad (4.45)$$

Thus

$$\int_b^{b+h_m} (u^z - v^z)^2 dz = \frac{h_m^3}{3} A^2 - h_m^2 AB + h_m B^2. \quad (4.46)$$

Then, using the definition of  $\overline{p_{f_m}^e}$  in (4.15),

$$-h_m \overline{p_{f_m}^e} \bar{\Phi} - h_m \overline{p_{f_m}^e} \nabla_{\mathbf{x}} \cdot \left( (1 - \bar{\varphi})(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \right) = -h_m \overline{p_{f_m}^e} (1 - \bar{\varphi}) A = \bar{\beta} h_m A \left( \frac{h_m^2}{3} A - \frac{h_m}{2} B \right). \quad (4.47)$$

With the definition of  $(p_{f_m}^e)_{|b}$  in (4.15) it yields

$$\begin{aligned} & -h_m \overline{p_{f_m}^e} \bar{\Phi} - h_m \overline{p_{f_m}^e} \nabla_{\mathbf{x}} \cdot \left( (1 - \bar{\varphi})(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) \right) + (1 - \bar{\varphi}) B (p_{f_m}^e)_{|b} \\ &= \bar{\beta} h_m A \left( \frac{h_m^2}{3} A - \frac{h_m}{2} B \right) - \bar{\beta} B \left( \frac{h_m^2}{2} A - h_m B \right) \\ &= \bar{\beta} \left( \frac{h_m^3}{3} A^2 - h_m^2 AB + h_m B^2 \right) = \bar{\beta} \int_b^{b+h_m} (u^z - v^z)^2 dz. \end{aligned} \quad (4.48)$$

This can be written equivalently

$$R_e = \int_b^{b+h_m} \left( -p_{f_m}^e \bar{\Phi} - \bar{\beta}(u^z - v^z)^2 \right) dz, \quad (4.49)$$

proving the claimed identity. This identity means that instead of having an approximation of  $R_e$  up to  $\mathcal{O}(\epsilon^3)$  errors, it is now up to  $\mathcal{O}(\epsilon^5)$  errors.

We conclude that in any case (I) or (II), the right-hand side  $R$  of (4.41) represents the integral of the energy dissipation of the original 3D model. In order to evaluate more accurately the term corresponding to  $(p_s - p_c)\Phi$  in (2.25), we can write the right-hand side of (4.41) as

$$R = (\bar{p}_s - p_c(\bar{\varphi}))h_m\bar{\Phi} + h_m\overline{p_{f_m}^e}\bar{\Phi} + R_e - |\bar{v}^{\mathbf{x}}| \tan \overline{\delta_{\text{eff}}} (\bar{\varphi}(\rho_s - \rho_f)g \cos \theta h_m - (p_{f_m}^e)_{|b})_+ - \bar{\beta}h_m|\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}|^2 - k_i|\bar{u}_f^{\mathbf{x}} - \bar{u}^{\mathbf{x}}|^2 - k_b|\bar{u}^{\mathbf{x}}|^2, \quad (4.50)$$

with

$$\bar{p}_s = \frac{1}{2}(\rho_s - \rho_f)\bar{\varphi}g \cos \theta h_m - \overline{p_{f_m}^e}, \quad (4.51)$$

and where

$$h_m\overline{p_{f_m}^e}\bar{\Phi} + R_e = -\bar{\beta} \int_b^{b+h_m} (u^z - v^z)^2 dz \quad \text{in case (II)}, \quad (4.52)$$

while further error in  $\mathcal{O}(\epsilon^3)$  need to be added in (4.52) in case (I) (the error is indeed  $R_e$  itself).

We mention finally that the same energy balance equation (4.41) and the same analysis hold for the simplified model of Subsection 4.3, one just has to set  $\overline{u_f^{\mathbf{x}}} = \bar{u}^{\mathbf{x}}$ . For the model of Subsection 4.4 it is not so clear.

## 4.6 Parameter settings and discussion

Here we would like to discuss the values of the parameters of our model and the consequences of these values on the nature of the system to be solved. We recall that the model has three scalar unknowns  $\bar{\varphi}$ ,  $h_m$ ,  $h_f$ , and three vector unknowns  $\bar{v}^{\mathbf{x}}$ ,  $\bar{u}^{\mathbf{x}}$ ,  $\overline{u_f^{\mathbf{x}}}$ , and is defined by (4.10)-(4.13), with the average pore fluid pressure  $\overline{\nabla_{\mathbf{x}} p_{f_m}^e}$  defined by (3.63), and either (4.14) for case (I) or (4.15) for case (II). Alternatively, for the simplified two-velocity model of Subsection 4.3, the model has only two vector unknowns  $\bar{v}^{\mathbf{x}}$ ,  $\bar{u}^{\mathbf{x}}$ , and is defined by (4.29) and (4.30). In any case the energy equation (4.41) holds, with the identity (4.50) on energy dissipation.

The value of  $\bar{\beta}$  defined in (3.37) was already discussed at the end of Subsection 3.3. According to (2.20) and (2.17), the effective bottom solid friction coefficient is given by

$$\tan \overline{\delta_{\text{eff}}} = \tan \delta + K(\bar{\varphi} - \bar{\varphi}_c). \quad (4.53)$$

We recall also the definition of  $\bar{\Phi}$  in (3.31),

$$\bar{\Phi} = K\bar{\gamma}(\bar{\varphi} - \bar{\varphi}_c). \quad (4.54)$$

The dimensionless constant  $K$  in (4.54) characterizes the strength of the dilatancy effects. As in [26], it should be of the order of unity. The formula (4.53) describes the effect of enforcing the solid friction when  $\bar{\varphi} > \bar{\varphi}_c$  and diminishing it when  $\bar{\varphi} < \bar{\varphi}_c$ . Note that in the solid equation (4.10b) of our model, the solid friction term is not only proportional to  $\tan \overline{\delta_{\text{eff}}}$ , but also to  $p_{s|b}$ , which contains the excess term  $-(p_{f_m}^e)_{|b}$ . According to (4.14) or (4.15), this excess term itself contains  $\bar{\Phi}$ , which according to (4.54) has the same effect of enforcing the solid friction when  $\bar{\varphi} > \bar{\varphi}_c$ . Thus both factors  $\tan \overline{\delta_{\text{eff}}}$  and

$p_{s|b}$  contribute to the same effect. When  $(p_{f_m}^e)_{|b}$  becomes larger than the hydrostatic fluid pressure  $\bar{\varphi}(\rho_s - \rho_f)g \cos \theta h_m$ , the solid pressure  $p_{s|b}$  vanishes because of the positive part in the bottom friction of (4.10b), and the granular material is totally fluidized.

The value of  $\bar{\varphi}_c$  in (4.54) is a key issue for the energy consistency of our model. It would be possible to define  $\bar{\varphi}_c$  in terms of the inertial number as in [26, 8, 16], but then there is no reason to get a nonpositive energy dissipation  $R$  in (4.50). Thus we prefer to propose a definition of  $\bar{\Phi}$  that is more consistent than (4.54). For this, we put the same structure as that of the energy dissipation (2.25) of the 3D model in the energy dissipation  $R$  in (4.50). This means defining  $\bar{\varphi}_c = \varphi_c(\bar{p}_s)$ , where as in Subsection 2.4, the function  $\varphi = \varphi_c(p)$  is the inverse of the function  $p = p_c(\varphi)$ . This definition leads automatically to a nonpositive first term in the formula (4.50) for  $R$ . However, (4.54) then means defining  $\bar{\Phi}$  as a function of  $\bar{p}_s$ , which by (4.51) and (4.14) or (4.15) itself depends on  $\bar{\Phi}$ . To avoid dealing with a nonlinear equation, we can use the closure (2.21) instead of (2.18), which means that we replace (4.54) by

$$\bar{\Phi} = K_p \bar{\gamma} (p_c(\bar{\varphi}) - \bar{p}_s). \quad (4.55)$$

Then with the relations (4.51) and (4.14) or (4.15), we obtain the value of  $\bar{\Phi}$ ,

$$\begin{aligned} \left(1 + \frac{\bar{\beta} K_p \bar{\gamma} h_m^2}{(1 - \bar{\varphi})^2 3}\right) \bar{\Phi} = & K_p \bar{\gamma} \left( p_c(\bar{\varphi}) - \frac{1}{2}(\rho_s - \rho_f) \bar{\varphi} g \cos \theta h_m \right. \\ & \left. - \frac{\bar{\beta}}{1 - \bar{\varphi}} \left( \frac{h_m^2}{3(1 - \bar{\varphi})} \nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}})) - \frac{h_m}{2} (\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}) \cdot \nabla_{\mathbf{x}} b \right) \right). \end{aligned} \quad (4.56)$$

This formula is for case (II), i.e. (4.15). Otherwise for case (I) and (4.14) the second line must be removed in the right-hand side of (4.56). With this formula for case (II), we get  $R \leq 0$ , i.e. full dissipativity of the model.

Concerning well-posedness of the model, case (I) shows a quasilinear system with an energy identity that has a formally small right-hand side, but that contains derivatives because of the term  $\nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}))$  in  $R_e$ . Thus we do not obtain a mathematical entropy for the system.

In case (II) the system contains second-order terms because of the term  $\overline{\nabla_{\mathbf{x}} p_{f_m}^e}$  in (4.10b) and (4.11b), defined by (3.63) where  $\overline{p_{f_m}^e}$  involves  $\nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}))$  in (4.15). However, the right-hand side  $R$  of the energy equation contains (4.52) that can be expressed with (4.46) as a positive definite quadratic form in terms of  $A, B$  defined in (4.45). We therefore have bounds on  $A, B$  and thus also on  $\overline{p_{f_m}^e}$ . With (4.55) this gives a bound on  $\bar{\Phi}$ , and thus also on  $\nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}))$  since  $A$  is bounded. This means that we have a mathematical entropy which is compatible with the system, providing bounds on the second-order terms. Indeed, the term  $\nabla_{\mathbf{x}} \cdot ((1 - \bar{\varphi})(\bar{u}^{\mathbf{x}} - \bar{v}^{\mathbf{x}}))$  is the depth-averaged counterpart of  $\nabla \cdot v$  in (2.25), (2.26).

Concerning the value of  $\dot{\gamma}$ , if we consider that  $\dot{\gamma} = |Dv|$ , with  $Dv = (\nabla v + (\nabla v)^t)/2$ , the shear components  $\partial_z v^{\mathbf{x}}$  and  $\nabla_{\mathbf{x}} v^z$  are  $\mathcal{O}(\epsilon)$  according to the asymptotic assumptions, while  $\partial_z v^z$  is of the order of  $\nabla_{\mathbf{x}} \cdot v^{\mathbf{x}}$  because of (3.4). Thus a typical value is  $\bar{\dot{\gamma}} = |D_{\mathbf{x}} \bar{v}^{\mathbf{x}}|$ . However, such a definition would lead to possible difficulties for numerical resolution. A possible approximation is  $\bar{\dot{\gamma}} = |\bar{v}^{\mathbf{x}}|/L$ , which is less singular than the approximation of  $\dot{\gamma}$  in (4.18) that could be taken if  $\bar{v}^{\mathbf{x}} = \mathcal{O}(\epsilon)$ . With these approximations of  $\dot{\gamma}$ , the 3D and depth-averaged models look more like bulk viscous than viscoplastic models. Nevertheless, the value of  $\dot{\gamma}$  from (4.18), i.e.  $\dot{\gamma} = 3|\bar{v}^{\mathbf{x}}|/h$ , can be used if strong shear is

present for real flows, since the good properties of the model are still valid with this closure.

Concerning the choice between Model (I), i.e. (4.14), or Model (II), i.e. (4.15), we recall that they have been derived under the assumptions of (3.38)-(3.39) or (3.40) respectively. However, whatever are the scaling assumptions (i.e. (3.38)-(3.39) or (3.40)), the closure (4.15) (thus Model (II)) is always more accurate, since in this case we retain all the terms in the expansions of Subsection 3.5. The advantage of Model (I) is that it is simpler since it involves only first-order derivatives. On the contrary, Model (II) involves second-order viscoplastic-like terms. The variable  $\overline{u^x} - \overline{v^x}$  obeys an equation that includes a diffusion very similar to the one in the George and Iverson model, but which results from different physical assumptions (see Subsection 4.2). In particular, the time derivative simply results here from the mass and momentum equations while it is related to mixture compressibility in Iverson and Georges.

The sign of the excess pore pressure  $p_{f_m}^e$  is always the same as that of the normal relative velocity  $u^z - v^z$  because of (3.50). According to (4.14) for Model (I), this sign is opposite to  $\bar{\Phi}$ , which corresponds exactly to the pore pressure feedback described in Subsection 2.3. However, other terms involving the tangential relative velocity  $\overline{u^x} - \overline{v^x}$  also come into play in (4.15) for Model (II) to determine if the fluid is transferred into or out of the mixture.

The dynamical behaviour of our model is naturally induced by the dilatancy closure of Roux and Radjai. Namely, in the absence of external inflow, there is convergence to the hydrostatic equilibrium over a sufficient long time. This can be seen in terms of the volume fraction by its evolution equation (4.6) that holds for any variant of the model considered. In this equation, the right-hand side  $\bar{\Phi}$  is defined according to one of the closures (4.54) or (4.55)-(4.56). Thus we conclude that either  $\bar{\gamma}$  tends to zero, or  $\bar{\varphi}$  tends to  $\bar{\varphi}_c \equiv \varphi_c(\bar{p}_s)$ , which is an attractive value for (4.6). In any case  $\bar{\Phi}$  tends to zero. In the case of Model (II) with closure (4.55)-(4.56), the fully dissipative nature of the system, as seen on the right-hand side (4.50) of the energy equation, leads to normal dissipation (4.46) tending to zero and thus  $A$ ,  $B$  defined in (4.45) also tending to zero. In any case, we conclude that  $(p_{f_m}^e)_{|b}$ ,  $\overline{p_{f_m}^e}$  in (4.14) or (4.15) tend to zero. This means that  $p_{f_m}^e$  tends to zero, and the pressures becomes hydrostatic.

Finally, note that several limit systems can be obtained from our model for particular values of the parameters. The first is obtained by taking  $\bar{\varphi} \equiv 0$ , leading simply to the standard shallow water system for height  $h_m + h_f$  and velocity  $\overline{u^x}$ . A second system is obtained for  $\bar{\varphi} \equiv 1$ , leading to  $\bar{\Phi} \equiv 0$  and the usual two-layer shallow water system. Finally, a third system is obtained by taking  $\rho_f = 0$ . It leads to  $\bar{\beta} = 0$ ,  $p_{f_m}^e \equiv 0$ , thus (I) and (II) are identical. The term  $\bar{\Phi}$  can be taken either from (4.54) or (4.55). This yields an apparently new thin-layer model for a dry granular material with dilatancy effects, described with the unknowns  $\bar{\varphi}$ ,  $h_m$  and  $\overline{v^x}$ .

## 5 Conclusion

We have proposed a depth-averaged model describing mass and momentum conservation for a two-phase mixture layer of solid granular material and fluid, topped with an upper single fluid layer. The existence of fluid transfer between these two layers makes it possible to describe the relative motion between the fluid and solid phases. As physically expected, this transfer of fluid is directly related to dilation-contraction of the granular phase, described here using a dilatancy closure proposed by Roux and Radjai. This closure relates dilation and contraction to the existence of a critical volume fraction. Our thin-layer approximation shows that the pore pressure is not hydrostatic. An excess pore pressure



term appears, that is related to the dilatancy closure equation. Two approximations (I) or (II) of the excess terms have been proposed, the first being simpler and the second more accurate. Assuming that the critical volume fraction is directly related to the solid pressure, we have interpreted the Roux and Radjai closure as a compressible rheological law with some sort of viscoplastic dissipation. Accordingly, the depth-averaged model satisfies an energy balance identity, which has a rigorously dissipative right-hand side with viscoplastic dissipation in case (II). We have compared our model with existing models in the literature that include dilatancy effects, and shown that our model is more complete, especially concerning the mass balance equations. Our model (II) contains a diffusion equation in the relative velocity which is very similar to that of the George and Iverson model, but with a different physical interpretation. However we did not include shear stresses in our model, because of the lack of a thermodynamically consistent and mathematically well-behaved rheology including dilatancy and inertial number.

In a forthcoming paper we shall address numerical simulations of the proposed model and compare results with experimental data.

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## Appendix: boundary conditions at the interface

In this appendix we show in details the calculations corresponding to the jump relations that we have considered at the interface between the mixture and the fluid in Subsection 2.5.

- Conservation of the total momentum.

The momentum conservation for the fluid-only layer is given by

$$\partial_t(\rho_f u_f) + \nabla \cdot (\rho_f u_f \otimes u_f) = -\nabla \cdot T_f + \rho_f \mathbf{g}, \quad (\text{A.1})$$

and the total momentum conservation of the mixture is (2.10). In order to ensure that the total momentum is conserved across the interface, we impose the Rankine-Hugoniot condition, which gives

$$\begin{aligned} & (\rho_s \varphi^* v + \rho_f (1 - \varphi^*) u) \tilde{N}_t + (\rho_s \varphi^* v \otimes v + \rho_f (1 - \varphi^*) u \otimes u) \tilde{N}_X + (T_s + T_{f_m}) \tilde{N}_X \\ & = (\rho_f u_f) \tilde{N}_t + (\rho_f u_f \otimes u_f) \tilde{N}_X + T_f \tilde{N}_X. \end{aligned} \quad (\text{A.2})$$

Taking into account the kinematic condition for the solid phase (2.34), the two terms containing  $v$  disappear in (A.2). We observe that the fluid mass conservation across the interface (2.35) gives two possible definitions for  $\mathcal{V}_f$ . The first definition is used for the terms containing  $u_f$  in the right-hand side of (A.2), and gives  $\rho_f u_f \mathcal{V}_f$ . The second definition is used for the terms containing  $u$  in the left-hand side of (A.2), and gives  $\rho_f u \mathcal{V}_f$ . Thus, from (A.2) we obtain (2.36).

- Energy balance.

We first notice that the fluid conservation (2.35) gives the relation

$$(u_f - u) \cdot \tilde{N}_X = -\varphi^*(\tilde{N}_t + u \cdot \tilde{N}_X) = -\frac{\varphi^*}{1 - \varphi^*} \mathcal{V}_f. \quad (\text{A.3})$$

This means in particular that  $\mathcal{V}_f$  has the sign of  $(u - u_f) \cdot \tilde{N}_X$ . The energy equation in the fluid-only layer is (2.28) and the total energy equation in the mixture is given by (2.12) or (2.25). In order for the energy to be decreasing across the interface, we write the Rankine-Hugoniot inequality (eliminating the  $v$  terms because of (2.34), and the gravity terms because of (2.35))

$$\begin{aligned} & \left( \rho_f(1 - \varphi^*) \frac{|u|^2}{2} - \rho_f \frac{|u_f|^2}{2} \right) \tilde{N}_t \\ & + \left( \rho_f(1 - \varphi^*) \frac{|u|^2}{2} u - \rho_f \frac{|u_f|^2}{2} u_f + \varphi^* p_{f_m}(v - u) + T_s v + T_{f_m} u - T_f u_f \right) \cdot \tilde{N}_X \geq 0. \end{aligned}$$

Note that the sense of the inequality is related to the assumed upward orientation of  $\tilde{N}_X$ . We rearrange this inequality under the form

$$\begin{aligned} & \rho_f \left( \frac{|u|^2}{2} - \frac{|u_f|^2}{2} \right) \tilde{N}_t - \rho_f \varphi^* \frac{|u|^2}{2} \tilde{N}_t + \rho_f \left( \frac{|u|^2}{2} - \frac{|u_f|^2}{2} \right) u_f \cdot \tilde{N}_X + \rho_f \frac{|u|^2}{2} ((1 - \varphi^*)u - u_f) \cdot \tilde{N}_X \\ & + \varphi^* p_{f_m}(v - u) \cdot \tilde{N}_X + (T_s v) \cdot \tilde{N}_X + ((T_{f_m} - T_f)u_f) \cdot \tilde{N}_X + (T_{f_m}(u - u_f)) \cdot \tilde{N}_X \geq 0. \end{aligned}$$

Because of (A.3), two terms in  $|u|^2$  in the first line disappear. From (A.3) and the conservation of the solid mass (2.34) we get  $\varphi^*(v - u) \cdot \tilde{N}_X + (u - u_f) \cdot \tilde{N}_X = 0$ . For the other terms we use the symmetry of the stresses to get

$$\begin{aligned} & \rho_f \left( \frac{|u|^2}{2} - \frac{|u_f|^2}{2} \right) (\tilde{N}_t + u_f \cdot \tilde{N}_X) - p_{f_m}(u - u_f) \cdot \tilde{N}_X \\ & + (T_s \tilde{N}_X) \cdot v + ((T_{f_m} - T_f) \tilde{N}_X) \cdot u_f + (T_{f_m} \tilde{N}_X) \cdot (u - u_f) \geq 0. \end{aligned} \quad (\text{A.4})$$

We look now for a boundary condition of the form

$$T_s \tilde{N}_X = p_s^* \tilde{N}_X, \quad (\text{A.5})$$

where  $p_s^*$  is a scalar that will be chosen so that the energy inequality holds. We have using (2.34) and (2.35) that

$$(T_s \tilde{N}_X) \cdot v = p_s^* \tilde{N}_X \cdot v = -p_s^* \tilde{N}_t = p_s^*(u_f \cdot \tilde{N}_X - \mathcal{V}_f). \quad (\text{A.6})$$

Therefore, taking the scalar product of (2.36) with  $u_f$  and subtracting the result to (A.4) yields

$$\begin{aligned} & \rho_f \left( \frac{|u|^2}{2} - \frac{|u_f|^2}{2} \right) (\tilde{N}_t + u_f \cdot \tilde{N}_X) - p_{f_m}(u - u_f) \cdot \tilde{N}_X \\ & - p_s^* \mathcal{V}_f - \rho_f \mathcal{V}_f (u - u_f) \cdot u_f + (T_{f_m} \tilde{N}_X) \cdot (u - u_f) \geq 0. \end{aligned}$$

Using the formula for  $\mathcal{V}_f$  in (2.35), this can be written

$$\left( \frac{\rho_f}{2} |u - u_f|^2 - p_s^* \right) \mathcal{V}_f + \left( (T_{f_m} - p_{f_m} \text{Id}) \tilde{N}_X \right) \cdot (u - u_f) \geq 0. \quad (\text{A.7})$$

Next, we write the tangential part of (2.36) using (A.5), that gives

$$((T_{f_m} - T_f)\tilde{N}_X)_\tau = \rho_f \mathcal{V}_f(u_f - u)_\tau. \quad (\text{A.8})$$

Together with (2.38) it yields

$$\begin{aligned} (T_{f_m}\tilde{N}_X)_\tau &= \frac{1}{2}\rho_f \mathcal{V}_f(u_f - u)_\tau - k_i(u_f - u)_\tau, \\ (T_f\tilde{N}_X)_\tau &= -\frac{1}{2}\rho_f \mathcal{V}_f(u_f - u)_\tau - k_i(u_f - u)_\tau, \end{aligned} \quad (\text{A.9})$$

or equivalently

$$\begin{aligned} T_{f_m}\tilde{N}_X &= (T_{f_m}\tilde{N}_X) \cdot \tilde{N}_X \frac{\tilde{N}_X}{|\tilde{N}_X|^2} + \frac{1}{2}\rho_f \mathcal{V}_f(u_f - u)_\tau - k_i(u_f - u)_\tau, \\ T_f\tilde{N}_X &= (T_f\tilde{N}_X) \cdot \tilde{N}_X \frac{\tilde{N}_X}{|\tilde{N}_X|^2} - \frac{1}{2}\rho_f \mathcal{V}_f(u_f - u)_\tau - k_i(u_f - u)_\tau. \end{aligned} \quad (\text{A.10})$$

Plugging this in (A.7) we get the energy dissipation condition

$$\left( \frac{\rho_f}{2} \left( (u - u_f) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|} \right)^2 - p_s^* \right) \mathcal{V}_f + \left( (T_{f_m}\tilde{N}_X) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|^2} - p_{f_m} \right) \tilde{N}_X \cdot (u - u_f) + k_i |(u_f - u)_\tau|^2 \geq 0. \quad (\text{A.11})$$

Now, the term proportional to  $k_i \geq 0$  is a dissipation. The other terms are proportional to  $\mathcal{V}_f$  because of (A.3). Since  $\mathcal{V}_f$  can be positive or negative, we write that its factor vanishes, that is

$$\frac{\rho_f}{2} \left( (u - u_f) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|} \right)^2 - p_s^* + \left( (T_{f_m}\tilde{N}_X) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|^2} - p_{f_m} \right) \frac{\varphi^*}{1 - \varphi^*} = 0. \quad (\text{A.12})$$

This gives the value of  $p_s^*$ , and reporting this in (A.5) we finally obtain the interface condition (2.37).

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