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NEARBY SLOPES AND BOUNDEDNESS FOR ℓ -ADIC SHEAVES IN POSITIVE CHARACTERISTIC

by

Jean-Baptiste Teyssier

Introduction

Let S be a strictly henselian trait of equal characteristic $p > 0$. As usual, s denotes the closed point of S , k its residue field, $\eta = \text{Spec } K$ the generic point of S , \overline{K} an algebraic closure of K and $\overline{\eta} = \text{Spec } \overline{K}$. Let $f : X \rightarrow S$ be a morphism of finite type, $\ell \neq p$ a prime number, \mathcal{F} an object of the derived category $D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$ of ℓ -adic complexes with bounded and constructible cohomology.

Let $\psi_f^t : D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell) \rightarrow D_c^b(X_s, \overline{\mathbb{Q}}_\ell)$ be the moderate nearby cycle functor. We say that $r \in \mathbb{R}_{\geq 0}$ is a *nearby slope of \mathcal{F} associated to f* if one can find $N \in \text{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ with slope r such that $\psi_f^t(\mathcal{F} \otimes f^*N) \neq 0$. We denote by $\text{Sl}_f^{\text{nb}}(\mathcal{F})$ the set of nearby slopes of \mathcal{F} associated to f .

The main result of [Tey15] is a boundedness theorem for the set of nearby slopes of a complex holonomic \mathcal{D} -module. The goal of the present (mostly programmatic) paper is to give some motivation for an analogue of this theorem for ℓ -adic sheaves in positive characteristic.

For complex holonomic \mathcal{D} -modules, regularity is preserved by push-forward. On the other hand, for a morphism $C' \rightarrow C$ between smooth curves over k , a tame constructible sheaf on C' may acquire wild ramification by push-forward. If $0 \in C$ is a closed point, the failure of $C' \rightarrow C$ to preserve tameness above 0 is accounted for by means of the ramification filtration on the absolute Galois group of the function field of the strict henselianization C_0^{sh} of C at 0 . Moreover, the Swan conductor at 0 measures to which extent an ℓ -adic constructible sheaf on C fails to be tame at 0 .

In higher dimension, both these measures of wild ramification (for a morphism and for a sheaf) are missing in a form that would give a precise meaning to the following question raised in [Tey14]

Question 1. — *Let $g : V_1 \rightarrow V_2$ be a morphism between schemes of finite type over k , and $\mathcal{G} \in D_c^b(V_1, \overline{\mathbb{Q}}_\ell)$. Can one bound the wild ramification of $Rg_*\mathcal{G}$ in terms of the wild ramification of \mathcal{G} and the wild ramification of $g|_{\text{Supp } \mathcal{G}}$?*

Note that in an earlier formulation, "wild ramification of $g|_{\text{Supp } \mathcal{G}}$ " was replaced by "wild ramification of g ", which cannot hold due to the following example that we owe to Alexander Beilinson: take $f : \mathbb{A}_S^1 \rightarrow S$, $P \in S[t]$ and $i_P : \{P = 0\} \hookrightarrow \mathbb{A}_S^1$. Then $i_{P*} \overline{\mathbb{Q}}_\ell$ is tame but $f_*(i_{P*} \overline{\mathbb{Q}}_\ell)$ has arbitrary big wild ramification as P runs through the set of Eisenstein polynomials.

If $f : X \rightarrow S$ is proper, proposition 2.2.1 shows that $\text{Sl}_f^{\text{nb}}(\mathcal{F})$ controls the slopes of $H^i(X_{\overline{\eta}}, \mathcal{F})$ for every $i \geq 0$. It is thus tempting to take for "wild ramification of \mathcal{G} " the nearby slopes of \mathcal{G} .

So Question 1 leads to the question of bounding nearby slopes of constructible ℓ -adic sheaves. Note that this question was raised imprudently in [Tey15]. It has a negative answer as stated in *loc. cit.* since already the constant sheaf $\overline{\mathbb{Q}}_\ell$ has arbitrary big nearby slopes. This is actually good news since for curves, these nearby slopes keep track of the aforementioned ramification filtration⁽¹⁾. Hence, one can use them in higher dimension to quantify the wild ramification of a morphism and in Question 1 take for "wild ramification of $g|_{\text{Supp } \mathcal{G}}$ " the nearby slopes of $\overline{\mathbb{Q}}_\ell$ on $\text{Supp } \mathcal{G}$ associated with $g|_{\text{Supp } \mathcal{G}}$ (at least when V_2 is a curve).

To get a good boundedness statement, one has to correct the nearby slopes associated with a morphism by taking into account the maximal nearby slope of $\overline{\mathbb{Q}}_\ell$ associated with the same morphism. That such a maximal slope exists in general is a consequence of the following

Theorem 1. — *Let $f : X \rightarrow S$ be a morphism of finite type and $\mathcal{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$. The set $\text{Sl}_f^{\text{nb}}(\mathcal{F})$ is finite.*

The proof of this theorem follows an argument due to Deligne [Del77, Th. finitude 3.7]. For a \mathcal{D} -module version, let us refer to [Del07]. Thus, $\text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell)$ makes sense if $\text{Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell)$ is not empty. Otherwise, we set $\text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell) = +\infty$. Proposition 2.3.4 suggests and gives a positive answer to the following question for smooth curves

Question 2. — *Let V/k be a scheme of finite type and $\mathcal{F} \in D_c^b(V, \overline{\mathbb{Q}}_\ell)$. Is it true that the following set*

$$(0.0.1) \quad \{r/(1 + \text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell)), \text{ for } r \in \text{Sl}_f^{\text{nb}}(\mathcal{F}) \text{ and } f \in \mathcal{O}_V\}$$

is bounded?

Let us explain what $\text{Sl}_f^{\text{nb}}(\mathcal{F})$ means in this global setting. A function $f \in \Gamma(U, \mathcal{O}_V)$ reads as $f : U \rightarrow \mathbb{A}_k^1$. If S is the strict henselianization of \mathbb{A}_k^1 at a geometric point over the origin, we set $\text{Sl}_f^{\text{nb}}(\mathcal{F}) := \text{Sl}_{f_S}^{\text{nb}}(\mathcal{F}_{U_S})$ where the subscripts are synonyms of pull-back.

For smooth curves, the main point of the proof of boundedness is the concavity of Herbrand φ functions. In case f has generalized semi-stable reduction (see 1.4), the above weighted slopes are the usual nearby slopes. This is the following

1. see 2.1.2 (3) for a precise statement.

Theorem 2. — *Suppose that $f : X \rightarrow S$ has generalized semi-stable reduction. Then we have $\mathrm{SI}_f^{\mathrm{nb}}(\overline{\mathbb{Q}}_\ell) = \{0\}$.*

We owe the proof of this theorem to Joseph Ayoub. For the vanishing of $\mathcal{H}^0 \psi_f^t$, we also give an earlier argument based on the geometric connectivity of the connected components of the moderate Milnor fibers in case of generalized semi-stable reduction.

As a possible application of a boundedness theorem in the arithmetic setting, let us remark that for every compactification $j : V \rightarrow \overline{V}$, one could define a separated decreasing $\mathbb{R}_{\geq 0}$ -filtration on $\pi_1(V)$ by looking for each $r \in \mathbb{R}_{\geq 0}$ at the category of ℓ -adic local systems L on V such that the weighted slopes (0.0.1) of $j_! L$ are $\leq r$.

Let us also remark that on a smooth curve C , the tameness of $\mathcal{F} \in \mathrm{Sh}_c(C, \overline{\mathbb{Q}}_\ell)$ at $0 \in C$ is characterized by $\mathrm{SI}_f^{\mathrm{nb}}(\mathcal{F}) \subset [0, \mathrm{Max SI}_f^{\mathrm{nb}}(\overline{\mathbb{Q}}_\ell)]$ for every $f \in \mathcal{O}_C$ vanishing only at 0. This suggests a notion of tame complex in any dimension that may be of interest.

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1. Notations

1.1. For a general reference on wild ramification in dimension 1, let us mention [Ser68]. Let η_t be the point of S corresponding to the tamely ramified closure K_t of K in \overline{K} and $P_K := \mathrm{Gal}(\overline{K}/K_t)$ the wild ramification group of K . We denote by $(G_K^r)_{r \in \mathbb{R}_{\geq 0}}$ the *upper-numbering ramification filtration* on G_K and define

$$G_K^{r+} := \overline{\bigcup_{r' > r} G_K^{r'}}$$

If L/K is a finite extension, we denote by S_L the normalization of S in L and v_L the valuation on L associated with the maximal ideal of S_L .

If moreover L/K is separable, we denote by $q : G_K \rightarrow G_K/G_L$ the quotient morphism and define a decreasing separated $\mathbb{R}_{\geq 0}$ -filtration on the set G_K/G_L by $(G_K/G_L)^r := q(G_K^r)$. We also define $(G_K/G_L)^{r+} := q(G_K^{r+})$.

In case L/K is Galois, this filtration is the upper numbering ramification filtration on $\mathrm{Gal}(L/K)$. If L/K is non separable trivial, the *jumps* of L/K are the $r \in \mathbb{R}_{\geq 0}$ such that $(G_K/G_L)^{r+} \subsetneq (G_K/G_L)^r$. If L/K is trivial, we say by convention that 0 is the only jump of $\mathrm{Gal}(L/K)$.

1.2. For $M \in D_c^b(\eta, \overline{\mathbb{Q}}_\ell)$, we denote by $\mathrm{SI}(M) \subset \mathbb{R}_{\geq 0}$ the set of *slopes* of M as defined in [Kat88, Ch 1]. We view M in an equivalent way as a continuous representation of G_K .

1.3. Let $f : X \rightarrow S$ be a morphism of finite type and $\mathcal{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$. Consider the following diagram with cartesian squares

$$\begin{array}{ccccc} X_s & \xrightarrow{i} & X & \xleftarrow{\bar{j}} & X_{\bar{\eta}} \\ \downarrow & & \downarrow f & & \downarrow \\ s & \longrightarrow & S & \longleftarrow & \bar{\eta} \end{array}$$

Following [DK73, XIII], we define the *nearby cycles* of \mathcal{F} as

$$\psi_f \mathcal{F} := i^* R\bar{j}_* \bar{j}^* \mathcal{F}$$

By [Del77, Th. finitude 3.2], the complex $\psi_f \mathcal{F}$ is an object of $D_c^b(X_s, \overline{\mathbb{Q}}_\ell)$ endowed with a continuous G_K -action. Define $X_t := X \times_S \eta_t$ and $j_t : X_t \rightarrow X$ the projection. Following [Gro72, I.2], we define the *moderate nearby cycles* of \mathcal{F} as

$$\psi_f^t \mathcal{F} := i^* Rj_{t*} j_t^* \mathcal{F}$$

It is a complex in $D_c^b(X_s, \overline{\mathbb{Q}}_\ell)$ endowed with a continuous G/P_K -action. Since P_K is a pro- p group, we have a canonical identification

$$\psi_f^t \mathcal{F} \simeq (\psi_f \mathcal{F})^{P_K}$$

Note that by proper base change [AGV73, XII], ψ_f^t and ψ_f are compatible with proper push-forward.

1.4. By a *generalized semi-stable reduction* morphism, we mean a morphism $f : X \rightarrow S$ of finite type such that etale locally on X , f has the form

$$S[x_1, \dots, x_n] / (\pi - x_1^{a_1} \cdots x_m^{a_m}) \rightarrow S$$

where π is a uniformizer of S and where the $a_i \in \mathbb{N}^*$ are prime to p .

1.5. If X is a scheme, $x \in X$ and if \bar{x} is a geometric point of X lying over x , we denote by X_x^{sh} the strict henselization of X at x .

2. Nearby slopes in dimension one

2.1. We show here that nearby slopes associated with the identity morphism are the usual slopes as in [Kat88, Ch 1].

Lemma 2.1.1. — *For every $M \in \text{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$, we have*

$$\text{Sl}_{\text{id}}^{\text{nb}}(M) = \text{Sl}(M)$$

Proof. — We first remark that ψ_{id}^t is just the "invariant under P " functor. Suppose that $r \in \text{Sl}(M)$. Then M has a non zero quotient N purely of slope r . The dual N^\vee has pure slope r . Since N is non zero, the canonical map

$$N \otimes N^\vee \longrightarrow \overline{\mathbb{Q}}_\ell$$

is surjective. Since taking P -invariants is exact, we obtain that the maps in

$$(M \otimes N^\vee)^P \longrightarrow (N \otimes N^\vee)^P \longrightarrow \overline{\mathbb{Q}}_\ell$$

are surjective. Hence $(M \otimes N^\vee)^P \neq 0$, so $r \in \text{Sl}_{\text{id}}^{\text{nb}}(M)$.

If r is not a slope of M , then for any N of slope r , the slopes of $M \otimes N$ are non zero. This is equivalent to $(M \otimes N)^P = 0$. \square

We deduce the following

Lemma 2.1.2. — *Let $f : X \rightarrow S$ be a finite morphism with X local and $\mathcal{F} \in \text{Sh}_c(X_\eta, \overline{\mathbb{Q}}_\ell)$.*

(1) $\text{Sl}_f^{\text{nb}}(\mathcal{F}) = \text{Sl}(f_*\mathcal{F})$.

(2) *Suppose that X is regular connected and let L/K be the extension of function fields induced by f . Suppose that L/K is separable. Then $\text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell)$ is the highest jump in the ramification filtration on G_K/G_L .*

(3) *Suppose further in (2) that L/K is Galois and set $G := \text{Gal}(L/K)$. Then $\text{Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell)$ is the union of $\{0\}$ with the set of jumps in the ramification filtration on G .*

Proof. — Point (1) comes from 2.1.1 and the compatibility of ψ_f^t with proper push-forward.

From point (1) and $f_*\overline{\mathbb{Q}}_\ell \simeq \overline{\mathbb{Q}}_\ell[G_K/G_L]$, we deduce

$$\text{Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell) = \text{Sl}(\overline{\mathbb{Q}}_\ell[G_K/G_L])$$

If L/K is trivial, (2) is true by our definition of jumps in that case. If L/K is non trivial, $r_{\text{max}} = \text{Max Sl}(\overline{\mathbb{Q}}_\ell[G_K/G_L])$ is characterized by the property that $G_K^{r_{\text{max}}}$ acts non trivially on $\overline{\mathbb{Q}}_\ell[G_K/G_L]$ and $G_K^{r_{\text{max}}+}$ acts trivially. On the other hand, the highest jump r_0 in the ramification filtration on G_K/G_L is such that $q(G_K^{r_0}) \neq \{G_L\}$ and $q(G_K^{r_0+}) = \{G_L\}$, that is $G_K^{r_0} \not\subset G_L$ and $G_K^{r_0+} \subset G_L$. The condition $G_K^{r_0} \not\subset G_L$ ensures that $G_K^{r_0}$ acts non trivially on $\overline{\mathbb{Q}}_\ell[G_K/G_L]$. If $h \in G_K^{r_0+}$, then for every $g \in G_K$

$$h \cdot (gG_L) = hgG_L = gg^{-1}hgG_L = gG_L$$

where the last equality comes from the fact that since $G_K^{r_0+}$ is a normal subgroup in G_K , we have $g^{-1}hg \in G_K^{r_0+} \subset G_L$. So (2) is proved.

Let S be the union of $\{0\}$ with the set of jumps in the ramification filtration of G . To prove (3), we have to prove $\text{Sl}(\overline{\mathbb{Q}}_\ell[G]) = S$. If $r \in \mathbb{R}_{\geq 0}$ does not belong to S , we can find an open interval J containing r such that $G^{r'} = G^r$ for every $r' \in J$. In particular, the image of $G_K^{r'}$ by $G_K \rightarrow \text{GL}(\overline{\mathbb{Q}}_\ell[G])$ does not depend on r' for every $r' \in J$. So r is not a slope of $\overline{\mathbb{Q}}_\ell[G]$.

Reciprocally, $\overline{\mathbb{Q}}_\ell[G]$ contains a copy of the trivial representation, so $0 \in \text{Sl}(\overline{\mathbb{Q}}_\ell[G])$. Let $r \in S \setminus \{0\}$. The projection morphism $G \rightarrow G/G^{r+}$ induces a surjection of G_K -representations

$$\overline{\mathbb{Q}}_\ell[G] \longrightarrow \overline{\mathbb{Q}}_\ell[G/G^{r+}] \longrightarrow 0$$

So $\mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G/G^{r+}]) \subset \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G])$. Note that G^{r+} acts trivially on $\overline{\mathbb{Q}}_\ell[G/G^{r+}]$. By definition $G^{r+} \subsetneq G^r$, so G^r acts non trivially on $\overline{\mathbb{Q}}_\ell[G/G^{r+}]$. So $r = \mathrm{Max} \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G/G^{r+}])$ and point (3) is proved. \square

2.2. Let us draw a consequence of 2.1.1. We suppose that $f : X \rightarrow S$ is proper. Let $\mathcal{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$. The G_K -module associated to $R^k f_* \mathcal{F} \in D_c^b(\eta, \overline{\mathbb{Q}}_\ell)$ is $H^k(X_{\overline{\eta}}, \mathcal{F})$. From 2.1.1, we deduce

$$\begin{aligned} \mathrm{Sl}(H^k(X_{\overline{\eta}}, \mathcal{F})) &= \mathrm{Sl}_{\mathrm{id}}^{\mathrm{nb}}(R^k f_* \mathcal{F}) \\ &\subset \mathrm{Sl}_{\mathrm{id}}^{\mathrm{nb}}(Rf_* \mathcal{F}) \end{aligned}$$

where the inclusion comes from the fact that taking P_K -invariants is exact. For every $N \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$, the projection formula and the compatibility of ψ_f^t with proper push-forward gives

$$\begin{aligned} \psi_{\mathrm{id}}^t(Rf_* \mathcal{F} \otimes N) &\simeq \psi_{\mathrm{id}}^t(Rf_*(\mathcal{F} \otimes f^* N)) \\ &\simeq Rf_* \psi_f^t(\mathcal{F} \otimes f^* N) \end{aligned}$$

Hence we have proved the following

Proposition 2.2.1. — *Let $f : X \rightarrow S$ be a proper morphism, and let $\mathcal{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$. For every $i \geq 0$, we have*

$$\mathrm{Sl}(H^i(X_{\overline{\eta}}, \mathcal{F})) \subset \mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{F})$$

2.3. Boundedness. — We first need to see that the upper-numbering filtration is unchanged by purely inseparable base change. This is the following

Lemma 2.3.1. — *Let K'/K be a purely inseparable extension of degree p^n . Let L/K be finite Galois extension, $L' := K' \otimes_K L$ the associated Galois extension of K' . Then, the isomorphism*

$$(2.3.2) \quad \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(L'/K')$$

$$(2.3.3) \quad g \longrightarrow \mathrm{id} \otimes g$$

is compatible with the upper-numbering filtration.

Proof. — Note that for every $g \in \mathrm{Gal}(L/K)$, $\mathrm{id} \otimes g \in \mathrm{Gal}(L'/K')$ is determined by the property that its restriction to L is g .

Let π be a uniformizer of S and π_L a uniformizer of S_L . We have $K \simeq k((\pi))$ and $L \simeq k((\pi_L))$. Since k is perfect and since K'/K and L'/L are purely inseparable of degree p^n , we have $K' = k((\pi^{1/p^n}))$ and $L' = k((\pi_L^{1/p^n}))$. So π_L^{1/p^n} is a uniformizer of $S_{L'}$. For every $\sigma \in \mathrm{Gal}(L'/K')$ we have

$$(\sigma(\pi_L^{1/p^n}) - \pi_L^{1/p^n})^{p^n} = \sigma|_L(\pi_L) - \pi_L$$

so

$$\begin{aligned} v_{L'}(\sigma(\pi_L^{1/p^n}) - \pi_L^{1/p^n}) &= \frac{1}{p^n} v_{L'}(\sigma|_L(\pi_L) - \pi_L) \\ &= v_L(\sigma|_L(\pi_L) - \pi_L) \end{aligned}$$

So (2.3.2) commutes with the lower-numbering filtration. Hence, (2.3.2) commutes with the upper-numbering filtration and lemma 2.3.1 is proved. \square

Boundedness in case of smooth curves over k is a consequence of the following

Proposition 2.3.4. — *Let S_0 be an henselian trait over k , let $\eta_0 = \text{Spec } K_0$ be the generic point of S_0 and $M \in \text{Sh}_c(\eta_0, \overline{\mathbb{Q}}_\ell)$. There exists a constant $C_M \geq 0$ depending only on M such that for every finite morphism $f : S_0 \rightarrow S$, we have*

$$(2.3.5) \quad \text{Sl}_f^{\text{nb}}(M) \subset [0, \text{Max}(C_M, \text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell))]$$

In particular, the quantity

$$\text{Max Sl}_f^{\text{nb}}(M)/(1 + \text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell))$$

is bounded uniformly in f .

Proof. — By 2.1.2 (1), we have to bound $\text{Sl}(f_*M)$ in terms of $\text{Max Sl}(f_*\overline{\mathbb{Q}}_\ell)$. Using [Kat88, I 1.10], we can replace $\overline{\mathbb{Q}}_\ell$ by \mathbb{F}_λ , where $\lambda = \ell^n$. Hence, G_{K_0} acts on M via a finite quotient $H \subset \text{GL}_{\mathbb{F}_\lambda}(M)$. Let L/K_0 be the corresponding finite Galois extension and $f_M : S_L \rightarrow S_0$ the induced morphism. We have $H = \text{Gal}(L/K_0)$. Let us denote by r_M the highest jump in the ramification filtration of H . Using Herbrand functions [Ser68, IV 3], we will prove that the constant $C_M := \psi_{L/K_0}(r_M)$ does the job.

Using 2.3.1, we are left to treat the case where K_0/K is separable. The adjunction morphism

$$M \longrightarrow f_{M*} f_M^* M$$

is injective. Since $f_M^* M \simeq \mathbb{F}_\lambda^{\text{rg } M}$, we obtain by applying f_* an injection

$$f_* M \longrightarrow \mathbb{F}_\lambda[\text{Gal}(L/K)]^{\text{rg } M}$$

So we are left to bound the slopes of $\mathbb{F}_\lambda[\text{Gal}(L/K)]$ viewed as a G_K -representation, that is by 2.1.2 (2) the highest jump in the upper-numbering ramification filtration of $\text{Gal}(L/K)$. By 2.1.2 (2), $r_0 := \text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell)$ is the highest jump in the ramification filtration of $\text{Gal}(L/K)/H$. Choose $r > \text{Max}(r_0, \varphi_{L/K} \psi_{L/K_0}(r_M))$. We have

$$\begin{aligned} \text{Gal}(L/K)^r &= H \cap \text{Gal}(L/K)^r \\ &= H \cap \text{Gal}(L/K)_{\psi_{L/K}(r)} \\ &= H_{\psi_{L/K}(r)} \\ &= H^{\varphi_{L/K_0} \psi_{L/K}(r)} \\ &= \{1\} \end{aligned}$$

The first equality comes from $r > r_0$. The third equality comes from the compatibility of the lower-numbering ramification filtration with subgroups. The last equality comes from the fact that $r > \varphi_{L/K} \psi_{L/K_0}(r_M)$ is equivalent to $\varphi_{L/K_0} \psi_{L/K}(r) > r_M$. Hence,

$$\mathrm{SI}_f^{\mathrm{nb}}(M) \subset [0, \mathrm{Max}(r_0, \varphi_{L/K} \psi_{L/K_0}(r_M))]$$

Since $\varphi_{L/K} : [-1, +\infty[\rightarrow \mathbb{R}$ is concave, satisfies $\varphi_{L/K}(0) = 0$ and is equal to the identity on $[-1, 0]$, we have

$$\varphi_{L/K} \psi_{L/K_0}(r_M) \leq \psi_{L/K_0}(r_M)$$

and we obtain (2.3.5) by setting $C_M := \psi_{L/K_0}(r_M)$. \square

3. Proof of Theorem 1

3.1. Preliminary. — Let us consider the affine line $\mathbb{A}_S^1 \rightarrow S$ over S . Let s' be the generic point of \mathbb{A}_S^1 and S' the strict henselianization of \mathbb{A}_S^1 at s' . We denote by \overline{S} the normalization of S in $\overline{\eta}$, by κ the function field of the strict henselianization of \mathbb{A}_S^1 at s' , and by $\overline{\kappa}$ an algebraic closure of κ . We have $\kappa \simeq K' \otimes_K \overline{K}$ and

$$(3.1.1) \quad G_K \simeq \mathrm{Gal}(\kappa/K')$$

Let L/K be a finite Galois extension of K in \overline{K} . Set $L' := K' \otimes_K L$. At finite level, (3.1.1) reads

$$(3.1.2) \quad \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(L'/K')$$

$$(3.1.3) \quad g \rightarrow \mathrm{id} \otimes g$$

Since a uniformizer in S_L is also a uniformizer in $S'_{L'}$, we deduce that (3.1.2) is compatible with the lower-numbering ramification filtration on $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(L'/K')$. Hence, (3.1.2) is compatible with the upper-numbering ramification filtration on $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(L'/K')$. We deduce that through (3.1.1), the canonical surjection $G_{K'} \rightarrow G_K$ is compatible with the upper-numbering ramification filtration.

3.2. The proof. — We can suppose that \mathcal{F} is concentrated in degree 0. In case $\dim X = 0$, there is nothing to prove. We first reduce the proof of Theorem 1 to the case where $\dim X = 1$ by arguing by induction on $\dim X$.

Since the problem is local on X , we can suppose that X is affine. We thus have a digram

$$(3.2.1) \quad \begin{array}{ccccc} X & \longrightarrow & \mathbb{A}_S^n & \longrightarrow & \mathbb{P}_S^n \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

Let \overline{X} be the closure of X in \mathbb{P}_S^n and let $j : X \hookrightarrow \overline{X}$ be the associated open immersion. Replacing (X, \mathcal{F}) by $(\overline{X}, j_* \mathcal{F})$, we can suppose X/S projective. Then Theorem 1 is a consequence of the following assertions

- (A) There exists a finite set $E_A \subset \mathbb{R}_{\geq 0}$ such that for every $N \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ with slope not in E_A , the support of $\psi_f^t(\mathcal{F} \otimes f^* N)$ is punctual.

(B) There exists a finite set $E_B \subset \mathbb{R}_{\geq 0}$ such that for every $N \in \text{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ with slope not in E_B , we have

$$R\Gamma(X_s, \psi_f^t(\mathcal{F} \otimes f^*N)) \simeq 0$$

Let us prove (A). This is a local statement on X , so we can suppose X to be a closed subset in \mathbb{A}_S^n and consider the factorisations

$$\begin{array}{ccc} X & \xrightarrow{p_i} & \mathbb{A}_S^1 \\ & \searrow f & \downarrow \\ & & S \end{array}$$

where p_i is the projection on the i -th factor of \mathbb{A}_S^n . Using the notations in 3.1, let X'/S' making the upper square of the following diagram

$$\begin{array}{ccc} X' & \xrightarrow{\lambda} & X \\ p'_i \downarrow & & \downarrow p_i \\ S' & \longrightarrow & \mathbb{A}_S^1 \\ & \searrow h & \downarrow \\ & & S \end{array}$$

cartesian. Let us set $\mathcal{F}' := \lambda^*\mathcal{F}$ and $N' := h^*N$. From [Del77, Th. finitude 3.4], we have

$$(3.2.2) \quad \lambda^*\psi_f(\mathcal{F} \otimes f^*N) \simeq \psi_{hp'_i}(\mathcal{F}' \otimes p_i'^*N') \simeq \psi_{p'_i}(\mathcal{F}' \otimes p_i'^*N')^{G_\kappa}$$

where G_κ is a pro- p group sitting in an exact sequence

$$1 \longrightarrow G_\kappa \longrightarrow G_{K'} \longrightarrow G_K \longrightarrow 1$$

In particular, G_κ is a subgroup of the wild-ramification group $P_{K'}$ of $G_{K'}$. So applying the $P_{K'}$ -invariants on (3.2.2) yields

$$(3.2.3) \quad \lambda^*\psi_f^t(\mathcal{F} \otimes f^*N) \simeq \psi_{p'_i}^t(\mathcal{F}' \otimes p_i'^*N')$$

If N has pure slope r , we know from 3.1 that N' has pure slope r as a sheaf on η' . Applying the recursion hypothesis gives a finite set $E_i \subset \mathbb{R}_{\geq 0}$ such that the right-hand side of (3.2.3) is 0 for N of slope not in E_i . The union of the E_i for $1 \leq i \leq n$ is the set E_A sought for in (A).

To prove (B), we observe that the compatibility of ψ_f^t with proper morphisms and the projection formula give

$$R\Gamma(X_s, \psi_f^t(\mathcal{F} \otimes f^*N)) \simeq \psi_{\text{id}}^t(Rf_*\mathcal{F} \otimes N)$$

By 2.1.1, the set $E_B := \text{Sl}(Rf_*\mathcal{F})$ has the required properties.

We are thus left to prove Theorem 1 in the case where $\dim X = 1$. At the cost of localizing, we can suppose that X is local and maps surjectively on S . Let x be the closed point of X . Note that $k(x)/k(s)$ is of finite type but may not be finite. Choosing a transcendence basis of $k(x)/k(s)$ yields a factorization $X \longrightarrow S' \longrightarrow S$

satisfying $\text{trdeg}_{k(s')} k(x) = \text{trdeg}_{k(s)} k(x) - 1$.

So we can further suppose that $k(x)/k(s)$ is finite. Since $k(s)$ is algebraically closed, we have $k(x) = k(s)$. If \widehat{S} denotes the completion of S at s , we deduce that $X \times_S \widehat{S}$ is finite over \widehat{S} . By faithfully flat descent [Gro71, VIII 5.7], we obtain that X/S is finite. We conclude the proof of Theorem 1 with 2.1.2 (1).

4. Proof of Theorem 2

4.1. That $0 \in \text{Sl}_f(\overline{\mathbb{Q}}_\ell)$ is easy by looking at the smooth locus of f . We are left to prove that for every $N \in \text{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ with slope > 0 , the following holds

$$(4.1.1) \quad \psi_f^t f^* N \simeq 0$$

Since the problem is local on X for the étale topology, we can suppose that $X = S[x_1, \dots, x_n]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$ and we have to prove (4.1.1) at the origin $0 \in X_s$. Let a be the lowest common multiple of the a_i and define $b_i = a/a_i$. Note that a and the b_i are prime to p . Hence the morphism h defined as

$$\begin{aligned} Y := S[t_1, \dots, t_n]/(\pi - t_1^a \cdots t_m^a) &\longrightarrow X \\ (t_1, \dots, t_n) &\longrightarrow (t_1^{b_1}, \dots, t_m^{b_m}, t_{m+1}, \dots, t_n) \end{aligned}$$

is finite surjective and finite étale above η with Galois group G . Set $g = fh$. Then

$$(\mathcal{H}^i \psi_f^t f^* N)_{\overline{0}} \simeq (\mathcal{H}^i \psi_g^t g^* N)_{\overline{0}}^G$$

for every $i \geq 0$, so we can suppose $a_1 = \cdots = a_m = a$. Since a is prime to p , the map of absolute Galois groups induced by $S[\pi^{1/a}] \rightarrow S$ induces an identification at the level of the ramification groups. By compatibility of nearby cycles with change of trait [Del77, Th. finitude 3.7], we can suppose $a = 1$.

Let us now reduce the proof of Theorem 2 to the case where $m = 1$. We argue by induction on m . The case $m = 1$ follows from the compatibility of nearby cycles with smooth morphisms. We thus suppose that Theorem 2 is true for $m < n$ with all a_i equal to 1 and prove it for $m + 1$ with all a_i equal to 1. Let $h : \widetilde{X} \rightarrow X$ be the blow-up of X along $x_m = x_{m+1} = 0$. Define $g := fh$ and denote by E the exceptional divisor of \widetilde{X} . Since h induces an isomorphism on the generic fibers, and since ψ_f^t is compatible with proper push-forward, we have

$$(4.1.2) \quad Rh_* \psi_g^t g^* N \simeq \psi_f^t f^* N \simeq 0$$

By proper base change, (4.1.2) gives

$$(4.1.3) \quad R\Gamma(h^{-1}(0), (\psi_g^t g^* N)|_{h^{-1}(0)}) \simeq 0$$

The scheme \widetilde{X} is covered by a chart U affine over S given by

$$S[(u_i)_{1 \leq i \leq n}]/(\pi - u_1 \cdots u_m)$$

with $E \cap U$ given by $u_m = 0$, and a chart U' affine over S given by

$$S[(u_i)_{1 \leq i \leq n}]/(\pi - u_1 \cdots u_{m+1})$$

with $E \cap U'$ given by $u_{m+1} = 0$. By recursion hypothesis, $(\psi_g^t g^* N)|_{h^{-1}(0)}$ is a skyscraper sheaf supported at the origin 0 of U' . Hence, (4.1.3) gives

$$(\psi_g^t g^* N)_{\overline{0}} \simeq 0$$

This finishes the induction, and thus the proof of Theorem 2.

4.2. Let us give a geometric-flavoured proof of

$$\mathcal{H}^0 \psi_f^t f^* N \simeq 0$$

in case $X = S[x_1, \dots, x_n]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$. By constructibility [Del77, Th. finitude 3.2], it is enough to work at the level of germs at a geometric point \bar{x} lying over a closed point $x \in X$.

Hence, we have to prove $H^0(C, f^* N) \simeq 0$ for every connected component C of $X_{x, \eta_t}^{\text{sh}}$. For such C , denote by $\rho_C : \pi_1(C) \rightarrow \pi_1(\eta_t) = P_K$ the induced map. Then $H^0(C, f^* N) \simeq N^{\text{Im } \rho_C}$. Since by definition $N^{P_K} = 0$, it is enough to prove that ρ_C is surjective. From V 6.9 and IX 3.4 of [Gro71], we are left to prove that C is geometrically connected. To do this, we can always replace X_x^{sh} by its formalization $\widehat{X}_x = \text{Spec } R[[\underline{x}]]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$.

By hypothesis, $d := \gcd(a_1, \dots, a_m)$ is prime to p , so π has a d -root in K_t . Hence \widehat{X}_{x, η_t} is a direct union of d copies of

$$\text{Spec } K_t \otimes_R R[[\underline{x}]]/(\pi^{1/d} - x_1^{a'_1} \cdots x_m^{a'_m})$$

where $a_i = da'_i$. So we have to prove the following

Lemma 4.2.1. — *Let $a_1, \dots, a_m, d \in \mathbb{N}^*$ with $\gcd(a_1, \dots, a_m) = 1$. Then*

$$(4.2.2) \quad \text{Spec } \overline{K} \otimes_R R[[\underline{x}]]/(\pi^{1/d} - x_1^{a_1} \cdots x_m^{a_m})$$

is connected.

Proof. — One easily reduces to the case $d = 1$. If R' is the normalization of R in a Galois extension of K in \overline{K} , it is enough to prove that $\text{Spec } R'[[\underline{x}]]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$ is irreducible. If π' is a uniformizer of R' , we have $R' \simeq k[[\pi']]$, we write $\pi = P(\pi')$ where $P \in k[[X]]$ and then we are left to prove that $f_{a,P} := P(\pi') - x_1^{a_1} \cdots x_m^{a_m}$ is irreducible in $k[[x_1, \dots, x_n, \pi']]$. This follows from $\gcd(a_1, \dots, a_m) = 1$ via Lypkovski's indecomposability criterion [Lip88, 2.10] for the Newton polyhedron associated to $f_{a,P}$. □

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