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# NEARBY SLOPES AND BOUNDEDNESS FOR $\ell$ -ADIC SHEAVES IN POSITIVE CHARACTERISTIC

*by*

Jean-Baptiste Teyssier

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## Introduction

Let  $S$  be a strictly henselian trait of equal characteristic  $p > 0$ . As usual,  $s$  denotes the closed point of  $S$ ,  $k$  its residue field,  $\eta = \operatorname{Spec} K$  the generic point of  $S$ ,  $\overline{K}$  an algebraic closure of  $K$  and  $\overline{\eta} = \operatorname{Spec} \overline{K}$ . Let  $f : X \rightarrow S$  be a morphism of finite type,  $\ell \neq p$  a prime number,  $\mathcal{F}$  an object of the derived category  $D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$  of  $\ell$ -adic complexes with bounded and constructible cohomology.

Let  $\psi_f^t : D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell) \rightarrow D_c^b(X_s, \overline{\mathbb{Q}}_\ell)$  be the moderate nearby cycle functor. We say that  $r \in \mathbb{R}_{\geq 0}$  is a *nearby slope of  $\mathcal{F}$  associated to  $f$*  if one can find  $N \in \operatorname{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$  with slope  $r$  such that  $\psi_f^t(\mathcal{F} \otimes f^*N) \neq 0$ . We denote by  $\operatorname{Sl}_f^{\text{nb}}(\mathcal{F})$  the set of nearby slopes of  $\mathcal{F}$  associated to  $f$ .

The main result of [Tey15] is a boundedness theorem for the set of nearby slopes of a complex holonomic  $\mathcal{D}$ -module. The goal of the present (mostly programmatic) paper is to give some motivation for an analogue of this theorem for  $\ell$ -adic sheaves in positive characteristic.

For complex holonomic  $\mathcal{D}$ -modules, regularity is preserved by push-forward. On the other hand, for a morphism  $C' \rightarrow C$  between smooth curves over  $k$ , a tame constructible sheaf on  $C'$  may acquire wild ramification by push-forward. If  $0 \in C$  is a closed point, the failure of  $C' \rightarrow C$  to preserve tameness above 0 is accounted for by means of the ramification filtration on the absolute Galois group of the function field of the strict henselianization  $C_0^{\text{sh}}$  of  $C$  at 0. Moreover, the Swan conductor at 0 measures to which extent an  $\ell$ -adic constructible sheaf on  $C$  fails to be tame at 0.

In higher dimension, both these measures of wild ramification (for a morphism and for a sheaf) are missing in a form that would give a precise meaning to the following question raised in [Tey14]

**Question 1.** — *Let  $g : V_1 \rightarrow V_2$  be a morphism between schemes of finite type over  $k$ , and  $\mathcal{G} \in D_c^b(V_1, \overline{\mathbb{Q}}_\ell)$ . Can one bound the wild ramification of  $Rg_*\mathcal{G}$  in terms of the wild ramification of  $\mathcal{G}$  and the wild ramification of  $g|_{\operatorname{Supp} \mathcal{G}}$ ?*

Note that in an earlier formulation, "wild ramification of  $g|_{\text{Supp } \mathcal{G}}$ " was replaced by "wild ramification of  $g$ ", which cannot hold due to the following example that we owe to Alexander Beilinson: take  $f : \mathbb{A}_S^1 \rightarrow S$ ,  $P \in S[t]$  and  $i_P : \{P = 0\} \hookrightarrow \mathbb{A}_S^1$ . Then  $i_{P*}\overline{\mathbb{Q}}_\ell$  is tame but  $f_*(i_{P*}\overline{\mathbb{Q}}_\ell)$  has arbitrary big wild ramification as  $P$  runs through the set of Eisenstein polynomials.

If  $f : X \rightarrow S$  is proper, proposition 2.2.1 shows that  $\text{Sl}_f^{\text{nb}}(\mathcal{F})$  controls the slopes of  $H^i(X_{\overline{\eta}}, \mathcal{F})$  for every  $i \geq 0$ . It is thus tempting to take for "wild ramification of  $\mathcal{G}$ " the nearby slopes of  $\mathcal{G}$ .

So Question 1 leads to the question of bounding nearby slopes of constructible  $\ell$ -adic sheaves. Note that this question was raised imprudently in [Tey15]. It has a negative answer as stated in *loc. cit.* since already the constant sheaf  $\overline{\mathbb{Q}}_\ell$  has arbitrary big nearby slopes. This is actually good news since for curves, these nearby slopes keep track of the aforementioned ramification filtration<sup>(1)</sup>. Hence, one can use them in higher dimension to quantify the wild ramification of a morphism and in Question 1 take for "wild ramification of  $g|_{\text{Supp } \mathcal{G}}$ " the nearby slopes of  $\overline{\mathbb{Q}}_\ell$  on  $\text{Supp } \mathcal{G}$  associated with  $g|_{\text{Supp } \mathcal{G}}$  (at least when  $V_2$  is a curve).

To get a good boundedness statement, one has to correct the nearby slopes associated with a morphism by taking into account the maximal nearby slope of  $\overline{\mathbb{Q}}_\ell$  associated with the same morphism. That such a maximal slope exists in general is a consequence of the following

**Theorem 1.** — *Let  $f : X \rightarrow S$  be a morphism of finite type and  $\mathcal{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$ . The set  $\text{Sl}_f^{\text{nb}}(\mathcal{F})$  is finite.*

The proof of this theorem follows an argument due to Deligne [Del77, Th. finitude 3.7]. For a  $\mathcal{D}$ -module version, let us refer to [Del07]. Thus,  $\text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell)$  makes sense if  $\text{Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell)$  is not empty. Otherwise, we set  $\text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell) = +\infty$ . Proposition 2.3.4 suggests and gives a positive answer to the following question for smooth curves

**Question 2.** — *Let  $V/k$  be a scheme of finite type and  $\mathcal{F} \in D_c^b(V, \overline{\mathbb{Q}}_\ell)$ . Is it true that the following set*

$$(0.0.1) \quad \{r/(1 + \text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell)), \text{ for } r \in \text{Sl}_f^{\text{nb}}(\mathcal{F}) \text{ and } f \in \mathcal{O}_V\}$$

*is bounded?*

Let us explain what  $\text{Sl}_f^{\text{nb}}(\mathcal{F})$  means in this global setting. A function  $f \in \Gamma(U, \mathcal{O}_V)$  reads as  $f : U \rightarrow \mathbb{A}_k^1$ . If  $S$  is the strict henselianization of  $\mathbb{A}_k^1$  at a geometric point over the origin, we set  $\text{Sl}_f^{\text{nb}}(\mathcal{F}) := \text{Sl}_{f_S}^{\text{nb}}(\mathcal{F}_{U_S})$  where the subscripts are synonyms of pull-back.

For smooth curves, the main point of the proof of boundedness is the concavity of Herbrand  $\varphi$  functions. In case  $f$  has generalized semi-stable reduction (see 1.4), the above weighted slopes are the usual nearby slopes. This is the following

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1. see 2.1.2 (3) for a precise statement.

**Theorem 2.** — *Suppose that  $f : X \rightarrow S$  has generalized semi-stable reduction. Then we have  $\mathrm{Sl}_f^{\mathrm{nb}}(\overline{\mathbb{Q}}_\ell) = \{0\}$ .*

We owe the proof of this theorem to Joseph Ayoub. For the vanishing of  $\mathcal{H}^0 \psi_f^t$ , we also give an earlier argument based on the geometric connectivity of the connected components of the moderate Milnor fibers in case of generalized semi-stable reduction.

As a possible application of a boundedness theorem in the arithmetic setting, let us remark that for every compactification  $j : V \rightarrow \overline{V}$ , one could define a separated decreasing  $\mathbb{R}_{\geq 0}$ -filtration on  $\pi_1(V)$  by looking for each  $r \in \mathbb{R}_{\geq 0}$  at the category of  $\ell$ -adic local systems  $L$  on  $V$  such that the weighted slopes (0.0.1) of  $j_! L$  are  $\leq r$ .

Let us also remark that on a smooth curve  $C$ , the tameness of  $\mathcal{F} \in \mathrm{Sh}_c(C, \overline{\mathbb{Q}}_\ell)$  at  $0 \in C$  is characterized by  $\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{F}) \subset [0, \mathrm{Max} \mathrm{Sl}_f^{\mathrm{nb}}(\overline{\mathbb{Q}}_\ell)]$  for every  $f \in \mathcal{O}_C$  vanishing only at 0. This suggests a notion of tame complex in any dimension that may be of interest.

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## 1. Notations

**1.1.** For a general reference on wild ramification in dimension 1, let us mention [Ser68]. Let  $\eta_t$  be the point of  $S$  corresponding to the tamely ramified closure  $K_t$  of  $K$  in  $\overline{K}$  and  $P_K := \mathrm{Gal}(\overline{K}/K_t)$  the wild ramification group of  $K$ . We denote by  $(G_K^r)_{r \in \mathbb{R}_{\geq 0}}$  the *upper-numbering ramification filtration* on  $G_K$  and define

$$G_K^{r+} := \overline{\bigcup_{r' > r} G_K^{r'}}$$

If  $L/K$  is a finite extension, we denote by  $S_L$  the normalization of  $S$  in  $L$  and  $v_L$  the valuation on  $L$  associated with the maximal ideal of  $S_L$ .

If moreover  $L/K$  is separable, we denote by  $q : G_K \rightarrow G_K/G_L$  the quotient morphism and define a decreasing separated  $\mathbb{R}_{\geq 0}$ -filtration on the set  $G_K/G_L$  by  $(G_K/G_L)^r := q(G_K^r)$ . We also define  $(G_K/G_L)^{r+} := q(G_K^{r+})$ .

In case  $L/K$  is Galois, this filtration is the upper numbering ramification filtration on  $\mathrm{Gal}(L/K)$ . If  $L/K$  is non separable trivial, the *jumps* of  $L/K$  are the  $r \in \mathbb{R}_{\geq 0}$  such that  $(G_K/G_L)^{r+} \subsetneq (G_K/G_L)^r$ . If  $L/K$  is trivial, we say by convention that 0 is the only jump of  $\mathrm{Gal}(L/K)$ .

**1.2.** For  $M \in D_c^b(\eta, \overline{\mathbb{Q}}_\ell)$ , we denote by  $\mathrm{Sl}(M) \subset \mathbb{R}_{\geq 0}$  the set of *slopes* of  $M$  as defined in [Kat88, Ch 1]. We view  $M$  in an equivalent way as a continuous representation of  $G_K$ .

**1.3.** Let  $f : X \longrightarrow S$  be a morphism of finite type and  $\mathcal{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$ . Consider the following diagram with cartesian squares

$$\begin{array}{ccccc} X_s & \xrightarrow{i} & X & \xleftarrow{\bar{j}} & X_{\overline{\eta}} \\ \downarrow & & \downarrow f & & \downarrow \\ s & \longrightarrow & S & \longleftarrow & \overline{\eta} \end{array}$$

Following [DK73, XIII], we define the *nearby cycles* of  $\mathcal{F}$  as

$$\psi_f \mathcal{F} := i^* R\bar{j}_* \bar{j}^* \mathcal{F}$$

By [Del77, Th. finitude 3.2], the complex  $\psi_f \mathcal{F}$  is an object of  $D_c^b(X_s, \overline{\mathbb{Q}}_\ell)$  endowed with a continuous  $G_K$ -action. Define  $X_t := X \times_S \eta_t$  and  $j_t : X_t \longrightarrow X$  the projection. Following [Gro72, I.2], we define the *moderate nearby cycles* of  $\mathcal{F}$  as

$$\psi_f^t \mathcal{F} := i^* Rj_{t*} j_t^* \mathcal{F}$$

It is a complex in  $D_c^b(X_s, \overline{\mathbb{Q}}_\ell)$  endowed with a continuous  $G/P_K$ -action. Since  $P_K$  is a pro- $p$  group, we have a canonical identification

$$\psi_f^t \mathcal{F} \simeq (\psi_f \mathcal{F})^{P_K}$$

Note that by proper base change [AGV73, XII],  $\psi_f^t$  and  $\psi_f$  are compatible with proper push-forward.

**1.4.** By a *generalized semi-stable reduction* morphism, we mean a morphism  $f : X \longrightarrow S$  of finite type such that etale locally on  $X$ ,  $f$  has the form

$$S[x_1, \dots, x_n] / (\pi - x_1^{a_1} \cdots x_n^{a_n}) \longrightarrow S$$

where  $\pi$  is a uniformizer of  $S$  and where the  $a_i \in \mathbb{N}^*$  are prime to  $p$ .

**1.5.** If  $X$  is a scheme,  $x \in X$  and if  $\overline{x}$  is a geometric point of  $X$  lying over  $x$ , we denote by  $X_x^{\text{sh}}$  the strict henselization of  $X$  at  $x$ .

## 2. Nearby slopes in dimension one

**2.1.** We show here that nearby slopes associated with the identity morphism are the usual slopes as in [Kat88, Ch 1].

**Lemma 2.1.1.** — *For every  $M \in \text{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ , we have*

$$\text{Sl}_{\text{id}}^{\text{nb}}(M) = \text{Sl}(M)$$

*Proof.* — We first remark that  $\psi_{\text{id}}^t$  is just the "invariant under  $P$ " functor. Suppose that  $r \in \text{Sl}(M)$ . Then  $M$  has a non zero quotient  $N$  purely of slope  $r$ . The dual  $N^\vee$  has pure slope  $r$ . Since  $N$  is non zero, the canonical map

$$N \otimes N^\vee \longrightarrow \overline{\mathbb{Q}}_\ell$$

is surjective. Since taking  $P$ -invariants is exact, we obtain that the maps in

$$(M \otimes N^\vee)^P \longrightarrow (N \otimes N^\vee)^P \longrightarrow \overline{\mathbb{Q}}_\ell$$

are surjective. Hence  $(M \otimes N^\vee)^P \neq 0$ , so  $r \in \mathrm{Sl}_{\mathrm{id}}^{\mathrm{nb}}(M)$ .

If  $r$  is not a slope of  $M$ , then for any  $N$  of slope  $r$ , the slopes of  $M \otimes N$  are non zero. This is equivalent to  $(M \otimes N)^P = 0$ .  $\square$

We deduce the following

**Lemma 2.1.2.** — *Let  $f : X \longrightarrow S$  be a finite morphism with  $X$  local and  $\mathcal{F} \in \mathrm{Sh}_c(X_\eta, \overline{\mathbb{Q}}_\ell)$ .*

(1)  $\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{F}) = \mathrm{Sl}(f_*\mathcal{F})$ .

(2) *Suppose that  $X$  is regular connected and let  $L/K$  be the extension of function fields induced by  $f$ . Suppose that  $L/K$  is separable. Then  $\mathrm{Max} \mathrm{Sl}_f^{\mathrm{nb}}(\overline{\mathbb{Q}}_\ell)$  is the highest jump in the ramification filtration on  $G_K/G_L$ .*

(3) *Suppose further in (2) that  $L/K$  is Galois and set  $G := \mathrm{Gal}(L/K)$ . Then  $\mathrm{Sl}_f^{\mathrm{nb}}(\overline{\mathbb{Q}}_\ell)$  is the union of  $\{0\}$  with the set of jumps in the ramification filtration on  $G$ .*

*Proof.* — Point (1) comes from 2.1.1 and the compatibility of  $\psi_f^t$  with proper push-forward.

From point (1) and  $f_*\overline{\mathbb{Q}}_\ell \simeq \overline{\mathbb{Q}}_\ell[G_K/G_L]$ , we deduce

$$\mathrm{Sl}_f^{\mathrm{nb}}(\overline{\mathbb{Q}}_\ell) = \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G_K/G_L])$$

If  $L/K$  is trivial, (2) is true by our definition of jumps in that case. If  $L/K$  is non trivial,  $r_{\max} = \mathrm{Max} \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G_K/G_L])$  is characterized by the property that  $G_K^{r_{\max}}$  acts non trivially on  $\overline{\mathbb{Q}}_\ell[G_K/G_L]$  and  $G_K^{r_{\max}+}$  acts trivially. On the other hand, the highest jump  $r_0$  in the ramification filtration on  $G_K/G_L$  is such that  $q(G_K^{r_0}) \neq \{G_L\}$  and  $q(G_K^{r_0+}) = \{G_L\}$ , that is  $G_K^{r_0} \not\subset G_L$  and  $G_K^{r_0+} \subset G_L$ . The condition  $G_K^{r_0} \not\subset G_L$  ensures that  $G_K^{r_0}$  acts non trivially on  $\overline{\mathbb{Q}}_\ell[G_K/G_L]$ . If  $h \in G_K^{r_0+}$ , then for every  $g \in G_K$

$$h \cdot (gG_L) = hgG_L = gg^{-1}hgG_L = gG_L$$

where the last equality comes from the fact that since  $G_K^{r_0+}$  is a normal subgroup in  $G_K$ , we have  $g^{-1}hg \in G_K^{r_0+} \subset G_L$ . So (2) is proved.

Let  $S$  be the union of  $\{0\}$  with the set of jumps in the ramification filtration of  $G$ . To prove (3), we have to prove  $\mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G]) = S$ . If  $r \in \mathbb{R}_{\geq 0}$  does not belong to  $S$ , we can find an open interval  $J$  containing  $r$  such that  $G^{r'} = G^r$  for every  $r' \in J$ . In particular, the image of  $G_K^{r'}$  by  $G_K \longrightarrow \mathrm{GL}(\overline{\mathbb{Q}}_\ell[G])$  does not depend on  $r'$  for every  $r' \in J$ . So  $r$  is not a slope of  $\overline{\mathbb{Q}}_\ell[G]$ .

Reciprocally,  $\overline{\mathbb{Q}}_\ell[G]$  contains a copy of the trivial representation, so  $0 \in \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G])$ . Let  $r \in S \setminus \{0\}$ . The projection morphism  $G \longrightarrow G/G^{r+}$  induces a surjection of  $G_K$ -representations

$$\overline{\mathbb{Q}}_\ell[G] \longrightarrow \overline{\mathbb{Q}}_\ell[G/G^{r+}] \longrightarrow 0$$

So  $\mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G/G^{r+}]) \subset \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G])$ . Note that  $G^{r+}$  acts trivially on  $\overline{\mathbb{Q}}_\ell[G/G^{r+}]$ . By definition  $G^{r+} \subsetneq G^r$ , so  $G^r$  acts non trivially on  $\overline{\mathbb{Q}}_\ell[G/G^{r+}]$ . So  $r = \mathrm{Max} \mathrm{Sl}(\overline{\mathbb{Q}}_\ell[G/G^{r+}])$  and point (3) is proved.  $\square$

**2.2.** Let us draw a consequence of 2.1.1. We suppose that  $f : X \rightarrow S$  is proper. Let  $\mathcal{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$ . The  $G_K$ -module associated to  $R^k f_* \mathcal{F} \in D_c^b(\eta, \overline{\mathbb{Q}}_\ell)$  is  $H^k(X_{\overline{\eta}}, \mathcal{F})$ . From 2.1.1, we deduce

$$\begin{aligned} \mathrm{Sl}(H^k(X_{\overline{\eta}}, \mathcal{F})) &= \mathrm{Sl}_{\mathrm{id}}^{\mathrm{nb}}(R^k f_* \mathcal{F}) \\ &\subset \mathrm{Sl}_{\mathrm{id}}^{\mathrm{nb}}(Rf_* \mathcal{F}) \end{aligned}$$

where the inclusion comes from the fact that taking  $P_K$ -invariants is exact. For every  $N \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$ , the projection formula and the compatibility of  $\psi_f^t$  with proper push-forward gives

$$\begin{aligned} \psi_{\mathrm{id}}^t(Rf_* \mathcal{F} \otimes N) &\simeq \psi_{\mathrm{id}}^t(Rf_*(\mathcal{F} \otimes f^* N)) \\ &\simeq Rf_* \psi_f^t(\mathcal{F} \otimes f^* N) \end{aligned}$$

Hence we have proved the following

**Proposition 2.2.1.** — *Let  $f : X \rightarrow S$  be a proper morphism, and let  $\mathcal{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$ . For every  $i \geq 0$ , we have*

$$\mathrm{Sl}(H^i(X_{\overline{\eta}}, \mathcal{F})) \subset \mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{F})$$

**2.3. Boundedness.** — We first need to see that the upper-numbering filtration is unchanged by purely inseparable base change. This is the following

**Lemma 2.3.1.** — *Let  $K'/K$  be a purely inseparable extension of degree  $p^n$ . Let  $L/K$  be finite Galois extension,  $L' := K' \otimes_K L$  the associated Galois extension of  $K'$ . Then, the isomorphism*

$$(2.3.2) \quad \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(L'/K')$$

$$(2.3.3) \quad g \longrightarrow \mathrm{id} \otimes g$$

*is compatible with the upper-numbering filtration.*

*Proof.* — Note that for every  $g \in \mathrm{Gal}(L/K)$ ,  $\mathrm{id} \otimes g \in \mathrm{Gal}(L'/K')$  is determined by the property that its restriction to  $L$  is  $g$ .

Let  $\pi$  be a uniformizer of  $S$  and  $\pi_L$  a uniformizer of  $S_L$ . We have  $K \simeq k((\pi))$  and  $L \simeq k((\pi_L))$ . Since  $k$  is perfect and since  $K'/K$  and  $L'/L$  are purely inseparable of degree  $p^n$ , we have  $K' = k((\pi^{1/p^n}))$  and  $L' = k((\pi_L^{1/p^n}))$ . So  $\pi_L^{1/p^n}$  is a uniformizer of  $S_{L'}$ . For every  $\sigma \in \mathrm{Gal}(L'/K')$  we have

$$(\sigma(\pi_L^{1/p^n}) - \pi_L^{1/p^n})^{p^n} = \sigma|_L(\pi_L) - \pi_L$$

so

$$\begin{aligned} v_{L'}(\sigma(\pi_L^{1/p^n}) - \pi_L^{1/p^n}) &= \frac{1}{p^n} v_{L'}(\sigma|_L(\pi_L) - \pi_L) \\ &= v_L(\sigma|_L(\pi_L) - \pi_L) \end{aligned}$$

So (2.3.2) commutes with the lower-numbering filtration. Hence, (2.3.2) commutes with the upper-numbering filtration and lemma 2.3.1 is proved.  $\square$

Boundedness in case of smooth curves over  $k$  is a consequence of the following

**Proposition 2.3.4.** — *Let  $S_0$  be an henselian trait over  $k$ , let  $\eta_0 = \text{Spec } K_0$  be the generic point of  $S_0$  and  $M \in \text{Sh}_c(\eta_0, \overline{\mathbb{Q}}_\ell)$ . There exists a constant  $C_M \geq 0$  depending only on  $M$  such that for every finite morphism  $f : S_0 \rightarrow S$ , we have*

$$(2.3.5) \quad \text{Sl}_f^{\text{nb}}(M) \subset [0, \text{Max}(C_M, \text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell))]$$

In particular, the quantity

$$\text{Max Sl}_f^{\text{nb}}(M)/(1 + \text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell))$$

is bounded uniformly in  $f$ .

*Proof.* — By 2.1.2 (1), we have to bound  $\text{Sl}(f_*M)$  in terms of  $\text{Max Sl}(f_*\overline{\mathbb{Q}}_\ell)$ . Using [Kat88, I 1.10], we can replace  $\overline{\mathbb{Q}}_\ell$  by  $\mathbb{F}_\lambda$ , where  $\lambda = \ell^n$ . Hence,  $G_{K_0}$  acts on  $M$  via a finite quotient  $H \subset \text{GL}_{\mathbb{F}_\lambda}(M)$ . Let  $L/K_0$  be the corresponding finite Galois extension and  $f_M : S_L \rightarrow S_0$  the induced morphism. We have  $H = \text{Gal}(L/K_0)$ . Let us denote by  $r_M$  the highest jump in the ramification filtration of  $H$ . Using Herbrand functions [Ser68, IV 3], we will prove that the constant  $C_M := \psi_{L/K_0}(r_M)$  does the job.

Using 2.3.1, we are left to treat the case where  $K_0/K$  is separable. The adjunction morphism

$$M \longrightarrow f_{M*} f_M^* M$$

is injective. Since  $f_M^* M \simeq \mathbb{F}_\lambda^{\text{rg } M}$ , we obtain by applying  $f_*$  an injection

$$f_* M \longrightarrow \mathbb{F}_\lambda[\text{Gal}(L/K)]^{\text{rg } M}$$

So we are left to bound the slopes of  $\mathbb{F}_\lambda[\text{Gal}(L/K)]$  viewed as a  $G_K$ -representation, that is by 2.1.2 (2) the highest jump in the upper-numbering ramification filtration of  $\text{Gal}(L/K)$ . By 2.1.2 (2),  $r_0 := \text{Max Sl}_f^{\text{nb}}(\overline{\mathbb{Q}}_\ell)$  is the highest jump in the ramification filtration of  $\text{Gal}(L/K)/H$ . Choose  $r > \text{Max}(r_0, \varphi_{L/K} \psi_{L/K_0}(r_M))$ . We have

$$\begin{aligned} \text{Gal}(L/K)^r &= H \cap \text{Gal}(L/K)^r \\ &= H \cap \text{Gal}(L/K)_{\psi_{L/K}(r)} \\ &= H_{\psi_{L/K}(r)} \\ &= H^{\varphi_{L/K_0} \psi_{L/K}(r)} \\ &= \{1\} \end{aligned}$$



The first equality comes from  $r > r_0$ . The third equality comes from the compatibility of the lower-numbering ramification filtration with subgroups. The last equality comes from the fact that  $r > \varphi_{L/K} \psi_{L/K_0}(r_M)$  is equivalent to  $\varphi_{L/K_0} \psi_{L/K}(r) > r_M$ . Hence,

$$\mathrm{Sl}_f^{\mathrm{nb}}(M) \subset [0, \mathrm{Max}(r_0, \varphi_{L/K} \psi_{L/K_0}(r_M))]$$

Since  $\varphi_{L/K} : [-1, +\infty[ \rightarrow \mathbb{R}$  is concave, satisfies  $\varphi_{L/K}(0) = 0$  and is equal to the identity on  $[-1, 0]$ , we have

$$\varphi_{L/K} \psi_{L/K_0}(r_M) \leq \psi_{L/K_0}(r_M)$$

and we obtain (2.3.5) by setting  $C_M := \psi_{L/K_0}(r_M)$ .  $\square$

### 3. Proof of Theorem 1

**3.1. Preliminary.** — Let us consider the affine line  $\mathbb{A}_S^1 \rightarrow S$  over  $S$ . Let  $s'$  be the generic point of  $\mathbb{A}_S^1$  and  $S'$  the strict henselianization of  $\mathbb{A}_S^1$  at  $s'$ . We denote by  $\overline{S}$  the normalization of  $S$  in  $\overline{\eta}$ , by  $\kappa$  the function field of the strict henselianization of  $\mathbb{A}_S^1$  at  $s'$ , and by  $\overline{\kappa}$  an algebraic closure of  $\kappa$ . We have  $\kappa \simeq K' \otimes_K \overline{K}$  and

$$(3.1.1) \quad G_K \simeq \mathrm{Gal}(\kappa/K')$$

Let  $L/K$  be a finite Galois extension of  $K$  in  $\overline{K}$ . Set  $L' := K' \otimes_K L$ . At finite level, (3.1.1) reads

$$(3.1.2) \quad \mathrm{Gal}(L/K) \xrightarrow{\sim} \mathrm{Gal}(L'/K')$$

$$(3.1.3) \quad g \rightarrow \mathrm{id} \otimes g$$

Since a uniformizer in  $S_L$  is also a uniformizer in  $S'_L$ , we deduce that (3.1.2) is compatible with the lower-numbering ramification filtration on  $\mathrm{Gal}(L/K)$  and  $\mathrm{Gal}(L'/K')$ . Hence, (3.1.2) is compatible with the upper-numbering ramification filtration on  $\mathrm{Gal}(L/K)$  and  $\mathrm{Gal}(L'/K')$ . We deduce that through (3.1.1), the canonical surjection  $G_{K'} \rightarrow G_K$  is compatible with the upper-numbering ramification filtration.

**3.2. The proof.** — We can suppose that  $\mathcal{F}$  is concentrated in degree 0. In case  $\dim X = 0$ , there is nothing to prove. We first reduce the proof of Theorem 1 to the case where  $\dim X = 1$  by arguing by induction on  $\dim X$ .

Since the problem is local on  $X$ , we can suppose that  $X$  is affine. We thus have a digram

$$(3.2.1) \quad \begin{array}{ccccc} X & \longrightarrow & \mathbb{A}_S^n & \hookrightarrow & \mathbb{P}_S^n \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

Let  $\overline{X}$  be the closure of  $X$  in  $\mathbb{P}_S^n$  and let  $j : X \hookrightarrow \overline{X}$  be the associated open immersion. Replacing  $(X, \mathcal{F})$  by  $(\overline{X}, j_* \mathcal{F})$ , we can suppose  $X/S$  projective. Then Theorem 1 is a consequence of the following assertions

- (A) There exists a finite set  $E_A \subset \mathbb{R}_{\geq 0}$  such that for every  $N \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$  with slope not in  $E_A$ , the support of  $\psi_f^t(\mathcal{F} \otimes f^* N)$  is punctual.

(B) There exists a finite set  $E_B \subset \mathbb{R}_{\geq 0}$  such that for every  $N \in \text{Sh}_c(\eta, \overline{\mathbb{Q}}_\ell)$  with slope not in  $E_B$ , we have

$$R\Gamma(X_s, \psi_f^t(\mathcal{F} \otimes f^*N)) \simeq 0$$

Let us prove (A). This is a local statement on  $X$ , so we can suppose  $X$  to be a closed subset in  $\mathbb{A}_S^n$  and consider the factorisations

$$\begin{array}{ccc} X & \xrightarrow{p_i} & \mathbb{A}_S^1 \\ & \searrow f & \downarrow \\ & & S \end{array}$$

where  $p_i$  is the projection on the  $i$ -th factor of  $\mathbb{A}_S^n$ . Using the notations in 3.1, let  $X'/S'$  making the upper square of the following diagram

$$\begin{array}{ccc} X' & \xrightarrow{\lambda} & X \\ p'_i \downarrow & & \downarrow p_i \\ S' & \longrightarrow & \mathbb{A}_S^1 \\ & \searrow h & \downarrow \\ & & S \end{array}$$

cartesian. Let us set  $\mathcal{F}' := \lambda^*\mathcal{F}$  and  $N' := h^*N$ . From [Del77, Th. finitude 3.4], we have

$$(3.2.2) \quad \lambda^*\psi_f(\mathcal{F} \otimes f^*N) \simeq \psi_{hp'_i}(\mathcal{F}' \otimes p_i'^*N') \simeq \psi_{p'_i}(\mathcal{F}' \otimes p_i'^*N')^{G_\kappa}$$

where  $G_\kappa$  is a pro- $p$  group sitting in an exact sequence

$$1 \longrightarrow G_\kappa \longrightarrow G_{K'} \longrightarrow G_K \longrightarrow 1$$

In particular,  $G_\kappa$  is a subgroup of the wild-ramification group  $P_{K'}$  of  $G_{K'}$ . So applying the  $P_{K'}$ -invariants on (3.2.2) yields

$$(3.2.3) \quad \lambda^*\psi_f^t(\mathcal{F} \otimes f^*N) \simeq \psi_{p'_i}^t(\mathcal{F}' \otimes p_i'^*N')$$

If  $N$  has pure slope  $r$ , we know from 3.1 that  $N'$  has pure slope  $r$  as a sheaf on  $\eta'$ . Applying the recursion hypothesis gives a finite set  $E_i \subset \mathbb{R}_{\geq 0}$  such that the right-hand side of (3.2.3) is 0 for  $N$  of slope not in  $E_i$ . The union of the  $E_i$  for  $1 \leq i \leq n$  is the set  $E_A$  sought for in (A).

To prove (B), we observe that the compatibility of  $\psi_f^t$  with proper morphisms and the projection formula give

$$R\Gamma(X_s, \psi_f^t(\mathcal{F} \otimes f^*N)) \simeq \psi_{\text{id}}^t(Rf_*\mathcal{F} \otimes N)$$

By 2.1.1, the set  $E_B := \text{Sl}(Rf_*\mathcal{F})$  has the required properties.

We are thus left to prove Theorem 1 in the case where  $\dim X = 1$ . At the cost of localizing, we can suppose that  $X$  is local and maps surjectively on  $S$ . Let  $x$  be the closed point of  $X$ . Note that  $k(x)/k(s)$  is of finite type but may not be finite. Choosing a transcendence basis of  $k(x)/k(s)$  yields a factorization  $X \longrightarrow S' \longrightarrow S$

satisfying  $\mathrm{trdeg}_{k(s')} k(x) = \mathrm{trdeg}_{k(s)} k(x) - 1$ .

So we can further suppose that  $k(x)/k(s)$  is finite. Since  $k(s)$  is algebraically closed, we have  $k(x) = k(s)$ . If  $\hat{S}$  denotes the completion of  $S$  at  $s$ , we deduce that  $X \times_S \hat{S}$  is finite over  $\hat{S}$ . By faithfully flat descent [Gro71, VIII 5.7], we obtain that  $X/S$  is finite. We conclude the proof of Theorem 1 with 2.1.2 (1).

#### 4. Proof of Theorem 2

**4.1.** That  $0 \in \mathrm{Sl}_f(\overline{\mathbb{Q}_\ell})$  is easy by looking at the smooth locus of  $f$ . We are left to prove that for every  $N \in \mathrm{Sh}_c(\eta, \overline{\mathbb{Q}_\ell})$  with slope  $> 0$ , the following holds

$$(4.1.1) \quad \psi_f^t f^* N \simeq 0$$

Since the problem is local on  $X$  for the étale topology, we can suppose that  $X = S[x_1, \dots, x_n]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$  and we have to prove (4.1.1) at the origin  $0 \in X_s$ . Let  $a$  be the lowest common multiple of the  $a_i$  and define  $b_i = a/a_i$ . Note that  $a$  and the  $b_i$  are prime to  $p$ . Hence the morphism  $h$  defined as

$$\begin{aligned} Y := S[t_1, \dots, t_n]/(\pi - t_1^a \cdots t_m^a) &\longrightarrow X \\ (t_1, \dots, t_n) &\longrightarrow (t_1^{b_1}, \dots, t_m^{b_m}, t_{m+1}, \dots, t_n) \end{aligned}$$

is finite surjective and finite étale above  $\eta$  with Galois group  $G$ . Set  $g = fh$ . Then

$$(\mathcal{H}^i \psi_f^t f^* N)_{\overline{0}} \simeq (\mathcal{H}^i \psi_g^t g^* N)_{\overline{0}}^G$$

for every  $i \geq 0$ , so we can suppose  $a_1 = \cdots = a_m = a$ . Since  $a$  is prime to  $p$ , the map of absolute Galois groups induced by  $S[\pi^{1/a}] \longrightarrow S$  induces an identification at the level of the ramification groups. By compatibility of nearby cycles with change of trait [Del77, Th. finitude 3.7], we can suppose  $a = 1$ .

Let us now reduce the proof of Theorem 2 to the case where  $m = 1$ . We argue by induction on  $m$ . The case  $m = 1$  follows from the compatibility of nearby cycles with smooth morphisms. We thus suppose that Theorem 2 is true for  $m < n$  with all  $a_i$  equal to 1 and prove it for  $m + 1$  with all  $a_i$  equal to 1. Let  $h : \tilde{X} \longrightarrow X$  be the blow-up of  $X$  along  $x_m = x_{m+1} = 0$ . Define  $g := fh$  and denote by  $E$  the exceptional divisor of  $\tilde{X}$ . Since  $h$  induces an isomorphism on the generic fibers, and since  $\psi_f^t$  is compatible with proper push-forward, we have

$$(4.1.2) \quad Rh_* \psi_g^t g^* N \simeq \psi_f^t f^* N \simeq 0$$

By proper base change, (4.1.2) gives

$$(4.1.3) \quad R\Gamma(h^{-1}(0), (\psi_g^t g^* N)_{|h^{-1}(0)}) \simeq 0$$

The scheme  $\tilde{X}$  is covered by a chart  $U$  affine over  $S$  given by

$$S[(u_i)_{1 \leq i \leq n}]/(\pi - u_1 \cdots u_m)$$

with  $E \cap U$  given by  $u_m = 0$ , and a chart  $U'$  affine over  $S$  given by

$$S[(u_i)_{1 \leq i \leq n}]/(\pi - u_1 \cdots u_{m+1})$$

with  $E \cap U'$  given by  $u_{m+1} = 0$ . By recursion hypothesis,  $(\psi_g^t g^* N)_{|h^{-1}(0)}$  is a skyscraper sheaf supported at the origin 0 of  $U'$ . Hence, (4.1.3) gives

$$(\psi_g^t g^* N)_{\overline{0}} \simeq 0$$

This finishes the induction, and thus the proof of Theorem 2.

**4.2.** Let us give a geometric-flavoured proof of

$$\mathcal{H}^0 \psi_f^t f^* N \simeq 0$$

in case  $X = S[x_1, \dots, x_n]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$ . By constructibility [Del77, Th. finitude 3.2], it is enough to work at the level of germs at a geometric point  $\bar{x}$  lying over a closed point  $x \in X$ .

Hence, we have to prove  $H^0(C, f^* N) \simeq 0$  for every connected component  $C$  of  $X_{x, \eta_t}^{\text{sh}}$ . For such  $C$ , denote by  $\rho_C : \pi_1(C) \longrightarrow \pi_1(\eta_t) = P_K$  the induced map. Then  $H^0(C, f^* N) \simeq N^{\text{Im } \rho_C}$ . Since by definition  $N^{P_K} = 0$ , it is enough to prove that  $\rho_C$  is surjective. From V 6.9 and IX 3.4 of [Gro71], we are left to prove that  $C$  is geometrically connected. To do this, we can always replace  $X_x^{\text{sh}}$  by its formalization  $\hat{X}_x = \text{Spec } R[[\underline{x}]]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$ .

By hypothesis,  $d := \gcd(a_1, \dots, a_m)$  is prime to  $p$ , so  $\pi$  has a  $d$ -root in  $K_t$ . Hence  $\hat{X}_{x, \eta_t}$  is a direct union of  $d$  copies of

$$\text{Spec } K_t \otimes_R R[[\underline{x}]]/(\pi^{1/d} - x_1^{a'_1} \cdots x_m^{a'_m})$$

where  $a_i = da'_i$ . So we have to prove the following

**Lemma 4.2.1.** — *Let  $a_1, \dots, a_m, d \in \mathbb{N}^*$  with  $\gcd(a_1, \dots, a_m) = 1$ . Then*

$$(4.2.2) \quad \text{Spec } \overline{K} \otimes_R R[[\underline{x}]]/(\pi^{1/d} - x_1^{a_1} \cdots x_m^{a_m})$$

*is connected.*

*Proof.* — One easily reduces to the case  $d = 1$ . If  $R'$  is the normalization of  $R$  in a Galois extension of  $K$  in  $\overline{K}$ , it is enough to prove that  $\text{Spec } R'[[\underline{x}]]/(\pi - x_1^{a_1} \cdots x_m^{a_m})$  is irreducible. If  $\pi'$  is a uniformizer of  $R'$ , we have  $R' \simeq k[[\pi']]$ , we write  $\pi = P(\pi')$  where  $P \in k[[X]]$  and then we are left to prove that  $f_{a, P} := P(\pi') - x_1^{a_1} \cdots x_m^{a_m}$  is irreducible in  $k[[x_1, \dots, x_n, \pi']]$ . This follows from  $\gcd(a_1, \dots, a_m) = 1$  via Lypkovski's indecomposability criterion [Lip88, 2.10] for the Newton polyhedron associated to  $f_{a, P}$ . □

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