Liouville Quantum Gravity on the unit disk

Yichao Huang, Rémi Rhodes, Vincent Vargas

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Abstract

Our purpose is to pursue the rigorous construction of Liouville Quantum Field Theory on Riemann surfaces initiated by F. David, A. Kupiainen and the last two authors in the context of the Riemann sphere and inspired by the 1981 seminal work by Polyakov. In this paper, we investigate the case of simply connected domains with boundary. We also make precise conjectures about the relationship of this theory to scaling limits of random planar maps with boundary conformally embedded onto the disk.

Key words or phrases: Liouville Quantum Gravity, quantum field theory, Gaussian multiplicative chaos, KPZ formula, KPZ scaling laws, Polyakov formula, conformal anomaly.
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Contents

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1 Introduction

Let us begin this introduction with a soft attempt of explanation for mathematicians of what is Liouville Quantum Field Theory (LQFT). This theory may be better understood if we first briefly recall the Feynman path integral representation of the Brownian motion on $\mathbb{R}^d$. Denoting by Σ the space of paths $\sigma : [0,1] \to \mathbb{R}^d$ starting from $\sigma(0) = 0$, we define the action functional on Σ by

$$\forall \sigma \in \Sigma, \quad S_{\mathrm{BM}}(\sigma) = \frac{1}{2}\int_0^T |\dot{\sigma}(r)|^2 \, dr. \tag{1.1}$$

It is nowadays rather well understood that Brownian motion, call it B, can be understood in terms of Feynman path integrals via the relation

$$\mathbb{E}[F((B_s)_{s \leqslant T})] = \frac{1}{Z}\int_\Sigma F(\sigma) e^{-S_{BM}(\sigma)} D\sigma \tag{1.2}$$

where $D\sigma$ stands for a formal uniform measure on Σ and Z is a renormalization constant. The Brownian motion is also often said to be the canonical uniform random path in $\mathbb{R}^d$: this terminology is due to the fact the Brownian motion is the scaling limit ($n \to \infty$) of the simple random walk, i.e. the uniform measure on the n-step paths joining neighboring sites of the regular Euclidean lattice.

The reader may try to guess what could be the above picture if, instead of "canonical random path", we ask for a "canonical random Riemann surface". The answer is Liouville quantum gravity. As in the case of the Brownian motion, there are two ways to give sense to this theory: directly in the continuum in terms of Feynmann surface integrals or as scaling limit of suitable discrete models called Random Planar Maps (RPM). This picture is nowadays well understood in physics literature since the pioneering work by Polyakov [23]. The reader is referred to [18, 22] for physics reviews, to [23, 6, 8, 20] for founding papers in physics and to [7] for a brief introduction for mathematicians and a rigorous construction on the Riemann sphere.

In this paper, we will construct the Liouville quantum field theory on Riemann surfaces with boundary directly in the continuum in the spirit of Feynman surface integrals. More precisely we consider a (strict) simply connected domain D of $\mathbb{R}^2$ with a simple boundary equipped with a Riemannian metric g. Similar to the action (1.1) for Brownian motion, we must consider the Liouville action functional on such a Riemannian manifold. It is defined for each function $X : \overline{D} \to \mathbb{R}$ by

$$S(X,g) := \frac{1}{4\pi}\int_D \left(|\partial^g X|^2 + QR_g X + 4\pi\mu e^{\gamma X}\right)\lambda_g + \frac{1}{2\pi}\int_{\partial D}\left(QK_g X + 2\pi\mu_\partial e^{\frac{\gamma}{2}X}\right)\lambda_{\partial g} \tag{1.3}$$

where ∂^g, R_g, K_g, λ_g and $\lambda_{\partial g}$ respectively stand for the gradient, Ricci scalar curvature, geodesic curvature (along the boundary), volume form and line element along ∂D in the metric g: see section 2.1 for the definitions. The parameters $\mu, \mu_\partial \geqslant 0$ (with $\mu + \mu_\partial > 0$) are respectively the bulk and boundary cosmological constants and Q, γ are real parameters.

Before going into further details of the quantum field theory, let us first make a detour in Riemann geometry to explain why the roots of LQFT are deeply connected to the the theory of uniformization of Riemann surfaces. Indeed, a fundamental problem in geometry is to uniformize the surface $(\overline{D}, g)$: this means that we look for a metric g' on $\overline{D}$ conformally equivalent to g, i.e. $g' = e^u g$ for some smooth function u on $\overline{D}$, with constant Ricci scalar curvature in D and constant geodesic curvature on ∂D. Under appropriate assumptions, the unknown function u is a minimizer of the Liouville action functional (1.3). Indeed, for the particular value

$$Q = \frac{2}{\gamma}, \tag{1.4}$$

the saddle points X of this functional with Neumann boundary condition $\partial_{n_g}(\frac{\gamma}{2}X) + K_g = -\frac{\pi\mu_\partial\gamma^2}{2}e^{\frac{\gamma}{2}X}$, where ∂_{n_g} stands for the Neumann operator along ∂D, solve (if exists) the celebrated *Liouville equation*

$$-\triangle_g(\gamma X) + R_g = -2\pi\mu\gamma^2 e^{\gamma X} \quad \text{on } D, \quad \partial_{n_g}(\frac{\gamma}{2}X) + K_g = -\frac{\pi\mu_\partial\gamma^2}{2}e^{\frac{\gamma}{2}X} \quad \text{on } \partial D. \tag{1.5}$$

Setting $u = \gamma X$ and defining a new metric $g' = e^u g$, the metric g' satisfies the relations

$$R_{g'} = -2\pi\mu\gamma^2 \quad \text{and} \quad K_{g'} = -\frac{\pi\mu_\partial\gamma^2}{2},$$

hence providing a solution to the uniformization problem of the Riemann surface $(\overline{D}, g)$. Let us further mention that, for the value of Q given by (1.4), this theory is *conformally invariant*: this means that if we choose a conformal map $\psi : \widetilde{D} \mapsto D$ then the couple (X, g) solves (1.5) on D if and only if $(X\circ\psi + Q\ln|\psi'|, g\circ\psi)$ solves (1.5) on $\widetilde{D}$ [1]. These are the foundations of the theory of uniformization of surfaces with boundary in $2d$, also called *Classical Liouville field theory*.

In quantum (or probabilistic) Liouville field theory, one looks for the construction of a random field X with law given heuristically in terms of a functional integral

$$\mathbb{E}[F(X)] = Z^{-1}\int F(X)e^{-S(X,g)}DX \tag{1.6}$$

where Z is a normalization constant and DX stands for a formal uniform measure on some space of maps $X : \overline{D} \to \mathbb{R}$. This expression is in the same spirit as for the Brownian motion (1.2). This formalism describes the law of the log-conformal factor X of a formal random metric of the form $e^{\gamma X}g$ on D. Of course, this description is purely formal and giving a mathematical description of this picture is a longstanding problem since the work of Polyakov [23]. It turns out that for the particular values

$$\gamma \in]0, 2], \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2},$$

[1] Let us prove this for the Neumann boundary condition; the other equation can be dealt with similarly. Since ψ is an isometry from $(\widetilde{D}, g\circ\psi|\psi'|^2)$ to (D, g), we have $K_{g\circ\psi|\psi'|^2} = K_g\circ\psi$. Now applying formula (2.3) which is valid in great generality, we get that $K_{g\circ\psi} = |\psi'|(K_g\circ\psi - \frac{1}{(g\circ\psi)^{1/2}|\psi'|}\ln|\psi'|)$. Hence we get that

$$\partial_{n_{g\circ\psi}}(\frac{\gamma}{2}(X\circ\psi + Q\ln|\psi'|)) + K_{g\circ\psi} = |\psi'|(\frac{1}{g\circ\psi}\frac{\partial(\frac{\gamma}{2}X)}{\partial n}\circ\psi + K_g\circ\psi) = -\frac{\pi\mu_\partial\gamma^2}{2}e^{\frac{\gamma}{2}(X\circ\psi + Q\ln|\psi'|)}$$

this field theory is expected to become a *Conformal Field Theory* (see [16] for a background on this topic). The aim of this paper is to make rigorous sense of the above heuristic picture and thereby defining a canonical random field X inspired by Feynman surface integrals. A noticeable difference with the example of the Brownian motion where there is only one canonical random path (up to reparametrization) is that there is a whole family of canonical random Riemann surfaces indexed by a single parameter $\gamma \in]0,2]$. Conformal Field Theories are characterized by their central charge $c \in \mathbb{R}$ that reflects the way the theory reacts to conformal changes of the background metric g defined on D (see section 3.3). For the Liouville quantum theory on the disk, we will establish that the central charge is $c = 1 + 6Q^2$: thus it can range continuously in the interval $[25, +\infty[$ and this is one of the interesting features of this theory. We will also study the conformal covariance (KPZ formula) and μ, μ_∂-dependence of this theory. Once constructed, the Liouville (random) field X allows us to define the Liouville measure, which can be thought of as the volume form associated to the random metric tensor $e^{\gamma X} g$. We will state a precise mathematical conjecture on the relationship between the Liouville measure and the scaling limit of random planar maps with a simple boundary conformally embedded onto the unit disk.

To conclude, let us stress that the thread of the paper is inspired by [7]. The main input is here to understand the phenomena related to the presence of a boundary; in particular, part of the construction relies on the theory of Gaussian multiplicative chaos (GMC) and the presence of the boundary requires to integrate against GMC measures functions that are not integrable with respect to Lebesgue measure when approaching the boundary (these technical difficulties do not appear in the case of the sphere [7] where there is no boundary): see proposition 2.3 for instance.

Remark 1.1. *The authors of [10, 30] developed a theory of quantum surfaces on domains with two marked points on the boundary. In this context, the measures live in a quotient space where two measures are equivalent if one is the image of the other by a conformal map which fixes the two marked points on the boundary (in the case of the upper half plane with* 0 *and* ∞ *as marked points, two measures* M *and* N *are equivalent if there exists some* $\lambda > 0$ *such that* $M(.) = N(\lambda .)$*). In our paper, we develop a theory with three (or more) marked points on the boundary of the domain. Basically, these three points are what you need to fix the degree of freedom with respect to the automorphisms of the disk.*

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2 Background and preliminary results

In order to facilitate the reading of the manuscript, we gather in this section the basics in Riemann geometry and probability theory that we will use throughout the paper.

2.1 Metrics on the unit disk

Let us denote by $\mathbb{D}$ the unit disk and $\partial\mathbb{D}$ its boundary. We consider the Laplace-Beltrami operator $\triangle$ and Neumann operator ∂_n on $\mathbb{D}$ and $\partial\mathbb{D}$ equipped with the Euclidean metric. More generally, we say that a metric $g = g(x)dx^2$ on the unit disk is conformally equivalent to the Euclidean metric if $g(x) = e^{\varphi(x)}$ for some function $\varphi : \bar{\mathbb{D}} :\to \mathbb{R}$ of class $C^1(\mathbb{D}) \cap C^0(\bar{\mathbb{D}})$ such that

$$\int_{\mathbb{D}} |\partial \varphi|^2 \, d\lambda < +\infty. \tag{2.1}$$

Notice that we use the same notation g for the metric tensor and the function which defines it but this should not lead to confusions. In that case, the Laplace-Beltrami operator $\triangle_g$ and Neumann operator ∂_{n_g} in the metric g are given by

$$\triangle_g = g^{-1} \triangle, \qquad \text{and} \qquad \partial_{n_g} = g^{-1/2} \partial_n.$$

We denote respectively by R_g and K_g the Ricci scalar curvature and geodesic curvature K_g in the metric g. If $g' = e^{\varphi} g$ is another metric on the unit disk conformally equivalent to the flat metric, we get the following rules for the changes of (geodesic) curvature under such a conformal change of metrics

$$R_{g'} = e^{-\varphi}(R_g - \triangle_g \varphi) \qquad \text{on } \mathbb{D}, \tag{2.2}$$

$$K_{g'} = e^{-\varphi/2}(K_g + \partial_{n_g} \varphi/2) \qquad \text{on } \partial\mathbb{D}. \tag{2.3}$$

For instance, when equipped with the Euclidean metric, the unit disk has Ricci scalar curvature 0 and geodesic curvature 1 along its boundary. Combining these data with the rules (2.2)+(2.3), one can recover the explicit expressions of R_g and K_g for any metric g conformally equivalent to the Euclidean metric. We will also consider the volume form λ_g on $\mathbb{D}$, the line element $\lambda_{\partial g}$ on $\partial\mathbb{D}$, and the gradient ∂^g associated to the metric g.

Let us further recall the Gauss-Bonnet theorem

$$\int_{\mathbb{D}} R_g \, d\lambda_g + 2 \int_{\partial\mathbb{D}} K_g \, d\lambda_{\partial g} = 4\pi \chi(\mathbb{D}), \tag{2.4}$$

where $\chi(\mathbb{D})$ is the Euler characteristics of the disk (that is $\chi(\mathbb{D}) = 1$), and the Green-Riemann formula

$$\int_{\mathbb{D}} \psi \triangle_g \varphi \, d\lambda_g + \int_{\mathbb{D}} \partial^g \varphi \cdot \partial^g \psi \, d\lambda_g = \int_{\partial\mathbb{D}} \partial_{n_g} \varphi \psi \, d\lambda_{\partial g}. \tag{2.5}$$

We will denote by $m_\nu(f)$ and $m_{\partial\nu}(f)$ the mean value of f respectively in the disk $\mathbb{D}$ or the boundary $\partial\mathbb{D}$ with respect to a measure ν on $\mathbb{D}$ or $\partial\mathbb{D}$, that is

$$m_\nu(f) = \frac{1}{\nu(\mathbb{D})} \int_{\mathbb{D}} f \, d\nu \qquad \text{or} \qquad \frac{1}{\nu(\partial\mathbb{D})} \int_{\partial\mathbb{D}} f \, d\nu.$$

If the measure ν is the volume form (or the line element on $\partial\mathbb{D}$) of some metric g, we will use the notation $m_g(f)$ (or $m_{\partial g}(f)$). When no reference to the metric g is given ($m(f)$ or $m_\partial(f)$) this means that we work with the Euclidean metric.

The Sobolev space $H^1(\mathbb{D})$ is defined as the closure of the space of smooth functions on $\bar{\mathbb{D}}$ with respect to the inner product

$$\int_{\mathbb{D}} (fh + \partial f \cdot \partial h) \, d\lambda.$$

We denote by $H^{-1}(\mathbb{D})$ its dual.

Finally, we introduce the Green function G of the Neumann problem on $\mathbb{D}$

$$G(x,y) = \ln \frac{1}{|x-y||1-x\bar{y}|}. \tag{2.6}$$

It is the unique function satisfying

1. $x \mapsto G(x,y)$ is harmonic on $\mathbb{D} \setminus \{y\}$,
2. $x \mapsto G(x,y) + \ln|y-x|$ is harmonic on $\mathbb{D}$ for all $y \in \bar{D}$,
3. $\partial_n G(x,y) = -1$ for $x \in \partial\mathbb{D}$, $y \in \mathbb{D}$,
4. $G(x,y) = G(y,x)$ for $x, y \in \mathbb{D}$ and $x \neq y$,
5. $m_\partial G(x,\cdot) = 0$ for all $x \in \mathbb{D}$.

Recall that (2.5) combined with the properties of G implies that for all $f \in C^2(\mathbb{D}) \cap C^1(\bar{\mathbb{D}})$

$$-2\pi(f(x) - m_{\partial\mathbb{D}}(f)) = \int_{\mathbb{D}} G(x,y) \triangle f(y)\, \lambda(dy) - \int_{\partial\mathbb{D}} G(x,y) \partial_n f(y)\, \lambda_\partial(dy). \tag{2.7}$$

It is quite important to observe here that G is positive definite on $\mathbb{D}$.

2.2 Möbius transforms of the unit disk

The Möbius transforms of the unit disk are given by $\psi(x) = e^{i\alpha} \frac{x-a}{1-\bar{a}x}$ with $|a| < 1$. Recall that

$$\psi'(x) = e^{i\alpha} \frac{1-|a|^2}{(1-\bar{a}x)^2}$$

from which one gets

$$\psi(y) - \psi(x) = (\psi'(y))^{1/2} (\psi'(x))^{1/2} (y-x), \quad 1 - \psi(x)\overline{\psi(y)} = (\psi'(x))^{1/2} (\psi'(y))^{1/2} (1 - x\overline{y}). \tag{2.8}$$

The Green function for the Neumann problem defined above thus verifies

$$G(\psi(x), \psi(y)) = G(x,y) - \ln|\psi'(x)| - \ln|\psi'(y)|. \tag{2.9}$$

2.3 Gaussian Free Field with Neumann boundary conditions

We consider on $\mathbb{D}$ a Gaussian Free Field (GFF) $X_{\partial\mathbb{D}}$ with Neumann boundary conditions and vanishing mean along the boundary, namely $m_\partial(X_{\partial\mathbb{D}}) = 0$ (see [9, 29] for more details about GFF). This field is a Gaussian centered distribution (in the sense of Schwartz) with covariance kernel given by the Green function of the Neumann problem with vanishing mean along the boundary

$$\mathbb{E}[X_{\partial\mathbb{D}}(x) X_{\partial\mathbb{D}}(y)] = G(x,y). \tag{2.10}$$

It can be shown that this Gaussian random distribution (in the sense of Schwartz) lives almost surely in $H^{-1}(\mathbb{D})$ (same argument as in [9]).

As a distribution, the field $X_{\partial\mathbb{D}}$ cannot be understood as a fairly defined function. To remedy this problem, we will need to consider some regularizations of this field in order to deal with nice (random) functions. Thus, we introduce the regularized field $X_{\partial\mathbb{D},\epsilon}$ as follows. For $\epsilon > 0$, we let $l_\epsilon(x)$ be the length of the arc $A_\epsilon(x) = \{z \in \mathbb{D}; |z-x| = \epsilon\}$ (computed with the Euclidean line element ds on the boundary of the disk centered at x and radius ϵ). Then we set

$$X_{\partial\mathbb{D},\epsilon}(x) = \frac{1}{l_\epsilon(x)} \int_{A_\epsilon(x)} X_{\partial\mathbb{D}}(x+s)ds.$$

A similar regularization was considered in [11] and the reader can check that this field has a locally Hölder version both in the variables x and ϵ. Let us mention that we have the following two options: either $x \in \mathbb{D}$ and then for $\epsilon < \text{dist}(x, \partial\mathbb{D})$ we obtain

$$X_{\partial\mathbb{D},\epsilon}(x) = \frac{1}{2\pi} \int_0^{2\pi} X_{\partial\mathbb{D}}(x + \epsilon e^{i\theta})d\theta,$$

or $x \in \partial\mathbb{D}$ and then $X_{\partial\mathbb{D},\epsilon}(x)$ is intuitively the same as above except that we integrate along the "half-circle" centered at x with radius ϵ contained in $\mathbb{D}$.

Proposition 2.1. *Let us denote by g_P the Poincaré metric over the unit disk*

$$g_P = \frac{1}{(1-|x|^2)^2} dx^2. \tag{2.11}$$

We claim
1) As $\epsilon \to 0$, the convergence $\mathbb{E}[X_{\partial\mathbb{D},\epsilon}(x)^2] + \ln \epsilon \to \frac{1}{2} \ln g_P(x)$ holds uniformly over the compact subsets of $\mathbb{D}$.
2) As $\epsilon \to 0$, the convergence $\mathbb{E}[X_{\partial\mathbb{D},\epsilon}(x)^2] + 2\ln \epsilon \to -1$ holds uniformly over $\partial\mathbb{D}$.
3) Consider a Möbius transform ψ of the disk. Denote by $X_{\partial\mathbb{D}} \circ \psi_\epsilon$ the ϵ-circle average of the field $X_{\partial\mathbb{D}} \circ \psi$. Then as $\epsilon \to 0$, we have the convergence

$$\mathbb{E}[X_{\partial\mathbb{D}} \circ \psi_\epsilon(x)^2] + \ln \epsilon \to \frac{1}{2} \ln g_P(\psi(x)) - 2\ln|\psi'(x)|$$

uniformly over the compact subsets of $\mathbb{D}$ and the convergence

$$\mathbb{E}[X_{\partial\mathbb{D}} \circ \psi_\epsilon(x)^2] + 2\ln \epsilon \to -1 - 2\ln|\psi'(x)|$$

uniformly over $\partial\mathbb{D}$.

Proof. To prove the first statement results, apply the ϵ-circle average regularization to the Green function G in (2.6) and use the fact that the following integral vanishes

$$\int_0^{2\pi} \int_0^{2\pi} \ln \frac{1}{|e^{i\theta} - e^{i\theta'}|} d\theta d\theta' = 0$$

to get the uniform convergence over compact subsets of $\mathbb{E}[X_{\partial,\epsilon}(x)^2] + \ln \epsilon$ towards

$$x \mapsto \frac{1}{2} \ln g_P(x).$$

The strategy is similar for the second statement except that you get π^{-2} times the integral $\int_0^\pi \int_0^\pi \ln \frac{1}{|e^{i\theta}-e^{i\theta'}|} d\theta d\theta'$, which does not vanish anymore and yields the constant -1. The third claim results from (2.9). ∎

2.4 Gaussian Multiplicative Chaos

Gaussian multiplicative chaos theory was introduced in [19]. The reader is referred to [24] for a review on the topic. Here, we deal with convolution of the GFF so that as a straightforward combination of the main result in [28] and Proposition 2.2, we claim

Proposition 2.2. *For $\gamma \in [0,2[$ and λ, λ_∂ the volume form and line element on $\mathbb{D}$, $\partial\mathbb{D}$ of the Euclidean metric, the random measures $e^{\gamma X_{\partial\mathbb{D}}}d\lambda$, $e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}d\lambda_\partial$ are defined as the limits in probability*

$$e^{\gamma X_{\partial\mathbb{D}}}d\lambda = \lim_{\epsilon\to 0}\epsilon^{\frac{\gamma^2}{2}}e^{\gamma X_{\partial\mathbb{D},\epsilon}}d\lambda \qquad e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}d\lambda_{\partial\mathbb{D}} = \lim_{\epsilon\to 0}\epsilon^{\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}X_{\partial\mathbb{D},\epsilon}}d\lambda_\partial$$

in the sense of weak convergence of measures over $\mathbb{D}$, $\partial\mathbb{D}$. These limiting measures are non trivial and are two standard Gaussian Multiplicative Chaos (GMC) on $\mathbb{D}$, $\partial\mathbb{D}$, namely

$$e^{\gamma X_{\partial\mathbb{D}}}d\lambda = e^{\gamma X_{\partial\mathbb{D}}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}(x)^2]}g_P(x)^{\frac{\gamma^2}{4}}d\lambda \qquad e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}d\lambda_{\partial\mathbb{D}} = e^{-\frac{\gamma^2}{8}}e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}(x)-\frac{\gamma^2}{8}\mathbb{E}[X_{\partial\mathbb{D}}(x)^2]}d\lambda_\partial.$$

Actually, the main issue is to show that these measures give almost surely finite mass respectively to the disk and its boundary. This turns out to be obvious for the boundary measure as the expectation of the total mass of $\partial\mathbb{D}$ is finite. Concerning the bulk measure, this statement is not straightforward: observe for instance that the expectation is infinite

$$\mathbb{E}[\int_{\mathbb{D}}e^{\gamma X_{\partial\mathbb{D}}}d\lambda] = \int_{\mathbb{D}}g_P(x)^{\frac{\gamma^2}{4}}d\lambda$$

as soon as $\gamma^2 \geqslant 2$. Yet, we show in the following proposition that the random variable $\int_{\mathbb{D}}e^{\gamma X_{\partial\mathbb{D}}}d\lambda$ is almost surely finite for all values of $\gamma \in]0,2[$.

Proposition 2.3. *For $\gamma \in]0,2[$, the quantities below are almost surely finite*

$$\int_{\mathbb{D}}e^{\gamma X_{\partial\mathbb{D}}}d\lambda \quad \text{and} \quad \int_{\partial\mathbb{D}}e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}d\lambda_{\partial\mathbb{D}}.$$

Proof. As explained above, we only need to focus on the bulk measure. Observe first that its expectation is finite in the case $\gamma^2 < 2$. For $\gamma^2 \geqslant 2$ (in fact the argument below works for $\gamma > 1$), we prove that it has moments of small order $\alpha > 0$, which entails the a.s. finiteness of the total mass of the interior of the disk.

Recall the sub-additivity inequality for $\alpha \in]0,1[$: if $(a_j)_{1\leqslant j\leqslant n}$ are positive real numbers then

$$(a_1+\cdots+a_n)^\alpha \leqslant a_1^\alpha+\cdots+a_n^\alpha.$$

Therefore we can write

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\mathbb{D}}e^{\gamma X_{\partial\mathbb{D}}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]}\frac{1}{(1-|x|^2)^{\gamma^2/2}})d\lambda\Big)^\alpha\Big]\\
&\quad=\mathbb{E}\Big[\Big(\sum_{n\in\mathbb{N}}\int_{1-2^{-n}\leqslant|x|^2\leqslant 1-2^{-n-1}}e^{\gamma X_{\partial\mathbb{D}}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]}\frac{1}{(1-|x|^2)^{\gamma^2/2}}d\lambda\Big)^\alpha\Big]\\
&\quad\leqslant\sum_{n\in\mathbb{N}}2^{n\alpha\frac{\gamma^2}{2}}\mathbb{E}\Big[\Big(\int_{1-2^{-n}\leqslant|x|^2\leqslant 1-2^{-n-1}}e^{\gamma X_{\partial\mathbb{D}}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]}d\lambda\Big)^\alpha\Big].
\end{aligned}$$

Now we trade the GFF $X_{\partial\mathbb{D}}$ for a log-correlated field that possesses a nicer structure of correlations with the help of Kahane convexity inequality [19]. More precisely, we consider any log-correlated field on $\mathbb{R}^2$ with a white noise decomposition and invariant under rotation. For instance, let us consider a star scale invariant kernel with compact support (see [3]): we choose a positive definite isotropic positive function k with compact support of class C^2 and we set

$$K_\epsilon(x)=\int_1^{\epsilon^{-1}}\frac{k(ux)}{u}\,du.$$

We consider a family of Gaussian processes $(Y_\epsilon(x))_\epsilon$ such that (see [3] for the details of the construction of such fields)

$$\forall x,y\in\mathbb{R}^2,\quad \mathbb{E}[Y_\epsilon(x)Y_{\epsilon'}(y)]=K_{\max(\epsilon,\epsilon')}(x-y).$$

The reader may check that for all $r,r'\in]0,1]$ such that $1-2^{-n}\leqslant r^2,r'^2\leqslant 1-2^{-n-1}$ and $\theta,\theta'\in[0,2\pi]$

$$\mathbb{E}[X_{\partial\mathbb{D}}(re^{i\theta})X_{\partial\mathbb{D}}(r'e^{i\theta'})]\geqslant 2\mathbb{E}[Y_{2^{-n}}(e^{i\theta})Y_{2^{-n}}(e^{i\theta'})]-A$$

for some constant A independent of n,θ. This inequality of covariances allows us to use Kahane's convexity inequality (see [19] or [24, Theorem 2.1]). Indeed, because the map $x\mapsto x^\alpha$ is concave, we have for some standard Gaussian random variable N independent of everything

$$\begin{aligned}\mathbb{E}\Big[\Big(e^{\gamma A^{1/2}N-A\gamma^2/2}\int_{1-2^{-n}\,\leqslant\,|x|^2\,\leqslant\,1-2^{-n-1}}e^{\gamma X_{\partial\mathbb{D}}-\frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]}d\lambda\Big)^\alpha\Big]&\\ \leqslant\mathbb{E}\Big[\Big(\int_0^{2\pi}\int_{(1-2^{-n})^{1/2}}^{(1-2^{-n-1})^{1/2}}&e^{\gamma\sqrt{2}Y_{2^{-n}}(e^{i\theta})-\gamma^2\mathbb{E}[Y_{2^{-n}}(e^{i\theta})^2]}drd\theta\Big)^\alpha\Big]\\ =C2^{-n\alpha}\mathbb{E}\Big[\Big(\int_0^{2\pi}&e^{\gamma\sqrt{2}Y_{2^{-n}}(e^{i\theta})-\gamma^2\mathbb{E}[Y_{2^{-n}}(e^{i\theta})^2]}d\theta\Big)^\alpha\Big]\end{aligned}$$

for some constant C independent of everything. By using the comparison to Mandelbrot's multiplicative cascades as explained in [12, Appendix B.1] to use a moment estimate in [21, Proposition 2.1 and the remark just after], we have that for any $\alpha<\gamma^{-1}$ and some other constant $C>0$

$$\sup_n\mathbb{E}\Big[\Big(n^{\frac{3\gamma}{2}}2^{n(\gamma-1)^2}\int_0^{2\pi}e^{\gamma\sqrt{2}Y_{2^{-n}}(e^{i\theta})-\gamma^2\mathbb{E}[Y_{2^{-n}}(e^{i\theta})^2]}d\theta\Big)^\alpha\Big]\leqslant C.$$

Combining we get (up to changing the value of C to absorb the constant $\mathbb{E}[e^{\alpha\gamma A^{1/2}N-\alpha A\gamma^2/2}]$)

$$\mathbb{E}\Big[\Big(\int_{\mathbb{D}}e^{\gamma X_{\partial\mathbb{D}}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]}\frac{1}{(1-|x|^2)^{\gamma^2/2}})d\lambda\Big)^\alpha\Big]\leqslant C\sum_{n\in\mathbb{N}}2^{n\alpha\left(\frac{\gamma^2}{2}-1-(\gamma-1)^2\right)}n^{-\frac{3\gamma}{2}\alpha},$$

which is finite when $\gamma\in]1,2[$. $\square$

3 Liouville Quantum Gravity on the disk

We are now in a position to give the precise definition of the LQFT on the disk with marked points: n points in the bulk $\mathbb{D}$ and n' points on the boundary $\mathbb{D}$. In what follows, we will first give a necessary and sufficient condition (Seiberg's bound) on these marked points in order that

LQFT is well defined. This will allow us to give the definitions of the Liouville field and measure. Finally, we will explain how these objects behave under conformal changes of background metrics and conformal reparametrization of the domain. Basically, the approach is the same as in [7] but there are some technical differences in order to treat the interactions bulk/boundary.

3.1 Definition and existence of the partition function

LQFT on the disk will be defined in terms of three parameters γ, μ, μ_∂, respectively the coupling constant and the bulk/boundary cosmological constants, and marked points. In this section, we will assume that the parameters γ, μ, μ_∂ satisfy

$$\gamma \in]0,2[, \quad \mu, \mu_\partial \geqslant 0 \quad \text{and} \quad \mu + \mu_\partial > 0. \tag{3.1}$$

Concerning the marked points, we fix a set of n points $(z_i)_{1 \leqslant i \leqslant n}$ in the interior of $\mathbb{D}$ together with n weights $(\alpha_i)_{1 \leqslant i \leqslant n} \in \mathbb{R}^n$ and n' points $(s_j)_{1 \leqslant j \leqslant n'}$ on the boundary $\partial\mathbb{D}$ together with n' weights $(\beta_j)_{1 \leqslant j \leqslant n'} \in \mathbb{R}^{n'}$. The family $(z_i, \alpha_i)_i$ will be called bulk marked points and the family $(s_j, \beta_j)_j$ boundary marked points.

Consider any metric $g = e^{\varphi} dx^2$ on the unit disk conformally equivalent to the Euclidean metric in the sense of (2.1).

Our purpose is now to define the partition function $\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(\epsilon, g, F)$ of LQFT applied to a functional F. This partition function formally corresponds to the Feynmann surface integral (1.6) with action (1.3). Yet, a rigorous approach requires the regularization procedure. This is the reason why we define the regularized partition function for all $\epsilon \in]0,1]$ and bounded continuous functional F on $H^{-1}(\mathbb{D})$ by

$$\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(\epsilon, g, F) \\
=&e^{\frac{1}{96\pi}\left(\int_{\mathbb{D}} |\partial \ln g|^2 \, d\lambda + \int_{\partial\mathbb{D}} 4 \ln g \, d\lambda_\partial\right)} \int_{\mathbb{R}} \mathbb{E}\Big[F(X_{\partial\mathbb{D}} + c + Q/2 \ln g) \prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i (c + X_{\partial\mathbb{D},\epsilon} + Q/2 \ln g)(z_i)} \\
&\times \prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(c + X_{\partial\mathbb{D},\epsilon} + Q/2 \ln g)(s_j)} \exp\Big(-\frac{Q}{4\pi} \int_{\mathbb{D}} R_g (c + X_{\partial\mathbb{D}}) \, d\lambda_g - \mu e^{\gamma c} \epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{D}} e^{\gamma(X_{\partial\mathbb{D},\epsilon} + Q/2 \ln g)} \, d\lambda\Big) \\
&\exp\Big(-\frac{Q}{2\pi} \int_{\partial\mathbb{D}} K_g (c + X_{\partial\mathbb{D}}) \, d\lambda_{\partial g} - \mu_\partial e^{\frac{\gamma}{2}c} \epsilon^{\frac{\gamma^2}{4}} \int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}(X_{\partial\mathbb{D},\epsilon} + Q/2 \ln g)} \, d\lambda_\partial\Big)\Big] \, dc.
\end{aligned} \tag{3.2}$$

The first natural question is to inquire whether the limit

$$\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g, F) := \lim_{\epsilon \to 0} \Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(\epsilon, g, F). \tag{3.3}$$

exists and is not trivial. Existence and non triviality will be phrased in terms of the following three conditions

$$\sum_i \alpha_i + \frac{1}{2}\beta_j > Q, \tag{3.4}$$

$$\forall i \quad \alpha_i < Q, \tag{3.5}$$

$$\forall j \quad \beta_j < Q. \tag{3.6}$$

We claim

Theorem 3.1. (Seiberg bounds) *We have the following alternatives*

1. *Assume $\mu > 0$ and $\mu_\partial \geqslant 0$. The partition function $\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,1)$ converges and is non trivial if and only if (3.4)+(3.5)+(3.6) hold.*

2. *Assume $\mu = 0$ and $\mu_\partial > 0$. The partition function $\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,1)$ converges and is non trivial if and only if (3.4) +(3.6) hold.*

3. *In all other cases, we have*

$$\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,1) = 0 \quad \text{or} \quad \Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,1) = +\infty.$$

Along the computations involved in Theorem 3.1, we get the expression below for the partition function when the metric g is the Euclidean metric. Notice that considering the only Euclidean metric is not a restriction because we will see later that there is an explicit procedure to express the partition function in any background metric g in terms of that in the Euclidean metric (Weyl anomaly, subsection 3.3).

Proposition 3.2. (Partition function) *Assume g is the Euclidean metric dx^2. Then, in each case of Theorem 3.1 ensuring existence and non triviality, we have*

$$\begin{aligned}\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,F) =&\Big(\prod_i g_P(z_i)^{-\frac{\alpha_i^2}{4}}\Big)e^{C(\mathbf{z},\mathbf{s})}\int_{\mathbb{R}} e^{\big(\sum_i \alpha_i+\sum_j \frac{\beta_j}{2}-Q\big)c} \\ &\mathbb{E}\Big[F(X_{\partial\mathbb{D}}+H+c)\exp\Big(-\mu e^{\gamma c}\int_{\mathbb{D}} e^{\gamma H}e^{\gamma X_{\partial\mathbb{D}}}\,d\lambda-\mu_\partial e^{\frac{\gamma}{2}c}\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}H}e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}\,d\lambda_\partial\Big)\Big]\,dc,\end{aligned} \tag{3.7}$$

where

$$H(x) = \sum_i \alpha_i G(x,z_i) + \sum_j \frac{\beta_j}{2}G(x,s_j),$$

$$C(\mathbf{z},\mathbf{s}) = \sum_{i<i'}\alpha_i\alpha_{i'}G(z_i,z_{i'}) + \sum_{j<j'}\frac{\beta_j\beta_{j'}}{4}G(s_j,s_{j'}) + \sum_{i,j}\frac{\alpha_i\beta_j}{2}G(z_i,s_j) - \sum_j \frac{\beta_j^2}{8}.$$

Proof of Theorem 3.1 and Proposition 3.2. We begin with the Seiberg bound. Because the conformal factor φ of $g = e^{\varphi}dx^2$ is assumed to be smooth (i.e. of class C^1), we can assume without loss of generality that $\varphi = 0$. The main lines of the argument will be similar to [7, Section 3], up to a few modifications that we explain below. First observe that Propositions 2.2 and 2.3 ensure that the interaction terms

$$\lim_{\epsilon\to 0}\epsilon^{\frac{\gamma^2}{4}}\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}X_{\partial\mathbb{D},\epsilon}}\,d\lambda_\partial \quad \text{and} \quad \lim_{\epsilon\to 0}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{D}} e^{\gamma X_{\partial\mathbb{D},\epsilon}}\,d\lambda$$

are non trivial provided that $\gamma \in]0,2[$. Hence, following [7, Section 3], $\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,1) < +\infty$ if and only if (3.4) holds: roughly speaking, recall that basically this amounts to claiming that the integral (A, A' are two strictly positive constants)

$$\int_{\mathbb{R}} e^{\big(\sum_i \alpha_i+\frac{1}{2}\sum_j \beta_j-Q\big)c}e^{-\mu e^{\gamma c}A-\mu_\partial A'}\,dc$$

is converging if and only if (3.4) holds.

Recall then that the remaining part of the proof in [7, Section 3] consists in determining when a marked point causes the blowing up of the interaction measure, in which case $\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,F)=0$. The reason why a marked point may cause the blowing up of the interaction measure is that these marked points are handled with the Girsanov transform and this amounts to determining whether the bulk/boundary measures integrates some singularities of the type $\frac{1}{|x-z_i|^{\alpha_i\gamma}}$ or $\frac{1}{|x-s_j|^{\frac{\beta_j}{2}\gamma}}$. This is what we study in more details below.

Here we have two types of marked points (in the bulk or along the boundary) and two interaction measures: boundary $e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}\,d\lambda_\partial$ or bulk $e^{\gamma X_{\partial\mathbb{D}}}\,d\lambda$. A marked point (z_i,α_i) in the bulk questions whether the bulk measure integrates the singularity $x\mapsto e^{\alpha_i\gamma G(x,z_i)}$. This is exactly the same situation as in [7, Section 3]. Therefore the conclusion is the same: α_i must be strictly less than Q. The same argument settles the case of the effect of a boundary marked point (s_j,β_j) on the boundary measure: β_i must be strictly less than Q.

What is not treated in [7, Section 3] is the effect of boundary marked points on the bulk measure: namely we have to determine when the measure $e^{\gamma X_{\partial\mathbb{D}}}\,d\lambda$ integrates the singularity $x\mapsto e^{\frac{\beta_j}{2}\gamma G(x,s_j)}$ for some s_j belonging to the boundary $\partial\mathbb{D}$. Observe that the situation is more complicated as the behavior of the bulk measure is highly perturbed when approaching the boundary: recalling the expression of the bulk measure in Proposition 2.2, we see that on the one hand the deterministic density $g_P(x)^{\frac{\gamma^2}{4}}$ blows up along the boundary and on the other hand the field $X_{\partial\mathbb{D}}$ acquires more and more correlations, which become maximal along the boundary: as x approaches the boundary, $G(x,y)$ tends to behave like $2\ln\frac{1}{|x-y|}$ rather than $\ln\frac{1}{|x-y|}$.

Let us now analyze the situation. We want to prove that the singularity is integrable if and only if $\beta_j<Q$. Without loss of generality, we assume that $s_j=1$. In what follows, C stands for some generic constant, which may change along the lines and does not depend on relevant quantities.

Let us first assume that the singularity is integrable, more precisely for some δ fixed small enough

$$\lim_{\epsilon\to 0}\int_{\mathbb{D}\cap B(1,\delta)} e^{\frac{\beta_j}{2}\gamma G_\epsilon(\cdot,1)}\epsilon^{\frac{\gamma^2}{2}}e^{\gamma X_{\partial\mathbb{D},\epsilon}}\,d\lambda<+\infty \tag{3.8}$$

where

$$G_\epsilon(x,y)=\mathbb{E}[X_{\partial\mathbb{D},\epsilon}(x)X_{\partial\mathbb{D},\epsilon}(y)].$$

For each $\epsilon>0$ small enough, we denote by D_ϵ the small disk centered at $1-2\epsilon$ with radius ϵ. Notice that for ϵ small enough, this disk is contained in $B(1,\delta)\cap\mathbb{D}$. Therefore, we have the obvious relation

$$\int_{\mathbb{D}\cap B(1,\delta)} e^{\frac{\beta_j}{2}\gamma G_\epsilon(\cdot,1)}\epsilon^{\frac{\gamma^2}{2}}e^{\gamma X_{\partial\mathbb{D},\epsilon}}\,d\lambda\geqslant\int_{D_\epsilon} e^{\frac{\beta_j}{2}\gamma G_\epsilon(\cdot,1)}e^{\gamma X_{\partial\mathbb{D},\epsilon}-\frac{\gamma^2}{2}\mathbb{E}[X^2_{\partial\mathbb{D},\epsilon}]}e^{\frac{\gamma^2}{2}(\mathbb{E}[X^2_{\partial\mathbb{D},\epsilon}]-\ln\frac{1}{\epsilon})}\,d\lambda.$$

It is then plain to check that, for some constant C independent of ϵ and uniformly with respect to $x\in D_\epsilon$,

$$|\mathbb{E}[X_{\partial\mathbb{D},\epsilon}(x)^2]-2\ln\frac{1}{\epsilon}|\leqslant C,\quad |G_\epsilon(x,1)-2\ln\frac{1}{\epsilon}|\leqslant C.$$

We deduce

$$\int_{\mathbb{D}\cap B(1,\delta)} e^{\frac{\beta_j}{2}\gamma G_\epsilon(\cdot,1)}\epsilon^{\frac{\gamma^2}{2}}e^{\gamma X_{\partial\mathbb{D},\epsilon}}\,d\lambda\geqslant C\epsilon^{-\beta_j\gamma-\frac{\gamma^2}{2}}\int_{D_\epsilon} e^{\gamma X_{\partial\mathbb{D},\epsilon}-\frac{\gamma^2}{2}\mathbb{E}[X^2_{\partial\mathbb{D},\epsilon}]}\,d\lambda.$$

If we can establish the following estimate

$$\text{in probability,}\quad \limsup_{\epsilon\to 0} \epsilon^{-2-\gamma^2}\int_{D_\epsilon} e^{\gamma X_{\partial\mathbb{D},\epsilon}-\frac{\gamma^2}{2}\mathbb{E}[X^2_{\partial\mathbb{D},\epsilon}]}\,d\lambda=+\infty, \tag{3.9}$$

we deduce that necessarily $\beta_j<Q$ in order for (3.8) to hold.

To establish (3.9), observe (see subsection 6.2) that, for some deterministic constant C independent of ϵ,

$$\sup_{\epsilon>0}\sup_{x\in D_\epsilon}|G_\epsilon(x,x)+2\ln\epsilon|<+\infty,$$

in such a way that

$$\int_{D_\epsilon} e^{\gamma X_{\partial\mathbb{D},\epsilon}-\frac{\gamma^2}{2}\mathbb{E}[X^2_{\partial\mathbb{D},\epsilon}]}\,d\lambda\geqslant C\epsilon^{2+\gamma^2}e^{\gamma X_{\partial\mathbb{D},\epsilon}(1)}e^{\min_{x\in D_\epsilon}X_{\partial\mathbb{D},\epsilon}(x)-X_{\partial\mathbb{D},\epsilon}(1)}. \tag{3.10}$$

Next, we estimate the min in the above expression. Observe that ($D(2,1)$ stands for the disk centered at 2 with radius 1)

$$\min_{x\in D_\epsilon}X_{\partial\mathbb{D},\epsilon}(x)-X_{\partial\mathbb{D},\epsilon}(1)=\min_{u\in D(2,1)}Y_\epsilon(u)$$

where the Gaussian process Y_ϵ is defined by

$$Y_\epsilon(u)=X_{\partial\mathbb{D},\epsilon}(1-\epsilon u)-X_{\partial\mathbb{D},\epsilon}(1).$$

The key point is to estimate the fluctuations of the Gaussian process Y_ϵ. The reader may check (see subsection 6.2) that the variance of $Y_\epsilon(2)$ is bounded independently of ϵ and that for all $z,z'\in D(2,1)$

$$\mathbb{E}[(Y_\epsilon(z)-Y_\epsilon(z'))^2]\leqslant C|z-z'|,$$

uniformly in $0<\epsilon\leqslant 1$. Recall the Kolmogorov criterion

Theorem 3.3. (Kolmogorov criterion) *Let X be a continous stochastic process on $D(1,2)$. If, for some $\beta,\alpha,C>0$:*

$$\forall x,z\in D(1,2),\quad \mathbb{E}[|X_x-X_z|^q]\leqslant C|x-z|^{2+\beta}.$$

For all $\delta\in]0,\frac{\beta}{q}[$, we set $L=\sup_{x\neq z}\frac{|X_x-X_z|}{|x-z|^\delta}$. Then, for all $p<q$, $\mathbb{E}[L^\beta]\leqslant 1+\frac{Cp2^{\beta-q\delta}}{(q-p)(2^{\beta-q\delta}-1)}$.

One can then deduce that the family of processes $(Y_\epsilon)_\epsilon$ is tight in the space of continuous functions over $D(2,1)$ for the topology of uniform convergence. We deduce that for each subsequence, we can find R large enough such that $\min_{x\in D_\epsilon}X_{\partial\mathbb{D},\epsilon}(x)-X_{\partial\mathbb{D},\epsilon}(1)\geqslant -R$ with probability arbitrarily close to 1. Finally, we observe that the process $\epsilon\mapsto X_{\partial\mathbb{D},\epsilon}(1)$ behaves like a Brownian motion at time $2\ln\frac{1}{\epsilon}$ (see [11, section 6.1]), we can use the law of the iterated logarithm in (3.10) to complete the proof of (3.9).

Now it remains to show that the condition $\beta_j<Q$ is sufficient to have integrability. Now it remains to show that the condition $\beta_j<Q$ is sufficient to have integrability. For each $r>0$, we denote

$$A_r=\{z\in\mathbb{D};|z-1|\leqslant r\}.$$

We consider $0 < \alpha < 1$. Observe that for $z, z' \in A_r$, we have the relation

$$G(z,z') \geqslant \mathbb{E}[(X_{\partial\mathbb{D},r}(1-r))^2] - C$$

for some constant C independent of r. Because of this relation and the concavity of the map $x \mapsto x^\alpha$, we can apply Kahane's convexity inequality and get for some standard Gaussian random variable N independent of everything

$$\mathbb{E}\Big[\Big(\int_{A_r} e^{\sqrt{C}N - C/2} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]}\, d\lambda\Big)^\alpha\Big] \leqslant \mathbb{E}\Big[\Big(\int_{A_r} e^{\gamma X_{\partial\mathbb{D},r}(1-r) - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D},r}(1-r)^2]}\, d\lambda\Big)^\alpha\Big].$$

The last expectation can be easily explicitly computed as it is just the Laplace transform of some Gaussian random variable. Given the fact that $\mathbb{E}[X_{\partial\mathbb{D},r}(1-r)^2] \sim -2\ln r$, we get for some other constant C,

$$\mathbb{E}\Big[\Big(\int_{A_r} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]}\, d\lambda\Big)^\alpha\Big] \leqslant C r^{(2+\gamma^2)\alpha - \gamma^2\alpha^2}.$$

Let $\eta > 0$. We deduce

$$\mathbb{P}\Big(\int_{A_r} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]}\, d\lambda > r^{2+\gamma^2-\eta}\Big) \leqslant r^{-\alpha(2+\gamma^2-\eta)}\mathbb{E}\Big[\Big(\int_{A_r} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]}\, d\lambda\Big)^\alpha\Big] \leqslant C r^{\eta\alpha - \alpha^2\gamma^2}.$$

Choosing $\alpha > 0$ small enough, we have $\eta\alpha - \alpha^2\gamma^2 > 0$. We can then use the Borel-Cantelli lemma to deduce that there exists a random constant R, which is finite almost surely, such that

$$\sup_{r \in]0,1]} r^{-(2+\gamma^2-\eta)} \int_{A_r} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]}\, d\lambda \leqslant R. \tag{3.11}$$

Now we introduce the sets for $n \geqslant 1$ ($\delta > 0$ is fixed)

$$B_n = \{z \in \mathbb{D}; |z-1| \leqslant \delta\,, 2^{-n} \leqslant (1-|z|^2) \leqslant 2^{-n+1}\}.$$

Finally, we get

$$\begin{aligned}
\int_{\mathbb{D}\cap B(1,\delta)} e^{\gamma\frac{\beta_j}{2}G(\cdot,1)} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]} g_P^{\frac{\gamma^2}{4}}\, d\lambda &= \sum_{n\geqslant 1}\int_{B_n} e^{\gamma\frac{\beta_j}{2}G(\cdot,1)} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]} g_P^{\frac{\gamma^2}{4}}\, d\lambda \\
&\leqslant C\sum_{n\geqslant 1} 2^{n\frac{\gamma^2}{2}} 2^{\gamma\beta_j n} \int_{A_{2^{-n+1}}} e^{\gamma X_{\partial\mathbb{D}} - \frac{\gamma^2}{2}\mathbb{E}[X_{\partial\mathbb{D}}^2]}\, d\lambda \\
&\leqslant CR\sum_{n\geqslant 1} 2^{n\frac{\gamma^2}{2}} 2^{\gamma\beta_j n} 2^{-n(2+\gamma^2-\eta)}.
\end{aligned}$$

The proof of Theorem 3.1 is complete provided that we choose $0 < \eta < \gamma(Q-\beta_j)$. Once the Seiberg bounds are established, the computation of the partition function (i.e. Proposition 3.2) follows the same lines as in [7, Theorem 3.2]. ∎

3.2 Definitions of the Liouville field, Liouville measure and boundary Liouville measure.

As long as one of the two conditions of Theorem 3.1 is satisfied, one may define the joint law of the Liouville field ϕ together with the Liouville measure $Z(\cdot)$ and boundary Liouville measure $Z_\partial(\cdot)$. In spirit, the situation is that the convergence of the partition function entails that we get a non trivial probability law for the field $\phi = c + X_{\partial\mathbb{D}} + \frac{Q}{2}\ln g$ under the probability measure defined by the partition function. This field formally corresponds to the log-conformal factor of some random metric $e^{\gamma\phi}g$ conformally equivalent to g. Yet, observe that the field ϕ is in H^{-1} almost surely so that a rigorous description of this metric is not straightforward, at least clearly not standard. The Liouville measure that we construct below is a random measure that can be thought of as the volume form of this formal metric tensor whereas the boundary Liouville measure corresponds to the line element along the boundary. Let us mention that we could construct as well the Liouville Brownian motion by using the construction made in [14, 15] but a rigorous construction of a distance function associated to the metric tensor $e^{\gamma\phi}g$ remains an open question.

Given a measured space E, we denote by $\mathcal{R}(E)$ the space of Radon measures on E equipped with the topology of weak convergence. The joint law of (ϕ, Z, Z_∂) is defined for all continuous bounded functional F on $H^{-1}(\bar{\mathbb{D}}) \times \mathcal{R}(\mathbb{D}) \times \mathcal{R}(\partial\mathbb{D})$ by

$$
\begin{aligned}
&\mathbb{E}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu,g}\left[F(\phi, Z, Z_\partial)\right] \\
=&\frac{e^{\frac{1}{96\pi}\left(\int_{\mathbb{D}}|\partial \ln g|^2\,d\lambda + \int_{\partial\mathbb{D}} 4\ln g\,d\lambda_\partial\right)}}{\Pi^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu}(g,1)} \lim_{\epsilon\to 0}\int_{\mathbb{R}}\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)(z_i)} \prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)(s_j)} \\
&E\Big[F\Big(X_{\partial\mathbb{D}} + c + Q/2\ln g, e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}} e^{\gamma(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)}\,d\lambda, e^{\frac{\gamma}{2}c}\epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)}\,d\lambda_\partial\Big) \\
&\exp\Big(-\frac{Q}{4\pi}\int_{\mathbb{D}} R_g(c+X_{\partial\mathbb{D}})\,d\lambda_g - \mu e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{D}} e^{\gamma(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)}\,d\lambda\Big) \\
&\exp\Big(-\frac{Q}{2\pi}\int_{\partial\mathbb{D}} K_g(c+X_{\partial\mathbb{D}})\,d\lambda_{\partial g} - \mu_\partial e^{\frac{\gamma}{2}c}\epsilon^{\frac{\gamma^2}{4}}\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)}\,d\lambda_\partial\Big)\Big]\,dc.
\end{aligned}
$$

We denote by $\mathbb{P}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu,g}$ the associated probability measure. In the following subsections, we will mention several interesting properties satisfied by these objects.

3.3 Conformal changes of metric and Weyl anomaly

Here we want to determine the dependence of the partition function (3.3) (as well as the Liouville field/measures) on the metric g conformally equivalent to the Euclidean metric

Theorem 3.4. (Weyl anomaly)

1. *Given two metrics g, g' conformally equivalent to the flat metric and $g' = e^{\varphi}g$, we have*

$$
\ln\frac{\Pi^{(z_i,\alpha_i)_i,(z_i',\alpha_i')_i}_{\gamma,\mu_{\partial D},\mu}(g',F)}{\Pi^{(z_i,\alpha_i)_i,(z_i',\alpha_i')_i}_{\gamma,\mu_{\partial D},\mu}(g,F)} = \frac{1+6Q^2}{96\pi}\Big(\int_{\mathbb{D}}|\partial^g\varphi|^2\,d\lambda_g + \int_{\mathbb{D}} 2R_g\varphi\,d\lambda_g + 4\int_{\partial\mathbb{D}} K_g\varphi\,d\lambda_{\partial_g}\Big).
$$

2. *The law of the triple (ϕ, Z, Z_∂) under $\mathbb{P}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial,\mu,g}$ does not depend on the metric g in the conformal equivalence class of the Euclidean metric.*

Proof. In (3.3), we use the Girsanov transform to the exponential term

$$\exp\Big(-\frac{Q}{4\pi}\int_{\mathbb{D}} R_g X_{\partial\mathbb{D}}\,d\lambda_g - \frac{Q}{2\pi}\int_{\partial\mathbb{D}} K_g X_{\partial\mathbb{D}}\,d\lambda_{\partial g}\Big),$$

which has the effect of shifting the field X by

$$-\frac{Q}{4\pi}\int_{\mathbb{D}} R_g G_{\partial\mathbb{D}}(\cdot,z)\,\lambda_g(dz) - \frac{Q}{2\pi}\int_{\partial\mathbb{D}} G_{\partial\mathbb{D}}(\cdot,z) K_g\,d\lambda_{\partial g}.$$

Then we use the rules (2.2)+(2.3)+(2.7) to see that this shift is equal to

$$-\frac{Q}{2}(\ln g - m_\partial(\ln g)).$$

Due to the Girsanov renormalization, the whole partition function will be multiplied by the exponential of the variance of the field $\frac{Q}{4\pi}\int_{\mathbb{D}} R_g X_{\partial\mathbb{D}}\,d\lambda_g + \frac{Q}{2\pi}\int_{\partial\mathbb{D}} K_g X_{\partial\mathbb{D}}\,d\lambda_{\partial g}$, which can be computed with (2.2)+(2.3)+(2.7) and is given by

$$\frac{Q^2}{16\pi}\int_{\mathbb{D}} |\partial \ln g|^2\,d\lambda.$$

Hence, by making the changes of variables $v = c + \frac{Q}{2} m_\partial(\ln g)$, we get

$$\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,F)\\
&= e^{\frac{6Q^2}{96\pi}\int_{\mathbb{D}}|\partial\ln g|^2\,d\lambda + \frac{Q^2}{2} m_\partial(\ln g)} e^{\frac{1}{96\pi}\left(\int_{\mathbb{R}^2}|\partial \ln g|^2\,d\lambda + \int_\partial 4\ln g\,d\lambda_\partial\right)}\\
&\quad \lim_{\epsilon\to 0}\int_{\mathbb{R}} e^{\left(\sum_i\alpha_i + \frac12\sum_j\beta_j - Q\right)v}\,\mathbb{E}\Big[F(X_{\partial\mathbb{D}}+v, e^{\gamma v}e^{\gamma X_{\partial\mathbb{D}}}\,d\lambda, e^{\frac{\gamma}{2}v}e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}\,d\lambda_\partial)\\
&\quad \prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i X_{\partial\mathbb{D},\epsilon}(z_i)} \prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}X_{\partial\mathbb{D},\epsilon}(s_j)}\\
&\quad \exp\Big(-\mu e^{\gamma v}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{D}} e^{\gamma X_{\partial\mathbb{D},\epsilon}}\,d\lambda - \mu_\partial e^{\frac{\gamma}{2}v}\epsilon^{\frac{\gamma^2}{4}}\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}X_{\partial\mathbb{D},\epsilon}}\,d\lambda_\partial\Big)\Big]\,dc\\
&= e^{\frac{1+6Q^2}{96\pi}\left(\int_{\mathbb{D}}|\partial\ln g|^2\,d\lambda + \int_{\partial\mathbb{D}} 4\ln g\,d\lambda_\partial\right)}\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,F).
\end{aligned}$$

To complete the proof for two metrics g, g' conformally equivalent to the Euclidean metric, say $g' = e^{\varphi} g$, we apply twice the above result to get

$$\begin{aligned}
&\ln\frac{\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g',F)}{\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,F)}\\
&= \frac{1+6Q^2}{96\pi}\Big(\int_{\mathbb{D}}|\partial\ln g'|^2\,d\lambda + \int_{\partial\mathbb{D}} 4\ln g'\,d\lambda_\partial - \int_{\mathbb{D}}|\partial \ln g|^2\,d\lambda - \int_{\partial\mathbb{D}} 4\ln g\,d\lambda_\partial\Big)\\
&= \frac{1+6Q^2}{96\pi}\Big(\int_{\mathbb{D}}|\partial\varphi|^2\,d\lambda + 2\int_{\mathbb{D}}\partial\varphi\cdot\partial\ln g\,d\lambda + \int_{\partial\mathbb{D}} 4\varphi\,d\lambda_\partial\Big).
\end{aligned}$$

Now we use (2.5)+(2.2) to get

$$\ln\frac{\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g',F)}{\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,F)} = \frac{1+6Q^2}{96\pi}\Big(\int_{\mathbb{D}}|\partial\varphi|^2\,d\lambda + 2\int_{\mathbb{D}}\varphi R_g\,d\lambda_g + 4\int_{\partial\mathbb{D}}\varphi(1+\frac12\partial_n\ln g)\,d\lambda_\partial\Big).$$

We complete the proof with (2.3). ∎

3.4 Conformal covariance and KPZ formula

Now we want to establish the conformal covariance of the partition function, i.e. to determine its behavior under the action of Möbius transforms on the marked points. We focus here on the case when the background metric is the Euclidean one: as shown by the Weyl anomaly (Theorem 3.4), this is not a restriction. One thus looks at

$$\begin{aligned}&\Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(dx^2,F) \qquad (3.12)\\ =&\lim_{\epsilon\to 0}\int_{\mathbb{R}} e^{\left(\sum_i\alpha_i+\frac{1}{2}\sum_j\beta_j-Q\right)c}\mathbb{E}\Big[F(X_{\partial\mathbb{D},\epsilon}+c)\prod_i \epsilon^{\frac{\alpha_i^2}{2}}e^{\alpha_i X_{\partial\mathbb{D},\epsilon}(\psi(z_i))}\prod_j \epsilon^{\frac{\beta_j^2}{4}}e^{\frac{\beta_j}{2}X_{\partial\mathbb{D},\epsilon}(\psi(s_j))}\\ &\exp\Big(-\mu e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{D}} e^{\gamma X_{\partial\mathbb{D},\epsilon}}\,d\lambda-\mu_\partial e^{\frac{\gamma}{2}c}\epsilon^{\frac{\gamma^2}{4}}\int_{\partial\mathbb{D}} e^{\frac{\gamma}{2}X_{\partial\mathbb{D},\epsilon}}\,d\lambda_\partial\Big)\Big]\,dc.\end{aligned}$$

where ψ is a Möbius transform of the unit disk.

We use the following convention for the rest of this section. If M is a measure on a measurable space E and $\psi : E \to E$ is a bi-measurable bijection then the measure $M \circ \psi$ is defined by the relation $M \circ \psi(A) = M(\psi(A))$ for all measurable set $A \subset E$.

Theorem 3.5. *Let ψ be a Möbius transform of the disk. Then*

$$\Pi_{\gamma,\mu}^{(\psi(z_i),\alpha_i)_i}(dx^2,1)=\prod_i|\psi'(z_i)|^{-2\Delta_{\alpha_i}}\prod_j|\psi'(s_j)|^{-\Delta_{\beta_j}}\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(dx^2,1)$$

where the conformal weights $\triangle_\alpha$ are defined by

$$\triangle_\alpha=\frac{\alpha}{2}(Q-\frac{\alpha}{2}).$$

Furthermore the law of the triple (ϕ,Z,Z_∂) under $\mathbb{P}_{\gamma,\mu_\partial,\mu,dx^2}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}$ is the same as that of the triple $\big(\phi\circ\psi+Q\ln|\psi'|,Z\circ\psi,Z_\partial\circ\psi\big)$ under $\mathbb{P}_{\gamma,\mu_\partial,\mu,dx^2}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}$.

Proof. To facilitate the comprehension, we take only into consideration the law of the Liouville field and we leave to the reader the details of the whole proof for the triple (ϕ, Z, Z_∂).
We first study the behavior of the measure under the Möbius transform ϕ:

Lemma 3.6. *For any $f \in C^2(\overline{\mathbb{D}})$, we have*

$$\begin{aligned}&(X_{\partial\mathbb{D}}\circ\psi,\lim_{\epsilon\to0}\int_{\mathbb{D}} f\epsilon^{\frac{\gamma^2}{2}}e^{\gamma X_{\partial\mathbb{D},\epsilon}}d\lambda,\lim_{\epsilon\to0}\int_{\mathbb{D}} f\epsilon^{\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}X_{\partial\mathbb{D},\epsilon}}d\lambda_\partial)\\ \overset{law}{=}&(X_{\partial\mathbb{D}}+m_\partial(X_{\partial\mathbb{D}}\circ\psi),\lim_{\epsilon\to0}\int_{\mathbb{D}} f\circ\psi e^{\gamma(X_{\partial\mathbb{D},\epsilon}+m_\partial(X_{\partial\mathbb{D}}\circ\psi))}|\psi'|^{Q\gamma}d\lambda,\lim_{\epsilon\to0}\int_{\mathbb{D}} f\circ\psi e^{\frac{\gamma}{2}(X_{\partial\mathbb{D},\epsilon}+m_\partial(X_{\partial\mathbb{D}}\circ\psi))}|\psi'|^{\frac{Q\gamma}{2}}d\lambda_\partial)\end{aligned}$$

Proof of Lemma 3.6. Using Proposition 2.1, we have that

$$\lim_{\epsilon\to0}\mathbb{E}[X_{\partial\mathbb{D},\epsilon}(\psi(x))^2]-\mathbb{E}[(X_{\partial\mathbb{D}}\circ\psi)_{\frac{\epsilon}{|\phi'(x)|}}(x)^2]=0$$

on $\mathbb{D}$ and on $\partial\mathbb{D}$.
As $|\phi'(x)|$ is always larger than a constant that is strictly positif, we can use the result in [28] to show that the measures

$$(\frac{\epsilon}{|\phi'|})^{\frac{\gamma^2}{2}}e^{\gamma X_{\partial\mathbb{D},\epsilon}\circ\psi}d\lambda$$

and

$$\epsilon^{\frac{\gamma^2}{2}} e^{\gamma (X_{\partial\mathbb{D}}\circ\psi)_\epsilon} d\lambda$$

converge in probability to the same random measure on $\mathbb{D}$.
Similarly,

$$(\frac{\epsilon}{|\phi'|})^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} X_{\partial\mathbb{D},\epsilon}\circ\psi} d\lambda_\partial$$

and

$$\epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} (X_{\partial\mathbb{D}}\circ\psi)_\epsilon} d\lambda_\partial$$

converge in probability to the same limit measure on $\partial\mathbb{D}$.
We also have, by change of variables in the integrand

$$\int_{\mathbb{D}} f\ \epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{\partial\mathbb{D},\epsilon}} d\lambda = \int_{\mathbb{D}} f\circ\psi\ \epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{\partial\mathbb{D},\epsilon}\circ\psi} |\psi'|^2 d\lambda = \int_{\mathbb{D}} f\circ\psi\ (\frac{\epsilon}{|\psi'|})^{\frac{\gamma^2}{2}} e^{\gamma X_{\partial\mathbb{D},\epsilon}\circ\psi} |\psi'|^{Q\gamma} d\lambda$$

and similarly

$$\int_{\mathbb{D}} f\ \epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} X_{\partial\mathbb{D},\epsilon}} d\lambda_\partial = \int_{\mathbb{D}} f\circ\psi\ \epsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} X_{\partial\mathbb{D},\epsilon}\circ\psi} |\psi'| d\lambda_\partial = \int_{\mathbb{D}} f\circ\psi\ (\frac{\epsilon}{|\psi'|})^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} X_{\partial\mathbb{D},\epsilon}\circ\psi} |\psi'|^{\frac{Q\gamma}{2}} d\lambda_\partial$$

Combining the above arguments, we conclude the proof by recalling the change of metric formula

$$X_{\partial\mathbb{D}}\circ\psi - m_\partial(X_{\partial\mathbb{D}}\circ\psi) \overset{law}{=} X_{\partial\mathbb{D}}, \tag{3.13}$$

which can be verified using the definition of m_∂ and the Green function. ☐

Anticipating the formula (3.13), we use the change of variables $\overline{c} = c + m_\partial(X_\partial\circ\psi)$ to write

$$\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(dx^2,F)\\
&=\lim_{\epsilon\to0}\int_{\mathbb{R}} e^{\left(\sum_i\alpha_i+\frac12\sum_j\beta_j-Q\right)(\overline{c}-m_\partial(X_{\partial\mathbb{D}}\circ\psi))}\mathbb{E}\Big[F(X_{\partial\mathbb{D},\epsilon}+\overline{c}-m_\partial(X_{\partial\mathbb{D}}\circ\psi))\prod_i\epsilon^{\frac{\alpha_i^2}{2}}e^{\alpha_iX_{\partial\mathbb{D},\epsilon}(\psi(z_i))}\\
&\prod_j\epsilon^{\frac{\beta_j^2}{4}}e^{\frac{\beta_j}{2}X_{\partial\mathbb{D},\epsilon}(\psi(s_j))}\exp\Big(-\mu e^{\gamma\overline{c}}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{D}}e^{\gamma(X_{\partial\mathbb{D},\epsilon}-m_\partial(X_{\partial\mathbb{D}}\circ\psi))}\,d\lambda-\mu_\partial e^{\frac{\gamma}{2}\overline{c}}\epsilon^{\frac{\gamma^2}{4}}\int_{\mathbb{D}}e^{\frac{\gamma}{2}(X_{\partial\mathbb{D},\epsilon}-m_\partial(X_{\partial\mathbb{D}}\circ\psi))}\,d\lambda_{\partial g}\Big)\Big]\,dc.
\end{aligned}$$

We now apply the Girsanov transform to the factor $e^{Qm_\partial(X_{\partial\mathbb{D}}\circ\psi)}$. This will shift the law of the field $X_{\partial\mathbb{D}}$, which becomes

$$X_{\partial\mathbb{D}} + \frac{Q}{2\pi}\int_{\partial\mathbb{D}} G(\cdot,\psi(z))\lambda_\partial(dz)$$

We now introduce a useful constant in the following calculation

$$D_\psi = \int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} G(\psi(y),\psi(z))\lambda_\partial(dy)\lambda_\partial(dz) = 4\pi^2\mathbb{E}[m_\partial(X_{\partial\mathbb{D}}\circ\psi)^2].$$

We also introduce the function

$$H(y) = \int_{\partial\mathbb{D}} G(\psi(y),\psi(z))\lambda_\partial(dz)$$

so that $D_\psi = \int_{\partial\mathbb{D}} H(y)\lambda_\partial(dy)$. Recall that $\int_{\partial\mathbb{D}} G(y,z)\lambda_\partial(dz) = 0$ for all y.

Under the Girsanov transform $X_{\partial\mathbb{D}}(x) - m_\partial(X_{\partial\mathbb{D}} \circ \psi)$ becomes $X_{\partial\mathbb{D}}(x) - m_\partial(X_{\partial\mathbb{D}} \circ \psi) + \frac{Q}{2\pi} H(\psi^{-1}(x)) - \frac{Q}{4\pi^2} D_\psi$ and we get

$$
\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(dx^2, F) \\
&= \lim_{\epsilon\to 0} e^{\frac{Q^2}{8\pi} D_\psi} \int_{\mathbb{R}} e^{\left(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j - Q\right)\bar{c}} \mathbb{E}\Big[F(X_{\partial\mathbb{D},\epsilon} + \bar{c} - m_\partial(X_{\partial\mathbb{D}} \circ \psi) + \frac{Q}{2\pi} H(\psi^{-1}(\cdot)) - \frac{Q}{4\pi^2} D_\psi) \\
&\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i (X_{\partial\mathbb{D},\epsilon}(\psi(z_i)) - m_\partial(X_{\partial\mathbb{D}}\circ\psi) + \frac{Q}{2\pi} H(z_i) - \frac{Q}{4\pi^2} D_\psi)} \prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2} (X_{\partial\mathbb{D},\epsilon}(\psi(s_j)) - m_\partial(X_{\partial\mathbb{D}}\circ\psi) + \frac{Q}{2\pi} H(s_j) - \frac{Q}{4\pi^2} D_\psi)} \\
&\exp\Big(- \mu e^{\gamma \bar{c}} \epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{D}} e^{\gamma (X_{\partial\mathbb{D}}(x) - m_\partial(X_{\partial\mathbb{D}}\circ\psi) + \frac{Q}{2\pi} H(\psi^{-1}(x)) - \frac{Q}{4\pi^2} D_\psi)}\, d\lambda \\
&- \mu_\partial e^{\frac{\gamma}{2}\bar{c}} \epsilon^{\frac{\gamma^2}{4}} \int_{\mathbb{D}} e^{\frac{\gamma}{2} (X_{\partial\mathbb{D}}(x) - m_\partial(X_{\partial\mathbb{D}}\circ\psi) + \frac{Q}{2\pi} H(\psi^{-1}(x)) - \frac{Q}{4\pi^2} D_\psi)}\, d\lambda_{\partial g} \Big) \Big]\, dc.
\end{aligned}
$$

Notice the relation (consequence of (2.9))

$$
\frac{Q}{2\pi} H(x) = Q \ln \frac{1}{|\psi'(x)|} + \frac{Q}{8\pi^2} D_\psi
$$

the D_ψ part with cancel out the first exponential term in the above expression when we do the change of variables $c = \bar{c} - \frac{Q}{8\pi^2} D_\psi$.
Now using (3.13), (2.2) and Lemma 3.6, we finally have

$$
\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(dx^2, F) \\
&= \prod_i |\psi'(z_i)|^{\alpha^2/2 - \alpha Q} \prod_j |\psi'(s_j)|^{\beta^2/4 - \beta Q/2} \\
&\lim_{\epsilon\to 0} \int_{\mathbb{R}} e^{\left(\sum_i \alpha_i + \frac{1}{2}\sum_j \beta_j - Q\right) c} \mathbb{E}\Big[F(X_{\partial\mathbb{D},\epsilon} \circ \psi^{-1} - Q \ln |\psi'(\psi^{-1}(\cdot))| + c) \prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i (X_{\partial\mathbb{D},\epsilon}(z_i))} \\
&\prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(X_{\partial\mathbb{D},\epsilon}(s_j))} \exp\Big(- \mu e^{\gamma c} \epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{D}} e^{\gamma X_{\partial\mathbb{D},\epsilon}}\, d\lambda - \mu_\partial e^{\frac{\gamma}{2} c} \epsilon^{\frac{\gamma^2}{4}} \int_{\mathbb{D}} e^{\frac{\gamma}{2} X_{\partial\mathbb{D},\epsilon}}\, d\lambda_{\partial g} \Big) \Big]\, dc.
\end{aligned}
$$

This completes the proof of the theorem. □

3.5 Conformal changes of domains

In this section, we explain how to construct the LQFT on domains that are conformally equivalent to the unit disk. Basically, the idea is to find a conformal map sending this domain to the unit disk and to use the conformal covariance property of the LQFT.

Let D be a simply connected (strict) domain of $\mathbb{C}$, say with a C^1 Jordan boundary. From the Riemann mapping theorem, we can consider a conformal map $\psi : D \to \mathbb{D}$. If we further consider marked points (z_i, α_i) in D and boundary marked points $(s_j, \beta_j)_j$ in ∂D, they will be sent respectively to $(\psi(z_i), \alpha_i)$ in $\mathbb{D}$ and to the boundary marked points $(\psi(s_j), \beta_j)_j$ in ∂D. Finally, the uniformization theorem tells us that there is no restriction if we assume that D is equipped with a metric of the type $g_\psi = |\psi'|^2 g(\psi)$ for some metric g on $\partial\mathbb{D}$.

The Liouville partition function on (D, g_ψ) applied to a functional F reads

$$\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(D, g_\psi, F) \tag{3.14}$$

$$=e^{\frac{1}{96\pi}\left(\int_{\mathbb{D}}|\partial \ln g|^2\,d\lambda+\int_{\partial\mathbb{D}} 4\ln g\,d\lambda_\partial\right)} \lim_{\epsilon\to 0}\int_{\mathbb{R}} \mathbb{E}\Big[F(X_\nu + c + Q/2\ln g_\psi)\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(c+X_{\nu,\epsilon}+Q/2\ln g_\psi)(z_i)}$$

$$\prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(c+X_{\nu,\epsilon}+Q/2\ln g_\psi)(s_j)} \exp\Big(-\frac{Q}{4\pi}\int_D R_{g_\psi}(c+X_\nu)\,d\lambda_{g_\psi} - \mu e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}}\int_D e^{\gamma(X_{\nu,\epsilon}+Q/2\ln g_\psi)}\,d\lambda\Big)$$

$$\exp\Big(-\frac{Q}{2\pi}\int_{\partial D} K_{\partial g_\psi}(c+X_\nu)\,d\lambda_{\partial g_\psi} - \mu_\partial\, e^{\frac{\gamma}{2}c}\epsilon^{\frac{\gamma^2}{4}}\int_{\partial D} e^{\frac{\gamma}{2}(X_{\nu,\epsilon}+Q/2\ln g_\psi)}\,d\lambda_\partial\Big)\Big]\,dc,$$

where X_ν is a GFF on D with Neumann boundary condition and vanishing ν-mean. By shift invariance of the Lebesgue measure, the choice of ν is irrelevant and it will be convenient to take $X_\nu = X_{\partial\mathbb{D}}\circ\psi$, which is free boundary GFF with vanishing mean for the line element on ∂D in the metric $|\psi'|^2dx^2$ on $\bar{D}$.

Proposition 3.7. *Let D be a simply connected (strict) domain of $\mathbb{C}$ with a C^1 Jordan boundary. Then we have the relation*

$$\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(D, g_\psi, F(\phi, Z, Z_\partial))$$
$$=\prod_i |\psi'(z_i)|^{2\triangle_{\alpha_i}}\prod_j |\psi'(s_j)|^{\triangle_{\beta_j}}\Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(\mathbb{D}, g, F(\phi\circ\psi + Q\ln|\psi'|, Z\circ\psi, Z_\partial\circ\psi)).$$

In particular:

1. *we have the following relation between the partition functions ($F = 1$)*

$$\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(D, g_\psi, 1) = \prod_i |\psi'(z_i)|^{2\triangle_{\alpha_i}}\prod_j |\psi'(s_j)|^{\triangle_{\beta_j}}\Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(\mathbb{D}, g, 1).$$

2. *The law of the triple (ϕ, Z, Z_∂) under $\mathbb{P}_{(D,g_\psi),\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}$ is the same as $(\phi\circ\psi + Q\ln|\psi'|Z, Z_\partial)$ under $\mathbb{P}_{(\mathbb{D},g),\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}$.*

Proof. Again we only treat a functional F depending only on the Liouville field for simplicity. Applying lemma 3.6 and using that $R_{g_\psi}(x) = R_g(\psi(x))$ and $K_{g_\psi}(x) = K_g(\psi(x))$ (because ψ is a conformal map), (3.14) is equal to

$$\begin{aligned}
&\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(D,g_\psi,F)\\
=&e^{\frac{1}{96\pi}\Big(\int_{\mathbb{D}}|\partial\ln g|^2\,d\lambda+\int_{\partial\mathbb{D}}4\ln g\,d\lambda_\partial\Big)}\prod_i|\psi'(z_i)|^{Q\alpha_i-\frac{\alpha_i^2}{2}}\prod_j|\psi'(s_j)|^{Q\frac{\beta_j}{2}-\frac{\beta_j^2}{4}}\\
&\lim_{\epsilon\to 0}\int_{\mathbb{R}}\mathbb{E}\Big[F((X_{\partial\mathbb{D}}+c+Q/2\ln g)\circ\psi+Q\ln|\psi'|)\\
&\prod_i\epsilon^{\frac{\alpha_i^2}{2}}e^{\alpha_i(c+X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)(\psi(z_i))}\prod_j\epsilon^{\frac{\beta_j^2}{4}}e^{\frac{\beta_j}{2}(c+X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)(\psi(s_j))}\\
&\exp\Big(-\frac{Q}{4\pi}\int_D R_g(\psi)(c+X_{\partial\mathbb{D}}\circ\psi)g(\psi)|\psi'|^2\,d\lambda-\mu e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{D}}e^{\gamma(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)}\,d\lambda\Big)\\
&\exp\Big(-\frac{Q}{2\pi}\int_{\partial D}K_g(\psi)(c+X_{\partial\mathbb{D}}\circ\psi)|\psi'|g^{1/2}(\psi)\,d\lambda_\partial-\mu_\partial e^{\frac{\gamma}{2}c}\epsilon^{\frac{\gamma^2}{4}}\int_{\partial\mathbb{D}}e^{\frac{\gamma}{2}(X_{\partial\mathbb{D},\epsilon}+Q/2\ln g)}\,d\lambda_\partial\Big)\Big]\,dc.\\
=&\prod_i|\psi'(z_i)|^{2\triangle_{\alpha_i}}\prod_j|\psi'(s_j)|^{\triangle_{\beta_j}}\Pi_{\gamma,\mu_\partial,\mu}^{(\psi(z_i),\alpha_i)_i,(\psi(s_j),\beta_j)_j}(\mathbb{D},g,F(\phi\circ\psi+Q\ln|\psi'|)).
\end{aligned}\tag{3.15}$$

This completes the proof. ☐

3.6 Law of the volume of space/boundary

We want to express here the (joint) law of the volume of bulk/boundary on the unit disk equipped with the Euclidean metric. It will be convenient to express this law in terms of the couple of random measures $(Z_0(\cdot),Z_0^\partial(\cdot))$ under $\mathbb{P}$ respectively defined on $\mathbb{D}$ and $\partial\mathbb{D}$ by (recall Proposition 2.2)

$$Z_0(\cdot)=e^{\gamma H_{\mathbb{D}}}e^{\gamma X_{\partial\mathbb{D}}}\,d\lambda,\quad Z_0^\partial(\cdot)=e^{\frac{\gamma}{2}H_{\partial\mathbb{D}}}e^{\frac{\gamma}{2}X_{\partial\mathbb{D}}}\,d\lambda_\partial\tag{3.16}$$

with

$$H_{\mathbb{D}}(x)=\sum_i\alpha_iG(x,z_i),\quad H_{\partial\mathbb{D}}(x)=\sum_j\frac{\beta_j}{4}G(x,s_i).\tag{3.17}$$

We further introduce the ratio

$$R=\frac{Z_0(\mathbb{D})}{Z_0(\partial\mathbb{D})^2}.$$

By definition of the law of the bulk/boundary Liouville measures, we have

$$\begin{aligned}
&\mathbb{E}_{\gamma,\mu_\partial,\mu,dx^2}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}\big[F(Z,Z_\partial)\big]\\
=&(\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,1))^{-1}\int_{\mathbb{R}}e^{\big(\sum_i\alpha_i+\frac12\sum_j\beta_j-Q\big)c}\mathbb{E}\Big[F(e^{\gamma c}Z_0(dx),e^{\frac{\gamma}{2}c}Z_0^\partial(dx))\\
&\exp\Big(-\mu e^{\gamma c}Z_0(\mathbb{D})-\mu_\partial e^{\frac{\gamma}{2}c}Z_0^\partial(\partial\mathbb{D})\Big)\Big]\,dc\\
=&\frac{2}{\gamma}(\Pi_{\gamma,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(dx^2,1))^{-1}\int_0^\infty y^{\frac{2}{\gamma}(\sum_i\alpha_i+\frac12\sum_j\beta_j-Q)-1}\\
&\mathbb{E}\Big[F\Big(y^2R\frac{Z_0(dx)}{Z_0(\mathbb{D})},y\frac{Z_0^\partial(dx)}{Z_0^\partial(\partial\mathbb{D})}\Big)\exp\Big(-\mu y^2R-\mu_\partial y\Big)Z_0^\partial(\partial\mathbb{D})^{-\frac{2}{\gamma}(\sum_i\alpha_i+\frac12\sum_j\beta_j-Q)}\Big]\,dy.
\end{aligned}$$

This is the general formula. It may be useful to state as a particular example the case $\mu_\partial=0$ as it often arises in the study of random planar maps with a boundary.

Corollary 3.8. *Assume $\mu_\partial = 0$. The joint law of the bulk/boundary Liouville measures are given by*

$$\begin{aligned}&\mathbb{E}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial=0,\mu,dx^2}\big[F(Z,Z_\partial)\big]\\ =&Z^{-1}\int_{\mathbb{R}} u^{\frac{1}{\gamma}\big(\sum_i\alpha_i+\frac{1}{2}\sum_j\beta_j-Q\big)-1}\mathbb{E}\Big[F\Big(u\frac{Z_0(dx)}{Z_0(\mathbb{D})},u^{\frac{1}{2}}\frac{Z_0^\partial(dx)}{Z_0(\mathbb{D})^{\frac{1}{2}}}\Big)Z_0(\mathbb{D})^{-\frac{1}{\gamma}(\sum_i\alpha_i+\frac{1}{2}\sum_j\beta_j-Q)}\Big]e^{-\mu u}\,du.\end{aligned}$$

where Z is a renormalization constant to get a probability measure. In particular, the law of the volume of space follows a Gamma law with parameters $\big(\frac{\sum_i\alpha_i+\frac{1}{2}\sum_j\beta_j-Q}{\gamma},\mu\big)$ and the random variable $Z(\mathbb{D})$ is independent of the random measures $(\frac{Z(dx)}{Z(\mathbb{D})},\frac{Z_\partial(dx)}{Z(\mathbb{D})^{\frac{1}{2}}})$.

Remark 3.9. *Since the geometrical KPZ formula established in [25] has been established almost surely with respect to the GFF expectation, it holds for the Liouville measure in our context almost surely too.*

4 Liouville QFT at $\gamma = 2$

Here we explain how to construct LQFT on the unit disk in the important case $\gamma = 2$. The reason why this case is so specific is that it is no more superrenormalizable at small scales. In other words the interaction terms $e^{2X_{\partial\mathbb{D}}}d\lambda$ or $e^{X_{\partial\mathbb{D}}}d\lambda_\partial$ can no more be obtained as a Wick ordering, i.e. a subcritical Gaussian multiplicative chaos: it corresponds to the phase transition in Gaussian multiplicative chaos theory. Indeed, the standard renormalizations

$$\epsilon^2\, e^{2X_{\partial\mathbb{D},\epsilon}(x)}d\lambda \quad \text{and} \quad \epsilon\, e^{X_{\partial\mathbb{D},\epsilon}(x)}d\lambda_\partial$$

yield vanishing limiting measures. To get a non trivial limit, an extra push $\sqrt{\ln\frac{1}{\epsilon}}$ is necessary, which is called the Seneta-Heyde norming. For Gaussian multiplicative chaos, this has been investigated in [13] for a white noise decomposition of the GFF, which does not exactly correspond to our framework as we work with convolution cutoff approximations. So, we explain in this section how to generalize the results in [13] to convolutions.

We first claim

Theorem 4.1. *The family of boundary approximation measures on $\partial\mathbb{D}$*

$$\sqrt{\ln\frac{1}{\epsilon}}\,\epsilon\, e^{X_{\partial\mathbb{D},\epsilon}}d\lambda_\partial$$

converges in probability as ϵ goes to 0 towards a non trivial limiting measure, which has moments of order q for all $q < 1$.

Theorem 4.2. *The family of bulk approximation measures on $\mathbb{D}$*

$$\sqrt{\ln\frac{1}{\epsilon}}\,\epsilon^2\, e^{X_{\partial\mathbb{D},\epsilon}}d\lambda$$

converges in probability as ϵ goes to 0 towards a non trivial limiting measure, which has moments of order q for all $q < 1$.

Remark 4.3. *Actually, our proof for the two above theorems establishes convergence in probability for a large class of cutoff approximations with mollifying family, not only the circle average family.*

Proof of Theorem 4.1. The strategy is the following: first we show the convergence in probability of a specific family of white noise cutoff approximations. Then we will show that this entails the convergence in probability for a whole class of convolution approximations, including circle averages.

Recall that if we consider a centered Gaussian distribution X on the boundary of the unit disk with the following covariance structure

$$\mathbb{E}[X(e^{i\theta})X(e^{i\theta})] = 2\ln\frac{1}{|e^{i\theta}-e^{i\theta'}|},$$

then the law on the boundary of the GFF $X_{\partial\mathbb{D}}$ is given by

$$X_{\partial\mathbb{D}} = X - \frac{1}{2\pi}\int_0^{2\pi} X(e^{i\theta})d\theta.$$

Our first step is to construct $X_{\partial\mathbb{D}}$ as a function of some white noise W and of a smooth Gaussian process Y. This decomposition will be convenient to establish convergence in probability of the approximating measures based on martingale techniques. We will recover the situation of approximations based on convolution of $X_{\partial\mathbb{D}}$ after that.

Recall the following decomposition (see [27])

$$\forall x\in\mathbb{R}^2,\quad \ln_+\frac{1}{|x|} = 2\int_0^1 (t-|x|^{\frac12})_+\frac{dt}{t^2} + 2(1-|x|^{\frac12})_+\,.$$

Now we construct two Gaussian distributions: the first one $\bar{X}$ will have the covariance structure of the first term in the above right-hand side and the second one Y the second term.

Lemma 4.4. *There exists a white noise W on $[1,+\infty[\times\partial\mathbb{D}$ and a family of centered Gaussian processes $(\bar{X}_\epsilon)_{\epsilon\in]0,1]}$ on $\partial\mathbb{D}$, which are measurable functions of this white noise, such that*

$$\forall 0<\epsilon<\epsilon'\leqslant 1,\quad \bar{X}_\epsilon-\bar{X}_{\epsilon'} \text{ is independent of } \sigma\{X_u(e^{i\theta});\epsilon'\leqslant u\leqslant 1,\theta\in[0,2\pi]\} \tag{4.1}$$

and

$$\mathbb{E}[\bar{X}_\epsilon(e^{i\theta})\bar{X}_\epsilon(e^{i\theta'})] = 2\int_{\sqrt{\epsilon}}^1 (t-|e^{i\theta}-e^{i\theta'}|^{\frac12})_+\frac{dt}{t^2} = \int_1^{\frac{1}{\epsilon}}(1-|v(e^{i\theta}-e^{i\theta'})|^{\frac12})_+\frac{dv}{v}. \tag{4.2}$$

The limiting distribution $\bar{X}=\lim_{\epsilon\to0}\bar{X}_\epsilon$ is a centered Gaussian distribution with covariance structure

$$\mathbb{E}[\bar{X}(e^{i\theta})\bar{X}(e^{i\theta'})] = 2\int_0^1 (t-|e^{i\theta}-e^{i\theta'}|^{\frac12})_+\frac{dt}{t^2}.$$

Finally, for any smooth function R on $[1,+\infty[\times\partial\mathbb{D}$ with compact support, the function

$$z\in\partial\mathbb{D}\mapsto T_\epsilon(R)(z) := \mathbb{E}[\bar{X}_\epsilon(z)W(R)]$$

is a continuous function which converges uniformly as $\epsilon\to0$ towards

$$z\in\partial\mathbb{D}\mapsto T(R)(z) := \mathbb{E}[\bar{X}(z)W(R)].$$

This lemma is proved in appendix 6.3. Then we consider a centered Gaussian field Y independent of $(\bar{X}_\epsilon)_{\epsilon\in]0,1]}$ with covariance given by

$$\mathbb{E}[Y(e^{i\theta})Y(e^{i\theta'})] = (1-|e^{i\theta}-e^{i\theta'}|^{\frac{1}{2}})_+ \,.$$

Recall that such a kernel is indeed positive definite [17].

Now we can set

$$X_{\partial\mathbb{D}} = \bar{X} + Y - \frac{1}{2\pi}\int_0^{2\pi} (\bar{X}(e^{i\theta}) + Y(e^{i\theta}))d\theta.$$

This is a construction of $X_{\partial\mathbb{D}}$ as a function of (W, Y). Now we would like to use [13] to show that the random measures

$$\sqrt{\ln\frac{1}{\epsilon}}\,\epsilon\, e^{\bar{X}_\epsilon} d\lambda_\partial \tag{4.3}$$

converges in probability to a non trivial limiting random measure as $\epsilon \to 0$. To this purpose, observe that the covariance $(k_\epsilon)_{\epsilon\in]0,1]}$ kernels of the family $(\bar{X}_\epsilon)_{\epsilon\in]0,1]}$ can be written as

$$k_\epsilon(e^{i\theta}, e^{i\theta'}) = \int_1^{\frac{1}{\epsilon}} \frac{k(v, e^{i\theta}, e^{i\theta'})}{v}\,dv \quad \text{ with } \quad k(v, e^{i\theta}, e^{i\theta'}) = (1-|v(e^{i\theta}-e^{i\theta'})|^{\frac{1}{2}})_+.$$

Such a kernel k satisfies the properties

A.1 k is nonnegative, continuous.

A.2 k is Hölder on the diagonal, more precisely $\forall \theta, \theta'$, $\forall v \geqslant 1$,

$$|k(v, e^{i\theta}, e^{i\theta}) - k(v, e^{i\theta}, e^{i\theta'})| \leqslant v^{1/2}|e^{i\theta}-e^{i\theta'}|^{1/2}$$

A.3 k satisfies the integrability condition

$$\sup_{\theta,\theta'} \int_{\frac{1}{|e^{i\theta}-e^{i\theta'}|}}^{\infty} \frac{k(v, e^{i\theta}, e^{i\theta'})}{v}\,dv < +\infty.$$

A.4 for all $\epsilon \in]0,1]$, $\int_1^{\frac{1}{\epsilon}} \frac{k(v,e^{i\theta},e^{i\theta})}{v}\,dv = \ln\frac{1}{\epsilon}$,

A.5 $k(v, e^{i\theta}, e^{i\theta'}) = 0$ for $|e^{i\theta}-e^{i\theta'}| \geqslant v^{-1}$.

Observe in particular that [A.2] implies that

$$|\ln\frac{1}{\epsilon} - k_\epsilon(e^{i\theta}, e^{i\theta'})| \leqslant \int_1^{1/\epsilon} \frac{|e^{i\theta}-e^{i\theta'}|^{1/2}}{v^{1/2}}dv \leqslant C(|e^{i\theta}-e^{i\theta'}|/\epsilon)^{1/2}$$

for some constant C (independent of ϵ). In particular we have the property

$$|e^{i\theta}-e^{i\theta'}| \leqslant \epsilon \quad \Rightarrow \quad |\ln\frac{1}{\epsilon} - k_\epsilon(e^{i\theta}, e^{i\theta'})| \leqslant C. \tag{4.4}$$

These properties are the only assumptions used in [26] to construct the derivative martingale and in [13] to prove the Seneta-Heyde norming. Therefore the family of random measures 4.3 converges in probability towards a non trivial random measures, which has moments of order q for all $q<1$ (see [13]).

Hence, if $X_\epsilon=\bar{X}_\epsilon+Y-\frac{1}{2\pi}\int_0^{2\pi}(\bar{X}_\epsilon(e^{i\theta})+Y(e^{i\theta}))d\theta$, then

$$M_\epsilon=\sqrt{\ln\frac{1}{\epsilon}}\,\epsilon\, e^{X_\epsilon(e^{i\theta})}d\lambda_\partial$$

converges in probability to a random measure M' which is a measurable function of the white noise W and the process Y, call it $F(W,Y)$.

Now, we show convergence in probability of $\sqrt{\ln\frac{1}{\epsilon}}\,\epsilon\, e^{X_{\partial\mathbb{D},\epsilon}}d\lambda_\partial$, where $X_{\partial\mathbb{D},\epsilon}$ is the circle average approximation of $X_{\partial\mathbb{D}}$ towards the same limit M'. The ideas in the following stem from the techniques developed in [24] along with some some variant of lemma 49 in [28] (we will not recall lemma 49 as our proof will be self contained).

For this, we introduce $X^1_{\partial\mathbb{D}}$, an independent copy of $X_{\partial\mathbb{D}}$, and $X^1_{\partial\mathbb{D},\epsilon}$ its circle average approximation. Let us define for $t\in[0,1]$ and $\theta\in[0,2\pi]$

$$Z_\epsilon(t,e^{i\theta})=\sqrt{t}X^1_{\partial\mathbb{D},\epsilon}(e^{i\theta})+\sqrt{1-t}X_\epsilon(e^{i\theta}).$$

Now, we set

$$M^1_\epsilon=\sqrt{\ln\frac{1}{\epsilon}}\,\epsilon\, e^{X^1_{\partial\mathbb{D},\epsilon}(e^{i\theta})}d\lambda_\partial.$$

We first show that M^1_ϵ converges in distribution to $M'=F(W,Y)$. From [27, Proof of Theorem 2.1], one gets that for all $\alpha<1$

$$\begin{aligned}
&\overline{\lim_{\epsilon\to0}}\big|\mathbb{E}[M^1_\epsilon(B)^\alpha]-\mathbb{E}[M_\epsilon(B)^\alpha]\big|\\
&\leqslant c\frac{\alpha(1-\alpha)}{2}C_A\overline{\lim_{\epsilon\to0}}\int_0^1\mathbb{E}\Big[\Big(\sqrt{\ln\frac{1}{\epsilon}}\int_{\partial\mathbb{D}}e^{Z_\epsilon(t,\cdot)-\frac12\mathbb{E}[Z_\epsilon(t,\cdot)^2]}\,d\lambda_\partial\Big)^\alpha\Big]\,dt\\
&+c\,\overline{C}_A\overline{\lim_{\epsilon\to0}}\int_0^1\mathbb{E}\Big[\Big(\sup_{0\leqslant i<\frac{1}{A\epsilon}}\sqrt{\ln\frac{1}{\epsilon}}\int_{2iA\epsilon}^{2(i+1)A\epsilon}e^{Z_\epsilon(t,e^{i\theta})-\frac12\mathbb{E}[Z_\epsilon(t,e^{i\theta})^2]}d\theta\Big)^\alpha\Big]dt,
\end{aligned}$$

where

$$C_A=\overline{\lim_{\epsilon\to0}}\sup_{|e^{i\theta}-e^{i\theta'}|\geqslant A\epsilon}|\mathbb{E}[X^1_{\partial\mathbb{D},\epsilon}(e^{i\theta})X^1_{\partial\mathbb{D},\epsilon}(e^{i\theta'})]-\mathbb{E}[X_\epsilon(e^{i\theta})X_\epsilon(e^{i\theta'})]|$$

and

$$\overline{C}_A=\overline{\lim_{\epsilon\to0}}\sup_{|e^{i\theta}-e^{i\theta'}|\leqslant A\epsilon}|\mathbb{E}[X^1_{\partial\mathbb{D},\epsilon}(e^{i\theta})X^1_{\partial\mathbb{D},\epsilon}(e^{i\theta'})]-\mathbb{E}[X_\epsilon(e^{i\theta})X_\epsilon(e^{i\theta'})]|.$$

The reader can check that $\overline{C}_A$ is bounded independently of A and $\lim_{A\to\infty}C_A=0$. Since $\mathbb{E}\Big[\Big(\sqrt{\ln\frac{1}{\epsilon}}\int_0^1e^{Z_\epsilon(t,u)-\frac12\mathbb{E}[Z_\epsilon(t,u)^2]}\,du\Big)^\alpha\Big]$ is also bounded independently of everything (by comparison with Mandelbrot's multiplicative cascades as explained in the [12, appendix] and [13, appendix B.4]), we are done if we can show that for all $t\in[0,1]$

$$\overline{\lim_{\epsilon\to0}}\,\mathbb{E}\Big[\Big(\sup_{0\leqslant i<\frac{1}{A\epsilon}}\sqrt{\ln\frac{1}{\epsilon}}\int_{2iA\epsilon}^{2(i+1)A\epsilon}e^{Z_\epsilon(t,e^{i\theta})-\frac12\mathbb{E}[Z_\epsilon(t,e^{i\theta})^2]}d\theta\Big)^\alpha\Big]=0.\tag{4.5}$$

Notice that this quantity is less than

$$\big(\ln\frac{1}{\epsilon}\big)^{\alpha/2}\epsilon^{\alpha}\mathbb{E}\Big[\Big(e^{\sup_{\theta\in[0,2\pi]} Z_\epsilon(t,e^{i\theta})-\frac{1}{2}\mathbb{E}[Z_\epsilon(t,e^{i\theta})^2]}\Big)^{\alpha}\Big]. \tag{4.6}$$

To estimate this quantity, we use the main result of [1]: more precisely, setting

$$m_\epsilon = 2\ln\frac{1}{\epsilon}-\frac{3}{2}\ln\ln\frac{1}{\epsilon},$$

we claim that there exist two constants $C, c > 0$ such that for ϵ small enough

$$\forall x \geqslant 0,\quad \mathbb{P}\Big(\Big|\max_{\theta\in[0,2\pi]} Z_\epsilon(t,e^{i\theta})-m_\epsilon\Big|\geqslant x\Big)\leqslant Ce^{-cx}.$$

In particular we get that for $\alpha < c$

$$\sup_\epsilon \mathbb{E}\Big[\Big(e^{\sup_{\theta\in[0,2\pi]} Z_\epsilon(t,e^{i\theta})}\Big)^{\alpha}\Big]<\infty.$$

Plugging this estimate into (4.6), we see that the quantity (4.6) is less than

$$C'\big(\ln\frac{1}{\epsilon}\big)^{\alpha/2}\epsilon^{2\alpha}e^{\alpha m_\epsilon}=C'\big(\ln\frac{1}{\epsilon}\big)^{-\alpha}.$$

for some constant $C' > 0$. This proves the claim (4.5), hence the convergence in law of the random measure M^1_ϵ towards $M' = F(W, Y)$.

Now we deduce that the family $(W, Y, M^1_\epsilon)_\epsilon$ converges in law. Take any smooth function R on $[1, +\infty[\times\partial\mathbb{D}$ with compact support, any continuous function g on $\partial\mathbb{D}$, any bounded continuous function G on $\mathbb{R}$ and $u \in \mathbb{R}$. We have by using the Girsanov transform

$$\mathbb{E}[e^{W(R)+uY}G(M^1_\epsilon(g))]=e^{\frac{1}{2}\mathrm{Var}[W(R)+uY]}\mathbb{E}[G(M^1_\epsilon(e^{T_\epsilon(R)}g)]$$

where $T_\epsilon(R)$ is defined in Lemma 4.4. The quantity in the right-hand side converges as $\epsilon \to 0$ towards

$$e^{\frac{1}{2}\mathrm{Var}[W(R)+uY]}\mathbb{E}[G(M'(e^{T(R)}g)]=\mathbb{E}[e^{W(R)+uY}G(M'(g))].$$

Hence our claim about the convergence in law of the triple $(W, Y, M^1_\epsilon)_\epsilon$ towards $(W, Y, M' = F(W, Y))$.

Now we consider the family $(W, Y, M^1_\epsilon, F(W, Y))_\epsilon$, which is tight. Even if it means extracting a subsequence, it converges in law towards some $(\mathcal{W}, \mathcal{Y}, \mathcal{M}, \bar{\mathcal{M}})$. We have just shown that the law of $(\mathcal{W}, \mathcal{Y}, \mathcal{M})$ is that of $(\mathcal{W}, \mathcal{Y}, F(\mathcal{W}, \mathcal{Y}))$, i.e. the same as the law of $(\mathcal{W}, \mathcal{Y}, \bar{\mathcal{M}})$. Hence $\mathcal{M} = \bar{\mathcal{M}}$ almost surely. Therefore $M^1_\epsilon - F(W, Y)$ converges in law towards 0, hence in probability. Since the convergence in probability of the family $(M^1_\epsilon)_\epsilon$ implies the convergence of probability of every family $(\widehat{M}_\epsilon)_\epsilon$ that has the same law as $(M^1_\epsilon)_\epsilon$, the proof of Theorem 4.1 is complete.

Finally, one can notice that instead of $X_{\partial\mathbb{D},\epsilon}$ we could have considered any smooth convolution approximation of X. □

Proof of Theorem 4.2. Let us consider the Poisson kernel on the unit disk

$$\forall 0\leqslant r<1, \forall\theta\in[0,2\pi],\quad P_r(\theta)=\sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta}.$$

We can then consider the harmonic extension inside the unit disk of the trace of the GFF $X_{\partial\mathbb{D}}$ along the boundary

$$P_X(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t) X_{\partial\mathbb{D}}(e^{it})\, dt.$$

It is plain to see that P_X is a continuous Gaussian process inside the unit disk. If we set

$$X^{\mathrm{Dir}} = X_{\partial\mathbb{D}} - P_X,$$

one can check that we get a GFF with Dirichlet boundary condition in the unit disk. Therefore, by continuity of P_X inside $\mathbb{D}$, the convergence in probability of the random measures $(\epsilon^2\, e^{X_{\partial\mathbb{D},\epsilon}(x)} d\lambda)_\epsilon$ boils down to showing the convergence in probability for the random measures

$$(\epsilon^2\, e^{X_\epsilon^{\mathrm{Dir}}(x)} d\lambda)_\epsilon$$

where $(X_\epsilon^{\mathrm{Dir}})_\epsilon$ stands for the circle average approximations of the GFF X^{Dir}. Given the fact that the Seneta-Heyde norming has been proved in [13] for a white noise decomposition of X^{Dir}, we can use the same argument as in the proof of Theorem 4.1 to show that convergence for the white noise approximation family entails the convergence in probability for the circle average approximations. $\square$

From now on, the construction of the Liouville LQG on the unit disk for $\gamma = 2$ follows the same lines as for $\gamma < 2$ by taking the limit as $\epsilon \to 0$ of the quantity

$$\begin{aligned}
&\Pi_{2,\mu_\partial,\mu}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}(g,F) \qquad (4.7)\\
=&e^{\frac{1}{96\pi}\left(\int_{\mathbb{D}} |\partial \ln g|^2\, d\lambda + \int_{\partial\mathbb{D}} 4\ln g\, d\lambda_\partial\right)} \lim_{\epsilon\to 0}\int_{\mathbb{R}} \mathbb{E}\Big[F(X_{\partial\mathbb{D}} + c + \ln g)\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(c + X_{\partial\mathbb{D},\epsilon} + \ln g)(z_i)}\\
&\prod_j \epsilon^{\frac{\beta_j^2}{4}} e^{\frac{\beta_j}{2}(c + X_{\partial\mathbb{D},\epsilon} + \ln g)(s_j)} \exp\Big(-\frac{2}{4\pi}\int_{\mathbb{D}} R_g(c + X_{\partial\mathbb{D}})\, d\lambda_g - \mu e^{2c}\sqrt{-\ln\epsilon}\,\epsilon^2 \int_{\mathbb{D}} e^{2(X_{\partial\mathbb{D},\epsilon} + \ln g)}\, d\lambda\Big)\\
&\exp\Big(-\frac{2}{2\pi}\int_{\partial\mathbb{D}} K_g(c + X_{\partial\mathbb{D}})\, d\lambda_{\partial g} - \mu_\partial e^c \sqrt{-\ln\epsilon}\,\epsilon \int_{\partial\mathbb{D}} e^{(X_{\partial\mathbb{D},\epsilon} + \ln g)}\, d\lambda_\partial\Big)\Big]\, dc.
\end{aligned}$$

defined for all continuous and bounded functional F on $H^{-1}(\mathbb{D})$. From now on, the properties of LQG (and their proofs) on the disk for $\gamma = 2$ are the same as for $\gamma < 2$ except Proposition 2.3, which needs some extra care that we treat now.

Proposition 4.5. *The quantities below are almost surely finite*

$$\int_{\mathbb{D}} e^{2X_{\partial\mathbb{D}}} d\lambda \quad \textit{and} \quad \int_{\partial\mathbb{D}} e^{X_{\partial\mathbb{D}}} d\lambda_{\partial\mathbb{D}}.$$

Proof. Recall the sub-additivity inequality for $\alpha \in]0,1[$: if $(a_j)_{1\leqslant j\leqslant n}$ are positive real numbers then

$$(a_1 + \cdots + a_n)^\alpha \leqslant a_1^\alpha + \cdots + a_n^\alpha.$$

Now we use Kahane's convexity inequality [24, Theorem 2.1] to compare the Gaussian multiplicative chaos with standard dyadic lognormal cascade (once again we refer to [12, Appendix B.1] for full details). We consider the dyadic tree with i.i.d. weights with Gaussian law $\mathcal{N}(0, \ln 2)$ on the edges of the tree and denote by Y_j^n the sum of these weights starting from the root up to the dyadic indexed by j at generation n. We denote by $(Z_j)_j$ an i.i.d sequence (independent of everything) standing for the mass of the dyadic cascade at criticality rooted at the dyadic j at generation n. From [21] these random variables have distribution tail $\mathbb{P}(Z_j > x) \leqslant \frac{C}{x}$ for some constant $C > 0$, and $\mathbb{E}[Z_j^q] < \infty$ for $q < 1$. Hence we get

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\mathbb{D}} e^{2X_{\partial\mathbb{D}}(x)-2\mathbb{E}[X_{\partial\mathbb{D}}^2]}\frac{1}{(1-|x|^2)^2})d\lambda\Big)^{\alpha}\Big]\\
&\leqslant \sum_{n\in\mathbb{N}} 2^{2n\alpha}\mathbb{E}\Big[\Big(\int_{1-2^{-n}\leqslant |x|^2\leqslant 1-2^{-n-1}} e^{2X_{\partial\mathbb{D}}(x)-2\mathbb{E}[X_{\partial\mathbb{D}}^2]}d\lambda\Big)^{\alpha}\Big]\\
&\leqslant \sum_{n\in\mathbb{N}} \mathbb{E}\Big[\Big(\sum_{j=1}^{2^n} Z_j e^{2\sqrt{2}(Y_j^n-\sqrt{2}\ln 2\, n)}\Big)^{\alpha}\Big]\\
&= \sum_{n\in\mathbb{N}} \frac{1}{n^{3\alpha}}\mathbb{E}\Big[\Big(\sum_{j=1}^{2^n} Z_j e^{2\sqrt{2}(Y_j^n-\sqrt{2}\ln 2\, n+\frac{3}{2\sqrt{2}}\ln n)}\Big)^{\alpha}\Big].
\end{aligned}$$

Let $\eta\in]0,1[$. By Jensen and for some constant $B>0$

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{j=1}^{2^n} Z_j e^{2\sqrt{2}(Y_j^n-\sqrt{2}\ln 2\, n+\frac{3}{2\sqrt{2}}\ln n)}\Big)^{\alpha}\Big] &= \mathbb{E}\Big[\Big(\sum_{j=1}^{2^n} \mathbb{E}[Z_j^{1-\eta}]e^{2\sqrt{2}(1-\eta)(Y_j^n-\sqrt{2}\ln 2\, n+\frac{3}{2\sqrt{2}}\ln n)}\Big)^{\frac{\alpha}{1-\eta}}\Big]\\
&\leqslant B\mathbb{E}\Big[\Big(\sum_{j=1}^{2^n} e^{2\sqrt{2}(1-\eta)(Y_j^n-\sqrt{2}\ln 2\, n+\frac{3}{2\sqrt{2}}\ln n)}\Big)^{\frac{\alpha}{1-\eta}}\Big]
\end{aligned}$$

From [21] again, this last expectation is bounded independently of n provided that we choose $2\alpha(1-\eta)<1$. In that case, up to changing the value of B, we get

$$\mathbb{E}\Big[\Big(\int_{\mathbb{D}} e^{2X_{\partial\mathbb{D}}(x)-2\mathbb{E}[X_{\partial\mathbb{D}}^2]}\frac{1}{(1-|x|^2)^2})d\lambda\Big)^{\alpha}\Big] \leqslant B\sum_{n\in\mathbb{N}}\frac{1}{n^{3\alpha}},$$

which can be obviously made finite provided that $\alpha > 1/3$. □

5 Conjectures related to planar quadrangulations with boundary

We consider $\overline{\mathcal{T}}_{n,p}$ the set of quandrangulations of size n with a simple boundary of length $2p$ with one marked point on the frontier and one marked point inside. Now to each quadrangulation T with marked point inside and a marked point on the boundary we associate a standard conformal structure (by gluing Euclidean squares along their edges as prescribed by the quadrangulation) and map it to the disk such that the interior point gets mapped to 0 and the frontier point to 1. We give volume a^2 to each quadrilateral and length a to each edge on the boundary: we denote $\nu_{T,a}$ the corresponding volume measure and $\nu_{T,a}^{\partial}(dx)$ the corresponding boundary length measure. Recall that we have the following asymptotics as $n,p\to\infty$ with $\frac{n}{p^2}$ fixed (see appendix):

$$|\overline{\mathcal{T}}_{n,p}| \sim e^{n\ln 12}e^{2p\ln\frac{3}{\sqrt{2}}}n^{-3/2}\frac{\sqrt{3p}}{2\pi}e^{-\frac{9(2p)^2}{16n}}$$

and we set $\mu^c = \ln 12$, $\mu_\partial^c = \ln \frac{3}{\sqrt{2}}$ (these two constants are not universal as they depend on the map one considers, i.e. are different for triangulations, etc...). Now, we consider the measures (ν_a, ν_a^∂) defined by the following expression for all F

$$\mathbb{E}^a[F(\nu_a(dx), \nu_a^\partial(dx))] = \frac{1}{Z_a} \sum_{N,p} e^{-\bar{\mu}N} e^{-\bar{\mu}_\partial 2p} \sum_{T \in \mathcal{T}_{N,p}} F(\nu_{T,a}(dx), \nu_{T,a}^\partial(dx)),$$

where the constants $\bar{\mu}, \bar{\mu}_\partial$ are functions of a defined by $\bar{\mu} = \mu_c + a^2\mu$, $\bar{\mu}_\partial = \mu_\partial^c + a\mu_\partial$ and Z_a is a normalization constant. We can now state a precise mathematical conjecture:

Conjecture 1. *The limit in law* $\lim_{a \to 0} (\nu_a(dx), \nu_a^\partial(dx))$ *exists in the product space of Radon measures equipped with the topology of weak convergence and is given (up to deterministic constants) by the Liouville measure of LQG with parameter* $\gamma = \sqrt{\frac{8}{3}}$, μ, μ_∂ *and* $\alpha_1 = \gamma$, $\beta_1 = \gamma$ *and points* $z_1 = 0$, $s_1 = 1$.

Here we give a few more details on the above conjecture. It states the existence of constants $C, c > 0$ such that

$$\lim_{a \to 0} \mathbb{E}^a[F(\nu_a(dx), \nu_a^\partial(dx))] = \mathbb{E}_{\gamma,\mu_\partial,\mu,dx^2}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}[F(CZ, cZ_\partial)]$$

where (Z, Z_∂) is given in subsection 3.6 with $\gamma = \sqrt{\frac{8}{3}}$, μ, μ_∂ and $\alpha_1 = \gamma$, $\beta_1 = \gamma$ and points $z_1 = 0$, $s_1 = 1$. The constants C, c are non universal in the sense that they depend on the planar map you consider. For instance, the constants C, c will be different if you consider triangulations instead of quadrangulations. This can be seen directly on the asymptotics of planar maps: the joint law of the volume and the boundary length will be given by the following density within the regime of conjecture 1 (see appendix 6.1)

$$\frac{1}{C_{\mu,\mu_\partial}} V^{-3/2} l^{1/2} e^{-\mu V} e^{-\mu_\partial l} e^{-\frac{9l^2}{16V}} dl dV.$$

In fact, the above distribution should be universal, i.e. should not depend on the planar map model, except for the $\frac{9}{16}$ constant in $e^{-\frac{9l^2}{16V}}$ which is specific to quadrangulations and in the case of triangulations (for instance) one should get a different constant than $\frac{9}{16}$. On the LQG side, it is natural to conjecture that for $\gamma = \sqrt{\frac{8}{3}}$ there exists some constant $C > 0$ such that for all function F

$$\mathbb{E}_{\gamma=\sqrt{\frac{8}{3}},\mu_\partial,\mu,dx^2}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}[F(Z(\mathbb{D}), Z_\partial(\partial\mathbb{D}))] = \frac{1}{C_{\mu,\mu_\partial,\gamma}} \int_0^\infty \int_0^\infty F(V,l) V^{-3/2} l^{1/2} e^{-\mu V} e^{-\mu_\partial l} e^{-C\frac{l^2}{V}} dV dl.$$

where $(z_i, \alpha_i)_i, (s_j, \beta_j)_j$ are as in conjecture 1. Finally, let us mention that one could also state similar conjectures with three distinct marked points on the boundary (instead of one interior marked point and one marked point on the boundary) or/and by conditioning on the measures to have fixed volume (instead of the Boltzmann weight setting of conjecture 1). For instance, within the framework of three marked points on the boundary $(s_j)_{1 \leqslant j \leqslant 3}$ each with weight γ and $\mu_\partial = 0$, one recovers the following very simple formula for the interior volume measure conditioned to have volume 1 (with the notations of section 3.6) that is the conjectured scaling limit of the corresponding volume measure of planar maps conditioned to have volume 1

$$\mathbb{E}^{(z_i,\alpha_i)_i,(s_j,\beta_j)_j}_{\gamma,\mu_\partial=0,\mu,dx^2}[F(Z)|Z(\mathbb{D})=1]=\frac{\mathbb{E}[F(\frac{Z_0(dx)}{Z_0(\mathbb{D})})Z_0(\mathbb{D})^{-(3/2-\frac{Q}{\gamma})}]}{\mathbb{E}[Z_0(\mathbb{D})^{-(3/2-\frac{Q}{\gamma})}]}.$$

6 Appendix

6.1 Asymptotics of quadrangulations with a boundary

Here we take material from [4] (see also [5]). Let $\mathcal{T}_{n,p}$ denote quandrangulations of size n with a simple boundary of length $2p$ and a marked point on the frontier. Then we have

$$|\mathcal{T}_{n,p}|=\frac{1}{3^p}\frac{(3p)!}{p!(2p-1)!}3^n\frac{(2n+p-1)!}{(n-p+1)!(n+2p)!}.$$

We are interested in the asymptotics of $|\mathcal{T}_{n,p}|$ as $n,p\to\infty$ with $\frac{p^2}{n}$ fixed. Notice that we have within this asymptotic:

$$\begin{aligned}(2n+p-1)! &\sim\sqrt{2\pi}2^{2n+p-1}e^{(2n+p-1)\ln n+p-1+\frac{p^2}{4n}-(2n+p-1)}\sqrt{2n},\\ (n-p+1)! &\sim\sqrt{2\pi}e^{(n-p+1)\ln n-p+1+\frac{p^2}{2n}-(n-p+1)}\sqrt{n},\\ (n+2p)! &\sim\sqrt{2\pi}e^{(n+2p)\ln n+2p+\frac{2p^2}{n}-(n+2p)}\sqrt{n}.\end{aligned}$$

Hence, we get that

$$\frac{(2n+p-1)!}{(n-p+1)!(n+2p)!}\sim\sqrt{\frac{1}{\pi}}n^{-5/2}2^{2n+p-1}e^{-\frac{9p^2}{4n}}.$$

Also,

$$\frac{(3p)!}{p!(2p-1)!}\sim\frac{\sqrt{3}}{\sqrt{\pi}}\sqrt{p}(\frac{27}{4})^p$$

in such way that we get

$$|\mathcal{T}_{n,p}|\sim 12^n(\frac{9}{2})^p n^{-5/2}\frac{\sqrt{3p}}{2\pi}e^{-\frac{9p^2}{4n}}.$$

Finally, if $\overline{\mathcal{T}}_{n,p}$ denotes the set of quandrangulations of size n with a simple boundary of length $2p$ with one marked point on the frontier and one marked point inside then we get

$$|\overline{\mathcal{T}}_{n,p}|\sim e^{n\ln 12}e^{2p\ln\frac{3}{\sqrt{2}}}n^{-3/2}\frac{\sqrt{3p}}{2\pi}e^{-\frac{9(2p)^2}{16n}}.$$

6.2 Some auxiliary estimates

Here we give hints for some estimates used in the proof of Theorem 3.1 and Proposition 3.2. We stick to the notations used in this proof.

Lemma 6.1. *On boundary behavior of the regularized Green function G_ϵ: remember that D_ϵ is the disk of radius ϵ centered at $1-2\epsilon$, we claim that*

$$\sup_{\epsilon>0}\sup_{x\in D_\epsilon}|G_\epsilon(x,x)+2\ln\epsilon|<+\infty.$$

As a consequence, one sees that if $x\in D_\epsilon$,

$$|\mathbb{E}[X_{\partial\mathbb{D},\epsilon}(x)^2]-2\ln\frac{1}{\epsilon}|\leqslant C,\quad |G_\epsilon(x,1)-2\ln\frac{1}{\epsilon}|\leqslant C.$$

Proof. Let us calculate $G_\epsilon(x,x)$ for $\epsilon>0$ small enough. Recall that the non-regularized Green function $G(x,y)$ is the sum of $\ln\frac{1}{|x-y|}$ and $\ln\frac{1}{|1-x\overline{y}|}$. We have already seen that the ϵ-regularization of $\ln\frac{1}{|x-y|}$ part of $G_\epsilon(x,x)$ will simply be $-\ln\epsilon$ as in the proof of proposition 2.1. Now for the $\ln\frac{1}{|1-x\overline{y}|}$ part, we remark a scaling relation: we can compare what is happening at ϵ with that at $\epsilon/2$ via the following observation (with $a,b>0$ both small of order ϵ)

$$\ln\frac{|a/2+b/2-ab/4|}{|a+b-ab|}-\ln\frac{1}{2}=\ln\frac{|a+b-ab/2|}{|a+b-ab|}\asymp\frac{|ab/2|}{|a+b-ab|}\leqslant|a|$$

By taking $a=1-(x+\epsilon e^{i\theta})$ and $b=1-(\overline{x}-\epsilon e^{i\theta'})$ we can establish

$$\sup_{\epsilon>0}\sup_{x\in D_\epsilon}|\frac{1}{4\pi^2}\int_{\mathbb{S}_1}\int_{\mathbb{S}_1}\ln\frac{1}{|1-(x+\epsilon e^{i\theta})(\overline{x}+e^{i\theta'})|}d\theta d\theta'+\ln\epsilon|<+\infty$$

Together we get the first part of the lemma.
The first inequality in the second part of the lemma comes as a direct consequence. The second inequality can be proved using a similar scaling relation as in the above proof. ☐

Now we establish another estimate concerning the process Y_ϵ. Recall that Y_ϵ is the Gaussian process defined as $Y_\epsilon(u)=X_{\partial\mathbb{D},\epsilon}(1-\epsilon u)-X_{\partial\mathbb{D},\epsilon}(1)$ and $D(2,1)$ is the disk centered at 2 with radius 1.

Lemma 6.2. *For all $z,z'\in D(2,1)$,*

$$\mathbb{E}[(Y_\epsilon(z)-Y_\epsilon(z'))^2]\leqslant C|z-z'|$$

uniformly in $0<\epsilon\leqslant 1$.

Proof. It suffices to prove that uniformly in ϵ,

$$|G_\epsilon(1-\epsilon z,1-\epsilon z)-G_\epsilon(1-\epsilon z,1-\epsilon z')|\leqslant C|z-z'|.$$

For the $\ln\frac{1}{|x-y|}$ part of G, it suffices to prove that the following function is Lipschitz in r for $r\in[0,2]$

$$f(r)=\frac{1}{4\pi^2}\int_{\mathbb{S}_1}\int_{\mathbb{S}_1}\ln\frac{1}{|e^{i\theta}-re^{i\theta'}|}d\theta d\theta'$$

notice that $f(1)=0$. But we have already seen that $f(r)=0$ when $r\leqslant 1$ and this implies that $f(r)=\ln r$ when $r>1$.
As of the $\ln\frac{1}{|1-x\overline{y}|}$ part, we will write the difference as

$$\frac{1}{4\pi^2}\int_{\mathbb{S}_1}\int_{\mathbb{S}_1}\ln\frac{|1-(x+\epsilon e^{i\theta})((\overline{y}-\overline{x})+\overline{x}+\epsilon e^{i\theta'})|}{|1-(x+\epsilon e^{i\theta})(\overline{x}+\epsilon e^{i\theta'})|}d\theta d\theta'$$

where $x = 1 - \epsilon z$ and $y = 1 - \epsilon z'$. Then we note $t = \overline{\frac{y-x}{\epsilon}}$ and this becomes

$$\frac{1}{4\pi^2}\int_{\mathbb{S}_1}\int_{\mathbb{S}_1} \ln \frac{|1/\epsilon^2 - (x/\epsilon + e^{i\theta})(t + \overline{x}/\epsilon + e^{i\theta'})|}{|1/\epsilon^2 - (x/\epsilon + e^{i\theta})(\overline{x}/\epsilon + e^{i\theta'})|} d\theta d\theta'$$

As the derivative with respect to t is continuous and uniformly bounded in ϵ for $|t| \leqslant 2$, our proof is complete. □

6.3 Proof of Lemma 4.4

We introduce the Fourier coefficients $\alpha_v(n) \geqslant 0$ for $n \in \mathbb{Z}, v \in [1, \infty[$ given by

$$|\alpha_v(n)|^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}(1 - |v(e^{i\theta} - 1)|^{\frac{1}{2}})_+ d\theta.$$

We consider a standard white noise W on $[1, \infty[\times \partial\mathbb{D}$ and we set

$$\bar{X}_\epsilon(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \alpha_v(n) e^{in\theta} \frac{1}{\sqrt{2\pi}} \int_1^{\frac{1}{\epsilon}} \int_0^{2\pi} \frac{e^{-inu}}{\sqrt{2\pi v}} W(dv, du)$$

Observe that $\alpha_v(n) = \alpha_v(-n)$ for $n \geqslant 0$ in such a way that $\bar{X}_\epsilon$ is real-valued. Then we can check that

$$\mathbb{E}[\bar{X}_\epsilon(e^{i\theta})\overline{\bar{X}_{\epsilon'}(e^{i\theta})}] = \sum_{n\in\mathbb{Z}} e^{in(\theta-\theta')} \int_1^{\frac{1}{\epsilon}} |\alpha_v(n)|^2 \frac{dv}{v} = \int_1^{\frac{1}{\epsilon}} (1 - |v(e^{i\theta} - e^{i\theta'})|^{\frac{1}{2}})_+ \frac{dv}{v}.$$

Also, notice that we have

$$\bar{X}(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \alpha_v(n) e^{in\theta} \int_1^{\infty} \int_0^{2\pi} \frac{e^{-inu}}{\sqrt{2\pi v}} W(dv, du).$$

Now we compute the correlations between the family $(\bar{X}_\epsilon)_\epsilon$ and the white noise W. We consider a smooth function $H : [1, +\infty[\to \mathbb{R}$ with compact support and a smooth function f on $\partial\mathbb{D}$: we set $F = H \otimes f$ and

$$W(F) = \frac{1}{\sqrt{2\pi}} \int_{[1,+\infty[\times[0,2\pi]} H(v) f(e^{iu}) W(dv, du).$$

Therefore, by considering the Fourier coefficients $(c_n(f))_n$ of f, we obtain

$$\begin{aligned} T_\epsilon(F)(e^{i\theta}) =& \mathbb{E}[\bar{X}_\epsilon(e^{i\theta}) W(H \otimes f)] \\ =& \sum_n \frac{1}{2\pi} \int_{[1,1/\epsilon[\times[0,2\pi]} \frac{\alpha_v(n)}{\sqrt{v}} e^{in\theta} f(e^{iu}) e^{-inu} H(v)\, dvdu \\ =& \sum_n c_n(f) e^{in\theta} \int_{[1,1/\epsilon[} \frac{\alpha_v(n)}{\sqrt{v}} H(v)\, dv. \end{aligned}$$

Because H has compact support, it is readily seen that this series defines a continuous function of θ, which converges uniformly as $\epsilon \to 0$ towards a continuous function given by

$$T(F)(e^{i\theta}) = \sum_n c_n(f) e^{in\theta} \int_{[1,\infty[} \frac{\alpha_v(n)}{\sqrt{v}} H(v)\, dv.$$

□

6.4 Backgrounds on fractional Brownian sheet

We look at the main theorem in [1] and we slightly modify the hypothesis (1.2).
Let $\{(Y_\epsilon^x : x \in [0,1]^d\}_{\epsilon>0}$ be a family of centerd Gaussian fields indexed by $[0,1]^d$ where d is the dimension of the space. We suppose that for some constant $0 < C_Y < \infty$,

$$\forall x, y \in [0,1]^d, \forall \epsilon > 0, |Cov(Y_\epsilon^x, Y_\epsilon^y) + \log(\max\{\epsilon, |x-y|\})| \leqslant C_Y \tag{6.1}$$

$$\mathbb{E}[(Y_\epsilon^x - Y_\epsilon^y)^2] \leqslant C_Y \epsilon^{-1/2}|x-y|^{1/2} \text{ if } |x-y| \leqslant \epsilon \tag{6.2}$$

where $|\cdot|$ is the Euclidean distance.
We claim that

Theorem 6.3. *There exist constants $0 < c, C < \infty$ and a small $\epsilon_0 > 0$ (all depending of C_Y and d) such that for all $0 < \epsilon \leqslant \epsilon_0$ and all $\lambda \geqslant 0$,*

$$\mathbb{P}[|\max_{x\in[0,1]^d} Y_\epsilon^x - m_\epsilon| \geqslant \lambda] \leqslant Ce^{-c\lambda}$$

We adapt the proof by introducing the fractional Brownian sheet. Recall that a (one-dimensional) fractional Brownian sheet $B_0^H = \{B_0^H, t \in \mathbb{R}^N\}$ with Hurst index $H = (H_1, \dots, H_N), 0 < H_j < 1$ is a real-valued centered Gaussian field with covariance structure

$$\mathbb{E}[B_0^H(s)B_0^H(t)] = \prod_{j=1}^N \frac{1}{2}(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}), s, t \in \mathbb{R}^N.$$

In particular B_0^H is self-similar, i.e. for all constants $c > 0$,

$$\left\{B_0^H(ct), t \in \mathbb{R}^N\right\} = \left\{c^{\sum_{j=1}^N H_j} B_0^H(t), t \in \mathbb{R}^N\right\}$$

in distribution.
In view of comparing with equation (6.2), we will choose a d-dimensional vector H with all H_j equal to $1/4$. Let us denote this particular fractional Brownian sheet by Φ.
We now define the field Φ_ϵ on $[0,\epsilon[^d$ by linearly shrinking the region $[p,2p[^d$ of Φ onto $[0,\epsilon[^d$, that is, $(\Phi_\epsilon(x), x \in [0,\epsilon[^d) = (\Phi(l(x)), l(x) \in [p,2p[^d)$ where l is the affine map from $[0,\epsilon[^d$ to $[p,2p[^d$. Notice that Φ_ϵ depends on the choice of p, and p can be chosen as large as desired.
Let us recall two estimations that are useful for the proof (compare with equations (2.7) and (2.8) in [1]):
Following the definition of fractional Brownian sheet:

$$p^{d/2} \leqslant Var(\Phi_\epsilon(x)) \leqslant (2p)^{d/2} \tag{6.3}$$

Combine self-similarity of Φ with lemma 3.4 from [2] we deduce that there exist $c, C > 0$ such that

$$cp^{d/2}\epsilon^{-1/2}|x-y|_2^{1/2} \leqslant \mathbb{E}[(\Phi_\epsilon(x) - \Phi_\epsilon(y))^2] \leqslant Cp^{d/2}\epsilon^{-1/2}|x-y|_2^{1/2} \tag{6.4}$$

where $|\cdot|_2$ is the 2-norm, which is equivalent to the Euclidean norm.
New following [1] we will divide $[0,1[^d$ into boxes of side length $\epsilon > 0$ and assign values to each box using independent copies of Φ_ϵ.
We first recover lemma 2.2 in [1]. We claim that

Lemma 6.4. *There exist constants $0 < c, C < \infty$ (depending on p and d) such that*

$$\sup_{v \in V_\epsilon} \mathbb{P}(\sup_{x \in \square_\epsilon^v} \Phi_\epsilon(x) \geqslant \lambda) \leqslant Ce^{-c\lambda^2} \tag{6.5}$$

To prove this lemma we use Fernique's majorizing measure argument. Notice that

$$B(x,r) := \left\{ y \in \square_\epsilon^v : \mathbb{E}[(\Phi_\epsilon(x) - \Phi_\epsilon(y))^2] \leqslant r^2 \right\} \supset \left\{ y \in \square_\epsilon^v : Cp^{d/2}\epsilon^{-1/2}|y-x|_2^{1/2} \leqslant r^2 \right\}$$

so that

$$\mu(B(x,r)) \geqslant Cr^{4d}$$

for some $C > 0$ depending on p and d.
Applying the majorizing measure technique we obtain

$$\mathbb{E}[\sup_{x \in \square_\epsilon^v} \Phi_\epsilon(x)] \leqslant C \int_0^\infty \sqrt{-\log(cr^{4d})}dr \leqslant C < \infty$$

then we complete the proof of lemma 2.2 by invoking Borell's inequality:

$$\mathbb{E}[\sup_{x \in \square_\epsilon^v} \Phi_\epsilon(x) \geqslant C + \lambda] \leqslant e^{-\lambda^2/2(2p)^{d/2}}$$

the quantity on the right results from (6.3).
We then follow exactly the same steps as in [1] (the only difference is to replace some d's by $d/2$'s because of (6.3)) to recover the main theorem.

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