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# ON THE ASYMPTOTIC BEHAVIOR OF MINIMAL SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

*by*

Benoît R. Kloeckner & Rafe Mazzeo

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**Abstract.** — We consider the asymptotic behaviour of properly embedded minimal surfaces in  $\mathbb{H}^2 \times \mathbb{R}$  taking into account the fact that there is more than one natural compactification of this space. This provides a better setting in which to consider the general problem of determining which curves at infinity are the asymptotic boundary of such minimal surfaces. We also construct some new examples of such surfaces and describe the boundary regularity.

## 1. Introduction

The study of minimal surfaces in  $\mathbb{H}^2 \times \mathbb{R}$  has received considerable attention in the past decade or more, beginning with the work of Nelli and Rosenberg [NR02, NR07]; we refer to [MRR11, Mas04, KKS09, ERR10, Mor11, Pyo11, MMR14] as well as other references cited below. This is part of a larger effort to understand minimal surfaces in each of the three-dimensional homogeneous (Thurston) geometries. The understanding of minimal surfaces in  $\mathbb{R}^3$ ,  $S^3$  and  $\mathbb{H}^3$  is fairly advanced, but by contrast, although  $\mathbb{H}^2 \times \mathbb{R}$  is the simplest non-constant-curvature geometry of this type, the behavior of minimal surfaces in it is much less well understood.

Our initial motivation in the work leading to this paper was to consider this from a much more general point of view, namely as a stepping stone toward the study of complete properly embedded minimal submanifolds in general symmetric spaces of noncompact type and arbitrary rank. The basic question is the asymptotic Plateau problem, in which one asks which curves on the asymptotic boundary of such a space can be “filled” by minimal surfaces. However, these spaces admit many useful but non-equivalent compactifications, which adds both ambiguity and complexity to this problem. Our goal here is to study some aspects of this problem in  $\mathbb{H}^2 \times \mathbb{R}$ , which is a particularly simple type of rank 2 symmetric space. Strictly speaking, this space is

reducible and so does not exhibit the true geometric complexity of a standard rank 2 space such as  $\mathrm{SL}(3)/\mathrm{SO}(3)$ ; however, it is hyperbolic in some directions and Euclidean in others, which leads to some surprising phenomena. Furthermore, it does have two interesting and natural compactifications: the ‘product’ compactification and the geodesic compactification. We explain these in some detail below, but briefly, the product compactification is the one obtained as the product of the compactifications of each of the factors,  $\overline{\mathbb{H}^2} \times \overline{\mathbb{R}}$ , while the geodesic compactification is a closed 3-ball obtained by attaching an endpoint to each infinite geodesic ray emanating from a given point  $p$ .

Nelli and Rosenberg [NR02] proved that every simple closed curve lying in the vertical boundary  $\partial\mathbb{H}^2 \times \mathbb{R}$  which is a vertical graph (over  $\partial\mathbb{H}^2 \times \{0\}$ ) is the asymptotic boundary of a minimal surface – or, as we shall say, is minimally fillable. On the other hand, [SET08] shows that many curves on this vertical boundary are not minimally fillable. We may also consider curves which reach  $\partial\mathbb{H}^2 \times \{\pm\infty\}$ , or which cross into the horizontal parts of this boundary (i.e., the caps at  $\partial\mathbb{R}$ ). We explain below a general result, Proposition 4.4, which guarantees fillability of a wide class of curves, but also present curves, in Theorem 4.5, which are minimally fillable but do not satisfy the hypotheses of Proposition 4.4.

We also consider the fillability question for curves in the geodesic boundary; this seems not to have been considered explicitly before, and was the original starting point of our work. It turns out that in fact almost no curve in this geodesic boundary is minimally fillable. Our second main result Theorem 5.1 shows that a minimal surface with a nontrivial portion of its asymptotic boundary contained in the geodesic boundary of  $\mathbb{H}^2 \times \mathbb{R}$  must oscillate so greatly that its boundary cannot be a curve, and often has nonempty interior. We present some examples which exhibit this behaviour in Theorem 5.7.

The research which led to this paper was undertaken some years ago, but for various reasons this manuscript was not completed. In the intervening time, an interesting new paper by Coskunuzer [Cos14] has appeared which addresses certain of these same questions, but from a somewhat different point of view. While there are obvious points of intersection between these two papers, the present paper contains a number of different results and perspectives which we hope will be of interest.

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## 2. Compactifications

In this first section we set some notation and describe in more detail the two compactifications of  $X$  mentioned above

**2.1. Notation.** — To simplify notation, we often write  $\mathbb{H}^2 \times \mathbb{R}$  as  $X$ . Let  $g$  be the standard product metric on this space, and  $d$  the induced distance function. The distance in the horizontal factor  $\mathbb{H}$  is denoted by  $d_{\mathbb{H}}$ .

We consider two compactifications of  $X$ : the product compactification  $\overline{X}^\times$  and the geodesic compactification  $\overline{X}^g$ . Each is obtained by adjoining to  $X$  an asymptotic boundary,  $\partial_\times X$  and  $\partial_g X$ , respectively. Given any properly embedded surface  $\Sigma \subset X$ , we let  $\overline{\Sigma}^\times$  denote the closure of  $\Sigma$  in  $\overline{X}^\times$  and then write  $\partial_\times \Sigma := \overline{\Sigma}^\times \cap \partial_\times X$ . We use the corresponding notation for the analogous sets in the geodesic compactification.

The two factors in  $X$  have rank one (i.e., the maximal flat totally geodesic subspaces are one-dimensional), hence all of their classical compactifications are equivalent. We denote these by

$$\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \partial\mathbb{H}^2 \quad \text{and} \quad \overline{\mathbb{R}} = \mathbb{R} \cup \partial\mathbb{R}.$$

**2.2. The product compactification of  $\mathbb{H}^2 \times \mathbb{R}$ .** — The product compactification of  $X$  is the product

$$\overline{X}^\times = \overline{\mathbb{H}^2} \times \overline{\mathbb{R}}$$

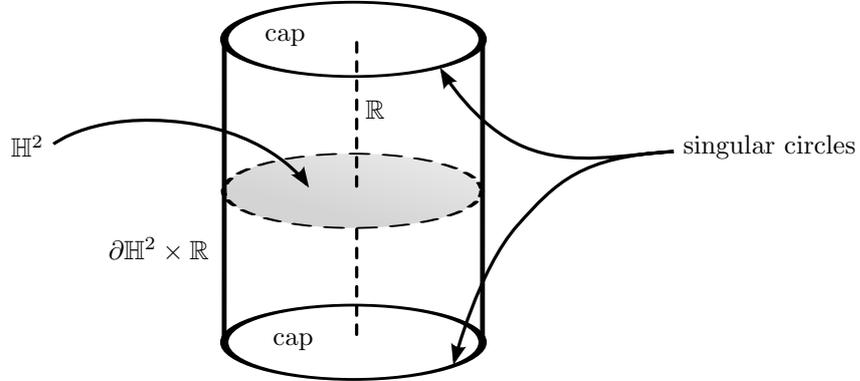
with the product topology (see Figure 1). Its boundary  $\partial_\times X = \overline{X}^\times \setminus X$  is the disjoint union of three parts:

$$\partial_\times X = (\partial\mathbb{H}^2 \times \mathbb{R}) \cup (\partial\mathbb{H}^2 \times \partial\mathbb{R}) \cup (\mathbb{H}^2 \times \partial\mathbb{R})$$

The first component, which is an open cylinder, is called the vertical boundary; the third is the union of two open disks, which we call the upper and lower caps. The middle component is the product of the boundaries of each factor and is the union of two circles. In particular,  $\overline{X}^\times$  is a manifold with boundary and corners of codimension 2.

**Remark 2.1.** — This product compactification only makes sense for a product of rank one spaces. For  $\mathbb{H}^2 \times \mathbb{R}$ , it is essentially isomorphic to more classical compactifications (Satake-Furstenberg, Chabauty, ...), but this identification is not quite true because of issues related to the one-dimensional Euclidean factor.

To be more specific, the Chabauty compactification is defined as follows: each point in  $X$  is identified with its isotropy group, and the compactification is obtained by taking the closure of the image of this map with respect to the so-called Chabauty topology on closed subgroups of the isometry group. This procedure does not distinguish between the two caps,  $\mathbb{H}^2 \times \{+\infty\}$  and

FIGURE 1. The product compactification of  $\mathbb{H}^2 \times \mathbb{R}$ .

$\mathbb{H}^2 \times \{-\infty\}$ , so these are glued together and the resulting compactification is a solid torus. Other compactifications have different undesirable features, again because of this Euclidean factor. This product compactification, although perhaps ad hoc, serves our purposes well.

**2.3. The geodesic compactification of  $\mathbb{H}^2 \times \mathbb{R}$ .** — Now recall the geodesic compactification. This is defined for any simply connected, non-positively curved manifold, see [Ebe96], but we discuss it only for  $X$ .

*2.3.1. Construction.* — We work only with geodesics with constant speed parametrizations, and include the case of speed zero, which are constant maps. A geodesic of speed 1 is called a *unit geodesic*. A *ray* is a geodesic defined on  $\mathbb{R}^+$ . The set of all rays in  $X$  is denoted by  $\mathcal{R}$  and the subset of unit rays is written  $\mathcal{R}_1$ .

Two rays  $\gamma, \sigma$  are *asymptotic* at  $+\infty$ ,  $\gamma \sim \sigma$ , if they remain at bounded distance from one from another in the future:

$$\sup_{t \geq 0} d(\gamma(t), \sigma(t)) < \infty.$$

This is an equivalence relation, and the class of a ray  $\gamma$  is denoted  $[\gamma]$ . Given  $x \in X$  and  $u \in T_x X$ , then

$$\gamma(x, u)(t) := \exp_x(tu)$$

is the unique ray with initial position  $x$  and velocity  $u$ .

The *geodesic boundary* of  $X$  is the set of classes of unit rays:

$$\partial_g X := \mathcal{R}_1 / \sim$$

endowed with the following topology. For any  $x \in X$ , there is a natural map from the unit sphere in the tangent space to this geodesic boundary:

$$\begin{aligned} \pi_x := S_x X &\rightarrow \partial_g X \\ u &\mapsto [\gamma(x, u)] \end{aligned}$$

This is a bijection, so we endow  $\partial_g X$  with the topology of  $S_x X$  under this identification. For any two points  $x, y$ ,  $\pi_x^{-1} \circ \pi_y$  is a homeomorphism, so this topology does not depend on the choice of  $x$ .

The *geodesic compactification* of  $X$  is the disjoint union

$$\overline{X}^g := X \cup \partial_g X;$$

with topology determined by the requirements that  $X$  is open in  $\overline{X}^g$  and inherits its usual topology, and a sequence  $x_n$  in  $X$  converges to a boundary point if and only if for some (and hence all)  $y \in X$ ,  $d(y, x_n) \rightarrow \infty$  and the sequence  $u_n$  of unit tangent vectors at  $y$  defined by  $x_n \in \gamma(y, u_n)$  converges to some  $u$ ; then the limit of  $x_n$  is  $[\gamma(y, u)]$ , which does not depend on  $y$ .

The topology defined earlier on  $\partial_g X$  coincides with the induced topology. With this topology,  $\overline{X}^g$  is homeomorphic to a closed ball, and the isometry group of  $X$  extend to an action by homeomorphisms on the compactification.

*2.3.2. Structure.* — Each unit ray  $\gamma$  in  $X$  can be written as  $(\gamma_1, \gamma_2)$  where  $\gamma_1$  and  $\gamma_2$  are rays in  $\mathbb{H}^2$  and  $\mathbb{R}$  with speeds  $s_1$  and  $s_2$  respectively, where  $s_1^2 + s_2^2 = 1$ . If  $\gamma' = (\gamma'_1, \gamma'_2)$  is another unit ray together with projections of speed  $s'_i$ , then  $\gamma \sim \gamma'$  if and only if  $\gamma_i \sim \gamma'_i$  for  $i = 1, 2$ . This implies that  $s_i = s'_i$  (in particular, any two constant geodesics are asymptotic to one other).

When  $s_1 = 0$ , then  $\gamma$  projects to a constant in the  $\mathbb{H}^2$  factor and we say that  $\gamma$  is *vertical*. There are two classes of vertical geodesic, those that go up and those that go down, corresponding to the two points of  $\partial\mathbb{R}$ . These are called the *poles* and denoted  $p^\pm$ .

imilarly, when  $s_2 = 0$ , then  $\gamma$  is called *horizontal*. Since  $(\gamma_1, t_0) \sim (\gamma_1, t'_0)$  for any  $t_0, t'_0 \in \mathbb{R}$ , asymptotic classes of horizontal geodesics are in bijection with points of  $\partial\mathbb{H}^2$ . The set of all horizontal classes comprises the *equator* of  $\partial_g X$ .

These special subsets of  $\partial_g X$  are artifacts of the product structure of  $X$ , and are related to the fact that this geodesic boundary has the structure of a joint. This means the following. For each  $(q, p^*) \in \partial\mathbb{H}^2 \times \partial\mathbb{R}$ ,  $* = \pm$ , there is a segment in  $\partial_g X$  which consists of all classes  $[(\gamma_1, \gamma_2)]$  where either  $[\gamma_1] = q$  or  $\gamma_1$  is constant, and in addition,  $[\gamma_2] = p^*$  or  $\gamma_2$  is constant. This segment is parametrized by the ratio of speeds  $\rho = s_2/s_1 \in \overline{\mathbb{R}^+}$  and connects a point of the equator to one of the poles. It is called a *Weyl chamber* and denoted  $W^*(q)$ ,  $* = \pm$ . The set of all Weyl chambers is the union of two circles, the

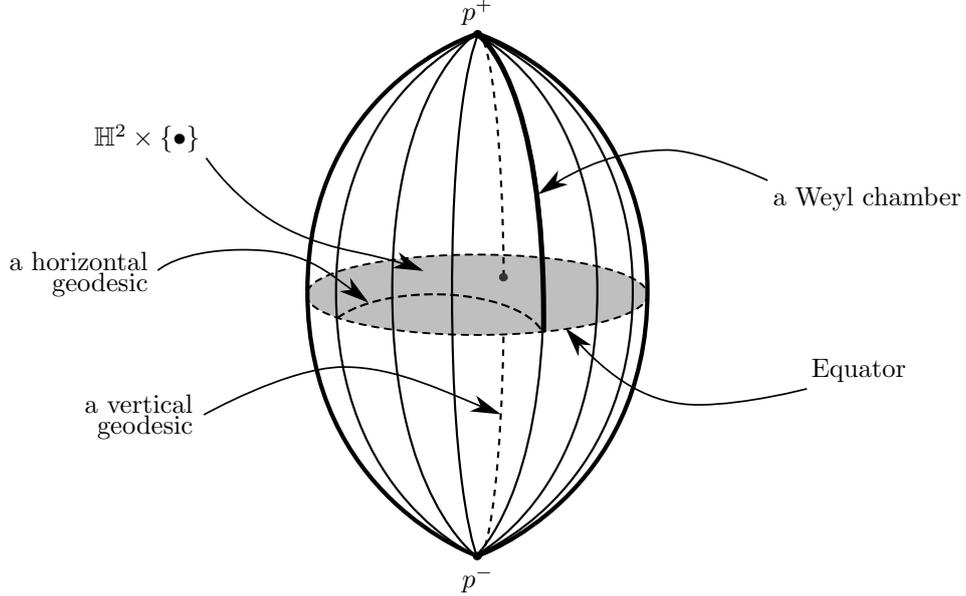


FIGURE 2. The geodesic compactification of  $\mathbb{H}^2 \times \mathbb{R}$ . Note that all horizontal planes  $\mathbb{H}^2 \times \{\bullet\}$  are asymptotic, i.e. have their boundary equal to the same curve, the equator.

set of midpoints of Weyl chambers, where  $\rho = 1$ , is again the union of two circles, and can be identified with the Furstenberg boundary.

One fact which will play an important role later is that vertical translation acts trivially on  $\partial_g X$ .

*2.3.3. Relationship between  $\overline{X}^\times$  and  $\overline{X}^g$ .* — These two compactifications are related in the following way. Briefly,  $\overline{X}^g$  is obtained from  $\overline{X}^\times$  by blowing up the corner and blowing down the top and bottom caps as well as the vertical cylinder. Similarly, to get to  $\overline{X}^\times$  from  $\overline{X}^g$ , we blow up the equator and the poles and blow down each of the joint segments to points.

In slightly more detail, the blowdown of the top and bottom caps of  $\overline{X}^\times$  is the space where sequences  $q_n \in X$  converging to any point of the top cap in the product compactification are identified with one another, and hence correspond to sequences converging to  $p^+$  in the geodesic compactification. In a similar way, any point  $(q, t) \in \partial\mathbb{H} \times \mathbb{R}$  is mapped to the point  $q$  on the equator of  $\partial_g X$ . In the other direction, points in the corner  $\partial\mathbb{H} \times \partial\mathbb{R}$  are stretched into the Weyl chambers, which thus captures the direction of approach to infinity (i.e., the ratio of speeds) for geodesics  $(\gamma_1, \gamma_2)$  which escape to infinity in both factors.

From all of this, we see that points of  $\partial_\times X$  and  $\partial_g X$  correspond to different ways of distinguishing classes of diverging geodesics.

### 3. Embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$

In this section we write out the PDE for minimal surfaces in  $X$  in terms of two natural graphical representations, and explain certain boundary regularity theorems. After that we recall a number of important examples of these surfaces which will enter our later constructions.

**3.1. Equations of horizontal and vertical minimal graphs.** — In the following, use the half-space model for  $\mathbb{H}^2$  with coordinates  $z = (x, y)$ ,  $x > 0$ ,  $y \in \mathbb{R}$  and metric  $(dx^2 + dy^2)/x^2$ , as well as the corresponding coordinates  $(x, y, t) \in X$  and metric

$$g = \frac{dx^2 + dy^2}{x^2} + dt^2$$

on  $X$ .

There are two obvious ways to represent a surface in  $X$ , namely as a graph over a horizontal slice  $\mathbb{H} \times \{0\}$ , or as a graph over a vertical flat  $F = \gamma \times \mathbb{R}$  where  $\gamma$  is a geodesic in  $\mathbb{H}^2$ . In the former case, we write

$$\Sigma = \{(z, u(z)) : z \in \mathbb{H}^2\},$$

while in the later, if  $F = \{y = 0\}$ , then

$$\Sigma = \{(x, u(x, t), t) : x > 0, t \in \mathbb{R}\}.$$

The only nonvanishing Christoffel symbols in these coordinates are

$$\Gamma_{xx}^x = -\frac{1}{x}, \quad \Gamma_{yy}^x = \frac{1}{x}, \quad \Gamma_{yx}^y = \Gamma_{xy}^y = -\frac{1}{x}.$$

*3.1.1. Vertical graphs.* — First suppose that  $\Sigma$  is the graph of some function  $u(x, y)$  defined over an open set  $\mathcal{U} \subset \mathbb{H}^2$ . Then it is standard, see [SET08] that the graph of  $u$  is minimal if and only if

$$\operatorname{div}^g \left( \frac{\nabla^g u}{\sqrt{1 + |\nabla^g u|_g^2}} \right) = 0.$$

where the divergence and gradient are taken with respect to the hyperbolic metric. Expanding in terms of the Euclidean metric  $g_0$ , this is the same as

$$(1) \quad \Delta_{g_0} u (1 + x^2 |\nabla^{g_0} u|_{g_0}^2) - x^2 \sum_{i,j} u_{ij} u_i u_j - x u_x |\nabla^{g_0} u|_{g_0}^2 = 0,$$

or finally, in upper half-space coordinates,

$$(2) \quad u_{xx}(1 + x^2 u_y^2) + u_{yy}(1 + x^2 u_x^2) - 2x^2 u_{xy} u_x u_y - x u_x (u_y^2 + u_x^2) = 0.$$

Solutions of this equation which are defined over all of  $\mathbb{H}^2$  are called *vertical graphs*; this parlance is perhaps confusing inasmuch as the ends of these are horizontal.

Since (2) is uniformly elliptic, even near  $x = 0$ , we deduce the following, see [GT98, Theorem 15.11].

**Proposition 3.1.** — *Let  $\mathcal{U}$  be an open set in  $\overline{\mathbb{H}^2}$  which intersects  $\partial\mathbb{H}^2$  in an open interval. Let  $u(x, y)$  be a function defined on  $\mathcal{U}$  which satisfies (1). If  $u(0, y) \in C^{k, \alpha}$  for any  $k \in \mathbb{N}$  and  $0 < \alpha < 1$ , then  $u$  is  $C^\infty$  in  $\mathcal{U} \cap \mathbb{H}^2$  and  $C^{k, \alpha}$  on  $\mathcal{U}$  up to  $x = 0$ .*

*3.1.2. Horizontal graphs.* — Now suppose that  $\Sigma$  is a graph over a flat  $F = \gamma \times \mathbb{R}$ , where as before,  $F = \{y = 0\}$ . We write this initially as  $\{(x, y, t) : y = v(x, t)\}$ . Very similar calculations to the ones above show that this graph is minimal if and only if

$$(3) \quad v_{xx}(x^2 + v_t^2) + v_{tt}(1 + v_x^2) - 2v_{xt}v_xv_t - xv_x(1 + v_x^2) = 0.$$

Solutions are called horizontal graphs (noting, however, that these surfaces have vertical ends).

This equation is nondegenerate at  $x = 0$  only when  $v_t(0, t) \neq 0$ . However, if this condition were to hold, then the surface could also be written locally as a vertical graph, and its regularity near such portions of the boundary would therefore follow already from Proposition 3.1.

We restrict, therefore, to consideration of the special case where  $v(0, t) \equiv 0$  for  $t$  lying in some interval  $\mathcal{I}$ . This excludes cases where  $v_t(0, t_0) = 0$  at some isolated point  $t_0$ , as well as endpoints of vertical boundary regions, i.e. values  $t_0$  such that  $v(0, t) = 0$  for  $t_0 - \epsilon < t \leq t_0$  and  $v'(0, t) \neq 0$  for  $t_0 < t < t_0 + \epsilon$ . If  $|\mathcal{I}| > \pi$ , then there is an immediate a priori estimate for the behavior of  $v$  near  $x = 0$  which is obtained by trapping  $\Sigma$  between barriers on either side. As barriers we use the ‘tall rectangles’ which are described in the next section. (These are minimal disks which intersect  $\partial_\times X$  in the union of two vertical lines of height greater than  $\pi$  and two circular arcs connecting their endpoints; these make a variable angle of contact along the vertical portions of their boundary.) This geometric argument yields a Lipschitz bound for  $v$  as an immediate corollary.

**Proposition 3.2.** — *Suppose that some portion of the boundary of the minimal surface  $\Sigma$  is a vertical line  $\mathcal{I} \subset \partial_\times X$  with  $|\mathcal{I}| > \pi$ . Writing  $\Sigma$  as a horizontal graph of a function  $v$  defined on a vertical flat which contains  $\mathcal{I}$  in its boundary, then for each compact subinterval  $\mathcal{I}'$  in  $\mathcal{I}$  there exists a constant  $C > 0$  such that  $|v(x, t)| \leq Cx$ .*

Proposition 4.4 below provides a large class of examples of minimal surfaces which have boundary containing a vertical line segment.

Unfortunately it seems difficult to show that  $\Sigma$  is actually smooth up to a vertical boundary segment. The reason is that the particular type of degeneracy in equation (3) only permits good regularity results in certain restricted cases, typically where the variable  $t$  lies in a compact manifold with boundary (e.g. the circle) or else if we already know quite a bit more about the values of the function  $v(x, t)$  along the curves  $t = t_0$  and  $t = t_1$  at the top and bottom of the interval  $\mathcal{I}$ . We can, however, show that  $v$  is *conormal* at  $x = 0$ , which is slightly weaker regularity statement. We explain this now.

First let us reparametrize  $\Sigma$  by writing it as the exponential of  $w\nu$ , where  $\nu = x\partial_y$  is the *unit* normal to  $F$ . It is not hard to check that  $w = xv + \mathcal{O}(x^2)$ . The Lipschitz bound in Proposition 3.2 shows that  $\Sigma$  remains a bounded distance from  $F$  even up to  $x = 0$ , so  $w(x, t)$  satisfies  $|w| \leq C$ . Next, change variables by setting  $s = -\log x$ , which means that we regard  $w$  as a function of  $(s, t)$ . Thus  $w$  satisfies the minimal surface equation  $\mathcal{M}(w) = 0$  over an infinite region  $\mathcal{S} = [s_0, \infty) \times [t_0, t_1]$ .

In these coordinates, the minimal surface operator  $\mathcal{M}$  is a quasilinear elliptic operator with uniformly bounded coefficients, see [GT98], and it follows from classical estimates (applied on any ball of some fixed radius  $r_0$  in the  $(s, t)$  coordinates) there that the function  $w$  and all its derivatives are uniformly bounded, i.e.,  $|\partial_s^p \partial_t^q w| \leq C_{p,q}$  for every  $p, q \in \mathbb{N}$ . Equivalently, since  $\partial_s = x\partial_x$ ,  $|(x\partial_x)^p \partial_t^q w| \leq C_{p,q}$  for all  $p, q$ . However, this collection of estimates is just the definition of what is known as *conormal* regularity of order 0 at the boundary  $x = 0$ . The graph function  $v$  is conormal of order 1, i.e.,  $|(x\partial_x)^p \partial_t^q v| \leq C_{p,q}x$  for all  $p, q$ . We summarize all of this in the

**Proposition 3.3.** — *Let  $\Sigma$  be a complete minimal surface which contains a vertical interval  $\mathcal{I}$  in its asymptotic boundary. Write  $\Sigma$  as a horizontal graph, with graph function  $v(x, t)$ , and assume that  $|v(x, t)| \leq Cx$  (this condition is automatic by the barrier construction if  $|\mathcal{I}| > \pi$ ). Then for all  $p, q \in \mathbb{N}$ ,*

$$|(x\partial_x)^p \partial_t^q v(x, t)| \leq C_{p,q}x,$$

*i.e.,  $v$  is conormal of order 1.*

Now consider the problem of determining whether  $\Sigma$  is actually smooth up to  $\partial_\times X$ ; this corresponds to the assertion that  $w$  has an asymptotic expansion  $\sim \sum_{j \geq 0} e^{-js} w_j(t)$  where each  $w_j \in \mathcal{C}^\infty$ . Suppose that we are in the even more restricted case where  $\Sigma$  is locally trapped between portions of two flats which intersect along  $\mathcal{I}$ . This means that  $|w| \leq Ce^{-s}$ . We can then study the minimal surface equation for  $w$  perturbatively; in other words, we use the Taylor expansion of  $\mathcal{M}$  at 0, which can be written as

$$Lw = Q(w, \nabla w, \nabla^2 w),$$

where  $L = \partial_s^2 + \partial_t^2 - 1$  is the Jacobi operator along  $F$  and  $Q$  is a quadratic remainder term. The same classical local elliptic theory (in the  $(s, t)$  coordinates) shows that all higher derivatives of  $w$  satisfy the same bounds as  $w$  itself, i.e.,  $|\partial_s^p \partial_t^q w| \leq C_{p,q} e^{-s}$  uniformly in any semi-infinite strip  $\mathcal{S}' \subset \mathcal{S}$ . Relabelling  $\mathcal{S}'$  as  $\mathcal{S}$ , we study this by first considering the inhomogeneous problem  $Lw = f$  where  $|f| \leq C e^{-2s}$  (along with similar estimates for its higher derivatives). Simplifying even further, consider the homogeneous problem

$$Lw = 0, \quad w(t_0, s) = w_0(s), \quad w(t_1, s) = w_1(s).$$

If we happen to know that  $w_0, w_1$  admit asymptotic expansions in integer powers of  $e^{-s}$  as  $s \nearrow \infty$ , then it is straightforward to show that the same is true for  $w(s, t)$  for  $t \in \mathcal{I}$ . A small modification of this argument gives the same conclusion for the nonlinear equation. However, our a priori information is only that  $|w_0|, |w_1| \leq C e^{-s}$ , and this is not enough information to reach the desired conclusion, even in a slightly smaller strip. In fact, it is not hard to show that if  $w_0$  and  $w_1$  decay exponentially but do not have such smooth expansions, then neither does the solution  $w$ . However, it is unclear whether such a phenomenon can happen in our setting, i.e., when  $\Sigma$  is a complete minimal surface, and it is entirely possible that  $\Sigma$  is actually smooth up to vertical boundaries, but this remains an open question.

Notice that we are not making any claim about the regularity of this graph at the horizontal caps  $\mathbb{H}^2 \times \{\pm\infty\}$ . We discuss this point later.

**3.2. Examples.** — We review here the basic examples of properly embedded minimal surfaces in  $X$ . These give some intuition about boundary behavior of more general surfaces of this type, and some of these will also be used as barriers in constructions below.

*3.2.1. Horizontal disks and vertical planes.* — The simplest examples of properly embedded minimal surfaces are the ones which respect the product structure of  $X$ . These are the horizontal disks

$$\mathbb{H}^2 \times \{a\},$$

for any  $a \in \mathbb{R}$ , and the vertical planes

$$F = \gamma \times \mathbb{R},$$

where  $\gamma$  is a geodesic in  $\mathbb{H}^2$ . Note that each of these is not only minimal, but totally geodesic. The horizontal planes have Gauss curvature  $K \equiv -1$ , while the flats have  $K \equiv 0$ .

*3.2.2. Tall rectangles.* — The next example is a two-parameter family of minimal surfaces in  $X$ , each element of which has asymptotic boundary which is a finite ‘rectangle’ in the vertical boundary of  $\partial_\times X$ . Limiting elements of this family are semi-infinite or infinite rectangles; the semi-infinite ones have

asymptotic boundary which includes an entire geodesic in one of the two horizontal components of  $\partial_\times X$ , while the infinite ones are simply flats,  $F = \gamma \times \mathbb{R}$ . This family was initially described by Hauswirth [Hau06] and independently Sa Earp and Toubiana [SET08]. These surfaces were first used as a very useful set of barriers in [MRR14]. The name ‘tall’ is due to Coskunuzer [Cos14] and refers to the fact that these only exist when their height is greater than  $\pi$ .

Fix any arc  $c \subset \partial\mathbb{H}^2$  and denote its endpoints by  $q_1$  and  $q_2$ ; choose  $a, b \in \overline{\mathbb{R}}$  with  $a < b$  and  $\ell := b - a > \pi$ . There is a unique minimal surface  $H = H(c, a, b) \subset X$  which is a horizontal graph over the rectangle  $c \times [a, b]$ . This surface  $H$  has the following properties. Let  $\gamma$  be the geodesic in  $\mathbb{H}^2$  which terminates at  $q_1$  and  $q_2$ . Then  $H$  is contained in the half-space bounded by  $F = \gamma \times \mathbb{R}$ . It is invariant under the one-parameter group of isometries of  $\mathbb{H}^2$  which fix  $q_1$  and  $q_2$ ; these are isometries of hyperbolic type, and extend in a natural way to isometries of  $X$  leaving each level  $\{t = \text{const}\}$  fixed.

Because of this isometry invariance, it is not hard to see that the intersection of  $H$  with any horizontal slice  $\mathbb{H}^2 \times \{t\}$  is a curve in that copy of  $\mathbb{H}^2$  which is equidistant from the geodesic  $\gamma$ . Denoting its distance from  $\gamma$  by  $\rho(t)$ , then  $\rho(t) = \infty$  at  $t = a, b$ , corresponding to the fact that  $H \cap (\mathbb{H}^2 \times \{a\}) = c \times \{a\}$ , and similarly at  $t = b$ . For the limiting case  $a = -\infty, b = \infty$ , we can take  $\rho(t) \equiv 0$ , so that  $H_{c, -\infty, \infty}$  is the flat  $F = \gamma \times \mathbb{R}$ . When  $-\infty < a$  but  $b = \infty$ , then  $\rho(t)$  decreases monotonically from  $\infty$  to 0 as  $t$  increases from  $a$  to  $\infty$ . When  $a$  and  $b$  are both finite, then  $\rho(t)$  is proper and convex on the interval  $(a, b)$ , and obviously symmetric around the midpoint  $t = (a + b)/2$ . Finally, this minimum value of  $\rho$  tends to  $\infty$  as the overall height  $\ell$  of the boundary rectangle decreases to  $\pi$ .

All of this can be deduced from an explicit expression for the function  $\rho(t)$  involving integrals, see [SET08]. Figure 3 illustrates some members of this family of surfaces.

If  $a$  and  $b$  are finite, then the asymptotic boundary of  $H(c, a, b)$  in the product compactification is the rectangle with four segments:

$$\partial_\times H(c, a, b) = (c \times \{a, b\}) \cup (\{q_1, q_2\} \times [a, b]).$$

Similarly,

$$\partial_\times H(c, a, \infty) = (c \times \{a\}) \cup (\{q_1, q_2\} \times [a, \infty]) \cup (\gamma \times \{\infty\}),$$

and

$$\partial_\times H(c, -\infty, \infty) = (\{q_1, q_2\} \times \overline{\mathbb{R}}) \cup (\gamma \times \{\pm\infty\}).$$

In the geodesic compactification, on the one hand when  $a$  and  $b$  are finite,  $\partial_g H(c, a, b)$  is simply the arc  $c$  on the equator. On the other hand  $\partial_g H(c, a, \infty)$  is the union of  $c$  and of the two Weyl chambers connecting the endpoints of  $c$  to the north pole  $p^+$ ,  $\partial_g H(c, -\infty, b)$  is the union of  $c$  and of the two Weyl

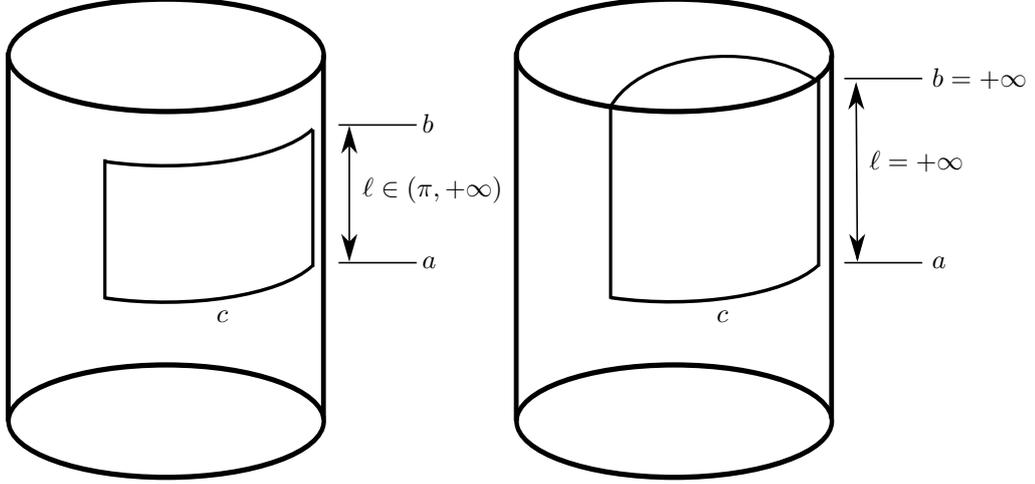


FIGURE 3. The boundary of tall rectangles in the product compactification.

chambers connecting points of  $c$  both to  $p^+$  and to  $p^-$ , and  $\partial_g H(c, -\infty, \infty)$  is the union of the four Weyl chambers having  $q_1$  or  $q_2$  as endpoints (including  $p^\pm, q_1, q_2$ ).

The expressions for these surfaces are written out in the disk model in [SET08], but we find it more useful here to write them in the half-plane model. We do so using the minimal surface equation for vertical graphs.

Use coordinates  $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$  and suppose that the boundary arc is the segment between  $q_1 = (0, 1)$  and  $q_2 = (0, -1)$ . Also let  $a = 0$ . The circular arcs with endpoints  $q_1, q_2$ , which correspond to curves equidistant from the geodesic connecting  $q_1$  and  $q_2$ , are described by

$$s(x, y) := \frac{1 - (x^2 + y^2)}{2x} = \text{const.}$$

We seek graphs  $t = u(x, y)$  with  $u(x, y) = f(s(x, y))$ . When  $b < \infty$ ,  $u$  is multi-valued, since the entire surface is a bigraph over a concave region in  $\mathbb{H}^2$  bounded by an equidistant curve. When  $b = \infty$ , the surface is a single-valued graph over a half-space in  $\mathbb{H}^2$ .

One computes

$$\begin{aligned} s_x &= -1 - \frac{s}{x}, & s_y &= -\frac{y}{x} \\ u_x &= \left(-1 - \frac{s}{x}\right) f'(s), & u_y &= -\frac{y}{x} f'(s) \end{aligned}$$

Then

$$\begin{aligned} u_{xy} &= \frac{y}{x^2}f'(s) + \frac{y}{x}\left(1 + \frac{s}{x}\right)f''(s) \\ u_{xx} &= \frac{1}{x}\left(1 + 2\frac{s}{x}\right)f'(s) + \left(1 + \frac{s}{x}\right)^2f''(s) \\ u_{yy} &= -\frac{1}{x}f'(s) + \frac{y^2}{x^2}f''(s). \end{aligned}$$

Plugging this into (2) and using that  $y^2 + x^2 + 2xs = 1$ , we obtain

$$(4) \quad 0 = 2sf' + s(1 + s^2)f'^3 + (1 + s^2)f''.$$

This is a Bernoulli equation in  $f'$ , which has solutions

$$f'(s) = \frac{-1}{\sqrt{C - 1 + (2C - 1)s^2 + Cs^4}}.$$

where  $C \in (0, 1)$ . This is impossible to integrate in explicit terms, except when  $C = 1$ , which corresponds to  $b = \infty$ , and in that case

$$f(s) = \tanh^{-1} (1 + s^2)^{-\frac{1}{2}}.$$

*3.2.3. Horizontal and vertical catenoids.* — As a final set of examples, we list the horizontal and vertical catenoids.

The horizontal catenoids  $\mathcal{C}_H(p, a, b)$  were first described by Pedrosa and Ritoré [PR99], but independently and more explicitly by Nelli and Rosenberg [NR02, NR07]. Each one of these is a surface of rotation in  $X$  around a vertical axis  $\{p\} \times \mathbb{R}$ , where the point  $p$  can be arbitrary in  $\mathbb{H}^2$ , and contained in the slab  $a \leq t \leq b$ .

Assuming that  $p = 0$  and that the catenoid is symmetric with respect to  $(t = 0)$ , i.e.  $a = -b$ , a parametrization of  $\mathcal{C}_H(p, -b, b)$  is

$$\{(r(t) \cos \theta, r(t) \sin \theta, t)\}, \quad |t| < b, \quad \theta \in S^1,$$

where  $r(t)$ , the radial coordinate in the disk model of  $\mathbb{H}^2$ , satisfies

$$r' = \pm \sqrt{Cr^2 - \frac{1 + r^4}{4}};$$

here  $C$  is a constant determined by the height  $2b$  [NR02]. These solutions exist if and only if  $C > 1/2$ , which corresponds to the height limitation  $2b < \pi$ . If  $\varphi : \mathbb{H}^2 \rightarrow \mathbb{H}^2$  is any isometry, then its extension  $(\varphi, \text{Id})$  to  $X$  acts on the space of horizontal catenoids, sending  $\mathcal{C}_H(p, a, b)$  to  $\mathcal{C}_H(\varphi(p), a, b)$ . If  $\varphi_\lambda$  is a family of hyperbolic isometries which fixes two points  $q_\pm \in \partial\mathbb{H}^2$ , then the limit of  $\mathcal{C}_H(\varphi_\lambda(p), a, b)$  is the union of two horizontal hyperbolic planes  $\mathbb{H}^2 \times \{a, b\}$ ; the neck has disappeared at infinity. There is a different limit if one lets  $\lambda \rightarrow \infty$  and simultaneously  $b - a \nearrow \pi$ . This is called Daniel's surface [Dan09], and is a disk which has boundary along the union of two circles at distance  $\pi$  from

one another and a straight line connecting these circles. Daniel's surface can also be obtained as the limit of tall rectangles as  $b - a \searrow \pi$  and the arc  $c$  converges to the whole circle.

The vertical catenoids  $\mathcal{C}_V$  are parametrized, by contrast, by pairs of geodesics  $\gamma_{\pm} \subset \mathbb{H}^2$  such that  $\text{dist}(\gamma_+, \gamma_-)$  is less than some critical value  $\eta_0$ . Each of these geodesics determines a flat  $F_{\pm} = \gamma_{\pm} \times \mathbb{R}$ , which is a vertical plane, and  $\mathcal{C}_V(\gamma_+, \gamma_-)$  is the annular surface which is asymptotic to this pair of flats. These were first constructed in [MR12] and [Pyo11], but see also [MMR14] for a careful explanation of their geometrical properties.

In contrast to the other surfaces described above, the horizontal and vertical catenoids have disconnected boundaries.

#### 4. Minimally fillable curves on the product boundary

We now describe broader classes of properly embedded minimal surfaces in  $X$  which generalize the examples above in various ways. Much of this summarizes previously known results, but we present a few new results too.

The general motivation for these questions here is the basic one, to determine which curves in  $\partial_{\times} X$  or  $\partial_g X$  occur as boundaries of properly embedded minimal surfaces  $\Sigma \subset X$ . We call any such curve *minimally fillable*. By curve, we implicitly mean a finite union of disjoint *Jordan curve*, i.e., of topological embedding of the circle; the word *arc* signifies a continuous image of an interval.

We also discuss the more restrictive problem of characterizing curves  $\sigma$  for which the filling is not only minimal but area-minimizing, which means that any compact portion of this surface is absolutely area-minimizing.

**4.1. Barriers and minimally fillable curves.** — Most of the existence theorems rely on some version of the following folklore result:

**Proposition 4.1.** — *Let  $\sigma \subset \partial_{\times} X$  be a curve, and assume that every  $p \in \partial_{\times} X \setminus \sigma$  can be separated from  $\sigma$  by the boundary  $\partial_{\times} \Sigma_p$  of a properly immersed minimal surface  $\Sigma_p$ . Then  $\sigma$  is minimally fillable.*

The proof of this proceeds as follows. Let  $\sigma_j$  be a sequence of curves in  $X$  which approach  $\sigma$ , and for each  $j$ , let  $\Sigma_j$  be a solution of the Plateau problem (or indeed any other minimal surface) with  $\partial \Sigma_j = \sigma_j$ . The key step in showing that the  $\Sigma_j$  converge to a solution of our problem is to ensure that  $\Sigma_j$  does not leave every compact set as  $j \rightarrow \infty$ . However, regarding  $\Sigma_j$  in  $\bar{X}^{\times}$ , we see that it is impossible for this sequence to have any limit points  $p$  which do not lie on  $\sigma$ , since the surfaces  $\Sigma_p$  serve as barriers which prevent  $\Sigma_j$  from having  $p$  as a limit point. Since  $\sigma$  is a curve, any sequence of surfaces escaping all

compacts and whose boundaries approach  $\sigma$  must have an accumulation point outside  $\sigma$ , completing the proof.

A recent result by Coskunuzer [Cos14] settles part of the problem of characterizing curves which are contained in the vertical boundary of  $\partial_\times X$  and which are fillable by *minimizing* rather than just *minimal* surfaces. To state his result, we recall his terminology that a curve  $\sigma \subset \partial\mathbb{H}^2 \times \mathbb{R}$  is called *tall* if  $(\partial\mathbb{H}^2 \times \mathbb{R}) \setminus \sigma$  is a union of tall rectangles, i.e., rectangles of the form  $c \times [t_1, t_2]$ , where  $c$  is an arc in  $\partial\mathbb{H}^2$  and  $t_2 - t_1 > \pi$ . Next, define the height  $h(\sigma)$  of the curve  $\sigma$  to be the infima of lengths of the bounded components of  $(\{\theta_0\} \times \mathbb{R}) \cap \sigma$ . He proves the following

**Proposition 4.2** ([Cos14]). — *A (possibly disconnected) curve  $\sigma \subset \partial\mathbb{H}^2 \times \mathbb{R}$  with  $h(\sigma) \neq \pi$  is the boundary of a properly embedded area-minimizing surface  $\Sigma$  if and only if  $\sigma$  is tall.*

The proof of existence when  $\sigma$  is tall is much the same as above: take a sequence of curves  $\sigma_i \subset X$  converging to  $\sigma$ , and find an area-minimizing surface  $\Sigma_i$  with  $\partial\Sigma_i = \sigma_i$  for each  $i$ . One then uses tall rectangles as barriers to show that some subsequence of the  $\Sigma_i$  converge to a solution of the problem. Nonexistence when  $\sigma$  is short is proved by a cut and paste argument which uses strongly the fact that one is seeking a minimizing filling  $\Sigma$ .

This result does not guarantee that the surface  $\Sigma$  has only one component; for example, if  $\sigma$  consists of a pair of horizontal circles which are sufficiently far apart, then the (unique) minimizing surface bounded by them is the pair of horizontal disks.

The criteria for existence of minimal fillings are clearly different: to return to the example where  $\sigma = S^1 \times \{t_1, t_2\}$ , suppose now that  $t_2 - t_1 < \pi$  so that  $\sigma$  is short. Then there is no minimizing filling, but on the other hand there is a minimal catenoid  $\Sigma$  with  $\partial_\times \Sigma = \sigma$ .

Many interesting questions remain open. We discuss a few of these after presenting several types of existence results.

**4.2. Vertical and horizontal graphs.** — The direct generalization of the family of horizontal disks  $\mathbb{H}^2 \times \{a\}$  are the vertical graphs. It is reasonable to expect a general existence theorem for solutions of the minimal surface equation for vertical graphs over  $\mathbb{H}^2 \times \{0\}$ , based on the well-known solvability of the asymptotic Plateau problem for harmonic functions on  $\mathbb{H}^2$ , and this is indeed the case. Let  $u_0 : S^1 \rightarrow \mathbb{R}$  be any  $\mathcal{C}^0$  function. Nelli and Rosenberg [NR02] proved that there exists a solution  $u : \mathbb{H}^2 \rightarrow \mathbb{R}$  to (2), such that  $u$  extends continuously to  $\overline{\mathbb{H}^2}$  with  $u|_{\partial\mathbb{H}^2} = u_0$ . This solution is unique. By Proposition 3.1,  $u \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{H}^2}) \cap \mathcal{C}^\infty(\mathbb{H}^2)$  if  $u_0 \in \mathcal{C}^{k,\alpha}(S^1)$ .

The analogous problem of finding solutions which are graphs over vertical flats  $F = \gamma \times \mathbb{R}$  is less developed. One point is that the possibly minimally fillable boundaries are more constrained. We have already quoted Coskunuzer's result, Proposition 4.2, but an earlier result by Sa Earp and Toubiana [SET08] presents a rather odd restriction, that if  $\sigma$  has a 'thin tail', then  $\sigma$  has no minimal filling. By definition a thin tail in  $\sigma$  is an open subarc  $c \subset \sigma$  which lies entirely on one side of some vertical line  $\{p\} \times \mathbb{R}$ ,  $p \in \partial\mathbb{H}^2$  except at some interior point or segment of the arc where it intersects this line, and such that  $c$  lies within a horizontal slab  $\partial\mathbb{H}^2 \times [t_1, t_2]$  with  $t_2 - t_1 < \pi$ .

Another restriction concerns the behavior of horizontal graphs near the top and bottom caps  $\mathbb{H}^2 \times \{\pm\infty\}$ .

**Proposition 4.3.** — *Suppose that  $\sigma \subset \partial_\times X$  is a curve, and denote by*

$$\sigma^\pm = \{p \in \mathbb{H}^2 \mid (p, \pm\infty) \in \sigma\}$$

*the intersections of  $\sigma$  with the upper and lower caps. If  $\sigma$  is minimally fillable, then  $\sigma^\pm$  are unions of disjoint geodesics in  $\mathbb{H}^2$ .*

*Proof.* — Let  $\Sigma$  be an embedded minimal surface with  $\sigma = \partial_\times \Sigma$ . Setting  $T_s(x, y, t) = (x, y, t - s)$ , then write  $\Sigma_s = T_s(\Sigma)$ .

By definition of  $\bar{X}^\times$ , if  $p$  is any point in  $\sigma^+$  and  $s_j$  any sequence which tends to infinity, then there exist  $p_j \in \mathbb{H}^2$  such that  $(p_j, 0) \in \Sigma_{s_j}$  and  $p_j \rightarrow p$ . Clearly,  $\Sigma_{s_j}$  converges to a complete minimal surface  $\Sigma^+$  which contains  $(p, 0)$ , and since this is true for any  $p \in \sigma^+$  for the same sequence  $s_j$ , the surface  $\Sigma^+$  contains  $\sigma^+$ . Replacing the sequence  $s_j$  by  $s_j + a$  for any fixed  $a \in \mathbb{R}$ , we deduce that  $\Sigma^+$  equals  $\sigma^+ \times \mathbb{R}$ . It is straightforward to check that this product is minimal if and only if  $\sigma^+$  be a union of disjoint geodesics.

The same conclusion obviously holds for  $\sigma^-$ . □

The two restrictions we have now seen are equivalent to the following. Using upper half-plane coordinates on  $\mathbb{H}^2$ , if  $\Sigma$  is a minimal graph over  $F = \{y = 0\}$  with graph function  $v(x, t)$ ,  $v(\cdot, t)$  tends to a hyperbolic geodesic as  $|t| \rightarrow \infty$ , and moreover, the restriction  $v(0, t)$  cannot have any local maxima or minima in any interval  $[t_1, t_2]$  with  $t_2 - t_1 < \pi$  (this is the nonexistence of thin tails). These suggest that it may not be easy to formulate sharp conditions for the existence of minimal horizontal graphs.

There are, however, some interesting nontrivial solutions.

**Proposition 4.4.** — *Suppose that  $\sigma \subset \partial_\times X$  is a curve such that each of the arcs  $\sigma^\pm$  at the top and bottom caps are either empty or a single complete geodesic in  $\mathbb{H}^2$ , and in addition, for each vertical line  $L \subset \partial_\times X$ , the components of  $L \setminus \sigma$  are intervals of length greater than  $\pi$ , then  $\sigma$  is minimally fillable.*

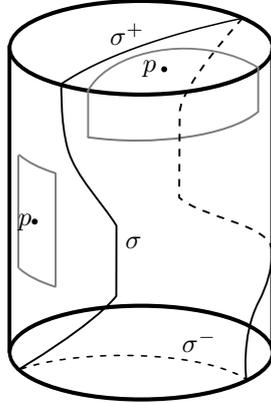


FIGURE 4. The curve  $\sigma$  is fillable, since the points  $p \notin \sigma$  can be enclosed by tall rectangles.

As an example,  $\sigma$  could be the union of two geodesics on the upper and lower caps along with two arcs lying along  $\partial\mathbb{H}^2 \times \mathbb{R}$  which are monotone with respect to  $t \in \mathbb{R}$  and which connect the respective endpoints of these geodesics, and which do not have thin tails, see Figure 4.

*Proof.* — We apply Proposition 4.1. To do so, we must show that if  $p \in \partial_\times X \setminus \sigma$ , then there is a curve which separates  $p$  from  $\sigma$  and which is minimally fillable.

If  $p \in \partial\mathbb{H}^2 \times \mathbb{R}$ , let  $L$  be the vertical line containing  $p$  and  $\{q\} \times (a, b) \subset L$  the connected component of  $L \setminus \sigma$  containing  $p$ . By hypothesis  $b - a > \pi$ , so there is a small arc  $c \ni q$  and a tall rectangle  $\partial_\times H(c, a + \varepsilon, b - \varepsilon)$  separating  $p$  from  $\sigma$ , as desired.

Next, assume  $p \in \overline{\mathbb{H}^2} \times \{\infty\}$ , say. If  $\sigma^+$  is empty, we can separate  $p$  from  $\sigma$  by a horizontal circle  $\partial_\times \mathbb{H}^2 \times \{t\}$ , which is fillable by the horizontal disk. Otherwise, let  $\gamma$  be a geodesic in  $\mathbb{H}^2$  separating  $p$  from  $\sigma^+$ . If  $c$  is the arc of  $\partial\mathbb{H}^2$  which joins the endpoints of  $\gamma$  without meeting  $\sigma^+$ , then for sufficiently large  $a$ , the the semi-infinite tall rectangle  $\partial_\times H(c, a, \infty)$  separates  $p$  from  $\sigma$ .  $\square$

Note that the examples given by Proposition 4.4 are different than the various generalized helicoids that have been previously constructed. Indeed, there is greater flexibility here, e.g. the winding of the vertical arcs around  $\partial\mathbb{H}^2 \times \mathbb{R}$  is relatively unconstrained. While on the other hand, helicoids can cut the vertical lines into short intervals. One important difference is that the boundary of helicoids in  $\partial_\times X$  contains the entire caps at  $t = \pm\infty$ ; the examples here only make a finite number of turns, which avoids this issue.

**4.3. Contractible curves on the vertical boundary.** — Nullhomotopic curves  $\sigma$  on  $\partial\mathbb{H}^2 \times \mathbb{R}$  exhibit rather different minimal fillability properties. The tall rectangles and their fillings discussed above are basic models in this class. We have already described Coskunuzer's result, characterizing which of these have minimizing fillings, and also pointed out the basic obstruction that any curve with a thin tail has no minimal filling.

We describe here another class of examples, which we call butterfly curves. These illustrate that in this setting too it may be difficult to fully characterize the fillable nullhomologous curves. Our discovery of these curves below was one of the starting points of the present work; however, in the intervening time, Coskunuzer's paper appeared and it contains a similar class of examples.

Fix positive reals  $\ell < \pi < L$ , numbers  $a, b$  such that  $a < b$ ,  $b + \ell < a + L$  and four cyclically ordered points  $q_1, \dots, q_4$  on  $\partial\mathbb{H}^2$ . We define the butterfly curve associated to these numbers to be the concatenation  $\sigma$  of

- the vertical segments  $\{q_1\} \times [a, a + L]$  and  $\{q_4\} \times [a, a + L]$ ;
- the four horizontal arcs

$$q_1 \widehat{q_2} \times \{a, L\}, \quad q_3 \widehat{q_4} \times \{a, L\};$$

- the four vertical segments

$$\{q_2, q_3\} \times [a, b], \quad \{q_2, q_3\} \times [b + \ell, a + L];$$

- the two horizontal arcs

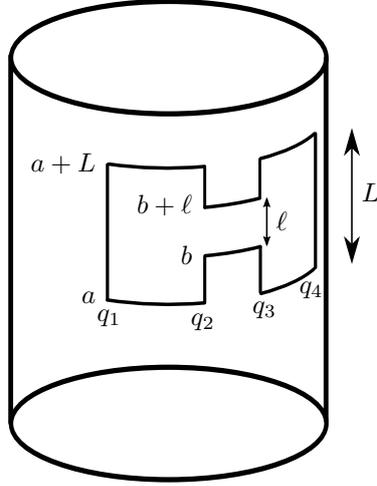
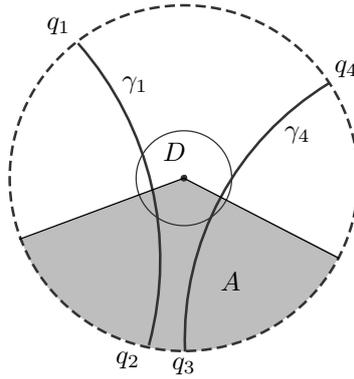
$$q_2 \widehat{q_3} \times \{b, b + \ell\}.$$

See Figure 5 for an illustration. Observe that this curve does not satisfy the criterion in Theorem 4.4 because the vertical distance between the two last horizontal arcs is less than  $\pi$ .

**Theorem 4.5.** — *For every  $\ell \in (0, \pi)$ , there exists an  $L > \pi$  and a butterfly curve with the parameters  $(\ell, L)$  which is minimally fillable.*

*Proof.* — We use a slight variant of Proposition 4.1, using a composite barrier formed by two tall rectangles and a portion of a catenoid, the interior boundary of which lies in the tall rectangles. This barrier prevents convergence of approximations to points on  $\partial\mathbb{H}^2 \times \mathbb{R}$  inside  $\sigma$ ; the tall rectangles alone provide similar barriers for points outside  $\sigma$ .

So, let  $\gamma_1$  and  $\gamma_2$  be the geodesics in  $\mathbb{H}^2$  which connect  $q_1, q_2$ , and  $q_3, q_4$ , respectively. Next, choose a horizontal catenoid  $C$  of height  $\ell$ , the projection of which onto  $\mathbb{R}$  is the interval  $[b, b + \ell]$  and such that the disk  $D \subset \mathbb{H}^2$  which is the complement of the projection of  $C$  to the  $\mathbb{H}^2$  factor intersects both  $\gamma_1$  and  $\gamma_2$ , see Figure 6. Note that such choices can be made as soon as the hyperbolic diameter of  $D$  is larger than the distance between  $\gamma_1$  and  $\gamma_2$ , so that there is a


 FIGURE 5. An  $(\ell, L)$  butterfly curve.

 FIGURE 6. Construction of the composite barrier, viewed projected in  $\mathbb{H}^2$ .

trade-off between the parameter  $\ell$  and the cross-ratio of  $q_1, q_2, q_3, q_4$ : to realize a small  $\ell$ , one has to start with  $q_1$  close to  $q_4$  or  $q_2$  close to  $q_3$ .

Finally, let  $A$  be the angular section with the same center as  $D$ , and such that the interior boundary of  $C \cap (A \times \mathbb{R})$  lies on the same side of the flats  $\gamma_1 \times \mathbb{R}$  and  $\gamma_4 \times \mathbb{R}$  as the arcs  $\widehat{q_1 q_2}$  and  $\widehat{q_3 q_4}$ , respectively.

Now choose  $L$  sufficiently large so that the tall rectangles  $H, H'$  of height  $L$  with the same plane of symmetry as  $C$  approximate these flats closely enough so that these too separate the interior boundary of  $C'$  from the center of  $D$ . This is always possible, but depending on the choices of  $\ell$  and  $q_1, q_2, q_3, q_4$ , the

diameter of  $D$  can be barely larger than the distance between  $\gamma_1$  and  $\gamma_2$ , so that it might be necessary to take a very large  $L$ .

The union of  $C'$ ,  $H$  and  $H'$  has the required properties and has this  $(\ell, L)$  butterfly curve as its product boundary.  $\square$

Note that for each  $\ell < \pi$ , one could in principle compute an explicit optimal bound  $L \geq f(\ell)$ , because the hyperbolic diameter  $d$  of  $D$  can be computed in terms of  $\ell$ , and the distance  $r$  from  $H(c, a, b)$  to its limiting flat can be computed in terms of  $L$ . Choosing  $q_1, q_2, q_3, q_4$  adequately, we only need  $2r < D$ ; but these computation would probably not yield an explicit bound, as both  $d$  and  $r$  are obtained from  $\ell$  and  $L$  through integrals without closed form.

**4.4. Horizontal annuli.** — The horizontal catenoids discussed earlier are minimal, but not minimizing, surfaces which have boundary  $\sigma = \partial\mathbb{H}^2 \times \{t_1, t_2\}$ , where  $|t_2 - t_1| < \pi$ . Conversely, any minimal surface  $\Sigma$  with  $\partial\Sigma$  equal to a pair of horizontal circles must be one of these horizontal catenoids, or a translate of one by a horizontal isometry of  $X$ , i.e., an isometry which fixes the  $t$  component, see [FMMR].

This suggests that there should be a reasonable existence theory for minimal surfaces which have boundary equal to a union of disjoint two curves  $\sigma_1$  and  $\sigma_2$ , which are each vertical graphs on  $\partial\mathbb{H}^2$ , and which are not too far apart. Recent work by the second author, Ferrer, Martin and Rodriguez [FMMR] shows that this is indeed the case. We say that the two curves  $\sigma_1$  and  $\sigma_2$  are a short pair if the separation between them along any vertical line on  $\partial\mathbb{H}^2$  is less than  $\pi$ . Then [FMMR] shows that there is an open dense set in the space of short pairs which bound a minimal annulus. However, not every short pair is minimally fillable. One example, explained in [FMMR], is if the  $\sigma_j$  are the intersections of  $S^1 \times \mathbb{R}$  with a pair of planes, one sloping up and the other down, separated by a vertical distance less than  $\pi$ .

**4.5. Vertical surfaces of finite total curvature.** — Another class of minimal surfaces in  $X$  are those which have vertical ends, i.e., ends which are asymptotic to unions of vertical flats. The prototype and building blocks for these surfaces are the vertical catenoids. The following result is proved in [MMR14].

**Proposition 4.6.** — *For each  $g \geq 0$  there is a  $k_0 = k_0(g)$  such that if  $k \geq k_0$ , then there exists a properly embedded minimal surface with finite total curvature and vertical ends in  $\mathbb{H}^2 \times \mathbb{R}$ , with genus  $g$  and  $k$  ends.*

These surfaces are obtained by gluing together vertical catenoids along vertical strips which are positioned far from the neck regions of each catenoid.

It is not yet clear what the sharp constant  $k_0(g)$  should equal. Using different gluing constructions, it now seems possible that there might exist vertical

minimal surfaces with any genus  $g$  and with only 3 ends, but this remains conjectural at present.

## 5. Minimally fillable sets on the geodesic boundary

We now study the asymptotic behavior of properly embedded minimal surfaces in  $X$  at the geodesic boundary, i.e., we study  $\partial_g \Sigma = \overline{\Sigma}^g \cap \partial_g X$ , where  $\overline{\Sigma}^g$  is the closure of  $\Sigma$  in  $\overline{X}^g$ . The main result is that  $\partial_g \Sigma$  can be quite wild, and in particular can be quite far from being a curve. We exhibit a significant restriction as to which closed subsets of  $\partial_g X$  can appear as  $\partial_g \Sigma$  for some  $\Sigma$ ; we also obtain a complete classification of the subsets of this type which are embedded curves. We also give examples, some of which show that the extreme types of behavior allowed by Theorem 5.1 actually do occur.

### 5.1. Oscillations of linear minimal surfaces. —

**Theorem 5.1.** — *Let  $\Sigma$  be a complete, properly embedded minimal surface in  $X$ . Then for any  $q \in \partial\mathbb{H}^2$ , the intersection of  $\partial_g \Sigma$  with the Weyl chamber  $W^\pm(q)$  is either empty, equal only to the pole  $p^\pm$  at the end of that Weyl chamber, or else is a closed interval containing  $q$ .*

What this result means is that if there exists a diverging sequence of points  $(q_n, t_n) \in \Sigma$  with  $\lim_n t_n/d_{\mathbb{H}}(q_n, q_0) = r$ , then for any  $r' \in [0, r]$ , there must exist another diverging sequence of points  $(q'_n, t'_n) \in \Sigma$  with  $\lim_n t'_n/d_{\mathbb{H}}(q'_n, q'_0) = r'$ . This may happen above all equatorial points  $q$  lying on some arc  $c \subset \partial\mathbb{H}^2$ , which means that  $\Sigma$  must then oscillate quite wildly in the corresponding sector of  $X$ . This behavior may seem surprising, but is known to happen for orbits of certain discrete groups of isometries. We use this observation to construct minimal surfaces exhibiting similar oscillations.

A somewhat similar oscillation phenomenon has already been noticed by Wolf [Wol07]. He produces examples of minimal surfaces which attain a given height above a ray in  $\mathbb{H}^2$  infinitely many times. The oscillations here are slightly different since the heights are growing linearly rather than being just bounded. However, it would not be surprising if the two phenomena were related.

*Proof.* — Assume that  $\partial_g \Sigma$  meets the interior of  $W^+(q)$ .

We first show that  $\partial_g \Sigma$  must contain  $q$ . If this were not the case, then there would exist a small equatorial arc  $c$  containing  $q$  and a neighborhood  $U$  of  $c$  in  $\overline{X}^g$  which does not intersect  $\Sigma$ .

Consider a hyperbolic barrier  $H = H(c, 0, \ell)$  with  $\ell$  sufficiently close to  $\pi$  to ensure that  $H$  stays far from the flat  $F$  defined by  $c$ , and hence ensures that  $H \subset U$ , hence  $H \cap \Sigma = \emptyset$ .

Now, consider the vertical translates  $H_t = H(c, t, t + \ell)$  of  $H$ . Any first contact between  $H_t$  and  $\Sigma$  would have to happen in the interior, since the geodesic boundaries of these surfaces are independent of  $t$  and hence ‘uniformly disjoint’. By the maximum principle, this interior contact is impossible, so  $\Sigma$  does not meet any of the  $H_t$ . The union  $\cup_t H_t$  equals the set of all points of distance at least  $\rho_0$  from  $F$  on the side of  $c$ . This is incompatible with the fact that  $\partial_g \Sigma$  meets the interior of  $W^+(q)$ .

We can adapt this same argument to prove that in fact,  $\partial_g \Sigma$  must meet  $W^+(q)$  along an interval containing  $q$  (see Figure 7). Indeed, otherwise there would exist some  $(q, r) \in W^+(q) \setminus \partial_g \Sigma$  and some  $r' > r$  with  $(q, r') \in \partial_g \Sigma$ . As before, there would then be a neighborhood  $\mathcal{V}$  of  $(q, r)$  in  $\overline{X^g}$  which does not intersect  $\Sigma$ . Fix an arc  $c \subset \partial \mathbb{H}^2$  containing  $q$  such that  $c \times \{r\} \subset \mathcal{V}$ . Define, for any fixed  $p_0 \in \mathbb{H}^2$ ,

$$D = \{(p, t) \mid t \geq rd_{\mathbb{H}}(p, p_0)\}, \quad \text{so that} \quad \partial D = \{(p, t) : tz = rd_{\mathbb{H}}(p, p_0)\}.$$

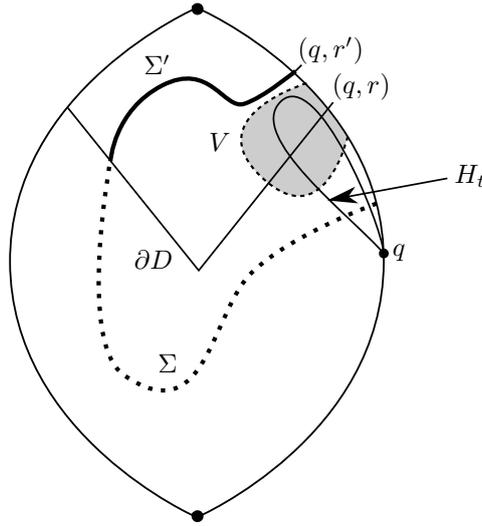


FIGURE 7. Touching the upper part of  $\Sigma$  with a hyperbolic barrier.

Replace  $\Sigma$  by  $\Sigma' = \Sigma \cap D$ ; this has an interior boundary contained in  $\partial D$  which lies outside  $\mathcal{V}$ . In particular  $\partial_g \Sigma'$  intersects  $\partial_g X$  only ‘above’ slope  $r$ , while the interior boundary of  $\Sigma'$  is disjoint from the half space determined by some subarc  $c' \subset c$ ,  $c' \ni q$ . Considering the same translating family of hyperbolic barriers  $H_t = H(c', t, t + \ell)$  as before, now with  $\ell$  small enough to ensure that  $H_0 \cap \Sigma = \emptyset$ , we see that there must be an interior first point of contact between  $\Sigma'$  and one of the  $H_t$ . This is a contradiction.  $\square$

As a straightforward consequence, we classify the curves  $\sigma \subset \partial_g X$  which are minimally fillable. As before, by curve we implicitly mean a finite union of disjoint *Jordan curve*, i.e., of topological embedding of the circle.

**Corollary 5.2.** — *Any minimally fillable curve in  $\partial_g X$  is one of the following:*

1. *the equator  $\partial\mathbb{H}^2$ ;*
2. *the union of an equatorial arc  $c = \widehat{q_1 q_2}$  and two Weyl chambers  $W^\epsilon(q_1)$ ,  $W^\epsilon(q_2)$ ,  $\epsilon \in \{+, -\}$ ;*
3. *the union of four Weyl chambers  $W^+(q_1)$ ,  $W^+(q_2)$ ,  $W^-(q_3)$  and  $W^-(q_4)$  and two disjoint equatorial arcs  $\widehat{q_2 q_3}$   $\widehat{q_4 q_1}$ , which possibly reduce to single points;*
4. *the union of two curves of type 2 with different  $\epsilon$  and disjoint  $c$ .*

In this result, the notation  $\widehat{q_1 q_2}$  denotes either of the two geodesic arcs joining  $q_1$  and  $q_2$ , and no orientation is specified. Note also that there is no requirement for the minimal surface filling a curve to be connected.

*Proof.* — We deduce first from Theorem 5.1 that any connected, minimally fillable curve  $\sigma \subset \partial_g X$  is of type 1, 2 or 3. If  $\sigma$  does not meet any Weyl chamber except along the equator, then it must be contained in the equator. Since  $\sigma$  is assumed to be a Jordan curve, it must be equal to the equator, hence we are in case 1.

Otherwise,  $\sigma$  contains an interior point of a Weyl chamber,  $(q_1, r) \in W^\epsilon(q_1)$  for some  $q_1 \in \partial\mathbb{H}^2$ ,  $\epsilon \in \{+, -\}$  and  $r \in (0, 1)$ . By Theorem 5.1,  $\sigma$  thus contains the segment  $[0, r]$  in that Weyl chamber. Since  $\sigma$  is a curve, it cannot terminate or turn (for in that case, it would contain a union of segments in Weyl chambers, and hence have interior), so it contains the entire Weyl chamber  $W^\epsilon(q_1)$ . Once it reaches  $p^\epsilon$ , it must follow another Weyl chamber  $W^\epsilon(q_2)$ . If these are the only Weyl chambers which meet  $\sigma$ , then we are in case 2; namely, to close up,  $\sigma$  must follow an equatorial arc to return from  $q_2$  to  $q_1$ .

Suppose then that  $\sigma$  meets (and hence contains) more than two Weyl chambers. Suppose we start at  $q_1$  and travel up to  $p^+$  with  $W^+(q_1)$ , then traverse  $W^+(q_2)$  down to the equator. After possibly following an equatorial arc to  $q_3$ ,  $\sigma$  must contain the chamber  $W^-(q_3)$ ; it cannot contain  $W^+(q_3)$  since in that case  $\sigma$  would self-intersect at  $p^+$ . It then contains some other  $W^-(q_4)$ , and once again to avoid self-intersections cannot contain any other Weyl chamber, hence the only alternative is that it contains the final equatorial arc connecting  $q_4$  to  $q_1$ . Thus we are in case 3.

If  $\sigma$  is not connected, then each one of its connected components is one of the types 1, 2 or 3. If one component were the equator then the other component would have to intersect it. Similarly, if one component is of type

3, then any other component would have to intersect it at one of the poles. The only possibility is that  $\sigma$  is a union of two disjoint curves of type 2. It is clearly impossible for  $\sigma$  to have more than two components.

It remains to show that each of these cases is actually realized as  $\partial_g \Sigma$ .

The geodesic boundary of any horizontal slice  $\mathbb{H}^2 \times \{t\}$  achieves case 1.

Next, case 2 is realized by the semi-infinite tall rectangles  $H(c, a, +\infty)$ , or  $H(c, -\infty, b)$ . The union of two tall rectangles with  $b < a$  is a disconnected embedded minimal surface which fills the disjoint union of two curves of this type.

Finally, noting the relationship between the product and geodesic boundaries, the minimal surfaces described in Proposition 4.4 realize case 3.  $\square$

**5.2. Examples.** — We recall a number of examples, all previously known, which illustrate Theorem 5.1 in different ways.

*Example 5.3.* — First recall the Scherk-type minimal surfaces constructed first in [NR02, Section 4], and latter by [MRR11]. These are obtained by solving a Dirichlet problem on a regular polygon with angles  $\pi/2$ , imposing either finite or infinite boundary conditions on each side, and symmetrizing (by reflection across the boundary) to obtain complete embedded surfaces. For the surfaces constructed in this way,  $\partial_g \Sigma = \partial_g X$ . For example, the surface described in of [NR02, Remark 3] contains vertical lines above every point of a lattice  $\Lambda \subset \mathbb{H}^2$ , the limit set of which is the entire boundary  $\partial\mathbb{H}^2$ , from which the assertion about  $\partial_g \Sigma$  follows.

Note that in these examples, the product boundary is also quite degenerate:  $\partial_\times \Sigma$  contains  $\partial\mathbb{H}^2 \times \mathbb{R}$ .

*Example 5.4.* — Next consider the helicoid  $\Sigma$  in  $X$ . This surface is invariant by a one-parameter group of isometries which acts by rotations on  $\mathbb{H}^2$  and simultaneously by translations in  $\mathbb{R}$ . It is not hard to see that  $\partial_g \Sigma = \partial_g X$ , but on the other hand,  $\partial_\times \Sigma$  is the union of a smooth helix in  $\partial\mathbb{H}^2 \times \mathbb{R}$  and the two caps.

Other examples with less extreme behavior have geodesic boundary which is the union of an equator and of a number of arcs in Weyl chambers.

*Example 5.5.* — Fix a one-parameter group of isometries  $\varphi_t$  of  $X$ , such that  $\varphi_t$  acts by translation along a geodesic  $\gamma$  on  $\mathbb{H}^2$ , and by (nontrivial) translation on  $\mathbb{R}$ . This is a ‘diagonal’ translation of  $X$ . Let  $\sigma$  be a geodesic in  $\mathbb{H}^2$  which is orthogonal to  $\gamma$ , and let  $\Sigma = \sqcup_t \varphi_t(\sigma \times \{0\})$ .

The fact that  $\Sigma$  is minimal can be argued as follows. It is invariant with respect to the geodesic symmetry around each translate of  $\sigma$ , hence its mean curvature vector is invariant under these symmetries, and must therefore vanish. One checks easily that  $\partial_g \Sigma$  is the union of the equator and of two arcs in

the Weyl chambers lying over the endpoints of  $\gamma$ ; the lengths of these arcs are equal and are determined by the ratio of the translation lengths of the vertical and horizontal parts of  $\varphi_t$ .

**Example 5.6.** — By the method of [NR02], one can construct a minimal graph over  $\mathbb{H}^2$  which has boundary values given by the graph of an unbounded function  $\sigma : \partial\mathbb{H}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$  which is finite and smooth on  $\partial\mathbb{H}^2 \setminus \{q\}$ , and with  $\sigma(q) = +\infty$ . Adjusting the rate of divergence of  $\sigma$  near  $q$ , one can obtain a properly embedded minimal disk  $\Sigma$  such that  $\partial_g \Sigma$  is the union of the equator and of an arc of any length in the Weyl chamber  $W^+(q)$ . It is also possible to find such surfaces where this arc is empty or equal to the entire Weyl chamber.

We now give a class of examples showing that  $\partial_g \Sigma$  can be as wild as in the conclusion of Theorem 5.1 without being equal to all of  $\partial_g X$ .

**Theorem 5.7.** — *For any  $r_0 > 0$ , there exists a complete minimal surface  $\Sigma \subset X$  such that  $\partial_g \Sigma$  is the entire strip  $\{(q, r) : |r| \leq r_0, q \in \partial\mathbb{H}^2\}$ .*

*Proof.* — We shall construct  $\Sigma$  to be periodic with respect to some discrete subgroup  $\Gamma$  of isometries of  $X$ . Indeed, let  $\Gamma = \pi_1(S_g)$ , where  $S_g$  is a compact, orientable genus  $g$  surface, and choose any discrete and faithful representation  $\alpha : \Gamma \rightarrow \mathrm{PSL}(2; \mathbb{R})$ .

There is a standard presentation of  $\Gamma$  as the quotient of the free group  $\mathcal{F}$  on generators  $a_1, \dots, a_{2g}$  by a normal subgroup  $R$  generated by some product of commutators  $[a_i, a_j] = a_i^{-1}a_j^{-1}a_i a_j$ . Any homomorphism  $\mathcal{F} \rightarrow \mathbb{R}$  sends all commutators to 0, and hence descends to a homomorphism  $\Gamma \rightarrow \mathbb{R}$ . We apply this to the homomorphism which sends  $a_1$  to 1 and all other  $a_i, i > 1$ , to 0, and denote the induced homomorphism on  $\Gamma$  by  $\beta$ , so  $\beta(a_1) = 1$  and all other  $\beta(a_i) = 0$ . We identify  $a_i$  with its image under the quotient map.

Since  $\alpha(a_1)$  is a hyperbolic element of  $\mathrm{PSL}(2; \mathbb{R})$ , it fixes a unique geodesic  $\gamma_1$  in  $\mathbb{H}^2$ , and acts by a translation of  $\tau$  along this geodesic. Consider now the action of  $\Gamma$  on  $X$  defined by

$$A : \Gamma \rightarrow \mathrm{Isom}(X), \quad A(\gamma) = (\alpha(\gamma), \frac{r_0}{\tau} \cdot \beta(\gamma)).$$

The quotient  $X/\Gamma$  associated to this action is diffeomorphic to the product  $S_g \times \mathbb{R}$ . Now choose another representation  $B : \mathbb{Z} \rightarrow \mathrm{Isom}(X)$  which commutes with  $A$ , and is such that the associated quotient  $X/(\Gamma \times \mathbb{Z})$  is diffeomorphic to  $S_g \times S^1$ . For example, we can simply let  $B(k)(p, t) = (p, t + 1)$ . With respect to this identification, consider the incompressible embedding  $f : S_g \hookrightarrow S_g \times \{0\} \subset S_g \times S^1$ . According to a theorem of Schoen and Yau [SY79], there is an immersion  $\hat{f} : S_g \rightarrow S_g \times S^1$  which is isotopic to  $f$  and such that the image of  $\hat{f}$  has least area in this isotopy class. Let us denote this surface by  $\hat{S}$ . By a further result of Freedman-Hass-Scott [FHS83],  $\hat{S}$  is actually embedded.

It is then clear that we may pass to the  $\mathbb{Z}$  cover and consider  $\hat{S}$  as a compact embedded minimal surface in  $X/\Gamma$ . Finally, let  $\Sigma$  be the inverse image of  $\hat{S}$  in the universal cover  $X$ . Thus  $\Sigma$  is a minimal surface which is invariant under the action of  $\Gamma$ .

Now consider the behavior of  $\Sigma$  above the geodesic  $\gamma_1$ , or in other words, the behaviour of the intersection  $(\gamma_1 \times \mathbb{R}) \cap \Sigma$ . Denote the endpoints of  $\gamma_1$  by  $q_1, q_2$ . If  $(z_0, t_0) \in \Sigma$  with  $z_0 \in \gamma_1$ , then the entire sequence  $(z_n, t_n) := A(a_1^n) \cdot (z_0, t_0)$  lies in  $\Sigma$ . Observe that  $t_n = (r_0/\tau)n$  and  $d_{\mathbb{H}^2}(z_0, z_n) = n\tau$ ,  $n \in \mathbb{Z}$ . It follows that  $\partial_g \Sigma$  contains both  $(q_1, -r_0)$  and  $(q_2, r_0)$ .

Suppose finally that  $q$  is any point on  $\partial\mathbb{H}^2$ . Since  $\Gamma$  acts cocompactly on  $\mathbb{H}^2$ ,  $q$  lies in its limit set so we can find a sequence  $h_n \in \Gamma$  such that the endpoints  $q_1^n, q_2^n$  of the geodesic  $\alpha(h_n) \cdot \gamma_1$  both limit to  $q$ . The asymptotic behavior of  $\Sigma$  over the geodesic  $\alpha(h_n) \cdot \gamma_1$  is conjugate to its behavior over  $\gamma_1$ , hence  $\partial_g \Sigma$  contains  $(q_1^n, -r_0)$  and  $(q_2^n, r_0)$  for all  $n$ . Since  $\partial_g \Sigma$  is closed, it must contain  $(q, -r_0)$  and  $(q, r_0)$ ; using the previous Theorem, it must contain the whole interval  $q \times [-r_0, r_0]$ .

We may of course assume that  $\tau$  is the systole of  $\alpha$ , i.e., the smallest translation length amongst all elements of  $\alpha(\Gamma)$ . Then it is clear from the construction that  $\partial_g \Sigma$  is contained in the strip  $\{|r| \leq r_0\}$ , which completes the proof.  $\square$

Theorems 5.1 and 5.7 were inspired by Benoist's article [Ben97], where a similar behavior is exhibited in some linear groups. It is natural to ask if any similar statements hold for minimal hypersurfaces (or minimal submanifolds of codimension greater than one) in more general symmetric spaces. Since  $X$  has a Euclidean factor, an equatorial point may be interpreted either as a boundary point for a Weyl chamber as we do here, or as the center of a larger chamber. It is not clear which of these would be part of the most natural to conjecture in the higher rank setting.

## 6. Open questions: minimal submanifolds in nonpositively curved symmetric spaces

We conclude with some remarks and open questions which are meant to convey the range of unexplored territory.

**6.1. Minimal surfaces of  $\mathbb{H}^2 \times \mathbb{R}$ .** — The results quoted and proved above leave open some questions about fillability of boundary curves in  $\partial\mathbb{H}^2 \times \mathbb{R}$ .

**Problem 1.** — *Characterize the curves of  $\partial_{\times} \mathbb{H}^2 \times \mathbb{R}$  which are minimally fillable.*

In particular, what can one say about curves which are connected and tall in the vertical part of the boundary, and for which the intersection with each

cap is a pair of disjoint geodesics? Some of these can be filled by minimal surfaces of Jenkins-Serrin type, cf. [NR02], but that method does not give as much flexibility in the choice of vertical components as in Proposition 4.4.

**Problem 2.** — *Characterize the closed subsets of  $\partial_g \mathbb{H}^2 \times \mathbb{R}$  which arise as  $\partial_g \Sigma$  for an embedded minimal surface  $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ .*

Theorem 5.1 provides a strong restriction, but is far from definitive.

**6.2. Higher-dimensional symmetric spaces.** — As noted in the introduction, the starting point for our work was to consider  $\mathbb{H}^2 \times \mathbb{R}$  as a particular type of nonpositively curved symmetric spaces. This perspective leads naturally to the question of understanding the properly embedded minimal submanifolds and the asymptotic Plateau problem for other such symmetric spaces.

The next simplest case is the product of two hyperbolic planes, possibly with different curvatures,  $\mathbb{H}_\kappa^2 \times \mathbb{H}_\lambda^2$ . It is likely that the minimal surfaces in these might have rather different behavior depending on whether the curvatures  $\kappa$ ,  $\lambda$  are equal to one another or not.

The space  $\mathbb{H}_\kappa^2 \times \mathbb{H}_\lambda^2$  has both a product and a geodesic boundary. The paper [MV02] explores the role of these two boundaries in linear elliptic theory. The natural question in the present setting is then

**Problem 3.** — *Determine which curves or which closed surfaces in*

$$\partial_\times(\mathbb{H}_\kappa^2 \times \mathbb{H}_\lambda^2) \quad \text{and} \quad \partial_g(\mathbb{H}_\kappa^2 \times \mathbb{H}_\lambda^2)$$

*are minimally fillable by surfaces or 3-dimensional submanifolds.*

We also pose the

**Problem 4.** — *Determine which curves of  $\partial\mathbb{H}_\kappa^2 \times \partial\mathbb{H}_\lambda^2$  are the boundary of an embedded minimal surface of  $\mathbb{H}_\kappa^2 \times \mathbb{H}_\lambda^2$ .*

While the complete characterization may be very difficult, one might construct examples using pairs of representations of a surface group  $\Gamma$  into  $\mathrm{PSL}(2, \mathbb{R})$  to construct periodic examples, similar to what we did in the proof of Theorem 5.7.

Similar questions can be posed in other symmetric spaces, e.g.  $\mathrm{SL}(3)/\mathrm{SO}(3)$ . A number of different compactifications have been studied, see [BJ06], [MV05].

**Problem 5.** — *Determine a compactification of a nonpositively curved symmetric space which is particularly well suited for the study of the asymptotic Plateau problem.*

This is of course only loosely stated, since it is not clear what ‘well suited’ should mean. Presumably this entails that many submanifolds of that compactification are minimally fillable.

Note that the rank one case is basically covered by a more general result of Lang [Lan03].

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