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# On least action principles for discrete quantum scales

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**Abstract.** We consider variational problems where the velocity depends on a scale. After recalling the fundamental principles that lead to classical and quantum mechanics, we study the dynamics obtained by replacing the velocity by some physical observable at a given scale into the expression of the Lagrangian function. Then, discrete Euler-Lagrange and Hamilton-Jacobi equations are derived for a continuous model that incorporates a real-valued discrete velocity. We also examine the paradigm for complex-valued discrete velocity, inspired by the scale relativity of Nottale. We present also rigorous definitions and preliminary results in this direction.

**Keywords:** quantum operators, scale relativity.

## 1 Some philosophical principles for Physics

In this contribution, we first introduce some general philosophical hypotheses that are also widely discussed by several authors (see *e.g.* Bitbol [1], d’Espagnat [4], Filk and von Müller [5] among others). We set three hypotheses. The two first ones are of ontological type and the third one is concerned with experiments.

**(H1)-Principle of reality.** It exists a reality which is independent of any observer.

**(H2)-Continuous space-time.** The space-time is a continuous manifold on which the movement of particles can be described by continuous trajectories.

**(H3)-Measurement and scale.** The measurement of a physical quantity (time, space, velocity, energy, *etc*) involves a notion of scale.

- In classical physics, hypothesis (H2) is more constrained: trajectories are supposed to be differentiable or more regular. In this case, the particle velocity is uniquely defined by  $v = \frac{dq}{dt}$  which is independent of the scale. Observe that if the trajectory is not regular (continuous but nowhere differentiable) or if some general hypothesis of continuous but non-differentiable space-time is done (as in scale relativity [12]), hypothesis (H3) remains true but the previous velocity has no meaning. On the contrary, a discrete velocity associated with a given scale can still be well-defined.

- This framework leads to a first paradigm (labelled by the letter “a” in table 1) of continuous classical physics. We recall in section 2 the main point about Euler-Lagrange and Hamilton-Jacobi equations. As noticed by Gondran [7], a complexification of the Hamilton-Jacobi framework provides a natural introduction to the Schrödinger equation. This second paradigm (letter “b” in table 1) is shortly displayed in section 3. As a consequence the differentiability of the trajectories is lost and they can be interpreted in terms of Brownian motion (see *e.g.* Nelson [11]).
- In this contribution, we develop a scale point of view based on the analysis of reality associated with observations at a given discrete scale. We develop in section 4 a paradigm (labeled with the letter “c” in table 1) based on the knowledge of real-valued discrete velocities. In other words, the velocity at a given scale remains a real number. The idea of introducing discrete operators as fundamental principles of mechanics and quantum mechanics has been proposed by several authors as Greenspan [8], Friedberg and Lee [6] and recently by Khrennikov *et al.* [9, 10] as well as Otake and Sasaki [13]. Nevertheless, our approach does not follow the paradigms suggested by the above references. Our objective is to develop our understanding of the ideas of Nottale [12] who introduced a set of discrete complex velocities (see the label “d” in table 1). We propose some preliminary remarks in this direction in section 5.

	Continuous Geometry	Given Scale Geometry
Classical Physics	Ⓐ Hamilton-Jacobi	Ⓒ Real-valued velocity
Quantum Physics	Ⓑ Schrödinger	Ⓓ Complex-valued velocity

**Table 1.** Proposition of four paradigms.

## 2 Some classical results on Hamilton-Jacobi equations

In order to reduce the notations, a Lagrangian function  $L(x, v)$  independent of the time is given. To fix the ideas, this Lagrangian can be chosen as

$$L(x, v) = \frac{1}{2} m v^2 - \varphi(x). \quad (1)$$

The potential energy  $\varphi(x)$  structures the space-time with objects governed by physical laws (H1), whereas the kinetic energy  $K(v) \equiv \frac{mv^2}{2}$  catches the dynamics through the velocity. Consider a regular trajectory  $\theta \mapsto X(\theta)$  for  $0 \leq \theta \leq t$  and the associated action

$$A(t, X(\bullet)) = \int_0^t L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) d\theta.$$

For an arbitrary variation  $\delta t$  and for all  $C^1$ -functions  $X$  and associated variations  $\delta X$ , we introduce the variation  $\delta A$  of the action. It is given by:

$$\begin{aligned} \delta A(t, X(\bullet)) &= L\left(X(t), \frac{d}{d\theta}X(t)\right) \delta t \\ &\quad + \int_0^t \left[ \partial_x L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) \delta X(\theta) + \partial_v L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) \frac{d}{d\theta} \delta X(\theta) \right] d\theta \end{aligned}$$

and after integrating by parts:

$$\begin{aligned} \delta A &= L\left(X(t), \frac{d}{d\theta}X(t)\right) \delta t + \int_0^t \frac{d}{d\theta} \left[ \partial_v L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) \delta X(\theta) \right] d\theta \\ &\quad + \int_0^t \left( \partial_x L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) - \frac{d}{d\theta} \left[ \partial_v L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) \right] \right) \delta X(\theta) d\theta. \end{aligned} \quad (2)$$

• Let  $q_0$  be fixed and consider the class of functions  $\mathcal{C}_{q_0}^1(0, t) = \{X \in \mathcal{C}^1(0, t) \text{ such that } X(0) = q_0\}$ . Notice that the difference between two functions of  $\mathcal{C}_{q_0}^1(0, t)$  belongs to  $\mathcal{C}_0^1(0, t)$ . Thus, if  $\delta X \in \mathcal{C}_0^1(0, t)$ , then  $\delta X(0) = 0$ . Vanishing the first variations of the action leads to the well-known Euler-Lagrange equation given by

$$\begin{aligned} \partial_x L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) - \frac{d}{d\theta} \left[ \partial_v L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) \right] &= 0, \\ \text{with } X(0) = q_0, X(t) = q. \end{aligned} \quad (3)$$

Moreover, for any arbitrary time  $t > 0$  and any arbitrary state  $q$ , let  $X^{\text{opt}}(\bullet; t, q)$  be the solution  $X(\bullet)$  in  $\mathcal{C}_{q_0}^1(0, t)$  of the Dirichlet boundary problem given by the Euler-Lagrange equation (3). Observe that  $X^{\text{opt}}(\bullet; t, q)$  is parameterized by the time of arrival  $t$  and the value  $q$ , as precised in (3). At fixed time  $t$  and position  $q$ , the optimal trajectory  $X^{\text{opt}}(\bullet; t, q)$  is supposed to exist and to be unique. We have the initial condition  $X^{\text{opt}}(0; t, q) = q_0$  and the final condition  $X^{\text{opt}}(t; t, q) = q$ . Moreover the trajectory  $\theta \mapsto X^{\text{opt}}(\theta; t, q)$  has a velocity at time  $t$  and position  $q$  equal to  $\partial_\theta X^{\text{opt}}(\theta; t, q) \Big|_{\theta=t}$  that can also be considered as a “natural” velocity  $\frac{dq}{dt}(t) = \partial_\theta X^{\text{opt}}(\theta; t, q) \Big|_{\theta=t}$ .

• Let the momentum  $p(t, q)$  be defined by

$$p(t, q) = \partial_v L(q, \partial_\theta X^{\text{opt}}(t; t, q)) \quad (4)$$

and the optimal action  $S(t, q)$  as the action along the optimal trajectory :

$$S(t, q) = A(t, X^{\text{opt}}(\bullet; t, q)). \quad (5)$$

At fixed time  $t$ , due to the Euler-Lagrange equation (3), we deduce from (2) that  $\delta A(t, X^{\text{opt}}(\bullet)) = \partial_v L(q, \partial_\theta X^{\text{opt}}(t; t, q)) = p(t, q)$ . In other words, the first variation of the optimal action with respect to the final state is the momentum, namely

$$\partial_q S(t, q) = p(t, q). \quad (6)$$

If time  $t$  is varying and considering the optimal trajectory  $\theta \mapsto X^{\text{opt}}(\theta; t, q)$ , we have  $\partial_t A(t, X^{\text{opt}}(\bullet)) = L(q, \partial_\theta X^{\text{opt}}(t; t, q))$ . Writing that this quantity is

the time variation of the optimal action (5) and taking into account the velocity of the optimal trajectory at the location  $q$ , we deduce

$$\partial_t S + \partial_q S \bullet \partial_\theta X^{\text{opt}}(t; t, q) = \partial_t A(t, X^{\text{opt}}(\bullet)) = L(q, \partial_\theta X^{\text{opt}}(t; t, q)). \quad (7)$$

• Introduce now the Legendre transform of the Lagrangian  $L$  relatively to the second variable  $v$ . Suppose that the function  $v \mapsto y = \partial_v L(x, v)$  is invertible and denote by  $V(x, y)$  its inverse. The Hamiltonian  $H(y, x)$  is classically defined by

$$H(y, x) = y \bullet V(x, y) - L(x, V(x, y)).$$

Observe that if (4) holds then  $\partial_\theta X^{\text{opt}}(t; t, q) = V(q, p)$  and  $H(p, q) = p \bullet V(q, p) - L(q, V(q, p))$ . We deduce from (7),

$$\begin{aligned} L(q, \partial_\theta X^{\text{opt}}(t; t, q)) &= \partial_t S + (\partial_q S) \bullet \partial_\theta X^{\text{opt}}(t; t, q) \\ &= \partial_t S + p(t, q) \bullet V(q, p(t, q)). \end{aligned}$$

This leads to the well-known Hamilton-Jacobi equation

$$\partial_t S + H(\partial_q S, q) = 0. \quad (8)$$

### 3 How to derive the Schrödinger equation ?

The “break through” from classical Hamilton-Jacobi equations to quantum dynamics is due to Schrödinger [14]. Introduce the wave function  $\psi$  according to

$$\psi = \exp\left(i \frac{S}{\hbar}\right) \quad (9)$$

and inject this relation into (4) and (8). We get  $\frac{i}{\hbar} dS = \frac{1}{\psi} d\psi$  and due to (6), we have  $p = \frac{\hbar}{i} \frac{1}{\psi} \partial_q \psi$ . Then Schrödinger transforms the momentum  $p$  into the so-called momentum operator  $P$  defined by  $P \bullet \psi \equiv -i \hbar \partial_q \psi$ . Observe that the momentum  $P$  becomes now a complex derivative operator. Starting from the usual Lagrangian, we observe that the good generalisation of quantum mechanic of  $v^2$  is not  $|v|^2$  (or  $PP^*$ ) but  $vv$  (or  $PP$  in the classical formalism). Then the Hamiltonian  $H$  takes the expression  $H = \frac{1}{2m} P^2 + \varphi(q) = -\frac{\hbar^2}{2m} \Delta + \varphi(q)$  and the Schrödinger equation

$$i \hbar \partial_t \psi = -\frac{\hbar^2}{2m} \Delta \psi + \varphi(q) \psi \quad (10)$$

is a direct consequence of the Hamilton-Jacobi equation (8).

• An other way to derive the Schrödinger equation has been proposed by Nottale [12]. The idea consists in replacing the classical trajectory derivative  $\frac{d}{dt} \equiv \partial_t + v \bullet \partial_q$  by the complex Dynkin operator  $\frac{d}{dt} \equiv \partial_t + v \bullet \partial_q - i \frac{\hbar}{2m} \Delta$ . Then, equation (7) takes the form:  $L = \partial_t S + v \bullet (\partial_q S) - i \frac{\hbar}{2m} \Delta S$  and  $\partial_t S + \frac{1}{m} (\partial_q S)^2 - i \frac{\hbar}{2m} \Delta S + \varphi(q) - \frac{m}{2} \left(\frac{1}{m} \partial_q S\right)^2 = 0$ . Following Gondran [7], one can derive a complex Hamilton-Jacobi equation

$$\partial_t S + \frac{1}{2m} (\partial_q S)^2 + \varphi(q) - i \frac{\hbar}{2m} \Delta S = 0. \quad (11)$$

If we decompose the complex optimal action  $S$  into its real and imaginary parts, *id est*  $S = \Sigma - i \hbar \log R$ , an elementary calculus allows to transform the complex Hamilton-Jacobi equation (11) into the form proposed by Bohm and Hiley [2]:

$$\partial_t \Sigma + \frac{1}{2m} (\partial_q \Sigma)^2 + \varphi(q) - \frac{\hbar}{2m} \frac{\partial_q R}{R} = 0, \quad \partial_t R^2 + \operatorname{div} \left( \frac{R^2}{m} \partial_q \Sigma \right) = 0. \quad (12)$$

The quantum potential  $Q \equiv -\frac{\hbar}{2m} \frac{\partial_q R}{R}$  is the quantity that has to be added to transform the classical Hamilton-Jacobi equation (8) into the real part of the complex Hamilton-Jacobi equation (11).

• Introduce now the change of variables (9) into the complex Hamilton-Jacobi equation (11). If we derive once again the relation  $\frac{i}{\hbar} \partial_q S = \frac{1}{\psi} \partial_q \psi$  towards the space variable  $q$ , we get  $\frac{i}{\hbar} \partial_q^2 S = -\frac{1}{\psi^2} (\partial_q \psi)^2 + \frac{1}{\psi} \partial_q^2 \psi$ . The left hand side of the complex Hamilton-Jacobi equation (11) is now equal to

$$\frac{1}{\psi} \left[ \frac{\hbar}{i} \partial_t \psi + \frac{1}{2m\psi} \left( \frac{\hbar}{i} \partial_q \psi \right)^2 + \varphi(q) \psi - i \frac{\hbar}{2m} \frac{\hbar}{i} \left( -\frac{1}{\psi} (\partial_q \psi)^2 + \partial_q^2 \psi \right) \right]$$

and the Schrödinger equation (10) is established.

#### 4 Real-valued discrete-measured velocity at a given scale

We consider now that the classical velocity is not a relevant observable. We introduce a given strictly positive scale parameter  $\varepsilon$ , a “fat” initial condition  $q_0 \in \mathcal{C}([-\varepsilon, 0])$  as a continuous function and the classical discrete so-called finite difference operators

$$(d_\varepsilon^- q)(\theta) \equiv \frac{1}{\varepsilon} (q(\theta) - q(\theta - \varepsilon)), \quad (d_\varepsilon^+ q)(\theta) \equiv \frac{1}{\varepsilon} (q(\theta + \varepsilon) - q(\theta)). \quad (13)$$

Let us notice that the velocity  $v_\varepsilon^\pm = d_\varepsilon^\pm q$  is now measured at the given scale  $\varepsilon$  by two possible schemes (13), as a consequence of the hypothesis (H3). We consider a given (final) time  $t$  strictly positive and a continuous trajectory  $([-\varepsilon, t] \ni \theta \mapsto q(\theta)) \in \mathcal{C}([-\varepsilon, t])$  with the initial condition  $q_0$ . This initial condition is not classical,  $q_0$  is not anymore given at a time  $t = 0$ , but on a small interval depending on the scale  $\varepsilon$ . It has to be considered in the following sense: restricted to the interval  $[-\varepsilon, 0]$ , function  $q$  is equal to the given function  $q_0$ . As in the classical case described in section 2, we introduce an action  $A$  based on a regular Lagrangian  $L(x, v)$  which is similar to that introduced at the relation (1):

$$A(t, q) \equiv \int_0^t L(q(\theta), d_\varepsilon^- q(\theta)) d\theta. \quad (14)$$

In the following, we examine the choice of  $d_\varepsilon^- q$  as the observed velocity. Thus the paradigm based on this choice and (14) is studied. We have just formally replaced velocity  $v$  in the second argument of the Lagrangian (1) by the discrete velocity  $v_\varepsilon = d_\varepsilon^- q$ . We have the following result.

• **Proposition 1. Variation of the discrete action**

The variation  $\delta A$  of the action  $A$  defined in (14) when trajectory  $q$  is varying by an increment  $\delta q$  and time by an increment  $\delta t$  is given by

$$\begin{aligned} \delta A &= L \delta t - \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \partial_v L(\theta + \varepsilon) \delta q(\theta) d\theta \\ &\quad + \int_{t-\varepsilon}^t \left[ \partial_x L + \frac{1}{\varepsilon} \partial_v L \right](\theta) \delta q(\theta) d\theta \\ &\quad + \int_0^{t-\varepsilon} \left[ \partial_x L - d_\varepsilon^+(\partial_v L) \right](\theta) \delta q(\theta) d\theta. \end{aligned} \quad (15)$$

The first integral in (15) is null *a priori* since initial condition  $q_0$  is supposed to be fixed between  $-\varepsilon$  and 0.

**Proof of Proposition 1.**

Since Lagrangian  $L$  in (1) is a regular function, differentiating (14) yields

$$\begin{aligned} \delta A &= L \delta t + \int_0^t (\partial_x L) \delta q(\theta) d\theta + \int_0^t (\partial_v L) \frac{1}{\varepsilon} (\delta q(\theta) - \delta q(\theta - \varepsilon)) d\theta \\ &= L \delta t + \int_0^t (\partial_x L) \delta q(\theta) d\theta + \frac{1}{\varepsilon} \int_0^t (\partial_v L) \delta q(\theta) d\theta - \frac{1}{\varepsilon} \int_{-\varepsilon}^{t-\varepsilon} (\partial_v L)(\theta + \varepsilon) \delta q(\theta) d\theta \\ &= L \delta t - \frac{1}{\varepsilon} \int_{-\varepsilon}^0 (\partial_v L)(\theta + \varepsilon) \delta q(\theta) d\theta \\ &\quad + \int_0^{t-\varepsilon} \left( \partial_x L - \frac{1}{\varepsilon} [(\partial_v L)(\theta + \varepsilon) - (\partial_v L)(\theta)] \right) \delta q(\theta) d\theta \\ &\quad + \int_{t-\varepsilon}^t \left[ \partial_x L + \frac{1}{\varepsilon} \partial_v L \right](\theta) \delta q(\theta) d\theta, \end{aligned}$$

so that (15) is a consequence of the definition of the operator  $d_\varepsilon^+$  given by (13).  $\square$

• We deduce from relation (15) that an optimal trajectory satisfies the discrete version of the Euler-Lagrange equation, that is

$$\partial_x L(q(\theta), d_\varepsilon^- q) - d_\varepsilon^+ [\partial_v L(q(\theta), d_\varepsilon^- q)] = 0, \quad 0 \leq \theta \leq t - \varepsilon. \quad (16)$$

This discrete-time dynamics is formally very similar to the classical Euler-Lagrange dynamics (3). Remark that it is nothing but an implicit finite difference scheme:

$$\begin{aligned} \partial_x L \left( q(\theta), \frac{1}{\varepsilon} (q(\theta) - q(\theta - \varepsilon)) \right) - \frac{1}{\varepsilon} \partial_v L \left( q(\theta + \varepsilon), \frac{1}{\varepsilon} (q(\theta + \varepsilon) - q(\theta)) \right) \\ + \frac{1}{\varepsilon} \partial_v L \left( q(\theta), \frac{1}{\varepsilon} (q(\theta) - q(\theta - \varepsilon)) \right) = 0. \end{aligned} \quad (17)$$

From (17), it is clear that the dynamics of the optimal trajectory is that of a delay system, and more precisely,

$$q(\theta) \text{ is a function of } \theta, q(\theta - \varepsilon), q(\theta - 2\varepsilon). \quad (18)$$

Function  $q$  is the solution of the two-step finite-difference scheme (17). Because  $q_0(\theta)$  is known for  $-\varepsilon \leq \theta \leq 0$ , the knowledge of  $q(\theta)$  for  $0 \leq \theta \leq \varepsilon$  is generically sufficient for solving the scheme (17) under the form (18). The knowledge

of  $q_0$  on  $[-\varepsilon, \varepsilon]$  is equivalent to the knowledge of the discrete derivative  $d_\varepsilon^+ q(\theta)$  for  $-\varepsilon \leq \theta \leq 0$ . Let us define this initial variation  $(d_\varepsilon^+ q)_0$  as

$$(d_\varepsilon^+ q)_0(\theta) = \frac{1}{\varepsilon} (q(\theta + \varepsilon) - q(\theta)), \quad -\varepsilon \leq \theta \leq 0. \quad (19)$$

• From the knowledge of  $q_0(\theta)$  and  $(d_\varepsilon^+ q)_0$  we construct *a priori* without major difficulty the continuous trajectory  $q$  solution of (17) of the type (18) for  $0 \leq \theta \leq t$ . We obtain in this way a “final state”  $q^f$  which is now a piece of trajectory  $q$ :

$$q^f(\theta) = q(t + \theta), \quad -\varepsilon \leq \theta \leq 0.$$

This leads to the functional  $Q_t : (d_\varepsilon^+ q)_0 \mapsto q^f = Q_t((d_\varepsilon^+ q)_0)$  defined from  $\mathcal{C}([-\varepsilon, 0])$  to  $\mathcal{C}([-\varepsilon, 0])$ , for  $q_0$  fixed. We suppose this functional to be one to one. In consequence, we can suppose the optimal trajectory parameterized by the final state  $q^f \in \mathcal{C}([-\varepsilon, 0])$ . We denote by  $S(t, q^f)$  the corresponding optimal action. We observe that at fixed  $q_0$ , it depends only on the final time  $t$  and the final state  $q^f$  whereas the action  $A$  is a functional of all the states along the whole trajectory.

• **Proposition 2. Derivative of the optimal action**

Under a variation  $\delta q^f$  of the final state, the optimal action admits a variation  $\delta S(t, q^f)$  given by

$$\delta S(t, q^f) \equiv \frac{\partial S}{\partial q^f} \bullet \delta q^f = \int_{t-\varepsilon}^t \left[ \partial_x L + \frac{1}{\varepsilon} \partial_v L \right] (q(\theta), (d_\varepsilon^- q)(\theta)) \delta q(\theta) d\theta. \quad (20)$$

**Proof of Proposition 2.**

Due to the discrete Euler-Lagrange equations (16), the optimal trajectory vanishes the third term of the right hand side of the relation (15). The first one is identically null because the initial condition  $q_0$  remains fixed. The result is then a simple consequence of the relation (15) when time  $t$  is fixed.  $\square$

• In the right hand side of relation (20) the final state is not explicit. In order to exhibit the variation  $\delta q^f$  we introduce

$$\Gamma(t, q^f)(\theta) \equiv (\partial_v L + \varepsilon \partial_x L)(q(t + \theta), (d_\varepsilon^- q)(t + \theta)), \quad -\varepsilon \leq \theta \leq 0. \quad (21)$$

Then,  $\Gamma(t, q^f) \in \mathcal{C}([-\varepsilon, 0])$  and relation (20) can be also written as

$$\frac{\partial S}{\partial q^f} \bullet \delta q^f = \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \Gamma(t, q^f)(\theta) \delta q^f(\theta) d\theta. \quad (22)$$

Let us observe that expression  $\Gamma(t, q^f)$  is a good candidate for a momentum variable analogous to the one that satisfies the relation (6) in differentiable mechanics.

• The natural question is now to determinate the “total variation” with time of the optimal action, *id est* the discrete analogous of the expression (7). This



is not possible if we restrict to solely continuous trajectories. Nevertheless we propose a result for a discrete variation in time of amplitude exactly equal to  $\varepsilon$ . We denote by  $\tilde{q}^f$  the trajectory obtained from the final state  $q^f$  after a time extension of amplitude  $\varepsilon$ :  $\tilde{q}^f(\theta) \equiv q(t + \varepsilon + \theta)$  for  $-\varepsilon \leq \theta \leq 0$ . Then we have a simple expression for the difference  $S(t + \varepsilon, \tilde{q}^f) - S(t, q^f)$  because the two integrals in (14) operates on the same optimal trajectory:

$$S(t + \varepsilon, \tilde{q}^f) - S(t, q^f) = \int_t^{t + \varepsilon} L(q(\theta), (d_\varepsilon^- q)(\theta)) d\theta. \quad (23)$$

• **Proposition 3. Discrete variation of the optimal action**

Let  $\xi$  be a continuous function in the space  $\mathcal{C}([-\varepsilon, 0])$ . We have

$$S(t, q^f + \xi) - S(t, q^f) = \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \left[ \int_0^1 \Gamma(t, q^f + \eta \xi)(\theta) d\eta \right] \xi(\theta) d\theta. \quad (24)$$

**Proof of Proposition 3.**

We introduce  $\Phi(\eta) \equiv S(t, q^f + \eta \xi)$  for  $0 \leq \eta \leq 1$ . It is a derivable function of the real variable  $\eta$  and we have

$$\begin{aligned} \frac{d\Phi}{d\eta} &= \frac{\partial S}{\partial q^f}(t, q^f + \eta \xi) \bullet \frac{d}{d\eta}(q^f + \eta \xi) = \frac{\partial S}{\partial q^f}(t, q^f + \eta \xi) \bullet \xi \\ &= \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \Gamma(t, q^f + \eta \xi)(\theta) \xi(\theta) d\theta. \end{aligned}$$

Then the relation (24) is obtained by integration relative to  $\eta \in [0, 1]$  and using Fubini theorem.  $\square$

Then, we present here the main result of this contribution.

• **Proposition 4. Discrete temporal variation of the optimal action**

Let  $\Gamma_\varepsilon(t, q^f)$  be a mean value at final time  $t$  of the momentum introduced in (21):

$$\Gamma_\varepsilon(t, q^f)(\theta) \equiv \int_0^1 \Gamma(t + \varepsilon, q^f + \varepsilon \eta (d_\varepsilon^- q)(t + \varepsilon + \theta))(\theta) d\eta, \quad -\varepsilon \leq \theta \leq 0. \quad (25)$$

The following discrete Hamilton-Jacobi type equation holds

$$\begin{aligned} d_\varepsilon^+ S + \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \Gamma_\varepsilon(t, q^f)(\theta) (d_\varepsilon^- q)(t + \varepsilon + \theta) d\theta \\ - \frac{1}{\varepsilon} \int_t^{t + \varepsilon} L(q(\tau), (d_\varepsilon^- q)(\tau)) d\tau = 0. \end{aligned} \quad (26)$$

**Proof of Proposition 4.**

We recall that  $d_\varepsilon^+ S \equiv \frac{1}{\varepsilon} (S(t + \varepsilon, q^f) - S(t, q^f))$ . Then we have the decomposition

$$\varepsilon d_\varepsilon^+ S = -(S(t + \varepsilon, \tilde{q}^f) - S(t + \varepsilon, q^f)) + (S(t + \varepsilon, \tilde{q}^f) - S(t, q^f)).$$

We remark also that  $\tilde{q}^f(\theta) - q^f(\theta) = q(t + \varepsilon + \theta) - q(t + \theta) = \varepsilon (d_\varepsilon^- q)(t + \varepsilon + \theta)$ . Then we have from (24) with  $\xi = \varepsilon (d_\varepsilon^- q)(t + \varepsilon + \theta)$ :

$$\begin{aligned} S(t + \varepsilon, \tilde{q}^f) - S(t + \varepsilon, q^f) &= \\ &= \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \left[ \int_0^1 \Gamma(t + \varepsilon, q^f + \varepsilon \eta (d_\varepsilon^- q)(t + \varepsilon + \theta))(\theta) d\eta \right] (\tilde{q}^f(\theta) - q^f(\theta)) d\theta \\ &= \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \left[ \int_0^1 \Gamma(t + \varepsilon, q^f + \varepsilon \eta (d_\varepsilon^- q)(t + \varepsilon + \theta))(\theta) d\eta \right] \varepsilon (d_\varepsilon^- q)(t + \varepsilon + \theta) d\theta \end{aligned}$$

and the second term of the left hand side of the relation (26) is clear. The end of the proof is a consequence of the decomposition of  $\varepsilon d_\varepsilon^+ S$  and relation (23).  $\square$

- The analogy between the classical Hamilton-Jacobi equation (8) and the discrete version (26) is clear. We observe that the Lagrangian is replaced by its mean value on an interval of size  $\varepsilon$ . Moreover the natural associated momentum  $\Gamma_\varepsilon(t, q^f)(\theta)$  defined at relation (25) is not *a priori* strictly equal to the momentum  $\Gamma(t, q^f)(\theta)$  introduced at relation (21). This splitting at the discrete scale of the moment  $p$  satisfying both relations (4) and (6) is a real difficulty that we will consider in a future contribution.

## 5 Towards complex-valued discrete-measured velocity

The discrete scaled velocity  $v_\varepsilon = d_\varepsilon^- q$  introduced in section 4 is purely real. We consider now a complex discrete velocity  $v_\varepsilon$ . Following an idea proposed by Nottale [12], we introduce a discrete complex derivation operator  $\square_\varepsilon$  according to

$$(\square_\varepsilon q)(\theta) \equiv \frac{1}{2\varepsilon} (q(\theta + \varepsilon) - q(\theta - \varepsilon)) + \frac{i\mu}{2\varepsilon} (q(\theta + \varepsilon) - 2q(\theta) + q(\theta - \varepsilon)), \quad (27)$$

with  $\mu^2 = 1$ . We decompose the discrete operator  $\square_\varepsilon q$  under the form  $\square_\varepsilon q \equiv \square_\varepsilon^r q + i\mu \square_\varepsilon^i q$ . We have

$$\begin{aligned} (\square_\varepsilon^r q)(\theta) &\equiv \frac{1}{2\varepsilon} (q(\theta + \varepsilon) - q(\theta - \varepsilon)) = \frac{1}{2} (d_\varepsilon^+ q(\theta) + d_\varepsilon^- q(\theta)) \\ (\square_\varepsilon^i q)(\theta) &\equiv \frac{1}{2\varepsilon} (q(\theta + \varepsilon) - 2q(\theta) + q(\theta - \varepsilon)) = \frac{1}{2} (d_\varepsilon^+ q(\theta) - d_\varepsilon^- q(\theta)). \end{aligned} \quad (28)$$

The real part  $\square_\varepsilon^r q$  is the standard time derivative for regular trajectories when  $\varepsilon$  goes to 0. The imaginary part  $\square_\varepsilon^i q$  is asymptotically null for a regular function and accounts for the slope jump at a given time. This framework has been proven to be well-posed by Cresson and Greff [3] introducing a limit when  $\varepsilon$  goes to zero in a well-defined projection functional space.

- As remarked previously, the appropriate generalization of the kinetic energy  $\frac{m}{2} v^2$  is obtained by taking the (complex) square of the momentum operator. So in the expression of the Lagrangian we have to replace  $v^2$  by  $(\square_\varepsilon q)^2$ . We set

$$K_\varepsilon \equiv \frac{m}{2} (\square_\varepsilon q)^2 = \frac{m}{2} [(\square_\varepsilon^r q)^2 - (\square_\varepsilon^i q)^2 + 2i\mu (\square_\varepsilon^r q) (\square_\varepsilon^i q)]. \quad (29)$$

If  $K_\varepsilon$  is real, *i.e.*  $\text{Im } K_\varepsilon = 0$ , the product  $(\square_\varepsilon^r q)(\square_\varepsilon^i q)$  is null and two cases occur.

(i) If  $K_\varepsilon \geq 0$ , then  $\square_\varepsilon^i q = 0$  and we have a natural reference to a regular trajectory.

(ii) If  $K_\varepsilon < 0$ , then  $\square_\varepsilon^r q = 0$ . The position  $q(\theta)$  is essentially unchanged during one  $\varepsilon$ -step but the jump is not null and the direction of the trajectory has changed abruptly.

If the kinetic energy is imaginary,  $\text{Re } K_\varepsilon = 0$  and we have  $(\square_\varepsilon^r q)^2 = (\square_\varepsilon^i q)^2$  that implies  $d_\varepsilon^+ q(\theta) = 0$  or  $d_\varepsilon^- q(\theta) = 0$ . The particle has not moved just before time  $t$  or just after!

• We consider now the iterate of the operator  $\square_\varepsilon$  with itself. This type of algebraic formula is natural for the extension of  $\frac{d}{dt}(m \frac{d}{dt})$  in the Euler-Lagrange equation. We emphasise the role of  $\mu^2 = 1$  when we consider the composed operator. We have

$$\begin{cases} \text{Re}((\square_\varepsilon \circ \square_\varepsilon)q) = (1 - \mu^2) (\square_\varepsilon^r \circ \square_\varepsilon^r)q + \mu^2 (\square_{\frac{\varepsilon}{2}}^r \circ \square_{\frac{\varepsilon}{2}}^r)q \\ \text{Im}((\square_\varepsilon \circ \square_\varepsilon)q) = 2\mu (\square_\varepsilon^r \circ \square_\varepsilon^i)q. \end{cases}$$

Roughly speaking the product “jump by jump” allows to recover some regularity at a smaller scale  $\varepsilon/2$ .

• We propose to introduce the following complex action for  $q \in \mathcal{C}([-\varepsilon, t + \varepsilon])$ :

$$A_\varepsilon(t, q) \equiv \int_0^t L(q(\theta), (\square_\varepsilon q)(\theta)) d\theta = \int_0^t \left[ \frac{m}{2} (\square_\varepsilon q)^2 - \varphi(q(\theta)) \right] d\theta.$$

Our working plan follows the ideas presented in sections 2 and 3. In an analogous way as the one proposed in section 3, we will consider the Euler-Lagrange optimality condition, introduce the optimal trajectories, derive a Hamilton-Jacobi like equation for the optimal value of the action. Then make the change of variable (9) to transform the evolution equation (26) into a Schrödinger type equation.

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