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MODELLING AND NUMERICAL SIMULATION OF STRINGS BASED ON LIE GROUPS AND ALGEBRAS APPLICATIONS TO THE NONLINEAR DYNAMICS OF REISSNER BEAMS

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ABSTRACT

A nonlinear string can be modelled as a prestressed beam using Reissner's assumptions. Namely, that plane sections normal to the neutral axis remain plane but that their displacements and rotations can be arbitrarily large. The strains are assumed to be small enough to neglect material nonlinearity, which means that each section is a rigid body. The nonlinear analysis of such a system can lead to a complex formulation when second Piola-Kirchoff stress and Green-Lagrange strain tensors are used.

Alternatively, Lie groups and algebras offer a efficient formulation by considering the space of mechanical system transformations (the SE3 group) instead of the generalised co-ordinates space (R3). The intrinsic nonlinearities due to the curvature of the group SE3 (geometric nonlinearities) are correctly handled and this leads to a compact and exact form of the nonlinear equilibrium equations from which further models can easily be derived. As evidence, a linearisation in the neighbourhood of the prestressed beam can be written taking into account tension, flexion, shear, rotation and coupling phenomena. The general nonlinear problem can also be solved using pure numerical methods or semi-analytical Volterra series.

INTRODUCTION

In the context of musical acoustics, physical model of musical instruments have to be more and more sophisticated. For string model realism is obtained by taking into account tension, flexion, shear, rotation and coupling phenomena but also nonlinear effects.

Since the nineteen century, Kirchhoff [10], Carrier [5], Anand [1] or more recently Watzky [14], Chaigne[6] or Bilbao[3] have all studied this problem but they all made the assumption that the strain is represented by the Green-Lagrange tensor expressed in terms of a displacement function. This leads to a certain complexity but the more complex the models, the slower the sound synthesis.

As an alternative, the Lie groups and algebras offert an elegant way to express the strain vector and tensor in terms of group elements. Properties of Lie groups can then be applied to mechanics.

Following the work of D. Primault [11], the first part of this article establishes a nonlinear model for Reissner Beam. The Reissner's definitions for kinematics of beam suitable for large displacement and small strain is adopted. In that context, Hamilton's principle can be applied with the help of Lie groups and algebras. It gives rise, in a second part, to an application to string instruments where the dynamic equations suitable for a prestressed beam are studied.

NONLINEAR MODEL FOR REISSNER BEAM

Reissner kinematic and Lie groups and algebras

A beam of length L, with cross-sectional area A and mass per unit volume ρ is considered. Following the Reissner kinematic, each section of the beam is supposed to be a rigid body. The beam configuration can be described by a position $\mathbf{r}(X,t)$ and a rotation $\mathbf{R}(X,t)$ of each section. The coordinate X corresponds to the position of the section in the reference configuration Σ_0 (see figure 1).

Any material point M which is located at $\mathbf{x}(X,0) = \mathbf{r}(X,t) + \mathbf{w}_0 = X\mathbf{E}_1 + \mathbf{w}_0$ in the reference configuration ($t = 0$) have a new position (at time t) $\mathbf{x}(X,t) = \mathbf{r}(X,t) + \mathbf{R}\mathbf{w}_0$ with velocity $\dot{\mathbf{x}} = \dot{\mathbf{r}} + \dot{\mathbf{R}}\mathbf{w}_0$.

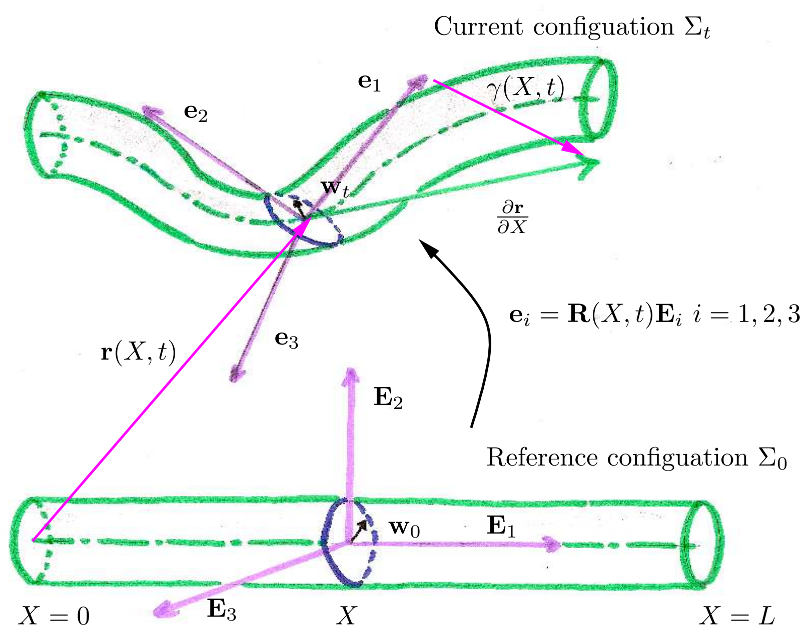

FIG. 1: Reference and current configuration of a beam. Each section, located at position X in the reference configuration Σ_0, is parametrized by a translation $\mathbf{r}(X,t)$ and a rotation $\mathbf{R}(X,t) \in SO_3$ in the current configation Σ_t.

Since the three-dimensional rotation, $\mathbf{R}$, belongs to the Lie group

$$SO_3 = \{R \in gl(3,\mathcal{R}) | \quad \mathbf{R}^T\mathbf{R} = \mathbb{I} \text{ et } det(\mathbf{R}) = 1\}, \tag{1}$$

where $gl(3,\mathcal{R})$ is the vectorial space of 3×3 matrices with real coefficients, some interesting properties can be used. Matrix $\mathbf{R} \in SO_3$ checks

$$\frac{d}{dt}(\mathbf{R}^T\mathbf{R}) = 0 = \dot{\mathbf{R}}^T\mathbf{R} + \mathbf{R}^T\dot{\mathbf{R}}. \tag{2}$$

The angular velocity, $\hat{\Omega} = \mathbf{R}^T\dot{\mathbf{R}}$, is then a skew-symmetric matrix ($\hat{\Omega}^T = -\hat{\Omega}$) with vanishing trace that can be associated to an axial vector

$$\Omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \quad \text{such that } \hat{\Omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \tag{3}$$

Thus, $\forall \mathbf{u} \in \mathcal{R}^3$, once have $\hat{\Omega}\mathbf{u} = \Omega \wedge \mathbf{u}$.

To every Lie group G, we can associate a Lie algebra, whose underlying vector space is the tangent space of G at the identity element, which completely captures the local structure of the group. The matrices $\hat{\Omega} = \mathbf{R}^T\dot{\mathbf{R}}$ belong to the space so_3 tangent to SO_3.

Multiplying on the left the definition $\hat{\Omega} = \mathbf{R}^T\dot{\mathbf{R}}$ by the rotation $\mathbf{R}$, it comes the differential equation $\dot{\mathbf{R}} - \mathbf{R}\hat{\Omega} = 0$ whose solution is

$$\mathbf{R} = \mathbf{R}(0)e^{\hat{\Omega}t} \tag{4}$$

This is called the exponential map and it maps the Lie algebra $\mathfrak{so}_3$ into the Lie group SO_3. It provides a diffeomorphism between a neighborhood of 0 in $\mathfrak{so}_3$ and a neighborhood of the idendity in SO_3.

Since two rotations do not commute, the group SO_3 is curved, which is not the case of the "flat" physical space $\mathcal{R}^3$. According to Primault [11] mechanical models based on the material physical space can "lead to a certain complexity due, not to the intrinsic nonlinearity of the system (the curvature of the group) but rather to the parameter setting, that is to say to the flattening operation which it implies".

The dynamics of a mechanical system will not be described in terms of the evolution of its material components (material points in $\mathcal{R}^3$) but rather in terms of the transformations (translations and rotations) that they may endure. This subtle distinction makes it possible to obtain an exact dynamic model of nonlinear Reissner beams.

Hamilton's principle

Hamilton's principle states that the beam configuration $\mathbf{u}(X,t) = (\mathbf{r}(X,t), \mathbf{R}(X,t)$ must maximize the quantity

$$A = \int_{t_1}^{t_2} (T - V)dt, \quad \text{for all interval } [t_1, t_2]$$

where T and V represent kinetic and total potential energies. To obtain the dynamic equations of the beam an infinitesimal variation

$$\mathbf{r}_\epsilon = \mathbf{r} + \epsilon\delta\mathbf{r}, \quad \mathbf{R}_\epsilon = \mathbf{R}e^{\epsilon\delta\hat{\boldsymbol{\psi}}} \tag{5}$$

is introduced in the neighborhood of the solution $\mathbf{u}$. If the action A is a extremun for the solution $\mathbf{u}$, its Lie derivative should vanish :

$$\delta A = \frac{d}{d\epsilon}\Big|_{\epsilon=0} A_\epsilon = \lim_{\epsilon \to 0} \frac{A_\epsilon - A}{\epsilon} = \int_{t_1}^{t_2} (\delta T - \delta V)dt = 0. \tag{6}$$

So, the variations δT and δV, in the neighborhood of the solution $\mathbf{u}$, due to the variations (5) have to be determined in order to obtain a mathematical model for the beam dynamics. In the following, the potential energy V will be splited into internal and external parts : $V = U_{int} - U_{ext}$.

Contribution to Hamilton's principle due to kenetic energy

Using Lie algebras tools, the kinetic energy density $1/2\rho\dot{\mathbf{x}}^T\dot{\mathbf{x}}$ can be evaluated with the velocity expression, $\dot{\mathbf{x}}(X,t) = \dot{\mathbf{r}}(X,t) + \mathbf{R}(X,t)\hat{\Omega}.\mathbf{w}_0$, to give

$$1/2\rho\dot{\mathbf{x}}^T\dot{\mathbf{x}} = 1/2\rho\big[\dot{\mathbf{r}}^T\dot{\mathbf{r}} + 2\dot{\mathbf{r}}^T\mathbf{R}\hat{\Omega}\mathbf{w}_0 + \Omega^T\hat{\mathbf{w}}_0{}^T\hat{\mathbf{w}}_0\Omega\big].$$

Integrating over the beam, and taking into account the center of mass property[1], the kinetic energy and its variation are finally

$$T = 1/2\int_0^L \rho A\dot{\mathbf{r}}^T\dot{\mathbf{r}}dX + 1/2\int_0^L \rho\Omega^T J\Omega dX \text{ and } \delta T = \int_0^L \rho A\delta\dot{\mathbf{r}}^T\dot{\mathbf{r}}dX + \int_0^L \rho\delta\Omega^T J\Omega dX \tag{7}$$

where J is the inertial tensor[2]of the section.

Equation (7) shows that the variation, $\delta\Omega$, consequence of the variations (5) must be computed. To do so, lets calculate $\delta\hat{\Omega} = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\hat{\Omega}_\epsilon = \lim_{\epsilon\to 0}\frac{\hat{\Omega}_\epsilon - \hat{\Omega}}{\epsilon}$ with $\hat{\Omega}_\epsilon = \mathbf{R}_\epsilon^T\dot{\mathbf{R}}_\epsilon$ and

$$\begin{aligned}
\mathbf{R}_\epsilon &= \mathbf{R}e^{\epsilon\delta\hat{\boldsymbol{\psi}}} \simeq \mathbf{R}(\mathbb{I} + \epsilon\delta\hat{\boldsymbol{\psi}}) \\
\dot{\mathbf{R}}_\epsilon &= \dot{\mathbf{R}}e^{\epsilon\delta\hat{\boldsymbol{\psi}}} + \epsilon\mathbf{R}\delta\dot{\hat{\boldsymbol{\psi}}}e^{\epsilon\delta\hat{\boldsymbol{\psi}}} \simeq \dot{\mathbf{R}} + \epsilon(\dot{\mathbf{R}}\delta\hat{\boldsymbol{\psi}} + \mathbf{R}\delta\dot{\hat{\boldsymbol{\psi}}}) \\
\mathbf{R}_\epsilon^T\dot{\mathbf{R}}_\epsilon &= \mathbf{R}^T\dot{\mathbf{R}} + \epsilon[\mathbf{R}^T\dot{\mathbf{R}}\delta\hat{\boldsymbol{\psi}} + \delta\hat{\boldsymbol{\psi}}^T\mathbf{R}^T\dot{\mathbf{R}} + \delta\dot{\hat{\boldsymbol{\psi}}}] + O(\epsilon^2).
\end{aligned}$$

[1] In the current configuration, the fiber located by $\mathbf{r}(X,t)$ is the a line which joins the center of mass of each section. The kinetic energy expression is different for any other definition : the term $\int_0^L\int_S \rho\dot{\mathbf{r}}^T\mathbf{R}\hat{\Omega}\mathbf{w}_0 dSdX$ could not vanish. In the general case (variable cross-sectional area over the beam), this fiber must not be interpreted as the neutral axis (fiber without any deformation commonly used in mechanics).

[2] It is important to see that, as a consequence of the Lie algebra's rules, the inertial tensor

$$J = \int_S \hat{\mathbf{w}}_0{}^T\hat{\mathbf{w}}_0 dS = \int_S \begin{pmatrix} Y^2+Z^2 & 0 & 0 \\ 0 & Z^2 & -YZ \\ 0 & -YZ & Y^2 \end{pmatrix} dS$$

has been obtain from the $\mathbf{w}_0$ expression

$$\mathbf{w}_0 = \begin{pmatrix} 0 \\ Y \\ Z \end{pmatrix} \to \hat{\mathbf{w}}_0 = \begin{pmatrix} 0 & -Z & Y \\ Z & 0 & 0 \\ -Y & 0 & 0 \end{pmatrix}.$$

that is to say

$$\begin{aligned}\delta\hat{\Omega} &= \hat{\Omega}\delta\hat{\psi} + \delta\hat{\psi}^T\hat{\Omega} + \delta\hat{\dot{\psi}} = \hat{\Omega}\delta\hat{\psi} - \delta\hat{\psi}\hat{\Omega} + \delta\hat{\dot{\psi}} \\ &= [\hat{\Omega};\delta\hat{\psi}] + \delta\hat{\dot{\psi}} = \widehat{\Omega\wedge\delta\psi} + \delta\hat{\dot{\psi}}.\end{aligned}$$

In the last line the Lie crochet, $[A;B] = AB - BA$, has been used with the property $[\hat{A};\hat{B}] = \widehat{A\wedge B}$. In term of axial vector, the variation of the angular velocity is finally

$$\delta\Omega = \Omega\wedge\delta\psi + \delta\dot{\psi} \tag{8}$$

Putting this expression into variation (7) and integrating by part, the kinetic contribution to the Hamilton equilibrium equation (6) is obtained

$$\int_{t_1}^{t_2}\delta T dt = \int_0^L [\rho A\delta\mathbf{r}^T\dot{\mathbf{r}} + \delta\psi^T\rho J\Omega]_{t_1}^{t_2} dX - \int_{t_1}^{t_2}\int_0^L \begin{pmatrix}\delta\dot{\mathbf{r}}^T & \delta\psi^T\end{pmatrix}\begin{pmatrix}\rho A\delta\mathbf{r}^T\ddot{\mathbf{r}} \\ \rho J\dot{\Omega} + \Omega\wedge\rho J\Omega)\end{pmatrix} dXdt \tag{9}$$

Contribution to Hamilton's principle due to potential energy

In order to calculate the strain energy, two quantities are introduced : the strain vector Γ and the curvature tensor $\hat{\Pi}$. The first one distinguishes the longitudinal and tangential strain. The coordinate Γ_1 denotes the axial strain while the Γ_2 et Γ_3 coordinates measure the shear strain. This vector is obtain by the difference between the tangent to the axial fiber and the normal to the section, that is

$$\gamma = \partial_X\mathbf{r} - \mathbf{e}_1.$$

This quantity is evaluated in the moving referencial $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2)$. To convert it in the fixed referencial a rotation must be applied (see fig. 1),

$$\Gamma = \mathbf{R}^T\gamma = \mathbf{R}^T\partial_X\mathbf{r} - \mathbf{E}_1, \quad \text{written in the followings } \mathbf{R}^T\mathbf{r}' - \mathbf{E}_1 \tag{10}$$

This is an objective vector because it is invariant by any rigid body motion of the beam (global translation or rotation). The curvature tensor, $\hat{\Pi}$, is obtained, as the angular velocity $\hat{\Omega}$, by a derivation of the definition (1) of $SO3$, but this time a spatial derivation is used (see section)

$$\hat{\Pi} = \mathbf{R}^T\partial_X\mathbf{R}, \quad \text{written in the followings } \hat{\Pi} = \mathbf{R}^T\mathbf{R}' \tag{11}$$

This skew symmetric tensor is also an objective one and can be represented by an axial vector

$$\Pi = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}, \quad \text{such that } \hat{\Pi} = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix}$$

This pseudo vector takes into account strain due to the torsion (component k_1) and to the bending (k_2 et k_3). Assuming a linear stress-strain relation, those definitions (10) et (11) give a expression to the internal force $F = \mathbb{H}_d\Gamma$ and torque $M = \mathbb{H}_r\Pi$, where the Hooke tensors are

$$\mathbb{H}_d = \begin{pmatrix} EA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & GA \end{pmatrix}, \quad \mathbb{H}_r \begin{pmatrix} GI_\rho & 0 & 0 \\ 0 & EI_a & 0 \\ 0 & 0 & EI_a \end{pmatrix} \tag{12}$$

with E, G and A are the Young's modulus, shear coefficient and cross-sectional area respectively. The value $I_\rho = \int_S (Y^2 + Z^2)dS$ is the polar moment of inertia, while I_a represent the axial moment of inertia (for a cicular section $I_a = \int_S Y^2 dS = \int_S Z^2 dS$).

Under that circumstances, the strain energy and its variation are given in a quadratic form

$$U_{int} = 1/2\int_0^L \Gamma^T\mathbb{H}_d\Gamma dX + 1/2\int_0^L \Pi^T\mathbb{H}_r\Pi dX \text{ and } \delta U_{int} = \int_0^L \delta\Gamma^T\mathbb{H}_d\Gamma dX + \int_0^L \delta\Pi^T\mathbb{H}_r\Pi dX \tag{13}$$

The last expression shows that $\delta\Gamma$ and $\delta\Pi$ must be also computed. The same calculus as (8) can be achived to obtain

$$\delta\Gamma = \mathbf{R}^T\mathbf{r}'\wedge\delta\psi + \mathbf{R}^T\delta\mathbf{r}' \tag{14}$$

$$\delta\Pi = \Pi\wedge\delta\psi + \delta\psi' \tag{15}$$

which follow the elementary variations (5).

Again putting (14) and (15) in Hamilton's principle, integrating by part and taking into account the virtual work produced by external forces and moments $\delta U_{ext} = \delta\mathbf{r}^T\bar{F} + \delta\boldsymbol{\psi}^T\bar{M}$, a potential contribution is obtained

$$\begin{aligned}\int_{t_1}^{t_2}(\delta U_{int} - \delta U_{ext})dt &= \int_{t_1}^{t_2}\left(\left[\begin{pmatrix}\delta\mathbf{r}^T & \delta\boldsymbol{\psi}^T\end{pmatrix}\begin{pmatrix}\mathbf{R}\mathbb{H}_d\Gamma - \bar{F}\\ \mathbb{H}_r\Pi - \bar{M}\end{pmatrix}\right]_0^L\right.\\ &\quad - \left.\int_0^L\begin{pmatrix}\delta\mathbf{r}^T & \delta\boldsymbol{\psi}^T\end{pmatrix}\begin{pmatrix}(\mathbf{R}\mathbb{H}_d\Gamma)' + \bar{F}\\ \mathbf{R}^T\mathbf{r}'\wedge\mathbb{H}_d\Gamma + \mathbf{R}^T(\mathbf{R}\mathbb{H}_r\Pi)' + \bar{M}\end{pmatrix}dX\right)dt.\end{aligned} \tag{16}$$

The difference of two terms (9) and (16) consitute the principle of virtual work applied to a Reissner Beam.

Nonlinear equations of movement

The nonlinear equilibrium equations can finally be obtained from (9) and (16) considering that the variations $\left(\delta\mathbf{r}^T, \delta\boldsymbol{\psi}^T\right)$ are virtuals

$$\begin{cases}\text{constitutives equations for } X \in]0, L[& \\ (\mathbf{R}\mathbb{H}_d\Gamma)' + \bar{F} = \rho A\ddot{\mathbf{r}} & \text{translation}\\ \mathbf{R}^T\mathbf{r}'\wedge\mathbb{H}_d\Gamma + \mathbf{R}^T(\mathbf{R}\mathbb{H}_r\Pi)' + \bar{M} = \rho J\dot{\Omega} + \Omega\wedge\rho J\Omega & \text{rotation}\\ \text{boundaries conditions for } X = 0 \text{ and} X = L & \\ \mathbf{R}\mathbb{H}_d\Gamma = \bar{F} & \text{forces}\\ \mathbb{H}_r\Pi = \bar{M} & \text{torques}\end{cases} \tag{17}$$

The strains definitions of Γ and Π must be added to these relations , as functions of the variables $\mathbf{r}$ et $\mathbf{R}$ given by (10) and (11) ; the angular velocity is related to the rotation $\mathbf{R}$ by

$$\hat{\Omega} = \mathbf{R}^T\dot{\mathbf{R}} \tag{18}$$

DYNAMIC BEHAVIOR OF A PRESTRESSED BEAM

In this section an axial prestressed beam is studied. First a trivial static equlibrium solution is obtain from the nonlinear equations (17). In a second time, an linearization of this model in the neighbourhood of the static solution is outlined leading to a linear model of a string where stretching, bending, shearing, rotating and coupling phenomena are taking into account. Finally a second order model is obtained to catch the first nonlinear phenomena of a string.

Static equilibrium

A beam is subjected to a Dirichlet boundary condition at $X = 0$ and to a static force $\bar{F}_L$ to its extremity $X = L$. Introducing a tension coefficient α, it is easy to see that the couple $(\mathbf{r}_0(X), \mathbf{R}_0)$ such that

$$\mathbf{r}_0(X) = (\alpha + 1)\mathbf{E}_1 X, \quad \text{and } \mathbf{R}_0 = \mathbb{I} \tag{19}$$

is a static solution of (17) with $\bar{F}_L = \alpha EA\mathbf{E}_1$.

String : first and second order model

A linear and a second order model are then obtained, in the neighbourhood of the static solution, by introducing small variations $(\Delta\mathbf{r}, \Delta\boldsymbol{\psi})$ in the form

$$\mathbf{r}_\epsilon = \mathbf{r}_0 + \epsilon\Delta\mathbf{r}, \quad \mathbf{R}_\epsilon = e^{\epsilon\Delta\hat{\boldsymbol{\psi}}} = \mathbb{I} + \epsilon\Delta\hat{\boldsymbol{\psi}} + \epsilon^2/2\Delta\hat{\boldsymbol{\psi}}\Delta\hat{\boldsymbol{\psi}} + \ldots \tag{20}$$

This leads to the constitutives linear equations for $X \in]0, L[$

$$\begin{cases}\mathbb{H}_d\Delta\mathbf{r}'' + \mathbb{H}_c\Delta\boldsymbol{\psi}' + \Delta\bar{F} = \rho A\Delta\ddot{\mathbf{r}} & \text{translation}\\ \mathbb{H}_r\Delta\boldsymbol{\psi}'' + \mathbb{H}_c\Delta\mathbf{r}' + \mathbb{H}_\psi\Delta\boldsymbol{\psi} + \Delta\bar{M} = \rho J\Delta\ddot{\boldsymbol{\psi}} & \text{rotation}\end{cases} \tag{21}$$

where $\mathbb{H}_c = A\big[(\alpha+1)G - \alpha E\big]\hat{\mathbf{E}}_1$ and $\mathbb{H}_\psi = (\alpha+1)\mathbb{H}_c\hat{\mathbf{E}}_1$ .The skew-symmetric matrix $\hat{\mathbf{E}}_1$ is made from $\mathbf{E}_1$ considered as an axial vector. This leads to the matrix $\mathbb{H}_c$, also skew-symmetric, which traduces the coupling between translation and rotation. Dynamic external forces and moments applied to the string are handled by

$\Delta\bar{F}$ and $\Delta\bar{M}$ repectively. The configuration of the beam can be restituted by (20) with the choice $\epsilon = 1$. The constitutives equations (21) change at the second order to give

$$\begin{cases} \mathbb{H}_d\Delta\mathbf{r}'' + \mathbb{H}_c\Delta\boldsymbol{\psi}' + \Delta\bar{F} - \rho A\Delta\ddot{\mathbf{r}} + \mathbb{H}_{\psi\psi}\big[\Delta\boldsymbol{\psi}\wedge(\mathbf{E}_1\wedge\Delta\boldsymbol{\psi})\big]' + \big([\Delta\hat{\boldsymbol{\psi}};\mathbb{H}_d]\Delta\mathbf{r}'\big)' = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{translation} \\ \mathbb{H}_r\Delta\boldsymbol{\psi}'' + \mathbb{H}_c\Delta\mathbf{r}' + \mathbb{H}_\psi\Delta\boldsymbol{\psi} + \Delta\bar{M} - \rho J\Delta\ddot{\boldsymbol{\psi}} \\ +(\mathbf{E}_1\wedge\Delta\boldsymbol{\psi})\wedge\mathbb{H}_e\Delta\mathbf{r}' + \Delta\mathbf{r}'\wedge\mathbb{H}_d\Delta\mathbf{r}' + \mathbb{H}_c(\Delta\mathbf{r}'\wedge\Delta\boldsymbol{\psi}) - \mathbb{H}_f(\Delta\boldsymbol{\psi}\wedge(\mathbf{E}_1\wedge\Delta\boldsymbol{\psi})) \\ +\Delta\boldsymbol{\psi}'\mathbb{H}_r\Delta\boldsymbol{\psi}' + \frac{1}{2}\mathbb{H}_r(\Delta\boldsymbol{\psi}''\wedge\Delta\boldsymbol{\psi}) - \frac{1}{2}\rho J(\Delta\ddot{\boldsymbol{\psi}}\wedge\Delta\boldsymbol{\psi}) - \Delta\dot{\boldsymbol{\psi}}\wedge\rho J\Delta\dot{\boldsymbol{\psi}} = 0 \qquad \text{rotation} \end{cases} \tag{22}$$

where et $\mathbb{H}_{\psi\psi} = A\big[(\alpha+1)G - \frac{\alpha}{2}E\big]\mathbb{I} - \frac{\alpha+1}{2}\mathbb{H}_d$, $\mathbb{H}_e = (\alpha+1)(\mathbb{H}_d - GA\mathbb{I})$ et $\mathbb{H}_f = \frac{(\alpha+1)^2}{2}\mathbb{H}_c$

CONCLUSIONS AND PERSPECTIVES

The linear model (21) can be easily solved using numerical methods. In the thesis [2], a modal method base on finite element is proposed and can lead to real time sound synthesis. For the second order model Volterra series can be evoked. This technic is usually used to solve algrebraic nonlinear differential equations but an extension to weakly nonlinear partial differential equations have been made by T. Helie and M. Hasler in [8]. Since Helie [9] has adapted the method to treat acoustic propagation in tubes and Roze [12] for nonlinear string. Under some hypothesis (relatively small variations) the system of equations (22) are suitable for such a method saving computation time compare to the general Newmark procedure proposed by Boyer and Primault in [4] or [11]. Sound synthesis simulation are in progress and will be presented at the conference.

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