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# On Jacobi fields and canonical connection in sub-Riemannian geometry

Davide Barilari<sup>b</sup> and Luca Rizzi<sup>#</sup>

ABSTRACT. In sub-Riemannian geometry the coefficients of the Jacobi equation define curvature-like invariants. We show that these coefficients can be interpreted as the curvature of a canonical Ehresmann connection associated to the metric, first introduced in [14]. We show why this connection is naturally nonlinear, and we discuss some of its properties.

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## 1. Introduction

A key tool for comparison theorems in Riemannian geometry is the Jacobi equation, i.e. the differential equation satisfied by Jacobi fields. Assume  $\gamma_\varepsilon$  is a one-parameter family of geodesics on a Riemannian manifold  $(M, g)$  satisfying

$$(1) \quad \ddot{\gamma}_\varepsilon^k + \Gamma_{ij}^k(\gamma_\varepsilon) \dot{\gamma}_\varepsilon^i \dot{\gamma}_\varepsilon^j = 0.$$

The corresponding Jacobi field  $J = \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \gamma_\varepsilon$  is a vector field defined along  $\gamma = \gamma_0$ , and satisfies the equation

$$(2) \quad \ddot{J}^k + 2\Gamma_{ij}^k J^i \dot{\gamma}^j + \frac{\partial \Gamma_{ij}^k}{\partial x^\ell} J^\ell \dot{\gamma}^i \dot{\gamma}^j = 0.$$

The Riemannian curvature is hidden in the coefficients of this equation. To make it appear explicitly, however, one has to write (2) in terms of a parallel transported frame  $X_1(t), \dots, X_n(t)$  along  $\gamma(t)$ . Letting  $J(t) = \sum_{i=1}^n J_i(t) X_i(t)$  one gets the following normal form:

$$(3) \quad \ddot{J}_i + R_{ij}(t) J_j = 0.$$

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Indeed the coefficients  $R_{ij}$  are related with the curvature  $R^\nabla$  of the unique linear, torsion free and metric connection  $\nabla$  (Levi-Civita) as follows

$$R_{ij} = g(R^\nabla(X_i, \dot{\gamma})\dot{\gamma}, X_j).$$

Eq. (3) is the starting point to prove many results in Riemannian geometry. In particular, bounds on the curvature (i.e. on the coefficients  $R$ , or its trace) have deep consequences on the analysis and the geometry of the underlying manifold.

In the sub-Riemannian setting this construction cannot be directly generalized. Indeed, the analogue of the Jacobi equation is a first-order system on the cotangent bundle that cannot be written as a second-order equation on the manifold. Still one can put it in a normal form, analogue to (3), and study its coefficients [14]. These appear to be the correct objects to bound to control the behavior of the geodesic flow and get comparison-like results (see for instance [10, 7]). Nevertheless one can wonder if these coefficients can arise, as in the Riemannian case, as the curvature of a suitable connection. We answer to this question, by showing that these coefficients are part of the curvature of a nonlinear canonical Ehresmann connection associated with the sub-Riemannian structure. In the Riemannian case this reduces to the classical, linear, Levi-Civita connection.

**1.1. The general setting.** A sub-Riemannian structure is a triple  $(M, \mathcal{D}, g)$  where  $M$  is smooth manifold,  $\mathcal{D}$  is a smooth, completely non-integrable vector sub-bundle of  $TM$  and  $g$  is a smooth scalar product on  $\mathcal{D}$ . Riemannian structures are included in this definition, taking  $\mathcal{D} = TM$ . The sub-Riemannian distance is the infimum of the length of absolutely continuous admissible curves joining two points. Here admissible means that the curve is almost everywhere tangent to the distribution  $\mathcal{D}$ , in order to compute its length via the scalar product  $g$ . The totally non-holonomic assumption on  $\mathcal{D}$  implies, by the Rashevskii-Chow theorem, that the distance is finite on every connected component of  $M$ , and the metric topology coincides with the one of  $M$ . A more detailed introduction on sub-Riemannian geometry can be found in [12, 2, 13, 8].

In Riemannian geometry, it is well-known that the geodesic flow can be seen as a Hamiltonian flow on the cotangent bundle  $T^*M$ , associated with the Hamiltonian

$$H(p, x) = \frac{1}{2} \sum_{i=1}^n \langle p, X_i(x) \rangle^2, \quad (p, x) \in T^*M,$$

where  $X_1, \dots, X_n$  is any local orthonormal frame for the Riemannian structure. In the sub-Riemannian case, the Hamiltonian is defined by the same formula, where the sum is taken over a local orthonormal frame  $X_1, \dots, X_k$  for  $\mathcal{D}$ , with  $k = \text{rank } \mathcal{D}$ . The restriction of  $H$  to each fiber is a degenerate quadratic form, but Hamilton's equations are still defined. These can be written as a flow on  $T^*M$

$$\dot{\lambda} = \vec{H}(\lambda), \quad \lambda \in T^*M,$$

where  $\vec{H}$  is the Hamiltonian vector field associated with  $H$ . This system cannot be written as a second order equation on  $M$  as in (1). The projection  $\pi : T^*M \rightarrow M$  of its integral curves are geodesics, i.e. locally minimizing curves. In the general case, some geodesics may not be recovered in this way. These are the so-called strictly abnormal geodesics [11], and they are related with hard open problems in sub-Riemannian geometry [1].

In what follows, with a slight abuse of notation, the term “geodesic” refers to the not strictly abnormal ones.

An integral line of the Hamiltonian vector field  $\lambda(t) = e^{t\vec{H}}(\lambda) \in T^*M$ , with initial covector  $\lambda$  is called *extremal*. Notice that the same geodesic may be the projection of two different extremals. For these reasons, it is convenient to see the

Jacobi equation as a first order equation for vector fields on  $T^*M$ , associated with an extremal, rather than a second order system on  $M$ , associated with a geodesic.

## 2. Jacobi equation revisited

For any vector field  $V(t)$  along an extremal  $\lambda(t)$  of the sub-Riemannian Hamiltonian flow, a dot denotes the Lie derivative in the direction of  $\vec{H}$ :

$$\dot{V}(t) := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} e_*^{-\varepsilon \vec{H}} V(t + \varepsilon).$$

A vector field  $\mathcal{J}(t)$  along  $\lambda(t)$  is called a *sub-Riemannian Jacobi field* if it satisfies

$$(4) \quad \dot{\mathcal{J}} = 0.$$

The space of solutions of (4) is a  $2n$ -dimensional vector space. The projections  $J = \pi_* \mathcal{J}$  are vector fields on  $M$  corresponding to one-parameter variations of  $\gamma(t) = \pi(\lambda(t))$  through geodesics; in the Riemannian case, they coincide with the classical Jacobi fields.

We intend to write (4) using the natural symplectic structure  $\sigma$  of  $T^*M$ . First, observe that on  $T^*M$  there is a natural smooth sub-bundle of Lagrangian<sup>1</sup> spaces:

$$\mathcal{V}_\lambda := \ker \pi_*|_\lambda = T_\lambda(T_{\pi(\lambda)}^*M).$$

We call this the *vertical subspace*. Then, pick a Darboux frame  $\{E_i(t), F_i(t)\}_{i=1}^n$  along  $\lambda(t)$ . It is natural to assume that  $E_1, \dots, E_n$  belong to the vertical subspace. To fix the ideas, one can think at the canonical basis  $\{\partial_{p_i}|_{\lambda(t)}, \partial_{x_i}|_{\lambda(t)}\}$  induced by a choice of coordinates  $(x_1, \dots, x_n)$  on  $M$ .

In terms of this frame,  $\mathcal{J}(t)$  has components  $(p(t), x(t)) \in \mathbb{R}^{2n}$ :

$$\mathcal{J}(t) = \sum_{i=1}^n p_i(t) E_i(t) + x_i(t) F_i(t).$$

The elements of the frame satisfy

$$(5) \quad \begin{pmatrix} \dot{E} \\ \dot{F} \end{pmatrix} = \begin{pmatrix} C_1(t)^* & -C_2(t) \\ R(t) & -C_1(t) \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix},$$

for some smooth families of  $n \times n$  matrices  $C_1(t), C_2(t), R(t)$ , where  $C_2(t) = C_2(t)^*$  and  $R(t) = R(t)^*$ . We stress that the particular structure of the equations is implied solely by the fact that the frame is Darboux, that is

$$\sigma(E_i, E_j) = \sigma(F_i, F_j) = \sigma(E_i, F_j) - \delta_{ij} = 0, \quad i, j = 1, \dots, n.$$

Moreover,  $C_2(t) \geq 0$  as a consequence of the non-negativity of the sub-Riemannian Hamiltonian. To see this, for a bilinear form  $B : V \times V \rightarrow \mathbb{R}$  and  $n$ -tuples  $v, w \in V$  let  $B(v, w)$  denote the matrix  $B(v_i, w_j)$ . With this notation

$$C_2(t) = \sigma(\dot{E}, E)|_{\lambda(t)} = 2H(E, E)|_{\lambda(t)} \geq 0,$$

where we identified  $\mathcal{V}_{\lambda(t)} \simeq T_{\gamma(t)}^*M$  and we see the Hamiltonian as a symmetric bilinear form on fibers. In the Riemannian case,  $C_2(t) > 0$ . In turn, the Jacobi equation, written in terms of the components  $(p(t), x(t))$ , becomes

$$(6) \quad \begin{pmatrix} \dot{p} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} -C_1(t) & -R(t) \\ C_2(t) & C_1(t)^* \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}.$$

<sup>1</sup>A Lagrangian subspace  $L \subset \Sigma$  of a symplectic vector space  $(\Sigma, \sigma)$  is a subspace with  $\dim L = \dim \Sigma / 2$  and  $\sigma|_L = 0$ .

### 3. The Riemannian case

In the Riemannian case one can choose a suitable frame to simplify as much as possible (6). Let  $X_1, \dots, X_n$  be a parallel transported frame along the geodesic  $\gamma(t)$ . Let  $h_i : T^*M \rightarrow \mathbb{R}$  be the fiber-wise linear functions, defined by  $h_i(\lambda) := \langle \lambda, X_i \rangle$ . Indeed  $h_1, \dots, h_n$  define coordinates on each fiber, and the vectors  $\partial_{h_i}$ . We define a moving frame along the extremal  $\lambda(t)$  as follows

$$E_i := \partial_{h_i}, \quad F_i := -\dot{E}_i.$$

One can recover the original parallel transported frame by projection, namely  $\pi_* F_i|_{\lambda(t)} = X_i|_{\gamma(t)}$ . We state here the properties of the moving frame.

PROPOSITION 3.1. *The smooth moving frame  $\{E_i, F_i\}_{i=1}^n$  satisfies:*

(i)  $\pi_* E_i|_{\lambda(t)} = 0$ .

(ii) *It is a Darboux basis, namely*

$$\sigma(E_i, E_j) = \sigma(F_i, F_j) = \sigma(E_i, F_j) - \delta_{ij} = 0, \quad i, j = 1, \dots, n.$$

(iii) *The frame satisfies the structural equations*

$$\dot{E}_i = -F_i, \quad \dot{F}_i = \sum_{j=1}^n R_{ij}(t)E_j,$$

for some smooth family of  $n \times n$  symmetric matrices  $R(t)$ .

If  $\{\tilde{E}_i, \tilde{F}_j\}_{i=1}^n$  is another smooth moving frame along  $\lambda(t)$  satisfying (i)-(iii), for some matrix  $\tilde{R}(t)$  then there exist a constant, orthogonal matrix  $O$  such that

$$\tilde{E}_i|_{\lambda(t)} = \sum_{j=1}^n O_{ij}E_j|_{\lambda(t)}, \quad \tilde{F}_i|_{\lambda(t)} = \sum_{j=1}^n O_{ij}F_j|_{\lambda(t)}, \quad \tilde{R}(t) = OR(t)O^*.$$

Thanks to this proposition, the symmetric matrix  $R(t)$  induces a well defined quadratic form  $\mathfrak{R}_{\lambda(t)} : T_{\gamma(t)}M \times T_{\gamma(t)}M \rightarrow \mathbb{R}$

$$\mathfrak{R}_{\lambda(t)}(v, v) := \sum_{i,j=1}^n R_{ij}(t)v_i v_j, \quad v = \sum_{i=1}^n v_i X_i|_{\gamma(t)}.$$

Indeed one can prove that

$$(7) \quad \mathfrak{R}_{\lambda(t)}(v, v) = g(R^\nabla(v, \dot{\gamma})\dot{\gamma}, v), \quad v \in T_{\gamma(t)}M.$$

The proof is a standard computation that can be found, for instance, in [7, Appendix C]. Then, in the Jacobi equation (6), one has  $C_1(t) = 0$ ,  $C_2(t) = \mathbb{I}$  (in particular, they are constant matrices), and the only non-trivial block  $R(t)$  is the curvature operator along the geodesic:

$$\dot{x} = p, \quad \dot{p} = -R(t)x,$$

### 4. The sub-Riemannian case

The problem of finding a the set of Darboux frames normalizing the Jacobi equation has been first studied by Agrachev-Zelenko in [5, 6] and subsequently completed by Zelenko-Li in [14] in the general setting of curves in the Lagrange Grassmannian. A dramatic simplification, analogue to the Riemannian one, cannot be achieved in the general sub-Riemannian setting. Nevertheless, it is possible to find a normal form of (6) where the matrices  $C_1$  and  $C_2$  are constant. Moreover, the very block structure of these matrices depends on the geodesic and already contains important geometric invariants, that we now introduce.

**4.1. Geodesic flag and Young diagram.** Let  $\gamma(t)$  be a geodesic. Recall that  $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$  for every  $t$ . Consider a smooth admissible extension of the tangent vector, namely a vector field  $\mathbb{T} \in \Gamma(\mathcal{D})$  such that  $\mathbb{T}|_{\gamma(t)} = \dot{\gamma}(t)$ .

DEFINITION 4.1. The *flag of the geodesic*  $\gamma(t)$  is the sequence of subspaces

$$\mathcal{F}_{\gamma(t)}^i := \text{span}\{\mathcal{L}_{\mathbb{T}}^j(X)|_{\gamma(t)} \mid X \in \Gamma(\mathcal{D}), j \leq i-1\} \subseteq T_{\gamma(t)}M, \quad \forall i \geq 1,$$

where  $\mathcal{L}_{\mathbb{T}}$  denotes the Lie derivative in the direction of  $\mathbb{T}$ .

By definition, this is a filtration of  $T_{\gamma(t)}M$ , i.e.  $\mathcal{F}_{\gamma(t)}^i \subseteq \mathcal{F}_{\gamma(t)}^{i+1}$ , for all  $i \geq 1$ . Moreover,  $\mathcal{F}_{\gamma(t)}^1 = \mathcal{D}_{\gamma(t)}$ . Definition 4.1 is well posed, namely does not depend on the choice of the admissible extension  $\mathbb{T}$  (see [3, Sec. 3.4]). The *growth vector* of the geodesic  $\gamma(t)$  is the sequence of integer numbers

$$\mathcal{G}_{\gamma(t)} := \{\dim \mathcal{F}_{\gamma(t)}^1, \dim \mathcal{F}_{\gamma(t)}^2, \dots\}.$$

A geodesic  $\gamma(t)$ , with growth vector  $\mathcal{G}_{\gamma(t)}$ , is said

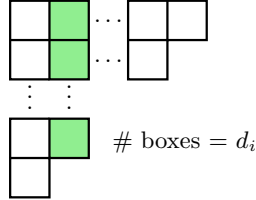
- *equiregular* if  $\dim \mathcal{F}_{\gamma(t)}^i$  does not depend on  $t$  for all  $i \geq 1$ ,
- *ample* if for all  $t$  there exists  $m \geq 1$  such that  $\dim \mathcal{F}_{\gamma(t)}^m = \dim T_{\gamma(t)}M$ .

Equiregular (resp. ample) geodesics are the microlocal counterpart of equiregular (resp. bracket-generating) distributions. Let  $d_i := \dim \mathcal{F}_{\gamma}^i - \dim \mathcal{F}_{\gamma}^{i-1}$ , for  $i \geq 1$ , be the increment of dimension of the flag of the geodesic at each step (with the convention  $\dim \mathcal{F}^0 = 0$ ).

LEMMA 4.2 ([3]). *For an equiregular, ample geodesic,  $d_1 \geq d_2 \geq \dots \geq d_m$ .*

The generic geodesic is ample and equiregular. More precisely, the set of points  $x \in M$  such that there exists a non-empty Zariski open set  $A_x \subseteq T_x^*M$  of initial covectors for which the associated geodesic is ample and equiregular with the same (maximal) growth vector, is open and dense in  $M$ . See [3, 14] for more details.

For an ample, equiregular geodesic we can build a tableau  $D$  with  $m$  columns of length  $d_i$ , for  $i = 1, \dots, m$ , as follows:



Indeed  $\sum_{i=1}^m d_i = n = \dim M$  is the total number of boxes in  $D$ .

Consider an ample, equiregular geodesic, with Young diagram  $D$ , with  $k$  rows, of length  $n_1, \dots, n_k$ . Indeed  $n_1 + \dots + n_k = n$ . The moving frame we are going to introduce is indexed by the boxes of the Young diagram. The notation  $ai \in D$  denotes the generic box of the diagram, where  $a = 1, \dots, k$  is the row index, and  $i = 1, \dots, n_a$  is the progressive box number, starting from the left, in the specified row. We employ letters  $a, b, c, \dots$  for rows, and  $i, j, h, \dots$  for the position of the box in the row.

We collect the rows with the same length in  $D$ , and we call them *levels* of the Young diagram. In particular, a level is the union of  $r$  rows  $D_1, \dots, D_r$ , and  $r$  is called the *size* of the level. The set of all the boxes  $ai \in D$  that belong to the same column and the same level of  $D$  is called *superbox*. We use Greek letters  $\alpha, \beta, \dots$  to denote superboxes. Notice that two boxes  $ai, bj$  are in the same superbox if and only if  $ai$  and  $bj$  are in the same column of  $D$  and in possibly distinct row but with same length, i.e. if and only if  $i = j$  and  $n_a = n_b$  (see Fig. 1).

The following theorem is proved in [14].

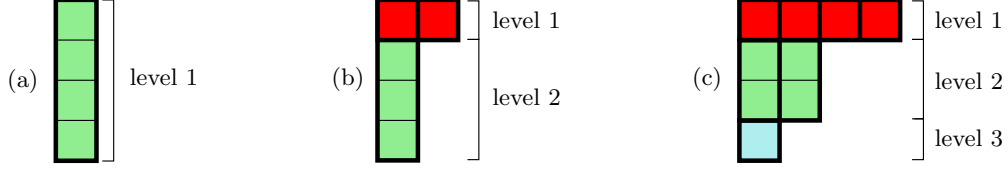


FIGURE 1. Levels (shaded regions) and superboxes (delimited by bold lines) for the Young diagram of (a) Riemannian, (b) contact, (c) a more general structure. The Young diagram for any Riemannian geodesic has a single level and a single superbox.

**THEOREM 4.3.** *Assume  $\lambda(t)$  is the lift of an ample and equiregular geodesic  $\gamma(t)$  with Young diagram  $D$ . Then there exists a smooth moving frame  $\{E_{ai}, F_{ai}\}_{ai \in D}$  along  $\lambda(t)$  such that*

- (i)  $\pi_* E_{ai}|_{\lambda(t)} = 0$ .
- (ii) *It is a Darboux basis, namely*

$$\sigma(E_{ai}, E_{bj}) = \sigma(F_{ai}, F_{bj}) = \sigma(E_{ai}, F_{bj}) = \delta_{ab} \delta_{ij}, \quad ai, bj \in D.$$

- (iii) *The frame satisfies structural equations*

$$(8) \quad \begin{cases} \dot{E}_{ai} = E_{a(i-1)} & a = 1, \dots, k, \quad i = 2, \dots, n_a, \\ \dot{E}_{a1} = -F_{a1} & a = 1, \dots, k, \\ \dot{F}_{ai} = \sum_{bj \in D} R_{ai, bj}(t) E_{bj} - F_{a(i+1)} & a = 1, \dots, k, \quad i = 1, \dots, n_a - 1, \\ \dot{F}_{an_a} = \sum_{bj \in D} R_{an_a, bj}(t) E_{bj} & a = 1, \dots, k, \end{cases}$$

for some smooth family of  $n \times n$  symmetric matrices  $R(t)$ , with components  $R_{ai, bj}(t) = R_{bj, ai}(t)$ , indexed by the boxes of the Young diagram  $D$ . The matrix  $R(t)$  is normal in the sense of [14] (see Appendix A).

If  $\{\tilde{E}_{ai}, \tilde{F}_{ai}\}_{ai \in D}$  is another smooth moving frame along  $\lambda(t)$  satisfying (i)-(iii), with some normal matrix  $\tilde{R}(t)$ , then for any superbox  $\alpha$  of size  $r$  there exists an orthogonal constant  $r \times r$  matrix  $O^\alpha$  such that

$$\tilde{E}_{ai} = \sum_{bj \in \alpha} O_{ai, bj}^\alpha E_{bj}, \quad \tilde{F}_{ai} = \sum_{bj \in \alpha} O_{ai, bj}^\alpha F_{bj}, \quad ai \in \alpha.$$

**REMARK 4.4.** For  $a = 1, \dots, k$ , the symbol  $E_a$  denotes the  $n_a$ -dimensional column vector  $E_a = (E_{a1}, E_{a2}, \dots, E_{an_a})^*$ , with analogous notation for  $F_a$ . Similarly,  $E$  denotes the  $n$ -dimensional column vector  $E = (E_1, \dots, E_k)^*$ , and similarly for  $F$ . Then, we rewrite the system (8) as follows (compare with (5))

$$(9) \quad \begin{pmatrix} \dot{E} \\ \dot{F} \end{pmatrix} = \begin{pmatrix} C_1^* & -C_2 \\ R(t) & -C_1 \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix},$$

where  $C_1 = C_1(D)$ ,  $C_2 = C_2(D)$  are  $n \times n$  matrices, depending on the Young diagram  $D$ , defined as follows: for  $a, b = 1, \dots, k$ ,  $i = 1, \dots, n_a$ ,  $j = 1, \dots, n_b$ :

$$[C_1]_{ai, bj} := \delta_{ab} \delta_{i, j-1}, \quad [C_2]_{ai, bj} := \delta_{ab} \delta_{i1} \delta_{j1}.$$

It is convenient to see  $C_1$  and  $C_2$  as block diagonal matrices:

$$C_i(D) := \begin{pmatrix} C_i(D_1) & & \\ & \ddots & \\ & & C_i(D_k) \end{pmatrix}, \quad i = 1, 2,$$

the  $a$ -th block being the  $n_a \times n_a$  matrices

$$C_1(D_a) := \begin{pmatrix} 0 & \mathbb{I}_{n_a-1} \\ 0 & 0 \end{pmatrix}, \quad C_2(D_a) := \begin{pmatrix} 1 & 0 \\ 0 & 0_{n_a-1} \end{pmatrix},$$

where  $\mathbb{I}_m$  is the  $m \times m$  identity matrix and  $0_m$  is the  $m \times m$  zero matrix. Notice that the matrices  $C_1, C_2$  satisfy the Kalman rank condition

$$(10) \quad \text{rank}\{C_2, C_1 C_2, \dots, C_1^{n-1} C_2\} = n.$$

Analogously, the matrices  $C_i(D_a)$  satisfy (10) with  $n = n_a$ .

Let  $\{X_{ai}\}_{ai \in D}$  be the moving frame along  $\gamma(t)$  defined by  $X_{ai}|_{\gamma(t)} = \pi_* F_{ai}|_{\lambda(t)}$ , for some choice of a canonical Darboux frame. Theorem 4.3 implies that the following definitions are well posed.

DEFINITION 4.5. The *canonical splitting* of  $T_{\gamma(t)}M$  is

$$T_{\gamma(t)}M = \bigoplus_{\alpha} S_{\gamma(t)}^{\alpha}, \quad S_{\gamma(t)}^{\alpha} := \text{span}\{X_{ai}|_{\gamma(t)} \mid ai \in \alpha\},$$

where the sum is over the superboxes  $\alpha$  of  $D$ . Notice that the dimension of  $S_{\gamma(t)}^{\alpha}$  is equal to the size  $r$  of the level to which the superbox  $\alpha$  belongs.

DEFINITION 4.6. The *sub-Riemannian directional curvature*, is the quadratic form  $\mathfrak{R}_{\lambda(t)} : T_{\gamma(t)}M \times T_{\gamma(t)}M \rightarrow \mathbb{R}$  whose representative matrix, in terms of the basis  $\{X_{ai}\}_{ai \in D}$ , is  $R_{ai,bj}(t)$ . In other words

$$\mathfrak{R}_{\lambda(t)}(v, v) := \sum_{ai, bj \in D} R_{ai, bj}(t) v_{ai} v_{bj}, \quad v = \sum_{ai \in D} v_{ai} X_{ai}|_{\gamma(t)} \in T_{\gamma(t)}M.$$

We denote the restrictions of  $\mathfrak{R}_{\lambda(t)}$  on the appropriate subspaces by:

$$\mathfrak{R}_{\lambda(t)}^{\alpha\beta} : S_{\gamma(t)}^{\alpha} \times S_{\gamma(t)}^{\beta} \rightarrow \mathbb{R}.$$

The Jacobi equation, written in terms of the components  $(p(t), x(t))$  with respect to a canonical Darboux frame  $\{E_{ai}, F_{ai}\}_{ai \in D}$ , becomes

$$\begin{pmatrix} \dot{p} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} -C_1 & -R(t) \\ C_2 & C_1^* \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}.$$

This is the sub-Riemannian generalization of the classical Jacobi equation seen as first-order equation for fields on the cotangent bundle. Its structure depends on the Young diagram of the geodesic through the matrices  $C_i(D)$ , while the remaining invariants are contained in the curvature matrix  $R(t)$ . Notice that this includes the Riemannian case, where  $D$  is the same for every geodesic, with  $C_1 = 0$  and  $C_2 = \mathbb{I}$ .

**4.2. Homogeneity properties.** For all  $c > 0$ , let  $H_c := H^{-1}(c/2)$  be the Hamiltonian level set. In particular  $H_1$  is the unit cotangent bundle: the set of initial covectors associated with unit-speed geodesics. Since the Hamiltonian function is fiber-wise quadratic, we have the following property for any  $c > 0$

$$(11) \quad e^{t\vec{H}}(c\lambda) = ce^{ct\vec{H}}(\lambda).$$

Let  $P_c : T^*M \rightarrow T^*M$  the dilation along the fibers  $P_c(\lambda) = c\lambda$ . Indeed  $\alpha \mapsto P_c^\alpha$  is a one-parameter group of diffeomorphisms. Its generator is the *Euler vector field*  $\mathfrak{e} \in \Gamma(\mathcal{V})$ , and is characterized by  $P_c = e^{(\ln c)\mathfrak{e}}$ . We can rewrite (11) as the following commutation rule for the flows of  $\vec{H}$  and  $\mathfrak{e}$ :

$$e^{t\vec{H}} \circ P_c = P_c \circ e^{ct\vec{H}}.$$

Observe that  $P_c$  maps  $H_1$  diffeomorphically on  $H_c$ . Let  $\lambda \in H_1$  be associated with an ample, equiregular geodesic with Young diagram  $D$ . Clearly also the geodesic



associated with  $\lambda^c := c\lambda \in H_c$  is ample and equiregular, with the same Young diagram. This corresponds to a reparametrization of the same curve: in fact  $\lambda^c(t) = e^{t\bar{H}}(c\lambda) = c(\lambda(ct))$ , hence  $\gamma^c(t) = \pi(\lambda^c(t)) = \gamma(ct)$ .

**THEOREM 4.7** (Homogeneity properties of the directional curvature). *For any superbox  $\alpha \in D$ , let  $|\alpha|$  denote the column index of  $\alpha$ . Denoting  $\lambda^c(t) = e^{t\bar{H}}(c\lambda)$  we have, for any  $c > 0$*

$$\mathfrak{R}_{\lambda^c(t)}^{\alpha\beta} = c^{|\alpha|+|\beta|}\mathfrak{R}_{\lambda(ct)}^{\alpha\beta},$$

**REMARK 4.8.** In the Riemannian setting,  $D$  has only one superbox with  $|\alpha| = 1$  (see Fig. 1). Then  $\mathfrak{R}_\lambda := \mathfrak{R}_{\lambda(0)}^{\alpha\alpha}$  is homogeneous of degree 2 as a function of  $\lambda$ .

Theorem 4.7 follows directly from the next result and Definition 4.6.

**PROPOSITION 4.9.** *Let  $\lambda \in H_1$  and  $\{E_{ai}, F_{ai}\}_{ai \in D}$  be the associated canonical frame along the extremal  $\lambda(t)$ . Let  $c > 0$  and define, for  $ai \in D$*

$$E_{ai}^c(t) := \frac{1}{c^i}(d_{\lambda(ct)}P_c)E_{ai}(ct), \quad F_{ai}^c(t) := c^{i-1}(d_{\lambda(ct)}P_c)F_{ai}(ct).$$

*The moving frame  $\{E_{ai}^c(t), F_{ai}^c(t)\}_{ai \in D} \in T_{\lambda^c(t)}(T^*M)$  is a canonical frame associated with the initial covector  $\lambda^c = c\lambda \in H_c$ , with curvature matrix*

$$(12) \quad R_{ai,bj}^{\lambda^c}(t) = c^{i+j}R_{ai,bj}^\lambda(ct).$$

**PROOF.** We check all the relations of Theorem 4.3. Indeed  $P_\alpha$  sends fibers to fibers, hence (i) is trivially satisfied. For what concerns (ii), let  $\theta$  be the Liouville one-form, and  $\sigma = d\theta$ . Indeed  $P_c^*\theta = c\theta$ . Hence  $P_c^*\sigma = c\sigma$ . It follows that  $\{E_{ai}^c(t), F_{ai}^c(t)\}_{ai \in D}$  is a Darboux frame at  $\lambda^c(t)$ :

$$\sigma_{\lambda^c(t)}(E_{ai}^c(t), F_{bj}^c(t)) = \frac{1}{c}(P_c^*\sigma)_{\lambda(t)}(E_{ai}(t), F_{bj}(t)) = \delta_{ab}\delta_{ij},$$

and similarly for the others Darboux relations.

For what concerns (iii) (the structural equations), let  $\xi(t)$  be any vector field along  $\lambda(t)$ , and  $(d_{\lambda(t)}P_c)\xi(ct)$  be the corresponding vector field along  $\lambda^c(t)$ . Then

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} e_*^{-\varepsilon\bar{H}} \circ (d_{\lambda(t)}P_c)\xi(c(t+\varepsilon)) &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} (e^{-\varepsilon\bar{H}} \circ P_c)_*\xi(c(t+\varepsilon)) \\ &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} (P_c \circ e^{-c\varepsilon\bar{H}})_*\xi(c(t+\varepsilon)) \\ &= c \frac{d}{d\tau} \Big|_{\tau=0} (P_c \circ e^{-\tau\bar{H}})_*\xi(ct+\tau) \\ &= c(d_{\lambda(ct)}P_c)\dot{\xi}(ct). \end{aligned}$$

Applying the above identity to compute the derivatives of the new frame, and using (8), one finds that  $\{E_{ai}^c(t), F_{ai}^c(t)\}_{ai \in D}$  satisfies the structural equations, with curvature matrix given by (12). For example

$$\begin{aligned} \dot{F}_{ai}^c(t) &= c^{i-1}c(d_{\lambda(ct)}P_c)\dot{F}_{ai}(ct) \\ &= c^i(d_{\lambda(ct)}P_c)[R_{ai,bj}^\lambda(ct)E_{bj}(ct) - F_{a(i+1)}(ct)] \\ &= c^i[c^j R_{ai,bj}^\lambda(ct)E_{bj}^c(t) - c^{-i}F_{a(i+1)}^c(t)] \\ &= c^{i+j}R_{ai,bj}^\lambda(ct)E_{bj}^c(t) - F_{a(i+1)}^c(t), \end{aligned}$$

where we suppressed a summation over  $bj \in D$ .  $\square$

Proposition 4.3 defines not only a curvature, but also a (non-linear) connection, in the sense of Ehresmann, that we now introduce.

### 5. Ehresmann curvature and curvature operator

For any vector bundle  $N$  over  $M$ , the notation  $\Gamma(N)$  denotes the sections of  $N$ . Recall that  $\mathcal{V} := \ker \pi_* \subset T(T^*M)$  is the *vertical distribution*. An *Ehresmann connection* on  $T^*M$  is a smooth distribution  $\mathcal{H} \subset T(T^*M)$  such that

$$T(T^*M) = \mathcal{H} \oplus \mathcal{V}.$$

We call  $\mathcal{H}$  the *horizontal distribution*<sup>2</sup>. An Ehresmann connection  $\mathcal{H}$  is *linear* if  $\mathcal{H}_{c\lambda} = (d_\lambda P_c)\mathcal{H}_\lambda$  for every  $\lambda \in T^*M$  and  $c > 0$ .

For any  $X \in \Gamma(TM)$  there exists a unique *horizontal lift*  $\nabla_X$  in  $\Gamma(\mathcal{H})$  such that  $\pi_* \nabla_X = X$ .

REMARK 5.1. A function  $h \in C^\infty(T^*M)$  is fiber-wise linear if it can be written locally as  $h_Y(\lambda) = \langle \lambda, Y \rangle$ , for some  $Y \in \Gamma(TM)$ . A connection  $\nabla$  is linear if, for every  $X \in \Gamma(TM)$ , the derivation  $\nabla_X$  maps fiber-wise linear functions to fiber-wise linear functions. In this case, we recover the classical notion of covariant derivative by defining  $\nabla_X Y = Z$  if  $\nabla_X h_Y = h_Z$ , where  $Y, Z \in \Gamma(TM)$ .

We recall the definition of curvature of an Ehresmann connection [9].

DEFINITION 5.2. The *Ehresmann curvature* of the connection  $\nabla$  is the  $C^\infty(M)$ -linear map  $R^\nabla : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(\mathcal{V})$  defined by

$$R^\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}, \quad X, Y \in \Gamma(TM).$$

$R^\nabla$  is skew-symmetric, namely  $R^\nabla(X, Y) = -R^\nabla(Y, X)$ . Notice that  $R^\nabla = 0$  if and only if  $\mathcal{H}$  is involutive.

**5.1. Canonical connection.** Let  $\gamma(t)$  be a fixed ample and equiregular geodesic with Young diagram  $D$ , projection of the extremal  $\lambda(t)$ , with initial covector  $\lambda$ . Let  $\{E_{ai}(t), F_{ai}(t)\}$  be a canonical frame along  $\lambda(t)$ . For  $t = 0$ , this defines a subspace at  $\lambda \in T^*M$ , namely

$$(13) \quad \mathcal{H}_\lambda := \text{span}\{F_{ai}|_\lambda\}_{ai \in D}, \quad \lambda \in T^*M.$$

Indeed this definition makes sense on the subset of covectors  $N \subset T^*M$  associated with ample and equiregular geodesics. In the Riemannian case, every non-trivial geodesic is ample and equiregular, with the same Young diagram. Hence  $N = T^*M \setminus H^{-1}(0)$ . A posteriori one can show that this connection is linear and can be extended smoothly on the whole  $T^*M$ . In the sub-Riemannian case,  $N \subset T^*M \setminus H^{-1}(0)$ .

In general, using the results of [3, Section 5.2] and [14, Section 5], one can prove that  $N$  is open and dense in  $T^*M$ . Moreover, the elements of the frame depend rationally (in charts) on the point  $\lambda$ , hence  $\mathcal{H}$  is smooth on  $N$ .

For simplicity of notation, we assume that it is possible to extend  $\mathcal{H}$  to a smooth distribution on the whole  $T^*M$ . This is indeed possible in some cases of interest: on contact sub-Riemannian structures [4] or on corank 1 structures with symmetries [10]. In the general case, we must replace  $T^*M$  with  $N$ .

DEFINITION 5.3. The *canonical Ehresmann connection* associated with the sub-Riemannian structure is the horizontal distribution  $\mathcal{H} \subset T(T^*M)$  defined by (13).

As a consequence of Proposition 4.9,  $\mathcal{H}$  is non-linear, in general. However, if the structure is Riemannian, one has  $\mathcal{H}_{c\lambda} = (d_\lambda P_c)\mathcal{H}_\lambda$  and the connection is linear.

PROPOSITION 5.4. *Let  $H$  be the sub-Riemannian Hamiltonian and  $\mathcal{H}$  the canonical connection. Then  $\nabla_X H = 0$  for every  $X \in \Gamma(TM)$ . Equivalently,  $\vec{H} \in \mathcal{H}$ .*

<sup>2</sup> Note that this is a distribution on  $T^*M$ , i.e. a sub-bundle of  $T(T^*M)$  and should not be confused with the sub-Riemannian distribution  $\mathcal{D}$ , that is a subbundle of  $TM$ .

REMARK 5.5. The above condition is the compatibility of the canonical connection with the sub-Riemannian metric. In the Riemannian setting,  $\mathcal{H}$  is linear and this condition can be rewritten, in the sense of covariant derivative, as  $\nabla g = 0$ .

PROOF. The equivalence of the two statements follows from the definition of Hamiltonian vector field and the fact that  $\mathcal{H}$  is Lagrangian, by construction. Indeed

$$\nabla_X H = dH(\nabla_X) = \sigma(\vec{H}, \nabla_X).$$

Then we prove that  $\vec{H} \in \mathcal{H}$ .

LEMMA 5.6. *Let  $\mathfrak{e}$  be the Euler vector field. Then  $\dot{\mathfrak{e}} = -\vec{H}$ .*

PROOF OF LEMMA 5.6. Let  $P_s = e^{(\ln s)\mathfrak{e}}$  be the dilation along the fibers. We have the following commutation rule for the flows of  $\vec{H}$  and  $\mathfrak{e}$

$$P_{-s} \circ e^{-t\vec{H}} \circ P_s = e^{-ts\vec{H}}.$$

Computing the derivative w.r.t  $t$  and  $s$  at  $(t, s) = (0, 1)$  we obtain  $[\vec{H}, \mathfrak{e}] = -\mathfrak{e}$ , that implies the statement.  $\square$

LEMMA 5.7. *Since  $\mathfrak{e}$  is vertical, then  $\mathfrak{e} = v(t)^* E(t)$  for some smooth  $v(t) \in \mathbb{R}^n$ . Accordingly with the decomposition of Remark 4.4, we set*

$$v(t) = (v_1(t), \dots, v_k(t))^*, \quad \text{with } v_a(t) = (v_{a1}(t), \dots, v_{an_a}(t))^*.$$

Then  $v(t)$  is constant and we have

$$\mathfrak{e} = \sum_{\substack{ai \in D \\ n_a = 1}} v_{ai} E_{ai}.$$

PROOF OF LEMMA 5.7. As a consequence of Lemma 5.6,  $\ddot{\mathfrak{e}} = 0$ . Using the structural equations (9), we obtain

$$(14) \quad C_1^* C_2 v - C_2 C_1 v - 2C_2 \dot{v} = 0,$$

$$(15) \quad \ddot{v} + 2C_1 \dot{v} + C_1^2 v - RC_2 v = 0.$$

We show that for any row index of the Young diagram  $a = 1, \dots, k$

$$v_a = \begin{cases} (0, \dots, 0)^* & n_a > 1, \\ \text{constant} & n_a = 1. \end{cases}$$

Let us focus on (14). For each  $a = 1, \dots, k$ , we take its  $a$ -th block. By the block structure of  $C_1$  and  $C_2$ , this is

$$(16) \quad C_1^* C_2 v_a - C_2 C_1 v_a - 2C_2 \dot{v}_a = 0, \quad \forall a = 1, \dots, k,$$

where here  $C_1 = C_1(D_a)$  and  $C_2 = C_2(D_a)$ . If  $n_a = 1$ , then  $C_1 = 0$  and  $C_2 = 1$ . In this case (16) implies  $v_a(t) = v_a$  is constant. Now let  $n_a > 1$ . In this case, the particular form of  $C_1, C_2$  for (16) yields

$$C_1^* C_2 v_a = 0, \quad \text{and} \quad C_2 C_1 v_a + 2C_2 \dot{v}_a = 0, \quad (n_a > 1).$$

Indeed the kernel of  $C_1^*$  is orthogonal to the image of  $C_2$ . Hence  $C_1^* C_2 v_a = 0$  implies  $C_2 v_a = 0$ . In particular (16) is equivalent to

$$(17) \quad C_2 v_a = 0, \quad C_2 C_1 v_a = 0, \quad (n_a > 1).$$

More explicitly,  $v_a = (0, 0, v_{a3}, \dots, v_{an_a})$ . For the case  $n_a = 2$  this is sufficient to completely determine  $v_a$ . In all the other cases, let us turn to (15). The latter does not split immediately, as the curvature matrix  $R$  is not block-diagonal. However,

let us consider a copy of (15) multiplied by  $C_2C_1^i$ . For each  $a$  such that  $n_a > 2$  we consider its  $a$ -th block, obtaining the following:

$$C_2C_1^i\ddot{v}_a + 2C_2C_1^{i+1}\dot{v}_a + C_2C_1^{i+2}v_a - [C_2C_1^iRC_2v]_a = 0, \quad (n_a > 2).$$

We claim that  $[C_2C_1^iRC_2v]_a = 0$  if  $n_a > 2$  and  $i < n_a - 2$ .

By setting the matrix  $[R_{ab}]_{ij} := R_{ai,bj}$ , with  $ai, bj \in D$  (this is a block of  $R$ , corresponding to the rows  $a, b$  of the Young diagram  $D$ ), we compute

$$\begin{aligned} [C_2C_1^iRC_2v]_a &= \sum_{b,c,d=1}^k [C_2C_1^i]_{ab}R_{bc}[C_2]_{cd}v_d = \sum_{b=1}^k (C_2C_1^i)_{ab}R_{ab}(C_2v_b) \\ &= \sum_{n_b=1} (C_2C_1^i)_{ab}R_{ab}(C_2v_b) = \sum_{n_b=1} R_{a(i+1),b1}v_{b1}, \end{aligned}$$

where we used the block structure of the  $C_i$ 's and (17). The last sum involves only  $R_{a(i+1),b1}$  with  $n_b = 1$  and  $n_a > 2$ . If  $i < n_a - 2$ , then  $R_{a(i+1),b1}$  is *not* in the last  $2n_b = 2$  elements of Table 1, and vanishes by the normal conditions (see Appendix A). Thus we have:

$$(18) \quad C_2C_1^i\ddot{v}_a + 2C_2C_1^{i+1}\dot{v}_a + C_2C_1^{i+2}v_a = 0, \quad (n_a > 2, \quad i < n_a - 2).$$

In particular using (17), and taking  $i = 0, \dots, n_a - 3$  we see that (18) is equivalent to  $C_2C_1^{i+2}v_a = 0$  for all  $i = 0, \dots, n_a - 3$ . Combining all the cases

$$v_a \in \ker\{C_2, C_2C_1, C_2C_1^2, \dots, C_2C_1^{n_a-1}\}, \quad (n_a > 1).$$

This yields  $v_a = 0$ , by Kalman rank condition (10).  $\square$

Lemma 5.7 implies our statement since

$$\vec{H} = -\dot{\mathbf{c}} = - \sum_{\substack{ai \in D \\ n_a=1}} v_{ai}\dot{E}_{ai} = \sum_{\substack{ai \in D \\ n_a=1}} v_{ai}F_{ai} \in \mathcal{H},$$

where we used the structural equations (8) for the  $E_{ai}$ 's with  $n_a = 1$ .  $\square$

**5.2. Relation with the canonical curvature.** We now discuss the relation between the curvature of the canonical Ehresmann connection and the sub-Riemannian curvature operator. In what follows we denote by  $\mathfrak{R}_\lambda := \mathfrak{R}_{\lambda(0)}$ , where  $\lambda(t)$  is the extremal with initial datum  $\lambda$ . Then  $\mathfrak{R}$  extends to a well defined map

$$(19) \quad \begin{aligned} \mathfrak{R} : \Gamma(T^*M) \times \Gamma(TM) \times \Gamma(TM) &\rightarrow C^\infty(M), \\ (\lambda, X, Y) &\mapsto \mathfrak{R}_\lambda(X, Y). \end{aligned}$$

We stress that here the first argument is a section  $\lambda \in \Gamma(T^*M)$ .

Although  $\mathfrak{R}$  is  $C^\infty(M)$ -linear in the last two arguments by construction, it is in general non-linear in the first argument, so it does not define a  $(1, 2)$  tensor. Nevertheless, for any fixed section  $\lambda \in \Gamma(T^*M)$ , the restriction  $\mathfrak{R}_\lambda : \Gamma(TM) \times \Gamma(TM) \rightarrow C^\infty(M)$  is a  $(0, 2)$  symmetric tensor.

**THEOREM 5.8.** *Let  $R^\nabla : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(\mathcal{V})$  be the curvature of the canonical Ehresmann connection, and let  $\mathfrak{R} : \Gamma(T^*M) \times \Gamma(TM) \times \Gamma(TM) \rightarrow C^\infty(M)$  be the canonical curvature map (19). Then*

$$(20) \quad \mathfrak{R}_\lambda(X, Y) = \sigma_\lambda(R^\nabla(\mathbb{T}, X), \nabla_Y), \quad \forall \lambda \in \Gamma(T^*M), \quad X, Y \in \Gamma(TM),$$

where  $\mathbb{T} = \pi_*\vec{H}|_\lambda \in \Gamma(TM)$ .

PROOF. We evaluate the right hand side of (20) at the point  $x$ , for any fixed section  $\lambda = \lambda(x) \in \Gamma(T^*M)$ . By linearity, it is sufficient to take  $X = X_{ai}$  and  $Y = Y_{bj}$ , projections of a canonical frame  $F_{ai}|_\lambda, F_{bj}|_\lambda$  at  $t = 0$ . Indeed, by definition,  $\nabla_{X_{ai}}|_\lambda = F_{ai}|_\lambda$ . Then

$$\begin{aligned} \sigma_\lambda(R^\nabla(\mathbb{T}, X_{ai}), \nabla_{X_{bj}}) &= \sigma_\lambda([\nabla_{\mathbb{T}}, F_{ai}], F_{bj}) = \sigma_\lambda([\vec{H}, F_{ai}], F_{bj}) \\ &= \sigma_\lambda(\dot{F}_{ai}, F_{bj}) = R_{ai,bj}^\lambda(0). \end{aligned}$$

Here we used the structural equations and that  $\vec{H} \in \mathcal{H}$ , thus  $\nabla_{\mathbb{T}} = \vec{H}$ . By definition of canonical curvature map, we obtain the statement.  $\square$

REMARK 5.9. For  $\lambda \in \Gamma(T^*M)$ , the corresponding tangent field  $\mathbb{T} \in \Gamma(\mathcal{D}) \subsetneq \Gamma(TM)$ . Therefore,  $\mathfrak{R}$  recovers only part of the whole Ehresmann connection.

REMARK 5.10 (On the Riemannian case). As we proved in (7), we have

$$\mathfrak{R}_\lambda(X, Y) = R^\nabla(\mathbb{T}, X, Y, \mathbb{T}),$$

where  $\mathbb{T} = \pi_*\vec{H}|_\lambda$  is the tangent vector associated with the covector  $\lambda$ . For completeness, let us recover the same formula by the r.h.s. of (20). Indeed, for any vertical vector  $V \in \mathcal{V}_\lambda$  and  $W \in T_\lambda(T^*M)$ , we have  $\sigma_\lambda(V, W) = V(h_{\pi_*W})|_\lambda$  as one can check from a direct computation. Thus the r.h.s. of (20) is

$$\begin{aligned} \sigma_\lambda([\nabla_{\mathbb{T}}, \nabla_X] - \nabla_{[\mathbb{T}, X]}, \nabla_Y) &= (\nabla_{\mathbb{T}}\nabla_X(h_Y) - \nabla_X\nabla_{\mathbb{T}}(h_Y) - \nabla_{[\mathbb{T}, X]}(h_Y))|_\lambda \\ &= h_{\nabla_{\mathbb{T}}\nabla_X Y - \nabla_X\nabla_{\mathbb{T}} Y - \nabla_{[\mathbb{T}, X]} Y}(\lambda) \\ &= \langle \lambda, \nabla_{\mathbb{T}}\nabla_X Y - \nabla_X\nabla_{\mathbb{T}} Y - \nabla_{[\mathbb{T}, X]} Y \rangle \\ &= g(\nabla_{\mathbb{T}}\nabla_X Y - \nabla_X\nabla_{\mathbb{T}} Y - \nabla_{[\mathbb{T}, X]} Y, \mathbb{T}) \\ &= R^\nabla(\mathbb{T}, X, Y, \mathbb{T}). \end{aligned}$$

### Appendix A. Normal condition for the canonical frame

Here we rewrite the *normal* condition for the matrix  $R_{ai,bj}$  mentioned in Theorem 4.3 (and defined in [14]) according to our notation.

DEFINITION A.1. The matrix  $R_{ai,bj}$  is *normal* if it satisfies:

(i) global symmetry: for all  $ai, bj \in D$

$$R_{ai,bj} = R_{bj,ai}.$$

(ii) partial skew-symmetry: for all  $ai, bi \in D$  with  $n_a = n_b$  and  $i < n_a$

$$R_{ai,b(i+1)} = R_{bi,a(i+1)}.$$

(iii) vanishing conditions: the only possibly non vanishing entries  $R_{ai,bj}$  satisfy

(iii.a)  $n_a = n_b$  and  $|i - j| \leq 1$ ,

(iii.b)  $n_a > n_b$  and  $(i, j)$  belong to the last  $2n_b$  elements of Table 1.

TABLE 1. Vanishing conditions.

$i$	1	1	2	$\dots$	$\ell$	$\ell$	$\ell + 1$	$\dots$	$n_b$	$n_b + 1$	$\dots$	$n_a - 1$	$n_a$
$j$	1	2	2	$\dots$	$\ell$	$\ell + 1$	$\ell + 1$	$\dots$	$n_b$	$n_b$	$\dots$	$n_b$	$n_b$

The sequence is obtained as follows: starting from  $(i, j) = (1, 1)$  (the first boxes of the rows  $a$  and  $b$ ), each next even pair is obtained from the previous one by increasing  $j$  by one (keeping  $i$  fixed). Each next odd pair is obtained from the previous one by increasing  $i$  by one (keeping  $j$  fixed). This stops when  $j$  reaches its maximum, that is  $(i, j) = (n_b, n_b)$ . Then, each next pair is obtained from the previous one by increasing  $i$  by one (keeping  $j$  fixed), up to  $(i, j) = (n_a, n_b)$ . The total number of pairs appearing in the table is  $n_b + n_a - 1$ .

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