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# Finite group actions and cyclic branched covers of knots in $\mathbf{S}^3$

M. Boileau\*, C. Franchi, M. Mecchia†, L. Paoluzzi\*, and B. Zimmermann†

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## Abstract

We show that a hyperbolic 3-manifold can be the cyclic branched cover of at most fifteen knots in  $\mathbf{S}^3$ . This is a consequence of a general result about finite groups of orientation preserving diffeomorphisms acting on 3-manifolds. A similar, although weaker, result holds for arbitrary irreducible 3-manifolds: an irreducible 3-manifold can be the cyclic branched cover of odd prime order of at most six knots in  $\mathbf{S}^3$ .

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*Keywords:* Finite group actions, homology spheres, geometric structures in dimension 3, cyclic branched covers of knots.

## 1 Introduction

A classical way of presenting certain closed orientable 3-manifolds is by taking cyclic covers of  $\mathbf{S}^3$  branched along knots. This article deals with the question of understanding in how many ways a given manifold can be presented as the total space of a cyclic branched cover of some knot. Given a closed orientable 3-manifold  $M$ , these different presentations are in one-to-one correspondence with conjugacy classes of the corresponding groups of deck transformations. These are finite cyclic groups generated by periodic diffeomorphisms with connected and non-empty fixed-point set, and orbit space homeomorphic to the 3-sphere. We call such diffeomorphisms *hyperelliptic rotations*.

For example, the 3-sphere is the  $n$ -fold cyclic branched cover of the trivial knot for any integer  $n \geq 2$ . On the other hand, due to the orbifold theorem, it is known that for a closed 3-manifold which is not homeomorphic to  $\mathbf{S}^3$ , the order of a hyperelliptic rotation is bounded by a constant depending on the manifold, see [Ko1]. So when  $M$  is not the 3-sphere  $\mathbf{S}^3$  one may ask whether there is a universal upper bound for the number of distinct conjugacy classes of groups of deck transformations associated to cyclic branched covers. We show that this is indeed the case if the manifold is hyperbolic. For the general case of

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irreducible manifolds we obtain a universal bound only for the groups of deck transformations that have odd prime order.

Establishing such a universal upper bound for hyperbolic 3-manifolds boils down to bounding the number of conjugacy classes of cyclic groups of isometries generated by hyperelliptic rotations inside the finite group of isometries of the manifold. Since any finite group acts on some hyperbolic 3-manifold [Ko3], [CL] we have to consider any finite group action on a 3-manifold. The proof, however, only uses that the manifold is not  $\mathbf{S}^3$ , so our main result is the following

**Theorem 1.** *Let  $M$  be a closed orientable connected 3-manifold which is not homeomorphic to  $\mathbf{S}^3$ . Let  $G$  be a finite group of orientation preserving diffeomorphisms of  $M$ . Then  $G$  contains at most six conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation of order not a power of 2.*

Thanks to previous work of one of the authors ([Mec]) and of M. Reni ([Re]) on hyperelliptic rotations of orders a power of 2 (see Section 7, Theorem 9), we deduce:

**Corollary 1.** *Let  $M$  be a closed orientable connected 3-manifold which is not homeomorphic to  $\mathbf{S}^3$ . Let  $G$  be a finite group of orientation preserving diffeomorphisms of  $M$ . Then  $G$  contains at most fifteen conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation. Moreover, the set of orders of such subgroups has cardinality at most nine.*

The above corollary applies in particular to the orientation preserving group of isometries  $Isom^+(M)$  of a closed hyperbolic 3-manifold  $M$  which is finite. Since any finite group of orientation preserving diffeomorphisms acting on  $M$  is conjugate to a subgroup of  $Isom^+(M)$ , the following is a direct consequence of Corollary 1

**Corollary 2.** *A closed orientable connected hyperbolic 3-manifold  $M$  is a cyclic cover of  $\mathbf{S}^3$  along a knot for at most fifteen distinct knots. The possible branching orders are at most nine.*

From Corollary 1 it is also possible to derive a characterisation of the 3-sphere:

**Corollary 3.** *A closed, connected, orientable 3-manifold  $M$  is homeomorphic to  $\mathbf{S}^3$  if and only if there is a finite group  $G$  of orientation preserving diffeomorphisms of  $M$  such that  $G$  contains sixteen conjugacy classes of subgroups generated by hyperelliptic rotations.*

One interesting aspect of the proof of Theorem 1 is the substantial use of finite group theory and the classification of finite simple groups. In particular, it relies on previous works on finite groups acting on 3-manifolds by two of the authors in [Mec2] and [MZ]. A crucial point is to control the rank of the  $p$ -Sylow subgroup of a finite group acting on a 3-manifold. More precisely, the proof of Theorem 1 splits into various cases, according to the structures of the normalisers of the Sylow  $p$ -subgroups containing an element of order  $p$  that is a power of a hyperelliptic rotation, where  $p$  is an odd prime number. This structure is reflected in the symmetry group of the branching knot in the quotient  $\mathbf{S}^3$  of the manifold by the action of the hyperelliptic rotation, see Proposition 5 in Section 2.

Note that, although the proof of Theorem 1 uses the classification of finite simple groups, it is enough to know that there is only a finite number of sporadic simple groups (that is, groups that are not alternating or of Lie type) to conclude on the existence of a uniform bound on the number of conjugacy classes. Since, as we have already observed, every finite group acts on some closed orientable 3-manifold (see also Section 10) there is no way to avoid relying on the classification of finite simple groups to prove the above results. If we only consider 3-manifolds that are  $\mathbb{Z}/2$ -homology spheres though, the proof of Theorem 1 (with the very same bounds) only relies on the Gorenstein-Harada classification of simple groups with sectional 2-rank  $\leq 4$ , [G], see Corollary 7. Indeed, the homological condition provides control on the size of the Sylow 2-subgroups of finite groups that can act on such manifolds. For other topological conditions that impose constraints on the finite groups that can act on a manifold see [Mec2].

It follows from the proof of Theorem 1 that the existence of at least four distinct conjugacy classes of hyperelliptic rotations of orders not powers of 2 in a finite group  $G$  of orientation preserving diffeomorphisms of a closed 3-manifold imposes constraints on the structure of  $G$ . A description of the structure of  $G$ , which is somehow technical, is given in Proposition 11.

#### *Irreducible 3-manifolds*

Theorem 1 has several geometric consequences, notably on the different ways a given manifold can appear as the total space of a cyclic cover of  $\mathbf{S}^3$  branched along a knot, without requiring that the manifold is hyperbolic.

**Theorem 2.** *Let  $M$  be a closed orientable connected irreducible 3-manifold. Then there are at most six inequivalent knots in  $\mathbf{S}^3$  having  $M$  as cyclic branched cover of odd prime order*

Theorem 2 follows from Theorem 1 and the existence of an equivariant geometric decomposition for a closed irreducible 3-manifold  $M$  with finite effective and non free group action, according to the orbifold theorem, see for example [BoP], and also [BLP], [CHK], [KL].

**Remark 1. a)** Note that one cannot hope to extend the result of Theorem 2 to arbitrary prime orders. Indeed, although a universal bound exists on the number of knots that can be double covered by a given hyperbolic manifold by work of Reni [Re], no such bound exists in general. In fact, given any integer  $n$  there are infinitely many closed orientable Seifert fibred 3-manifolds with at least  $n$  conjugacy classes of hyperelliptic rotations of order 2, see Remark 14 in Section 9.1. As in the case of Seifert fibred manifolds, other types of toroidal manifolds, that is manifolds with a non trivial JSJ-decomposition, can also admit arbitrarily many conjugacy classes of hyperelliptic rotations of order 2. As a consequence the finiteness result of Theorem 2 does not hold for 2-fold branched covers in general.

**b)** Also, one cannot extend the result of Theorem 2 to arbitrary orders that are not powers of 2, like in Theorem 1: in Section 9.1 we give examples of closed, irreducible, orientable Seifert fibered 3-manifolds with an arbitrarily large number of conjugacy classes of cyclic groups generated by hyperelliptic rotations of odd, but not prime, orders, see Proposition 14. Our examples are

not rational homology spheres, while the primeness assumption for the orders of the hyperelliptic rotations forces the manifold  $M$  to be a rational homology sphere, see Remark 2 in Section 2.

#### *Arbitrary 3-manifolds*

Using the decomposition of a closed orientable connected 3-manifold as a connected sum of prime manifolds, we deduce from Theorem 2 an upper bound on the number of odd primes that can occur as orders of hyperelliptic rotations of arbitrary closed 3-manifolds. Remark that for any  $k \in \mathbb{N}$  and every odd prime  $p$  there are non prime 3-manifolds that are the  $p$ -fold cyclic covers of at least  $k$  non equivalent knots, so that Theorem 2 does not generalise to arbitrary manifolds.

**Corollary 4.** *Let  $M$  be a closed (orientable) connected 3-manifold which is not homeomorphic to the 3-sphere  $\mathbf{S}^3$ . Then  $M$  is a  $p$ -fold cyclic cover of  $\mathbf{S}^3$  branched along a knot for at most six distinct odd prime numbers  $p$ .*

In [BPZ] three of the authors showed that the upper bound is 3 for the number of odd prime orders of hyperelliptic rotations acting on an integral homology sphere. In that case, the proof relies heavily on the restrictions on finite groups acting on integral homology 3-spheres. Such restrictions cannot exist in general since, as we observed at the beginning, every finite group does act on some closed orientable 3-manifold. Even if we only consider cyclic branched covers of knots of prime orders, which are rational homology spheres, no restriction can be obtained a priori, for every finite group acts on some rational homology sphere [CL]: the actions constructed in [CL] are free, but the same proof shows that for each finite group  $G$  it is also possible to construct a faithful and non free action of  $G$  on some rational homology sphere (see the Section 10 for more details on this).

At this point we do not know if the bound of Corollary 4 is sharp. So far, only examples of manifolds that are  $p$ -fold cyclic covers of  $\mathbf{S}^3$  branched along a knot for three distinct odd primes  $p$  and for  $p = 2$  are known. Examples are provided, for instance, by Brieskorn spheres of type  $\Sigma(p, q, r)$ , where  $p$ ,  $q$ , and  $r$  are three pairwise different odd primes:  $\Sigma(p, q, r)$  is the  $p$ -fold (resp.  $q$ -fold and  $r$ -fold) cyclic cover of  $\mathbf{S}^3$  branched along the torus knot  $T(q, r)$  (resp.  $T(p, r)$  and  $T(p, q)$ ) as well as the double branched cover of a Montesinos knot. Examples of hyperbolic manifolds that are  $p$ -fold cyclic covers of  $\mathbf{S}^3$  branched along a knot for three distinct odd primes  $p$  can also be exhibited [RZ2].

Corollary 4 gives yet another characterisation of the 3-sphere,  $\mathbf{S}^3$ :

**Corollary 5.** *A closed orientable connected 3-manifold  $M$  is homeomorphic to  $\mathbf{S}^3$  if and only if  $M$  is a  $p$ -cyclic cover of  $\mathbf{S}^3$  branched along a knot for at least seven distinct odd prime numbers.*

The paper is organised as follows. In Section 2 we define what (*hyperelliptic*) rotations are and determine different types of algebraic properties and constraints that they must satisfy. In Section 3, we give a proof of Theorem 1 under the extra hypothesis that the group  $G$  is solvable. In this case it is not difficult to show that, up to conjugacy, all hyperelliptic rotations must commute.

The central part of the paper is consacrated to the proof of a weaker version of Theorem 1, in which hyperelliptic rotations are assumed to have odd prime order, namely:

**Theorem 3.** *Let  $M$  be a closed orientable connected 3-manifold which is not homeomorphic to  $\mathbf{S}^3$ . Let  $G$  be a finite group of orientation preserving diffeomorphisms of  $M$ . Then  $G$  contains at most six conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation of odd prime order.*

In Section 4 we recall some standard definitions in finite group theory and provide several preliminary results that will be used in the following. The actual proof of Theorem 3 is split into two cases according to different properties of the hyperelliptic rotations contained in  $G$  and their associated quotient knots. In Section 5 we assume that at least one of the knots is not *self-symmetric* (see Section 2 for a definition and the translation of this property into algebraic terms), while in Section 6 we assume that none is. In Section 7 we explain how the proof of Theorem 1 can be reduced to that of Theorem 3. The reduction of the proof relies basically on the fact that the Sylow  $p$ -subgroup containing a non-trivial power of a hyperelliptic rotation has the same structure as that of a Sylow  $p$ -subgroup containing a hyperelliptic rotation of order  $p$ . In particular, all the key ingredients and essential difficulties are already encountered in the proof of Theorem 3. In this section we also give the proof of Corollary 1. In Section 8 we show that Theorems 1 and 3 are trivially fulfilled if the manifold is reducible, for no interesting finite group action exists on such manifolds. Section 9 is devoted to the proof of Theorem 2 and Corollary 4. More precisely, in Section 9.1 we show that Theorem 2 holds for Seifert fibred manifolds, while in Section 9.2 Theorem 2 is deduced from Theorem 3 for the remaining irreducible 3-manifolds, that is hyperbolic 3-manifolds, and 3-manifolds admitting a non trivial JSJ-decomposition. Finally, in Section 10, we see why every finite group acts non freely on some rational homology sphere.

## 2 Rotations and their properties

In this section we shall introduce certain special periodic diffeomorphisms and establish some of their properties. Note that all through the paper, unless otherwise stated, *3-manifold* will mean orientable, connected, closed 3-manifold. Also, all finite group actions by diffeomorphisms will be faithful and orientation preserving.

**Definition 1.** Let  $\psi : M \rightarrow M$  be a finite order diffeomorphism of a 3-manifold  $M$ . We shall say that  $\psi$  is a *rotation* if it preserves the orientation of  $M$ ,  $Fix(\psi)$  is non-empty and connected, and  $Fix(\psi) = Fix(\psi^k)$  for all non trivial powers  $\psi^k$  of  $\psi$ .  $Fix(\psi)$  will be referred to as the *axis of the rotation*. Note that if  $\psi$  is a periodic diffeomorphism of prime order, then  $\psi$  is a rotation if and only if  $Fix(\psi) = \mathbf{S}^1$ .

We shall say that a rotation  $\psi$  is *hyperelliptic* if the space of orbits  $M/\psi$  of its action is  $\mathbf{S}^3$ .

**Remark 2.** Assume that  $\psi$  is a hyperelliptic rotation acting on a 3-manifold  $M$  then:

1. The natural projection from  $M$  to the space of orbits  $M/\psi$  of  $\psi$  is a cyclic cover of  $\mathbf{S}^3$  branched along a knot  $K = \text{Fix}(\psi)/\psi$ . The converse is also true, that is any deck transformation generating the automorphism group of a cyclic covering of  $\mathbf{S}^3$  branched along a knot is a hyperelliptic rotation.
2. If the order of  $\psi$  is a prime  $p$ , then  $M$  is a  $\mathbb{Z}/p$ -homology sphere [Go].

We start with a somehow elementary remark which is however central to determine constraints on finite groups acting on 3-manifolds.

**Remark 3.** Let  $G \subset \text{Diff}^+(M)$  be a finite group of diffeomorphisms acting on a 3-manifold  $M$ . One can choose a Riemannian metric on  $M$  which is invariant by  $G$  and with respect to which  $G$  acts by isometries. Let now  $\psi \in G$  be a rotation. Since the normaliser  $\mathcal{N}_G(\langle\psi\rangle)$  of  $\psi$  in  $G$  consists precisely of those diffeomorphisms that leave the circle  $\text{Fix}(\psi)$  invariant, we deduce that  $\mathcal{N}_G(\langle\psi\rangle)$  is a finite subgroup of  $\mathbb{Z}/2 \times (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z})$ , where the nontrivial element in  $\mathbb{Z}/2$  acts by conjugation sending each element of  $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$  to its inverse. Note that the elements of  $\mathcal{N}_G(\langle\psi\rangle)$  are precisely those that rotate about  $\text{Fix}(\psi)$ , translate along  $\text{Fix}(\psi)$ , or invert the orientation of  $\text{Fix}(\psi)$ ; in the last case the elements have order 2 and non empty fixed-point set meeting  $\text{Fix}(\psi)$  in two points.

Note that if  $M \neq \mathbf{S}^3$  and  $\psi$  is a hyperelliptic rotation of order  $n > 2$ , then its centraliser  $\mathcal{C}_G(\langle\psi\rangle)$  in  $G$  satisfies  $1 \rightarrow \langle\psi\rangle \rightarrow \mathcal{C}_G(\langle\psi\rangle) \rightarrow H \rightarrow 1$ , where  $H$  is cyclic, possibly trivial. This follows easily from the positive solution to Smith's conjecture which implies that any group of symmetries of a non-trivial knot  $K$  (that is, any finite group of orientation-preserving diffeomorphisms of  $\mathbf{S}^3$  acting on the pair  $(\mathbf{S}^3, K)$ ) is either cyclic or dihedral.

**Definition 2.** With the notation of the above remark, we shall call *Fix*( $\psi$ )-rotations the elements of  $\mathcal{N}_G(\langle\psi\rangle)$  that preserve the orientation of  $\text{Fix}(\psi)$  and *Fix*( $\psi$ )-inversions those that reverse it.

**Lemma 1.** *Consider two rotations contained in a finite group of orientation preserving diffeomorphisms of a 3-manifold  $M$ .*

1. *A non trivial power of the first rotation commutes with a non trivial power of the second, both of orders different from 2, if and only if the two rotations commute.*
2. *Assume  $M \neq \mathbf{S}^3$ . If the two rotations are hyperelliptic and their fixed-point sets coincide, then they generate the same cyclic group (in particular they have the same order).*

**Proof.**

**Part 1** The sufficiency of the condition being obvious, we only need to prove the necessity. Remark that we can assume that both rotations act as isometries for some fixed Riemannian metric on the manifold. To fix ideas, denote by  $\psi$  and  $\varphi$  the two rotations and by  $g$  and  $f$  respectively their non trivial powers. Note that, by definition,  $\text{Fix}(\psi) = \text{Fix}(g)$  and  $\text{Fix}(\varphi) = \text{Fix}(f)$ . Since  $g$  and  $f$  commute,  $g$  leaves invariant  $\text{Fix}(\varphi) = \text{Fix}(f)$  and thus normalises every rotation about  $\text{Fix}(\varphi)$ . Moreover  $g$  and  $\varphi$  commute, for the order of  $g$  is not 2 (see Remark 3). In particular  $\varphi$  leaves  $\text{Fix}(\psi) = \text{Fix}(g)$  invariant and normalises every rotation about  $\text{Fix}(\psi)$ . The conclusion follows.

**Part 2** Reasoning as in Part 1, one sees that the two rotations commute. Assume, by contradiction, that the subgroups they generate are different. Under this assumption, at least one of the two subgroups is not contained in the other. Without loss of generality we can assume that  $\langle \varphi \rangle \not\subset \langle \psi \rangle$ . Take the quotient of  $M$  by the action of  $\psi$ . The second rotation  $\varphi$  induces a non-trivial rotation of  $\mathbf{S}^3$  which leaves the quotient knot  $K = \text{Fix}(\psi)/\psi \subset \mathbf{S}^3$  invariant. Moreover, this induced rotation fixes pointwise the knot  $K$ . The positive solution to Smith's conjecture implies now that  $K$  is the trivial knot and thus  $M = \mathbf{S}^3$ , against the hypothesis.  $\square$

## 2.1 Rotations on $\mathbb{Z}/p$ -homology spheres

Let  $M$  be a  $\mathbb{Z}/p$ -homology sphere,  $p$  a prime number. If  $\psi$  is a periodic diffeomorphism of order  $p$  acting on  $M$  then, according to Smith's theory, either  $\psi$  acts freely on  $M$  or it is a rotation. A generalisation of this fact is the following result whose proof can be found in [MZ, Prop 4, page 679]:

**Lemma 2.** *Let  $H \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$  act faithfully on a  $\mathbb{Z}/p$ -homology sphere. Then either  $H$  contains precisely two cyclic subgroups generated by rotations or  $p = 2$  and all three cyclic subgroups are generated by rotations.*

The previous remark and the above lemma have the following consequence:

**Proposition 1.** *Assume that the odd prime  $p$  is the order of a rotation  $f$  inside a finite group of diffeomorphisms  $G$  acting faithfully on and preserving the orientation of a  $\mathbb{Z}/p$ -homology sphere  $M$ . The Sylow  $p$ -subgroup  $S_p$  of  $G$  is either cyclic or of the form  $\mathbb{Z}/p^\alpha \oplus \mathbb{Z}/p^\beta$ .*

**Proof.**

Up to conjugacy we can assume that  $f \in S_p$ . According to Remark 3, the normaliser  $N = \mathcal{N}_{S_p}(\langle f \rangle)$  of  $\langle f \rangle$  in  $S_p$  is either cyclic or of the form  $\mathbb{Z}/p^\alpha \oplus \mathbb{Z}/p^\beta$ , for  $p$  is odd. We want to show that  $N = S_p$ . According to [Su1, 1.5, page 88], either  $N = S_p$  or there exists  $x \in \mathcal{N}_{S_p}(N) \setminus N$ . If  $N$  is cyclic, then every element of  $\mathcal{N}_{S_p}(N)$  must normalise the group generated by  $f$  so that  $N = S_p$ . We can thus assume that  $N \cong \mathbb{Z}/p^\alpha \oplus \mathbb{Z}/p^\beta$  and consider its characteristic subgroup  $H = \mathbb{Z}/p \oplus \mathbb{Z}/p$  which contains  $f$ .  $H$  acts on  $M$  which is a  $\mathbb{Z}/p$ -homology sphere hence, according to Lemma 2,  $H$  contains precisely two cyclic subgroups generated by rotations (one being generated by  $f$  and the other, say, by  $f'$ ). Assume now, by contradiction, that  $N \neq S_p$ . We can then choose  $x$  as described above. Since the order of  $x$  is odd,  $x$  cannot exchange  $\text{Fix}(f)$  and  $\text{Fix}(f')$ , so  $x$  must normalise the group generated by  $f$  and a contradiction is reached, proving the proposition.  $\square$

**Proposition 2.** *Let  $G \subset \text{Diff}^+(M)$  be a finite group acting on a  $\mathbb{Z}/p$ -homology sphere  $M$ . Let  $f \in G$  be a rotation of odd prime order  $p$  and  $S$  be a Sylow  $p$ -subgroup of  $G$  containing  $f$ . Then  $\mathcal{N}_G(S)$  contains  $\mathcal{N}_G(\langle f \rangle)$  with index at most 2 and contains an abelian subgroup of rank at most 2 with index at most 4. In particular  $\mathcal{N}_G(S)$  is solvable.*

**Proof.**

The Sylow  $p$ -subgroup  $S$  is described in Proposition 1 and  $\mathcal{N}_G(\langle f \rangle)$  is described in Remark 3: since  $S$  is abelian we clearly have  $S \subset \mathcal{N}_G(\langle f \rangle)$ . The



normaliser of  $\langle f \rangle$  in  $G$  contains  $Fix(f)$ -rotations and  $Fix(f)$ -inversions, hence it is contained in  $\mathcal{N}_G(S)$ . If each element in  $\mathcal{N}_G(S)$  normalises  $\langle f \rangle$  (in particular if  $S$  is cyclic), we are done. Otherwise  $S$  has rank two and contains precisely two subgroups of order  $p$  generated by a rotation (the group generated by  $f$  and another one; see Lemma 2). Each element contained in the complement  $\mathcal{N}_G(S) \setminus \mathcal{N}_G(\langle f \rangle)$  exchanges by conjugation these two cyclic groups, so the index of  $\mathcal{N}_G(\langle f \rangle)$  in  $\mathcal{N}_G(S)$  is two.  $\square$

## 2.2 Hyperelliptic rotations

The results of the previous subsection apply in particular to hyperelliptic rotations of odd prime order  $p$ . Indeed, in Propositions 1 and 2 the condition that  $M$  is a  $\mathbb{Z}/p$ -homology sphere can be replaced by the condition that  $f$  is a hyperelliptic rotation of order  $p$ , according to part 2 of Remark 2. Better still, in what follows we will show that it suffices that  $f$  is a power of odd prime order of a hyperelliptic rotation to reach the same conclusions.

**Lemma 3.** *Let  $p$  be an odd prime and assume that  $H \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$  acts faithfully by orientation preserving diffeomorphisms on a 3-manifold  $M$ . Assume, moreover, that  $H$  contains a non-trivial power  $f$  of a hyperelliptic rotation  $\varphi$  of  $M$ . Then  $H$  contains at most two subgroups generated by elements with non-empty fixed-point sets.*

**Proof.**

First of all, it is not restrictive to assume that  $M$  is not homeomorphic to  $\mathbf{S}^3$ , otherwise we are done by Lemma 2.

We need to prove that, besides the cyclic group generated by the rotation  $f$ ,  $H$  contains at most one other cyclic subgroup generated by an element  $g$  so that  $Fix(g) \neq \emptyset$ .

Reasoning as in the proof of the first part of Lemma 1, the elements of  $H$  commute with  $\varphi$  and thus induce symmetries of the non trivial knot  $K = Fix(\varphi)/\varphi$ . Also, since  $\varphi$  commutes with every subgroup of  $H$  it leaves setwise invariant each fixed-point set of cyclic subgroups of  $H$ .

We observe now that if  $G$  is a finite group acting by orientation preserving diffeomorphisms on a manifold and  $x, y \in G$  are two elements such that  $Fix(x) = Fix(y) \neq \emptyset$  then  $\langle x, y \rangle$  is cyclic; in particular if  $x$  and  $y$  have the same order they generate the same subgroup. This can be easily seen by considering a  $G$ -equivariant Riemmanian structure on  $M$ .

As a consequence different cyclic subgroups inside  $H$  acting non freely must have different fixed-point sets. Since  $H/\langle f \rangle$  maps to a rotation of  $\mathbf{S}^3$  this implies that at most one cyclic group other than  $\langle f \rangle$  has non-empty fixed-point set.  $\square$

The following proposition is just a restatement of Propositions 1 and 2 in this setting and it is also proved in the very same way using the fact that by Lemma 3 there are at most two cyclic subgroups of order  $p$  having non-empty fixed-point set.

**Proposition 3.** *Let  $G \subset Diff^+(M)$  be a finite group acting on a 3-manifold  $M$ . Let  $f \in G$  be an element of odd prime order  $p$  which is a non-trivial power of a hyperelliptic rotation. Let  $S$  be a Sylow  $p$ -subgroup of  $G$  containing  $f$ . Then*

- the Sylow  $p$ -subgroup  $S$  is either cyclic or the product of two cyclic groups, and
- its normaliser  $\mathcal{N}_G(S)$  contains  $\mathcal{N}_G(\langle f \rangle)$  with index at most 2, and contains an abelian subgroup of rank at most 2 with index at most 4. In particular  $\mathcal{N}_G(S)$  is solvable.

□

This general observation will be useful in the future.

**Remark 4.** Under the hypotheses of Proposition 2 or of Proposition 3, the normaliser of  $S$  contains the normaliser of  $\langle f \rangle$  with index 2 if and only if the two cyclic subgroups generated by the two rotations in  $S$  are conjugate. This case happens if and only if  $\mathcal{N}_G(S)$  contains elements of order a power of 2 which do not act in the same way on all elements of order  $p$  in  $S$ . Indeed, all elements in  $\mathcal{N}_G(\langle f \rangle)$  either commute with all elements of order  $p$  or act dihedrally. On the other hand, any element  $g$  in  $\mathcal{N}_G(S) \setminus \mathcal{N}_G(\langle f \rangle)$  conjugates  $f$  to a rotation  $f' = gfg^{-1}$  generating a different cyclic group, so that (up to perhaps taking a power of  $g$ ) one has  $gf'fg^{-1} = f'f$  and  $gf'f^{-1}g^{-1} = (f'f^{-1})^{-1}$ , i.e.  $g$  acts dihedrally on some elements of order  $p$  while it commutes with others.

### 2.3 Rotations and symmetries of knots

**Definition 3.** A *rotation of a knot  $K$*  in  $\mathbf{S}^3$  is a rotation  $\psi$  of  $\mathbf{S}^3$  such that  $\psi(K) = K$  and  $K \cap \text{Fix}(\psi) = \emptyset$ . We shall say that  $\psi$  is a *full rotation* if  $K/\psi$  in  $\mathbf{S}^3 = \mathbf{S}^3/\psi$  is the trivial knot.

**Remark 5.** Let  $\psi$  and  $\varphi$  be two commuting rotations acting on some manifold  $M$  whose orders are different from 2 and whose axes are distinct. Assume that  $\psi$  is hyperelliptic. In this case  $\varphi$  induces a rotation  $\phi$  of  $K = \text{Fix}(\psi)/\psi$ , for  $\text{Fix}(\psi) \cap \text{Fix}(\varphi) = \emptyset$  (see Remark 3). We have that  $\varphi$  is hyperelliptic if and only if  $\phi$  is a full rotation. This can be seen by considering the quotient of  $M$  by the action of the group generated by  $\psi$  and  $\varphi$ . This quotient is  $\mathbf{S}^3$  and the projection onto it factors through  $M/\varphi$ , which can be seen as a cyclic cover of  $\mathbf{S}^3$  branched along  $K/\phi$ . By the positive solution to the Smith conjecture,  $M/\varphi$  is  $\mathbf{S}^3$  if and only if  $K/\phi$  is the trivial knot.

The following finiteness result about commuting rotations of a non trivial knot in  $\mathbf{S}^3$  is one of the main ingredients in the proof of Theorem 10 (see [BoPa, Proposition 2], and [BoPa, Theorem 2] for a stronger result where commutativity is not required).

**Proposition 4.** *Let  $K$  be a non-trivial knot in  $\mathbf{S}^3$ . Then there are at most two non conjugate cyclic subgroups of  $\text{Diff}^+(S^3, K)$  generated by pairwise commuting full rotations.*

**Proof.**

Assume by contradiction that there are three non conjugate cyclic groups generated by commuting full rotations of  $K$ ,  $\varphi$ ,  $\psi$  and  $\rho$  respectively. If two of them -say  $\varphi$ ,  $\psi$ - have the same axis, then by hypothesis they cannot have the same order. Fix the one with smaller order -say  $\psi$ -: the quotient  $K/\psi$  is

the trivial knot, and  $\varphi$  induces a rotation of  $K/\psi$  which is non trivial since  $\varphi$  commutes with  $\psi$  and its order is distinct from that of  $\psi$ . The axis  $A$  of this induced symmetry is the image of  $Fix(\psi)$  in the quotient  $\mathbf{S}^3/\psi$  by the action of  $\psi$ . In particular  $K/\psi$  and  $A$  form a Hopf link and  $K$  is the trivial knot: this follows from the equivariant Dehn lemma, see [Hil].

We can thus assume that the axes are pairwise disjoint. Since the rotations commute, even if one of them has order 2, it cannot act as a strong inversion on the axes of the other rotations. Therefore we would have that the axis of  $\rho$ , which is a trivial knot, admits two commuting rotations,  $\varphi$  and  $\psi$ , with distinct axes, which is impossible: this follows, for instance, from the fact (see [EL, Thm 5.2]) that one can find a fibration of the complement of the trivial knot which is equivariant with respect to the two symmetries.  $\square$

Observe that the proof of the proposition shows that two commuting full rotations of a non trivial knot either generate the same cyclic group or have disjoint axes.

**Remark 6.** If a knot  $K \subset \mathbf{S}^3$  admits a full rotation, then it is a prime knot, see [BoPa, Lemma 2].

Let now  $G \subset Diff^+(M)$  be a finite group of diffeomorphisms acting on  $M \neq \mathbf{S}^3$ . Assume that  $G$  contains a hyperelliptic rotation  $\varphi$  admitting a non-trivial power  $f$  of odd prime order  $p$ . Let  $K$  be the non trivial knot  $Fix(\varphi)/\langle \varphi \rangle$ . The proof of Theorems 3 and 1 will be divided into different cases according to the structure of the normalisers of the Sylow  $p$ -subgroups of  $G$  containing (non trivial powers of) hyperelliptic rotations.

In what follows we will provide a dictionary translating between algebraic properties of the structure of the normaliser of  $\langle f \rangle$  in  $G$  and of the Sylow  $p$ -subgroup containing it, and the existence of special symmetries of  $K$ .

We need the following definition.

**Definition 4.** A 2-component link is called *exchangeable* if there exists an orientation-preserving diffeomorphism of  $\mathbf{S}^3$  which exchanges the two components of the link.

Let  $K$  be a knot and  $\psi$  a rotation of  $K$  of order  $n$  and with axis  $A$ . Consider the 2-component link  $\overline{K} \cup \overline{A}$  consisting of the images of the knot  $K$  and of the axis  $A$  in the quotient  $\mathbf{S}^3/\psi$  of the 3-sphere by the action of  $\psi$ . Note that at least one component of this link (i.e.  $\overline{A}$ ) is trivial. We call  $K$  *n-self-symmetric* if  $\overline{K} \cup \overline{A}$  is exchangeable. In this case  $\psi$  is a full rotation of  $K$ .

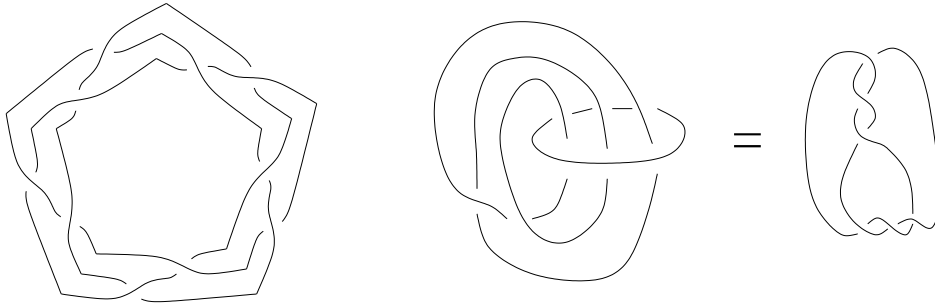


Figure 1: A 5-self-symmetric knot on the left, and its exchangeable quotient link on the right.

Since the structure of the normaliser of  $\langle f \rangle$  and of its centraliser, only depend on the symmetries of  $K$  that lift to  $G$ , we introduce the following definitions:

**Definition 5.** Let  $G$  be a finite group of orientation preserving diffeomorphisms of a closed connected 3-manifold  $M$ . Let  $\varphi$  be a hyperelliptic rotation contained in  $G$  with quotient knot  $K$ . We say that  $K$  is *strongly invertible with respect to  $G$*  if  $K$  admits a strong inversion that lifts to  $G$ . Similarly we say that  $K$  is *self-symmetric with respect to  $G$*  if  $G$  contains an element  $\varphi'$  conjugate to  $\varphi$  such that the subgroup  $\langle \varphi, \varphi' \rangle$  is abelian of rank 2, i.e. not cyclic. Remark that in the latter case  $K$  is  $n$ -self-symmetric, where  $n$  is the order of  $\varphi$ .

**Proposition 5.** Let  $K$ ,  $\varphi$ ,  $f$ , and  $G$  be as above.

- The centraliser of  $\langle f \rangle$  in  $G$  is contained with index 2 in its normaliser if and only if  $K$  is strongly invertible with respect to  $G$ .
- The normaliser of  $\langle f \rangle$  in  $G$  is contained with index 2 in the normaliser of  $C_G(f)$  in  $G$  if and only if  $K$  is self-symmetric with respect to  $G$ .

Moreover, if  $M$  is hyperbolic and  $G = Iso^+(M)$ ,  $K$  is strongly invertible if and only if it is strongly invertible with respect to  $G$ , and it is self-symmetric with respect to  $G$  if and only if it is  $n$ -self-symmetric, where  $n$  is the order of the hyperelliptic rotation  $\varphi$ .

**Proof.**

The necessity of the two conditions follows readily from Definition 5, so we only need to prove their sufficiency.

Any element in the normaliser of  $\langle f \rangle$  leaves  $Fix(f)$  setwise invariant and so belongs to the normaliser of  $\langle \varphi \rangle$ . As a consequence any element in the normaliser of  $\langle f \rangle$  induces a symmetry of  $K$ . If the normaliser of  $\langle f \rangle$  is not abelian, it must contain  $Fix(f)$ -inversions which induce strong inversions of the knot  $K$ . This proves the first part of the proposition.

For the second part, we have already noticed that the centraliser of  $f$  must contain a second cyclic subgroup generated by a rotation  $f'$  conjugate to  $f$ . The element  $g$  of  $G$  conjugating  $f$  to  $f'$  must map  $Fix(f)$  to  $Fix(f')$ . As a consequence,  $g$  conjugates  $\varphi$  to a hyperelliptic rotation  $\varphi'$  such that  $Fix(\varphi') = Fix(f')$ , having  $f'$  as non trivial power. The first part of Lemma 1 tells us that

$\varphi$  and  $\varphi'$  commute so that  $\varphi'$  induces a full  $n$ -rotation of  $K$ , where  $n$  is the order of  $\varphi$  and  $\varphi'$ . To see that  $K$  is  $n$ -self-symmetric it suffices to remark that the element  $g$ , which normalises  $\langle \varphi, \varphi' \rangle$ , induces a symmetry of the quotient link since it exchanges  $Fix(\varphi)$  and  $Fix(\varphi')$ .

When  $M$  is hyperbolic,  $K$  is a hyperbolic knot and it has a finite group of symmetries whose lift to  $M$  is contained in  $Iso^+(M)$ .  $\square$

### 3 The finite solvable case

In this section we prove a stronger version of Theorem 3 for a finite solvable subgroup of orientation preserving diffeomorphisms of a closed orientable 3-manifold. This is a key result not only for the proof of Theorem 3, but also to reduce the proof of Theorem 10 to the one of Theorem 3.

**Proposition 6.** *Let  $G$  be a finite group acting on a 3-manifold  $M$  and let  $\psi_i \in G$  be  $n$  rotations of  $M$  with odd prime orders. Assume that, up to conjugacy, the  $\psi_i$ s are contained in a solvable subgroup of  $G$ . Then, up to conjugacy, the rotations  $\psi_i$  commute.*

**Proof.**

We can assume that  $G$  itself is solvable. Then, applying [Su2, Thm 5.6, page 104], up to conjugacy all the rotations belong to the same subgroup of maximal odd order. In particular we may assume that  $G$  itself has odd order. Let  $\psi$  be a rotation of prime order  $p$  and let  $S$  be a Sylow  $p$ -subgroup of  $G$  containing  $\psi$ . Remark 3 implies that  $\mathcal{N}_G(S)$  is abelian, for the order of  $G$  is odd. In particular,  $S$  is central in its normalizer and [Su2, Thm 2.10, page 144] implies that  $G$  is the semidirect product of a characteristic subgroup  $U$  and  $S$ . Let  $\varphi$  be another rotation. If  $\varphi$  has order  $p$ , then up to conjugacy,  $\varphi$  sits inside  $S$  and thus it commutes with  $\psi$  according to Proposition 1. So let us assume that  $\varphi$  has order a prime  $q \neq p$ . Then  $\varphi$  belongs to  $U$ . Since  $p$  is coprime to the order of  $U$ ,  $\psi$  must normalise at least one Sylow  $q$ -subgroup  $T$  of  $U$ . Up to conjugacy, we may assume that  $\varphi$  is contained in  $T$ . Reasoning as above, we see that  $\psi$  must normalise at least one of the  $G$ -conjugates of  $\langle \varphi \rangle$ . By Remark 3, this implies that, up to conjugacy,  $\psi$  and  $\varphi$  commute.  $\square$

**Corollary 6.** *Let  $G$  be a finite group acting on a 3-manifold  $M$  and let  $f_i \in G$  be  $n$  rotations of  $M$  with orders that are not powers of 2. Assume that, up to conjugacy, the  $f_i$ s are contained in a solvable subgroup of  $G$ , then, up to conjugacy, the rotations  $f_i$  commute.*

**Proof.**

For each  $i = 1, \dots, n$  let  $\psi_i$  be a power of  $f_i$  of odd prime order. Apply Proposition 6 to the  $n$  rotations  $\psi_i$ , then the first part of Lemma 1 to reach the desired conclusion.  $\square$

**Theorem 4.** *Let  $G$  be a finite solvable group acting on a 3-manifold  $M$  which is not homeomorphic to  $\mathbf{S}^3$ . Up to conjugacy,  $G$  contains at most three cyclic subgroups generated by hyperelliptic rotations of orders which are not powers of 2. Moreover, either their orders are pairwise distinct or there are at most two such rotations.*

**Proof.**

Assume that there are  $n$  such rotations. According to Corollary 6 we can assume that all these  $n$  rotations commute. By Remark 5, we then have a non trivial knot (for  $M \neq \mathbf{S}^3$ ) admitting  $n - 1$  commuting full rotations. It follows immediately from Proposition 4 that  $n \leq 3$ . Note that, by the positive solution to Smith's conjecture, a non trivial knot cannot admit two distinct and commuting cyclic groups of symmetries of the same order. This proves the latter part of the theorem.  $\square$

## 4 Finite group actions

The goal of this section is to prove some useful results about hyperelliptic rotations in a finite subgroup of orientation preserving diffeomorphisms of a closed orientable 3-manifold. Again, all finite groups will be assumed to act faithfully by orientation preserving diffeomorphisms on some closed, connected 3-manifold  $M$ .

**Definition 6.** Let  $G \subset \text{Diff}^+(M)$  be a finite group acting on a closed orientable 3-manifold  $M$ . We say that an odd prime number  $p$  is *hyperelliptic* for  $G$  if  $G$  contains a hyperelliptic rotation of order  $p$ .

We remark once more that, if  $p$  is hyperelliptic for  $G$ , then the 3-manifold  $M$  is a  $\mathbb{Z}/p$ -homology 3-sphere. Moreover, if  $f$  is a hyperelliptic rotation of order  $p$  its normaliser and centraliser are determined in Remark 3.

**Remark 7.** If  $G$  is a finite group with a normal subgroup  $N$  and  $f \in G$  is an element of prime order not dividing the order of  $N$ , then, by a standard argument in group theory (see for example [K, Theorem 1.6.2]) the centraliser of  $f$  in  $G$  and the centraliser of  $fN$  in  $G/N$  are related by the following formula:

$$\mathcal{C}_G(f)/(N \cap \mathcal{C}_G(f)) \cong (\mathcal{C}_G(f)N)/N = \mathcal{C}_{G/N}(fN).$$

An analogous formula holds for the normaliser of the group generated by  $f$ :

$$\mathcal{N}_G(\langle f \rangle)/(N \cap \mathcal{N}_G(\langle f \rangle)) \cong (\mathcal{N}_G(\langle f \rangle)N)/N = \mathcal{N}_{G/N}(\langle fN \rangle).$$

**Proposition 7.** Let  $G \subset \text{Diff}^+(M)$  be a finite group acting on a  $\mathbb{Z}/p$ -sphere  $M$  and let  $N$  be a normal subgroup of  $G$ . If  $f$  is a rotation of order  $p$  such that  $p$  is coprime with the order of  $N$ , then the normaliser of a Sylow  $p$ -subgroup of  $G/N$  contains (up to conjugacy)  $\mathcal{N}_{G/N}(\langle fN \rangle)$  with index at most 2. In particular, the normaliser of a Sylow  $p$ -subgroup of  $G/N$  contains an abelian subgroup of rank at most 2 with index at most 4.

**Proof.**

Let  $S$  be a Sylow  $p$ -subgroup of  $G$ : up to conjugacy we can suppose that  $S$  contains  $f$ . We denote by  $\bar{S}$  the projection of  $S$  to  $G/N$ , which is a Sylow  $p$ -subgroup of  $G/N$ .

Since  $S$  is a normal subgroup of  $\mathcal{N}_G(\langle f \rangle)$ , by Remark 7 we obtain that  $\bar{S}$  is normal in  $\mathcal{N}_{G/N}(\langle fN \rangle)$ , thus  $\mathcal{N}_{G/N}(\bar{S}) \supseteq \mathcal{N}_{G/N}(\langle fN \rangle)$ .

The group  $G/N$  acts on the manifold  $M/N$ . Indeed, although  $N$  may not act freely, the space of orbits  $M/N$  is a manifold for the diffeomorphisms in  $G$  preserve the orientation of  $M$ . Let  $g \in G$  be a non-trivial element of  $S$  such that  $gN$  has order  $p$  and has non-empty fixed-point set with respect to its action on  $M/N$ . This can be rephrased by saying that there is  $x \in M$  and  $n \in N$  such that  $g(x) = n(x)$ , so that  $x$  maps to a fixed point of  $gN$  in  $M/N$ . We thus see that there exists  $g_1 = gn^{-1}$  acting on  $M$  with non-empty fixed-point set, where  $g_1N = gN$ . Since  $g_1$  projects to  $gN$ , we obtain that the order of  $g_1$  is a multiple of  $p$ . Let  $g_1^k$  be a non trivial power of  $g_1$  of order  $p$ . Note that, since  $\emptyset \neq \text{Fix}(g_1) \subseteq \text{Fix}(g_1^k)$ ,  $g_1^k$  has non-empty fixed-point set. The element  $g_1^k$  is contained in  $SN$ , the subgroup of  $G$  generated by  $S$  and  $N$ . Since by Proposition 1 the Sylow  $p$ -subgroup  $S$  is abelian,  $g_1^k$  is conjugated by an element of  $N$  to an element of  $S$  of order  $p$ . We can conclude that the cyclic group generated by  $gN$  is the projection of one cyclic subgroup of  $S$  of order  $p$  generated by elements with non-empty fixed-point set. Since  $S$  contains at most two of these subgroups (Proposition 1),  $\bar{S}$  has the same property and the thesis follows.  $\square$

**Remark 8.** Obviously, Proposition 7 applies whenever  $G$  contains a hyperelliptic rotation  $f$  of odd prime order  $p$  (see Remark 2). More generally, though, its proof is valid whenever the Sylow  $p$ -subgroup of  $G$  contains at most two cyclic subgroups generated by elements with non-empty fixed-point sets and at least one generated by a rotation, just as in the setting of Proposition 3, without any condition on the homology of  $M$ .

If  $M$  is a  $\mathbb{Z}/p$ -homology sphere we can give a different and more geometric proof by observing that the manifold  $M/N$  on which  $G/N$  acts is again a  $\mathbb{Z}/p$ -homology sphere. Proposition 7 then follows from Remark 7 and Proposition 1.

To see that  $M/N$  is a  $\mathbb{Z}/p$ -homology sphere, consider the following exact sequence

$$1 \longrightarrow \pi_1(M) \longrightarrow \pi_1^{\text{or}}(M/N) \longrightarrow N \longrightarrow 1$$

where  $\pi_1^{\text{or}}(M/N)$  denotes the orbifold fundamental group of the orbifold structure of  $M/N$ . The above exact sequence gives:

$$1 \longrightarrow H_1(M) \longrightarrow \pi_1^{\text{or}}(M/N)/\pi_1(M)' \longrightarrow N \longrightarrow 1$$

for the group  $\pi_1(M)'$  is characteristic and contained in  $\pi_1^{\text{or}}(M/N)'$ . Since both  $H_1(M)$  and  $N$  are finite and of order prime with  $p$  so are  $\pi_1^{\text{or}}(M/N)/\pi_1(M)'$  and its quotient, the abelianisation of  $\pi_1^{\text{or}}(M/N)$ . We now observe that  $\pi_1^{\text{or}}(M/N)$  must surject onto  $\pi_1(M/N)$ , the fundamental group of the manifold  $M/N$ , so that the abelianisation of  $\pi_1^{\text{or}}(M/N)$  must surject on the homology of  $M/N$ , proving the assertion.

This argument shows something more precise than what claimed in the statement of Proposition 7, namely that the Sylow  $p$ -subgroup of  $G/N$ , which is isomorphic to  $S$ , acts on  $M/N$  precisely as  $S$  acts on  $M$ .

Recall that a finite group  $Q$  is *quasisimple* if it is perfect (the abelianised group is trivial) and the factor group  $Q/\mathcal{Z}$  of  $Q$  by its centre  $\mathcal{Z}$  is a nonabelian simple group (see [Su2, chapter 6.6]). A group  $E$  is *semisimple* if it is perfect and the factor group  $E/\mathcal{Z}(E)$  is a direct product of nonabelian simple groups. A semisimple group  $E$  is a central product of quasisimple groups which are

uniquely determined. Any finite group  $G$  has a unique maximal semisimple normal subgroup  $E(G)$  (maybe trivial), which is characteristic in  $G$ . The subgroup  $E(G)$  is called the *layer* of  $G$  and the quasisimple factors of  $E(G)$  are called the *components* of  $G$ .

The maximal normal nilpotent subgroup of a finite group  $G$  is called the *Fitting subgroup* and is usually denoted by  $F(G)$ . The Fitting subgroup commutes elementwise with the layer of  $G$ . The normal subgroup generated by  $E(G)$  and by  $F(G)$  is called the *generalised Fitting subgroup* and is usually denoted by  $F^*(G)$ . The generalised Fitting subgroup has the important property to contain its centraliser in  $G$ , which thus coincides with the centre of  $F^*(G)$ . For further properties of the generalised Fitting subgroup see [Su2, Section 6.6.].

Let  $G \subset \text{Diff}^+(M)$  be a finite group acting on a 3-manifold  $M$ . In the following it will be convenient to consider  $O'$ , the maximal normal solvable subgroup of odd order coprime with any hyperelliptic prime for  $G$  and  $O'_2$ , the maximal normal solvable subgroup of order coprime with any hyperelliptic prime for  $G$ , not necessarily odd (for the existence of  $O'$  and  $O'_2$  see [Su2, page 29]).

**Proposition 8.** *Let  $G \subset \text{Diff}^+(M)$  be a finite group acting on a 3-manifold  $M$  not homeomorphic to  $\mathbf{S}^3$ . Let  $O'$  and  $O'_2$  be defined as above. Let  $F(G/O')$  (respectively  $F(G/O'_2)$ ) be the Fitting subgroup of  $G/O'$  (respectively  $G/O'_2$ ) and let  $E(G/O')$  (respectively  $E(G/O'_2)$ ) be the layer of  $G/O'$  (respectively  $G/O'_2$ ). Assume that one of the following conditions is satisfied:*

- *The maximal normal subgroup of odd order of the Fitting subgroup  $F(G/O')$  (respectively  $F(G/O'_2)$ ) is not cyclic;*
- *The Fitting subgroup  $F(G/O')$  (respectively  $F(G/O'_2)$ ) is cyclic and the layer  $E(G/O')$  (respectively  $E(G/O'_2)$ ) is trivial;*

*then  $G$  is solvable.*

**Proof.**

By the maximality of  $O'$  we obtain that the odd primes dividing the order of  $F(G/O')$  are hyperelliptic. If the maximal subgroup of  $F(G/O')$  of odd order is not cyclic, then there is at least one hyperelliptic odd prime  $p$  such that the Sylow  $p$ -subgroup of  $F(G/O')$  is abelian of rank 2, according to Remark 7 and Proposition 1. Since  $F(G/O')$  is nilpotent, its Sylow subgroups are characteristic. As a consequence,  $G/O'$  normalises the elementary abelian  $p$ -group of rank 2 contained in  $F(G/O')$ . Such elementary abelian  $p$ -group contains a projection of a hyperelliptic rotation: by Proposition 7 we obtain that  $G/O'$ , and hence  $G$ , are solvable groups. The proof of the situation where  $F(G/O'_2)$  is not cyclic follows the same lines.

If  $E(G/O')$  is trivial, then the generalised Fitting subgroup of  $G/O'$  coincides with  $F(G/O')$ . In this case we also have that the Fitting subgroup coincides with its centraliser in  $G/O'$ . Since the automorphism group of a cyclic group is abelian, so is  $(G/O')/F^*(G/O')$  which injects into  $\text{Aut}(F^*(G/O'))$ . We deduce once more that  $G$  is solvable. The very same argument gives the proof in the case where  $G/O'$  is replaced with  $G/O'_2$ .  $\square$

**Remark 9.** The proof of the following proposition will rely on the full classification of finite simple groups and, more specifically, on the fact that the structure of their automorphism groups is known.



**Proposition 9.** *Let  $G \subset \text{Diff}^+(M)$  be a finite group acting on a closed orientable 3-manifold  $M$ . Let  $N$  be a normal subgroup of  $G$  and let  $f \in G$  be a rotation of odd prime order  $p$ , coprime with  $|N|$ . If  $E(G/N) \neq 1$ , then the projections to  $G/N$  of the rotations of order  $p$  in  $G$  are contained in the group generated by  $E(G/N)$  and by its centraliser in  $G/N$ , and every component of  $G/N$  has order divisible by  $p$ . Moreover, if  $F(G/N)$  is cyclic, then the centraliser of  $E(G/N)$  in  $G/N$  is solvable.*

**Proof.**

Note that by Remark 3 and by Remark 7 the centraliser of  $fN$  in  $G/N$  is abelian.

First we prove that each component of  $G/N$  is normalised by  $fN$ . Let  $E_1$  be a component of  $G/N$  and suppose by contradiction that  $E_1$  is not normalised by  $fN$ . We denote by  $\bar{f}$  the coset  $fN$  and by  $n+1 = p$  the minimum positive integer such that  $\bar{f}^{n+1}E_1\bar{f}^{-(n+1)} = E_1$ . We define the following subgroup:

$$E_c = \{\bar{x}\bar{f}\bar{x}\bar{f}^{-1} \dots \bar{f}^n\bar{x}\bar{f}^{-n} \mid \bar{x} \in E_1\}.$$

Since elements of  $G/N$  contained in different components commute, it is possible to prove that  $E_c$  is a subgroup of  $G/N$  isomorphic to a quotient of  $E_1$  by a central subgroup, i.e.  $E_c$  is a quasisimple group. Moreover, each element of  $E_c$  commutes with  $\bar{f}$  and this is a contradiction to the fact that  $\bar{f}$  has abelian centraliser in  $G/N$ . Hence  $fN$  normalises  $E_1$ . Moreover, for the very same reason, the action by conjugation of  $\bar{f}$  on  $E_1$  is not trivial. We remark that the automorphism group of  $E_1$  injects in the automorphism group of its simple quotient (see [GLS3, Corollary 5.1.4])

Assume that the simple quotient of  $E_1$  is either sporadic or alternating. Since the order of the outer automorphism group of any such simple group is a (possibly trivial) power of 2 (see [GLS3, Section 5.2 and 5.3]), we conclude that  $\bar{f}$  must induce an inner automorphism of  $E_1$ . In particular  $p$  divides the order of  $E_1$ .

We can thus assume that  $E_1$  is a central extension of a simple group of Lie type.

Recall that, by [GLS3, Theorem 2.5.12],  $\text{Aut}(E_1)$  injects to the semidirect product of a normal subgroup  $\text{Inndiag}(E_1)$ , containing the subgroup  $\text{Inn}(E_1)$  of inner automorphisms, and a group  $\Phi\Gamma$ , where, roughly speaking,  $\Phi$  is the group of automorphisms of  $E_1$  induced by the automorphisms of the defining field and  $\Gamma$  is the group of automorphisms of  $E_1$  induced by the symmetries of the Dinking diagram associated to  $E_1$  (see [GLS3] for the exact definition). By [GLS3, Theorem 2.5.12.(c)], every prime divisor of  $|\text{Inndiag}(E_1)|$  divides  $|E_1|$ . Thus we can assume that the automorphism induced by  $\bar{f}$  on  $E_1$  is not contained in  $\text{Inndiag}(E_1)$  and its projection  $\theta$  on  $\text{Aut}(E_1)/\text{Inndiag}(E_1) \cong \Phi\Gamma$  has order  $p$ . We will find a contradiction showing that in this case the centraliser of  $\bar{f}$  in  $E_1$  is not abelian.

Write  $\theta = \phi\gamma$ , with  $\phi \in \Phi$  and  $\gamma \in \Gamma$ . If  $\phi = 1$ , then  $\gamma$  is nontrivial and  $\bar{f}$  induces a graph automorphism according to [GLS3, Definition 2.5.13]. Since  $p$  is odd, the only possibility is that  $E_1$  is a central perfect extension of  $D_4(q)$  and  $p = 3$  (see [GLS3, Theorem 2.5.12 (e)]). The centraliser of  $\bar{f}$  in  $E_1$  is nonabelian by [GLS3, Table 4.7.3 and Proposition 4.9.2.]. If  $\phi \neq 1$  and  $E_1$  is not isomorphic to the group  ${}^3D_4(q)$ , then the structure of the centraliser of  $\bar{f}$

in  $E_1$  is described by [GLS3, Theorem 4.9.1], and it is nonabelian. Finally, if  $\phi \neq 1$  and  $E_1 \cong {}^3D_4(q)$ , the structure of the non abelian centraliser of  $\bar{f}$  in  $E_1$  follows from [GLS3, Proposition 4.2.4]. We proved that the automorphism induced by  $\bar{f}$  is contained in  $\text{Inndiag}(E_1)$  and  $p$  divides  $|E_1|$ .

Let us now assume by contradiction that the projection to  $G/N$  of a rotation of order  $p$  in  $G$  is not contained in the group generated by  $E(G/N)$  and its centraliser in  $G/N$ . Denote by  $g$  such rotation, by  $S$  the Sylow  $p$ -subgroup of  $G/N$  containing  $\bar{g} = gN$  and by  $C$  the centraliser of  $E(G/N)$  in  $G/N$ . Note that  $C \supset F(G/N)$ . We recall that, by Proposition 1,  $S$  is abelian of rank at most 2.

Since  $p$  divides the order of every component of  $G/N$  and  $S$  has rank at most 2, we get that  $E(G/N)$  has only one component,  $E_1$ , with cyclic Sylow  $p$ -subgroup. Moreover, by the first part of the proof, the automorphism induced by  $\bar{g}$  on  $E_1$  is inner-diagonal. If it is inner, we obtain  $\bar{g}$  as a product of an element that centralises  $E_1$  and an element in  $E_1$ ; otherwise, we get a contradiction, since, by [GLS3, Theorem 2.5.12] and [A, (33.14)], a group of Lie type with cyclic Sylow  $p$ -subgroup cannot have a diagonal automorphism of order  $p$ .

Since the centraliser of the generalised Fitting subgroup coincides with its centre  $\mathcal{Z}(F^*(G/N))$  and since  $C$  acts trivially on  $E(G/N)$  by definition, the quotient  $C/\mathcal{Z}(F^*(G/N))$  merges injectively in the automorphism group of  $F(G/N)$ . If the Fitting subgroup is cyclic, then its automorphism group is abelian and  $C$  is solvable.  $\square$

## 5 Branched coverings of non self-symmetric knots

In this section we prove Theorem 3 under the assumption that the finite group  $G$  of orientation preserving diffeomorphisms of the closed orientable 3-manifold  $M$  contains a hyperelliptic rotation whose order  $n$  is not a power of 2 and which corresponds to a cyclic cover of  $\mathbf{S}^3$  branched along a non-self symmetric knot  $K$ .

The main result of this section is the following:

**Theorem 5.** *Let  $M \neq \mathbf{S}^3$  be the  $n$ -fold cyclic covering of  $\mathbf{S}^3$  branched along a knot  $K \subset \mathbf{S}^3$  such that  $n$  is not a power of 2. Let  $G$  be a finite subgroup of  $\text{Diff}^+(M)$  which contains the group of deck transformations of the  $n$ -fold cyclic branched covering of  $K$ . Assume that  $K$  is not self-symmetric with respect to  $G$ . Then  $G$  contains at most six conjugacy classes of cyclic groups generated by a hyperelliptic rotation of odd prime order.*

**Remark 10.** According to Definition 5 and Proposition 5, the hypothesis that  $K$  is not self-symmetric with respect to  $G$  is equivalent to the following algebraic condition: for each odd prime  $p$  dividing  $n$  the normaliser of a Sylow  $p$ -subgroup of  $G$  coincides with the normaliser of a rotation of order  $p$  (see also Lemma 1). It is in fact this equivalent condition that is used in the proofs of Theorem 7 in [Mec2] and of Theorem 5.

The following two theorems, proved in [Mec2], are key results for the proof of Theorem 5. Let us remark that their proofs only rely on the Gorenstein-Harada theorem [G, page 6] and not on the whole classification of simple groups.

**Theorem 6.** [Mec2, Theorem 2] *Let  $G$  be a finite group of orientation-preserving diffeomorphisms of a closed orientable 3-manifold. Let  $O$  be its maximal normal subgroup of odd order and  $E(G/O)$  be the layer of  $G/O$ . If  $G$  contains a rotation of order 2,  $h$ , such that the coset  $hO$  is contained in  $E(G/O)$ , then  $G/O$  has a normal subgroup  $D$  isomorphic to one of the following groups:*

$$PSL_2(q), \quad PSL_2(q) \times \mathbb{Z}/2 \quad \text{or} \quad SL_2(q) \times_{\mathbb{Z}/2} SL_2(q')$$

where  $q$  and  $q'$  are odd prime powers greater than 4. The factor group  $(G/O)/D$  contains, with index at most 2, an abelian subgroup of rank at most 4.

Let  $M$  be the  $n$ -fold and  $m$ -fold cyclic branched cover of two knots  $K$  and  $K'$ , respectively. We denote by  $H$  and  $H'$  the corresponding cyclic groups of deck transformations for  $K$  and  $K'$ , respectively. We shall say that the knots  $K$  and  $K'$  arise from the standard abelian construction if the groups  $H$  and  $H'$  commute up to conjugacy. This happens if and only if  $K$  (respectively  $K'$ ) admits a full rotation  $h'$  induced by the generator of  $H'$  (respectively  $h$  induced by the generator of  $H$ ) of order  $m$  (respectively  $n$ ) (see also Remark 5) and such that  $(K \cup \text{Fix}(h'))/\langle h' \rangle = (K' \cup \text{Fix}(h))/\langle h \rangle$ .

The following result is stated in [Mec2] under the extra hypothesis that  $M$  is hyperbolic. Hyperbolicity of the manifold  $M$  is only used in the proof to ensure that the covering transformations for  $K$  and  $K'$  sit inside a finite group of diffeomorphisms and that  $K$  and  $K'$  are not trivial. The conditions given here replace thus the hyperbolicity requirement.

**Theorem 7.** [Mec2, Theorem 3] *Let  $M$  be a 3-manifold not homeomorphic to  $\mathbf{S}^3$ . Suppose that  $M$  is the  $n$ -fold and  $m$ -fold cyclic covering of  $\mathbf{S}^3$  branched along two distinct knots  $K$  and  $K'$ , respectively, such that  $m$  and  $n$  are not powers of 2. Let  $G$  be a finite subgroup of  $\text{Diff}^+(M)$  which contains the corresponding cyclic groups of deck transformations for  $K$  and  $K'$ , respectively. If the knot  $K$  is not self-symmetric with respect to  $G$ , then one of the following cases occurs:*

1.  $K$  and  $K'$  arise from the standard abelian construction;
2. If  $O \subset G$  is the maximal normal subgroup of odd order,  $G$  contains a rotation  $h$  of order 2 such that  $hO$  is contained in the layer of  $G/O$  (in particular Theorem 6 applies to  $G$ );
3. All prime divisors of  $n$  and  $m$  are contained in  $\{2, 3, 5, 7\}$  and there exists a normal subgroup  $N$  of  $G$  such that  $N$  is solvable and  $G/N$  is isomorphic to a subgroup of  $GL_4(2)$ .

To prove Theorem 5 the strategy is to “cover” the group  $G$  with solvable groups. With this in mind, we introduce the notion of solvable normal  $\pi$ -cover.

**Definition 7.** Let  $G$  be a finite group. Let  $\pi$  be a set of primes dividing  $|G|$ . We will call a collection  $\mathcal{C}$  of subgroups of  $G$  a *solvable normal  $\pi$ -cover* of  $G$  if every element of  $G$  of prime order  $p$  belonging to  $\pi$  is contained in an element of  $\mathcal{C}$  and for every  $g \in G$ ,  $H \in \mathcal{C}$  we have that  $H^g \in \mathcal{C}$ . We denote by  $\gamma_\pi^s(G)$  the smallest number of conjugacy classes of subgroups in a solvable normal  $\pi$ -cover of  $G$ . Note that, since Sylow subgroups are clearly solvable,  $\gamma_\pi^s(G) \leq |\pi|$ .

Notice that our definition of solvable normal  $\pi$ -cover differs from the more usual definition of *normal cover for non-cyclic finite groups* in that we only need to cover elements of prime order and not elements of arbitrary order. Also the subgroups of a solvable normal  $\pi$ -cover must be solvable but need not be proper.

The following remark better explains the reason why we are interested in “covering”  $G$  with solvable groups.

**Remark 11.** If  $G$  is a finite group of orientation-preserving diffeomorphisms of a closed orientable 3-manifold  $M$  and  $\pi$  is the set of hyperelliptic primes for  $G$ , then, by Theorem 4, the number  $3\gamma_\pi^s(G)$  bounds from above the number of non conjugate cyclic groups generated by a hyperelliptic rotation of odd prime order.

This algebraic lemma, whose proof is straightforward, will be useful.

**Lemma 4.** *Let  $G$  be a finite group, and let  $\pi$  be a set of primes dividing the order of  $G$ .*

- *If  $N$  is a solvable normal subgroup of  $G$ , then  $\gamma_\pi^s(G) \leq \gamma_\pi^s(G/N)$ .*
- *If  $\pi'$  is a subset of  $\pi$ , then  $\gamma_{\pi'}^s(G) \leq \gamma_\pi^s(G)$ .*

We are now ready to prove Theorem 5.

**Proof of Theorem 5.**

Let  $f$  be a hyperelliptic rotation of  $M$  generating the group of deck transformations of the  $n$ -fold cyclic branched covering of  $K$ . We can apply Theorem 7 to deduce that either all the other hyperelliptic rotations commute with  $f$ , up to conjugacy, or one of situations 2. or 3. occurs.

If all hyperelliptic rotations commute with  $f$ , it follows from Proposition 4 that their number, including  $f$ , is at most three.

If we are in situation 3, the factor  $G/N$  has, up to conjugacy, at most three Sylow subgroups of odd order. Choosing a Sylow subgroup of  $G/N$  for each prime in  $\{3, 5, 7\}$ , the preimages of these subgroups in  $G$  are three solvable subgroups containing  $N$  such that their conjugates give a solvable normal  $\pi$ -cover, where  $\pi$  is the set of odd primes dividing the order of  $G$ . Therefore, if a hyperelliptic rotation is contained in  $N$  all the other rotations of odd order commute with it and we are done.

Otherwise, the hyperelliptic primes are contained in the set  $\{3, 5, 7\}$ . By Lemma 2 and Proposition 1, there are in  $G$  at most six non conjugate cyclic groups generated by a hyperelliptic rotation of odd prime order.

Finally we assume that condition 2 of Theorem 7 holds and  $G$  is one of the groups described in Theorem 6.

As in Section 4 and Proposition 8 we will consider  $O'$ , the maximal normal solvable subgroup of  $G$  whose order is odd and coprime with any hyperelliptic prime. If  $p$  is a hyperelliptic prime, since  $|O'|$  and  $p$  are coprime, the projections of the rotations of order  $p$  to  $G/O'$  have abelian centraliser (see Remark 7). We have  $O' \subseteq O$ , the maximal normal subgroup of odd order, so the layer of  $G/O'$ , which may be trivial, projects to a normal semisimple subgroup of  $G/O$  with the same number of quasisimple factors as the layer of  $G/O'$ . Hence, any component of  $G/O'$  is a central perfect extension of  $PSL_2(q)$ ; moreover in  $G/O'$  we have

at most two components. Remark that, although the layer  $E(G/O)$  cannot be trivial for it contains  $hO$ , the layer  $E(G/O')$  might be trivial a priori.

Let us now consider the Fitting subgroup  $F(G/O')$  of  $G/O'$ . By Proposition 8, we can assume that the maximal normal subgroup of  $F(G/O')$  of odd order is cyclic. We will now prove that  $F(G/O')$  itself is cyclic. Since  $F(G/O')$  is nilpotent, this boils down to showing that the Sylow 2-subgroup of  $F(G/O')$  is cyclic (or possibly trivial). The Sylow 2-subgroup of  $F(G/O')$  projects injectively to a subgroup of the Fitting subgroup of  $G/O$ ; in the final step of the proof of Theorem 6 it is proved that  $F(G/O)$  is cyclic, which is enough to reach the desired conclusion. We provide here the proof of this fact for completeness.

We remark that, by the maximality of  $O$ ,  $F(G/O)$  is a 2-group and the layer of  $G/O$  contains  $hO$  where  $h$  is a rotation of order 2. The centraliser of  $h$  is described in Remark 3 and by Remark 7 we can deduce that the structure of the centraliser of  $hO$  in  $G/O$  is analogous. If the layer of  $G/O$  is isomorphic to  $PSL_2(q)$ , the Sylow 2-subgroup of  $E(G/O)$  is dihedral and  $hO$  is centralised by an elementary abelian 2-subgroup of rank 2, this implies that  $F(G/O)$  has order 2. If the layer of  $G/O$  is  $SL_2(q) \times_{\mathbb{Z}/2} SL_2(q')$ , the quotient of  $E(G/O)$  by its centre contains an elementary abelian 2-subgroup of rank 4 that is centralised by  $F(G/O)/Z(E(G/O))$ . Since each 2-subgroup of  $G$  is generated by at most four elements (see [RZ, Lemma 2.2.]), then  $F(G/O)$  coincides with the centre of the layer.

Since  $F(G/O)$  is a cyclic group of order a (possibly trivial) power of 2,  $F(G/O')$  is cyclic. By Proposition 8 we can assume that the layer  $E(G/O')$  is not trivial.

By Proposition 9 we can suppose that  $G/O'$  coincides with the group generated by  $E(G/O')$  and by a normal solvable subgroup (the centraliser of the layer). By Lemma 4 and Remark 11, we obtain that the number of conjugacy classes of subgroups of  $G$  generated by some hyperelliptic rotation of odd prime order is bounded by  $3\gamma_{\pi}^s(E(G/O'))$ , where  $\pi$  is the set of odd primes dividing the order of  $E(G/O')$ .

We recall that the number  $\gamma_{\pi}^s(PSL_2(q))$  is two. In fact the upper triangular matrices form a solvable subgroup of  $SL_2(q)$  of order  $(q-1)q$ , moreover  $SL_2(q)$  contains a cyclic subgroup of order  $q+1$  (see [H]). The conjugates of the projections of these two subgroups to  $PSL_2(q)$  give a solvable normal  $\pi$ -cover. If  $G/O'$  has one component, then we obtain that six is an upper bound for the number of conjugacy classes of cyclic subgroups of  $G$  generated by hyperelliptic rotations. If  $G/O'$  has two components, the same argument implies readily that twelve is an upper bound but in this case, with a small modification of our reasoning, we are able to get that there are at most four hyperelliptic primes.

Suppose that the components of  $G/O'$  are two; the layer of  $G/O'$  projects surjectively to the layer of  $G/O$ . A central perfect extension of  $PSL_2(q)$  that is not simple has a center of order 2, with the only exception of  $PSL(2, 9) \cong \mathbb{A}_6$ . In this case the centre might contain some element of order 3 but, if a component of  $G/O'$  is a central perfect extension of  $PSL(2, 9)$  with centre containing an element of order 3, then the Sylow 3-subgroup is not abelian (see [A, 33.15]), and hence 3 cannot be hyperelliptic; this contradicts the maximality of  $O'$ .

We thus obtain that the layer of  $G/O'$  is  $SL_2(q) \times_{\mathbb{Z}/2} SL_2(q')$  (we remark that  $q$  and  $q'$  might be equal). By Proposition 9 a hyperelliptic prime number  $p$  divides the order of both components, this implies that the Sylow  $p$ -subgroup of  $E(G/O')$  is abelian of rank 2, and all the projections of the rotations of order

$p$  to  $G/O'$  are contained in the layer of  $G/O'$ . We denote by  $U_q$  the subgroup of the upper triangular matrices of  $SL_2(q)$  and by  $C_q$  a cyclic subgroup of order  $q + 1$  of  $SL_2(q)$ . We consider the following set of solvable subgroups of  $SL_2(q) \times_{\mathbb{Z}/2} SL_2(q')$ :

$$\Delta = \{U_q \times_{\mathbb{Z}/2} U_{q'}, C_q \times_{\mathbb{Z}/2} U_{q'}, U_q \times_{\mathbb{Z}/2} C_{q'}, V_q \times_{\mathbb{Z}/2} U_{q'}\}.$$

This set has the property that each Sylow  $p$ -subgroup in  $G/O'$  with  $p$  odd is contained in a conjugate of one of the subgroups in  $\Delta$ . We consider the preimages of the subgroups in  $\Delta$  with respect to the projection  $G \rightarrow G/O'$ . We obtain a set  $\tilde{\Delta}$  containing four solvable subgroups of  $G$  with the same property (i.e. each Sylow  $p$ -subgroup of  $G$  with  $p$  odd is contained in a conjugate of one of the subgroups in  $\tilde{\Delta}$ ). In this situation, if  $p$  is a hyperelliptic prime, its Sylow  $p$ -subgroup has rank 2. By Theorem 4, if one subgroup of  $\tilde{\Delta}$  contains the Sylow  $p$ -subgroup of a hyperelliptic prime, then it cannot contain any other hyperelliptic rotation of odd prime order. Therefore we have at most four hyperelliptic primes. For each of them we have at most two non conjugate cyclic subgroups generated by a hyperelliptic rotation according to Lemma 2. We will prove that in every case we have only one conjugacy class. In general this would imply that four is an upper bound (we use this fact later in the proof of Lemma 5) but, under the assumption of this theorem that at least one knot is non self-symmetric, this fact implies also that this case does not occur.

If  $p$  is a hyperelliptic prime and  $p \neq s$  (where  $q = s^n$ ), the component  $SL_2(q)$  contains an involution acting dihedrally on the Sylow  $p$ -subgroup by conjugation. This involution commutes with the other component of  $E(G/O')$ , and in particular with the Sylow  $p$ -subgroup of  $SL_2(q')$ . By Remark 4, we obtain that the two cyclic groups of order  $p$  generated by elements with non-empty fixed-point set and contained in a Sylow  $p$ -subgroup are conjugate. Let  $S$  be a Sylow  $p$ -subgroup of  $G$ ; we denote by  $f$  and  $f'$  two rotations of order  $p$  that are contained in  $S$  and generate different groups. Their projections to  $G/O'$  generate also different cyclic groups. Replacing  $f$  with one of its powers, we can suppose that the projections of  $f$  and  $f'$  to  $G/O'$  are conjugate. Thus there exist  $n \in O'$  and  $g \in G$  such that  $gfg^{-1} = f'n$ . The subgroup generated by  $S$  and  $O'$  contains  $f'n$ , then, by the Sylow theorems there exist  $s \in S$  and  $m \in O'$  such that  $msgfg^{-1}s^{-1}m^{-1} = msf'ns^{-1}m^{-1}$  is contained in  $S$ . We remark that  $msgfg^{-1}s^{-1}m^{-1}$  has non-empty fixed-point set, so it is contained either in the group generated by  $f'$  or in the group generated by  $f$ . In the first case we obtain that the two cyclic groups of order  $p$  generated by rotations in  $S$  are conjugate. The second case cannot occur because, if  $msgfg^{-1}s^{-1}m^{-1} = f^k$  in the quotient  $G/O'$  we would have that  $f^kO' = msgfg^{-1}s^{-1}m^{-1}O' = sfgf^{-1}s^{-1}O'$  and  $f^kO' = gfg^{-1}O' = f'O'$  which is impossible. We have only one conjugacy class of subgroups generated by a hyperelliptic rotation of order  $p$ .

It remains the case when  $p = s$ ; since the Sylow  $p$ -subgroup has to be cyclic, in this case  $n = 1$ . Since  $p > 3$  we have again elements in  $SL_2(q)$  that normalise the Sylow  $p$ -subgroup but do not centralise it, and hence we have in  $G/O'$  one conjugacy class of cyclic groups of order  $p$  generated by rotations and the conclusion follows as in the previous case.  $\square$

**Remark 12.** The proof of Theorem 5 shows that if one of the knots is non self-symmetric with respect to  $G$  and we have more than three classes of cyclic

groups generated by hyperelliptic rotations of odd prime order, then either we are in Case 3 of Theorem 7 and any hyperelliptic prime is contained in  $\{3, 5, 7\}$  or the layer of  $G/O$  is  $PSL_2(q)$ .

Taking into account the list of all finite non solvable groups that can possibly act on  $\mathbb{Z}/2$ -homology 3-spheres obtained in [MZ, Theorem 1, page 677], the proof of Theorem 5 and its generalisation to arbitrary orders that are not powers of 2 (see Section 7) show readily the result below. We stress that the proof of [MZ, Theorem 1, page 677] is based on the Gorenstein-Harada theorem only.

**Corollary 7.** *Let  $M \neq \mathbf{S}^3$  be a  $\mathbb{Z}/2$ -homology sphere. Then  $G$  contains at most six conjugacy classes of cyclic groups generated by a hyperelliptic rotation of order different from a power of 2.*

## 6 Proof of Theorem 3

In this section we complete the proof of Theorem 3.

**Theorem 3.** *Let  $G$  be a finite group of orientation preserving diffeomorphisms of a closed orientable 3-manifold  $M$  not homeomorphic to  $\mathbf{S}^3$ , then  $G$  contains at most six conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of odd prime order.*

This general upper bound is given by Theorem 5 of the previous section and by the analysis of the structure of a finite group acting on a closed orientable manifold and containing several hyperelliptic rotations corresponding to deck transformations of cyclic branched coverings along self-symmetric knots.

**Lemma 5.** *Let  $G$  be a finite group acting by orientation preserving diffeomorphisms on a closed orientable 3-manifold  $M \neq \mathbf{S}^3$ . Either Theorem 3 holds for  $G$ , or there exists  $G_0$  a subgroup of  $G$  such that for any hyperelliptic prime  $p$  the following properties hold:*

1.  $G_0$  contains all rotations of order  $p$  up to conjugacy;
2.  $S_p$ , a Sylow  $p$ -subgroup of  $G_0$ , has rank two;
3. if  $f$  is a rotation of order  $p$  the normaliser  $\mathcal{N}_{G_0}(\langle f \rangle)$  is abelian of rank two;
4.  $\mathcal{N}_{G_0}(\langle f \rangle)$  is a subgroup of index two of  $\mathcal{N}_{G_0}(S_p)$ .

**Proof.**

We suppose the existence of at least one hyperelliptic prime, otherwise the theorem is automatically verified.

If for some hyperelliptic prime  $p$  and for some hyperelliptic rotation  $f$  of order  $p$  we have that  $\mathcal{N}_G(\langle f \rangle) = \mathcal{N}_G(S_p)$ , then we can apply Theorem 5 (see Remark 10) and thus Theorem 3 holds for  $G$ .

So we can suppose that for any hyperelliptic prime  $\mathcal{N}_G(\langle f \rangle)$  has index 2 in  $\mathcal{N}_G(S_p)$ , thus  $S_p$  is of rank 2 and it contains exactly two groups of odd prime order generated by hyperelliptic rotations.

Now suppose that  $\mathcal{N}_G(\langle f \rangle)$  is not abelian, hence it contains some  $Fix(f)$ -inversion of order 2. Let  $h$  be such an involution, it projects to a strong inversion

of the knot given by the projection of  $Fix(f)$  in  $M/\langle f \rangle \cong \mathbf{S}^3$ . The projection of  $h$  is a rotation and its axis meets the projection of  $Fix(f)$  in two points. This implies that  $h$  is a rotation, too. In [Mec2, Theorem 1] the structure of a finite group acting on a 3-manifold and containing a rotation of order 2 was studied:

**Theorem 8.** [Mec2] *Let  $G$  be a finite group of orientation-preserving diffeomorphisms of a closed orientable 3-manifold. Let  $O$  be the maximal normal subgroup of odd order and  $E(\tilde{G})$  be the layer of  $\tilde{G} = G/O$ . Suppose that  $G$  contains an involution which is a rotation.*

1. *If the semisimple group  $E(\tilde{G})$  is not trivial, it has at most two components and the factor group of  $\tilde{G}/E(\tilde{G})$  is solvable. Moreover, the factor group of  $E(\tilde{G})$  by its centre is either a simple group of sectional 2-rank at most 4 or the direct product of two simple groups of sectional 2-rank at most 2.*
2. *If  $E(\tilde{G})$  is trivial, there exists a normal subgroup  $H$  of  $G$  such that  $H$  is solvable and  $G/H$  is isomorphic to a subgroup of  $GL_4(2)$ , the general linear group of  $4 \times 4$  matrices over the finite field with 2 elements.*

If  $E(\tilde{G})$  is trivial we obtain at most six non conjugate groups of hyperelliptic rotations of odd prime orders (see the proof of Theorem 5, situation 3).

If the projection to  $\tilde{G}$  of a rotation of order 2 is contained in  $E(\tilde{G})$ , then Theorem 6 applies. Once again the proof of Theorem 5 implies that six is an upper bound.

We can therefore suppose that  $E(\tilde{G})$  is not trivial and the projection of any rotation of order 2 is not contained in  $E(\tilde{G})$ . The quotient  $\tilde{G}/E(\tilde{G})$  is solvable, by Hall's Theorem we have a subgroup  $\tilde{G}_0$  (maybe trivial) of  $\tilde{G}/E(\tilde{G})$  such that the order of  $\tilde{G}_0$  is divided only by hyperelliptic primes and the index of  $\tilde{G}_0$  in  $\tilde{G}/E(\tilde{G})$  is coprime with each hyperelliptic prime. We denote by  $G_0$  the preimage of  $\tilde{G}_0$  in  $G$  under the projection  $G \rightarrow \tilde{G} \rightarrow \tilde{G}/E(\tilde{G})$ : the subgroup  $G_0$  contains all the Sylow subgroups corresponding to hyperelliptic primes and does not contain any rotation of order 2. This implies that in  $G_0$  a hyperelliptic rotation does not admit inversions and the normaliser of the subgroup it generates is abelian. The subgroup  $G_0$  satisfies our requests, except possibly the fourth, but the case when  $\mathcal{N}_{G_0}(\langle f \rangle) = \mathcal{N}_{G_0}(S_p)$  was analysed at the beginning of the proof giving that Theorem 3 holds for  $G_0$  and hence for  $G$ , since  $G_0$  contains all the hyperelliptic rotations of  $G$ .  $\square$

**Proof of Theorem 3.** The proof follows from Proposition 10 below since either Theorem 3 holds for  $G$  or Lemma 5 allows to replace  $G$  by a subgroup  $G_0$ , which satisfies the hypotheses of Proposition 10.  $\square$

The rest of the section is devoted to the proof of Proposition 10.

**Proposition 10.** *Let  $G$  be a finite group acting by diffeomorphisms on a closed orientable 3-manifold  $M$  not homeomorphic to  $\mathbf{S}^3$ . If for any hyperelliptic prime  $p$  of  $G$  the following properties hold:*

1.  *$S_p$ , a Sylow  $p$ -subgroup of  $G$ , is abelian of rank 2;*
2. *if  $f$  is a rotation of order  $p$  the normaliser  $\mathcal{N}_G(\langle f \rangle)$  is abelian of rank 2;*
3.  *$\mathcal{N}_G(\langle f \rangle)$  is a subgroup of index 2 of  $\mathcal{N}_G(S_p)$*



then we have in  $G$  at most four non conjugate cyclic subgroups generated by hyperelliptic rotations of odd prime orders.

**Proof.**

If  $p$  is a hyperelliptic prime, then  $S_p$  has rank 2 and contains exactly two subgroups of order  $p$  generated by a rotation. We denote by  $F_p$  and  $F'_p$ , these subgroups. The quotient of  $M$  by each of them is homeomorphic to  $\mathbf{S}^3$  because they are conjugate. We have that  $\mathcal{N}_G(F_p) = \mathcal{N}_G(F'_p)$  is abelian of rank two and each element in the complement  $\mathcal{N}_G(S_p) \setminus \mathcal{N}_G(F_p)$  exchanges by conjugation the subgroups  $F_p$  and  $F'_p$ .

Let  $O'_2$  be the maximal normal solvable subgroup of  $G$  of order coprime with every hyperelliptic prime (note that the order of  $O'_2$  is not necessarily odd). We denote by  $\bar{G}$  the factor group of  $G$  by  $O'_2$ . If  $x$  is an element of  $G$  we denote by  $\bar{x}$  the projection of  $x$  to  $\bar{G}$ , analogously if  $H$  is a subgroup of  $G$ ,  $\bar{H}$  is the projection of  $H$  to  $\bar{G}$ .

**Claim 1.** *The Sylow  $p$ -subgroup  $\bar{S}_p$  of  $\bar{G}$  is abelian of rank 2. Its normaliser  $\bar{N} = \mathcal{N}_{\bar{G}}(\bar{S}_p)$  contains  $\mathcal{N}_{\bar{G}}(\bar{F}_p)$  with index at most 2;  $\mathcal{N}_{\bar{G}}(\bar{F}_p)$  is abelian of rank 2 and contains  $\bar{F}_p$  and  $\bar{F}'_p$ . The groups  $\bar{F}_p$  and  $\bar{F}'_p$  are the only two cyclic groups of order  $p$  in  $\bar{N}$  acting with non-empty fixed-point sets on the manifold  $M/O'_2$ . Moreover, either we have at most three non conjugate groups of rotations of odd prime order in  $G$  or  $\mathcal{N}_{\bar{G}}(\bar{F}_p)$  has index exactly 2 in  $\bar{N}$  and the elements of  $\bar{N}$  not in  $\mathcal{N}_{\bar{G}}(\bar{F}_p)$  exchange  $\bar{F}_p$  and  $\bar{F}'_p$ .*

**Proof.**

The first part of the claim is implied by Proposition 7 and its proof; it remains to prove the last statement. If  $\mathcal{N}_{\bar{G}}(\bar{F}_p) = \mathcal{N}_{\bar{G}}(\bar{S}_p)$ , then we can apply [Su2, Theorem 2.10.] and obtain a normal  $p$ -complement  $\bar{H}$ . The preimage of  $\bar{H}$ , with respect to the projection of  $G$  onto  $\bar{G}$ , is a normal  $p$ -complement of  $G$ . The argument used in the proof of Proposition 6 shows that in this case rotations of odd prime orders commute up to conjugacy. Thus, the proof of Proposition 4 shows that in this case we have at most three non conjugate cyclic subgroups generated by a hyperelliptic rotation of odd prime order.  $\square$

From now on we suppose that  $\mathcal{N}_{\bar{G}}(\bar{F}_p)$  has index exactly 2 in  $\bar{N}$ .

**Claim 2.** *Let  $F(\bar{G})$  be the Fitting subgroup of  $\bar{G}$  and let  $E(\bar{G})$  be the layer of  $\bar{G}$ . Either we have at most three non conjugate cyclic subgroups generated by a hyperelliptic rotation of odd prime order or  $F(\bar{G})$  is cyclic and  $E(\bar{G})$  is not trivial.*

**Proof.**

The claim is just Proposition 8.  $\square$

From now on we suppose that  $F(\bar{G})$  is cyclic and  $E(\bar{G})$  is not trivial.

**Claim 3.**  $(\bar{F}_p \times \bar{F}'_p) \cap \mathcal{Z}(\bar{N}) \neq 1$ .

**Proof.**

Since the elements of  $\bar{N} \setminus \mathcal{N}_{\bar{G}}(\bar{F}_p)$  exchange  $\bar{F}_p$  and  $\bar{F}'_p$ , they centralise a subgroup of order  $p$  of  $\bar{S}_p$  (see Remark 4) and that subgroup is contained in  $\mathcal{Z}(\bar{N})$  since  $\mathcal{N}_{\bar{G}}(\bar{F}_p)$  is abelian. Since every element of order  $p$  in  $\bar{S}_p$  is contained in  $(\bar{F}_p \times \bar{F}'_p)$ , we get the assertion.  $\square$

**Claim 4.** *If a prime  $p$  is hyperelliptic, then the Sylow  $p$ -subgroup of  $E(\bar{G})$  is cyclic and non-trivial. Moreover  $E(\bar{G})$  is a simple group.*

**Proof.**

By [Su2, Exercise 1, page 161],  $\bar{S}_p$  is the direct product of  $\bar{S}_p \cap \mathcal{Z}(\bar{N})$  and  $\bar{G}' \cap \bar{S}_p$ , where  $\bar{G}'$  is the commutator subgroup of  $\bar{G}$ . Since  $\bar{S}_p$  is abelian of rank 2 and its maximal elementary abelian subgroup is  $\bar{F}_p \times \bar{F}'_p$ , it follows that  $\bar{F}_p \times \bar{F}'_p$  is the direct product of  $(\bar{F}_p \times \bar{F}'_p) \cap \mathcal{Z}(\bar{N})$  and  $(\bar{F}_p \times \bar{F}'_p) \cap \bar{G}'$ . By Claim 3,  $(\bar{F}_p \times \bar{F}'_p) \cap \mathcal{Z}(\bar{N}) \neq 1$ . On the other hand, by Proposition 9,  $1 \neq E(\bar{G}) \cap (\bar{F}_p \times \bar{F}'_p) \subseteq (\bar{F}_p \times \bar{F}'_p) \cap \bar{G}'$ , since  $E(\bar{G})$  is perfect. Thus, both  $(\bar{F}_p \times \bar{F}'_p) \cap \mathcal{Z}(\bar{N})$  and  $(\bar{F}_p \times \bar{F}'_p) \cap \bar{G}'$  are cyclic groups of order  $p$ .

If  $q$  is a prime dividing the order of the centre of a component of  $\bar{G}$ , according to [A, (33.14)], the Sylow  $q$ -subgroup of the component cannot be cyclic, so that  $q$  is not hyperelliptic. The maximality of  $O'_2$  ensures that the centre of  $E(\bar{G})$  is trivial.

It follows that  $E(\bar{G})$  is a simple group with cyclic Sylow  $p$ -subgroups. If we had two components in  $\bar{G}$ , then  $p$  should divide the order of both by Proposition 9 and we would obtain a noncyclic  $p$ -subgroup in  $E(\bar{G})$ ; this is impossible and we have one component.  $\square$

**Claim 5.** *If  $p$  is hyperelliptic, then the centraliser in  $E(\bar{G})$  of the Sylow  $p$ -subgroup of  $E(\bar{G})$  is abelian and the normaliser in  $E(\bar{G})$  of the Sylow  $p$ -subgroup of  $E(\bar{G})$  contains an abelian subgroup of index 2.*

**Proof.** Let  $f$  be a hyperelliptic rotation of order  $p$ . By Proposition 9 we have that  $\bar{f}$  is contained in the subgroup generated by the the layer and its centraliser.

We obtain that  $\bar{f}$  is the product of an element of order  $p$  in  $E(\bar{G})$  with an element of order  $p$  that centralises  $E(\bar{G})$ . Since the centraliser of  $\bar{f}$  is abelian, the centraliser in  $E(\bar{G})$  of any Sylow  $p$ -subgroup of  $E(\bar{G})$  is abelian, too. Indeed, any element  $x \in E(\bar{G})$  commutes with any element in the centraliser of  $E(\bar{G})$ . As a consequence, if  $x$  commutes with an element of order  $p$  in  $E(\bar{G})$  it commutes with the whole elementary abelian group  $\bar{F}_p \times \bar{F}'_p$ , and in particular  $x$  commutes with  $\bar{f}$ . Let  $\bar{S}_p$  be a Sylow  $p$ -subgroup of  $\bar{G}$ . If an element of  $E(\bar{G})$  normalises the Sylow  $p$ -subgroup of  $E(\bar{G})$  contained in  $\bar{S}_p$ , then it normalises also the subgroup generated by the elements of order  $p$  in  $\bar{S}_p$ . The group  $\bar{S}_p$  contains precisely two cyclic subgroups of order  $p$  generated by a rotation ( $\bar{F}_p$  and  $\bar{F}'_p$ ). Therefore the normaliser of a Sylow  $p$ -subgroup of  $E(\bar{G})$  contains with index 2 an abelian subgroup (the normaliser cannot be abelian otherwise we have a  $p$ -complement).  $\square$

The proof of the following claim will exploit the classification of finite simple groups.

**Claim 6.** *The group  $G$  contains at most four non conjugate cyclic subgroups generated by hyperelliptic rotations of odd prime order.*

**Proof.**

If  $E(\bar{G})$  is sporadic, the primes dividing the order of the group do not satisfy the condition on the normaliser given by Claim 5 (see [GLS3, Section 5.3.]).

Suppose now  $E(\bar{G}) \cong \mathbf{A}_n$ . If the Sylow  $p$ -subgroup is cyclic, then  $p > n/2$  and so the Sylow  $p$ -subgroup is generated by a  $p$ -cycle. Since by Claim 5 the

centraliser is abelian, we have that  $p > n - 4$ . Thus we have at most two hyperelliptic odd primes and for each prime we have only one conjugacy class of cyclic groups of order  $p$  generated by a hyperelliptic rotation.

The only remaining case is that of simple groups of Lie type.

We note that each solvable subgroup of  $G$  contains at most one conjugacy class of cyclic groups generated by a hyperelliptic rotation of odd prime order. In fact, if  $g$  were a rotation of odd prime order  $q$  commuting with a rotation  $f$  of order  $p$ , then  $g$  should be a  $Fix(f)$ -rotation and centralise a Sylow  $p$ -subgroup  $S_p$ . We should obtain that  $S_p$  is a group of  $Fix(g)$ -rotations. Since  $Fix(g)$  is a simple closed curve, the action of  $S_p$  on  $Fix(g)$  cannot be faithful and  $Fix(g)$  should coincide either with the fixed point of  $F$  or with that of  $F'$ . This is impossible by part 2 of Lemma 1. Therefore, we have that the number of conjugacy classes of subgroups generated by hyperelliptic rotations of odd prime order in  $G$  is bounded by  $\gamma_\pi^s(G)$  where  $\pi$  is the set of hyperelliptic primes. Since the Fitting subgroup of  $G$  is cyclic, by Proposition 9 and Lemma 4 we have that  $\gamma_\pi^s(G) \leq \gamma_\pi^s(E(G))$ . The following Lemma concludes the proof:

**Lemma 6.** *Let  $K$  be a finite quasisimple group of Lie type. If  $\pi$  is the set of odd primes  $p$  such that  $K$  has cyclic Sylow  $p$ -subgroups and  $C_K(g)$  is abelian for every element  $g \in K$  of order  $p$ , then  $\gamma_\pi^s(K) \leq 4$ .*

**Proof.**

Let  $K \cong \Sigma_n(q)$  or  $K \cong {}^d\Sigma_n(q)$ , where  $q$  is a power of a prime  $s$ . Here we use the same notation as in [GLS3]: the symbol  $\Sigma(q)$  (resp.  ${}^d\Sigma(q)$ ) may refer to finite groups in different isomorphism classes, each of them is an untwisted (resp. twisted) finite group of Lie type with root system  $\Sigma$  (see [GLS3, Remark 2.2.5]). Any finite group of Lie type is quasisimple with the exception of the following groups:  $A_1(2)$ ,  $A_1(3)$ ,  ${}^2A_2(2)$ ,  ${}^2B_2(2)$ ,  $B_2(2)$ ,  $G_2(2)$ ,  ${}^2F_4(2)$  and  ${}^2G_2(3)$  (see [GLS3, Theorem 2.2.7]).

If  $s \in \pi$ , then by [GLS3, Theorem 3.3.3], either  $s = 3$  and  $K \cong ({}^2G_2(3))'$  or  $K \cong A_1(s)$ . In the former case the order of  ${}^2G_2(3)'$  is divided only by two odd primes, thus  $\gamma_\pi^s(K) \leq 2$ ; in the latter case we have  $\gamma_\pi^s(K) \leq 2$  (reason as in the proof of Theorem 5, or see for example [H]).

Assume now that  $s \notin \pi$ . By [GLS3, Paragraph 4.10], since the Sylow subgroups are cyclic, every element of order  $p \in \pi$  is contained in a maximal torus of  $K$ , and clearly a maximal torus is abelian.

Therefore, we need only to bound the number of conjugacy classes of cyclic maximal tori in  $K$  with abelian centraliser. Note that the number of conjugacy classes of maximal tori in  $K$  is bounded by the number of different cyclotomic polynomials evaluated in  $q$  appearing as factors of  $|K|$ . Moreover the power of a cyclotomic polynomial in the order of  $K$  gives the rank of the corresponding maximal torus (except possibly when the prime divides the order of the centre but in this case the Sylow subgroup is not cyclic, see [A, (33.14)])

Recall  $\Sigma$  is the root system associated to  $K$  as in [GLS3, 2.3.1]; let  $\Pi = \{\alpha_1, \dots, \alpha_n\}$  be a fundamental system for  $\Sigma$  as in Table 1.8 in [GLS3],  $\alpha_*$  be the lowest root relative to  $\Pi$  as defined in [GLS3, Paragraph 1.8] and set  $\Pi^* = \Pi \cup \{\alpha_*\}$ . We recall that  $|K|$  can be deduced from [GLS3, Table 2.2] and the Dynkin diagrams can be found in [GLS3, Table 1.8]. Observe that, by [GLS3, Proposition 2.6.2], if  $\Sigma_0$  is a root subsystem of  $\Sigma$ , then  $K$  contains a subsystem subgroup  $H$ , which is a central product of groups of Lie type corresponding to the irreducible constituents of  $\Sigma_0$ . In order to prove the lemma, we shall show

that for every group  $K$  and for every element  $g$  of order a prime  $r$  lying in a maximal torus belonging to any but four conjugacy classes of maximal tori, either the Sylow  $r$ -subgroup is not cyclic or we find a subsystem subgroup  $H$  that is a central product of two groups  $H_1$  and  $H_2$  such that  $H_1$  contains  $g$  and  $H_2$  is not abelian. Note that for every prime power  $q$ ,  $A_1(q)$  is a non-abelian group (see [GLS3, Theorem 2.2.7]).

Let  $K \cong A_n(q)$ . Let  $m$  be the minimum index  $i$  such that  $r$  divides  $q^{i+1} - 1$  and let  $\Sigma_0$  be generated by  $\Pi^* \setminus \{\alpha_1, \alpha_n\}$ . Then the corresponding subsystem subgroup is  $H = H_1 \cdot H_2$ , where  $H_1 \cong A_{n-2}(q)$  and  $H_2 \cong A_1(q)$ . Thus if  $m \leq n - 1$ , then  $H_1$  contains an element  $g$  of order  $r$  and  $C_K(g)$  contains  $H_2$  which is not abelian. Therefore, since  $g$  has an abelian centraliser,  $r$  may divide only  $(q^n - 1)(q^{n+1} - 1)$ , that is  $r$  divides  $\Phi_n(q)\Phi_{n+1}(q)$ . Hence we have at most two conjugacy classes of maximal tori with abelian centraliser.

Let  $K \cong {}^2A_n(q)$ . Let  $m$  be the minimum index  $i$  such that  $r$  divides  $q^{i+1} - (-1)^{i+1}$ . Let  $\Sigma_0$  be the root subsystem generated by  $\Pi^* \setminus \{\alpha_1, \alpha_n\}$ . Then, the corresponding subsystem subgroup  $H$  can be written as  $H = H_1 \cdot H_2$ , where  $H_1 \cong {}^2A_{n-2}$  and  $H_2 \cong A_1(q)$ . Thus if  $m \leq n - 1$ , then  $H_1$  contains an element  $g$  of order  $r$  and  $C_K(g)$  contains  $H_2$  which is not abelian. Therefore  $r$  may divide only  $(q^n - (-1^n))(q^{n+1} - (-1^{n+1}))$ , that is  $r$  divides either  $\Phi_{2n}(q)\Phi_{n+1}(q)$  when  $n$  is odd or  $\Phi_n(q)\Phi_{2(n+1)}(q)$  when  $n$  is even. Hence we have at most two conjugacy classes of maximal tori with abelian centraliser.

Let  $K \cong B_n(q)$  and let  $m$  be the minimum  $i$  such that  $r$  divides  $q^{2i} - 1$ . Let  $\Sigma_0$  be the root subsystem generated by  $\Pi \setminus \{\alpha_2\}$ . Thus

$$\Sigma_0 = B_{n-2} \times A_1.$$

We get  $H = H_1 \cdot H_2$ , where  $H_1 \cong B_{n-2}(q)$  and  $H_2 \cong A_1(q)$ . Thus if  $m \leq n - 2$ , then  $H_1$  contains an element  $g$  of order  $r$  and  $C_K(g)$  contains  $H_2$  which is not abelian. Therefore, every element of  $\pi$  may divide only  $(q^{2n} - 1)(q^{2(n-1)} - 1)$ , that is the elements of  $\pi$  are divisors either of  $\Phi_n(q)\Phi_{2n}(q)\Phi_{2(n-1)}(q)$  when  $n$  is odd or of  $\Phi_{n-1}(q)\Phi_{2n}(q)\Phi_{2(n-1)}(q)$  when  $n$  is even. Hence we have at most three conjugacy classes of maximal tori with abelian centraliser.

The case when  $K \cong C_n(q)$  can be treated with a similar argument.

Let  $K \cong D_n(q)$ . Let  $m$  be the minimum  $i$  such that  $r$  divides  $q^{2i} - 1$ . Let  $\Sigma_0$  be the root subsystem generated by  $\Pi \setminus \{\alpha_2\}$ . Hence

$$\Sigma_0 = D_{n-2} \times A_1$$

and  $H = H_1 \cdot H_2$ , where  $H_1 \cong D_{n-2}(q)$  and  $H_2 \cong A_1(q)$ . As in the previous cases, if  $m \leq n - 3$ , then  $H_1$  contains an element  $g$  of order  $r$  and  $C_K(g)$  contains  $H_2$  which is not abelian. Therefore, if  $n$  is even, every element of  $\pi$  may divide only  $(q^{2(n-1)} - 1)(q^{2(n-2)} - 1)$ , that is the elements of  $\pi$  are divisors either of  $\Phi_{n-1}(q)\Phi_{2(n-2)}(q)\Phi_{2(n-1)}(q)$ .

Now assume that  $n$  is odd. Then every element of  $\pi$  may divide only  $(q^{2(n-1)} - 1)(q^{2(n-2)} - 1)(q^n - 1)$ , or equivalently  $\Phi_{n-2}(q)\Phi_{2(n-1)}(q)\Phi_{2(n-2)}\Phi_n(q)$ . Hence every element of order a prime in  $\pi$  is contained in a conjugate of at most four maximal tori.

Let  $K \cong {}^2D_n(q)$ . Let  $m$  be the minimum  $i$  such that  $r$  divides  $q^{2i} - 1$ . Let  $\Sigma_0$  be the root subsystem generated by  $\Pi \setminus \{\alpha_2\}$ . Hence

$$\Sigma_0 = D_{n-2} \times A_1$$

and  $H = H_1 \cdot H_2$ , where  $H_1 \cong {}^2D_{n-2}(q)$  and  $H_2 \cong A_1(q)$ . As in the previous cases, if  $m \leq n - 3$ , then  $H_1$  contains an element  $g$  of order  $r$  and  $C_K(g)$  contains  $H_2$  which is not abelian. Therefore every element of  $\pi$  may divide only  $(q^{2(n-2)} - 1)(q^{2(n-1)} - 1)(q^n + 1)$ , that is the elements of  $\pi$  are divisors either of  $\Phi_{2n}(q)\Phi_{2(n-1)}(q)\Phi_{2(n-2)}(q)\Phi_{n-2}(q)$  when  $n$  is odd or of  $\Phi_{2n}(q)\Phi_{2(n-1)}(q)\Phi_{2(n-2)}(q)\Phi_{n-1}(q)$  when  $n$  is even. Hence every element of order a prime in  $\pi$  is contained in a conjugate of at most four maximal tori.

Let  $K \cong {}^3D_4(q)$ . Then  $|K|_{s'} = (q^2 - 1)(q^8 + q^4 + 1)(q^6 - 1) = \Phi_1(q)^2\Phi_2(q)^2\Phi_3(q)\Phi_4(q)\Phi_6(q)^2\Phi_{12}(q)$  and thus we have that a Sylow  $r$ -subgroup of  $K$  is cyclic if and only if  $r$  divides only  $\Phi_3(q)\Phi_4(q)\Phi_{12}(q)$ . Hence the elements of order a prime in  $\pi$  are contained in the conjugates of at most three maximal tori of  $K$ .

Let  $K \cong G_2(q)$ . Then  $|K|_{s'} = (q^2 - 1)(q^6 - 1) = \Phi_1(q)^2\Phi_2(q)^2\Phi_3(q)\Phi_6(q)$  and thus a Sylow  $r$ -subgroup of  $K$  is cyclic if and only if  $r$  divides only  $\Phi_3(q)\Phi_6(q)$ . Hence the elements of order a prime in  $\pi$  are contained in the conjugates of at most two maximal tori of  $K$ .

Let  $K \cong F_4(q)$ . Then  $|K|_{s'} = (q^2 - 1)(q^6 - 1)(q^8 - 1)(q^{12} - 1) = \Phi_1(q)^4\Phi_2(q)^4\Phi_3(q)^2\Phi_4(q)^2\Phi_6(q)^2\Phi_8(q)\Phi_{12}(q)$  and thus a Sylow  $r$ -subgroup of  $K$  is cyclic if and only if  $r$  divides only  $\Phi_8(q)\Phi_{12}(q)$ . Hence the elements of order a prime in  $\pi$  are contained in the conjugates of at most two maximal tori of  $K$ .

Let  $K \cong E_6(q)$ . Then  $|K|_{s'} = (q^2 - 1)(q^5 - 1)(q^6 - 1)(q^8 - 1)(q^9 - 1)(q^{12} - 1)$  and a Sylow  $r$ -subgroup of  $K$  is cyclic if and only if  $r$  divides only  $\Phi_5(q)\Phi_8(q)\Phi_9(q)\Phi_{12}(q)$ . Hence the elements of order a prime in  $\pi$  are contained in the conjugates of at most four maximal tori of  $K$ .

Let  $K \cong {}^2E_6(q)$ . Then  $|K|_{s'} = (q^2 - 1)(q^5 + 1)(q^6 - 1)(q^8 - 1)(q^9 + 1)(q^{12} - 1)$  and a Sylow  $r$ -subgroup of  $K$  is cyclic if and only if  $r$  divides only  $\Phi_8(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{18}(q)$ . Hence the elements of order a prime in  $\pi$  are contained in the conjugates of at most four maximal tori of  $K$ .

Let  $K \cong E_7(q)$ . Then  $|K|_{s'} = (q^2 - 1)(q^6 - 1)(q^8 - 1)(q^{10} - 1)(q^{12} - 1)(q^{14} - 1)(q^{18} - 1)$  and a Sylow  $r$ -subgroup of  $K$  is cyclic if and only if  $r$  divides only

$$\Phi_5(q)\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{18}(q).$$

Let  $H$  be the subgroup that arises from the root subsystem  $\Sigma_0$  of  $\Sigma$  generated by the set  $\Pi^* \setminus \{\alpha_1\}$ . Then  $\Sigma_0 = A_1 \times E_6$  and  $H = H_1 \cdot H_2$  where  $H_1 \cong A_1(q)$  and  $H_2 \cong E_6(q)$ . Now if  $r$  divides  $\Phi_5(q)\Phi_6(q)\Phi_8(q)\Phi_9(q)\Phi_{12}(q)$ , then the group  $H_2$  contains an element of order  $r$  whose centralizer is not abelian. Hence the elements of order a prime in  $\pi$  are contained in the conjugates of at most four maximal tori of  $K$ .

Let  $K \cong E_8(q)$ . Then  $|K|_{s'} = (q^2 - 1)(q^8 - 1)(q^{12} - 1)(q^{14} - 1)(q^{18} - 1)(q^{20} - 1)(q^{24} - 1)(q^{30} - 1)$  and a Sylow  $r$ -subgroup of  $K$  is cyclic if and only if  $r$  divides only

$$\Phi_7(q)\Phi_9(q)\Phi_{14}(q)\Phi_{15}(q)\Phi_{18}(q)\Phi_{20}(q)\Phi_{24}(q)\Phi_{30}(q).$$

Let  $H$  be the subgroup that arises from the root subsystem  $\Sigma_0$  of  $\Sigma$  generated by the set  $\Pi^* \setminus \{\alpha_1\}$ . Then  $\Sigma_0 = A_1 \times E_7$  and  $H = H_1 \cdot H_2$  where  $H_1 \cong A_1(q)$  and  $H_2 \cong E_7(q)$ . Now if  $r$  divides  $\Phi_7(q)\Phi_9(q)\Phi_{14}(q)\Phi_{18}(q)$ , then the group  $H_2$  contains an element of order  $r$  whose centralizer is not abelian. Hence the

elements of order a prime in  $\pi$  are contained in the conjugates of at most four maximal tori of  $K$ .

Let  $K \cong {}^2B_2(q)$ ,  $q$  even. Then  $|K|_{s'} = (q^2 + 1)(q - 1) = \Phi_1(q)\Phi_4(q)$ . Hence the elements of order an odd prime of  $K$  are contained in the conjugates of two maximal tori of  $K$ .

Let  $K \cong {}^2F_4(q)$ ,  $q$  even. Then  $|K|_{s'} = (q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1) = \Phi_1(q)^2\Phi_2(q)^2\Phi_4(q)^2\Phi_6(q)\Phi_{12}(q)$ . Hence a Sylow  $r$ -subgroup of  $K$  is cyclic if and only if  $r$  divides only  $\Phi_6(q)\Phi_{12}(q)$  and the elements of order a prime in  $\pi$  are contained in the conjugates of two maximal tori of  $K$ .

Let  $K \cong {}^2G_2(q)$ ,  $q$  a power of 3. Then  $|K|_{s'} = (q^3 + 1)(q - 1) = \Phi_1(q)\Phi_2(q)\Phi_6(q)$ . Hence the elements of  $K$  with order an odd prime distinct from 3 are contained in the conjugates of three maximal tori of  $K$ .  $\square$

## 7 Proof of Theorem 1

We proved Theorem 3 for hyperelliptic rotations of odd prime order. In Theorem 1 the assumption that the hyperelliptic rotations have orders that are not powers of 2, ensures that each hyperelliptic rotation has a non trivial power whose order is an odd prime. The proof of Theorem 1 reduces to that of Theorem 3 once we observe that powers of hyperelliptic rotations enjoy the same properties as hyperelliptic rotations of odd prime order provided that the powers have themselves odd prime order. As a consequence, the hyperelliptic rotations of odd prime order can be replaced by powers of odd prime order of hyperelliptic rotations in the proof.

**Theorem 1.** *Let  $M$  be a closed orientable connected 3-manifold which is not homeomorphic to  $\mathbf{S}^3$ . Let  $G$  be a finite subgroup of orientation preserving diffeomorphisms of  $M$ . Then  $G$  contains at most six conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation of order not a power of two.*

### Proof.

For each hyperelliptic rotation  $\varphi$  in  $G$  let  $p$  be an odd prime dividing its order and  $f$  a non trivial power of  $\varphi$  of order  $p$ . The key points to reduce the proof of Theorem 1 to that of Theorem 3 are the following:

1. As explained in Subsection 2.2, if  $\varphi$  is a hyperelliptic rotation in  $G$  and  $p$  is an odd prime dividing the order of  $\varphi$ , then a Sylow  $p$ -subgroup of  $G$  has the same algebraic properties as a Sylow  $p$ -subgroup where  $p$  is hyperelliptic. These properties are all that is needed in the proof of Theorem 3. If  $\varphi$  itself has order  $p$  and the Sylow  $p$ -subgroup has rank 2, then exactly two cyclic groups generated by a rotation of order  $p$  are contained in that Sylow  $p$ -subgroup. In the general case, that is, when  $f$  belongs to the Sylow  $p$ -subgroup, this is not true anymore but we have at most two cyclic groups of order  $p$  generated by elements with non-empty fixed-point sets, according to Lemma 3.
2. We remark that if two hyperelliptic rotations have a common non-trivial power, then their fixed-point sets coincide and, by Lemma 1, they generate the same cyclic group. This implies also that if two non-trivial powers

of two hyperelliptic rotations are conjugate, then so are the subgroups generated by the two rotations.

3. As noted in Remark 8 we know that Proposition 7 holds also in the general case. In the general case the subgroup  $O'$  can be replaced by the maximal normal solvable subgroup of odd order coprime with any prime that divides the order of a hyperelliptic rotation and  $O'_2$  can be replaced by the maximal normal solvable subgroup of order coprime with any odd prime that divides the order of a hyperelliptic rotation.
4. Theorem 4 and Proposition 4 ensure that there are at most three commuting cyclic subgroups of hyperelliptic rotations of order not a power of 2, also in this setting. Thus we can exploit the strategy to cover the group with solvable groups in the general case, too, and the results of Section 5 hold.
5. In Section 6 the proof is based essentially on the algebraic structure of the Sylow  $p$ -subgroups when  $p$  is hyperelliptic, so the arguments apply to cover the case of hyperelliptic rotations of order not a power of 2.

□

**Remark 13.** It is worth pointing out that items 1. and 2. above imply that if  $G$  is a finite group of orientation preserving diffeomorphisms acting on a manifold  $M \neq \mathbf{S}^3$  then, for any odd prime  $p$ , the cyclic subgroups generated by hyperelliptic rotations whose orders are divisible by  $p$  belong to at most two different conjugacy classes.

The proofs of Theorems 3 and 1 show that there are constraints on the structure of a finite group containing at least four conjugacy classes of hyperelliptic rotations with orders not powers of 2. We have:

**Proposition 11.** *Let  $G$  be a finite group of orientation preserving diffeomorphisms of a closed connected 3-manifold  $M \neq \mathbf{S}^3$ . Assume that  $G$  contains at least four conjugacy classes of hyperelliptic rotations whose orders are not powers of 2. Then, there is a solvable normal subgroup  $N$  of  $G$  such that:*

- *The layer  $E(G/N)$  has at most two components, which implies that so does  $G$ ;*
- *If the layer  $E(G/N)$  has two components, then  $E(G/N) = SL_2(q) \times_{\mathbb{Z}/2} SL_2(q')$ .*

*Moreover, if  $G/N$  is not a subgroup of  $GL_4(2)$ , then any odd prime dividing the order of a hyperelliptic rotation does not divide the order of  $N$ .*

The case of hyperelliptic rotations of order a power of 2 is considered in [Re] and [Mec]. We remark that, if a manifold admits a hyperelliptic rotation of order  $2^a$ , then the manifold is a  $\mathbb{Z}/2$ -homology sphere (see [Go]). Combining the results in [Re] and [Mec] we obtain the following result.

**Theorem 9.** *Let  $M$  be a closed orientable connected 3-manifold which is not homeomorphic to  $\mathbf{S}^3$ . Let  $G$  be a finite group of orientation preserving diffeomorphisms of  $M$ . Then  $G$  contains at most nine conjugacy classes of cyclic*

subgroups generated by a hyperelliptic rotation of order a power of 2 and at most two of these conjugacy classes contain cyclic subgroups of order strictly greater than 2.

Now Corollary 1 for hyperbolic manifold is a consequence of Theorems 1 and 9.

## 8 Finite group actions on reducible manifolds

In this short section we shall discuss finite groups of orientation preserving diffeomorphisms generated by hyperelliptic rotations and acting on reducible manifolds. We shall see that Theorem 3 is trivially fulfilled in this setting. The proof we give in the general case works also for the reducible one but here the situation is much simpler and is described by the following proposition.

**Proposition 12.** *Let  $M$  be a reducible manifold and let  $G \subset \text{Diff}^+(M)$  be a finite group generated by at least two hyperelliptic rotations with distinct orders. In this case  $G$  is isomorphic to a finite subgroup of  $SO(3)$ , and thus, up to conjugacy,  $G$  contains at most three cyclic groups generated by hyperelliptic rotations.*

**Proof.**

Since  $G$  is a finite group, we can assume that it acts by isometries with respect to some Riemannian metric on  $M$ . Moreover, by the equivariant sphere theorem, we can assume that there is a prime decomposition of  $M$  into irreducible components which is  $G$ -equivariant. Note that each hyperelliptic rotation must fix setwise each sphere of the decomposition since the space of orbits is  $\mathbf{S}^3$ , hence so must  $G$ . Since  $G$  acts by isometries, its action is determined by the action on any sphere of the decomposition. It follows that  $G$  is conjugate to a finite subgroup of  $SO(3)$ , that is cyclic, dihedral or a spherical triangular group.  $\square$

## 9 Proof of Theorem 2

The statement of Theorem 2 is equivalent to the following:

**Theorem 10.** *Let  $M$  be a closed, orientable, connected, irreducible 3-manifold which is not homeomorphic to  $\mathbf{S}^3$ , then the group  $\text{Diff}^+(M)$  of orientation preserving diffeomorphisms of  $M$  contains at most six conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation of odd prime order.*

Notice that generically one expects that two hyperelliptic rotations in the group  $\text{Diff}^+(M)$  generate an infinite subgroup.

### 9.1 Proof of Theorem 10 for Seifert manifolds

In this section we prove Proposition 13 which implies Theorem 10 for closed Seifert fibred 3-manifolds. We also show that the assumption that the hyperelliptic rotations have odd prime orders cannot be avoided in general by exhibiting examples of closed Seifert fibred 3-manifolds  $M$  such that  $\text{Diff}^+(M)$  contains an arbitrarily large number of conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of odd, but not prime, orders.



**Proposition 13.** *Let  $M$  be a closed Seifert fibred 3-manifold which is not homeomorphic to  $\mathbf{S}^3$ . Then the group  $\text{Diff}^+(M)$  of orientation preserving diffeomorphisms of  $M$  contains at most one conjugacy class of cyclic subgroups generated by a hyperelliptic rotation of odd prime order except if  $M$  is a Brieskorn integral homology sphere with 3 exceptional fibres. In this latter case  $\text{Diff}^+(M)$  contains at most three non conjugate cyclic subgroups generated by hyperelliptic rotations of odd prime orders.*

**Proof.**

By hypothesis  $M$  is a cyclic cover of  $S^3$  branched over a knot, so it is orientable and a rational homology sphere by Remark 2. Notably,  $M$  cannot be  $\mathbf{S}^1 \times \mathbf{S}^2$  nor a Euclidean manifold, except for the Hantzsche-Wendt manifold, see [Or, Chap. 8.2]. In particular, since  $M$  is prime it is also irreducible.

Consider a hyperelliptic rotation  $\psi$  on  $M$  of odd prime order  $p$  and let  $K$  be the image of  $\text{Fix}(\psi)$  in the quotient  $\mathbf{S}^3 = M/\psi$  by the action of  $\psi$ . The knot  $K$  must be hyperbolic or a torus knot, otherwise its exterior would be toroidal and have a non-trivial JSJ-collection of essential tori which would lift to a non trivial JSJ-collection of tori for  $M$ , since the order of  $\psi$  is  $p > 2$  (see [JS, J] and [BS]). By the orbifold theorem (see [BoP], [CHK]), the cyclic branched cover with order  $p \geq 3$  of a hyperbolic knot is hyperbolic, with a single exception for  $p = 3$  when  $K$  is the the figure-eight knot and  $M$  is the Hantzsche-Wendt Euclidean manifold. But then, by the orbifold theorem and the classification of 3-dimensional crystallographic groups,  $\psi$  is the unique, up to conjugacy, Euclidean hyperelliptic rotation on  $M$ , see for example [Dun], [Z].

So we can assume that  $M$  is the  $p$ -fold cyclic cover of  $\mathbf{S}^3$  branched along a non trivial torus knot  $K$  of type  $(a, b)$ , where  $a > 1$  and  $b > 1$  are coprime integers. Then  $M$  is a Brieskorn-Pham manifold  $M = V(p, a, b) = \{z^p + x^a + y^b = 0 \text{ with } (z, x, y) \in \mathbb{C}^3 \text{ and } |z|^2 + |x|^2 + |y|^2 = 1\}$ . A simple computation shows that  $M$  admits a Seifert fibration with 3,  $p$  or  $p + 1$  exceptional fibres and base space  $\mathbf{S}^2$ , see [Ko2, Lem. 2], or [BoPa, Lemma 6 and proof of Lemma 7]. In particular  $M$  has a unique Seifert fibration, up to isotopy: by [Wa], [Sco2] and [BOT] the only possible exception with base  $\mathbf{S}^2$  and at least 3 exceptional fibres is the double of a twisted  $I$ -bundle, which is not a rational homology sphere, since it fibers over the circle. We distinguish now two cases:

**Case 1:** *The integers  $a$  and  $b$  are coprime with  $p$ , and there are three singular fibres of pairwise relatively prime orders  $a$ ,  $b$  and  $p$ .* By the orbifold theorem any hyperelliptic rotation of  $M$  of order  $> 2$  is conjugate into the circle action  $S^1 \subset \text{Diff}^+(M)$  inducing the Seifert fibration, hence the uniqueness of the Seifert fibration, up to isotopy, implies that  $M$  admits at most 3 non conjugate cyclic groups generated by hyperelliptic rotations with odd prime orders belonging to  $\{a, b, p\}$ . Indeed  $M$  is a Brieskorn integral homology sphere, see [BPZ].

**Case 2:** *Either  $a = p$  and  $M$  has  $p$  singular fibres of order  $b$ , or  $a = a'p$  with  $a' > 1$ , and  $M$  has  $p$  singular fibres of order  $b$  and one extra singular fibre of order  $a'$ .* In both situations, there are  $p \geq 3$  exceptional fibres of order  $b$  which are cyclically permuted by the hyperelliptic rotation  $\psi$ . As before,  $M$  has a unique Seifert fibration, up to isotopy. Therefore, up to conjugacy,  $\psi$  is the only hyperelliptic rotation of order  $p$  on  $M$ , and by the discussion above  $M$  cannot

admit a hyperelliptic rotation of odd prime order  $q \neq p$ .  $\square$

**Remark 14.** The requirement that the rotations are hyperelliptic is essential in the proof of Proposition 13. The Brieskorn homology sphere  $\Sigma(p_1, \dots, p_n)$ ,  $n \geq 4$ , with  $n \geq 4$  exceptional fibres admits  $n$  rotations of pairwise distinct prime orders but which are not hyperelliptic.

The hypothesis that the orders of the hyperelliptic rotations are  $\neq 2$  cannot be avoided either.

Indeed, Montesinos' construction of fibre preserving hyperelliptic involutions on Seifert fibered rational homology spheres [Mon1], [Mon2], (see also [BS, Appendix A], [BZH, Chapter 12]), shows that for any given integer  $n$  there are infinitely many closed orientable Seifert fibered 3-manifolds with at least  $n$  conjugacy classes of hyperelliptic rotations of order 2.

On the other hand, the hypothesis that the orders are odd primes is sufficient but not necessary: A careful analysis of the Seifert invariants shows that if  $M \neq \mathbf{S}^3$  is a Seifert rational homology sphere, then  $M$  can be the cyclic branched cover of a knot in  $\mathbf{S}^3$  of order  $> 2$  in at most three ways.

The hypotheses of Proposition 13 cannot be relaxed further, though: Proposition 14 below shows that there exist closed 3-dimensional circle bundles with arbitrarily many conjugacy classes of hyperelliptic rotations of odd, but not prime, orders.

**Proposition 14.** *Let  $N$  be an odd prime integer. For any integer  $1 \leq q < \frac{N}{2}$  the Brieskorn-Pham manifold  $M = V((2^q + 1)(2^{(N-q)} + 1), 2^q + 1, 2^{(N-q)} + 1)$  is a circle bundle over a closed surface of genus  $g = 2^{N-1}$  with Euler class  $\pm 1$ . Hence, up to homeomorphism (possibly reversing the orientation),  $M$  depends only on the integer  $N$  and admits at least  $\frac{N-1}{2}$  conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of odd orders.*

**Proof.** We remark first that the integers  $q$  and  $N - q$  are relatively prime, because  $N$  is prime. If  $k$  is a common prime divisor of  $2^q + 1$  and  $2^{(N-q)} + 1$ , by Bezout identity we have  $2^1 = 2^{aq+b(N-q)} \equiv (-1)^{a+b} \pmod{k}$  which implies that  $k = 3$ . But then  $(-1)^q \equiv (-1)^{(N-q)} \equiv -1 \pmod{3}$  and thus  $(-1)^N \equiv 1 \pmod{3}$  which is impossible since  $N$  is odd. Hence  $2^q + 1$  and  $2^{(N-q)} + 1$  are relatively prime.

So the Brieskorn-Pham manifold  $M$  is the  $(2^q + 1)(2^{(N-q)} + 1)$ -fold cyclic cover of  $S^3$  branched over the torus knot  $K_q = T(2^q + 1, 2^{(N-q)} + 1)$ . It is obtained by Dehn filling the  $(2^q + 1)(2^{(N-q)} + 1)$ -fold cyclic cover of the exterior of the torus knot  $K_q$  along the lift of its meridian. The  $(2^q + 1)(2^{(N-q)} + 1)$ -fold cyclic cover of the exterior of  $K_q$  is a trivial circle bundle over a once punctured surface of genus  $g = 2^N - 1$ . On the boundary of the torus-knot exterior the algebraic intersection between a meridian and a fibre of the Seifert fibration of the exterior is  $\pm 1$  (the sign depends on a choice of orientation, see for example [BZH, Chapter 3]). So on the torus boundary of the  $(2^q + 1)(2^{(N-q)} + 1)$ -fold cyclic cover the algebraic intersection between the lift of a meridian of the torus knot and a  $\mathbf{S}^1$ -fiber is again  $\pm 1$ . Hence the circle bundle structure of the  $(2^q + 1)(2^{(N-q)} + 1)$ -fold cyclic cover of the exterior of the torus knot  $K_q$  can be extended with Euler class  $\pm 1$  to the Dehn filling along the lift of the meridian. So  $M$  is a circle bundle over a closed surface of genus  $g = 2^{N-1}$  with Euler class  $\pm 1$ .

Since the torus knots  $K_q = T(2^q + 1, 2^{(N-q)} + 1)$  are pairwise inequivalent for  $1 \leq q \leq \frac{N-1}{2}$ , the cyclic subgroups generated by the hyperelliptic rotations corresponding to the  $(2^q + 1)(2^{(N-q)} + 1)$ -fold cyclic branched covers of the knots  $K_q$  are pairwise not conjugate in  $Diff^+(M)$ .  $\square$

**Remark 15.** Note that the Seifert manifolds  $M$  and their hyperelliptic rotations constructed in Proposition 14 enjoy the following properties: If  $N > 8$ , then no hyperelliptic rotation can commute up to conjugacy with all the remaining ones (see Theorem 4 and [BoPa, Theorem 2]). If  $N > 14$  no finite subgroup of  $Diff^+(M)$  can contain up to conjugacy all hyperelliptic rotations of  $M$ , according to Theorem 1.

## 9.2 Reduction to the finite group action case

The fact that Theorem 10 implies Corollary 4 follows from the existence of a decomposition of a closed, orientable 3-manifold  $M$  as a connected sum of prime manifolds and the observation that a hyperelliptic rotation on  $M$  induces a hyperelliptic rotation on each of its prime summands. A 3-manifold admitting a hyperelliptic rotation must be a rational homology sphere, and so  $M$  cannot have  $\mathbf{S}^2 \times \mathbf{S}^1$  summands. Hence all prime summands are irreducible and at least one is not homeomorphic to  $\mathbf{S}^3$ , since  $M$  itself is not homeomorphic to  $\mathbf{S}^3$ . This is enough to conclude.

The remaining of this section is devoted to the proof that Theorem 3 implies Theorem 10.

We prove the following proposition:

**Proposition 15.** *If  $M$  is a closed, orientable, irreducible 3-manifold such that there are  $k \geq 7$  conjugacy classes of cyclic subgroups of  $Diff^+(M)$  generated by hyperelliptic rotations of odd prime order, then  $M$  is homeomorphic to  $\mathbf{S}^3$ .*

**Proof.**

Let  $M$  be a closed, orientable, irreducible 3-manifold such that  $Diff^+(M)$  contains  $k \geq 7$  conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of odd prime orders.

According to the orbifold theorem (see [BoP], [BMP], [CHK]), a closed orientable irreducible manifold  $M$  admitting a rotation has geometric decomposition. This means that  $M$  can be split along a (possibly empty) finite collection of  $\pi_1$ -injective embedded tori into submanifolds carrying either a hyperbolic or a Seifert fibered structure. This splitting along tori is unique up to isotopy and is called the JSJ-decomposition of  $M$ , see for example [NS], [BMP, chapter 3]. In particular, if its JSJ-decomposition is trivial,  $M$  admits either a hyperbolic or a Seifert fibered structure.

First we see that  $M$  cannot be hyperbolic. Indeed, if the manifold  $M$  is hyperbolic then, by the orbifold theorem, any hyperelliptic rotation is conjugate into the finite group  $Isom^+(M)$  of orientation preserving isometries of  $M$ . Hence, applying Theorem 3 to  $G = Isom^+(M)$ , we see that  $k \leq 6$  against the hypothesis.

If the manifold  $M$  is Seifert fibered, it follows readily from Proposition 13 of the previous section that  $M = \mathbf{S}^3$ . So we are left to exclude the case where the JSJ-decomposition of  $M$  is not empty.

Consider the JSJ-decomposition of  $M$ : each geometric piece admits either a complete hyperbolic structure with finite volume or a Seifert fibred product structure with orientable base. Moreover, the geometry of each piece is unique, up to isotopy.

Let  $\Psi = \{\psi_1, \dots, \psi_k, k \geq 7\}$  be the set of hyperelliptic rotations which generate non conjugate cyclic subgroups in  $Diff^+(M)$ . By the orbifold theorem [BoP], [BMP], [CHK], after conjugacy, one can assume that each hyperelliptic rotation preserves the JSJ-decomposition, acts isometrically on the hyperbolic pieces, and respects the product structure on the Seifert pieces. We say that they are geometric.

Let  $\Gamma$  be the dual graph of the JSJ-decomposition: it is a tree, for  $M$  is a rational homology sphere (in fact, the dual graph of a manifold which is the cyclic branched cover of a knot is always a tree, regardless of the order of the covering). Let  $H \subset Diff^+(M)$  be the group of diffeomorphisms of  $M$  generated by the set  $\Psi$  of geometric hyperelliptic rotations. By [BoPa, Thm 1], there is a subset  $\Psi_0 \subset \Psi$  of  $k_0 \geq 4$  hyperelliptic rotations with pairwise distinct odd prime order, say  $\Psi_0 = \{\psi_i, i = 1, \dots, k_0\}$ .

Let  $H_\Gamma$  denote the image of the induced representation of  $H$  in  $Aut(\Gamma)$ . Since rotations of finite odd order cannot induce an inversion on any edge of  $\Gamma$ , the finite group  $H_\Gamma$  must fix point-wise a non-empty subtree  $\Gamma_f$  of  $\Gamma$ .

The idea of the proof is now analogous to the ones in [BoPa] and [BPZ]: we start by showing that, up to conjugacy, the  $k_0 \geq 4$  hyperelliptic rotations with pairwise distinct odd prime orders can be chosen to commute on the submanifold  $M_f$  of  $M$  corresponding to the subtree  $\Gamma_f$ . We consider then the maximal subtree corresponding to a submanifold of  $M$  on which these hyperelliptic rotations commute up to conjugacy and prove that such subtree is in fact  $\Gamma$ . Then the conclusion follows as in the proof of Theorem 4, by applying Proposition 4.

The first step of the proof is achieved by the following proposition:

**Proposition 16.** *The hyperelliptic rotations in  $\Psi_0$  commute, up to conjugacy in  $Diff^+(M)$ , on the submanifold  $M_f$  of  $M$  corresponding to the subtree  $\Gamma_f$ .*

**Proof.**

Since the hyperelliptic rotations in  $\Psi_0$  have odd orders, either  $\Gamma_f$  contains an edge, or it consists of a single vertex. We shall analyse these two cases.

**Case 1:  $M_f$  contains an edge.**

**Claim 7.** *Assume that  $\Gamma_f$  contains an edge and let  $T$  denote the corresponding torus. The hyperelliptic rotations in  $\Psi_0$  commute, up to conjugacy in  $Diff^+(M)$ , on the geometric pieces of  $M$  adjacent to  $T$ .*

**Proof.**

The geometric pieces adjacent to  $T$  are left invariant by the hyperelliptic rotations in  $\Psi_0$ , since their orders are odd. Let  $V$  denote one of the two adjacent geometric pieces: each hyperelliptic rotation acts non-trivially on  $V$  with odd prime order. We distinguish two cases according to the geometry of  $V$ .

**$V$  is hyperbolic.** In this case all rotations act as isometries and leave a cusp invariant. Since their order is odd, the rotations must act as translations along horospheres, and thus commute.

Note that, even in the case where a rotation has order 3, its axis cannot meet a torus of the JSJ-decomposition of  $M$  for each such torus is separating and cannot meet the axis in an odd number of points.

**$V$  is Seifert fibred.** In this case we can assume that the hyperelliptic rotations in  $\Psi$  preserve the Seifert fibration with orientable base. Since their orders are odd and prime, each one preserves the orientation of the fibres and of the base. The conjugacy class of a fiber-preserving rotation of  $V$  with odd prime order depends only on its combinatorial behaviour, i.e. its translation action along the fibre and the induced permutation on cone points and boundary components of the base. In particular, two geometric rotations with odd prime order having the same combinatorial data are conjugate via a diffeomorphism isotopic to the identity.

Since the hyperelliptic rotations in  $\Psi_0$  have pairwise distinct odd prime orders, an analysis of the different cases described in Lemma 7 below shows that at most one among these hyperelliptic rotations can induce a non trivial action on the base of the fibration, and thus the remaining ones act by translation along the fibres and induce the identity on the base. Since the translation along the fibres commutes with every fiber-preserving diffeomorphism of  $V$ , the hyperelliptic rotations in  $\Psi_0$  commute on  $V$ .

Lemma 7 describes the Seifert fibred pieces of a manifold admitting a hyperelliptic rotation of odd prime order, as well as the action of the rotation on the pieces. Its proof can be found in [BoPa, Lemma 6 and proof of Lemma 7], see also [Ko2, Lem. 2].

**Lemma 7.** *Let  $M$  be an irreducible 3-manifold admitting a non trivial JSJ-decomposition. Assume that  $M$  admits a hyperelliptic rotation of prime odd order  $p$ . Let  $V$  be a Seifert piece of the JSJ-decomposition for  $M$ . Then the base  $B$  of  $V$  can be:*

1. *A disc with 2 cone points. In this case either the rotation freely permutes  $p$  copies of  $V$  or leaves  $V$  invariant and acts by translating along the fibres.*
2. *A disc with  $p$  cone points. In this case the rotation leaves  $V$  invariant and cyclically permutes the singular fibres, while leaving a regular one invariant.*
3. *A disc with  $p+1$  cone points. In this case the rotation leaves  $V$  and a singular fibre invariant, while cyclically permuting the remaining  $p$  singular fibres.*
4. *An annulus with 1 cone point. In this case either the rotation freely permutes  $p$  copies of  $V$  or leaves  $V$  invariant and acts by translating along the fibres.*
5. *An annulus with  $p$  cone points. In this case the rotation leaves  $V$  invariant and cyclically permutes the  $p$  singular fibres.*
6. *A disc with  $p-1$  holes and 1 cone point. In this case the rotation leaves  $V$  invariant and cyclically permutes all  $p$  boundary components, while leaving invariant the only singular fibre and a regular one.*
7. *A disc with  $k$  holes,  $k \geq 2$ . In this case either the rotation freely permutes  $p$  copies of  $V$  or leaves  $V$  invariant. In this latter case either the rotation*

acts by translating along the fibres, or  $k = p - 1$  and the rotation permutes all the boundary components (while leaving invariant two fibres), or  $k = p$  and the rotation permutes  $p$  boundary components, while leaving invariant the remaining one and a regular fibre.

□

We conclude that the rotations in  $\Psi_0$  can be chosen to commute on the submanifold  $M_f$  of  $M$  corresponding to  $\Gamma_f$  by using inductively at each edge of  $\Gamma_f$  the gluing lemma below (see [Lemma 6][BPZ]). We give the proof for the sake of completeness.

**Lemma 8.** *If the rotations preserve a JSJ-torus  $T$  then they commute on the union of the two geometric pieces adjacent to  $T$ .*

**Proof.**

Let  $V$  and  $W$  be the two geometric pieces adjacent to  $T$ . By Claim 7, after conjugacy in  $\text{Diff}^+(M)$ , the rotations in  $\Psi_0$  commute on  $V$  and  $W$ . Since they have pairwise distinct odd prime orders, their restrictions on  $V$  and  $W$  generate two cyclic groups of the same finite odd order. Let  $g_V$  and  $g_W$  be generators of these two cyclic groups. Since they have odd order, they both act by translation on  $T$ . We need the following result about the slope of translation for such periodic transformation of the torus:

**Claim 8.** *Let  $\psi$  be a periodic diffeomorphism of the product  $T^2 \times [0, 1]$  which is isotopic to the identity and whose restriction to each boundary torus  $T \times \{i\}$ ,  $i = 0, 1$ , is a translation with rational slopes  $\alpha_0$  and  $\alpha_1$  in  $H_1(T^2; \mathbb{Z})$ . Then  $\alpha_0 = \alpha_1$ .*

**Proof.**

By Meeks and Scott [MS, Thm 8.1], see also [BS, Prop. 12], there is a Euclidean product structure on  $T^2 \times [0, 1]$  preserved by  $\psi$  such that  $\psi$  acts by translation on each fiber  $T \times \{t\}$  with rational slope  $\alpha_t$ . By continuity, the rational slopes  $\alpha_t$  are constant. □

Now the the following claim shows that the actions of  $g_W$  and  $g_V$  can be glued on  $T$ .

**Claim 9.** *The translations  $g_V|_T$  and  $g_W|_T$  have the same slope in  $H_1(T^2; \mathbb{Z})$ .*

**Proof.**

Let  $\Psi_0 = \{\psi_i, i = 1, \dots, k_0\}$ . Let  $p_i$  the order of  $\psi_i$  and  $q_i = \prod_{j \neq i} p_j$ . Then the slopes  $\alpha_V$  and  $\alpha_W$  of  $g_V|_T$  and  $g_W|_T$  verify:  $q_i \alpha_V = q_i \alpha_W$  for  $i = 1, \dots, k_0$ , by applying Claim 8 to each  $\psi_i$ . Since the GCD of the  $q_i$  is 1, it follows that  $\alpha_V = \alpha_W$ . □

This finishes the proof of Lemma 8 and of Proposition 16 when  $M_f$  contains an edge. □

To complete the proof of Proposition 16 it remains to consider the case where  $\Gamma_f$  is a single vertex.

**Case 2:  $M_f$  is a vertex.**

**Claim 10.** *Assume that  $\Gamma_f$  consists of a single vertex and let  $V$  denote the corresponding geometric piece. Then the hyperelliptic rotations in  $\Psi_0$  commute on  $V$ , up to conjugacy in  $Diff^+(M)$ .*

**Proof.**

We consider again two cases according to the geometry of  $V$ .

The case where  $V$  is Seifert fibred follows once more from Lemma 7.

We consider now the case where  $V$  is hyperbolic.

In this case, the hyperelliptic rotations in  $\Psi$  act non trivially on  $V$  by isometries of odd prime orders. The restriction  $H|_V \subset Isom^+(V)$  of the action of the subgroup  $H$  that they generate in  $Diff^+(M)$  is finite.

If the action on  $V$  of the cyclic subgroups generated by two of the hyperelliptic rotations in  $\Psi$  are conjugate in  $H|_V$ , one can conjugate the actions in  $Diff^+(M)$  to coincide on  $V$ , since any diffeomorphism in  $H|_V$  extends to  $M$ . Then by [BoPa, Lemma 10] these actions must coincide on  $M$ , contradicting the hypothesis that the conjugacy classes of cyclic subgroups generated by the hyperelliptic rotations in  $\Psi$  are pairwise distinct in  $Diff^+(M)$ . Hence, the cyclic subgroup generated by the  $k \geq 7$  hyperelliptic rotations in  $\Psi$  are pairwise not conjugate in the finite group  $H|_V \subset Isom^+(V)$ .

Since the dual graph of the JSJ-decomposition of  $M$  is a tree, a boundary torus  $T \subset \partial V$  is separating and bounds a component  $U_T$  of  $M \setminus int(V)$ . Since, by hypothesis,  $\Gamma_f$  consists of a single vertex, no boundary component  $T$  is setwise fixed by the finite group  $H|_V$ . This means that there is a hyperelliptic rotation  $\psi_i \in \Psi$  of odd prime order  $p_i$  such that the orbit of  $U_T$  under  $\psi_i$  is the disjoint union of  $p_i$  copies of  $U_T$ . In particular  $U_T$  projects homeomorphically onto a knot exterior in the quotient  $\mathbf{S}^3 = M/\psi_i$ . Therefore on each boundary torus  $T = \partial U_T \subset \partial V$ , there is a simple closed curve  $\lambda_T$ , unique up to isotopy, that bounds a properly embedded incompressible and  $\partial$ -incompressible Seifert surface  $S_T$  in the knot exterior  $U_T$ .

By pinching the surface  $S_T$  onto a disc  $\mathbf{D}^2$ , in each component  $U_T$  of  $M \setminus int(V)$ , one defines a degree-one map  $\pi : M \rightarrow M'$ , where  $M'$  is the rational homology sphere obtained by Dehn filling each boundary torus  $T \subset V$  along the curve  $\lambda_T$ .

For each hyperelliptic rotation  $\psi_i$  in  $\Psi$ , of odd prime order  $p_i$ , the  $\psi_i$ -orbit of each component  $U_T$  of  $M \setminus int(V)$  consists of either one or  $p_i$  elements. As a consequence, by [Sa],  $\psi_i$  acts equivariantly on the set of isotopy classes of curves  $\lambda_T \subset \partial V$ . Hence, each  $\psi_i$  extends to periodic diffeomorphism  $\psi'_i$  of order  $p_i$  on  $M'$ . Moreover,  $M'$  is a  $\mathbb{Z}/p_i$ -homology sphere, since so is  $M$  and  $\pi : M \rightarrow M'$  is a degree-one map. According to Smith's theory, if  $Fix(\psi')$  is non-empty on  $M'$ , then  $\psi'_i$  is a rotation on  $M'$ . To see that  $Fix(\psi') \neq \emptyset$  on  $M'$  it suffices to observe that either  $Fix(\psi) \subset V$  or  $\psi_i$  is a rotation of some  $U_T$ ; in this latter case,  $\psi'_i$  must have a fixed point on the disc  $\mathbf{D}^2$  onto which the surface  $S_T$  is pinched. To show that  $\psi'_i$  is hyperelliptic it remains to show that the quotient  $M'/\psi'_i$  is homeomorphic to  $\mathbf{S}^3$ .

Since  $\psi_i$  acts equivariantly on the components  $U_T$  of  $M \setminus int(V)$  and on the set of isotopy classes of curves  $\lambda_T \subset \partial V$ , the quotient  $\mathbf{S}^3 = M/\psi_i$  is obtained from the compact 3-manifold  $V/\psi_i$  by gluing knot exteriors (maybe solid tori) to its boundary components, in such a way that the boundaries of the Seifert surfaces of the knot exteriors are glued to the curves  $\lambda_T/\psi_i \subset \partial V/\psi_i$ .

In the same way, the rotation  $\psi'_i$  acts equivariantly on the components  $M' \setminus$

$\text{int}(V)$  and on the set of isotopy classes of curves  $\lambda_T \subset \partial V$ . By construction, these components are solid tori, and either the axis of the rotation is contained in  $V$  or there exists a unique torus  $T \in \partial V$  such that the solid torus glued to  $T$  to obtain  $M'$  contains the axis. In the latter case, by [EL, Cor. 2.2], the rotation  $\psi'_i$  preserves a meridian disc of this solid torus and its axis is a core of  $W_T$ . It follows that the images in the quotient  $M'/\psi'_i$  of the the solid tori glued to  $\partial V$  are again solid tori. Hence  $M'/\psi'_i$  is obtained from  $\mathbf{S}^3$  by replacing each components of  $\mathbf{S}^3 \setminus V/\psi_i$  by a solid torus, in such way that boundaries of meridian discs of the solid tori are glued to the curves  $\lambda_T/\psi'_i \subset \partial V/\psi'_i$ . It follows that  $M'/\psi'_i$  is again  $\mathbf{S}^3$ .

So far we have constructed a closed orientable 3-manifold  $M'$  with a finite subgroup of orientation preserving diffeomorphisms  $H_V$  that contains at least seven conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of odd prime orders. Theorem 3 implies that  $M'$  must be  $\mathbf{S}^3$ , and thus by the orbifold theorem [BLP]  $H_V$  is conjugate to a finite subgroup of  $SO(4)$ . In particular the subgroup  $H_0 \subset H_V$  generated by the subset  $\Psi_0$  of at least 4 hyperelliptic rotations with pairwise distinct odd prime orders must be solvable. Therefore, by Proposition 6 the induced rotations commute on  $M'$  and, by restriction, the hyperelliptic rotations in  $\Psi_0$  commute on  $V$ .  $\square$

In the final step of the proof we extend the commutativity on  $M_f$  to the whole manifold  $M$ . The proof of this step is analogous to the one given in [BPZ], since the proof there was not using the homology assumption. We give the argument for the sake of completeness.

**Proposition 17.** *The  $k_0 \geq 4$  hyperelliptic rotations in  $\Psi_0$  commute, up to conjugacy in  $\text{Diff}^+(M)$ , on  $M$ .*

**Proof.**

Let  $\Gamma_c$  be the largest subtree of  $\Gamma$  containing  $\Gamma_f$ , such that, up to conjugacy in  $\text{Diff}^+(M)$ , the rotations in  $\Psi_0$  commute on the corresponding invariant submanifold  $M_c$  of  $M$ . We shall show that  $\Gamma_c = \Gamma$ . If this is not the case, we can choose an edge contained in  $\Gamma$  corresponding to a boundary torus  $T$  of  $M_c$ . Denote by  $U_T$  the submanifold of  $M$  adjacent to  $T$  but not containing  $M_c$  and by  $V_T \subset U_T$  the geometric piece adjacent to  $T$ .

Let  $H_0 \subset \text{Diff}^+(M)$  be the group of diffeomorphisms of  $M$  generated by the set of geometric hyperelliptic rotations  $\Psi_0 = \{\psi_i, i = 1, \dots, k_0\}$ . Since the rotations in  $\Psi_0$  commute on  $M_c$  and have pairwise distinct odd prime orders, the restriction of  $H_0$  on  $M_c$  is a cyclic group with order the product of the orders of the rotations. Since  $\Gamma_f \subset \Gamma_c$ , the  $H_0$ -orbit of  $T$  cannot be reduced to only one element. Moreover each rotation  $\psi \in \Psi_0$  either fixes  $T$  or acts freely on the orbit of  $T$  since its order is prime.

If no rotation in  $\Psi_0$  leaves  $T$  invariant, the  $H_0$ -orbit of  $T$  contains as many elements as the product of the orders of the rotations, for they commute on  $M_c$ . In particular, only the identity (which extends to  $U$ ) stabilises a torus in the  $H_0$ -orbit of  $T$ . Note that all components of  $\partial M_c$  are in the  $H_0$ -orbit of  $T$  and bound a manifold homeomorphic to  $U_T$ .

Since the rotation  $\psi_i$  acts freely on the  $H_0$ -orbit of  $U_T$ ,  $U_T$  is a knot exterior in the quotient  $M/\psi_i = \mathbf{S}^3$ . Hence there is a well defined meridian-longitude system on  $T = \partial U_T$  and also on each torus of the  $H_0$ -orbit of  $T$ . This set of meridian-longitude systems is cyclically permuted by each  $\psi_i$  and thus equivariant under the action of  $H_0$ .



Let  $M_c/H_0$  be the quotient of  $M_c$  by the induced cyclic action of  $H_0$  on  $M_c$ . Then there is a unique boundary component  $T' \subset \partial(M_c/H_0)$  which is the image of the  $H_0$ -orbit of  $T$ . We can glue a copy of the knot exterior  $U_T$  to  $M_c/H_0$  along  $T'$  by identifying the image of the meridian-longitude system on  $\partial U_T$  with the projection on  $T'$  of the equivariant meridian-longitude systems on the  $H_0$ -orbit of  $T$ . Denote by  $N$  the resulting manifold. For all  $i = 1, \dots, k_0$ , consider the cyclic (possibly branched) cover of  $N$  of order  $q_i = \prod_{j \neq i} p_j$  which is induced by the cover  $\pi_i : M_c/\psi_i \rightarrow M_c/H_0$ . This makes sense because  $\pi_1(T') \subset \pi_{i*}(\pi_1(M_c/\psi_i))$ . Call  $\tilde{N}_i$  the total space of such covering. By construction it follows that  $\tilde{N}_i$  is the quotient  $(M_c \cup H_0 \cdot U_T)/\psi_i$ . This implies that the  $\psi_i$ 's commute on  $M_c \cup H_0 \cdot U_T$  contradicting the maximality of  $\Gamma_c$ .

We can thus assume that some rotations fix  $T$  and some do not. Since all rotations commute on  $M_c$  and have pairwise distinct odd prime orders, we see that the orbit of  $T$  consists of as many elements as the product of the orders of the rotations which do not fix  $T$  and each element of the orbit is fixed by the rotations which leave  $T$  invariant. The rotations which fix  $T$  commute on the orbit of  $V_T$  according to Claim 7 and Lemma 8, and form a cyclic group generated by, say,  $\gamma$ . The argument for the previous case shows that the rotations acting freely on the orbit of  $T$  commute on the orbit of  $U_T$  and thus on the orbit of  $V_T$ , and form again a cyclic group generated by, say,  $\eta$ . To reach a contradiction to the maximality of  $M_c$ , we shall show that  $\gamma$ , after perhaps some conjugacy, commutes with  $\eta$  on the  $H_0$ -orbit of  $V_T$ , in other words that  $\gamma$  and  $\eta\gamma\eta^{-1}$  coincide on  $H_0 \cdot V_T$ . Since  $\eta$  acts freely and transitively on the  $H_0$ -orbit of  $V_T$  there is a natural and well-defined way to identify each element of the orbit  $H_0 \cdot V_T$  to  $V_T$  itself. Note that this is easily seen to be the case if  $V_T$  is hyperbolic: this follows from Claim 7 and Claim 8. We now consider the case when  $V_T$  is Seifert fibred.

**Claim 11.** *Assume that  $V_T$  is Seifert fibred and that the restriction of  $\gamma$  induces a non-trivial action on the base of  $V_T$ . Then  $\gamma$  induces a non-trivial action on the base of each component of the  $H_0$ -orbit of  $V_T$ . Moreover, up to conjugacy on  $H_0 \cdot V_T \setminus V_T$  by diffeomorphisms which are the identity on  $H_0 \cdot T$  and extend to  $M$ , we can assume that the restrictions of  $\gamma$  to these components induce the same permutation of their boundary components and the same action on their bases.*

**Proof.**

By hypothesis  $\gamma$  and  $\eta\gamma\eta^{-1}$  coincide on  $\partial M_c$ . The action of  $\gamma$  on the base of  $V_T$  is non-trivial if and only if its restriction to the boundary circle of the base corresponding to the fibres of the torus  $T$  is non-trivial. Therefore the action of  $\gamma$  is non-trivial on the base of each component of  $H_0 \cdot V_T$ .

By Lemma 7 and taking into account the fact that  $V_T$  is a geometric piece in the JSJ-decomposition of the knot exterior  $U_T$ , the only situation in which the action of  $\gamma$  on the base of  $V_T$  is non-trivial is when the base of  $V_T$  consists of a disc with  $p$  holes, where  $p$  is the order of one of the rotations that generate  $\gamma$ , the only one whose action is non trivial on the base of the fibration. Moreover, the restriction of  $\gamma$  to the elements of  $H_0 \cdot V_T$  cyclically permutes their boundary components which are not adjacent to  $M_c$ . Up to performing Dehn twists, along vertical tori inside the components of  $H_0 \cdot V_T \setminus V_T$ , which permute the boundary components, we can assume that the restriction of  $\gamma$  induces the same cyclic permutations on the boundary components of each element of the orbit  $H_0 \cdot V_T$ .

We only need to check that Dehn twists permuting two boundary components extend to the whole manifold  $M$ . This follows from the fact that the manifolds adjacent to these components are all homeomorphic and that Dehn twist act trivially on the homology of the boundary.

Since the actions of the restrictions of  $\gamma$  on the bases of the elements of  $H_0 \cdot V_T$  are combinatorially equivalent, after perhaps a further conjugacy by an isotopy, the different restrictions can be chosen to coincide on the bases.  $\square$

By Claim 7 and Claim 11 we can now deduce that the restrictions of  $\gamma$  and  $\eta\gamma\eta^{-1}$  to the  $H_0$ -orbit of  $V_T$  commute, up to conjugacy of  $\gamma$  which is the identity on the  $H_0$ -orbit of  $T$ . Since  $\gamma$  and  $\eta\gamma\eta^{-1}$  coincide on this  $H_0$ -orbit of  $T$ , we can conclude that they coincide on the  $H_0$ -orbit of  $V_T$ . This finishes the proof of Proposition 17.  $\square$

Since there are at least four hyperelliptic rotations with pairwise distinct odd prime orders in  $\Psi_0$ , Proposition 15 is consequence of Proposition 17 and Proposition 4, like in the solvable case, or by applying Theorem 4.  $\square$

**Remark 16.** As we have seen, the strategy to prove that an irreducible manifold  $M$  with non-trivial JSJ-decomposition cannot admit more than six conjugacy classes of subgroups generated by hyperelliptic rotations of odd prime order inside  $Diff^+(M)$  consists in modifying by conjugacy any given set of hyperelliptic rotations so that the new hyperelliptic rotations commute pairwise. Note that this strategy cannot be carried out in general if the orders are not pairwise coprime (see, for instance, [BoPa, Section 4.1] where the case of two hyperelliptic rotations of the same odd prime order, generating non conjugate subgroups, is considered). Similarly, for hyperelliptic rotations of arbitrary orders  $> 2$  lack of commutativity might arise on the Seifert fibred pieces of the decomposition, as it does for the Seifert fibred manifolds constructed in Proposition 14.

## 10 Appendix: non-free finite group actions on rational homology spheres

In this section we will show that every finite group  $G$  admits a faithful action by orientation preserving diffeomorphisms on some rational homology sphere so that some elements of  $G$  have non-empty fixed-point sets.

Cooper and Long's construction of  $G$ -actions on rational homology spheres in [CL] consists in starting with a  $G$ -action on some 3-manifold and then modifying the original manifold, notably by Dehn surgery, so that the new manifold inherits a  $G$ -action but has "smaller" rational homology. Since their construction does not require that the  $G$ -action is free, it can be used to prove the existence of non-free  $G$ -actions. We will thus start by exhibiting non-free  $G$ -actions on some 3-manifold before pointing out what need to be taken into account in this setting in order for Cooper and Long's construction to work.

Since every cyclic group acts as a group of rotations of  $\mathbf{S}^3$ , for simplicity we will assume that  $G$  is a finite non-cyclic group.

**Claim 12.** *Let  $G$  be a finite non-cyclic group. There is a closed, connected, orientable 3-manifold  $M$  on which  $G$  acts faithfully, by orientation preserving diffeomorphisms so that there are  $g \in G \setminus \{1\}$  with the property that  $Fix(g) \neq \emptyset$ .*

**Proof.**

Let  $k \geq 2$  and let  $\{g_i\}_{1 \leq i \leq k+1}$  be a system of generators for  $G$  satisfying the following requirements:

- for all  $1 \leq i \leq k + 1$ , the order of  $g_i$  is  $n_i \geq 2$ ;
- $g_{k+1} = g_1 g_2 \dots g_k$ .

Since  $G$  is not cyclic these conditions are not difficult to fulfill, and actually they can be fulfilled even in the case when  $G$  is cyclic for an appropriate choice of the set  $\{g_i\}_{1 \leq i \leq k+1}$ .

Consider the free group of rank  $k$  that we wish to see as the fundamental group of a  $(k + 1)$ -punctured 2-sphere: each generator  $x_i$  corresponds to a loop around a puncture of the sphere so that a loop around the  $k + 1$ st puncture corresponds to the element  $x_1 x_2 \dots x_k$ . Build an orbifold  $\mathcal{O}$  by compactifying the punctured-sphere with cone points so that the  $i$ th puncture becomes a cone point of order  $n_i$ . The resulting orbifold has (orbifold) fundamental group with the following presentation:

$$\langle x_1, x_2, \dots, x_k, x_{k+1} \mid \{x_i^{n_i}\}_{1 \leq i \leq k+1}, x_1 \dots x_k x_{k+1}^{-1} \rangle.$$

Clearly this group surjects onto  $G$ . Such surjection gives rise to an orbifold covering  $\Sigma \rightarrow \mathcal{O}$ , where  $\Sigma$  is an orientable surface on which  $G$  acts in such a way that each element  $g_i$  has non-empty fixed-point set. One can consider the 3-manifold  $\Sigma \times \mathbf{S}^1$ : the action of  $G$  on  $\Sigma$  extends to a product action of  $G$  on  $\Sigma \times \mathbf{S}^1$  which is trivial on the  $\mathbf{S}^1$  factor.  $\square$

To be able to repeat Cooper and Long's construction it is now easy to observe that it is always possible to choose  $G$ -equivariant families of simple closed curves in  $M$  so that they miss the fixed-point sets of elements of  $G$  and either their homology classes generate  $H_1(M; \mathbb{Q})$  (so that the hypothesis of [CL, Lemma 2.3] are fulfilled when we choose  $X$  to be the exterior of such families) or the family is the  $G$ -orbit of a representative of some prescribed homology class (as in the proof of [CL, Proposition 2.5]).

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AIX-MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M, UMR 7373, 13453 MARSEILLE, FRANCE  
 michel.boileau@univ-amu.fr

DIPARTIMENTO DI MATEMATICA E FISICA "NICCOLÒ TARTAGLIA", UNIVERSITÀ CATTOLICA DEL SACRO CUORE, VIA MUSEI 41, 25121 BRESCIA, ITALY  
 clara.franchi@unicatt.it

DIPARTIMENTO DI MATEMATICA E GEOSCIENZE, UNIVERSITÀ DEGLI STUDI DI TRIESTE, VIA VALERIO 12/1, 34127 TRIESTE  
 mmecchia@units.it

AIX-MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M, UMR 7373, 13453 MARSEILLE, FRANCE  
 luisa.paoluzzi@univ-amu.fr

DIPARTIMENTO DI MATEMATICA E GEOSCIENZE, UNIVERSITÀ DEGLI STUDI DI TRIESTE, VIA VALERIO 12/1, 34127 TRIESTE  
 zimmer@units.it