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► **To cite this version:**

van Hao Can. Super-exponential extinction time of the contact process on random geometric graphs. 2015. hal-01160633v2

HAL Id: hal-01160633

<https://hal.science/hal-01160633v2>

Preprint submitted on 29 Jul 2015 (v2), last revised 18 Jul 2017 (v3)

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SUPER-EXPONENTIAL EXTINCTION TIME OF THE CONTACT PROCESS ON RANDOM GEOMETRIC GRAPHS

VAN HAO CAN

ABSTRACT. In this paper, we prove lower and upper bounds for the extinction time of the contact process on random geometric graphs with connecting radius tending to infinity. We obtain that for any infection rate $\lambda > 0$, the contact process on these graphs survives a time super-exponential in the number of vertices.

1. INTRODUCTION

We will study the contact process on random geometric graphs (RGGs) in $d \geq 2$ dimensions with intensity g and connecting radius R , denoted by $G(n, R, g)$.

A RGG is constructed as follows. The vertex set is composed by the atoms of a Poisson point process with intensity g on $[0, \sqrt[d]{n}]^d$. Then for any two vertices $v \neq w$, we draw an edge between them if $\|v - w\| \leq R$, where $\|\cdot\|$ denotes the Euclidean norm in \mathbb{R}^d . We will assume in all this paper that there are positive constants b and B , such that

$$(1) \quad 0 < b \leq g(x) \leq B < +\infty \quad \text{for all } x.$$

The contact process is one of the most studied interacting particle systems and is also often interpreted as a model to describe the spread of a virus in a network (see for instance [13]). Mathematically, it can be defined as follows: given a locally finite graph $G = (V, E)$ and $\lambda > 0$, the contact process on G with infection rate λ is a pure jump Markov process $(\xi_t)_{t \geq 0}$ on $\{0, 1\}^V$. Vertices of V (also called sites) are regarded as individuals which are either infected (state 1) or healthy (state 0). By considering ξ_t as a subset of V via $\xi_t \equiv \{v : \xi_t(v) = 1\}$, the transition rates are given by

$$\begin{aligned} \xi_t &\rightarrow \xi_t \setminus \{v\} \text{ for } v \in \xi_t \text{ at rate } 1, \text{ and} \\ \xi_t &\rightarrow \xi_t \cup \{v\} \text{ for } v \notin \xi_t \text{ at rate } \lambda |\{w \in \xi_t : \{v, w\} \in E\}|, \end{aligned}$$

where $|A|$ is the cardinality of a set A .

Originally the contact process was studied on integer lattices or homogeneous trees. More recently, probabilists started investigating this process on some families of random networks like configuration models, or preferential attachment graphs, see for instance [1, 2, 3, 5, 16, 17].

Random geometric graphs have been extensively studied for a long time by many authors, see in particular Penrose's book [20]. Recently, these graphs have also been considered as models of wireless networks (see e.g. [12]). Therefore, there has been interest in processes occurring on it, including the contact process in both theoretical and practical approaches, see for example [8, 9, 21].

2010 *Mathematics Subject Classification.* 82C22; 60K35; 05C80.

Key words and phrases. Contact process; extinction time ; random geometric graph.

In this paper, we are in particular interested in the extinction time of the contact process,

$$\tau_n = \inf\{t : \xi_t^1 = \emptyset\},$$

where (ξ_t^1) is the contact process on $G(n, R, g)$ starting with all nodes infected.

We observe that w.h.p. the number of vertices in this graph is between $b(1 - o(1))n$ and $B(1 + o(1))n$. On the other hand, the contact process on finite graphs dies out a.s. i.e. $\tau_n < \infty$ a.s. Now, it is interesting to determine the order of magnitude of τ_n . For sparse graphs, i.e. graphs in which the number of edges is of order the number of vertices, it is shown that the extinction time is at most exponential in the number of vertices (see Lemma 3.4). On the other hand, we will show in Section 2.1 that the extinction time of the contact process on a complete graph is super-exponential in the number of vertices. In random geometric graphs with large connecting radius, the number of edges is much larger than the number of vertices. Moreover, the graph locally looks like a complete graph (all vertices in a ball of radius $R/2$ form a clique). Hence, we can expect that $\log \tau_n$ is super-linear in n as in the case of the complete graph. (Note that there are graphs which are not sparse but for which $\log \tau_n = \mathcal{O}(n)$, for example the configuration model with infinite mean degree, see Theorem 1.2 (ii) in [5]).

In [9], more precisely in Theorem 1.2, the author considers the contact process on an equivalent model of $G(n, R, g)$ in 2 dimensions. Translating to our model, he proves that if $R \rightarrow \infty$ and $R^2 = \mathcal{O}(\log n)$, then there exist positive constants $c = c(\lambda)$ and $C = C(\lambda)$, such that w.h.p. $Cn \log n \geq \log \tau_n \geq cnR^2 / \log n$.

In our main result, we will prove that in all dimensions larger than or equal to 2 if $R \rightarrow \infty$, then w.h.p. $\log \tau_n \asymp n \log(\lambda R^d)$.

Theorem 1.1. *Let $d \geq 2$ and τ_n be the extinction time of the contact process on the graph $G(n, R, g)$ with g satisfying (1) starting from full occupancy. There exist positive constants c, C and K depending only on d, b and B , such that the following statements hold.*

(i) *If $R^d \geq K/(\lambda \wedge 1)$, then*

$$\mathbb{P}(\tau_n \geq \exp(cn \log(\lambda R^d))) = 1 - o(1)$$

and

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n \rightarrow \infty]{(\mathcal{L})} \mathcal{E}(1),$$

with $\mathcal{E}(1)$ an exponential random variable with mean one.

(ii) *For all $R > 0$*

$$\mathbb{P}(\tau_n \leq \exp(Cn \log(\lambda R^d))) = 1 - o(1).$$

Part (i) implies that when R tends to infinity, the contact process survives a time super-exponential in n regardless the value of λ . We usually say that in this case the critical value of the infection rate is zero. On the other hand, recently in [19] the authors show that when R is *fixed*, there is a non-trivial phase transition of the contact process on *infinite* random geometric graphs (i.e. the vertices are atoms of a Poisson point process on the whole space \mathbb{R}^d). More precisely, they prove that there exists a constant $\lambda_c > 0$, such that if $\lambda < \lambda_c$, the contact process dies out a.s., whereas if $\lambda > \lambda_c$, it survives forever with positive probability.

There is a conjecture saying that the contact process on a sequence of finite graphs, say (G_n) , converging locally to some limiting graph, say G , exhibits a phase transition at the same critical value of infection rate as on the limit G : in the sub-critical regime, the contact process on G dies out a.s. (resp. τ_n -the extinction time of the process on G_n is of order $\log(|G_n|)$), whereas in the super-critical regime, the contact process survives forever with positive probability (resp. $\log \tau_n$ is of order $|G_n|$). This conjecture has been verified in some examples, see for instance [2, 3, 4, 16, 18].

Hence, for the contact process on $G(n, R, g)$ with R fixed, it is natural to guess that

$$\tau_n \asymp \log n \text{ when } \lambda < \lambda_c \text{ and } \log \tau_n \asymp n \text{ when } \lambda > \lambda_c,$$

with λ_c the critical value of the contact process on the infinite random geometric graphs as in [19].

We now make some comments on the proof of Theorem 1.1. To prove (i), we will find in $G(n, R, g)$ a subgraph composed by many adjacent complete graphs, see Lemma 3.1. Then we can compare the contact process with a super-critical oriented percolation. The proof of (ii) follows from a quite general argument: the extinction of the contact process on a graph $G = (V, E)$ is at most $\exp(C|V| \log(|E|/|V|))$, for some positive constant C .

The paper is organized as follows. In Section 2, we prove some preliminary results on the contact process on complete graphs, the oriented percolation in two dimensions and long paths in a site percolation. The main theorem is proved in Section 3. In the last section, we study some extensions: the case $d = 1$ and the equivalent model considered in [9].

We now fix some notation. We call size of a graph G the cardinality of its set of vertices and we denote it by $|G|$. For $\mu > 0$, we denote by $\text{Poi}(\mu)$ a Poisson random variable with mean μ and $\mathcal{E}(\mu)$ an exponential random variable with mean $1/\mu$. If f and g are two real functions, we write $f = \mathcal{O}(g)$ if there exists a constant $C > 0$, such that $f(x) \leq Cg(x)$ for all x ; $f \asymp g$ if $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$; $f = o(g)$ if $g(x)/f(x) \rightarrow 0$ as $x \rightarrow \infty$. The term w.h.p. means with probability tending to 1.

2. PRELIMINAIRES

2.1. Contact process on complete graphs. We denote by K_m the complete graph of size m . Similarly to the results for the contact process on star graphs in [3, 17], we prove the following.

Lemma 2.1. *Assume that $\lambda \leq 1$ and $m\lambda \geq 640$. Then the following assertions hold.*

(i) *Let (ξ_t) be the contact process on K_m . Then*

$$\mathbb{P} \left(\inf_{T_m/2 \leq t \leq T_m} |\xi_t| \geq m/4 \mid |\xi_0| \geq m/4 \right) \geq 1 - 2T_m^{-1},$$

with $T_m = \exp(m \log(\lambda m)/16)$.

(ii) *Let K_m^1 and K_m^2 be two disjoint complete graphs of size m , and $K_{m,m}$ be the graph formed by adding an edge between these two graphs. Let (ξ_t) be the contact process on $K_{m,m}$. Then*

$$\mathbb{P}(|\xi_{T_m} \cap K_m^2| \geq m/4 \mid |\xi_0 \cap K_m^1| \geq m/4) \geq 1 - 5T_m^{-1}.$$

Proof. Part (i) follows from the following claims

$$(2) \quad \mathbb{P} \left(\inf_{0 \leq t \leq T_m} |\xi_t| \geq m/4 \mid |\xi_0| \geq m/2 \right) \geq 1 - T_m^{-1},$$

$$(3) \quad \mathbb{P}(\exists t \leq T_m/2 : |\xi_t| \geq m/2 \mid |\xi_0| \geq m/4) \geq 1 - T_m^{-1}.$$

To prove (2), we note that $|\xi_t|$ increases by 1 with rate $\lambda|\xi_t|(m - |\xi_t|)$ and decreases by 1 with rate $|\xi_t|$. Therefore, when $|\xi_t| \in (m/4, 3m/4)$, the increasing rate is larger than $3\lambda m^2/16$ and the decreasing rate is less than $3m/4$.

Thus during the time that $|\xi_t| \in (m/4, 3m/4)$, it stochastically dominates a continuous time random walk (U_t) satisfying $U_0 = m/2$ and

$$\begin{aligned} U_t &\rightarrow U_t - 1 && \text{at rate } 3m/4, \\ U_t &\rightarrow U_t + 1 && \text{at rate } 3\lambda m^2/16. \end{aligned}$$

To understand (U_t) , let us consider its skeleton, a random walk X_r satisfying $X_0 = m/2$ and for $r \geq 0$

$$\begin{aligned} X_{r+1} &= X_r - 1 && \text{with probability } \frac{4}{\lambda m + 4}, \\ X_{r+1} &= X_r + 1 && \text{with probability } \frac{\lambda m}{\lambda m + 4}. \end{aligned}$$

Then θ^{X_r} is a martingale, where

$$\theta = \frac{4}{\lambda m}.$$

Let q be the probability that X_r goes below $m/4$ before hitting $3m/4$. It follows from the optional stopping theorem that

$$q\theta^{m/4} + (1 - q)\theta^{3m/4} \leq \theta^{m/2}.$$

Therefore

$$(4) \quad q \leq \theta^{m/4} = (4/\lambda m)^{m/4} \leq T_m^{-3}/(2m^2),$$

since $\lambda m \geq 640$.

Hence, the random walk (X_r) (and thus the continuous time random walk (U_t)) makes at least $\lceil m^2 T_m \rceil$ upcrossings between $m/2$ and $3m/4$ before hitting $m/4$ with probability larger than

$$(5) \quad 1 - \lceil m^2 T_m \rceil T_m^{-3}/(2m^2) \geq 1 - T_m^{-1}/2.$$

The law of the waiting time between two upcrossings of (U_t) stochastically dominates $\mathcal{E}(L)$, with $L = 3\lambda m^2/16 + 3m/4$, the waiting time between two stages of (U_t) .

Suppose that (U_t) makes more than $\lceil m^2 T_m \rceil$ consecutive upcrossings. Then the time that (U_t) stays above $m/4$ stochastically dominates S , the sum of $\lceil m^2 T_m \rceil$ i.i.d. exponential random variables with mean $1/L$. By applying Chebyshev's inequality, we get

$$(6) \quad \mathbb{P}(S < \lceil m^2 T_m \rceil / 2L) \leq 4/(\lceil m^2 T_m \rceil) \leq T_m^{-1}/2.$$

Since $L \leq m^2/2$ and $(|\xi_t|)$ stochastically dominates (U_t) , we deduce (2) from (5) and (6).

We now prove (3). Assume that $|\xi_0| \geq m/4$. Then during the time that $|\xi_t|$ is between $m/8$ and $m/2$, it stochastically dominates a continuous time random walk Y_t satisfying

$Y_0 = m/4$ and

$$\begin{aligned} Y_t &\rightarrow Y_t + 1 \text{ at rate } e_1 = 7\lambda m^2/64, \\ Y_t &\rightarrow Y_t - 1 \text{ at rate } e_2 = m/2. \end{aligned}$$

Let us define

$$\sigma_Y = \inf\{t : Y_t \geq m/2\} \quad \text{and} \quad \tilde{\sigma}_Y = \inf\{t : Y_t \leq m/8\}.$$

Then similarly to (4), we have

$$(7) \quad \mathbb{P}(\tilde{\sigma}_Y < \sigma_Y) \leq (32/(7\lambda m))^{m/8} \leq T_m^{-1}/3.$$

Since $Y_t - (e_1 - e_2)t$ is a martingale, it follows from the stopping time theorem that

$$m/4 = \mathbb{E}(Y_{\sigma_Y \wedge t}) - (e_1 - e_2)\mathbb{E}(\sigma_Y \wedge t) \leq m/2 - (e_1 - e_2)\mathbb{E}(\sigma_Y \wedge t).$$

Therefore using $m\lambda \geq 640$, we get

$$\mathbb{E}(\sigma_Y \wedge t) \leq \frac{m}{4(e_1 - e_2)} \leq 1/6.$$

Letting t go to infinity, we obtain

$$\mathbb{E}(\sigma_Y) \leq 1/6.$$

Thus using Markov inequality, we have

$$(8) \quad \mathbb{P}(\sigma_Y > T_m/2) \leq 2\mathbb{E}(\sigma_Y)/T_m \leq T_m^{-1}/3.$$

It follows from (7) and (8) that

$$\mathbb{P}(\sigma_Y \leq \min\{\tilde{\sigma}_Y, T_m/2\}) \geq 1 - T_m^{-1},$$

which proves (3) as $(|\xi_t|)$ stochastically dominates (Y_t) .

For (ii), let v and w be two vertices in K_m^1 and K_m^2 respectively, such that there is an edge between u and v . Let (ξ'_t) (resp. (ξ''_t)) be the contact process on K_m^1 (resp. K_m^2). By (i), we have

$$(9) \quad \mathbb{P}(\xi'_{T_m} \neq \emptyset \mid |\xi'_0| \geq m/4) \geq 1 - 2T_m^{-1}.$$

We now claim that

$$(10) \quad \mathbb{P}(\exists t \leq m/2 : |\xi''_t| \geq m/4 \mid \xi'_{m/2} \neq \emptyset) \geq e^{-m/4}/m.$$

To prove (10), it amounts to show that

$$(11) \quad \mathbb{P}(\exists t \leq m/4 : |\xi''_t| \geq m/4 \mid |\xi''_0| = 1) \geq e^{-m/4},$$

and

$$(12) \quad \mathbb{P}(w \text{ get infected before } m/4 \mid \xi'_{m/4} \neq \emptyset) \geq 1/m.$$

For (11), observe that when $|\xi''_t| \leq m/4$, it increases by 1 in the next stage with probability

$$\frac{\lambda|\xi''_t|(m - |\xi''_t|)}{\lambda|\xi''_t|(m - |\xi''_t|) + |\xi''_t|} \geq \frac{3\lambda m}{4\lambda m + 4} > 0.8,$$

as $\lambda m \geq 640$. Moreover, the waiting time to the next stage is an exponential random variable with mean less than 1. Therefore, the probability that in all the $[m/4] + 1$ first stages, $|\xi''_t|$ increases and the waiting time is less than 1, is larger than

$$(0.8(1 - e^{-1}))^{[m/4]+1} \geq e^{-m/4},$$

which implies (11).

For (12), we note that

$$\{\xi'_{m/4} \neq \emptyset\} \subset \bigcap_{i=1}^{\lfloor m/8 \rfloor} \mathcal{E}_i,$$

where

$$\mathcal{E}_i = \{\exists v_i \in K_m^1 : \xi'_{2i}(v_i) = 1\}.$$

We define

$\mathcal{I}_i = \{\text{there is no recovery at } v_i \text{ in } [2i, 2i + 1] \text{ and there is an infection spread from } v_i \text{ to } v \text{ in } [2i, 2i + 1], \text{ there is no recovery at } v \text{ in } [2i, 2i + 2] \text{ and there is an infection spread from } v \text{ to } w \text{ in } [2i + 1, 2i + 2]\}.$

If $v_i \equiv v$, we only consider the recovery in v and the infection spread from v to w . We see that if one of (\mathcal{I}_i) occurs then w gets infected before $m/4$ and for any $i = 0, \dots, \lfloor m/8 \rfloor$

$$(13) \quad \mathbb{P}(\mathcal{I}_i \mid \bigcap_{j=1}^i \mathcal{E}_j) \geq e^{-3}(1 - e^{-\lambda})^2 \geq \lambda^2/(4e^3),$$

as $\lambda \leq 1$. Therefore, by using induction we have

$$\begin{aligned} & \mathbb{P}(w \text{ is not infected before } m/4 \mid \xi'_{m/4} \neq \emptyset) \\ & \leq \mathbb{P}\left(\left(\bigcup_{i=1}^{\lfloor m/8 \rfloor} \mathcal{I}_i\right)^c \cap \left(\bigcap_{i=1}^{\lfloor m/8 \rfloor} \mathcal{E}_i\right)\right) \\ & \leq (1 - \lambda^2/(4e^3))^{\lfloor m/8 \rfloor} \\ (14) \quad & \leq 1/m, \end{aligned}$$

since $\lambda m \geq 640$. Thus (12) follows.

We now prove (ii) by using (10). Suppose that $\xi'_{T_m} \neq \emptyset$. We divide the time interval $[0, T_m/2]$ into $\lfloor T_m/m \rfloor$ small intervals of length $m/2$. Then (10) implies that in each interval with probability larger than $e^{-m/4}/m$, there is a time s , such that that $|\xi''_s| \geq m/4$. Hence, similarly to (14) we have

$$(15) \quad \mathbb{P}(\exists s \leq T_m/2 : |\xi''_s| \geq m/4 \mid \xi'_{T_m/2} \neq \emptyset) \geq 1 - (1 - e^{-m/4}/m)^{\lfloor T_m/m \rfloor} \geq 1 - T_m^{-1}.$$

Suppose that $|\xi''_s| \geq m/4$ with $s \leq T_m/2$. Then (i) implies that $|\xi'_{T_m}| \geq m/4$ with probability larger than $1 - 2T_m^{-1}$. Combining this with (9) and (15), we get (ii). \square

2.2. Oriented percolation on finite sets. For any positive integer ℓ , we consider an oriented percolation process on $[0, \ell]$ with parameter q defined as follows. Let

$$\Gamma = \{(i, k) \in [0, \ell] \times \mathbb{N} : i + k \text{ is even}\}.$$

We then independently draw an arrow from (i, k) to $(j, k + 1)$ with probability q , where $j = i - 1$ or $j = i + 1$. Given the initial configuration $A \subset [0, \ell]$, the oriented percolation $(\eta_t)_{t \geq 0}$ is defined by

$$\eta_t^A = \{i \in [0, \ell] : \exists j \in A \text{ s.t. } (j, 0) \leftrightarrow (i, t)\} \text{ for } t \in \mathbb{N},$$

where the notation $(j, 0) \leftrightarrow (i, t)$ means that there is an oriented path from $(j, 0)$ to (i, t) . If $A = \{x\}$, we simply write (η_t^x) . We call (η_t) a Bernoulli oriented percolation with parameter q .

The oriented percolation on \mathbb{Z} , denoted by $(\bar{\eta}_t)$, was investigated by Durrett in [6]. Using his results and techniques, we will prove the following.

Lemma 2.2. *Let (η_t) be the oriented percolation on $[0, \ell]$ with parameter q . Then there exist positive constants ε and c independent of q and ℓ , such that if $q \geq 1 - \varepsilon$ then the following statements hold.*

(i) For any ℓ ,

$$\mathbb{P}(\sigma_\ell \leq 2\ell) \geq c,$$

where

$$\sigma_\ell = \inf\{t : \eta_t^0(\ell) = 1\}.$$

(ii) For any ℓ ,

$$\mathbb{P}(\eta_{t_\ell}^1 \neq \emptyset) \geq 1 - 1/t_\ell,$$

where $t_\ell = \lceil (1-q)^{-c\ell} \rceil$ and (η_t^1) is the oriented percolation starting with $\eta_0^1 = [0, \ell]$.

(iii) There exist a positive constant $\beta \in (0, 1)$ and an integer $s_\ell \in [\exp(c\ell), 2\exp(c\ell)]$, such that

$$\mathbb{P}(|\eta_{s_\ell}^1 \cap [(1-\beta)\ell/2, (1+\beta)\ell/2]| \geq 3\beta\ell/4) \geq 1 - \exp(-c\ell).$$

Proof. Part (i) is Theorem B.24 (a) in [13] and (ii) can be proved using a contour argument as in Section 10 in [6].

We now prove (iii). Let $(\bar{\eta}_t)$ be the oriented percolation on \mathbb{Z} . Then

$$\alpha = \mathbb{P}(\bar{\eta}_t^0 \neq \emptyset \forall t) \rightarrow 1 \quad \text{as} \quad q \rightarrow 1.$$

Hence we can assume that $\alpha > 3/4$. Now we define

$$\ell_1 = \lfloor (2-\alpha)\ell/4 \rfloor, \quad \ell_2 = \lfloor (2+\alpha)\ell/4 \rfloor \quad \text{and} \quad \ell_3 = \lfloor \ell/4 \rfloor.$$

Then for all $t \leq \ell_3$,

$$\bar{\eta}_t^{[\ell_1, \ell_2]} \subset [\ell_1 - t, \ell_2 + t] \subset [\ell_1 - \ell_3, \ell_2 + \ell_3] \subset [0, \ell].$$

Therefore,

$$\left(\bar{\eta}_t^{[\ell_1, \ell_2]} \right)_{0 \leq t \leq \ell_3} \equiv \left(\eta_t^{[\ell_1, \ell_2]} \right)_{0 \leq t \leq \ell_3}.$$

Hence, to simplify notation, we use (η_t) for the both processes in the interval $[0, \ell_3]$. To prove (iii), our goal is to show that there exists a positive constant c , such that for any $A \subset [\ell_1, \ell_2]$ and $|A| \geq 3\alpha\ell/8$,

$$(16) \quad \mathbb{P}(|\eta_{\ell_3}^A| \geq 3\alpha\ell/8) \geq 1 - \exp(-c\ell).$$

Then repeatedly applying (16) implies (iii) with $\beta = \alpha/2$. To prove (16), it suffices to show that there exists a positive constant c , such that

$$(17) \quad \mathbb{P}(|\eta_{\ell_3}^x| \geq 3\alpha\ell/8 \mid \eta_{\ell_3}^x \neq \emptyset) \geq 1 - \exp(-c\ell) \quad \text{for all } x \in [\ell_1, \ell_2],$$

$$(18) \quad \mathbb{P}(\eta_{\ell_3}^A \neq \emptyset) \geq 1 - \exp(-c\ell) \quad \text{for all } A \subset [\ell_1, \ell_2] \text{ with } |A| \geq 3\alpha\ell/8.$$

To prove (17), we define for any $A \subset \mathbb{Z}$ and $t \geq 0$

$$\begin{aligned} r_t^A &:= \sup\{x : \exists y \in A, (y, 0) \leftrightarrow (x, t)\} \\ l_t^A &:= \inf\{x : \exists y \in A, (y, 0) \leftrightarrow (x, t)\}. \end{aligned}$$

Then (17) is a consequence of the following.

- it is not hard to see that if $[\ell_1, \ell_2] \subset [\ell_3^x, r_{\ell_3}^x]$, then

$$\eta_{\ell_3}^1 \cap [\ell_1, \ell_2] \equiv \eta_{\ell_3}^x \cap [\ell_1, \ell_2],$$

- $\mathbb{P}(|\eta_{\ell_3}^1 \cap [\ell_1, \ell_2]| \geq 3(\ell_2 - \ell_1)/4) \geq 1 - \exp(-c\ell)$, as $3/4 < \alpha$.
- $\mathbb{P}([\ell_3^x, r_{\ell_3}^x] \supset [\ell_1, \ell_2] \mid \eta_{\ell_3}^x \neq \emptyset) \geq 1 - \exp(-c\ell)$.

The second claim is a consequence of Theorem 1 in [7]. (Note that in [7] the result is proved for the contact process, but as mentioned by the author the proof works as well for oriented percolation). In fact, it still holds if we replace $3/4$ by any $\alpha' < \alpha$. To prove the third one, we observe that if $\eta_{\ell_3}^x \neq \emptyset$ then

$$r_{\ell_3}^x = r_{\ell_3}^{(-\infty, x]}.$$

Moreover, by the main result of Section 11 in [6], there is a positive constant c , such that for all integer x ,

$$\mathbb{P}\left(r_{\ell_3}^{(-\infty, x]} \leq x + \alpha\ell_3/2\right) \leq \exp(-c\ell).$$

Therefore if $x \in [\ell_1, \ell_2]$, then

$$\mathbb{P}\left(r_{\ell_3}^x \geq \ell_2 \mid \eta_{\ell_3}^x \neq \emptyset\right) \geq 1 - \exp(-c\ell),$$

since $x + \alpha\ell_3/2 \geq \ell_1 + \alpha\ell_3/2 \geq \ell_2$. Similarly

$$\mathbb{P}\left(\ell_3^x \leq \ell_1 \mid \eta_{\ell_3}^x \neq \emptyset\right) \geq 1 - \exp(-c\ell).$$

Then the third claim follows from the last two estimates.

To prove (18), we use the same argument as in Section 10 in [6]. We say that A is more spread out than B (and write $A \succ B$) if there is an increasing function φ from B into A such that $|\varphi(x) - \varphi(y)| \geq |x - y|$ for all $x, y \in B$. (Note that this implies $|A| \geq |B|$). In [6], Durrett proves that there is a coupling such that if $A \succ B$ then

$$\eta_t^A \succ \eta_t^B \text{ for all } t \geq 0,$$

and as a consequence $|\eta_t^A| \geq |\eta_t^B|$ for all t . Hence

$$\mathbb{P}(\eta_t^A = \emptyset) \leq \mathbb{P}(\eta_t^B = \emptyset).$$

On the other hand, by (ii)

$$\mathbb{P}(\eta_{\ell_3}^{[\ell_1, \ell_1 + \ell_4]} = \emptyset) \leq \exp(-c\ell),$$

with $\ell_4 = \lceil 3\alpha\ell/8 \rceil - 1$. We observe that $A \succ [\ell_1, \ell_1 + \ell_4]$ for any A with $|A| \geq 3\alpha\ell/8$. Thus (18) follows from the last two inequalities. \square

2.3. Long path in a site percolation. The Bernoulli site percolation on \mathbb{Z}^d with parameter p is defined as usual: designate each vertex in \mathbb{Z}^d to be open independently with probability p and closed otherwise. A path in \mathbb{Z}^d is called open if all its sites are open. Then there is a critical value $p_c^s(d) \in (0, 1)$, such that if $p > p_c^s(d)$, then a.s. there exists an infinite open path (cluster), whereas if $p < p_c^s(d)$, a.s. there is no infinite cluster.

Lemma 2.3. *Consider the Bernoulli site percolation on $[0, n]^d$ with $d \geq 2$ and $p > p_c^s(2)$. Then there exists a positive constant $\rho = \rho(p, d)$, such that w.h.p. there is an open path whose length is larger than ρn^d .*

Proof. We set $m = \lceil n^{1/4} \rceil$. For $n, d \geq 2$, we say that the box $[0, n]^d$ is ρ -**good** if the site percolation cluster on it satisfies:

there exist two vertices x in $\{m\} \times [0, n]^{d-1}$ and y in $\{n-m\} \times [0, n]^{d-1}$ and an open path composed of three parts: the first one included in $[0, m] \times [0, n]^{d-1}$ has length larger than m and ends at x ; the second one included in $[m, n-m] \times [0, n]^{d-1}$ has length larger than ρn^d , starts at x and ends at y ; the third one included in $[n-m, n] \times [0, n]^{d-1}$ starts at y and has length larger than m .

We now prove by induction on d that if $p > p_c^s(2)$, there is a positive constant $\rho_d = \rho(p, d)$, such that w.h.p. the box $[0, n]^d$ is ρ_d -good. Then Lemma 2.3 immediately follows.

When $d = 2$, the statement follows from the main result in [11]. We will prove it for $d = 3$, the proof for $d \geq 4$ is exactly the same and will not be reproduced here.

For $1 \leq i \leq n$, let $\Lambda_i = \{i\} \times [2m, n-2m]^2$. We define

$$n_1 = n - 4m \quad \text{and} \quad m_1 = \lceil n_1^{1/4} \rceil.$$

We say that the i^{th} plane is **nice** (or Λ_i is nice) if the site percolation on this plane satisfies: Λ_i is ρ_2 -good (we consider Λ_i as a box in \mathbb{Z}^2); in each of the rectangles $\{i\} \times [m, 2m+m_1] \times [0, n]$ and $\{i\} \times [n-2m-m_1, n-m] \times [0, n]$, there is a unique connected component of size larger than m_1 , see Figure 1 for a sample of a nice plane.

The result for $d = 2$ implies that w.h.p. Λ_i is ρ_2 -good. On the other hand, we know that w.h.p. in the percolation on a box of size n there is a unique open cluster having diameter larger than $C \log n$ for some C large enough (see for example Theorem 7.61 in [10]). Thus w.h.p. there is a unique open cluster of size larger $(C \log n)^d$. Hence Λ_i is nice w.h.p. for all $i = 1, \dots, n$. Moreover, the events $\{\Lambda_i \text{ is nice}\}$ are independent since the planes are disjoint. Therefore \mathcal{A}_n holds w.h.p. with

$$\mathcal{A}_n = \{\#\{i : \Lambda_i \text{ is nice}\} \geq n/2\}.$$

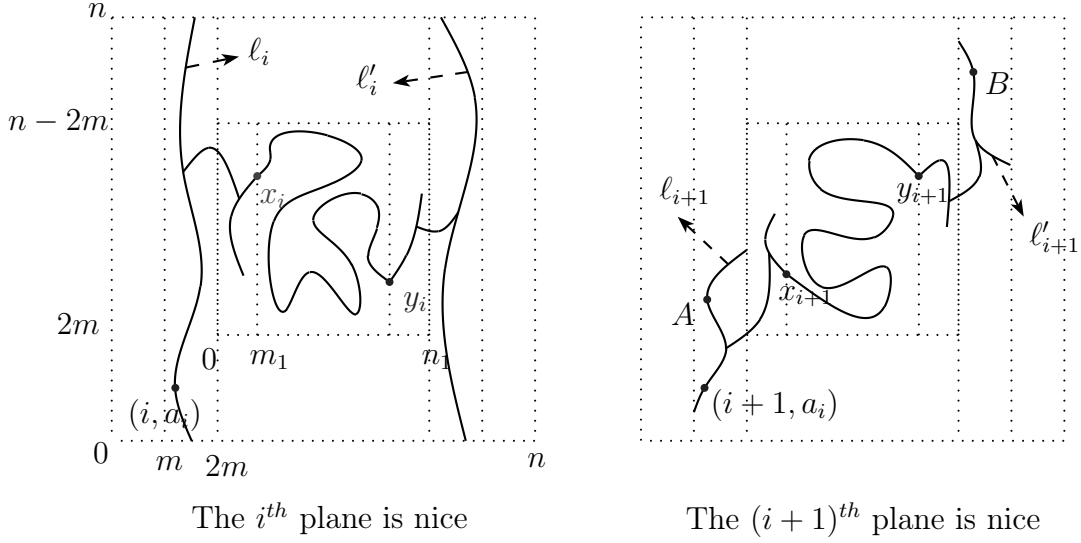
On \mathcal{A}_n , there are more than $n/2$ disjoint open paths (they are in disjoint planes), each of which has length larger than $\rho_2 n_1^2$. Thus, to obtain an open path of length of order n^3 , we will glue these long paths using shorter paths in good boxes of nice planes. To do that, we define

$\mathcal{B}_n = \{ \text{for all } 1 \leq i \leq \lfloor n/2 \rfloor, \text{ there exist open paths: } \ell_{2i-1} \subset \{2i-1\} \times [m, 2m] \times [0, n] \text{ whose end vertices are } u \text{ and } v \text{ with } u_3 = 0, v_3 = n; \ell'_{2i-1} \subset \{2i-1\} \times [n-2m, n-m] \times [0, n] \text{ whose end vertices are } u' \text{ and } v' \text{ with } u'_3 = 0, v'_3 = n; \ell_{2i} \subset \{2i\} \times [m, 2m] \times [0, n] \text{ whose end vertices are } z \text{ and } t \text{ with } z_2 = m, t_2 = 2m; \ell'_{2i} \subset \{2i\} \times [n-2m, n-m] \times [0, n] \text{ whose end vertices are } z' \text{ and } t' \text{ with } z'_2 = n-2m, t'_2 = n-m \}.$

We observe that ℓ_{2i-1} is a bottom-top crossing and ℓ_{2i} is a left-right crossing in two consecutive rectangles. Then they intersect when we consider only the last two coordinates, and the same holds for ℓ'_{2i-1} and ℓ'_{2i} . Hence on \mathcal{B}_n , for all $1 \leq i \leq n-2$, there are $a_i \in [m, 2m] \times [0, n]$, and $b_i \in [n-2m, n-m] \times [0, n]$ such that

$$\begin{aligned} (i, a_i) &\in \ell_i \text{ and } (i+1, a_i) \in \ell_{i+1}, \\ (i, b_i) &\in \ell'_i \text{ and } (i+1, b_i) \in \ell'_{i+1}. \end{aligned}$$

In other word, we can jump from the i^{th} plane to the next one in two ways. Moreover, on \mathcal{A}_n for all i such that the i^{th} plane is nice, the first part of the long open path in Λ_i is connected to ℓ_i (as these paths are in the same rectangle $\{i\} \times [m, 2m+m_1] \times [0, n]$ and have length larger than m_1), and similarly the third part is connected to ℓ'_i , see Figure 1.



Legend: $A = (i+1, a_{i+1}), B = (i+1, b_{i+1})$.

FIGURE 1. Gluing two long paths.

On $\mathcal{A}_n \cap \mathcal{B}_n$, we can find in $[m, n-m] \times [0, n]^2$ a path of length larger than $\rho_2 n^3/3$. Indeed, we start at the first nice plane, say the i^{th} plane, at an end point, from the right for example, of the long path, then go along this long path towards the other end point. Then we can go to l_i and arrive at (i, a_i) . Now we jump to $(i+1, a_i)$ (recall that it is a neighbor of (i, a_i)). If the $(i+1)^{\text{th}}$ plane is not nice, we go to $(i+1, a_{i+1})$ to jump to the next plane (note that both $(i+1, a_i)$ and $(i+1, a_{i+1})$ are in l_{i+1}). If the $(i+1)^{\text{th}}$ plane is nice, we now can touch and then go along to the long path in this plane and arrive at $(i+1, b_{i+1})$ to jump to the next plane. By continuing this procedure, we can go through all the long paths of nice planes. The resulting path is in $[m, n-m] \times [0, n]^2$.

Moreover, in the slabs $[0, m] \times [0, n]^2$ and $[n-m, n] \times [0, n]^2$, w.h.p. we can find two paths of length larger than m which are connected to the long path we have just found above. These paths form the required three-parts long path. Therefore on $\mathcal{A}_n \cap \mathcal{B}_n$, w.h.p. the box $[0, n]^3$ is ρ_3 -good with $\rho_3 = \rho_2/3$.

Now it remains to show that \mathcal{B}_n holds w.h.p. We observe that the probability of the existence of such a path l_i is larger than $1 - \exp(-cm)$ for some $c > 0$ (see for instance (7.70) in [10]). Thus \mathcal{B}_n holds w.h.p.

We summarize here the change of proving the induction from $d-1$ to d when $d \geq 4$. First, in the definition of a *nice* box, we consider

$$\Lambda_i = \{i\} \times [2m, n-2m]^{d-1},$$

and the uniqueness of the connected component of size larger than m_1 in the slabs $\{i\} \times [m, 2m+m_1] \times [0, n]^{d-2}$ and $\{i\} \times [n-2m-m_1, n-m] \times [0, n]^{d-2}$. Secondly, in the definition of \mathcal{B}_n , we consider $l_i \subset \{i\} \times \{m\}^{d-3} \times [0, n]^2$ and $l'_i \subset \{i\} \times \{n-m\}^{d-3} \times [0, n]^2$, two bottom-top (resp. left-right) crossings in the last two coordinates when i is odd (resp. even).

□

3. PROOF OF THEOREM 1.1

3.1. Proof of (i). For $\ell, m \in \mathbb{N}$, we denote by $\mathcal{C}(\ell, m)$ the graph obtained by attaching a complete graph of size m to each vertex in $[0, \ell]$.

Lemma 3.1. *Suppose that $d \geq 2$ and g satisfies (1). Then there exist positive constants c_1 and C_1 , such that if $R^d \geq C_1$ then w.h.p. $G(n, R, g)$ contains as a subgraph a copy of $\mathcal{C}([c_1 n R^{-d}], [c_1 R^d])$.*

Proof. If n/R^d is bounded from above, then w.h.p. $G(n, R, g)$ contains a clique of size of order n and thus the result follows. Indeed, by definition the vertices in $A = [0, R/\sqrt{d}]^d$ form a complete graph. Moreover the number of vertices in A is a Poisson random variable with mean $\int_A g(x)dx \asymp R^d \asymp n$, and hence w.h.p. it is of order n .

We now assume that n/R^d tends to infinity. Let $\ell = \lceil \sqrt[d]{n}/(R/2\sqrt{d}) \rceil$, we divide the box $[0, \sqrt[d]{n}]^d$ into ℓ^d smaller boxes of equal size, numerated by $(E_a)_{a \in [1, \ell]^d}$, whose side length is $R/(2\sqrt{d})$. We see that if v and w are in the same small box or in adjacent ones, then $\|v - w\| \leq R$, hence these two vertices are connected. This implies that the vertices on a small box form a clique and two adjacent cliques are connected.

For any $a \in [1, \ell]^d$, let us denote by

$$X_a = \#\{v : v \in E_a\}$$

the number of vertices located in E_a . Then (X_a) are independent and X_a is a Poisson random variable with mean

$$(19) \quad \mu_a = \int_{E_a} g(x)dx \geq b \left(\frac{R}{2\sqrt{d}} \right)^d =: \mu,$$

since $g(x) \geq b$ for all x . For any a , we define

$$Y_a = 1(\{X_a \geq \mu/2\}).$$

Since $\mathbb{P}(\text{Poi}(\mu) \geq \mu/2) \rightarrow 1$ as $\mu \rightarrow \infty$, it follows from (19) that $\mathbb{P}(Y_a = 1) \rightarrow 1$ as $R \rightarrow \infty$. Therefore there is a positive constant C_1 , such that if $R^d \geq C_1$, then

$$\mathbb{P}(Y_a = 1) \geq p := (1 + p_c^s(2))/2.$$

We note that the Bernoulli random variables (Y_a) are independent. Hence if we say the small box E_a open when $Y_a = 1$ and closed otherwise, then we get a site percolation on $[1, \ell]^d$ which stochastically dominates the Bernoulli site percolation on $[1, \ell]^d$ with parameter $p > p_c^s(2)$. Then Lemma 2.3 gives that w.h.p. there is an open path of length $c\ell^d \asymp nR^{-d}$. On the other hand, in each open box, there is a clique of size $\mu/2 \asymp R^d$ and these cliques in adjacent open boxes are connected. Hence, the result follows by taking c_1 small enough. \square

Lemma 3.2. *Let $\tau_{\ell, M}$ be the extinction time of the contact process on $\mathcal{C}(\ell, M)$ starting from full occupancy. Then there exist positive constants c and K independent of λ , such that if $\lambda M \geq K$, then*

$$(20) \quad \mathbb{P}(\tau_{\ell, M} \geq \exp(c\ell M \log(\lambda M))) \rightarrow 1 \quad \text{as } \ell \rightarrow \infty.$$

Proof. Let (ξ_t) be the contact process on $\mathcal{C}(\ell, M)$ with parameter $\lambda > 0$. It is sufficient to consider the case $\lambda \leq 1$, since the contact process is monotone in λ . We assume also that $M\lambda \geq 640$.

For $i \in [0, \ell]$, we say that it is **lit** at time t (the term is taken from [3]) if the number of infected vertices in its attached complete graph at time t is larger than $M/4$.

Let $T = \exp(M \log(\lambda M)/16)$. For $r \geq 0$ and $i, j \in [0, \ell]$ s.t. $|i - j| = 1$ and $i + r$ is even, we define

$$\begin{aligned} Z_{i,j}^r = & 1(\{i \text{ is not lit at time } rT\}) \\ & + 1(\{i \text{ is lit at time } rT \text{ and } i \text{ makes } j \text{ lit at time } (r+1)T\}), \end{aligned}$$

where " i makes j lit at time $(r+1)T$ " means that

$$\begin{aligned} & |\{y \in C(j) : \exists x \in C(i) \cap \xi_{rT} \text{ s.t. } (x, rT) \leftrightarrow (y, (r+1)T) \text{ inside } C(i) \cup C(j) \cup \{i, j\}\}| \\ & \geq M/4, \end{aligned}$$

with $C(i)$ the complete graph attached at i . Then $(Z_{i,j}^r)$ naturally define an oriented percolation by identifying

$$\{Z_{i,j}^r = 1\} \Leftrightarrow \{(i, r) \leftrightarrow (j, r+1)\}.$$

It follows from Lemma 2.1 (ii) that

$$\mathbb{P}(Z_{i,j}^r = 1 \mid \mathcal{F}_{rT}) \geq 1 - 5T^{-1} \quad \forall r \geq 0 \text{ and } |i - j| = 1,$$

where \mathcal{F}_t denotes the sigma-field generated by the contact process up to time t .

Moreover if $x \neq i$ and $y \neq j$, then $Z_{x,y}^r$ is independent of $Z_{i,j}^r$. Hence by a result of Liggett, Schonmann and Stacey [14] (see also Theorem B26 in [13]) the distribution of the family $(Z_{i,j}^r)$ stochastically dominates the measure of a Bernoulli oriented percolation with parameter

$$q \geq 1 - T^{-\gamma},$$

with $\gamma \in (0, 1)$. Moreover, if λM is large enough, then $1 - T^{-\gamma} > 1 - \varepsilon$, with ε as in Lemma 2.2.

In summary, when λM is large enough, the distribution of $(Z_{i,j}^r)$ stochastically dominates the one of an oriented percolation on $[0, \ell]$ with density close to 1. On the other hand, it follows from Lemma 2.2 (ii) that w.h.p. the oriented percolation process survives up to the step

$$[(1 - q)^{-c\ell}] \geq [T^{c\gamma\ell}] \geq \exp(c\gamma\ell M \log(\lambda M)),$$

for some constant $c > 0$. Combining these, we get the result. \square

We now prove a metastability result for connected graphs containing a copy of $\mathcal{C}(\ell, M)$.

Lemma 3.3. *Let (G_n^0) be a sequence of connected graphs, such that $|G_n^0| \leq n$, for all n . Let τ_n denote the extinction time of the contact process on G_n^0 starting from full occupancy. Assume that G_n^0 contains as a subgraph a copy of $\mathcal{C}(\ell_n, M)$. Then there exists a positive constant K , such that if $M \geq K/(\lambda \wedge 1)$ and*

$$(21) \quad \frac{\ell_n}{d_n \vee \log n} \rightarrow \infty,$$

with d_n the diameter of G_n^0 , then

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n \rightarrow \infty]{(\mathcal{L})} \mathcal{E}(1).$$

Proof. According to Proposition 1.2 in [15], it suffices to show that there exists a sequence (a_n) , such that $a_n = o(\mathbb{E}(\tau_n))$ and

$$(22) \quad \sup_{v \in V_n} \mathbb{P}(\xi_{a_n}^v \neq \xi_{a_n}, \xi_{a_n}^v \neq \emptyset) = o(1),$$

where $(\xi_t)_{t \geq 0}$ denotes the process starting from full occupancy.

Set $\bar{\lambda} = \lambda \wedge 1$. Using Lemma 3.2, we get

$$(23) \quad \mathbb{E}(\tau_n) \geq \exp(c\ell_n M \log(\bar{\lambda}M)),$$

with c as in this lemma. By (21), there is a sequence (φ_n) tending to infinity, such that

$$(24) \quad \frac{\ell_n}{k_n} \rightarrow \infty,$$

with

$$k_n = [(\log n \vee d_n)\varphi_n].$$

Now define

$$b_n = s_{k_n} T \quad \text{and} \quad a_n = 2b_n + 1,$$

with s_{k_n} as in Lemma 2.2 (iii) and $T = \exp(M \log(\bar{\lambda}M)/16)$.

Then (23) and (24) show that $a_n = o(\mathbb{E}(\tau_n))$, so it remains to prove (22) for this choice of (a_n) . To this end it is convenient to introduce the dual contact process. Given some positive real t and A a subset of the vertex set V_n of G_n , the dual process $(\hat{\xi}_s^{A,t})_{s \leq t}$ is defined by

$$\hat{\xi}_s^{A,t} = \{v \in V_n : (v, t-s) \longleftrightarrow A \times \{t\}\},$$

for all $s \leq t$. It follows from the graphical construction that for any v ,

$$(25) \quad \begin{aligned} & \mathbb{P}(\xi_{a_n}^v \neq \xi_{a_n}, \xi_{a_n}^v \neq \emptyset) \\ &= \mathbb{P}(\exists w \in V_n : \xi_{a_n}^v(w) = 0, \xi_{a_n}^v \neq \emptyset, \hat{\xi}_{a_n}^{w, a_n} \neq \emptyset) \\ &\leq \sum_{w \in V_n} \mathbb{P}\left(\xi_{a_n}^v \neq \emptyset, \hat{\xi}_{a_n}^{w, a_n} \neq \emptyset, \text{ and } \hat{\xi}_{a_n-t}^{w, a_n} \cap \xi_t^v = \emptyset \text{ for all } t \leq a_n\right), \end{aligned}$$

So let us prove now that the last sum above tends to 0 when $n \rightarrow \infty$.

By the hypothesis, there are vertices x_0, \dots, x_{k_n} together with complete graphs of size M , $C(x_0), \dots, C(x_{k_n})$, which form a graph isomorphic to $\mathcal{C}(k_n, M)$. Now we slightly change the definition of a lit vertex, and say that x_i is lit if the number of its infected neighbors in $C(x_i)$ is larger than $M/4$ for $i = 0, \dots, k_n$.

We first claim that for any v

$$(26) \quad \mathbb{P}(\mathcal{A}(v)^c, \xi_{b_n}^v \neq \emptyset) = o(1/n),$$

with

$$\mathcal{A}(v) = \{\xi_{b_n}^v \neq \emptyset, |\{i \in [(1-\beta)k_n/2, (1+\beta)k_n/2] : x_i \text{ is lit at time } b_n\}| \geq 3\beta k_n/4\},$$

with β as in Lemma 2.2.

Suppose for a moment that (26) holds. Then we also have

$$(27) \quad \mathbb{P}(\hat{\mathcal{A}}(w)^c, \hat{\xi}_{b_n}^{w, 2b_n+1} \neq \emptyset) = o(1/n),$$

with

$$\begin{aligned} \hat{\mathcal{A}}(w) &= \{\hat{\xi}_{b_n}^{w, 2b_n+1} \neq \emptyset, \exists S \subset [(1-\beta)k_n/2, (1+\beta)k_n/2] \text{ with } |S| \geq 3\beta k_n/4 \text{ and} \\ &\quad W_i \subset C(x_i) \text{ with } |W_i| \geq M/4 \forall i \in S : (x, b_n+1) \leftrightarrow (w, 2b_n+1) \forall x \in \cup_{i \in S} W_i\}. \end{aligned}$$

Note that $\mathcal{A}(v)$ and $\hat{\mathcal{A}}(w)$ are independent for all v and w . Moreover, on $\mathcal{A}(v) \cap \hat{\mathcal{A}}(w)$, there are more than $\beta k_n/2$ vertices which are lit in both the original and the dual processes.

More precisely, there is a set $S \subset [(1 - \beta)k_n/2, (1 + \beta)k_n/2]$ with $|S| \geq \beta k_n/2$ and sets $U_i, W_i \subset C(x_i)$ with $|U_i|, |W_i| \geq M/4$ for all $i \in S$, such that

$$\begin{aligned} (v, 0) &\leftrightarrow (x, b_n) && \text{for all } x \in \cup_{i \in S} U_i \\ (y, b_n + 1) &\leftrightarrow (w, 2b_n + 1) && \text{for all } y \in \cup_{i \in S} W_i. \end{aligned}$$

It is not difficult to show that there is a positive constant c , such that for any nonempty sets $U_i, W_i \subset C(x_i)$,

$$\mathbb{P}(U_i \times \{b_n\} \xleftrightarrow{C(x_i)} W_i \times \{b_n + 1\}) \geq c,$$

where the notation

$$U_i \times \{b_n\} \xleftrightarrow{C(x_i)} W_i \times \{b_n + 1\}$$

means that there is an infection path inside $C(x_i)$ from a vertex in U_i at time b_n to a vertex in W_i at time $b_n + 1$.

Moreover, conditionally on the sets U_i, W_i , these events are independent. Therefore,

$$\mathbb{P}(\exists i : U_i \times \{b_n\} \xleftrightarrow{C(x_i)} W_i \times \{b_n + 1\} \mid U_i, W_i) \geq 1 - (1 - c)^{\beta k_n/2} = 1 - o(1/n),$$

by our choice of k_n . This implies that

$$(28) \quad \mathbb{P}(\mathcal{A}(v), \hat{\mathcal{A}}(w), \hat{\xi}_{a_n-t}^{w, a_n} \cap \xi_t^v = \emptyset \text{ for all } t \leq a_n) = o(1/n).$$

Combining (26), (27) and (28) we obtain (25). Hence, now it amounts to prove (26).

To see this, we define an oriented percolation $(\tilde{\eta}_r)_{r \geq 0}$ similarly as in Lemma 3.2. For $0 \leq i, j \leq k_n$ and $r \geq 0$, such that $|i - j| = 1$ and $i + r$ is even, we let $Z_{i,j}^r = 1$ (or equivalently $(i, r) \leftrightarrow (j, r + 1)$) if either x_i is not lit at time rT or x_i is lit at time rT and x_i makes x_j lit at time $(r + 1)T$.

As in Lemma 3.2, there exists a positive constant K , such that if $M\bar{\lambda} \geq K$, then $(\tilde{\eta}_r)$ stochastically dominates a Bernoulli oriented percolation with parameter $1 - \varepsilon$, with ε as in Lemma 2.2.

Assume that d_n is even, if not we just take the smallest even integer larger than d_n . Then we set

$$\tilde{d}_n = d_n + 2k_n.$$

Now define for $k \geq 0$,

$$\tilde{\sigma}_{k_n}(k) = \inf\{r \geq k\tilde{d}_n + d_n : \tilde{\eta}_r^{0, k\tilde{d}_n + d_n}(k_n) = 1\},$$

where for any $A \subset \{0, \dots, k_n\}$ and $t \geq s \geq 0$

$$\tilde{\eta}_t^{A, s} = \{x : \exists y \in A, (y, s) \leftrightarrow (x, t)\}.$$

Note that $k\tilde{d}_n + d_n + 2k_n = (k + 1)\tilde{d}_n$. Then using Lemma 2.2 (i), we get

$$(29) \quad \mathbb{P}(\tilde{\sigma}_{k_n}(k) \leq (k + 1)\tilde{d}_n \mid \mathcal{F}_{k\tilde{d}_n + d_n}) \geq c.$$

We observe that if $\tilde{\sigma}_{k_n}(k) \leq (k + 1)\tilde{d}_n$, then there is a horizontal crossing before $(k + 1)\tilde{d}_n$. Hence,

$$\tilde{\eta}_r^1 = \tilde{\eta}_r^{0, k\tilde{d}_n + d_n} \quad \text{for all } r \geq (k + 1)\tilde{d}_n.$$

Define

$$\mathcal{E} = \{|\tilde{\eta}_{s_{k_n}}^1 \cap [(1 - \beta)k_n/2, (1 + \beta)k_n/2]| \geq 3\beta k_n/4\}.$$

On $\{\tilde{\sigma}_{k_n}(k) \leq (k+1)\tilde{d}_n\} \cap \mathcal{E}$, if $(k+1)\tilde{d}_n \leq s_{k_n}$ then

$$(30) \quad |\tilde{\eta}_{s_{k_n}}^{0, k\tilde{d}_n + d_n} \cap [(1-\beta)k_n/2, (1+\beta)k_n/2]| \geq 3\beta k_n/4.$$

Let $K_n = \lfloor s_{k_n}/\tilde{d}_n \rfloor$ and for any $0 \leq k \leq K_n - 1$

$$A_k := \{\xi_{k\tilde{d}_n}^v \neq \emptyset\},$$

and

$$B_k := \left\{ \xi_{k\tilde{d}_n}^v \times \{k\tilde{d}_n\} \longleftrightarrow (x_0, (k\tilde{d}_n + d_n - 1)T) \right\} \cap \{x_0 \text{ is lit at time } (k\tilde{d}_n + d_n)T\} \\ \cap \{\tilde{\sigma}_{k_n}(k) \leq (k+1)\tilde{d}_n\}.$$

We have

$$(31) \quad \{\xi_{b_n}^v \neq \emptyset\} \subset \bigcap_{k=0}^{K_n-1} A_k.$$

On the other hand, if x_0 is lit at time rT and $\tilde{\eta}_s^{0,r}(x_i) = 1$ for $s > r$, then x_i is lit at time sT . Hence by (30) on \mathcal{E} , if one of the events $(A_k \cap B_k)$ happens then $\mathcal{A}(v)$ occurs. Combing this with (31), we get

$$(32) \quad \{\xi_{b_n}^v \neq \emptyset\} \cap \mathcal{A}(v)^c \subset \mathcal{E}^c \cup \bigcap_{k=0}^{K_n-1} A_k \cap B_k^c.$$

Using Lemma 2.2 (iii), we obtain a bound for the first term

$$(33) \quad \mathbb{P}(\mathcal{E}^c) \leq \exp(-ck_n) = o(1/n),$$

by the choice of k_n . For the second term, by using a similar argument as for (13), we have

$$\mathbb{P}((z, t) \longleftrightarrow (z', t + (d_n - 1)T)) \geq \exp(-C(d_n - 1)T) \quad \text{for any } z, z' \in V_n \text{ and } t \geq 0,$$

for some constant $C > 0$. On the other hand, if x_i is infected at time t then it is lit at time $t + T$ with probability larger than $\exp(-CT)$. Therefore combing with (29), we get that for any $k \leq K_n - 1$,

$$\mathbb{P}(B_k^c \mid \mathcal{G}_k) \mathbf{1}(A_k) \leq 1 - c \exp(-Cd_n T),$$

where $\mathcal{G}_k = \mathcal{F}_{k\tilde{d}_n}$. Iterating this, we get

$$(34) \quad \mathbb{P}\left(\bigcap_{k=0}^{K_n-1} A_k \cap B_k^c\right) \leq (1 - c \exp(-Cd_n T))^{K_n-1} = o(1/n),$$

where the last equality follows from the definition of s_{k_n} . Combining (32), (33) and (34) we get (26) and finish the proof. \square

Proof of Theorem 1.1 (i). We first prove the lower bound on τ_n . By Lemma 3.1, there are positive constants c and K , such that if $R^d \geq K/(\lambda \wedge 1)$, then w.h.p. $G(n, R, g)$ contains as a subgraph a copy of $\mathcal{C}(\ell_n, M)$, with $\ell_n = \lfloor cnR^{-d} \rfloor$ and $M = \lfloor cR^d \rfloor$.

If ℓ_n is bounded (or $R^d \asymp n$), then $\mathcal{C}(\ell_n, M)$ contains a complete graph of size of order n . Then Lemma 2.1 (i) implies that w.h.p. the extinction time is larger than $\exp(cn \log(\lambda n))$, for some $c > 0$.

If ℓ_n tends to infinity, then the result follows from Lemma 3.2.

To prove the convergence in law of $\tau_n/\mathbb{E}(\tau_n)$, we recall some known results about the diameter of the giant component and the size of small components in RGGs. There is a positive constant R_0 , such that if $R > R_0$, then w.h.p.

(a) the diameter of the largest component is $d_n = \mathcal{O}(n^{1/d}/R)$,

(b) the size of the second largest component is $\mathcal{O}((\log n)^{d/(d-1)})$.

The first claim is proved in [8] (Corollary 6) and the second one is proved in [20] (Theorem 10.18) when $g \equiv 1$. It is not hard to generalize these results for our model with g bounded both from below and above.

The second claim together with Lemma 3.4 below show that w.h.p. the extinction of the contact process on G_n and on its largest component are equal. We are now in a position to complete the proof of (i).

- If $R^d = o(n/\log n)$, then by (a)

$$\frac{\ell_n}{d_n \vee \log n} \rightarrow \infty.$$

Therefore, Lemmas 3.1 and 3.3 imply the convergence in law of $\tau_n/\mathbb{E}(\tau_n)$.

- If $n/\log n = \mathcal{O}(R^d)$, then

$$\frac{D_{n,\max}}{d_n \vee \log n} \rightarrow \infty,$$

since $R^d = \mathcal{O}(D_{n,\max})$, with $D_{n,\max}$ the maximum degree in the largest component. Therefore, the result is proved by using Proposition 6.2 in [5] (see the latest version on arXiv).

□

3.2. Proof of (ii). We prove an upper bound on the extinction time of the contact process on an arbitrary graph.

Lemma 3.4. *Let τ_G be the extinction time of the contact process on a graph $G = (V, E)$ starting from full occupancy. Then*

$$(a) \mathbb{P}(\tau_G \leq F(|V|, |E|)) \geq 1 - \exp(-|V|),$$

$$(b) \mathbb{E}(\tau_G) \leq 2F(|V|, |E|)$$

with

$$F(|V|, |E|) = |V| \left(2 + \frac{4\lambda|E|}{|V|} \right)^{|V|}.$$

Proof. Observe that (b) is a consequence of (a) and the following. For any $s > 0$

$$\mathbb{E}(\tau_G) \leq \frac{s}{\mathbb{P}(\tau_G \leq s)}.$$

This result is Lemma 4.5 in [16].

We now prove (a). Let us denote by (ξ_t) the contact process on G starting with full occupancy. By using Markov's property and the monotonicity of the contact process, it suffices to show that

$$(35) \quad \mathbb{P}(\xi_1 = \emptyset) \geq \exp(-|V| \log(2 + 4\lambda|E|/|V|)).$$

Observe that the process dies at time 1 if for any vertex v , it heals before 1 and does not infect any neighbor. Let σ_v be the time of the first recovery at v , then $\sigma_v \sim \mathcal{E}(1)$. Let $\sigma_{v \rightarrow}$ be the time of the first infection spread from v to one of its neighbors. Then

it is the minimum of $\deg(v)$ i.i.d. exponential random variables with mean λ and thus $\sigma_{v \rightarrow} \sim \mathcal{E}(\lambda \deg(v))$. Moreover σ_v and $\sigma_{v \rightarrow}$ are independent. Therefore

$$\mathbb{P}(\sigma_v < \min\{\sigma_{v \rightarrow}, 1\}) = \frac{1 - e^{-(1+\lambda \deg(v))}}{1 + \lambda \deg(v)} \geq \frac{1}{2(1 + \lambda \deg(v))}.$$

On the other hand, these events $\{\sigma_v < \min\{\sigma_{v \rightarrow}, 1\}\}_v$ are independent. Then using Cauchy's inequality, we get that

$$\begin{aligned} \mathbb{P}(\xi_1 = \emptyset) &\geq \prod_{v \in V} (2 + 2\lambda \deg(v))^{-1} \\ &\geq \left(\frac{2|V| + 2\lambda \sum_{v \in V} \deg(v)}{|V|} \right)^{-|V|} \\ &= \left(2 + \frac{4\lambda|E|}{|V|} \right)^{-|V|}, \end{aligned}$$

which implies (35). □

Proof of Theorem 1.1 (ii). The upper bound on τ_n follows from Lemma 3.4 and the following: w.h.p. $G(n, R, g) = (V_n, E_n)$ with

- $|V_n| \leq 2Bn$
- $|E_n| \leq CnR^d$, for some $C = C(d, B)$.

The first claim is clear, since $|V_n|$ is a Poisson random variable with mean

$$\int_{[0, \sqrt[d]{n}]^d} g(x) dx \leq Bn.$$

For the second one, let $\ell = 1 + \lceil \sqrt[d]{n}/R \rceil$. We cover $[0, \sqrt[d]{n}]^d$ using translations by $R/2$ for each coordinate, accounting for $(2\ell - 1)^d$ boxes of volume R^d . We observe that points at distance larger than R are not connected. Hence, $|E_n|$ is less than the sum of the number of edges in the covering small boxes.

These small boxes are partitioned into 2^d groups such that each group contains at most ℓ^d disjoint boxes with the same volume R^d .

The number of vertices in each box is stochastically dominated by Z , a Poisson random variable with mean BR^d , since the integral of g on a box is smaller than BR^d (as $g(x) \leq B$ for all x). Hence, the number of edges in a box is stochastically dominated by Z^2 .

Moreover, in each group the numbers of edges are independent, as the boxes are disjoint. Therefore, using Chebyshev's inequality, the total number of edges in a group is w.h.p. smaller than

$$2\ell^d \mathbb{E}(Z^2) = 2\ell^d (BR^d)(BR^d + 1).$$

Hence, $|E_n|$ is w.h.p. less than

$$2^{d+1} \ell^d (BR^d)(BR^d + 1) \leq CnR^d,$$

for some $C = C(d, B)$ large enough. □

4. SOME EXTENSIONS

4.1. The one-dimensional case. When $d = 1$, RGGs are also called random interval graphs, see for instance [22]. We have the following result.

Proposition 4.1. *Let $d = 1$. Consider the contact process on one-dimensional random geometric graphs $G(n, R, g)$ with g satisfying (1). Then there exist positive constants \varkappa, K, c and C depending only on b and B , such that the following statements hold.*

(i) *If $R \leq \varkappa \log n$, then w.h.p. the number of vertices in the largest component is $o(n^{2/3})$. Thus $\log \tau_n = o(n)$ w.h.p.*

(ii) *If $R \geq K \log n$, then w.h.p. the graph is connected and*

$$cn \log(\lambda R) \leq \log(\tau_n) \leq Cn \log(\lambda R),$$

and

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n \rightarrow \infty]{(\mathcal{L})} \mathcal{E}(1).$$

Proof. For (i), it is sufficient to consider $R = \varkappa \log n$ with \varkappa chosen later. We divide $[0, n]$ into $[n/R]$ intervals of length R , denoted by $I_1, \dots, I_{[n/R]}$. Then the number of vertices in I_i is a Poisson random variable with mean $\int_{I_i} g(x) dx \asymp R$. Therefore

$$(36) \quad \mathbb{P}(\#\{\text{vertices in } I_i\} \leq R^2 \text{ for all } i = 1, \dots, [n/R]) = 1 - o(1),$$

as $R \asymp \log n$. On the other hand, since $\int_{I_i} g(x) dx \leq BR$ for all i , the probability for an interval to be empty is larger than e^{-BR} . Hence

$$(37) \quad \begin{aligned} \mathbb{P}(\text{there are at most } \lceil \sqrt{n} \rceil \text{ consecutive non-empty intervals}) &\geq 1 - [n/R](1 - e^{-BR})^{\lceil \sqrt{n} \rceil} \\ &= 1 - o(1), \end{aligned}$$

with $R = \varkappa \log n$ and \varkappa small enough.

We observe that if an interval is empty, then there is no edges between vertices in the left-hand side and the right-hand side of this interval. Thus, it follows from (36) and (37) that w.h.p. the number of vertices in any component is smaller than $R^2 \lceil \sqrt{n} \rceil = o(n^{2/3})$.

We now prove (ii). If $R \asymp n$, then the graph contains a complete graph of size of order n . Thus using Lemma 2.1 (i), we get the lower bound on τ_n . Assume that $R = o(n)$. We divide $[0, n]$ into $[2n/R]$ intervals of length $R/2$, denoted by $J_1, \dots, J_{[2n/R]}$. Then the numbers of vertices in these intervals form a sequence of independent Poisson random variables with mean larger than $bR/2$. For all $i \leq [2n/R]$, we define

$$\{J_i \text{ is good}\} = \{\text{the number of vertices in } J_i \geq bR/4\}.$$

We have

$$\mathbb{P}(\text{Poi}(bR/2) \geq bR/4) \geq 1 - \exp(-cR),$$

for some constant $c = c(b) > 0$. Therefore

$$(38) \quad \mathbb{P}(J_i \text{ is good for all } i \leq [2n/R]) \geq 1 - [2n/R]e^{-cR} = 1 - o(1),$$

with $R \geq K \log n$ and K large enough. This implies that w.h.p. $G(n, R, g)$ contains as a subgraph a copy of $\mathcal{C}([2n/R], [bR/4])$ (note that the vertices in the same interval or in adjacent ones are connected). Thus similarly to Theorem 1.1, we get the lower bound on τ_n . The upper bound also follows from the same argument as in Theorem 1.1.

For the connectivity, since all vertices in an interval J_i or in adjacent ones are connected, we observe that

$$\{J_i \text{ is good for all } i \leq [2n/R]\} \subset \{J_i \text{ is non-empty for all } i \leq [2n/R]\} \subset \{G_n \text{ is connected}\}.$$

Therefore by (38), when $R \geq K \log n$ with K large enough, w.h.p. G_n is connected. In addition, its diameter is $d_n \leq [2n/R] + 1$.

For the convergence in law, we note that $\mathcal{C}(\ell, M)$ always contains a copy of $\mathcal{C}(\ell[M/M_1], M_1)$ if $M \geq M_1$. Moreover, w.h.p. G_n contains a copy of $\mathcal{C}([2n/R], [bR/4])$. Therefore, w.h.p. G_n contains a copy of $\mathcal{C}([cn], M)$ for some $c > 0$ and M as in Lemma 3.3. Hence, G_n satisfies the hypothesis in Lemma 3.3 and the result follows. \square

4.2. An equivalent model. We consider another version of random geometric graphs with density function f and connecting radius r , denoted by $G'(n, r, f)$. It is defined as follows: place independently n points in $[0, 1]^d$ according to f , then connect two points u and v by an edge if $\|u - v\| \leq r$. Suppose that

$$(39) \quad 0 < b \leq f(x) \leq B < +\infty \quad \text{for all } x.$$

Similarly to the results for $G(n, R, g)$, we have the following.

Proposition 4.2. *The results of Theorem 1.1 and Proposition 4.1 hold for the graph $G'(n, r, f)$ with f satisfying (39) by replacing R^d by nr^d in the statements.*

Proof. First, we observe that the law of a Poisson point process with intensity g on a set A conditionally on its number vertices, say N , is the same as that of the process defined by placing independently N points in A with density $g/\int_A g$.

Therefore, the graph $G(n, R, g)$ conditionally on its size $|G|$ is isomorphic to $G'(|G|, r, f)$ with

$$r = \sqrt[d]{n}/R \quad \text{and} \quad f(x) = g(x\sqrt[d]{n}).$$

To prove the lower bound on τ_n , we consider $G_1 = G(n/(2B), R_1, g_1)$, where

$$R_1^d = nr^d/(2B) \quad \text{and} \quad g_1(x) = f\left(\frac{x}{\sqrt[d]{n}/(2B)}\right).$$

Since $|G_1|$ is a Poisson random variable with mean less than $n/2$, w.h.p. $|G_1|$ is less than n . Therefore w.h.p. G_1 can be coupled as a subgraph of $G'(n, r, f)$. This domination together with Theorem 1.1 and Proposition 4.1 imply the results for the lower bound on τ_n and the convergence in law of $\tau_n/\mathbb{E}(\tau_n)$ (note that the results of the connectivity and the diameter of the largest component or the size of the second largest component also hold in this model).

Similarly, for the upper bound of τ_n , we consider $G_2 = G(2n/b, R_2, g_2)$ with

$$R_2^d = (2n/b)r^d \quad \text{and} \quad g_2(x) = f\left(\frac{x}{\sqrt[d]{(2n/b)}}\right).$$

Then w.h.p. G_2 contains as a subgraph a copy of $G'(n, r, f)$. Thus by applying Theorem 1.1 and Proposition 4.1, we get desired results. \square

Acknowledgments. I am deeply grateful to my advisor Bruno Schapira for his help and many suggestions during the preparation of this work.

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