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The Ring of Support-Classes of $\text{SL}_2(\mathbb{F}_q)$

Roland Bacher*

June 4, 2015

Abstract: We introduce and study a subring $\mathcal{SC}$ of $\mathbb{Z}[\text{SL}_2(\mathbb{F}_q)]$ obtained by summing elements of $\text{SL}_2(\mathbb{F}_q)$ according to their support. The ring $\mathcal{SC}$ can be used for the construction of several association schemes.

1 Main results

Summing elements of the finite group $\text{SL}_2(\mathbb{F}_q)$ according to their support (locations of non-zero matrix coefficients), we get seven elements (six when working over $\mathbb{F}_2$) in the integral group-ring $\mathbb{Z}[\text{SL}_2(\mathbb{F}_q)]$.

Integral linear combinations of these seven elements form a subring $\mathcal{SC}$, called the ring of support classes, of the integral group-ring $\mathbb{Z}[\text{SL}_2(\mathbb{F}_q)]$. Supposing $q > 2$, we get thus a 7-dimensional algebra $\mathcal{SC}_K = \mathcal{SC} \otimes \mathbb{Z} K$ over a field $K$ when considering $K$-linear combinations.

This paper is devoted to the definition and the study of a few features of $\mathcal{SC}$.

More precisely, in Section 2 we prove that the ring of support-classes $\mathcal{SC}$ is indeed a ring by computing its structure-constants.

Section 3 describes the structure of $\mathcal{SC}_\mathbb{Q} = \mathcal{SC} \otimes \mathbb{Z} \mathbb{Q}$ as a semi-simple algebra independent of $q$ for $q > 2$.

In Section 4 we recall the definition of association schemes and use $\mathcal{SC}$ for the construction of hopefully interesting examples.

Finally, we study a few representation-theoretic aspects in Section 5.

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1
Given subsets $A, B, C, D$ of a finite field $\mathbb{F}_q$, we denote by
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \sum_{(a,b,c,d) \in A \times B \times C \times D, ad-bc=1} \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
the element of $\mathbb{Z}[\text{SL}_2(\mathbb{F}_q)]$ obtained by summing all matrices of $\text{SL}_2(\mathbb{F}_q)$ with coefficients $a \in A, b \in B, c \in C$ and $d \in D$.

Identifying 0 with the singleton subset $\{0\}$ of $\mathbb{F}_q$ and denoting by $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$ the set of all units in $\mathbb{F}_q$, we consider the seven elements
\[
A = \begin{pmatrix} \mathbb{F}_q^* & 0 \\ 0 & \mathbb{F}_q^* \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \mathbb{F}_q^* \\ \mathbb{F}_q^* & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \mathbb{F}_q^* & \mathbb{F}_q^* \\ \mathbb{F}_q^* & \mathbb{F}_q^* \end{pmatrix},
\]
\[
D_+ = \begin{pmatrix} \mathbb{F}_q^* & \mathbb{F}_q^* \\ 0 & \mathbb{F}_q^* \end{pmatrix}, \quad D_- = \begin{pmatrix} \mathbb{F}_q^* & 0 \\ \mathbb{F}_q^* & \mathbb{F}_q^* \end{pmatrix},
\]
\[
E_+ = \begin{pmatrix} \mathbb{F}_q^* & \mathbb{F}_q^* \\ \mathbb{F}_q^* & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & \mathbb{F}_q^* \\ \mathbb{F}_q^* & \mathbb{F}_q^* \end{pmatrix},
\]
corresponding to all possible supports of matrices in $\text{SL}_2[\mathbb{F}_q]$. The element $C$ is of course missing (and the remaining elements consist simply of all six matrices in $\text{SL}_2(\mathbb{F}_2)$) over $\mathbb{F}_2$. For the sake of concision, we will always assume that $q$ has more than 2 elements in the sequel (there is however nothing wrong with finite fields of characteristic 2 having at least 4 elements).

We denote by
\[
SC = \mathbb{Z}A + \mathbb{Z}B + \mathbb{Z}C + \mathbb{Z}D_+ + \mathbb{Z}D_- + \mathbb{Z}E_+ + \mathbb{Z}E_-
\]
the free $\mathbb{Z}$–module of rank seven spanned by these seven elements.

We call $SC$ the ring of support-classes of $\text{SL}_2(\mathbb{F}_q)$, a terminology motivated by our main result:

**Theorem 2.1** $SC$ is a subring of the integral group-ring $\mathbb{Z}[\text{SL}_2(\mathbb{F}_q)]$.

The construction of $SC$ can be carried over to the projective special groups $\text{PSL}_2(\mathbb{F}_q)$ without difficulties by dividing all structure-constants by 2 if $q$ is odd. The obvious modifications are left to the reader.

Another obvious variation is to work with matrices in $\text{GL}_2(\mathbb{F}_q)$. This multiplies all structure-constants by $(q-1)$ (respectively by $m$ if working with the subgroup of matrices in $\text{GL}_2(\mathbb{F}_q)$ having their determinants in a fixed multiplicative subgroup $M \subset \mathbb{F}_q^*$ with $m$ elements).

We hope to address a few other variations of our main construction in a future paper.
Products among generators of $\mathcal{SC}$ are given by

$$AX = XA = (q-1)X \text{ for } X \in \{A, B, C, D_\pm, E_\pm\},$$
$$B^2 = (q-1)A,$$
$$BC = CB = (q-1)C,$$
$$BD_+ = D_+B = (q-1)E_+,$$
$$BD_- = D_-B = (q-1)E_-,$$
$$BE_+ = E_+B = (q-1)D_-,$$
$$BE_- = E_-B = (q-1)D_+,$$
$$C^2 = (q-1)^2(q-2)(A+B) + (q-1)(q-3)(q-4)C + (q-1)(q-2)(D_++D_-+E_++E_-),$$
$$CD_+ = CE_- = (q-1)(q-3)C + (q-1)(q-2)(D_-+E_-),$$
$$CD_- = CE_+ = (q-1)(q-3)C + (q-1)(q-2)(D_++E_+),$$
$$D_+C = E_+C = (q-1)(q-3)C + (q-1)(q-2)(D_-+E_-),$$
$$D_-C = E_-C = (q-1)(q-3)C + (q-1)(q-2)(D_++E_+),$$
$$D_+^2 = E_+E_- = (q-1)^2A + (q-1)(q-2)D_+,$$
$$D_+D_- = E_+^2 = (q-1)(C+E_-),$$
$$D_-D_+ = E_-^2 = (q-1)(C+E_+),$$
$$D_+E_+ = E_+D_- = (q-1)^2B + (q-1)(q-2)E_+,$$
$$E_+D_+ = D_+E_- = (q-1)(C+D_-),$$
$$E_-D_+ = D_-E_+ = (q-1)^2B + (q-1)(q-2)E_-,$$
$$D_-^2 = E_-E_+ = (q-1)^2A + (q-1)(q-2)D_-,$$
$$D_-E_- = E_-D_+ = (q-1)(C+D_+).$$

Easy consistency checks of these formulae are given by the antiautomorphisms $\sigma$ and $\tau$ obtained respectively by matrix-inversion and matrix-transposition. Their composition $\sigma \circ \tau = \tau \circ \sigma$ is of course an involutive automorphism of $\mathbb{Z}[\mathcal{F}_q]$ which restricts to an automorphism of $\mathcal{SC}$. It coincides on $\mathcal{SC}$ with the action of the inner automorphism $X \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\mathbb{Z}[\text{SL}_2(\mathbb{F}_q)]$, fixes $A, B, C$ and transposes the elements of the two pairs $\{D_+, D_-\}$ and $\{E_+, E_-\}$.

**Remark 2.2** The construction of the ring $\mathcal{SC}$ described by Theorem 2.1 does not generalise to the matrix-algebra of all $2 \times 2$ matrices over $\mathbb{F}_q$. Indeed, $\begin{pmatrix} \mathbb{F}_q^* & 0 \\ \mathbb{F}_q^* & 0 \end{pmatrix} \begin{pmatrix} \mathbb{F}_q^* & \mathbb{F}_q^* \\ \mathbb{F}_q & 0 \end{pmatrix}$ equals, up to a factor $(q-1)$, to the sum of all $(q-1)^3$ possible rank 1 matrices with all four coefficients in $\mathbb{F}_q^*$. 

3
Square rank one matrices of any size behave however rather well: The set of all \((2^n - 1)^2\) possible sums of rank 1 matrices of size \(n \times n\) with prescribed support is a \(\mathbb{Z}\)-basis of a ring (defined by extending bilinearly the matrix product) after identifying the zero matrix with 0.

### 2.1 Proof of Theorem 2.1

We show that the formulae for the products are correct. We are however not going to prove all \(7^2 = 49\) possible identities but all omitted cases are similar and can be derived by symmetry arguments, use of the antiautomorphisms given by matrix-inversion and transposition, or (left/right)-multiplication by \(B\).

Products with \(A\) or \(B\) are easy and left to the reader.

We start with the easy product \(D^2_+\) (the products

\[ D_+ E_+, E_+ D_-, E_- D_+, D_- E_-, D^2_+, E_- E_+ \]

are similar and left to the reader). Since \(A + D_+\) is the sum of elements over the full group of all \(q(q-1)\) unimodular upper-triangular matrices, we have \((A + D_+)^2 = q(q-1)(A + D_+)\) showing that

\[
D^2_+ = (A + D_+)^2 - 2AD_+ - A^2 \\
= q(q-1)(A + D_+) - 2(q-1)D_+ - (q-1)A \\
= (q-1)^2A + (q-1)(q-2)D_+.
\]

For \(D_+ D_-\) we consider

\[
\begin{pmatrix} a_1 & b_1 \\ 0 & 1/a_1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ b_2 & 1/a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1b_2 & b_1/a_2 \\ b_2/a_1 & 1/(a_1a_2) \end{pmatrix}.
\]

Every unimodular matrix \(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\) with \(\beta, \gamma, \delta\) in \(\mathbb{F}_q^\ast\) can be realised as a summand in the product \(D_+ D_-\) in exactly \((q-1)\) different ways by choosing \(a_1\) freely in \(\mathbb{F}_q^\ast\) and by setting

\[
b_1 = \frac{\beta}{a_1\delta}, \quad a_2 = \frac{1}{a_1\delta}, \quad b_2 = a_1\gamma.
\]

This shows \(D_+ D_- = (q-1)(C + E_-)\). The products

\[
E^2_+, D_- D_+, E^2_-, E_+ D_+, D_+ E_-, D_- E_+, E_+ D_-
\]

are similar.

In order to compute \(D_+ C\), we consider \((D_+ + A)(C + D_- + E_+)\). Since

\[
\begin{pmatrix} a_1 & b_1 \\ 0 & 1/a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & (1 + b_2c_2)/a_2 \end{pmatrix}
\]

4
\[
\begin{pmatrix}
  a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 (1 + b_2 c_2)/a_2 \\
  c_2/a_1 & (1 + b_2 c_2)/(a_1 a_2)
\end{pmatrix},
\]

every unimodular matrix \(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\) with \(\gamma \in \mathbb{F}_q^*\) can be realised in exactly \((q - 1)^2\) ways as a summand in \((D_+ + A)(C + D_- + E_+ + E_-)\) by choosing \(a_1, a_2\) freely in \(\mathbb{F}_q^*\) and by setting
\[
b_1 = \frac{\alpha - a_1 a_2}{a_1 \gamma}, b_2 = \frac{a_1 a_2 \delta - 1}{a_1 \gamma}, c_2 = a_1 \gamma.
\]

This shows \((D_+ + A)(C + D_- + E_+ + E_-) = (q - 1)^2 (B + C + D_+ + E_+ + E_-)\). We get thus
\[
D_+ C = (D_+ + A)(C + D_- + E_+) - A(C + D_- + E_+) - D_+ D_- - D_+ E_+
\]
\[
= (q - 1)^2 (B + C + D_- + E_+) + (q - 1)(C + D_- + E_+)
\]
\[
- (q - 1)(C + E_-) - (q - 1)^2 B - (q - 1)(q - 2) E_+
\]
\[
= (q - 1)(q - 3) C + (q - 1)(q - 2)(D_- + E_-).
\]

The products
\[
CD_+, CE_-, CD_+, CE_+, E_+ C, D_- C, E_- C
\]
are similar.

Using all previous products, the formula for \(C^2\) can now be recovered from \((A + B + C + D_+ + D_- + E_+ + E_-)^2 = (q^3 - q)(A + B + C + D_+ + D_- + E_+ + E_-)\). We have indeed
\[
C^2 = (A + B + C + D_+ + D_- + E_+ + E_-)^2 - \sum_{(X,Y) \neq (C,C)} XY
\]
where the sum is over all elements of \(\{A, B, C, D_+, D_-, E_+, E_-\}^2 \setminus \{C, C\}\). All products of the right-hand-side are known and determine thus \(C^2\). Equivalently, structure-constants of \(C^2\) have to be polynomials of degree at most 3 in \(q\). They can thus also be computed by interpolating the coefficients in 4 explicit examples. (Using divisibility by \(q - 1\), computing 3 examples is in fact enough.)

The existence of these formulae proves Theorem 2.1. \(\square\)
2.2 Matrices for left-multiplication by generators

Left-multiplications by generators with respect to the basis $A, B, C, D_+, D_-, E_+, E_-$ of $SC$ are encoded by the matrices

$$M_B = (q - 1) \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$M_C = (q - 1) \begin{pmatrix} 0 & 0 & (q - 1)(q - 2) & 0 & 0 & 0 & 0 \\ 0 & 0 & (q - 1)(q - 2) & 0 & 0 & 0 & 0 \\ 1 & 1 & (q - 3)(q - 4) & q - 3 & q - 3 & q - 3 & q - 3 \\ 0 & 0 & (q - 2)(q - 3) & 0 & q - 2 & q - 2 & 0 \\ 0 & 0 & (q - 2)(q - 3) & q - 2 & 0 & 0 & q - 2 \\ 0 & 0 & (q - 2)(q - 3) & 0 & q - 2 & q - 2 & 0 \end{pmatrix}$$

$$M_{D_+} = (q - 1) \begin{pmatrix} 0 & 0 & 0 & q - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q - 1 & 0 \\ 0 & 0 & q - 3 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & q - 2 & 0 & 0 & 0 \\ 0 & 0 & q - 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & q - 2 & 0 \\ 0 & 0 & q - 2 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$M_{E_+} = (q - 1) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & q - 1 \\ 0 & 0 & 0 & q - 1 & 0 & 0 & 0 \\ 0 & 0 & q - 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & q - 2 \\ 0 & 0 & q - 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & q - 2 & 0 & 0 \\ 0 & 0 & q - 2 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The remaining matrices are given by $M_A = \frac{1}{q - 1}(M_B)^2$, $M_{D_-} = \alpha M_{D_+} \alpha$ and $M_{E_-} = \alpha M_{E_+} \alpha$ where

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$
is the matrix corresponding to the automorphism $\sigma \circ \tau$.

The map $\{A, B, C, D_+, D_-, E_+, E_-\} \ni X \mapsto M_X$ extends of course to an isomorphism between $SC$ and

$$ZM_A + ZM_B + ZM_C + ZM_{D_+} + ZM_{D_-} + ZM_{E_+} + ZM_{E_-}$$

which is a ring. Computations are easier and faster in this matrix-ring than in the subring $SC$ of $Z[SL_2(F_q)]$.

3 Algebraic properties of $SC_\mathbb{Q}$

3.1 $SC_\mathbb{Q}$ as a semisimple algebra

Theorem 3.1 The algebra $SC_\mathbb{Q} = SC \otimes_\mathbb{Z} \mathbb{Q}$ is a semi-simple algebra isomorphic to $\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q})$.

The structure of $SC_\mathbb{K} = SC \otimes_\mathbb{Z} \mathbb{K}$ is of course easy to deduce for any field of characteristic 0.

The algebra $SC_\mathbb{K}$ is also semi-simple (and has the same structure) over most finite fields.

Theorem 3.1 is an easy consequence of the following computations:

The center of $SC_\mathbb{Q}$ has rank 4. It is spanned by the 3 minimal central idempotents $\pi_1$, $\pi_2$, $\pi_3$ and by the central idempotent $\pi_4$ which is non-minimal among all idempotents. The three idempotents $\pi_1$, $\pi_2$, $\pi_3$ induce three different characters (homomorphisms from $SC_\mathbb{Q}$ into $\mathbb{Q}$). Identifying $1 \in \mathbb{Q}$ with $\pi_i$ in each case, the three homomorphisms are given by

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D_+$</th>
<th>$E_+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_1$</td>
<td>$q-1$</td>
<td>$q-1$</td>
<td>$(q-1)^2(q-2)$</td>
<td>$(q-1)^2$</td>
<td>$(q-1)^2$</td>
</tr>
<tr>
<td>$\pi_2$</td>
<td>$q-1$</td>
<td>$q-1$</td>
<td>$2(q-1)$</td>
<td>$1-q$</td>
<td>$1-q$</td>
</tr>
<tr>
<td>$\pi_3$</td>
<td>$q-1$</td>
<td>$1-q$</td>
<td>$0$</td>
<td>$1-q$</td>
<td>$1-q$</td>
</tr>
</tbody>
</table>

The idempotent $\pi_4$ is of course simply the augmentation map counting the number of matrices involved in each generator.
The idempotent \( \pi_1 + \pi_2 + \pi_3 + \pi_4 = \frac{1}{q-1}A \) is the identity of \( SC_Q \).

The idempotent \( \pi_4 \) projects \( SC_Q \) homomorphically onto a matrix-algebra of \( 2 \times 2 \) matrices.

\( \pi_4 \) can be written (not uniquely) as a sum of two minimal non-central idempotents. We have for example \( \pi_4 = M_{1,1} + M_{2,2} \) where

\[
M_{1,1} = \frac{1}{q^2-1}(A-B) + \frac{1}{2(q^2-1)}(D_+ + D_- - E_+ - E_-),
\]

\[
M_{2,2} = \frac{1}{q^2-1}(A+B) - \frac{2}{(q+1)(q-1)^2}C + \frac{q-3}{2(q+1)(q-1)^2}(D_+ + D_- + E_+ + E_-).
\]

Considering also

\[
M_{1,2} = \frac{1}{2(q-1)^2}(D_+ - D_- + E_+ - E_-),
\]

\[
M_{2,1} = \frac{1}{2(q^2-1)}(D_+ - D_- - E_+ + E_-),
\]

the elements \( M_{i,j} \) behave like matrix-units and the map

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto aM_{1,1} + bM_{1,2} + cM_{2,1} + dM_{2,2}
\]

defines thus an isomorphism from the ring of integral \( 2 \times 2 \) matrices into \( SC_Q = SC \otimes \mathbb{Z} \mathbb{Q} \). If \( F_q \) has odd characteristic, the elements \( M_{i,j} \) can be realised in \( SC_{F_q} = SC \otimes \mathbb{Z} F_q \). In particular, Formula (1) gives an “exotic”, non-unital embedding of \( SL_2(F_q) \) into the group-algebra \( F_q[SL_2(F_q)] \) (in fact, (1) gives an embedding of \( SL_2(F_r) \) into \( F_r[SL_2(F_q)] \) whenever the prime power \( r \) is coprime to \( 2(q^2-1) \)).

The nice non-central idempotent

\[
\pi_3 + M_{1,1} = \frac{1}{2(q-1)}(A-B)
\]

projects \( SC_Q \) onto the eigenspace of eigenvalue \( 1-q \) of the map \( X \mapsto BX \).

The projection of \( SC_Q \) onto the eigenspace of eigenvalue \( q-1 \) of the map \( X \mapsto BX \) is similarly given by

\[
\pi_1 + \pi_2 + M_{2,2} = \frac{1}{2(q-1)}(A+B).
\]
3.2 A few commutative subalgebras of $SC_Q$

The previous section shows that the dimension of a commutative subalgebra of $SC_Q$ cannot exceed 5.

The center of $SC_Q$ is of course of rank 4 and spanned by the four minimal central idempotents $\pi_1, \ldots, \pi_4$ of the previous Section.

Splitting $\pi_4 = M_{1,1} + M_{2,2}$, we get a maximal commutative subalgebra of rank 5 by considering the vector space spanned by the minimal idempotents $\pi_1, \pi_2, \pi_3, M_{1,1}, M_{2,2}$. Equivalently, this vector space is spanned by $A, B, C, D = D_+ + D_-, E = E_+ + E_-$, as shown by the formulae for $\pi_i$ and $M_{1,1}, M_{2,2}$.

In terms of $A, B, C, D, E$, minimal idempotents $\pi_1, \pi_2, \pi_3, M_{1,1}, M_{2,2}$ are given by

$$
\begin{align*}
\pi_1 &= \frac{1}{q^3 - q}(A + B + C + D + E), \\
\pi_2 &= \frac{q - 2}{2(q^2 - q)}(A + B) + \frac{1}{q(q - 1)^2}C - \frac{q - 2}{2q(q - 1)^2}(D + E), \\
\pi_3 &= \frac{1}{2(q + 1)}(A - B) + \frac{1}{2(q^2 - 1)}(-D + E), \\
M_{1,1} &= \frac{1}{q^2 - 1}(A - B) + \frac{1}{2(q^2 - 1)}(D - E), \\
M_{2,2} &= \frac{1}{q^2 - 1}(A + B) - \frac{2}{(q + 1)(q - 1)^2}C + \\
&\quad + \frac{q - 3}{2(q + 1)(q - 1)^2}(D + E).
\end{align*}
$$

They define five characters given by $\mathbb{Q}A + \cdots + \mathbb{Q}E \rightarrow \mathbb{Q}$ given by

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<thead>
<tr>
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<td>$0$</td>
<td>$2(q - 1)$</td>
<td>$2(q - 1)$</td>
</tr>
<tr>
<td>$M_{1,1}$</td>
<td>$q - 1$</td>
<td>$1 - q$</td>
<td>$0$</td>
<td>$(q - 1)^2$</td>
<td>$-(q - 1)^2$</td>
</tr>
<tr>
<td>$M_{2,2}$</td>
<td>$q - 1$</td>
<td>$q - 1$</td>
<td>$2(1 - q)(q - 2)$</td>
<td>$(q - 1)(q - 3)$</td>
<td>$(q - 1)(q - 3)$</td>
</tr>
</tbody>
</table>

Moreover, $A, B, C, F = D + E$ span a commutative 4-dimensional sub-
algebra. Minimal idempotents are

\[
\begin{align*}
\pi_1 &= \frac{1}{q^3 - q} (A + B + C + F), \\
\pi_2 &= \frac{q - 2}{2(q^2 - q)} (A + B) + \frac{1}{q(q - 1)^2} C - \frac{q - 2}{2q(q - 1)^2} F, \\
\pi_3 + M_{1,1} &= \frac{1}{2(q - 1)} (A - B), \\
M_{2,2} &= \frac{1}{q^2 - 1} (A + B) - \frac{2}{(q + 1)(q - 1)^2} C + \frac{q - 3}{2(q + 1)(q - 1)^2} F.
\end{align*}
\]

with character-table

<table>
<thead>
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<th>F</th>
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<tbody>
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<td>(q - 1)</td>
<td>(q - 1)</td>
<td>((q - 1)^2(q - 2))</td>
<td>(4(q - 1)^2)</td>
</tr>
<tr>
<td>(\pi_2)</td>
<td>(q - 1)</td>
<td>(q - 1)</td>
<td>(2(q - 1))</td>
<td>(4(1 - q))</td>
</tr>
<tr>
<td>(\pi_3 + M_{1,1})</td>
<td>(q - 1)</td>
<td>(q - 1)</td>
<td>(1 - q)</td>
<td>0</td>
</tr>
<tr>
<td>(M_{2,2})</td>
<td>(q - 1)</td>
<td>(q - 1)</td>
<td>(2(1 - q)(q - 2))</td>
<td>(2(q - 1)(q - 3))</td>
</tr>
</tbody>
</table>

\(I = A + B, C, F = E + D\) span a commutative 3–dimensional subalgebra of \(SC_2\). Generators of this last algebra are sums of elements in \(SL_2(\mathbb{F}_q)\) with supports of given cardinality. \(I\) is the sum of all elements with two non-zero coefficients, \(C\) contains all elements having only non-zero coefficients and \(F\) contains all elements with three non-zero coefficients.

Products are given by \(IX = XI = 2(q - 1)X\) for \(X \in \{I, C, F\}\) and

\[
\begin{align*}
C^2 &= (q - 1)^2(q - 2)I + (q - 1)(q - 3)(q - 4)C + (q - 1)(q - 2)(q - 3)F, \\
CF &= FC = 4(q - 1)(q - 3)C + 2(q - 1)(q - 2)F, \\
F^2 &= 4(q - 1)^2I + 8(q - 1)C + 2(q - 1)^2F.
\end{align*}
\]

Idempotents are given by

\[
\begin{align*}
\pi_1 &= \frac{1}{q^3 - q} (I + C + F), \\
\pi_2 &= \frac{q - 2}{2(q^2 - q)} I + \frac{1}{q(q - 1)^2} C - \frac{q - 2}{2q(q - 1)^2} F, \\
M_{2,2} &= \frac{1}{q^2 - 1} I - \frac{2}{(q + 1)(q - 1)^2} C + \frac{q - 3}{2(q + 1)(q - 1)^2} F.
\end{align*}
\]
with character-table

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>C</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>π₁</td>
<td>2(q - 1)</td>
<td>(q - 1)^2(q - 2)</td>
<td>4(q - 1)^2</td>
</tr>
<tr>
<td>π₂</td>
<td>2(q - 1)</td>
<td>2(q - 1)</td>
<td>4(1 - q)</td>
</tr>
<tr>
<td>M₂,₂</td>
<td>2(q - 1)</td>
<td>2(1 - q)(q - 2)</td>
<td>2(q - 1)(q - 3)</td>
</tr>
</tbody>
</table>

Working over \( \mathbb{C} \) (or over a suitable extension of \( \mathbb{Q} \)) and setting

\[
\tilde{I} = \frac{1}{2(q - 1)} I, \tilde{C} = \frac{1}{\sqrt{2(q - 1)^4(q - 2)}} C, \tilde{F} = \frac{1}{\sqrt{8(q - 1)^3}} F
\]

we get products \( XY = \sum_{Z \in \{ \tilde{I}, \tilde{C}, \tilde{F} \}} N_{X,Y,Z} Z \) defined by symmetric structure-constants \( N_{X,Y,Z} = N_{Y,X,Z} = N_{X,Z,Y} \) for all \( X, Y, Z \in \{ I, C, F \} \). Up to symmetric permutations, the structure-constants are given by

\[
\begin{align*}
N_{I,X,Y} &= \delta_{X,Y} \\
N_{C,C,C} &= \frac{(q - 3)(q - 4)}{\sqrt{2(q - 1)(q - 2)}} \\
N_{C,C,F} &= (q - 3)\sqrt{\frac{2}{q - 1}} \\
N_{C,F,F} &= 2(q - 2) \sqrt{\frac{q - 1}{q}} \\
N_{F,F,F} &= \sqrt{\frac{q - 1}{2}}
\end{align*}
\]

where \( \delta_{X,Y} = 1 \) if and only if \( X = Y \) and \( \delta_{X,Y} = 0 \) otherwise. The evaluation at \( q = 3 \) leads to particularly nice structure constants with values in \( \{ 0, 1 \} \).

Algebras with generating systems having symmetric structure-constants and a character taking real positive values on generators (satisfied by \( \pi_1 \) for \( q > 2 \)) are sometimes called algebraic fusion-algebras, see for example [2].

### 4 Association schemes and Bose-Mesner algebras

An association scheme is a set of \( d + 1 \) square matrices \( C_0, \ldots, C_d \) with coefficients in \( \{ 0, 1 \} \) such that \( C_0 \) is the identity-matrix, \( C_0 + \cdots + C_d \) is the all-one matrix and \( \mathbb{Z}C_0 + \cdots + \mathbb{Z}C_d \) is a commutative ring with (necessarily integral) structure constants \( p_{i,j}^k = p_{j,i}^k \) defined by \( C_i C_j = C_j C_i = \sum_{k=0}^d p_{i,j}^k C_k \). An association scheme is symmetric if \( C_1, \ldots, C_d \) are symmetric matrices. The algebra (over a field) generated by the elements \( C_i \) is called a Bose-Mesner algebra. See for example the monograph [1] for additional information.

Identifying an element \( g \) of \( \text{SL}_2(\mathbb{F}_q) \) with the permutation-matrix associated to left-multiplication by \( g \) we get a commutative association scheme
with $d = 5$ if $q \geq 4$ by setting
\[ C_0 = \text{Id}, \ C_1 = A - \text{Id}, \ C_2 = B, \ C_3 = C, \ C_4 = D_+ + D_-, \ C_5 = E_+ + E_- \]

where we consider sums of permutation-matrices. All matrices are symmetric. Products with $C_0, C_1$ are given by $C_0X = XC_0 = X, C_1^2 = (q - 2)C_0 + (q - 3)C_1, C_1Y = YC_1 = (q - 2)Y$ for $X \in \{C_0, \ldots, C_5\}$ and $Y \in \{C_2, \ldots, C_5\}$.

The remaining products are given by
\[
\begin{align*}
C_2^2 &= (q - 1)(C_0 + C_1), \\
C_2C_3 &= C_3C_2 = (q - 1)C_3 \\
C_2C_4 &= C_4C_2 = (q - 1)C_5 \\
C_2C_5 &= C_5C_2 = (q - 1)C_4 \\
C_3^2 &= (q - 1)^2(q - 2)(C_0 + C_1 + C_2) + (q - 1)(q - 3)(q - 4)(C_3) \\
&\quad + (q - 1)(q - 2)(q - 3)(C_4 + C_5) \\
C_3C_4 &= C_4C_3 = 2(q - 1)(q - 3)C_3 + (q - 1)(q - 2)(C_4 + C_5) \\
C_3C_5 &= C_5C_3 = 2(q - 1)(q - 3)C_3 + (q - 1)(q - 2)(C_4 + C_5) \\
C_4^2 &= 2(q - 1)^2(C_0 + C_1) + 2(q - 1)C_3 \\
&\quad + (q - 1)(q - 2)C_4 + (q - 1)C_5 \\
C_4C_5 &= C_5C_4 = 2(q - 1)^2C_2 + 2(q - 1)C_3 \\
&\quad + (q - 1)C_4 + (q - 1)(q - 2)C_5 \\
C_5^2 &= 2(q - 1)^2(C_0 + C_1) + 2(q - 1)C_3 \\
&\quad + (q - 1)(q - 2)C_4 + (q - 1)C_5
\end{align*}
\]

The reader should be warned that $C_1$ behaves not exactly like $(q - 2)C_0$.

We leave it to the reader to write down matrices for multiplications with basis-elements and to compute the complete list of minimal idempotents.

Additional association schemes are given by $C_0, C_1, C_2, C_3, C_4 + C_5$ and $C_0, C_1 + C_2, C_3, C_4 + C_5$. It is also possible to split $C_1$ and/or $C_2$ according to subgroups of $F_q^*$ into several matrices (or classes, as they are sometimes called).

We discuss now with a little bit more details the smallest interesting association scheme with classes $C_0, C_1 = C_1 + C_2, C_2 = C_3, C_3 = C_4 + C_5$.

Products are given by $C_0X = XC_0$ and
\[
\begin{align*}
\tilde{C}_0^2 &= (2q - 3)\tilde{C}_0 + 2(q - 2)\tilde{C}_1 \\
\tilde{C}_1\tilde{C}_2 &= \tilde{C}_2\tilde{C}_1 = (2q - 3)\tilde{C}_2 \\
\tilde{C}_1\tilde{C}_3 &= \tilde{C}_3\tilde{C}_1 = (2q - 3)\tilde{C}_3 \\
\tilde{C}_2^2 &= (q - 1)^2(q - 2)(\tilde{C}_0 + \tilde{C}_1) + (q - 1)(q - 3)(q - 4)\tilde{C}_2 \\
&\quad + (q - 1)(q - 2)(q - 3)\tilde{C}_3 \\
\tilde{C}_2\tilde{C}_3 &= \tilde{C}_3\tilde{C}_2 = 4(q - 1)(q - 3)\tilde{C}_2 + 2(q - 1)(q - 2)\tilde{C}_3 \\
\tilde{C}_3^2 &= 4(q - 1)^2(\tilde{C}_0 + \tilde{C}_1) + 8(q - 1)\tilde{C}_2 + 2(q - 1)^2\tilde{C}_3
\end{align*}
\]
Matrices $M_0, \ldots, M_3$ corresponding to multiplication by $\tilde{C}_0, \ldots, \tilde{C}_3$ are given by

\[
\tilde{C}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{C}_1 = \begin{pmatrix} 0 & 2q-3 & 0 & 0 \\ 1 & 2(q-2) & 0 & 0 \\ 0 & 0 & 2q-3 & 0 \\ 0 & 0 & 0 & 2q-3 \end{pmatrix}
\]

\[
\tilde{C}_2 = \begin{pmatrix} 0 & 0 & (q-1)^2(q-2) & 0 \\ 0 & 0 & (q-1)^2(q-2) & 0 \\ 1 & 2q-3 & (q-1)(q-3)(q-4) & 4(q-1)(q-3) \\ 0 & 0 & (q-1)(q-2)(q-3) & 2(q-1)(q-2) \end{pmatrix},
\]

\[
\tilde{C}_3 = \begin{pmatrix} 0 & 0 & 0 & 4(q-1)^2 \\ 0 & 0 & 0 & 4(q-1)^2 \\ 0 & 0 & 4(q-1)(q-3) & 8(q-1) \\ 1 & 2q-3 & 2(q-1)(q-2) & 2(q-1)^2 \end{pmatrix}
\]

and minimal idempotents are given by

\[
\beta_0 = \frac{1}{q^3-q}(\tilde{C}_0 + \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3),
\]

\[
\beta_1 = \frac{q-2}{2(q^2-q)}(\tilde{C}_0 + \tilde{C}_1) + \frac{1}{q(q-1)^2} \tilde{C}_2 - \frac{q-2}{2q(q-1)^2} \tilde{C}_3,
\]

\[
\beta_2 = \frac{2q-3}{2(q-1)} \tilde{C}_0 - \frac{1}{2(q-1)} \tilde{C}_1,
\]

\[
\beta_3 = \frac{1}{q^2-1}(\tilde{C}_0 + \tilde{C}_1) - \frac{2}{(q-1)^2(q+1)} \tilde{C}_2 + \frac{q-3}{2(q-1)^2(q+1)} \tilde{C}_3.
\]

The coefficient of $\tilde{C}_0$ multiplied by $(q^3-q)$ gives the dimension of the associated eigenspace. Eigenvalues of (with multiplicities) of generators, obtained by evaluating the characters $\beta_0, \ldots, \beta_4$ on $\tilde{C}_0, \ldots, \tilde{C}_3$, are given by

\[
\begin{array}{c|c|c|c|c}
\beta & \text{dim} & \tilde{C}_0 & \tilde{C}_1 & \tilde{C}_2 \\
\hline
\beta_0 & \frac{1}{q(q-1)(q+2)} & 1 & 2q-3 & (q-1)^2(q-2) & 4(q-1)^2 \\
\beta_1 & \frac{2(q-3)(q+1)}{q} & 1 & 2q-3 & 2(q-1) & 4(1-q) \\
\beta_2 & \frac{2}{q} & 1 & -1 & 0 & 0 \\
\beta_3 & \frac{1}{q} & 1 & 2q-3 & 2(1-q)(q-2) & 2(q-1)(q-3) \\
\end{array}
\]

**Remark 4.1** There exists a few more exotic variation of this construction. An example is given by partitioning the elements of $\text{SL}_2(\mathbb{F}_5)$ according to the 30 possible values of the Legendre symbol $(\frac{\cdot}{5})$ on entries. Since $-1$ is a square modulo 5, these classes are well-defined on $\text{PSL}_2(\mathbb{F}_5)$ which is isomorphic to the simple group $A_5$. Details (and a few similar examples) will hopefully appear in a future paper.
5 Representation-theoretic aspects

In this Section, we work over \( \mathbb{C} \) for the sake of simplicity.

5.1 Traces

Left-multiplication by the identity \( \frac{1}{q-1} A \) of \( \mathcal{S}\mathcal{C}_C \) defines an idempotent on \( \mathbb{Q}[\mathbb{SL}_2(\mathbb{F}_q)] \) whose trace is the dimension \( \frac{1}{q-1}(q^3-q) = q(q+1) \) of the non-trivial eigenspace. Indeed, every non-trivial element of \( \mathbb{SL}_2(\mathbb{F}_q) \) has trace 0 and the identity-matrix has trace \( q^3-q \) since it fixes all \( q^3-q \) elements of \( \mathbb{SL}_2(\mathbb{F}_q) \). A basis of the non-trivial \( q(q+1) \)-dimensional eigenspace of \( \frac{1}{q-1} A \) is given by sums over all matrices with rows representing two distinct fixed elements of the projective line over \( \mathbb{F}_q \).

The traces \( \text{tr}(\pi_1), \ldots, \text{tr}(\pi_4) \) of the four minimal central projectors \( \pi_0, \ldots, \pi_4 \) of \( \mathcal{S}\mathcal{C}_C \) are equal to \( q^3-q \) times the coefficient of \( A \) in \( \pi_i \). They are thus given by

\[
\begin{array}{cccc}
\text{trace} & \pi_1 & \pi_2 & \pi_3 & \pi_4 \\
1 & \frac{(q+1)(q-2)}{2} & \frac{q(q-1)}{2} & 2q \\
\end{array}
\]

and we have

\[ q(q+1) = \text{tr}(\pi_1) + \text{tr}(\pi_2) + \text{tr}(\pi_3) + \text{tr}(\pi_4), \]

as expected.

5.2 Characters

Since simple matrix-algebras of \( \mathbb{C}[\mathbb{SL}_2(\mathbb{F}_q)] \) are indexed by characters of \( \mathbb{C}[\mathbb{SL}_2(\mathbb{F}_q)] \), it is perhaps interesting to understand all irreducible characters involved in idempotents of \( \mathcal{S}\mathcal{C}_C \subset \mathbb{C}[\mathbb{SL}_2(\mathbb{F}_q)] \).

The algebra \( \mathcal{S}\mathcal{C}_C \) is in some sense almost “orthogonal” to the center of \( \mathbb{C}[\mathbb{SL}_2(\mathbb{F}_q)] \). The algebra \( \mathcal{S}\mathcal{C}_C \) should thus involve many different irreducible characters of \( \mathbb{SL}_2(\mathbb{F}_q) \). We will see that this is indeed the case.

We decompose first the identity \( \frac{1}{q-1} A \) according to irreducible characters of \( \mathbb{C}[\mathbb{SL}_2(\mathbb{F}_q)] \). We refine this decomposition to the minimal central idempotents \( \pi_1, \ldots, \pi_4 \) of \( \mathcal{S}\mathcal{C}_C \) in Section 5.4.

For simplicity we work over \( \mathbb{GL}_2(\mathbb{F}_q) \) which has essentially the same character-theory as \( \mathbb{SL}_2(\mathbb{F}_q) \). We work over \( \mathbb{C} \) and we identify (irreducible) characters with the corresponding (irreducible) representations.

In order to do this, we introduce \( F = \sum_{\lambda \in \mathbb{F}_q^*} \left( \begin{array}{cc} 1 & 0 \\ 0 & \lambda \end{array} \right) \in \mathbb{Z}[\mathbb{GL}_2(\mathbb{F}_q)] \). We have \( F^2 = (q-1)F \) and \( FX = XF \) for \( X \in \{ A, B, C, D_\pm, E_\pm \} \), considered as an element of \( \mathbb{Z}[\mathbb{GL}_2(\mathbb{F}_q)] \). The map \( X \mapsto \frac{1}{q-1}FX \) preserves traces and defines an injective homomorphism of \( \mathcal{S}\mathcal{C} \) into \( \mathbb{Q}[\mathbb{GL}_2(\mathbb{F}_q)] \).
We use the conventions of Chapter 5 of [3] for conjugacy classes of $GL_2(F_q)$. More precisely, we denote by $a_x$ conjugacy classes of central diagonal matrices with common diagonal value $x \in F_q^*$, by $b_x$ conjugacy classes given by multiplying unipotent matrices by a scalar $x \in F_q^*$, by $c_{x,y}$ conjugacy classes with two distinct eigenvalues $x, y \in F_q^*$ and by $d_\xi$ conjugacy classes with two conjugate eigenvalues $\xi, \xi^q \in F_{q^2} \setminus F_q$. The number of conjugacy classes of each type is given by

$$\frac{a_x}{q-1} \quad \frac{b_x}{q-1} \quad \frac{c_{x,y}}{(q-1)(q-2)} \quad \frac{d_\xi}{q(q-1)}.$$ 

The character table of $GL_2(F_q)$, copied from [3], is now given by

<table>
<thead>
<tr>
<th></th>
<th>$1$</th>
<th>$q^2-1$</th>
<th>$q^2+q$</th>
<th>$q^2-q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_\alpha$</td>
<td>$\alpha(x^2)$</td>
<td>$\alpha(x^2)$</td>
<td>$\alpha(xy)$</td>
<td>$\alpha(\xi^{q+1})$</td>
</tr>
<tr>
<td>$V_\alpha$</td>
<td>$q\alpha(x^2)$</td>
<td>$0$</td>
<td>$\alpha(xy)$</td>
<td>$-\alpha(\xi^{q+1})$</td>
</tr>
<tr>
<td>$W_{\alpha,\beta}$</td>
<td>$(q+1)\alpha(x)\beta(x)$</td>
<td>$\alpha(x)\beta(x)$</td>
<td>$\alpha(x)\beta(y) + \alpha(y)\beta(x)$</td>
<td>$0$</td>
</tr>
<tr>
<td>$X_\varphi$</td>
<td>$(q-1)\varphi(x)$</td>
<td>$-\varphi(x)$</td>
<td>$0$</td>
<td>$-(\varphi(\xi) + \varphi(\xi^q))$</td>
</tr>
</tbody>
</table>

where $\alpha, \beta$ are distinct characters of $F_q^*$ and where $\varphi$ is a character of $F_{q^2}^*$ with $\varphi^{q-1}$ non-trivial. The first row indicates the number of elements in a conjugacy class indicated by the second row. The remaining rows give the character-table. $U_\alpha$ are the one-dimensional representations factoring through the determinant. $V_\alpha = V \otimes U_\alpha$ are obtained from the permutation-representation $V = V_1 + U_1$ describing the permutation-action of $GL_2(F_q)$ on all $p+1$ points of the projective line over $F_q$. $W_{\alpha,\beta}$ (isomorphic to $W_{\beta,\alpha}$) are induced from non-trivial 1-dimensional representations of a Borel subgroup (given for example by all upper triangular matrices). $X_\varphi$ (isomorphic to $X_{\varphi^q}$) are the remaining irreducible representations $V_1 \otimes W_{\varphi|_{F_q},1} - W_{\varphi^q|_{F_q},1} - \text{Ind}_\varphi$ with $\text{Ind}_\varphi$ obtained by inducing a 1-dimensional representation $\varphi \neq \varphi^q$ of a cyclic subgroup isomorphic to $F_{q^2}^*$.

The trace of the idempotent $\pi = \frac{1}{(q-1)} F A$ in irreducible representations
of \( \text{GL}_2(\mathbb{F}_q) \) is now given by:

\[
U_\alpha : \frac{1}{(q-1)^2} \left( \sum_{x \in \mathbb{F}_q^*} \alpha(x) \right)^2 \\
V_\alpha : q \left( \frac{1}{q-1} \left( \sum_{x \in \mathbb{F}_q^*} \alpha(x^2) + \frac{1}{(q-1)^2} \left( \sum_{x \in \mathbb{F}_q^*} \alpha(x) \right)^2 \right) \right) \\
W_{\alpha,\beta} : (q+1) \left( \frac{1}{q-1} \sum_{x \in \mathbb{F}_q^*} \alpha(x)\beta(x) + \frac{2}{(q-1)^2} \left( \sum_{x \in \mathbb{F}_q^*} \alpha(x) \right) \left( \sum_{x \in \mathbb{F}_q^*} \beta(x) \right) \right) \\
X_\varphi : (q-1) \sum_{x \in \mathbb{F}_q^*} \varphi(x)
\]

The factors \( q, q \pm 1 \) are the multiplicities of the irreducible representations \( V_\alpha, W_{\alpha,\beta} \) and \( X_\varphi \) in the regular (left or right) representation. They are of course equal to the dimensions of \( V_\alpha, W_{\alpha,\beta} \) and \( X_\varphi \).

Irreducible representations \( U_\alpha \) are involved in \( \pi \) only if \( \alpha \) is the trivial character. This corresponds of course to the central idempotent \( \pi_1 \) of \( S\mathbb{C}_Q \).

For \( V_\alpha \) we get \( 2q \) for \( \alpha \) trivial, \( q \) for odd \( q \) if \( \alpha \) is the quadratic character (defined by the Legendre-symbol and existing only for odd \( q \)) and 0 otherwise.

For \( W_{\alpha,\beta} \) we get \( q+1 \) if \( \beta = \overline{\alpha}, \beta \neq \alpha \) and 0 otherwise. There are \( \frac{q-3}{2} \) such representations for odd \( q \) and \( \frac{q-2}{2} \) such representations for even \( q \).

For \( X_\varphi \) we get \( q-1 \) if the character \( \varphi \) of \( \mathbb{F}_q^* \) with non-trivial \( \varphi^{q-1} \) restricts to the trivial character of \( \mathbb{F}_q^* \) and 0 otherwise. Characters of \( \mathbb{F}_q^* \) with trivial restrictions to \( \mathbb{F}_q^* \) are in one-to-one correspondence with characters of the additive group \( \mathbb{Z}/(q+1)\mathbb{Z} \). The character \( \varphi^{q-1} \) is trivial for two of them (the trivial and the quadratic one) if \( q \) is odd. This gives thus \( \frac{q-1}{2} \) such irreducible representations for odd \( q \). For even \( q \) we get \( \frac{q}{2} \) irreducible representations.

All irreducible representations of \( \text{GL}_2(\mathbb{F}_q) \) involved in \( \pi = \frac{1}{(q-1)^2} FA \) have irreducible restrictions to \( \text{SL}_2(\mathbb{F}_q) \).

The following table sums up contributions of all different irreducible characters to the trace of \( \pi = \frac{1}{(q-1)^2} FA \):

<table>
<thead>
<tr>
<th>( q )</th>
<th>( U )</th>
<th>( V )</th>
<th>( W )</th>
<th>( X )</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd</td>
<td>1</td>
<td>3q</td>
<td>( (q+1)^{\frac{q-2}{2}} )</td>
<td>( (q-1)^{\frac{q-1}{2}} )</td>
</tr>
<tr>
<td>even</td>
<td>1</td>
<td>2q</td>
<td>( (q+1)^{\frac{q-2}{2}} )</td>
<td>( (q-1)^{\frac{q}{2}} )</td>
</tr>
</tbody>
</table>

and we have

\[
\text{tr}(\pi) = q(q+1) = 1 + 3q + (q+1)\frac{q-3}{2} + (q-1)\frac{q-1}{2},
\]

16
respectively
\[ \text{tr}(\pi) = q(q + 1) = 1 + 2q + (q + 1)\frac{q-2}{2} + (q - 1)\frac{q}{2}, \]
as expected.

5.3 Decompositions of \(\pi_1, \ldots, \pi_4\) for even \(q\)

Over a finite field of characteristic 2, conjugacy classes are involved in generators of \(SC\) as follows:

<table>
<thead>
<tr>
<th></th>
<th>(q - 1)</th>
<th>(q - 1)</th>
<th>(\frac{(q-1)(q-2)}{2})</th>
<th>(\frac{q(q-1)}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_x)</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>(b_x)</td>
<td>0</td>
<td>(q - 1)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(c_{x,y\neq z})</td>
<td>((q - 1)(q - 2))</td>
<td>((q - 1)(q - 4))</td>
<td>((q - 1)(q - 2))</td>
<td>0</td>
</tr>
<tr>
<td>(d_\xi)</td>
<td>0</td>
<td>2((q - 1))</td>
<td>0</td>
<td>((q - 1))</td>
</tr>
</tbody>
</table>

This implies the following decompositions of central idempotents of \(SC\):
The idempotent \(\pi_1\) of rank 1 is involved with multiplicity 1 in the trivial representation of type \(U\). The idempotent \(\pi_2\) of rank \(\frac{(q+1)(q-2)}{2}\) (in \(C[SL_2(F_q)]\)) is involved with multiplicity \(q + 1\) in all \(\frac{q}{2}\) relevant characters of type \(W\). The idempotent \(\pi_3\) of rank \(\frac{q(q-1)}{2}\) is involved with multiplicity \(q - 1\) in all \(\frac{q}{2}\) relevant characters of type \(\hat{U}\). Finally, the idempotent \(\pi_4\) is involved with multiplicity \(q\) in the irreducible representation \(V\).

5.4 Decompositions of \(\pi_1, \ldots, \pi_4\) for odd \(q\)

Over a finite field of odd characteristic, contributions of the different conjugacy classes to the elements \(FX\) (for \(X\) a generator of \(SC\)) are given by

<table>
<thead>
<tr>
<th></th>
<th>(q - 1)</th>
<th>(q - 1)</th>
<th>(\frac{q-1}{2})</th>
<th>(\frac{(q-1)(q-3)}{2})</th>
<th>(\frac{q-1}{2})</th>
<th>(\frac{(q-1)^2}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_x)</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(b_x)</td>
<td>0</td>
<td>(q - 1)</td>
<td>0</td>
<td>(q - 1)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(c_{x,y\neq z})</td>
<td>((q - 1)(q - 3))</td>
<td>((q - 1)(q - 4))</td>
<td>((q - 1)^2)</td>
<td>((q - 1)(q - 2))</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(d_\xi, \text{tr}(\xi) = 0)</td>
<td>0</td>
<td>2((q - 1))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(d_\xi, \text{tr}(\xi) \neq 0)</td>
<td>0</td>
<td>((q - 1)(q - 3))</td>
<td>((q - 1)(q - 4))</td>
<td>((q - 1)^2)</td>
<td>((q - 1)(q - 2))</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ F A \]
\[ F B \]
\[ F C \]
\[ F D_\pm \]
\[ F E_\pm \]

\[ 1 \]

\[ q^2 - 1 \]

\[ q^2 + q \]

\[ q^2 - q \]
and conjugacy classes of $\text{GL}_2(\mathbb{F}_q)$ are involved in projectors $\tilde{\pi}_1 = \frac{1}{q-1} \pi_1 F, \ldots, \tilde{\pi}_4 = \frac{1}{q-1} \pi_4 F$ with the following coefficients

<table>
<thead>
<tr>
<th>$q-1$</th>
<th>$q-1$</th>
<th>$q-1$</th>
<th>$q-1$</th>
<th>$q-1$</th>
<th>$q-1$</th>
<th>$q-1$</th>
<th>$q-1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_x$</td>
<td>$b_x$</td>
<td>$c_{x,-x}$</td>
<td>$c_{x,y \neq \pm x}$</td>
<td>$d_{\xi, \text{tr}(\xi) = 0}$</td>
<td>$d_{\xi, \text{tr}(\xi) \neq 0}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{\pi}_1$</td>
<td>$\frac{(q-1)(q+1)}{q-2}$</td>
<td>$\frac{q(q-1)}{q-2}$</td>
<td>$\frac{(q-1)^2}{2}$</td>
<td>$\frac{q(q-1)}{2}$</td>
<td>$\frac{q-1}{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{\pi}_2$</td>
<td>$\frac{2(q-1)^2}{2}$</td>
<td>$\frac{q(q-1)}{2}$</td>
<td>$\frac{q-1}{2}$</td>
<td>$\frac{q(q-1)}{2}$</td>
<td>$\frac{1}{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{\pi}_3$</td>
<td>$\frac{2(q-1)^2}{2}$</td>
<td>$\frac{2(q-1)}{2}$</td>
<td>$\frac{q-1}{2}$</td>
<td>$\frac{2(q-1)}{2}$</td>
<td>$\frac{q-1}{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{\pi}_4$</td>
<td>$\frac{(q-1)(q+1)}{2}$</td>
<td>$\frac{q(q-1)}{2}$</td>
<td>$\frac{(q-1)^2}{2}$</td>
<td>$\frac{q(q-1)}{2}$</td>
<td>$\frac{1}{2}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The idempotent $\tilde{\pi}_1$ (or equivalently the idempotent $\pi_1$ of $\text{SC}$) is only involved in the trivial representation with multiplicity 1. The idempotent $\tilde{\pi}_3$ appears (with multiplicity $q$) only in the unique non-trivial $q$-dimensional irreducible representation $V$ involved in the permutation-representation of $\text{GL}_2(\mathbb{F}_q)$ acting on all $q+1$ points of the projective line over $\mathbb{F}_q$.

The character $V_L = V \otimes \alpha_L$ where $\alpha_L$ is the quadratic character of $\mathbb{F}_q$ given by the Legendre-symbol has mean-values given by

$$
\begin{array}{c|cc|cccc}
q-1 & q-1 & q-1 & (q-1)(q-3)/2 & (q-1)/2 & (q-1)/2 & (q-3)/2 \\
a_x & b_x & c_{x,-x} & c_{x,y \neq \pm x} & d_{\xi, \text{tr}(\xi) = 0} & d_{\xi, \text{tr}(\xi) \neq 0} & \\
V_L & q & 0 & \left(-\frac{1}{q}\right) & \frac{-1+\left(-\frac{1}{q}\right)}{q-3} & \left(-\frac{1}{q}\right) & \frac{-1+\left(-\frac{1}{q}\right)}{q-3} \\
\end{array}
$$
on conjugacy-classes. The character $V_L$ involves thus $\pi_2$ (or $\pi_2$) with multiplicity $q$ if $q \equiv 1 \pmod{4}$ and $\pi_3$ (or $\pi_3$) with multiplicity $q$ if $q \equiv 3 \pmod{4}$.

Mean-values of a character $W_{\alpha,\beta}$ with non-real $\beta = \overline{\alpha}$ on conjugacy classes are given by

$$
\begin{array}{c|cc|cccc}
q-1 & q-1 & q-1 & (q-1)(q-3)/2 & (q-1)/2 & (q-1)/2 & (q-3)/2 \\
a_x & b_x & c_{x,-x} & c_{x,y \neq \pm x} & d_{\xi, \text{tr}(\xi) = 0} & d_{\xi, \text{tr}(\xi) \neq 0} & \\
W_{\alpha,\beta} & q+1 & 1 & 2\alpha(-1) & -\frac{2(q-1)(q-3)}{q-3} & 0 & 0 \\
\end{array}
$$
They depend on the value $\alpha(-1) \in \{\pm 1\}$. For $q \equiv 1 \pmod{4}$ there are $\frac{q-5}{4}$ such characters $W_{\alpha,\beta}$ with $\alpha(-1) = 1$ and $\frac{q-1}{4}$ such characters with $\alpha(-1) = -1$. For $q \equiv 3 \pmod{4}$, there are the same number $\frac{q-3}{4}$ of such characters for both possible values of $\alpha(-1)$. Such representations involve $\pi_2$ (or $\pi_2$ of $\text{SC}$) with multiplicity $q+1$ if $\alpha(-1) = 1$ and $\tilde{\pi}_3$ with multiplicity $q+1$ otherwise.

We consider now a non-real character $\varphi$ of $\mathbb{F}_q^{\times}$ with trivial restriction to $\mathbb{F}_q^{\times}$. Since $\xi^2$ belongs to $\mathbb{F}_q^{\times}$ if $\xi$ has trace 0, we have $\sigma = \varphi(\xi) \in \{\pm 1\}$.

Mean-values on conjugacy-classes of a character $X_\varphi$ with $\varphi$ as above are
thus given by

\[
\begin{array}{ccccccc}
   q - 1 & q - 1 & \frac{q-1}{2} & \frac{(q-1)(q-3)}{2} & \frac{q-1}{2} & \frac{(q-1)^2}{2} \\
   a_x & b_x & c_{x, -x} & c_{x, y \neq \pm x} & d_{\xi, \text{tr}(\xi)=0} & d_{\xi, \text{tr}(\xi) \neq 0} \\
   X_\phi & q - 1 & -1 & 0 & 0 & -2\sigma & \frac{2(1+\sigma)}{q-1} \\
\end{array}
\]

For \( q \equiv 1 \pmod{4} \), there are \( \frac{q-1}{4} \) such characters for both possible values of \( \sigma \). For \( q \equiv 3 \pmod{4} \), the value \( \sigma = 1 \) is achieved by \( \frac{q-3}{4} \) such representations and the value \( \sigma = -1 \) by \( \frac{q+1}{4} \) representations.

Each such representation with \( \sigma = 1 \) involves \( \tilde{\pi}_3 \) (or \( \pi_3 \)) with multiplicity \( q - 1 \) and each such representation with \( \sigma = -1 \) involves \( \tilde{\pi}_2 \) (or \( \pi_2 \)) with multiplicity \( q - 1 \).

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References


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